Planar Graphs With Bounded Treewidth and Their Upward Stack Number

Master's Thesis of

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I declare that I have developed and written the enclosed thesis completely by myself. I have not used any other than the aids that I have mentioned. I have marked all parts of the thesis that I have included from referenced literature, either in their original wording or paraphrasing their contents. I have followed the by-laws to implement scientific integrity at KIT.

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(Samuel Schneider)

Abstract

A *stack layout* of a directed graph G is a topological ordering of the vertices $V(G)$ together with a partition of the edges $E(G)$ into *stacks* $E_1, \ldots, E_k$ such that no two edges in the same stack cross. The *stack number* of a graph G is the smallest integer $k \in \mathbb{N}$ such that there exists a stack layout of G that uses k stacks. One of the biggest open questions regarding the stack number is whether upward planar graphs have bounded stack number. We advance the state of the art by showing that upward planar 2-trees, i. e. directed edge-maximal graphs with treewidth 2 that have an upward orientation, have constant stack number.

The second part of this thesis is dedicated to showing that every planar graph with treewidth at most 4 is a subgraph of a *planar quasi*-4-*tree.* This confirms a conjecture by Förster [50] from 2024. Furthermore, we generalize the notion of planar quasi-4-trees to *planar quasi-k-trees* and show that every planar graph with treewidth at most k is a subgraph of a planar quasi-k-tree. To the best of our knowledge, this is the first characterization for planar graphs with treewidth larger than 3.

Finally, we show bounds on the stack number for several subclasses of upward planar quasi-4-trees.

Zusammenfassung

Ein *Stack Layout* eines gerichteten Graphen G ist eine topologische Sortierung der Knoten $V(G)$ zusammen mit einer Partition in *Stacks* $E_1, \ldots, E_k$, sodass sich keine zwei Kanten in dem selben Stack kreuzen. Die *Stack Number* eines Graphen G ist das kleinste $k \in \mathbb{N}$, für das ein Stack Layout mit k Stacks von G existiert. Eine der größten offenen Fragen bezüglich der Stack Number ist ob *upward* planare Graphen konstante Stack Number haben. Diese Arbeit macht Fortschritt in der Beantwortung dieser Frage, indem gezeigt wird, dass upward planare 2-*Bäume*, also kantenmaximale Graphen mit Baumweite 2 mit einer upward Orientierung, konstante Stack Number haben.

Im zweiten Teil dieser Arbeit wird gezeigt, dass jeder planare Graph mit Baumweite höchstens 4 Teilgraph eines *planaren quasi*-4-*Baums* ist. Dies bestätigt eine Vermutung von Förster [50] von 2024. Außerdem werdem planare quasi-4-Bäume zu *planaren quasi-k-Bäumen* verallgemeinert und gezeigt, dass jeder planare Graph mit Baumweite k Teilgraph eines planaren quasi-k-Baums ist.

Des Weiteren werden obere Schranken für die Stack Number von mehreren Teilklassen von upward planaren quasi-4-Bäumen gezeigt.

Contents

1 Introduction

A *linear layout* of an undirected graph G is a total ordering $\prec$ of the vertices $V(G)$ together with a partition of the edges $E(G)$ into parts $E_1, \dots, E_k$ such that each part fulfills certain properties. Similarly, a linear layout of a *directed acyclic graph* G is a *topological* ordering $\prec$ of the vertices $V(G)$ together with a partition of the edges $E(G)$ into parts $E_1, \dots, E_k$ such that each part fulfills certain properties. These properties depend on the type of linear layout with the most important linear layouts being *stack layouts* and *queue layouts*. A stack layout is a linear layout such that no two edges in the same set *cross*, i. e. there is no set E_i with $1 \leq i \leq k$ such that there are two edges $ab, uv \in E_i$ with $a \prec u \prec b \prec v$. We refer to the sets $E_1, \dots, E_k$ of a stack layout as *stacks*. A queue layout is a linear layout such that no two edges in the same set *nest* i. e. there is no set E_i with $1 \leq i \leq k$ such that there are two edges $ab, uv \in E_i$ with $a \prec u \prec v \prec b$. We refer to the sets $E_1, \dots, E_k$ of a queue layout as *queue*. The *stack number* $\mathrm{sn}(G)$ (*queue number* $\mathrm{qn}(G)$) of G is the smallest number of stacks (queues) in a stack layout (queue layout) of G.

Linear layouts are mostly of theoretical interest as they improve our understanding of the interaction of three of the most fundamental data structures — graphs, stacks and queues. Notably, their investigation has led to structural insights into different graph classes including planar graphs due to the concept of *product structure*[1] [39]. However, there are also some applications that use linear layouts. This includes VLSI design [22], fault-tolerant arrays of processors [82], and three-dimensional drawings [31, 43].

Linear layouts have been studied extensively for different graph classes. On undirected graphs, the class of all planar graphs has been at the center of research. For both the stack number [20, 55, 89] and the queue number [5, 39] there are constant bounds known. For directed graphs, we first need to observe that linear layouts are only defined for graphs without directed cycles. This is the case as a graph has a topological ordering if and only if it does not have a directed cycle. A natural graph class that contains no graphs with directed cycles is the class of all *upward planar* graphs. A graph is upward planar if it has a planar embedding such that every edge is y-monotone. This immediately implies that there are no directed cycles in an upward planar graph. It is a long-standing open question whether upward planar graphs have bounded stack number [75].

There are several important subclasses of upward planar graphs where a constant upper bound on the stack number is known. This includes directed acyclic outerplanar graphs [66] and upward planar 3-trees[2] [52, 74]. However, the upper bound on upward planar 3-trees does not imply that all upward planar graphs of treewidth at most 3 have constant stack number. In this thesis, we show that upward planar 2-trees have bounded stack number as well. As directed trees have bounded stack number [60], this implies that all upward planar edge-maximal graphs with treewidth at most 3 have bounded stack number. These results and the lack of results for general upward planar graphs with treewidth at most 3 suggest that it might be of interest to look at upward planar edge-maximal graphs with treewidth k for k larger than 3. However, even in the undirected case, there is not a good understanding of

[1]For a definition of *product structure*, refer to Section 2.3.
[2]A 3-tree is an edge maximal graph with treewidth 3. We define 3-trees formally in Section 2.3.

edge-maximal planar graphs with treewidth k for $k > 3$. Due to the novel concept of product structure, a good understanding of planar graphs with treewidth 4 and 6 might translate to a better understanding of all planar graphs. Thus, it is of high interest to improve our understanding of these graphs.

Recently, Förster [50] introduced the concept of *planar quasi-*4*-trees* and conjectured that this class includes all edge-maximal planar graphs of treewidth 4. We answer their conjecture in the positive. Furthermore, we generalize the graph class to *planar quasi-k-trees* and show that every planar graph of treewidth at most k is a subgraph of a planar quasi-k-tree. We complement this with some bounds on the stack number for subclasses of upward planar quasi-4-trees.

1.1 Outline

We start by giving an overview of the existing results on the stack number and other linear layouts in Section 1.2 and a list of our results in Section 1.3. Then, we introduce the notation used throughout this thesis and define the relevant graph theoretic concepts in Chapter 2. In Chapter 3, we examine the stack number of upward planar 2-trees and give the first constant upper bound on their stack number. In Chapter 4, we introduce *planar quasi-k-trees* building on the definition of *planar quasi-*4*-trees* by Förster [50] and show that all planar graphs of treewidth at most k are a subgraph of a planar quasi-k-tree. In Chapter 5, we define *upward planar quasi-*4*-trees* and give upper bounds for two subclasses of these graphs.

Finally, we conclude this thesis in Chapter 6 with a discussion of our results and multiple open questions motivated by our findings.

1.2 Related Work

In this section, we give an overview of the related work on the stack number and some other types of linear layouts. For a survey of results on linear layouts that is regularly maintained, we refer to a survey by Pupyrev [79].

1.2.1 Stack Number

Undirected graphs. The *stack number*, also known as *page number* or *book thickness* of an undirected graph, was first introduced by Bernhart and Kainen [10] in 1979. They show that a graph has stack number 1 if and only if it is outerplanar. Furthermore, they show that a graph has stack number at most 2 if and only if it is a subgraph of a planar Hamiltonian graph and that the stack number of a graph is equal to the maximum stack number of its 2-connected components. They conjecture that the stack number of planar graphs is unbounded, which was disproved by Buss and Shor [20] showing that nine stacks suffice for planar graphs in 1984. Later, this was improved by Heath [55] to seven stacks and by Yannakakis [89] to four stacks. This bound then was shown to be tight independently by Bekos, Kaufmann, Klute, Pupyrev, Raftopoulou, and Ueckerdt [9] and Yannakakis [90] in 2020.

There are also several results on graphs that are not planar. For example, Chung, Leighton, and Rosenberg [23] show that every complete graph K_n has stack number at most $\lfloor \frac{n}{2} \rfloor$, Enomoto, Nakamigawa, and Ota [48] show that every complete bipartite graph $K_{n,n}$ has stack number at most $\lfloor \frac{2n}{3} \rfloor + 1$, and Malitz [70] shows that every graph G has stack number bounded by $\mathcal{O}(\sqrt{|E(G)|})$, which is achieved by complete graphs.

Blankenship [14] shows that for every proper minor-closed graph class $\mathcal{G}$, there is a $k \in \mathbb{N}$ such that every graph $G \in \mathcal{G}$ has stack number at most k. There are also results on graph classes that are not proper minor-closed. For example, Dujmović and Frati [37] show that every (g, d)-map graph has stack number in $\mathcal{O}(\log |V(G)|)$ for fixed g and d and every (g, k)-planar[3] graph has stack number in $\mathcal{O}(\log |V(G)|)$ for fixed g and k. Furthermore, Bekos, Da Lozzo, Griesbach, Gronemann, Montecchiani, and Raftopoulou [7] show that the stack number of k-map graphs is at most $6k + 7$ and at most 25 for optimal[4] 2-planar graphs. Their results are improved on by Brandenburg [18], who shows that the stack number of k-map graphs is at most $6\lfloor \frac{k}{2} \rfloor + 5$ and at most 17 for optimal 2-planar graphs. Furthermore, Brandenburg [19] shows that the stack number of 1-planar graphs is at most 10. Malitz [69] shows that every graph of genus g has stack number in $\mathcal{O}(\sqrt{g})$, proving a conjecture by Heath and Istrail [56] who showed that there are graphs with genus g that need $\Omega(\sqrt{g})$ stacks.

There are several results on k-trees[5]. Ganley and Heath [53] show that every k-tree has stack number at most $k + 1$ and that there exist k-trees that require k stacks. They conjecture that every k-tree has stack number at most k. Dujmovic and Wood [32] disprove this for every $k \geq 3$ by constructing a graph with treewidth k that requires $k + 1$ stacks. As Rengarajan and Veni Madhavan [80] show that 2-trees have stack number at most 2, this yields a tight bound. For the related concept of pathwidth, Togasaki and Yamazaki [83] show that graphs with pathwidth k have stack number at most k and graphs with strong pathwidth k have stack number between $\lceil \frac{3(k-1)}{2} \rceil$ and $3\lceil \frac{k}{2} \rceil$.

Directed graphs. The stack number of *directed* graphs was introduced by Nowakowski and Parker [75] in 1989 as a notion on posets. In 1999, Heath, Pemmaraju, and Trenk [59, 60] introduced it explicitly for directed graphs. Furthermore, Heath, Pemmaraju, and Trenk showed that a single stack suffices for all directed trees and two stacks suffice for all unicyclic directed acyclic graphs. They raise the question whether all upward planar graphs have bounded stack number. While Heath and Pemmaraju [58] show that there are directed planar graphs that have unbounded stack number, it is still unclear whether the stack number is bounded on upward planar graphs. Since then, the stack number has been extensively studied on several subclasses of upward planar graphs.

Bhore, Da Lozzo, Montecchiani, and Nöllenburg [11] show that several subclasses of upward outerplanar graphs have bounded stack number. Furthermore, there are results for several subclasses of upward planar graphs that need at most two stacks. This includes two terminal series-parallel digraphs [28], N-free graphs [72], and upward planar graphs whose faces have a special structure [13].

Frati, Fulek, and Ruiz-Vargas [52] show that every upward planar triangulation has stack number $o(n)$ if and only if every upward planar triangulation with maximum degree in $\mathcal{O}(\sqrt{n})$ has stack number in $o(n)$ and that upward planar 3-trees have bounded stack number. Nöllenburg and Pupyrev [74] improve their result on upward planar 3-trees by showing that their *twist number* is at most 5. They show that this bound on the twist number is tight and use a result by Davies [25] relating the stack number of a graph to its twist number to show that the stack number of upward planar 3-trees is at most 85. Furthermore, they improve upper bounds on several subclasses of upward outerplanar graphs.

[3]A graph is (g, k)-*planar* if and only if it has an embedding on a surface with Euler genus g where every edge is crossed at most k times.

[4]A k-planar graph is *optimal* if and only if it is a k-planar graph with the maximum number of edges.

[5]A k-tree is an edge maximal graph with treewidth k. We define k-trees formally in Section 2.3.

Jungeblut, Merker, and Ueckerdt [65] showed the first sublinear bound for all upward planar graphs in 2023. Namely, they show that every upward planar graph has stack number in $\mathcal{O}(n^{2/3}\log^{2/3}(n))$. Additionaly, they construct (independently of Nöllenburg and Pupyrev) an upward planar graph with twist number (and thus stack number) at least 5.

In a different paper, Jungeblut, Merker, and Ueckerdt [66] show that all outerplanar directed acyclic graphs have bounded stack number. Namely, they show that they have stack number at most 24 776 (the best currently known lower bound is 4 by Nöllenburg and Pupyrev [74]). Furthermore, Jungeblut, Merker, and Ueckerdt give a construction for a directed acyclic 2-tree that has unbounded stack number, improving on the existing construction with treewidth 3.

Algorithmic results. The first $\mathcal{NP}$-hardness result is due to Heath and Pemmaraju [59] who show that testing whether a directed graph admits a stack layout using six stacks is $\mathcal{NP}$-complete and conjecture that it is $\mathcal{NP}$-complete even for two stacks. Binucci, Da Lozzo, Di Giacomo, Didimo, Mchedlidze, and Patrignani [13] improve on this result and show that testing whether a directed graph admits a k-stack layout is $\mathcal{NP}$-complete for $k \geq 3$. For $k = 2$, they give an $\mathcal{O}(f(\beta)\cdot n + n^3)$ time algorithm with β being the branchwidth of the graph and f being a singly-exponential function on β. This means that the problem is FPT in β. Finally, Bekos, Da Lozzo, Frati, Gronemann, Mchedlidze, and Raftopoulou [6] show that testing whether a directed graph admits a 2-stack layout is $\mathcal{NP}$-complete and thus confirming the conjecture of Heath and Pemmaraju.

Bhore, Da Lozzo, Montecchiani, and Nöllenburg [11] show that the problem is FPT in the vertex cover number[6] τ of the graph and give an $\mathcal{O}(\tau^{\tau^{\mathcal{O}(\tau)}} + \tau\cdot n)$ time algorithm for it. Furthermore, Bhore, Da Lozzo, Montecchiani, and Nöllenburg show that the problem is $\mathcal{NP}$-hard for every fixed $k \geq 5$ even if the graph has domination number[7] in $\mathcal{O}(k)$.

1.2.2 Other Linear Layouts

Besides stack layouts, there are several other kinds of linear layouts, with the most prominent being queue layouts. They were first introduced by Heath and Rosenberg [61] in 1992. Heath and Rosenberg focus on graphs that admit 1-queue layouts and show that testing whether a graph admits a 1-queue layout is $\mathcal{NP}$-complete. Furthermore, they give a characterization of 1-queue graphs and show that every 1-queue graph admits a 2-stack layout and every 1-stack graph admits a 2-queue layout. In the same year, Heath, Leighton, and Rosenberg [57] conjectured that all planar graphs have bounded queue number. In 2020, this conjecture was answered in the positive by Dujmović, Joret, Micek, Morin, Ueckerdt, and Wood [39] using the novel concept of *product structure*[8]. They show that the queue number of planar graphs is at most 49. Furthermore, they show that graphs with genus g have queue number in $\mathcal{O}(g)$, proper minor-closed graph classes have constant queue number, and k-planar graphs have queue number in $\mathcal{O}(f(k))$ with f being exponential in k. Their result on planar graphs was then improved to 42 by Bekos, Gronemann, and Raftopoulou [5] by refining the proof of Dujmović, Joret, Micek, Morin, Ueckerdt, and Wood. For bipartite planar graphs, the best currently known upper bound is 28 due to Förster, Kaufmann, Merker, Pupyrev, and

[6]A *vertex cover* of a graph $G = (V, E)$ is a subset $C \subseteq V$ such that each edge in E is incident to at least one vertex in C. The vertex cover number $\tau(G)$ of G is the size of the smallest vertex cover of G.

[7]A *dominating set* of a graph $G = (V, E)$ is a subset $D \subseteq V$ such that every vertex in $V - D$ is adjacent to at least one vertex in D. The *domination number* $\gamma(G)$ of G is the size of its smallest dominating set.

[8]For a definition of *product structure*, refer to Section 2.3.

Raftopoulou [51]. There are also several results on non-planar graphs. This includes a tight bound of k for graphs with pathwidth k [86], a lower bound of $k + 1$ and an upper bound of $2^k - 1$ for graphs with treewidth k [85], and tight bounds of $\lfloor \frac{n}{2} \rfloor$ for complete graphs K_n and $\min\{\lceil \frac{n}{2} \rceil, \lceil \frac{m}{2} \rceil\}$ for complete bipartite graphs $K_{n,m}$ [61].

Other types of linear layouts include *track layouts* where every track has to be an independent set and edges between each pair of tracks must be non-crossing [4, 29, 42, 43, 78, 87] and *mixed page layouts* where every set of edges is either a stack or a queue [3, 24, 44, 47, 54, 62, 77].

1.3 Results

In this section, we present the main results of this thesis. For a definition of the notation that is used, refer to Chapter 2. For a graph G, we refer to the stack number of G as $\operatorname{sn}(G)$, the twist number of G as $\operatorname{tn}(G)$, and the treewidth of G as $\operatorname{tw}(G)$.

The first contribution of this thesis is a bound on the twist number of upward planar 2-trees that is only dependent on the twist number of outerplanar directed acyclic graphs.

Theorem 1.1: *The twist number of upward planar* 2*-trees is bounded by a constant. Moreover, for every upward planar* 2*-tree G, it holds that* $\operatorname{tn}(G) \leq 13(c + 2) + 1$, *with c being an upper bound on the twist number for the class of all outerplanar directed acyclic graphs.*

Using Theorem 1.1 together with the currently best upper bound on the twist number of outerplanar directed acyclic graphs by Jungeblut, Merker, and Ueckerdt [66], we get the following corollary.

Corollary 1.2: *The twist number of upward planar* 2*-trees is bounded by a constant. Moreover, for every upward planar* 2*-tree G, it holds that* $\operatorname{tn}(G) \leq 12\,611$.

Combining this with a result by Davies [25] relating the stack number to the twist number of a graph we get the following bound for the stack number of upward planar 2-trees.

Corollary 1.3: *The stack number of upward planar* 2*-trees is bounded by a constant. Moreover, for every upward planar* 2*-tree G, it holds that* $\operatorname{sn}(G) \leq 564\,728$.

The second contribution of this thesis is the definition of *planar quasi-k-trees* as a generalization of *planar quasi-*4*-trees*. Furthermore, we show two main results for planar quasi-k-trees. First, we show that planar quasi-k-trees are a subset of planar graphs with treewidth at most k.

Theorem 1.4: *Let G be a planar quasi-k-tree. Then, it holds that* $\operatorname{tw}(G) \leq k$.

The second — and arguably much more important — result is that every planar graph with treewidth at most k is a subgraph of a planar quasi-k-tree.

Theorem 1.5: *For every planar graph G with treewidth $k \geq 4$, there exists a planar quasi-k-tree H with $G \subseteq H$ and $|V(G)| = |V(H)|$. Furthermore, if $k = 4$, there exists such a planar quasi-k-tree H that is nice.*

This answers a conjecture by Förster [50] in the positive.

Finally, we give an upper bound for the stack number on a subclass of upward planar quasi-4-trees.

Theorem 1.6: *Let G be a $\{(Q2.2)^-, (Q3.4)^-, (Q4.4)^-\}$-free nice canonically upward planar quasi-*4*-tree. Then, it holds that* $\text{tn}(G) \leq 5d(G)$*, with $d(G)$ being the depth of G.*

For the definitions of *nice* planar quasi-k-trees and $\{(Q2.2)^-, (Q3.4)^-, (Q4.4)^-\}$-*free* planar quasi-4-trees, refer to Chapter 4 and Chapter 5, respectively.

2 Stack Number, Twist Number, Treewidth and (Upward) Planarity

We start this section by introducing the notation that is used throughout this thesis. Then, we formally define the *stack number* and the *twist number* and introduce several existing results that we use in our proofs. Finally, we formally define *treewidth* and *k-trees* and motivate their relevance in the context of planar graphs.

2.1 Notation

Let G be a graph. We refer to the set of vertices of G as $V(G)$ and to the set of edges of G as $E(G)$. If G is an undirected graph and $\{u, v\} \in E(G)$ is an edge of G, we refer to it as uv or vu. In a directed graph we refer to an edge from a vertex $u \in V(G)$ to a vertex $v \in V(G)$ as uv.

We denote *directed acyclic graphs* as *DAGs*. Let G be a planar graph with a fixed planar embedding and f be a face of G. We refer to the set of vertices incident to f as $V(f)$ and to the graph induced by all vertices incident to f as $G[f] \coloneqq G[V(f)]$. Let C be a cycle in G. We refer to the graph induced by all vertices in C or in the interior of C as $\mathrm{int}_G(C)$.

Let σ be an ordering of $V(G)$. We say a vertex v is to the left of a vertex u if and only if $v <_\sigma u$. Similarly, we say a vertex v is to the right of a vertex u if and only if $u <_\sigma v$.

2.2 Stack Number and Twist Number

We start by giving a formal definition of *stack layouts*.

Definition 2.1: *Let G be a graph, σ an ordering of $V(G)$ and $E_1, \dots, E_k$ a partition of the edges in G such that there are no two distinct edges $u_1v_1, u_2v_2 \in E_i$ with $1 \leq i \leq k$ and $u_1 <_\sigma u_2 <_\sigma v_1 <_\sigma v_2$. Then, $(\sigma, E_1, \dots, E_k)$ is a k-*stack layout *of G.*

Definition 2.2: *Let G be a graph and σ be an ordering of $V(G)$. Then, the* stack number $\mathrm{sn}_\sigma(G)$ of G in regard to σ *is the smallest $k \in \mathbb{N}$ such that there exists a k-stack layout of G with vertex ordering σ.*

Definition 2.3: *Let G be a graph. Then, the* stack number $\mathrm{sn}(G)$ *of G is the minimum over all $\mathrm{sn}_\sigma(G)$ with σ being an ordering of $V(G)$.*

The name *stack layout* hails from the fact that each part $E_1, \dots, E_k$ of $E(G)$ behaves like a stack in the sense that we can go along the ordering of $V(G)$ and push each edge onto its corresponding stack once we find its first endpoint and pop it from its respective stack once we find its second endpoint. Similarly, the parts $E_1, \dots, E_k$ in a *queue layout* behave like queues.

For both stack and queue layouts, there are certain patterns that force a large number of stacks and queues, respectively. For stack layouts, these are so-called *k-twists*.

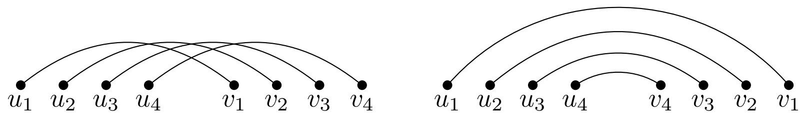


Figure 2.1: On the left is a 4-twist and on the right is a 4-rainbow.

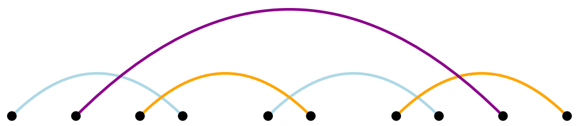


Figure 2.2: A vertex ordering with twist number 2 and stack number 3.

Definition 2.4: *Let G be a graph with an ordering σ of $V(G)$. A k-twist in σ is a set of k edges $u_1v_1, \ldots, u_kv_k$ such that $u_1 <_\sigma \cdots <_\sigma u_k <_\sigma v_1 <_\sigma \cdots <_\sigma v_k$.*

An example of a k-twist is depicted in Figure 2.1. In a k-twist, all k edges cross pairwise each other. We can define the *twist number* $\mathrm{tn}(G)$ of G as follows:

Definition 2.5: *Let G be a graph and σ be an ordering of $V(G)$. The* twist number $\mathrm{tn}_\sigma(G)$ *in regard to σ is the maximum $k \in \mathbb{N}$ such that there is a k-twist in σ. The* twist number $\mathrm{tn}(G)$ *is the minimum over all* $\mathrm{tn}_\sigma(G)$*.*

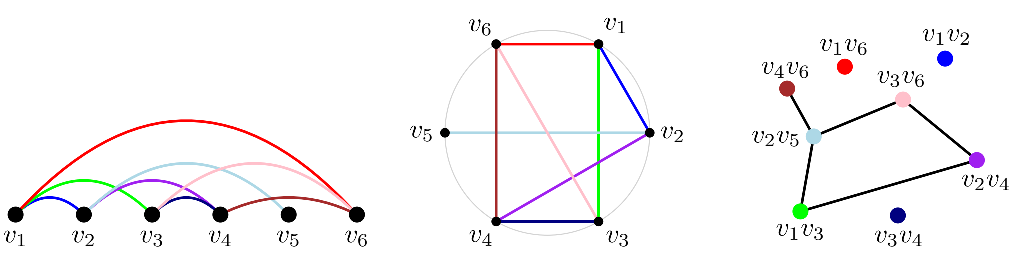
As every edge in a k-twist needs to be on a separate stack, we immediatly get the following lemma.

Lemma 2.6: *For all graphs G, it holds that* $\mathrm{tn}(G) \leq \mathrm{sn}(G)$*.*

Analogously to k-twists, a k-rainbow is a set of k edges $u_1v_1, \ldots, u_kv_k$ such that $u_1 \prec \cdots \prec u_k \prec v_k \cdots \prec v_1$ (see Figure 2.1). This means all k edges in a k-rainbow pairwise nest. Thus, any queue layout for a topological ordering of $V(G)$ that contains a k-rainbow needs at least k queues (one for each edge u_iv_i with $1 \leq i \leq k$). Heath and Rosenberg [62] show that for a fixed ordering $\prec$ of $V(G)$, the queue number of G for $\prec$ is the same as the size of the largest rainbow in $\prec$. Thus, the queue number of a graph is equal to the size, i. e. the number of edges, of the smallest rainbow in any topological vertex ordering of G.

While it is clear that a large twist implies a large stack number, we do not have an equality akin to the result of Heath and Rosenberg for stacks and twists (for an example of a vertex ordering that needs more stacks than the size of its largest twist refer to Figure 2.2). However, there is an upper bound on the stack number of a graph that is only dependent on the twist number of the graph. To see this, we take a slightly different view on stack layouts. For that, let G be a graph with a topological ordering σ of $V(G)$. We embed G by placing all vertices on a circle in the ordering σ and draw the edges as straight lines. This means all edges are chords of the same circle. For an example, refer to Figure 2.3.

Now, we look at the *intersection graph* of these chords. That is the graph that has a vertex for each chord and two vertices are adjacent if and only if their corresponding chords cross. Such a graph is called a *circle graph*. For an example, see Figures 2.3b and 2.3c. We refer to the corresponding circle graph of G with ordering σ as G_C^σ. Observe that two chords cross if and only if their endpoints are alternating on the circle and thus in σ. This means that their corresponding vertices are adjacent if and only if the two edges cross in σ. Thus, a proper coloring of the circle graph G_C^σ is equivalent to a stack layout of G (each color class corresponds to a stack). Furthermore, a k-twist in σ results in a k-clique in G_C^σ, as all edges



(a) The graph G. **(b)** G embedded on a circle. **(c)** The circle graph G_C^σ.

Figure 2.3: A graph G with an ordering $\sigma \coloneqq (v_1, \dots, v_6)$ and the resulting circle graph G_C^σ.

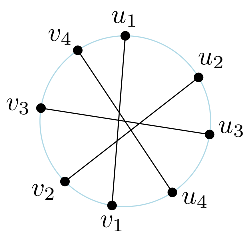


Figure 2.4: A 4-twist that is embedded on a circle. All edges of the twist cross each other pairwise.

in the twist cross each other pairwise as depicted in Figure 2.4. Thus, for the clique number $\omega(G)$ of G_C^σ we get that $\omega(G_C^\sigma) = \mathrm{tn}_\sigma(G)$. For circle graphs G, Davies [25] shows that the chromatic number $\chi(G)$ of G is bounded by a function of the clique number $\omega(G)$. Namely, Davies shows the following theorem:

Theorem 2.7 (Davies, 2022 [25]): *For all circle graphs G, it holds that*

$$\chi(G) \leq 2\omega(G)\log_2(\omega(G)) + 2\omega(G)\log_2(\log_2(\omega(G))) + 10\omega(G),$$

with $\chi(G)$ being the chromatic number of G and $\omega(G)$ being the clique number of G.

With the observations from before relating the clique number and the chromatic number to the twist number and the stack number, this immediately implies the following corollary:

Corollary 2.8: (Davies, 2022 [25]): *For all graphs G, it holds that*

$$\mathrm{sn}(G) \leq 2\,\mathrm{tn}(G)\log_2(\mathrm{tn}(G)) + 2\,\mathrm{tn}(G)\log_2(\log_2(\mathrm{tn}(G))) + 10\,\mathrm{tn}(G).$$

Davies also shows that this bound is asymptotically best possible. While this result is not as strong as the result of Heath and Rosenberg relating the size of the largest rainbow with the queue number of a graph, it is still immensely useful, as bounds on the twist number of a graph are sufficient to find bounds on the stack number. In particular, it means that any constant upper bound on the twist number implies a constant upper bound on the stack number.

2.3 Planar Graphs With Bounded Treewidth

Treewidth is a graph parameter that was first introduced by Robertson and Seymour [81] in 1986. Roughly speaking, it measures how close a graph is to a tree. Formally, we define treewidth as follows:

Let G be a graph. A *tree decomposition* of G is a pair $(\mathcal{X}, T)$, where $\mathcal{X} = \{X_1, \ldots, X_n\}$ is a family of subsets of $V(G)$ called bags and T is a tree with $V(T) = \mathcal{X}$. Furthermore, the following conditions must be satisfied:

(i) $V(G) = \bigcup_{i=1}^{n} X_i$,

(ii) For every edge $uv \in E(G)$, there is a bag X_i with $u \in X_i$ and $v \in X_i$ and

(iii) For $X_i, X_j, X_k \in \mathcal{X}$, if X_j is on the path from X_i to X_k in T, then $X_i \cap X_k \subseteq X_j$.

In particular, (iii) implies that for every vertex $v \in V(G)$ the bags in $\mathcal{X}$ that contain v induce a tree in T. The width of a tree decomposition $(\mathcal{X}, T)$ is $\max\{|X_i| \mid X_i \in \mathcal{X}\} - 1$. The *treewidth* of G is the minimum width of all of its possible tree decompositions.

The edge maximal graphs of treewidth k are called k-trees. In particular, 1-trees are trees. However, this is not an especially intuitive definition. Instead, we can also define k-trees as follows:

Definition 2.9: *A graph G is a k-tree if and only if it is either the complete graph on $k+1$ vertices K_{k+1} or if G can be constructed by adding a vertex v to a k-tree H such that v is incident to (and only to) a clique in H of size k.*

Before we consider the planar analogue to k-trees, we motivate why planar graphs of bounded treewidth are interesting. In 2020, Dujmović, Joret, Micek, Morin, Ueckerdt, and Wood [39] introduced the concept of *product structure*. Product structure gives a novel view on various graph classes including the class of all planar graphs. A graph class $\mathcal{G}$ admits product structure if there exists a constant $c \in \mathbb{N}$ such that for every graph $G \in \mathcal{G}$, there exists a graph H with treewidth at most c such that $G \subseteq H \boxtimes P$ for some path P. If such a constant c exists, it is called the *row treewidth* of $\mathcal{G}$.

The symbol $\boxtimes$ denotes the *strong product* of graphs. It is the combination of the Cartesian product and the tensor product of graphs. For two graphs G and H, we have $V(G \boxtimes H) := G \times H$. Furthermore, for two vertices $(u, v), (u'v') \in V(G \boxtimes V)$, there is an edge $(u, v)(u'v') \in E(G \boxtimes H)$ if and only if one of the following properties is fulfilled:

- $u = v$ and $u'v' \in E(H)$
- $u' = v'$ and $uv \in E(G)$
- $uv \in E(G)$ and $u'v' \in E(H)$

Since its introduction, product structure was shown to exist for a variety of graph classes. In particular the following theorems were shown for planar graphs:

Theorem 2.10 (Ueckerdt, Wood, and Yi, 2022 [84])**:** *For every planar graph G, there exists a planar graph H with treewidth at most 6 such that $G \subseteq H \boxtimes P$ for some path P.*

Theorem 2.11 (Dujmović, Joret, Micek, Morin, Ueckerdt, and Wood, 2020 [39])**:** *For every planar graph G, there exists a planar graph H with treewidth at most 4 such that $G \subseteq H \boxtimes P \boxtimes K_2$ for some path P.*

Theorem 2.12 (Dujmović, Joret, Micek, Morin, Ueckerdt, and Wood, 2020 [39])**:** *For every planar graph G, there exists a planar graph H with treewidth at most 3 such that $G \subseteq H \boxtimes P \boxtimes K_3$ for some path P.*

These theorems mean that it is of high interest to have a good understanding of planar graphs of treewidth 3, 4, and 6, as an improved understanding of these graphs might translate to an improved understanding of all planar graphs. Similar results to Theorems 2.10 to 2.12 exist for various other graph classes. This includes graphs with bounded Euler genus g, apex-minor-free graphs [39, 84], k-planar graphs, k-nearest-neighbor graphs, (g, k)-planar graphs, d-map graphs, (g, d)-map graphs [41], h-framed graphs [8], (g, δ)-string graphs [30, 41], k-th powers of planar graphs with bounded maximum degree [30, 63], fan-planar graphs, k-fan-bundle graphs [63], and K_w-free intersection graphs of unit disks in $\mathbb{R}^2$ [46].

Since its introduction, product structure has been used to improve the state of the art on several graph parameters. We highlight three of those parameters and show how bounds on planar graphs with bounded treewidth can improve the state of the art.

Product structure and queue layouts. Dujmović, Joret, Micek, Morin, Ueckerdt, and Wood [39] use product structure to show a bound of 49 on the queue number of planar graphs. In particular, they prove the following lemma:

Lemma 2.13 (Dujmović, Joret, Micek, Morin, Ueckerdt, and Wood, 2020 [39])**:** *Let G be a graph with $G \subseteq H \boxtimes P \boxtimes K_l$ with P being any path. Then, for the queue number of G, it holds that* $\operatorname{qn}(G) \leq 3l \operatorname{qn}(H) + \lfloor \frac{3}{2} l \rfloor$.

Using the three ways to describe planar graphs shown in Theorems 2.10 to 2.12, we get the following inequalities with c_k being a bound on the queue number of all planar graphs of treewidth k and G being a planar graph.

$$\operatorname{qn}(G) \leq 3c_6 \qquad \operatorname{qn}(G) \leq 6c_4 + 3 \qquad \operatorname{qn}(G) \leq 9c_3 + 4$$

As the best current upper bound on the queue number of planar graphs is 42 due to Bekos, Gronemann, and Raftopoulou [5], proving that $c_6 \leq 13$, $c_4 \leq 6$, or $c_3 \leq 4$ would improve the state of the art. Both the proof by Dujmović, Joret, Micek, Morin, Ueckerdt, and Wood and the proof by Bekos, Gronemann, and Raftopoulou leverage the good understanding of planar graphs with treewidth 3. Thus, it can be expected that a better understanding of planar graphs with treewidth 4 and 6 may proof equally useful. For planar graphs of treewidth at most 3, the best currently known upper bound is 5 due to Alam, Bekos, Gronemann, Kaufmann, and Pupyrev [1], who also give a lower bound of 4. Thus, we know that $4 \leq c_3 \leq 5$. For treewidth 4, the best upper bound is due to Wiechert [85] who shows that every graph of treewidth at most k has queue number at most $2^k - 1$. Thus, we have that $c_4 \leq 15$. For c_6, the best upper bound is the upper bound on general planar graphs. Thus, we have $c_6 \leq 42$. To the best of our knowledge, the best known lower bound for c_4 and c_6 is the same as for c_3.

Product structure and non-repetitive colorings. A coloring of a graph G is *non-repetitive* if for every path P in G, the first half of P is colored differently than the second half of P. Note that this implies that the coloring has to be proper as edges are paths of length 2 and thus adjacent vertices must be colored differently. The *non-repetitive chromatic number* $\pi(G)$ of G is the smallest integer k such that G has a non-repetitive coloring using k colors. Using product structure, Dujmović, Esperet, Joret, Walczak, and Wood [34] show that for planar graphs G, it holds that $\pi(G) \leq 768$. This answers the question whether planar graphs have bounded non-repetitive chromatic number, which was first asked by Alon, Grytczuk, Hałuszczak, and Riordan [2] in 2002, introducing this graph parameter for the first time. The proof by Dujmović, Esperet, Joret, Walczak, and Wood uses the following lemma that they proved:

Lemma 2.14 (Dujmović, Esperet, Joret, Walczak, and Wood, 2020 [34])**:** *Let G be a graph with $G \subseteq H \boxtimes P \boxtimes K_l$, with P being any path. Then, it holds that $\pi(G) \leq 4l\pi(H)$.*

Furthermore, Dujmović, Esperet, Joret, Walczak, and Wood show that every graph G with treewidth at most k has non-repetitive chromatic number at most 4^k. These two results combined yield their bound of 768 for all planar graphs. Once again, we can use the three ways of describing planar graphs with product structure to obtain the following inequalities, with c_k being a bound on the non-repetitive chromatic number of all planar graphs of treewidth k and G being a planar graph.

$$\pi(G) \leq 4c_6 \qquad \pi(G) \leq 8c_4 \qquad \pi(G) \leq 12c_3$$

Thus, showing that $c_6 < 192$, $c_4 < 96$, or $c_3 < 64$ would improve the state of the art for general planar graphs. The best currently known lower bound for general planar graphs is 11 due to Pascal Ochem (see [38] Appendix A). For a survey of the results on non-repetitive colorings, refer to this survey by Wood [88] from 2021.

Product structure and p-centered colorings. A coloring of a graph G is *p-centered* if the vertices of every connected subgraph H of G are colored with more than p different colors or there is a color that appears exactly once in H. The *p-centered chromatic number $\chi_p(G)$* of G is the minimum $k \in \mathbb{N}$ such that there is a p-centered coloring of G that uses k colors. Note that every p-centered coloring is proper, as connected subgraphs of size 2 ensure that adjacent vertices have distinct colors. The p-centered chromatic number was first introduced by Nešetřil and Ossona de Mendez [73] in 2006. In 2021, Dębski, Micek, Schröder, and Felsner used product structure to show that planar graphs have p-centered chromatic number in $\mathcal{O}(p^3 \log p)$. In particular, they show the following lemma:

Lemma 2.15 (Dębski, Micek, Schröder, and Felsner, 2021 [26])**:** *Let G be a graph with $G \subseteq H \boxtimes P \boxtimes K_l$, with P being any path. Then, it holds that $\chi_p(G) \leq (p+1)l\chi_p(H)$.*

Furthermore, they show that planar graphs with treewidth 3 have p-centered chromatic number in $\mathcal{O}(p^2 \log p)$ and that there are planar graphs with treewidth 3 with p-centered chromatic number in $\Omega(p^2 \log p)$. Combining this with Lemma 2.15 yields their bound on general planar graphs. Their lower bound on the p-centered chromatic number of planar graphs with treewidth 3 is also the best lower bound on general planar graphs. While Lemma 2.15 cannot be used to improve the upper bound asymptotically, it can potentially be used to improve the constants by proving bounds on planar graphs with treewidth 3, 4, or 6.

Product structure is used in proofs for various other graph parameters. Although there are no results as straight-forward as Lemmas 2.13 to 2.15 for those parameters, a better understanding of planar graphs with bounded treewidth might still prove useful to improve on these results. Graph parameters where product structure is used include adjacency labeling schemes [15, 33, 49], clustered colorings [21, 35, 36], vertex rankings [17], reduced bandwidth [16], comparable box dimension [45], neighborhood complexity [64], twin-width [8, 67], and odd-coloring numbers [40].

Planar quasi-k-trees Now that we have motivated why planar graphs with bounded treewidth are of interest, we look at how we can characterize these graphs. This is easy for non-planar graphs, as the edge maximal graphs with treewidth k are exactly k-trees. For planar graphs, this is only the case for $k \leq 2$. For $k \leq 2$, this holds because the class of all

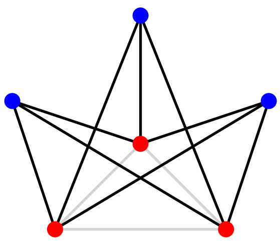


Figure 2.5: A 3-tree that contains $K_{3,3}$ as a subgraph.

graphs with treewidth 2 is the same as the class of all *planar* graphs with treewidth 2. It can be seen that all graphs of treewidth at most 2 are planar by observing that both $K_{3,3}$ and K_5 have treewidth larger than 2 and thus are not a minor of any graph with treewidth at most 2.

For $k = 3$, this does not hold, as there are 3-trees that contain $K_{3,3}$ as a subgraph and thus are not planar (for an example see Figure 2.5). However, El-Mallah and Colbourn [71] show that all planar graphs with treewidth at most 3 are a subgraph of a planar 3-tree. Furthermore, Kratochvíl and Vaner [68] show that every planar embedding of a graph with treewidth at most 3 can be extended to a planar embedding of a planar 3-tree. This means we can add edges to any planar embedding of a graph with treewidth at most 3 to obtain a planar embedding of a planar 3-tree.

It is well known that planar 3-trees can be constructed in the following way:

Lemma 2.16 (Biedl and Ruiz Velázquez, 2013 [12])**:** *A graph G is a planar 3-tree if and only if G is a triangle or if G can be constructed by placing a vertex v in an inner face f of a planar 3-tree H with a fixed planar embedding, with v being incident exactly to the vertices incident to f in H.*

An example for such a construction is depicted in Figure 2.6. Together with the result of El-Mallah and Colbourn, this gives us a nice characterization of planar graphs with treewidth at most 3 as well as a nice way to construct (a supergraph of) planar graphs with treewidth at most 3.

For $k > 3$, however, we do not have such a characterization of planar graphs with treewidth at most k. In particular, for all $k > 3$, there are planar graphs with treewidth at most k that are not a subgraph of a planar k-tree. To see this, consider a planar k-tree with $k \geq 4$. For $k \geq 4$, the base case of the construction of a k-tree contains the complete graph K_5 on five vertices as a subgraph. Thus, there is no planar k-tree for $k \geq 4$.

Recently, Förster [50] introduced the concept of *planar quasi-4-trees.*

Definition 2.17: (Förster, 2024 [50]): *A graph G is a planar quasi-4-tree if G is C_3 or C_4 or it can be constructed using one of the following rules:*

(i) G is obtained from a planar quasi-4-tree $G' = (V', E')$ by inserting a vertex $v \notin V'$ and edges vt_1 and vt_2 and possibly vt_3, where (t_1, t_2, t_3) is a triangular face of G'.

(ii) G is obtained from a planar quasi-4-tree $G' = (V', E')$ by inserting a vertex $v \notin V'$ and edges vq_1 and vq_3 and possibly vq_2 and/or vq_4, where (q_1, q_2, q_3, q_4) is a quadrangular face of G'.

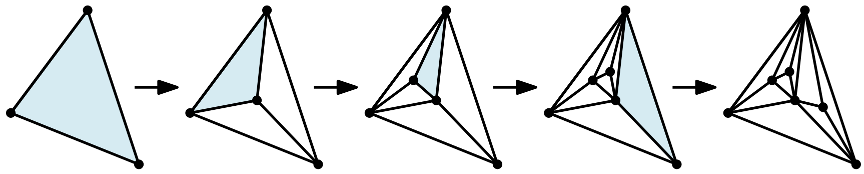


Figure 2.6: Construction of a planar 3-tree from left to right. The face where the next vertex is placed in is highlighted in blue.

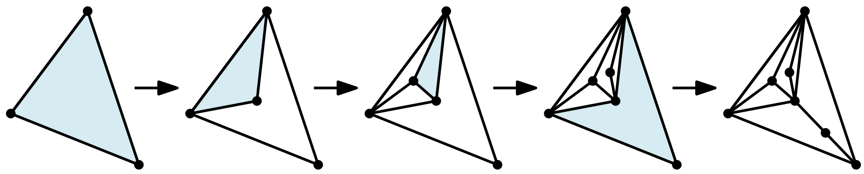


Figure 2.7: Construction of a planar quasi-4-tree from left to right. The face where the next vertex is placed in is highlighted in blue.

Planar quasi-4-trees provide a way of constructing some planar graphs with treewidth 4 that is similar to the construction of planar 3-trees described in Lemma 2.16 (compare Figure 2.6 to Figure 2.7). While Förster shows that all planar quasi-4-trees have treewidth at most 4, he does not show that every edge-maximal planar graph with treewidth 4 is a planar quasi-4-tree. Instead, Förster formulates the following conjecture:

Conjecture 2.18 (Förster, 2024 [50]): *Let G be a planar graph of treewidth 4. Then, G is a subgraph of a planar quasi-4-tree.*

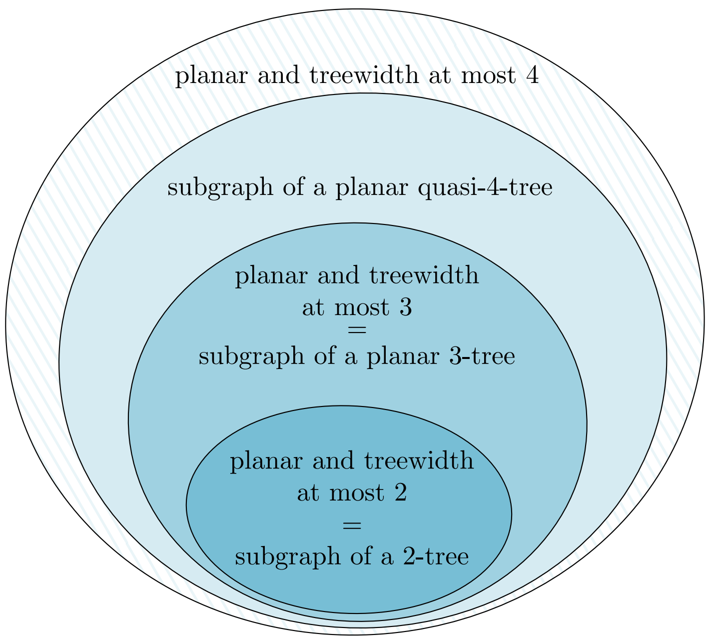


Figure 2.8: A Venn diagram relating planar graphs with treewidth at most 2, 3, and 4, planar 2-trees, planar 3-trees, and planar quasi-4-trees. The hatched area is empty if and only if Conjecture 2.18 holds, which we prove in Chapter 4.

To the best of our knowledge, there is no such construction for $k > 4$ previous to this thesis. We generalize this construction for $k > 4$ in Chapter 4. We conclude this section with a Venn diagram showing the relation between planar graphs with treewidth 2, 3, and 4, planar 2-trees, planar 3-trees, and planar quasi-4-trees in Figure 2.8.

2.4 Upward Planarity and Treewidth

A directed graph is *upward planar* if and only if it has a planar embedding such that every edge is y-monotone. Di Battista and Tamassia [27] show that a graph has such a drawing if and only if it has such a drawing such that every edge is additionally a straight line.

Similarly to planar k-trees, not every upward planar k-tree is a subgraph of an upward planar $k+1$-tree. However, unlike the result of El-Mallah and Colbourn [71] that every planar graph with treewidth at most 3 is a subgraph of a planar 3-tree, there are upward planar graphs with treewidth 2 that are not a subgraph of an upward planar 3-tree. An example for this is the graph depicted in Figure 2.9. For that graph, Jungeblut, Merker, and Ueckerdt [66] show that it is not a subgraph of an upward planar 3-tree. Similarly, there are upward planar graphs with treewidth 2 that are not a subgraph of an upward planar 2-tree. An example for this is the graph depicted in Figure 2.10. It can be seen that the embedding depicted in Figure 2.10 is the only combinatorial embedding that is upward. As there are no induced cycles of length greater than 3 in a 2-tree, we need to add one of the edges ca or db. It is not possible to do this while keeping the embedding upward planar. Thus, the graph depicted in Figure 2.10 is not a subgraph of an upward planar 2-tree. Note that with the same argument, the graph is not even a subgraph of an upward planar 3-tree.

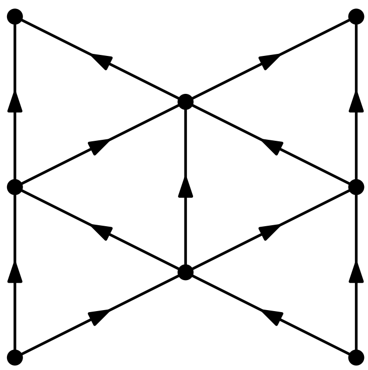


Figure 2.9: An upward planar 2-tree that is not a subgraph of any upward planar 3-tree.

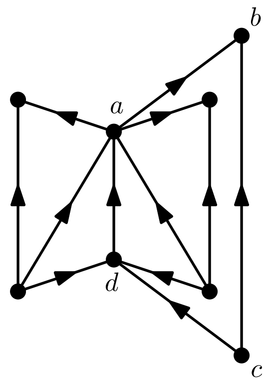


Figure 2.10: An upward planar graph with treewidth 2 that is not a subgraph of an upward planar 2-tree.

This means that a bound on the stack number of upward planar 3-trees is not a bound on all planar graphs with treewidth at most 3. Similarly, a bound on the stack number of upward planar 2-trees is not a bound on all planar graphs with treewidth at most 2. We conclude this section with a Venn diagram showing the relation between upward planar graphs with treewidth 2 and 3, upward planar 2-trees and upward planar 3-trees in Figure 2.11. We revisit this in Chapter 5, where we define upward planar quasi-4-trees.

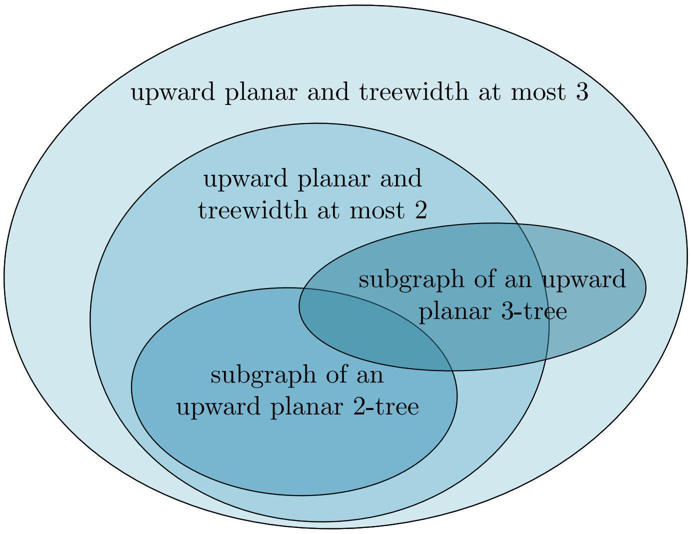


Figure 2.11: A Venn diagram relating upward planar graphs with treewidth 2 and 3, upward planar 2-trees, and upward planar 3-trees.

3 Upward Planar 2-Trees

In this chapter, we give a constant upper bound for the twist number of upward planar 2-trees. We do this by using a proof similar to the proof showing that upward planar 3-trees have constant twist number by Frati, Fulek, and Ruiz-Vargas [52].

We start by defining a method of *inserting* a vertex ordering σ' into another vertex ordering σ if certain conditions are met. Then, we show several properties of orderings that result from such an insertion and use these properties to construct a vertex ordering of any given upward planar 2-tree in Lemma 3.6. Finally, we show in Theorem 1.1 that this ordering only contains twists of constant size. This bound depends only on the twist number of outerplanar DAGs. Thus, improvements of the twist number of outerplanar DAGs directly improve our result for the twist number of upward planar 2-trees. We give a concrete upper bound for the twist number and stack number of upward planar 2-trees in Corollaries 1.2 and 1.3 by using the best known upper bound for the twist number of outerplanar DAGs by Jungeblut, Merker, and Ueckerdt [66] and a result relating the stack number of a graph to the twist number of a graph by Davies [25].

Definition 3.1: *Let G and G' be two graphs with $V(G) \cap V(G') = \{v_b, v_m, v_t\}$ and σ and σ' be vertex orderings of G and G' respectively such that $\sigma = (v_1, \dots, v_i = v_m, \dots, v_n)$ with $v_b <_\sigma v_m <_\sigma v_t$ and $\sigma' = (v'_1 = v_b, \dots, v'_j = v_m, \dots, v'_k = v_t)$. Then, we define the vertex ordering $\sigma' \curvearrowright \sigma$ resulting from inserting σ' into σ as $\sigma' \curvearrowright \sigma := (v_1, \dots, v_{i-1}, v'_2, \dots, v'_{k-1}, v_{i+1}, \dots, v_n)$.*

Intuitively, this means that we place all vertices of G' except v_b, v_t in a *block* around v_m and the ordering $\sigma' \curvearrowright \sigma$ looks like depicted in Figure 3.1. This will prove useful in later proofs, as this implies that edges in $G - \{v_b, v_m, v_t\}$ do not cross edges in $G' - \{v_b, v_m, v_t\}$.

Note that for $u, v \in V(G)$ we have $u <_\sigma v$ if and only if $u <_{\sigma \curvearrowright \sigma'} v$. Similarly, for $u, v \in V(G')$ we have $u <_{\sigma'} v$ if and only if $u <_{\sigma \curvearrowright \sigma'} v$. Thus, we can make the following observation that we use in the proof of Lemma 3.6.

Observation 3.2: *Let $G \cup G'$ be a graph with $V(G) \cap V(G') = \{v_b, v_m, v_t\}$ and σ and σ' be topological vertex orderings of G and G', respectively, such that $\sigma = (v_1, \dots, v_i = v_m, \dots, v_n)$ with $v_b <_\sigma v_m <_\sigma v_t$ and $\sigma' = (v'_1 = v_b, \dots, v'_j = v_m, \dots, v'_k = v_t)$. Then, $\sigma' \curvearrowright \sigma$ is a topological vertex ordering of $G \cup G'$. Furthermore, let $M \subseteq E(G)$ be a twist in $G \cup G'$ with respect to $\sigma' \curvearrowright \sigma$. Then, M is a twist in G with respect to σ.*

Now, we define a *separation tree* T of an upward planar 2-tree G. An example for this can be seen in Figure 3.2.

Definition 3.3: *Let $G = (V, E)$ be an upward planar 2-tree with a fixed upward planar embedding. We define the* separation tree $T = (V_T, E_T)$ *of G as follows:*

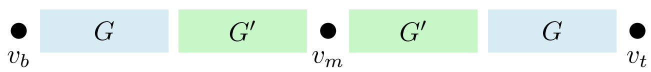


Figure 3.1: The ordering $\sigma' \curvearrowright \sigma$ as defined in Definition 3.1. The vertices of G are ordered according to σ, while the vertices of G' are ordered according to σ'.

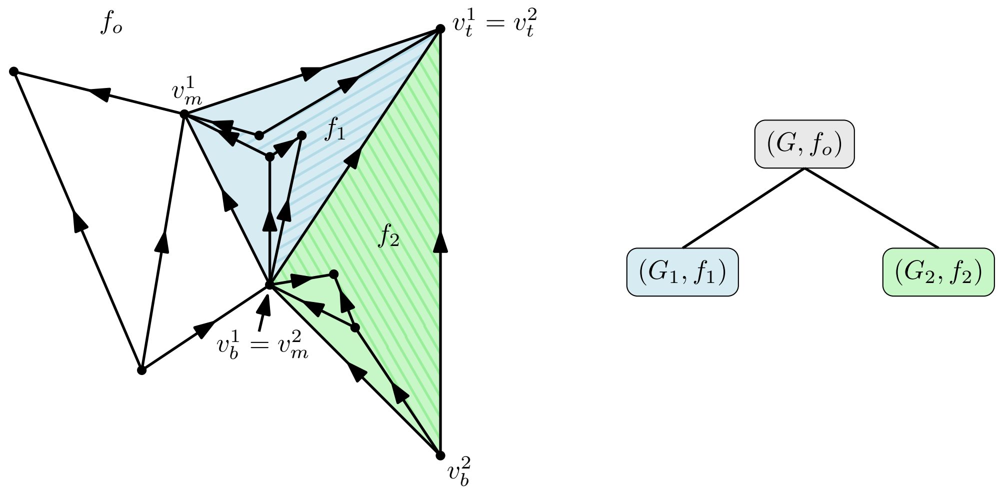


Figure 3.2: On the left an upward planar 2-tree G with outer face f_o. On the right the separation tree T of G. The root of T is (G, f_o). It has children (G_1, f_1) (blue) and (G_2, f_2) (green) in T. G_1 and G_2 have the bottom, the middle, and the top vertex v_b^i, v_m^i, and v_t^i, respectively.

- *The vertices V_T of T are tuples of a subgraph G' of G together with a face f of G'.*
- *The root of T is (G, f_o), with f_o being the outer face of G.*
- *Let $(G' \subseteq G, f)$ be a vertex of T and $t = (v_b, v_m, v_t)$ be a triangle in G' that is different from f with v_b, v_m, and v_t incident to f and with at least one vertex in its interior. Without loss of generality, let $v_b < v_m < v_t$ in any topological ordering of G and let $G'' = \mathrm{int}_{G'}(t)$. Note that G'' is bounded by the triangle t. As G'' is a 2-tree, it contains an inner face f' that is incident to v_b, v_m, and v_t. Then, (G'', f') is a child of (G', f) in T and we refer to v_b, v_m, and v_t as the bottom, the middle, and the top vertex of G'' and to the triangle t as the outer triangle of G''.*

We use this separation tree in Lemma 3.6 to construct a vertex ordering by induction on the structure of it. For that, we need to bound the twist number of the leaves of the separation tree. We do this by showing that the subgraphs corresponding to the leaves of the separation tree are outerplanar.

Lemma 3.4: *For every leaf (G, f) of T, G is an outerplanar graph.*

Proof. As G is a 2-tree, every induced cycle in G is a triangle. Furthermore, by the definition of T, we know that G does not contain a triangle with a vertex in its interior. Otherwise, (G, f) would not be a leaf. Thus, G is outerplanar, as every vertex of G is incident to f. □

For every vertex (G, f) in T that is not the root, the outer face of G is bounded by a triangle. Thus, we can make the following observation that we use in the proof of Lemma 3.6.

Observation 3.5: *For every vertex (G, f) in T that is not the root, the bottom vertex of G is a source and the top vertex of G is a sink.*

In the following lemma, we construct a topological vertex ordering for a given upward planar 2-tree. For several of the properties in Lemma 3.6, we require vertices to be part of $V(G' - G[f])$ with (G', f') being a child of (G, f) in the separation tree T of G. Recall that a child G' of G is a graph induced by the vertices contained or in the interior of a fixed triangle in G that is the outer triangle of G'. Thus, the graph $G' - G[f]$ is the graph that is induced by all vertices in the interior of the outer triangle of G' without the vertices contained in the outer triangle. In particular, this means that the bottom, the middle, and the top vertex of G' are not part of $G' - G[f]$.

Lemma 3.6: *Let G_0 be an upward planar 2-tree with a fixed upward planar embedding and a separation tree T. Then, there exists a topological ordering σ of the vertices of G_0 with the following properties:*

(i) Let (G, f) be a vertex of T. Then, $\mathrm{tn}_\sigma(G[f]) \leq c + 2$ with $c \in \mathbb{N}$ being an upper bound on the twist number of the class of all outerplanar DAGs.

(ii) Let (G, f) be a vertex of T and let (G_1, f_1) and (G_2, f_2) be two distinct children of (G, f) in T with respective middle vertices v_{m_1}, v_{m_2}. Then, there exist no edges $v_{m_1}u_1 \in E(G_1)$ and $v_{m_2}u_2 \in E(G_2)$ that cross in σ with $u_i \in V(G_i - G[f])$ for $i \in \{1, 2\}$.

(iii) Let (G, f) be a vertex of T and let (G_1, f_1) and (G_2, f_2) be two distinct children of (G, f) in T with different bottom vertices v_{b_1}, v_{b_2} and different middle vertices v_{m_1}, v_{m_2}, respectively. Furthermore, let $v_{b_1}u_1 \in E(G_1)$ and $v_{b_2}u_2 \in E(G_2)$ with $u_i \in V(G_i - G[f])$ for $i \in \{1, 2\}$ that cross in σ. Then, the edges $v_{b_1}v_{m_1}$ and $v_{b_2}v_{m_2}$ cross in σ as well.

(iv) Let (G, f) be a vertex of T and (G_1, f_1) and (G_2, f_2) be two distinct children of (G, f) in T. Then, there exist no edges u_1v_1 and u_2v_2 that cross in σ with $u_i, v_i \in V(G_i - G[f])$ for $i \in \{1, 2\}$.

Proof. We show the slightly stronger statement that there exists an ordering σ fulfilling Properties (i) to (iv) as well as the following property by induction on the separation tree of G as defined in Definition 3.3.

(v) Let (G, f) be a vertex of T with v_b and v_t being the bottom vertex and the top vertex of G. Then, for all vertices $u \in V(G)$, it holds that $v_b \leq_\sigma u \leq_\sigma v_t$.

Note that this is not the case for every topological ordering of an upward planar 2-tree, as upward planar 2-trees may have multiple sources and sinks. For the base case, let (G, f) be a leaf in T and v_b, v_m, and v_t be the bottom, the middle, and the top vertex of G. As G is a leaf, we know by Lemma 3.4 that G is outerplanar. Therefore, there exists a topological ordering

$$(v_1, \ldots, v_i = v_b, \ldots, v_j = v_m, \ldots, v_k = v_t, \ldots, v_n)$$

of $V(G)$ such that $\mathrm{tn}_\sigma(G)$ is at most c. Furthermore, as v_b is a global source in G and v_t is a global sink in G by Observation 3.5, the ordering

$$(v_i = v_b, v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{j-1}, v_m, \ldots, v_{k-1}, v_{k+1}, \ldots, v_n, v_k = v_t)$$

is a topological ordering of $V(G)$ that satisfies Property (v). Furthermore, this topological ordering has twist number at most $c + 2$, as we only changed the position of the two vertices v_b and v_t. Thus, Property (i) holds for this ordering. As (G, f) is a leaf, Properties (ii) to (iv) hold trivially.

Now, let (G, f) be an inner vertex of T with children $(G_1, f_1), \ldots, (G_k, f_k)$ in T. By induction, there exist topological vertex orderings $\sigma_1, \ldots, \sigma_k$ of $G_1, \ldots, G_k$, respectively, that fulfill Properties (i) to (v). Clearly, the graph $G[f]$ is outerplanar, as all vertices are incident to f. If $G \neq G_0$, let v_b be the bottom vertex of G and v_t the top vertex of G. If $G = G_0$, we choose any

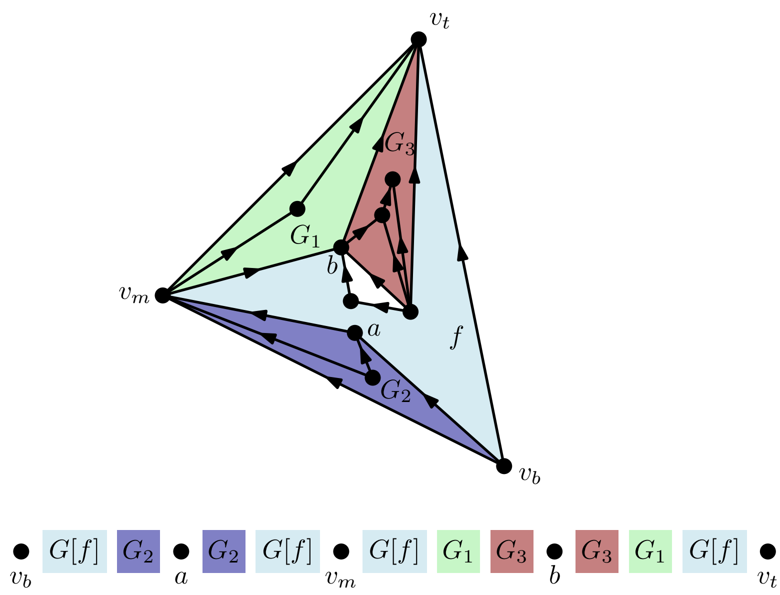


Figure 3.3: An upward planar 2-tree G. Together with the face f (light blue), the graph G is a node in a separation tree T. It has children (G_1, f_1) (light green), (G_2, f_2) (dark blue), and (G_3, f_3) (red) in T. G_1 and G_3 have the middle vertex b and G_2 has the middle vertex a. Below the graph is the ordering resulting from inserting orderings of G_1, G_2, and G_3 into an ordering of $G[f]$.

source of $G_0[f]$ as the bottom vertex and any sink of $G_0[f]$ as the top vertex. Then, similar to the base case, there exists a vertex ordering $\sigma_f = (v_1 = v_b, \ldots, v_n = v_t)$ of $G[f]$ with v_b and v_t being the bottom vertex and the top vertex of G and with $\text{tn}_{\sigma_f}(G[f]) \leq c + 2$. Now, we construct vertex orderings $\sigma'_1 \ldots, \sigma'_k$ with σ'_i being a vertex ordering of $G[f] \cup \bigcup_{j=1}^{i} G_j$ as follows: we set $\sigma'_1 := \sigma_1 \curvearrowright \sigma_f$ and $\sigma'_i := \sigma_i \curvearrowright \sigma'_{i-1}$ for all $2 \leq i \leq k$. For an example of this, refer to Figure 3.3. Note that we can insert the orderings according to Definition 3.1, as Property (v) ensures that the orderings $\sigma_1, \ldots, \sigma_k$ have the required form. We claim that $\sigma := \sigma'_k$ fulfills Properties (i) to (v). Property (v) follows directly from the definition of $\curvearrowright$ and Property (i) follows directly from the twist number of σ_f and Observation 3.2. Thus, it remains to show that Properties (ii) to (iv) hold.

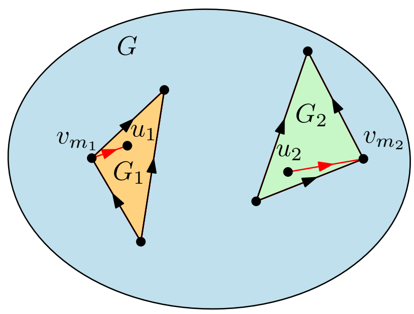


Figure 3.4: The situation described in Property (ii). The vertices v_{m_1} and v_{m_2} are the respective middle vertices of G_1 and G_2. The vertices u_1 and u_2 are in $V(G_1 - G[f])$ and $V(G_2 - G[f])$, respectively.

Property (ii). Let (G_1, f_1) and (G_2, f_2) be two distinct children of (G, f) in T with the respective middle vertices v_{m_1} and v_{m_2}. Let $v_{m_1}u_1 \in E(G_1)$ and $v_{m_2}u_2 \in E(G_2)$ be edges with $u_i \in V(G_i - G[f])$ for $i \in \{1, 2\}$. We show that $v_{m_1}u_1$ and $v_{m_2}u_2$ do not cross in σ. Clearly, this holds if $v_{m_1} = v_{m_2}$. Therefore, assume that $v_{m_1} \neq v_{m_2}$. Then, by the definition of $\curvearrowright$, we know that $v_{m_1} <_\sigma v_{m_2}$ if and only if $u_1 <_\sigma u_2$ and $u_1 <_\sigma v_{m_2}$. Thus, $v_{m_1}u_1$ and $v_{m_2}u_2$ do not cross and Property (ii) holds.

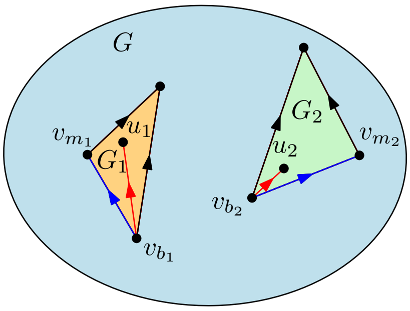


Figure 3.5: The situation described in Property (iii). The vertices v_{m_1} and v_{m_2} are the respective middle vertices of G_1 and G_2, v_{b_1} and v_{b_2} are the respective bottom vertices of G_1 and G_2. The vertices u_1 and u_2 are in $V(G_1 - G[f])$ and $V(G_2 - G[f])$, respectively.

Property (iii). Let (G, f) be a vertex of T and (G_1, f_1) and (G_2, f_2) be two distinct children of (G, f) in T with different bottom vertices v_{b_1} and v_{b_2} and different middle vertices v_{m_1} v_{m_2}, respectively. Furthermore, let $v_{b_1}u_1 \in E(G_1)$ and $v_{b_2}u_2 \in E(G_2)$ with $u_i \in V(G_i - G[f])$ for $i \in \{1, 2\}$ that cross in σ. We show that $v_{b_1}v_{m_1}$ and $v_{b_2}v_{m_2}$ cross in σ. For that, we assume without loss of generality that $v_{b_1} <_\sigma v_{b_2} <_\sigma u_1 <_\sigma u_2$. By the definition of the insertion operation $\curvearrowright$, we know that all vertices of G_i except its bottom vertex and its top vertex are placed in a block around its middle vertex v_{m_i} for $i \in \{1, 2\}$. As $v_{m_1} \neq v_{m_2}$, this directly implies that $v_{b_1} <_\sigma v_{b_2} <_\sigma v_{m_1} <_\sigma v_{m_2}$. Thus, $v_{b_1}v_{m_1}$ and $v_{b_2}v_{m_2}$ cross in σ and σ fulfills Property (iii).

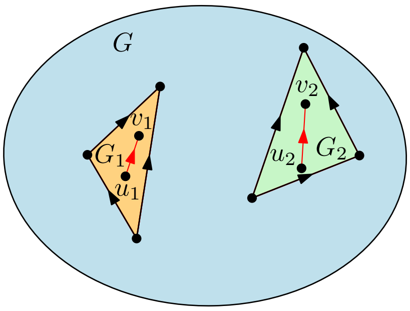


Figure 3.6: The situation described in Property (iv). The vertices u_1, v_1 and u_2, v_2 are in $V(G_1 - G[f])$ and $V(G_2 - G[f])$, respectively.

Property (iv). Let (G_1, f_1) and (G_2, f_2) be two distinct children of (G, f) in T and u_1v_1 and u_2v_2 edges with $u_i, v_i \in V(G_i - G[f])$, respectively. Note that this is almost the same case as in Property (ii), with the only difference being that v_1 and v_2 are not the middle vertex of G_1 and G_2, respectively. Assume, without loss of generality, that in the construction of σ, the vertex ordering of (G_1, f_1) was inserted before the vertex ordering of (G_2, f_2). Thus, if G_1 and G_2 have the same middle vertex, then u_1v_1 and u_2v_2 either nest in σ or one is entirely to the right of the other one. Therefore, assume that G_1 and G_2 have different middle vertices v_{m_1} and v_{m_2} and assume without loss of generality that $v_{m_1} <_\sigma v_{m_2}$. Then, by the definition of $\curvearrowright$, it holds that $u_1, v_1 <_\sigma u_2, v_2$. Thus, u_1v_1 and u_2v_2 do not cross in σ and σ fulfills Property (iv). This concludes the proof. □

Now, we have all the tools we need to show that the twist number of upward planar 2-trees is bounded by a constant and prove Theorem 1.1. The proof of Theorem 1.1 is nearly identical to the proof by Frati, Fulek, and Ruiz-Vargas [52] for upward planar 3-trees.

Theorem 1.1: *The twist number of upward planar 2-trees is bounded by a constant. Moreover, for every upward planar 2-tree G, it holds that $\mathrm{tn}(G) \leq 13(c+2)+1$, with c being an upper bound on the twist number for the class of all outerplanar directed acyclic graphs.*

Proof. Let $G = (V, E)$ be an upward planar 2-tree with a fixed upward planar embedding with outer face f. Furthermore, let T be a separation tree of G with root (G, f) — as defined in Definition 3.3 — and let σ be a topological ordering of the vertices of G satisfying Properties (i) to (iv) of Lemma 3.6.

Now, let M be a maximal twist in G with respect to the ordering σ. We assume that G is minimal in the sense that there exists no child (G', f') of (G, f) in the separation tree T of G such that G' contains a twist of the same size as M. Otherwise, we argue using that child instead.

We partition the twist M into three disjoint sets of edges $M = M_0 \cup M_1 \cup M_2$ with $M_i = \{e \in M : |e \cap V(G[f])| = i\}$ for $i \in \{0, 1, 2\}$, i. e. M_i contains the edges in M with i endpoints on the outer face of G. Thus, $\mathrm{tn}(G) \leq |M_0| + |M_1| + |M_2|$. Furthermore, since $|M_2| \leq \mathrm{tn}_\sigma(G[f])$ and $\mathrm{tn}_\sigma(G[f]) \leq c+2$ by Property (i) of the ordering σ, it suffices to show that $|M_0| + |M_1| \leq 12(c+2)+1$.

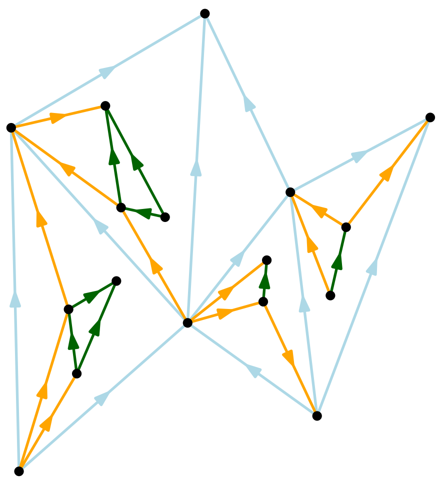


Figure 3.7: An upward planar 2-tree G. Edges with two endpoints on the outer face are colored light blue, edges with one endpoint on the outer face are colored orange, and edges with no endpoint on the outer face are colored dark green. The set M_2 is a subset of the light blue edges, M_1 is a subset of the orange edges, and M_0 is a subset of the dark green edges.

We start by showing that $M_0 = \emptyset$, i. e. every edge in M has at least one endpoint on the outer face of G. First, note that every edge in M_0 is contained in some child of G in T. Furthermore, M_0 cannot contain edges from two distinct children $(G_1, f_1), (G_2, f_2)$ of (G, f) in the separation tree T. This follows directly from Property (iv) of the ordering σ. Let (G', f') be the child of (G, f) in T that contains the edges in M_0 and let v_b, v_m and v_t be the bottom, the middle and the top vertex of G'. None of those three vertices are contained in edges in M_0 (otherwise, the edges would be in M_1 or in M_2). By the minimality of G, we know there exists an edge which is not contained in M_0 that has to cross all edges in M_0. By the definition of σ, this edge must contain v_m. However, there can only be one such edge in M. Then, however, the edge $v_b v_m$ also crosses the edges in M_0. Thus, $M_0 \cup \{v_b v_m\}$ is a twist with size equal to M. As $M_0 \cup \{v_b v_m\}$ is contained entirely in G', this contradicts the minimality of G. Thus, we get that $M_0 = \emptyset$.

It remains to show that $|M_1| \leq 12(c + 2) + 1$. By definition, every edge in M_1 is incident to either a middle, a bottom or a top vertex of a child of (G, f) in T. By Property (ii), there is at most one edge in M_1 that is incident to a middle vertex. As top vertices and bottom vertices are analogous to each other, it suffices to show that there are at most $6(c + 2)$ edges in M_1 incident to bottom vertices. For that, let M_1^b be the edges in M_1 that are incident to bottom vertices. Furthermore, let E_b be the set containing the edge $v_b v_m$ for every edge $v_b u \in M_1^b$ with $v_b u$ being an edge of some child (G', f') of (G, f) in T and v_m being the middle vertex of G' (see Figure 3.8). As every vertex can occur at most once in M, it holds that $|M_1^b| = |E_b|$. Furthermore, as each vertex is the middle vertex of at most two distinct children of (G, f), there are at most two distinct edges uv_m and $u'v_m$ in $|E_b|$ for every middle vertex v_m. For every vertex v_m for which two of those edges exist, we delete one of them to obtain E_b'. Then, it holds that $|M_1^b| \leq 2 \cdot |E_b'|$. Furthermore, in the graph induced by E_b', every vertex has degree at most two. This holds because for two distinct edges uv and $u'v'$ in E_b', we know that $v \neq v'$ (because of the deletions to obtain E_b' from E_b) and $u \neq u'$ (because every vertex may occur at most once in M). Therefore, there exists an independent subset $E_b'' \subseteq E_b'$ with $|E_b'| \leq 3 \cdot |E_b''|$. As M_1^b is a twist, E_b'' must be a twist as well (this follows directly from Property (iii) of σ). As E_b'' is a subset of $E(G[f])$, we know that $|E_b''| \leq \mathrm{tn}_\sigma(G[f]) \leq c + 2$ by Property (i) of σ. Thus, we also know that $|M_1^b| = |E_b| \leq 2 \cdot |E_b'| \leq 6 \cdot |E_b''| \leq 6(c + 2)$ and we have that $|M_1| \leq 12(c + 2) + 1$.

As $M_0 = \emptyset$, $|M_1| \leq 12(c + 2) + 1$ and $|M_2| \leq c + 2$, we have that $\mathrm{tn}(G) \leq |M| = |M_0| + |M_1| + |M_2| \leq 13(c + 2) + 1$. This concludes the proof. □

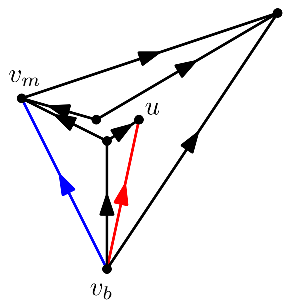


Figure 3.8: A graph containing an edge $v_b u \in M_1^b$ (red) and the edge $v_b v_m$ (blue) that is added to E_b instead.

The bound we just showed is dependent on the twist number of outerplanar DAGs. For the stack number of outerplanar DAGs, Jungeblut, Merker, and Ueckerdt [66] give a constant upper bound in form of the following theorem.

Theorem 3.7 (Jungeblut, Merker, and Ueckerdt, 2023 [66])**:** *The stack number of outerplanar DAGs is bounded by a constant. Moreover, for every outerplanar DAG G, it holds that* $\mathrm{sn}(G) \leq 24\,776$.

Our bound on the twist number of upward planar 2-trees depends on the twist number of outerplanar DAGs, not the stack number of outerplanar DAGs. In their proof, Jungeblut, Merker, and Ueckerdt use Lemma 2.6 by Davies [25] to get an upper bound on the stack number of monotone outerplanar DAGs by using the upper bound of 4 by Nöllenburg and Pupyrev [74] on the twist number of monotone outerplanar DAGs. Omitting this step yields the following result.

Theorem 3.8 (Jungeblut, Merker, and Ueckerdt, 2023 [66])**:** *The twist number of outerplanar DAGs is bounded by a constant. Moreover, for every outerplanar DAG G, it holds that* $\mathrm{tn}(G) \leq 968$.

We immediately get the following corollary by using 968 as the constant c in Theorem 1.1.

Corollary 1.2: *The twist number of upward planar 2-trees is bounded by a constant. Moreover, for every upward planar 2-tree G, it holds that* $\mathrm{tn}(G) \leq 12\,611$.

Recall the following result by Davies.

Corollary 2.8: (Davies, 2022 [25]): *For all graphs G, it holds that*

$$\mathrm{sn}(G) \leq 2\,\mathrm{tn}(G)\log_2(\mathrm{tn}(G)) + 2\,\mathrm{tn}(G)\log_2(\log_2(\mathrm{tn}(G))) + 10\,\mathrm{tn}(G).$$

Using Lemma 2.6 and Corollary 1.2, we immediately get the following corollary.

Corollary 1.3: *The stack number of upward planar 2-trees is bounded by a constant. Moreover, for every upward planar 2-tree G, it holds that* $\mathrm{sn}(G) \leq 564\,728$.

Note that this bound is most likely much larger than the best possible upper bound. The currently best known lower bound is 4 due to Nöllenburg and Pupyrev [74].

We conclude this chapter with an open question. We already discussed that there are upward planar graphs with treewidth 2 that are not a subgraph of an upward planar 2-tree in Section 2.4. Thus, the bound that we show for upward planar 2-trees is not sufficient to bound the stack number of general upward planar graphs with treewidth 2. This leads us to the following question.

Question 3.9: *Is there a constant $c \in \mathbb{N}$ such that $\mathrm{sn}(G) \leq c$ for all upward planar graphs G with treewidth 2?*

4 Planar Quasi-k-Trees

In this chapter, we answer Conjecture 2.18 by Förster in the positive. Moreover, we generalize the concept of planar quasi-4-trees to planar quasi-k-trees and show that every planar graph with treewidth at most k is a subgraph of a planar quasi-k-tree. We start by giving a definition of planar quasi-k-trees that differs from Definition 2.17. For that, recall Definition 2.17.

Definition 2.17: (Förster, 2024 [50]): *A graph G is a planar quasi-4-tree if G is C_3 or C_4 or it can be constructed using one of the following rules:*

(i) G is obtained from a planar quasi-4-tree $G' = (V', E')$ by inserting a vertex $v \notin V'$ and edges vt_1 and vt_2 and possibly vt_3, where (t_1, t_2, t_3) is a triangular face of G'.

(ii) G is obtained from a planar quasi-4-tree $G' = (V', E')$ by inserting a vertex $v \notin V'$ and edges vq_1 and vq_3 and possibly vq_2 and/or vq_4, where (q_1, q_2, q_3, q_4) is a quadrangular face of G'.

Unlike in Definition 2.17 it is not feasible to define planar quasi-k-trees by explicitly listing all possible ways of inserting a new vertex, as there are too many options for large k. Instead, we give the following slightly more abstract definition.

Definition 4.1: *Let G be a graph and $\sigma := (v_1, \ldots, v_n)$ be a vertex ordering of G. If there exists a planar embedding of G and an integer $l \leq k$ such that*

- *$G[\{v_1, \ldots, v_l\}]$ is a simple cycle that bounds the outer face of G,*
- *for every $l < i \leq n$, every face in $G_i := G[\{v_1, \ldots, v_i\}]$ is bounded by at most k vertices, and*
- *for every $l < i \leq n$, it holds that $\deg_{G_i}(v_i) \geq 1$,*

*we call σ a k-*construction sequence *of G and G a* planar quasi-k-tree*. Furthermore, if $\deg_{G_i}(v_i) \geq 2$ for every $l < i \leq n$, we call σ a* nice *k-construction sequence of G and G a* nice *planar quasi-k-tree. We refer to the planar embedding of G for that a (nice) k-construction sequence exists as a* (nice) canonical *embedding of G. We omit the k and refer to σ as a construction sequence of G if k is clear from the context.*

Note that Definition 4.1 is not the same as Definition 2.17 for $k = 4$. The difference between the definitions is that Definition 4.1 allows to insert a vertex with degree 1, while Definition 2.17 only allows to insert vertices with degree at least 2. However, our notion of *nice* planar quasi-4-trees is equivalent to the notion of planar quasi-4-trees by Förster.

We start by showing that every planar quasi-k-tree has treewidth at most k. We do this by constructing a tree decomposition with width k for a given planar quasi-k-tree.

Lemma 4.2: *Let G be a planar quasi-k-tree with a canonical planar embedding. Then, there exists a tree decomposition $(\mathcal{X}, T)$ of G with width at most k such that for every face f in G, there exists a distinct bag $X \in \mathcal{X}$ that contains exactly the vertices incident to f.*

Proof. We show the statement by induction on the number $n \in \mathbb{N}$ of vertices in G. First, note that a planar quasi-k-tree contains at least three vertices. For $n = 3$, the graph G has one inner face. Then, the tree decomposition containing two bags that each contain all vertices fulfills the statement. This holds, as every face (one inner and one outer face) in G contains all vertices.

Now, let G be a planar quasi-k-tree with $n > 3$ vertices and let the claim hold for all graphs G' with less than n vertices. Furthermore, let $\sigma = (v_1, \dots, v_n)$ be a construction sequence of G. Then, $G[\{v_1, \dots, v_{n-1}\}]$ is a planar quasi-k-tree with $n-1$ vertices. Thus, by induction, there exists a tree decomposition $(\mathcal{X}, T)$ of $G[\{v_1, \dots, v_{n-1}\}]$ with width at most k such that for every face f, there exists a bag $X \in \mathcal{X}$ that contains exactly the vertices incident to f. Let f be the face in $G[\{v_1, \dots, v_{n-1}\}]$ that v_n is placed in and let $X \in \mathcal{X}$ be the bag that contains exactly the vertices incident to f in $G[\{v_1, \dots, v_{n-1}\}]$. Then, it holds that $|X| \leq k$ because f is a face of a canonical embedding of a planar quasi-k-tree and thus is incident to at most k vertices. If v_n has degree 1 in G, we can add v_n to X. As every face is incident to at most k vertices, the tree decomposition still has width at most k. Thus, the claim holds.

Assume therefore, that v_n has degree $d \geq 2$ in G. Then, f is partitioned into faces $f_1, \dots, f_d$ in G. For each f_i, we add a bag X_i that contains the vertices incident to f_i to the tree decomposition and add the edge XX_i to T. These bags contain at most k vertices, as every face in G is incident to at most k vertices by the definition of planar quasi-k-trees. Finally, we add v_n to the bag X. This ensures that every X_i is a subset of X and thus that we still have a valid tree decomposition. As X contained at most k vertices before, the width of the tree decomposition is at most k. As the only bag we changed is X, all other faces still have a corresponding bag. Thus, the claim holds for G. This concludes the proof. □

Lemma 4.2 immediatly implies the following theorem.

Theorem 1.4: *Let G be a planar quasi-k-tree. Then, it holds that* $\operatorname{tw}(G) \leq k$.

Now that we have shown that planar quasi-k-trees are a subset of all planar graphs of treewidth at most k, the rest of this chapter is dedicated to showing that every planar graph of treewidth at most k is a subgraph of a planar quasi-k-tree.

We start by showing that under certain conditions planar graphs with at most $k+1$ vertices are planar quasi-k-trees. In Lemma 4.6 we use this statement as the base case of an induction. We show two slight variations of this statement. First, we show it for all $k \in \mathbb{N}$. Then, we show a slightly stronger statement for $k = 4$ in Lemma 4.4.

Lemma 4.3: *Let G be a connected planar graph with $|V(G)| \leq k+1$ that has a fixed planar embedding with no face of size $k+1$ and whose outer face is bounded by a simple cycle. Then, G is a planar quasi-k-tree and the fixed embedding of G is a canonical embedding.*

Proof. Let $v_1, \dots, v_{k+1}$ be some ordering of $V(G)$ such that $G[\{v_1, \dots, v_i\}]$ is connected for all $i \leq k$ and $v_1, \dots, v_l$ are the vertices of the outer face of G for some $l \leq k$. Then, for all $i \leq k$, the graph $G[\{v_1, \dots, v_i\}]$ is a planar quasi-k-tree, as every face contains at most $|V(G[\{v_1, \dots, v_i\}])| = i \leq k$ vertices and by the definition of the ordering $v_1, \dots, v_{k+1}$, we have $\deg_{G_i}(v_i) \geq 1$. Furthermore, placing v_{k+1} does not create a face of size $k+1$, as $G[\{v_1, \dots, v_{k+1}\}] = G$ and G does not contain such a face by assumption. Thus, σ is a construction sequence of G and G is a planar quasi-k-tree. □

Note that the requirement of $|V(G)| \leq k+1$ is indeed necessary, as the embedding of the graph G depicted in Figure 4.1 satisfies the requirements of Lemma 4.3 and is *not* a canonical embedding of a planar quasi-k-tree for $k > 4$. To see this, look at any vertex ordering σ of G

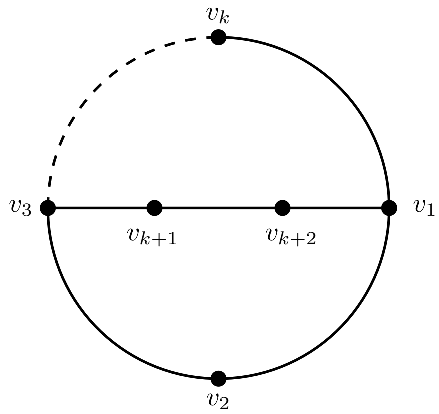


Figure 4.1: For $k > 4$, the depicted graph G is a connected planar graph with $|V(G)| = k + 2$. It has a planar embedding with no face of size $k + 1$. Its outer face is bounded by a simple cycle. The depicted embedding is *not* a canonical embedding of a planar quasi-k-tree.

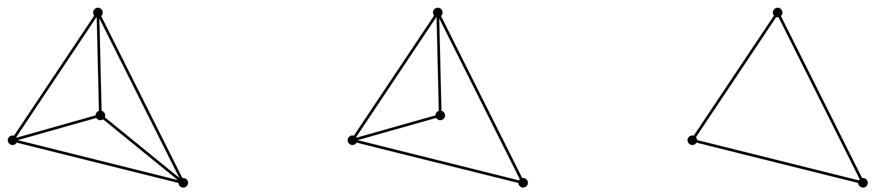


Figure 4.2: All 2-connected planar quasi-4-trees with at most four vertices. Adding at least one leaf to any vertex yields all planar quasi-4-trees with at most five vertices that are not *nice* planar quasi-4-trees. This holds because minimum degree at least 2 is clearly a necessary condition to be a *nice* planar quasi-4-tree.

that starts with the vertices incident to the outer face of the depicted embedding. Note that it is necessary for σ to start with the vertices incident to the outer face to be a construction sequence that produces the depicted embedding as a canonical embedding. Let $v \in V(G)$ be the last vertex in σ. Then, in $G - v$, there is a face that is incident to all $k + 1$ vertices. Thus, σ is not a construction sequence of G and therefore the depicted embedding is not canonical.

Now, we prove a slightly stronger statement for planar quasi-4-trees. Namely, we show that planar graphs with at most five vertices are *nice* planar quasi-4-trees if the graph is additionally 2-connected. Note that this additional requirement is necessary, as depicted in Figure 4.2. In fact, the nice planar quasi-4-trees with at most five vertices are exactly the 2-connected planar quasi-4-trees with at most five vertices.

Lemma 4.4: *Let G be a 2-connected planar graph with $|V(G)| = n \leq 5$ and a fixed planar embedding that has no face of size 5 and whose outer face is bounded by a simple cycle with length l. Then, G is a nice planar quasi-4-tree and the fixed embedding of G is a nice canonical embedding.*

Proof. As G has no face of size 5 or larger and G is bounded by a simple cycle, G is either bounded by C_3 or C_4. We do a case distinction based on that.

G is bounded by C_4: First, we show that the statement holds if G is bounded by $C_4 := (v_1, v_2, v_3, v_4)$. If $|V(G)| = 4$, the ordering $\sigma := (v_1, v_2, v_3, v_4)$ clearly is a nice construction sequence of G. Thus, assume now that $|V(G)| = 5$. Then, we claim that $\sigma := (v_1, v_2, v_3, v_4, v_5)$

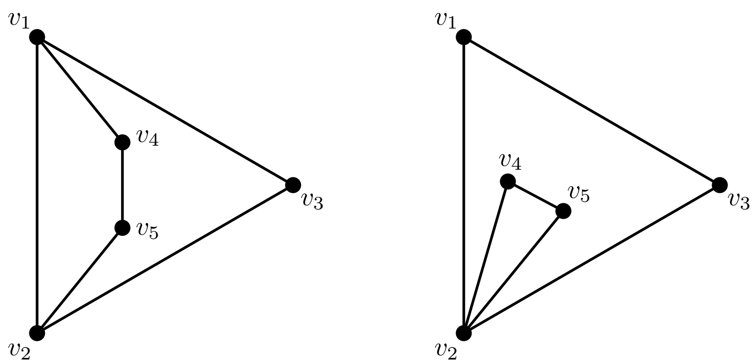


Figure 4.3: The two options (disregarding isomorphic graphs) for a graph bounded by $C_3 := (v_1, v_2, v_3)$ with two vertices v_4, v_5 adjacent to each other and one vertex in C_3 each.

is a nice construction sequence of G. For that, we need to show that G_5 does not contain a face of size 5 and that $\deg_{G_5}(v_5) \geq 2$. As $G_5 = G$ and G does not contain a face of size 5 by assumption, G_5 does not contain a face of size 5. Furthermore, as G is 2-connected, it holds that $\delta(G) \geq 2$. Thus, $\deg_{G_5}(v_5) = \deg_G(v_5) \geq \delta(G) \geq 2$ holds and σ is a nice construction sequence of G.

G is bounded by C_3: Assume now that G is bounded by $C_3 := (v_1, v_2, v_3)$. If $|V(G)| = 3$, the ordering $\sigma := (v_1, v_2, v_3)$ clearly is a nice construction sequence. If $|V(G)| = 4$, let $\sigma := (v_1, v_2, v_3, v_4)$. G_i does not contain a face of size at least 5 for $i \leq 4$, as $|V(G_i)| = i \leq 4$. Thus, we only need to show that $\deg_{G_4}(v_4) \geq 2$. As $G_4 = G$ and G is 2-connected, it holds that $\delta(G) \geq 2$. Thus, $\deg_{G_4}(v_4) \geq 2$ holds and σ is a nice construction sequence. Finally, assume that $|V(G)| = 5$ and that there exists no vertex ordering $\sigma := (v_1, v_2, v_3, v_4, v_5)$ such that $\deg_{G_4}(v_4) \geq 2$. That means for all orderings $\sigma := (v_1, v_2, v_3, v_4, v_5)$ with the vertices v_1, v_2, and v_3 being the vertices of the outer face, both v_4 and v_5 are incident to at most one vertex in $\{v_1, v_2, v_3\}$. As G is 2-connected, they are adjacent to exactly one of those vertices and furthermore adjacent to each other, as $\delta(G) \geq 2$. Thus, the graph G is isomorphic to one of the graphs depicted in Figure 4.3 and has a face incident to all five vertices. This contradicts the assumption that G has no such face. Therefore, there exists a vertex ordering $\sigma := (v_1, v_2, v_3, v_4, v_5)$ such that $\deg_{G_4}(v_4) \geq 2$. As G is 2-connected, it holds that $\deg_{G_5}(v_5) \geq 2$. Thus, σ is a nice construction sequence of G. This concludes the proof. □

Note that this lemma does not hold for $k \geq 5$. For example, the graph G with the embedding depicted in Figure 4.4 is a 2-connected planar graph with at most $k + 1$ vertices, has no face that contains $k + 1$ or more vertices, and is bounded by C_{k-1} for $k \geq 5$. By Lemma 4.3, G is a planar quasi-k-tree, but the depicted embedding is not a nice canonical embedding. To see this, look at any vertex ordering σ of G that starts with the vertices incident to the outer face, which is a necessary condition for the embedding to be canonical. Then, the first vertex that is added after the vertices incident to the outer face has degree 1 at the time of its insertion. This contradicts the construction sequence being nice. However, there is a supergraph of G that has a nice construction sequence. For example, such a supergraph can be obtained by internally triangulating G. It is an open question if there is a planar quasi-k-tree that is not a subgraph of a nice planar quasi-k-tree.

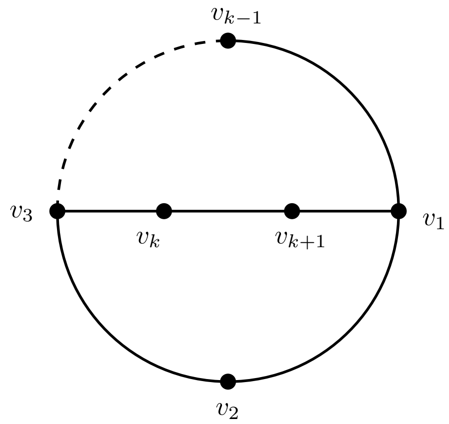


Figure 4.4: A planar quasi-k-tree that is not a *nice* planar quasi-k-tree for $k \geq 5$.

In the following lemma and theorem, we use tree decompositions with several properties. In order to properly reference those properties, we give tree decompositions fulfilling these properties a name.

Definition 4.5: *A tree decomposition $(\mathcal{X}, T)$ of a planar graph G with a fixed planar embedding is k-*nice *if it fulfills the following properties:*

(i) *$(\mathcal{X}, T)$ has width at most k.*

(ii) *There is a bag $X \in \mathcal{X}$ such that the vertices incident to the outer face of G are contained in X.*

(iii) *For all $XY \in E(T)$ the vertices in $X \cap Y$ induce a simple cycle in G.*

(iv) *For all $X \in \mathcal{X}$, the graph $G[X]$ is 2-connected.*

(v) *For all $X \in \mathcal{X}$, every inner face of $G[X]$ that is not a face in G is bounded by a cycle C that is induced by the vertices $X \cap Y$ for some neighbor Y of X in T.*

(vi) *For all $X \in \mathcal{X}$, the graph $G[X]$ does not contain a face of size $k + 1$ or larger.*

(vii) *For all $X \in \mathcal{X}$, the graph $G[X]$ is bounded by an induced simple cycle.*

(viii) *If $|\mathcal{X}| > 1$, the graph $G[X]$ contains more than one inner face for all $X \in \mathcal{X}$.*

(ix) *For all $XY \in E(T)$, the graph $G - C$ with $C := G[X \cap Y]$ has exactly two components A and B, with A being in the interior of C and B being in the exterior of C. Furthermore, there is a subtree $T' \subseteq T$ and a subset $\mathcal{X}' \subseteq \mathcal{X}$ such that $(\mathcal{X}', T')$ is a k-nice tree decomposition of $\mathrm{int}_G(C)$.*

Let us recall that we aim to show that every planar graph with treewidth at most k has a supergraph that is a planar quasi-k-tree. Before constructing k-nice tree decompositions, we show that they are useful, as every graph that has a k-nice tree decomposition is a planar quasi-k-tree.

Lemma 4.6: *Let G be a planar graph with a fixed planar embedding with a k-nice tree decomposition $(\mathcal{X}, T)$ of G. Then, G is a planar quasi-k-tree. Furthermore, if $k = 4$, the graph G is a nice planar quasi-k-tree.*

Proof. We show the statement by induction on the number $i \in \mathbb{N}$ of inner faces of G. For $i = 1$ and $k > 4$, the statement follows directly from Lemma 4.3, as G contains at most 3 vertices. For $i = 1$ and $k = 4$, the statement follows directly from Lemma 4.4.

Assume therefore, that $i > 1$ and that the statement holds for all graphs with a k-nice tree decomposition and at most $i - 1$ inner faces and let G be a graph with a k-nice tree decomposition $(\mathcal{X}, T)$ and i inner faces. Let $X \in \mathcal{X}$ be a bag that contains all vertices incident to the outer face of G. We know such a bag X exists because of Property (ii) of k-nice tree decompositions. Moreover, we know that $G[X]$ does not contain a face with size $k + 1$ or larger because of Property (vi) of k-nice tree decompositions. Thus, $G[X]$ has only faces of size at most k. Furthermore, $G[X]$ is bounded by a simple cycle because of Property (vii). Thus, $G[X]$ is a planar quasi-k-tree by Lemma 4.3. Furthermore, as $G[X]$ is 2-connected by Property (iv) of k-nice tree decompositions, if $k = 4$, then $G[X]$ is a nice planar quasi-k-tree by Lemma 4.4.

Let $f_1, \ldots, f_l$ be the inner faces in $G[X]$ that are not faces in G. We know that there is at least one such face, as otherwise we would be in the base case. Let f_i be such a face and C_i the simple cycle bounding it. We know that f_i is bounded by a simple cycle because $G[X]$ is 2-connected by Property (iv) of k-nice tree decompositions. Furthermore, we know that there exists a neighbor Y of X in T such that $X \cap Y$ induces C by Property (v) of k-nice tree decompositions. Thus, there exists a subtree $T' \subseteq T$ and a subset $\mathcal{X}' \subset \mathcal{X}$ such that $(\mathcal{X}', T')$ is a k-nice tree decomposition of $\text{int}_G(C_i)$. Recall that $G[X]$ contains multiple inner faces because of Property (viii) of k-nice tree decompositions and C_i bounds one of them. Thus, $\text{int}_G(C_i)$ has fewer inner faces than G. Therefore, by induction, $\text{int}_G(C_i)$ is a planar quasi-k-tree. Furthermore, if $k = 4$, by induction, $\text{int}_G(C_i)$ is a nice planar quasi-k-tree.

Let $\sigma_1, \ldots, \sigma_l$ be (nice if $k = 4$) construction sequences of the graphs $\text{int}_G(C_1), \ldots, \text{int}_G(C_l)$ minus the vertices in $C_1, \ldots, C_l$, respectively, and σ be a (nice if $k = 4$) construction sequence of $G[X]$. Then, $\sigma \cdot \sigma_1 \cdot \cdots \cdot \sigma_l$ is a (nice if $k = 4$) construction sequence of G. This concludes the proof. □

Now that we have shown that k-nice tree decompositions are useful, we prove that planar triangulations have k-nice tree decompositions. For that, we start by proving a key property of tree decompositions of triangulations in Lemmas 4.7 and 4.8.

We introduce the following notation for the next proof: Let $(\mathcal{X}, T)$ be a tree decomposition of a graph G and $A \subseteq V(G)$ a set of vertices of G. Then, we define $T[A] := T[(\bigcup_{X \in \mathcal{X}} X \cap A) - \emptyset]$. Note that $T[A]$ is connected if $G[A]$ is connected. Furthermore, let $X \in \mathcal{X}$. Then, X_A is the corresponding bag in $T[A]$. Let $(\mathcal{X}, T)$ be a tree decomposition of a graph G with width k. Then, we define $d_i((\mathcal{X}, T))$ as the number of edges $XY \in E(T)$ such that $|X \cap Y| - |C| = i$ with $C \subseteq X \cap Y$ being the smallest separator of G that is a subset of $X \cap Y$.

Lemma 4.7: *Let G be a graph with treewidth k such that every inclusion minimal separator of G separates G into exactly two components. Then, there exists a tree decomposition $(\mathcal{X}, T)$ of G with width k such that for every inclusion minimal separator $C \subseteq V(G)$ and every edge $XY \in E(T)$, it holds that the vertices $(X \cap Y) - C$ are in the same component of $G - C$.*

Proof. Let G be a triangulation and $(\mathcal{X}, T)$ be a tree decomposition of G with width k such that the tuple

$$[(k + 1, d_{k+1}((\mathcal{X}, T))), \ldots, (1, d_1((\mathcal{X}, T)))]$$

is the lexicographically smallest of any tree decomposition with width k of G.

We claim that $(\mathcal{X}, T)$ fulfils the claim. Assume that this is not the case. Then, there exists an inclusion minimal separator $C \subsetneq V(G)$, that separates G into two components A and B, and an edge $XY \in E(T)$ such that there are vertices $a, b \in (X \cap Y) - C$ with $a \in A$ and $b \in B$. Let $T_A := T[A \cup C]$, $T_B := T[B \cup C]$, and $T' := (V(T_A) \cup V(T_B), E(T_A) \cup E(T_B) \cup \{X_A X_B\})$. We claim that $(V(T'), T')$ is a tree decomposition of G. Note that the vertices in C are the only vertices that occur in both T_A and T_B. Thus, to show that $(V(T'), T')$ is a tree decomposition we only need to show that vertices in C occur in a subtree of T', as T_A and T_B are both tree decompositions. Clearly, $(A \cup C) \cap (B \cup C) = C$. Thus, it holds that $C \subsetneq X_A \cap X_B$ and $(V(T'), T')$ is a tree decomposition. Furthermore, T_A is tree decomposition of $C \cup A$ and T_B is a tree decomposition of $B \cup C$. Thus, $(V(T'), T')$ is a tree decomposition of G. We claim that

$$[(k+1, d_{k+1}((V(T)', T'))), \ldots, (1, d_1((V(T'), T')))]$$

is lexicographically smaller than

$$[(k+1, d_{k+1}((\mathcal{X}, T))), \ldots, (1, d_1((\mathcal{X}, T)))].$$

To see this, consider an edge $X'Y' \in E(T)$ and the corresponding edges $X_A Y_A \in E(T_A)$ and $X_B Y_B \in E(T_B)$. If neither $X_A \cap Y_A = X' \cap Y'$ nor $X_B \cap Y_B = X' \cap Y'$, we are done. Thus, assume without loss of generality, that $X_A \cap Y_A = X' \cap Y'$. Then, it holds that $X_B \cap Y_B = C$ (recall that $C \subseteq X \cap Y$ is a minimal separator), as all vertices in $X_A \cap Y_A = X' \cap Y'$ are in $A \cup C$ and it holds that $X_B \cap Y_B \subseteq B \cup C$. This means that every edge $X'Y' \in E(T)$ results either in two edges that are smaller, or in one edge that has the same size and one edge that that contains just C in T'. Additionally, recall that the edge $XY \in E(T)$ containes vertices $a, b \in (X \cap Y) - C$ with $a \in A$ and $b \in B$. Thus, the corresponding edges in T_A and T_B are smaller and

$$[(k+1, d_{k+1}((V(T)', T'))), \ldots, (1, d_1((V(T'), T')))]$$

is indeed lexicographically smaller than

$$[(k+1, d_{k+1}((\mathcal{X}, T))), \ldots, (1, d_1((\mathcal{X}, T)))].$$

This concludes the proof. □

With this property, we are now ready to show the following lemma.

Lemma 4.8: *Let G be a triangulation with treewidth k. Then, there exists a tree decomposition $(\mathcal{X}, T)$ of G that has width k such that for every edge $XY \in E(T)$, the vertices in $X \cap Y$ are an inclusion minimal separator of G.*

Proof. It is well known that every inclusion minimal separator in a triangulation induces a cycle. Let $\mathcal{C}$ be the set of all induced cycles in G with length at most k. This means that for all tree decompositions $(\mathcal{X}, T)$ of G and every edge $XY \in E(T)$, we have that $X \cap Y \subseteq V(C)$ for some $C \in \mathcal{C}$. We show the following stronger statement by induction on the number $n \in \mathbb{N}$ of vertices in G.

Claim: *Let G be an inner triangulation with treewidth k such that there is a tree decomposition $(\mathcal{X}, T)$ with width k of G with the following properties:*

(i) There is a bag $X \in \mathcal{X}$ such that every vertex incident to the outer face of G is in X.

(ii) For every edge $XY \in E(T)$, there is a subset $C \subseteq X \cap Y$ that induces a cycle in G.

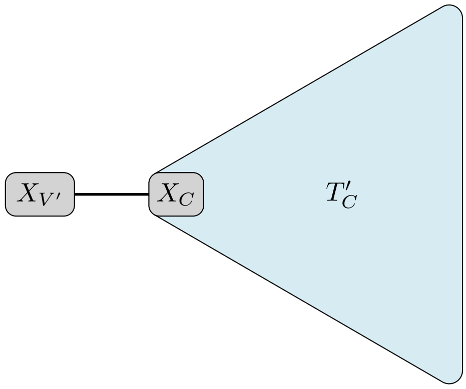


Figure 4.5: The tree resulting from adding the vertex $X_{V'}$ and the edge $X_{V'}X_C$ to T'_C.

(iii) *For every inclusion minimal separator $C \subseteq V(G)$ that induces a cycle in G and every edge $XY \in E(T)$, it holds that the vertices $(X \cap Y) - C$ are in the same component of $G - C$.*

Then, there exists a tree decomposition $(\mathcal{X}', T')$ of G that fulfills these properties and for every edge $XY \in E(T')$, the vertices in $X \cap Y$ induce a cycle in G.

By Lemma 4.7, there is a tree decomposition for a triangulation that fulfills Property (iii). The other two properties clearly hold for every tree decomposition of a triangulation.

For $n \leq k + 1$, the statement holds, as we can construct a tree decomposition with a single bag that contains all vertices.

Now, let G be an outerplanar graph with treewidth k and $(\mathcal{X}, T)$ be a tree decomposition with width k of G fulfilling Properties (i) to (iii). Furthermore, assume that the statement holds for all such graphs with at most $n - 1$ vertices. Let $\mathcal{C}$ be the set of all cycles that are induced by the subset of an edge in $E(T)$. Furthermore, let $V' \subseteq V(G)$ be the set of all vertices that are not in the interior of any cycle in $\mathcal{C}$. Then, $V' \subseteq X_{V'}$ holds for some bag $X_{V'} \in \mathcal{X}$. Otherwise, there would be an edge $XY \in E(T)$ such that $X \cap Y \subseteq V'$ and $X \cap Y$ separates V'. Thus, there is at least one vertex in V' that is in the interior of a cycle in $\mathcal{C}$. This contradicts that every vertex in V' is not in any cycle in $\mathcal{C}$. Let $\mathcal{C}' \subseteq \mathcal{C}$ be the set of non-empty cycles in $\mathcal{C}$ that are not dominated by another cycle in $\mathcal{C}$. By Property (iii), we have that the interiors of the cycles in $\mathcal{C}'$ are pairwise disjoint. Then, for every $C \in \mathcal{C}'$, we claim that the tree decomposition $(\mathcal{X}_C, T_C) := (V(T_C), T[V(\mathrm{int}_G(C))])$ of $\mathrm{int}_G(C)$ fulfills Properties (i) to (iii). Property (i) holds, as C bounds $\mathrm{int}_G(C)$ and is contained in a bag, because there is an edge in T that contains the vertices of C. Property (ii) holds, because otherwise there would be two cycles in $\mathcal{C}'$ that are not disjoint, which contradicts Property (iii) of $(\mathcal{X}, T)$. Finally, Property (iii) follows directly from Property (iii) holding for $(\mathcal{X}, T)$.

Thus, by induction, there is a tree decomposition $(\mathcal{X}'_C, T'_C)$ of $\mathrm{int}_G(C)$, such that for every edge $X'Y' \in E(T'_C)$ the vertices in $X' \cap Y'$ induce a cycle in $\mathrm{int}_G(C)$. Let $X_C \in \mathcal{X}'_C$ be the bag containing C. Then, the tree decomposition $(\mathcal{X}'_C \cup \{X_{V'}, T'_C + \{X_{V'}\} + \{X_{V'}X_C\})$ is the desired tree decomposition of $G[V' \cup V(\mathrm{int}_G(C)]$. The tree decomposition $(\mathcal{X}'_C \cup \{X_{V'}\}, T'_C + \{X_{V'}\} + \{X_{V'}X_C\})$ is depicted in Figure 4.5. Repeating this for all $C \in \mathcal{C}'$ yields the desired tree decomposition of G. $\square$

Now that we have shown this property, we have everything we need, to show that every triangulation has a k-nice tree decomposition.

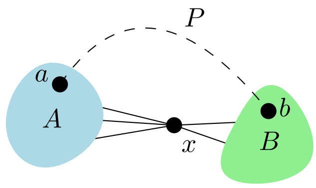


Figure 4.6: The graph $G[X]$ with a cut vertex x and two components A and B of $G[X] - x$. If there are vertices $a \in A$ and $b \in B$ and a neighbor Y of Y with $a, b \in Y$, there exists a path P connecting a and b in $G[X] - x$.

Lemma 4.9: *Let G be a planar triangulation with treewidth $k \geq 4$. Then, G has a k-nice tree decomposition.*

Proof. As the outer face of G is a triangle, every tree decomposition of minimal width $(\mathcal{X}, T)$ of H contains a bag that contains all vertices that are incident to the outer face proving Properties (i) and (ii) of k-nice tree decompositions.

By Lemma 4.8, there exists a tree decomposition $(\mathcal{X}, T)$ of G fulfilling Properties (i) to (iii) of k-nice tree decompositions.

Property (iv). Now, we show that Property (iv) of k-nice tree decompositions holds, i. e. that for every bag $X \in \mathcal{X}$, the graph $G[X]$ is 2-connected. Assume this is not the case and $G[X]$ is not 2-connected for some $X \in \mathcal{X}$. This means there exists a vertex $x \in X$ such that $G[X] - x$ is disconnected. Let $A, B \subset X$ be two different inclusion maximal components of $G[X] - x$ (see Figure 4.6). Using that every edge in T induces a cycle, we show that there exists no neighbor of X in T that contains a vertex $a \in A$ and a vertex $b \in B$. This holds, as every edge of T induces a cycle and thus A and B would not be distinct components of $G[X] - x$, as a and b would be connected by a path. Furthermore, as G is a triangulation and therefore is 2-connected, there is a path in $G - x$ connecting A and B. As X is the only bag containing vertices from A and B, this path exists in $G[X]$ as well. Thus, A and B are not distinct components of $G[X] - x$, $G[X]$ is 2-connected, and thus Property (iv) holds.

Property (v). Now we show that Property (v) of k-nice tree decompositions holds, i. e. that for all $X \in \mathcal{X}$ and every inner face f of $G[X]$ that is not a face in G, there exists a neighbor Y of X in T such that f is bounded by a cycle C that is induced by $X \cap Y$. For that, let $X \in \mathcal{X}$ be a bag and f be a face in $G[X]$ that is not a face in G. Property (iv) of k-nice tree decompositions gives us that $G[X]$ is 2-connected. Thus, f is bounded by a simple cycle $C := (v_1, \ldots, v_l)$. We aim to find a neighbor Y of X in T such that $X \cap Y$ induces C. As f is not a face in G, there exists at least one vertex v in the interior of f in G. Let T' be the component of $T - X$ such that there exists a bag in T' that contains v. Furthermore, let Y be the bag in T' that is adjacent to X in T and let $A \subset V(G)$ be the inclusion maximal component of G containing v that is separated by $C' := X \cap Y$. Our goal is to show that $C = C'$. First, as C' is an inclusion minimal separator in G, every vertex in C' is adjacent to a vertex in A. Thus, every vertex in C' is in C, as otherwise G would not be planar. Now, assume that there exists a vertex in C that is not part of C' and let $v_i, \ldots, v_j$ be a non-empty inclusion maximal interval of vertices in C that are not in C' as depicted in Figure 4.7. This means v_{i-1} and v_{j+1} are part of C'. Note that $v_{i-1} \neq v_{j+1}$, as all vertices of C' are part of C and C' contains at least three vertices, as it induces a simple cycle. Furthermore, as C bounds f, there is no edge $v_{i-1}v_{j+1}$ in the interior of f. Otherwise, f would not be a face in $G[X]$, as v_{i-1} and v_{j+1} are not consecutive in C. This

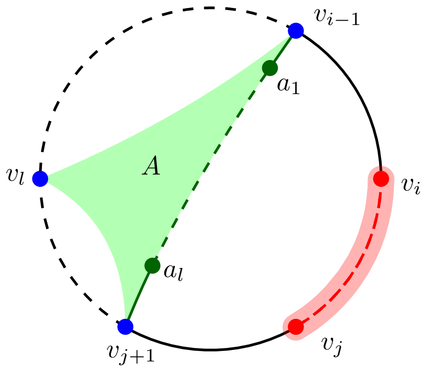


Figure 4.7: The cycle C. The vertices $v_i, \ldots, v_j$ (red) are not in C', while the vertices v_{i-1}, v_{j+1} and at least one more vertex v_l are in C'. The edge $v_{i-1}v_{j+1}$ can be drawn along the path $P := (v_{i-1}, a_1, \ldots, a_l, v_{j+1})$, which is part of A.

holds once again because C' contains at least three vertices. Let $P := (v_{i-1}, a_1, \ldots, a_l, v_{j+1})$ be a path with $a_1, \ldots, a_l \in A$. Furthermore, let P be minimal in the sense that there exists no other such path P' in $G[A \cup \{v_{i-1}, v_{j+1}\}]$ with $P' \subseteq \text{int}_G((v_{i-1}, a_1, \ldots, a_l, v_{j+1}, v_j, v_{j-1}, \ldots, v_i))$. This is depicted in Figure 4.7. As C' is a minimal separator, there is no vertex in $G - (A \cup V(C'))$ that is adjacent to A. Thus, we can draw an edge along P that connects v_{i-1} and v_{j+1}. This is a contradiction to G being a triangulation. Consequently, there exists no vertex in C that is not in C'. As we already showed that every vertex in C' is in C and C' is an induced cycle, we have that C is induced by $C' = X \cap Y$. Therefore, Property (v) holds.

Properties (v) and (vi). We can assume without loss of generality, that there are no two different bags in $\mathcal{X}$ that contain exactly the same vertices. Thus, for each $XY \in E(T)$, it holds that $|X \cap Y| \leq k$. Therefore, Property (v) immediately implies Property (vi).

Property (vii). Now, we show that Property (vii) of k-nice tree decompositions holds, i. e. that for all $X \in \mathcal{X}$, the graph $G[X]$ is bounded by a simple induced cycle. For the bag that contains the outer face of G, this holds, as G is bounded by a triangle. Together with Property (v), this immediately implies Property (vii).

Property (viii). Now, we show that Property (viii) of k-nice tree decompositions holds. Assume that Property (viii) does not hold and thus $|\mathcal{X}| > 1$ and $G[X]$ contains only one inner face for some $X \in \mathcal{X}$. We assume without loss of generality that the tree decomposition $(\mathcal{X}, T)$ is minimal in the sense that there exists no other tree decomposition fulfilling Properties (i) to (vi) of k-nice tree decompositions that contains fewer bags. We show that if $G[X]$ has only one inner face, then $(\mathcal{X}, T)$ is not minimal. As $G[X]$ contains only a single inner face, we know that all neighbors of X in T contain every vertex in X. Otherwise, there is a non-spanning induced cycle in $G[X]$ because of Property (iii). As $G[X]$ is 2-connected because of Property (iv), this would immediately contradict $G[X]$ having only one inner face. Let Y be a neighbor of X in T (such a neighbor exists as $|\mathcal{X}| > 1$) and let $(\mathcal{X} - X, T')$ be the tree decomposition that results from contracting XY in T. Let the vertex resulting from the contraction be Y' with the corresponding bag containing the vertices in $X \cup Y$. Note that $X \subseteq Y$ and thus $X \cup Y = Y$. As Properties (i), (ii), (iv), and (vi) are statements only regarding single bags and not the edges in T' and $\mathcal{X} - X \subseteq \mathcal{X}$, they still hold for $(\mathcal{X} - X, T')$. Furthermore, as $Z \cap X = Z \cap Y'$ for all neighbors $Z \neq Y$ of X in T, Properties (iii) and (v) hold as well. This contradicts the minimality of $(\mathcal{X}, T)$. Thus, Property (viii) holds.

Property (ix). It remains to show that Property (ix) of k-nice tree decompositions holds. For that, let $XY \in E(T)$ and $C := G[X \cap Y]$. We start by showing that $G - C$ has exactly two components A and B with A being in the interior of C and B being in the exterior of C. As every edge in T is an inclusion minimal separator, $G - C$ has at least 2 components. As C is an induced cycle because of Property (iii), every component of $G - C$ is either entirely in the interior of C or entirely in the exterior of C. Assume Property (ix) does not hold. Then, $G - C$ contains either two components that are both in the interior or both in the exterior of C, respectively. Without loss of generality, we assume that $G - C$ has at least two components A and B in the interior of C. As C is an inclusion minimal separator, every vertex in C is adjacent to some vertex in A and some vertex in B. Now, we look at the graph $G[A \cup B \cup V(C)]$ and add an additional vertex v that is adjacent to all vertices of C. We can add this vertex v in the exterior of C maintaining the planarity of the graph. However, contracting A and B into a single vertex a and b, respectively, yields a minor of $K_{3,3}$ with a, b, and v forming the vertices of one of the partitions and any three vertices in C forming the other partition (see Figure 4.8). This contradicts G being planar. Thus, $G - C$ has exactly one component in the interior of C and exactly one component in the exterior of C. Let A be the component of $G - C$ in the interior of C.

Let T' and T'' be the components of $T - XY$ with T' containing at least one bag that contains a vertex in the interior of C. Furthermore, let $\mathcal{X}' \subseteq \mathcal{X}$ and $\mathcal{X}'' \subseteq \mathcal{X}$ be the sets of bags that occur in T' and T'' respectively. We claim that $(\mathcal{X}', T')$ is a k-nice tree decomposition of $\mathrm{int}_G(C)$. We start by showing that $(\mathcal{X}', T')$ is a tree decomposition of $\mathrm{int}_G(C)$. Assume this is not the case. Then, both $(\mathcal{X}', T')$ and $(\mathcal{X}'', T'')$ contain vertices from the interior of C or both tree decompositions contain vertices from the exterior of C. Assume without loss of generality that both tree decompositions contain vertices from the interior of C. This contradicts that $G - C$ has exactly one component in the interior of C. Thus, $(\mathcal{X}', T')$ is a tree decomposition of $\mathrm{int}_G(C)$. It remains to show that $(\mathcal{X}', T')$ is k-nice.

For that, we first note that Properties (i), (iii), (iv), and (vi) to (ix) still hold. This follows directly from Properties (i), (iii), (iv), and (vi) to (ix) holding for $(\mathcal{X}, T)$.

Thus, it remains to show that Properties (ii) and (v) hold. As the outer face of $\mathrm{int}_G(C)$ is bounded by C and $V(C) \subseteq Y$, Property (ii) holds for $(\mathcal{X}', T')$. Regarding Property (v), it is clear that the only bag it might not hold for is Y, as all other bags in T' have the same neighbors in T' as in T. Assume Property (v) does not hold for Y. This means that there is an inner face in $G[Y]$ bounded by a cycle C' that is not a face in G such that there is no neighbor Z of Y in T' such that C' is induced by $Y \cap Z$. As the only neighbor of Y in T that is not in T' is X and Property (v) of k-nice tree decompositions holds for $(\mathcal{X}, T)$, it holds that $X = Z$. We know that $X \cap Y$ induces C. As C is an induced simple cycle that bounds $G[Y]$, this means that $G[Y]$ has only a single inner face. This contradicts Property (viii) of k-nice tree decompositions for $(\mathcal{X}, T)$. Thus, Property (v) holds for $(\mathcal{X}', T')$, $(\mathcal{X}', T')$ is a k-nice tree decomposition of $\mathrm{int}_G(C)$, and Property (ix) holds.

As we have shown that Properties (i) to (ix) hold, this concludes the proof. □

Finally, we use the following lemma by Biedl and Ruiz Velázquez to triangulate planar graphs without increasing the treewidth.

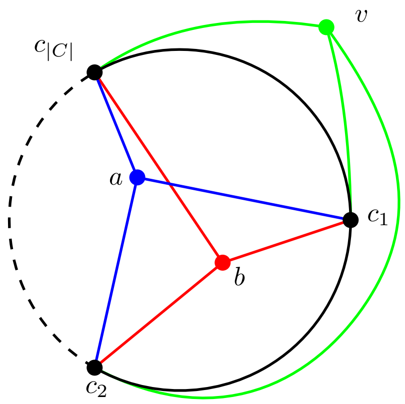


Figure 4.8: $K_{3,3}$ with one partition consisting of the vertices on a cycle $C := (c_1, \ldots, c_{|C|})$ and the other partition being $\{a, b, v\}$.

Lemma 4.10 (Biedl and Ruiz Velázquez, 2013[12])**:** *Let G be a planar graph with* $\mathrm{tw}(G) \leq k$ *with* $|V(G)| \geq 4$ *vertices. Then, there exists a triangulation* $G' \supseteq G$ *with* $|V(G')| = |V(G)|$ *and* $\mathrm{tw}(G') = \max\{\mathrm{tw}(G), 3\}$.

Now, we can prove Theorem 1.5, which is the main result of this chapter.

Theorem 1.5: *For every planar graph G with treewidth $k \geq 4$, there exists a planar quasi-k-tree H with $G \subseteq H$ and $|V(G)| = |V(H)|$. Furthermore, if $k = 4$, there exists such a planar quasi-k-tree H that is nice.*

Proof. By Lemma 4.10, there exists a triangulation H of G with the same treewidth as G and the same set of vertices as G. Thus, by Lemma 4.9, there exists a k-nice tree decomposition of G and, by Lemma 4.6, H is a planar quasi-k-tree. Furthermore, for $k = 4$, Lemma 4.6 gives us that H is a *nice* planar quasi-k-tree. As $|G| = |H|$, this concludes the proof. □

This proves Conjecture 2.18 in the positive, as the existence of a nice construction sequence is equivalent to the existence of a sequence of vertex insertions as defined in Definition 2.17. Furthermore, we now have a better understanding of what is sufficient to construct all edge maximal planar graphs with treewidth k. This motivates the following corollaries.

Corollary 4.11: *Every planar quasi-4-tree is a subgraph of a nice planar quasi-4-tree with the same set of vertices.*

Corollary 4.12: *For all $k \geq 4$, every edge maximal planar graph of treewidth at most k is a planar quasi-k-tree.*

Corollary 4.13: *For all $k \geq 4$, every planar graph with treewidth at most k is a subgraph of a planar quasi-k-tree. Furthermore, every planar graph with treewidth at most 4 is a subgraph of a* nice *planar quasi-4-tree.*

In particular, this means that every triangulation with treewidth at most 4 is a nice planar quasi-k-tree. For $k > 4$, it is not clear whether this holds. This leads us to the following open question.

Question 4.14: *Is there a planar quasi-k-tree that is not a subgraph of a nice planar quasi-k-tree?*

5 Upward Planar Quasi-4-Trees

In this chapter, we look at the stack number of upward planar quasi-4-trees. We start by definining upward planar quasi-4-trees and *canonically* upward planar-4-trees in Section 5.1. Then, we look at two different subclasses of upward planar quasi-4-trees in Sections 5.2 and 5.3.

5.1 Upward and Canonically Upward Planar Quasi-4-Trees

We say a directed graph G is an *upward planar quasi*-4-*tree* if the underlying undirected graph is a planar quasi-4-tree and G has an upward planar embedding.

However, to properly use the construction sequence of a planar quasi-4-tree, we need a canonical embedding, i. e. an embedding that results from a construction sequence. We call those graphs, i. e. upward orientations of canonical embeddings, *canonically upward planar quasi*-4-*trees*. We relate these two graph classes, as well as several other graph classes at the end of this section.

We can obtain canonically upward planar quasi-4-trees by using a construction sequence as defined in Definition 4.1 by simply replacing every occurrence of *planar* with *upward planar*. This results in the following definition.

Definition 5.1: *Let G be a directed graph and $\sigma \coloneqq (v_1, \dots, v_n)$ be a vertex ordering of G. If there exists an upward planar embedding of G and an integer $l \leq k$ such that*

- $G[\{v_1, \dots, v_l\}]$ *is a simple cycle that bounds the outer face of G,*
- *for every $l < i \leq n$, every face in $G_i \coloneqq G[\{v_1, \dots, v_i\}]$ is bounded by at most k vertices, and*
- *for every $l < i \leq n$, it holds that $\deg_{G_i}(v_i) \geq 1$,*

we call σ a k-construction sequence *of G and G a* canonically upward planar quasi-k-tree. *Furthermore, if $\deg_{G_i}(v_i) \geq 2$ for every $l < i \leq n$, we call σ a* nice *k-construction sequence of G and G a* nice *canonically upward planar quasi-k-tree. We refer to the upward planar embedding of G for that a (nice) k-construction sequence exists as a* (nice) canonical *embedding of G. We omit the k and refer to σ as a construction sequence of G if the k is clear from the context.*

However, as we only look at canonically upward planar quasi-4-trees and not at canonically upward planar quasi-k-trees for arbitrary k, we can instead use a more explicit definition akin to Definition 2.17 by listing all possible ways to insert a vertex in a given upward planar quasi-4-tree. This leads to the following definition of nice canonically upward planar quasi-4-trees.

Definition 5.2: *A graph G with a fixed upward planar embedding is a nice canonically upward planar quasi-*4*-tree if G is an upward planar embedding of C_3 or C_4 or it can be constructed using one of the following rules:*

Figure 5.1: On the left side, a graph resulting from applying the insertion rule (T2)$^-$ to the triangular face (t_1, t_2, t_3). On the right side, a graph resulting from applying the insertion rule (T2)$^+$

.

(i) *G is obtained from a nice canonically upward planar quasi-4-tree $G' = (V', E')$ with a fixed canonical upward planar embedding by inserting a vertex $v \notin V'$ into a triangular face (t_1, t_2, t_3) of G' and at least two edges incident to v and a vertex in $\{t_1, t_2, t_3\}$, with all inserted edges being upward.*

(ii) *G is obtained from a nice canonically upward planar quasi-4-tree $G' = (V', E')$ with a fixed canonical upward planar embedding by inserting a vertex $v \notin V'$ into a quadrangular face (q_1, q_2, q_3, q_4) of G' and an edge incident to v and q_1 as well as an edge incident to v and q_3 and potentially edges incident to v and a vertex in $\{q_2, q_4\}$, with all inserted edges being upward.*

In this section, especially in the proof of Theorem 1.6, we need to reference the specific way a vertex is inserted. Therefore, we give each way of inserting a vertex a name, depending on the orientation of the cycle bounding the face the vertex is inserted in and the orientation of the edges that are inserted. This results in the allowed vertex insertions (T1) to (T5), (Q1.1) to (Q1.4), (Q2.1) to (Q2.4), (Q3.1) to (Q3.4), and (Q4.1) to (Q4.4). These rules still allow some freedom, as there are optional edges that do not have to be added. For a rule R, we refer to the insertion where no optional edge is added as R^- and to the insertion where at least one optional edge is added as R^+. For an example, see Figure 5.1.

Definition 5.3: *A graph G with a fixed upward planar embedding is a nice canonically upward planar quasi-4-tree if G is an upward planar embedding of C_3 or C_4 or it can be constructed using one of the following rules:*

(i) *G is obtained from a nice canonically upward planar quasi-4-tree $G' = (V', E')$ with a fixed canonical upward planar embedding by inserting a vertex $v \notin V'$ and edges e_1, e_2, and possibly e_3, where (t_1, t_2, t_3) is a triangular face of G' and $t_1t_2, t_2t_3 \in E'$, with e_1, e_2, and e_3 fulfilling one of the following statements:*

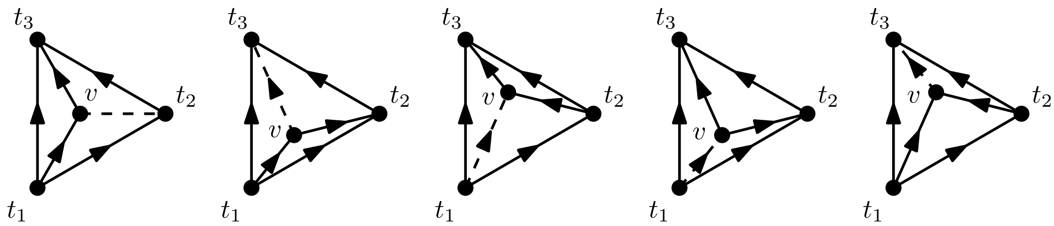
   (T1) *$e_1 = t_1v$, $e_2 = vt_3$, and $e_3 \in \{vt_2, t_2v\}$, as depicted in Figure 5.2a.*

   (T2) *$e_1 = t_1v$, $e_2 = vt_2$, and $e_3 = vt_3$, as depicted in Figure 5.2b.*

   (T3) *$e_1 = t_2v$, $e_2 = vt_3$, and $e_3 = t_1v$, as depicted in Figure 5.2c.*

   (T4) *$e_1 = vt_2$, $e_2 = vt_3$, and $e_3 = t_1v$, as depicted in Figure 5.2d.*

   (T5) *$e_1 = t_1v$, $e_2 = t_2v$, and $e_3 = vt_3$, as depicted in Figure 5.2e.*



(a) (T1) **(b)** (T2) **(c)** (T3) **(d)** (T4) **(e)** (T5)

Figure 5.2: All possible ways to insert a vertex v in a triangular face (t_1, t_2, t_3) of a nice canonically upward planar quasi-4-tree. Dashed edges may be omitted. Undirected dashed edges may be added in either direction.

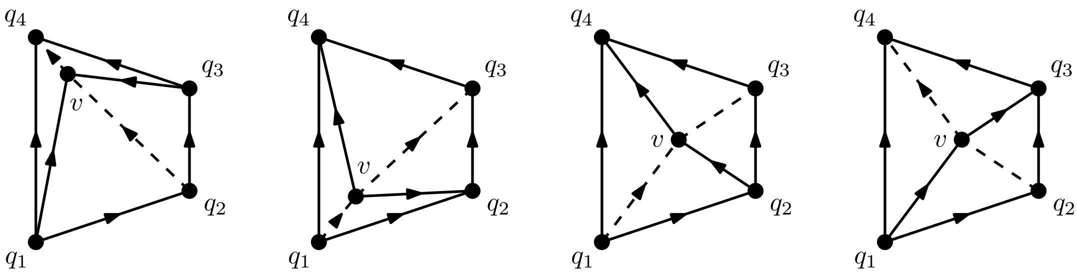
(ii) *G is obtained from a nice canonically upward planar quasi-4-tree $G' = (V', E')$ with a fixed canonical upward planar embedding by inserting a vertex $v \notin V'$ and edges e_1, e_2, and possibly e_3 and/or e_4, where (q_1, q_2, q_3, q_4) is a quadrangular face of G', $q_1q_2, q_1q_4, q_2q_3, q_3q_4 \in E'$, and e_1, e_2, e_3, and e_4 fulfill one of the following statements:*

(Q1.1) *$e_1 = q_1v$, $e_2 = q_3v$, $e_3 = q_2v$, and $e_4 = vq_4$, as depicted in Figure 5.3a.*

(Q1.2) *$e_1 = vq_2$, $e_2 = vq_4$, $e_3 = q_1v$, and $e_4 = vq_3$, as depicted in Figure 5.3b.*

(Q1.3) *$e_1 = q_2v$, $e_2 = vq_4$, $e_3 = q_1v$, and $e_4 = vq_3$, as depicted in Figure 5.3c.*

(Q1.4) *$e_1 = q_1v$, $e_2 = vq_3$, $e_3 = q_2v$, and $e_4 = vq_4$, as depicted in Figure 5.3d.*



(a) (Q1.1) **(b)** (Q1.2) **(c)** (Q1.3) **(d)** (Q1.4)

Figure 5.3: All possible ways to insert a vertex v in a quadrangular face (q_1, q_2, q_3, q_4) with $q_1q_2, q_1q_4, q_2q_3, q_3q_4 \in E(G')$ of a nice canonically upward planar quasi-4-tree G'. Dashed edges may be omitted. Undirected dashed edges may be added in either direction.

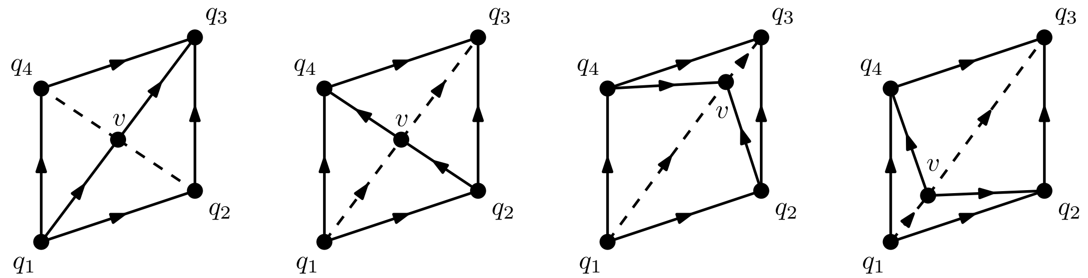
(iii) *G is obtained from a nice canonically upward planar quasi-4-tree $G' = (V', E')$ with a fixed canonical upward planar embedding by inserting a vertex $v \notin V'$ and edges e_1, e_2. and possibly e_3 and/or e_4, where (q_1, q_2, q_3, q_4) is a quadrangular face of G', $q_1q_2, q_1q_4, q_2q_3, q_4q_3 \in E'$, and e_1, e_2, e_3, and e_4 fulfill one of the following statements:*

(Q2.1) *$e_1 = q_1v$, $e_2 = vq_3$, $e_3 \in \{q_2v, vq_2\}$, and $e_4 \in \{vq_4, q_4v\}$, as depicted in Figure 5.4a.*

(Q2.2) *$e_1 = q_2v$, $e_2 = vq_4$, $e_3 = q_1v$, and $e_4 = vq_3$, as depicted in Figure 5.4b.*

(Q2.3) *$e_1 = q_2v$, $e_2 = q_4v$, $e_3 = q_1v$, and $e_4 = vq_3$, as depicted in Figure 5.4c.*

(Q2.4) *$e_1 = vq_2$, $e_2 = vq_4$, $e_3 = q_1v$, and $e_4 = vq_3$, as depicted in Figure 5.4d.*



(a) (Q2.1) **(b)** (Q2.2) **(c)** (Q2.3) **(d)** (Q2.4)

Figure 5.4: All possible ways to insert a vertex v in a quadrangular face (q_1, q_2, q_3, q_4) with $q_1q_2, q_1q_4, q_2q_3, q_4q_3 \in E(G')$ of a nice canonically upward planar quasi-4-tree G'. Dashed edges may be omitted. Undirected dashed edges may be added in either direction.

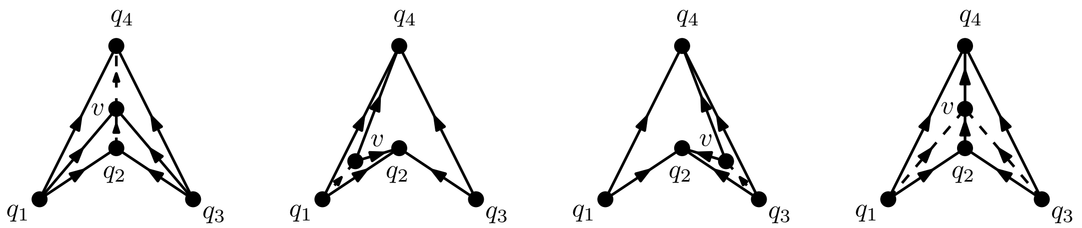
(iv) *G is obtained from a nice canonically upward planar quasi-4-tree $G' = (V', E')$ with a fixed canonical upward planar embedding by inserting a vertex $v \notin V'$ and edges e_1, e_2, and possibly e_3 and/or e_4, where (q_1, q_2, q_3, q_4) is a quadrangular face of G', $q_1q_2, q_1q_4, q_3q_2, q_3q_4 \in E'$, and e_1, e_2, e_3, and e_4 fulfill one of the following statements:*

(Q3.1) $e_1 = q_1v$, $e_2 = q_3v$, $e_3 = q_2v$, and $e_4 = vq_4$, as depicted in Figure 5.5a.

(Q3.2) $e_1 = vq_2$, $e_2 = vq_4$, $e_3 = q_1v$, and no e_4, as depicted in Figure 5.5b.

(Q3.3) $e_1 = vq_2$, $e_2 = vq_4$, $e_3 = q_3v$, and no e_4, as depicted in Figure 5.5c.

(Q3.4) $e_1 = q_2v$, $e_2 = vq_4$, $e_3 = q_1v$, and $e_4 = q_3v$, as depicted in Figure 5.5d.



(a) (Q3.1) **(b)** (Q3.2) **(c)** (Q3.3) **(d)** (Q3.4)

Figure 5.5: All possible ways to insert a vertex v in a quadrangular face (q_1, q_2, q_3, q_4) with $q_1q_2, q_1q_4, q_3q_2, q_3q_4 \in E(G')$ of a nice canonically upward planar quasi-4-tree G'. Dashed edges may be omitted.

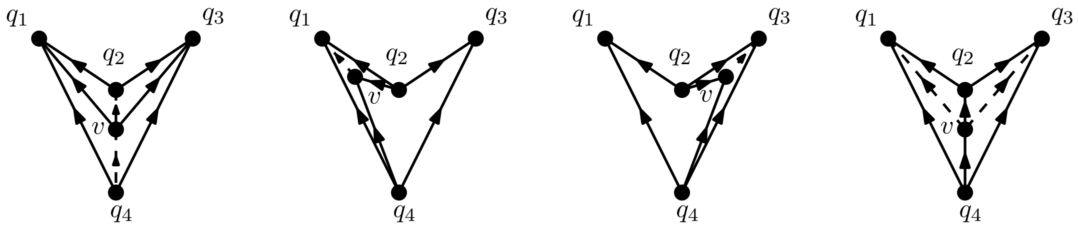
(v) *G is obtained from a nice canonically upward planar quasi-4-tree $G' = (V', E')$ with a fixed canonical upward planar embedding by inserting a vertex $v \notin V'$ and edges e_1, e_2, and possibly e_3 and/or e_4, where (q_1, q_2, q_3, q_4) is a quadrangular face of G', $q_4q_1, q_4q_3, q_2q_3, q_2q_1 \in E'$, and e_1, e_2, e_3, and e_4 fulfill one of the following statements:*

(Q4.1) $e_1 = vq_1$, $e_2 = vq_3$, $e_3 = vq_2$, and $e_4 = q_4v$, as depicted in Figure 5.6a.

(Q4.2) $e_1 = q_2v$, $e_2 = q_4v$, $e_3 = vq_1$, and no e_4, as depicted in Figure 5.6b.

(Q4.3) $e_1 = q_2v$, $e_2 = q_4v$, $e_3 = vq_3$, and no e_4, as depicted in Figure 5.6c.

(Q4.4) $e_1 = vq_2$, $e_2 = q_4v$, $e_3 = vq_1$, and $e_4 = vq_3$, as depicted in Figure 5.6d.



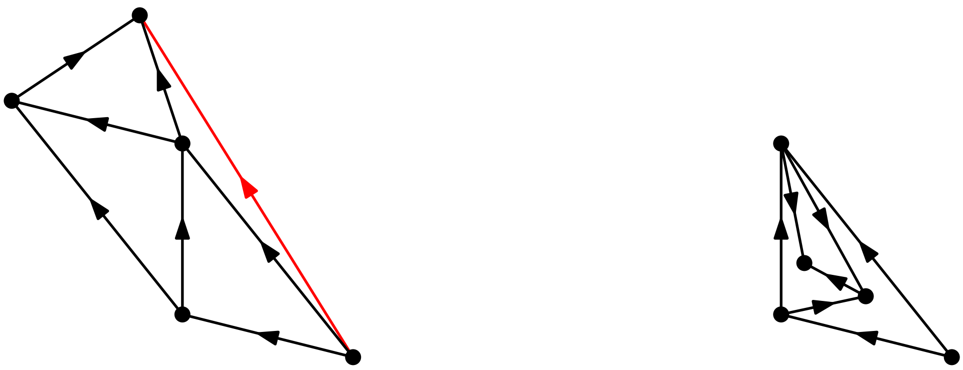
(a) (Q4.1) **(b)** (Q4.2) **(c)** (Q4.3) **(d)** (Q4.4)

Figure 5.6: All possible ways to insert a vertex v in a quadrangular face (q_1, q_2, q_3, q_4) with $q_4q_1, q_4q_3, q_2q_3, q_2q_1 \in E(G')$ of a nice canonically upward planar quasi-4-tree G'. Dashed edges may be omitted.

Relating upward planar graph classes. In Section 2.4 we discussed the relation between upward planar graphs with treewidth 2 and 3, upward planar 2-trees, and upward planar 3-trees. We want to revisit the Venn diagram showing these relations depicted in Figure 2.11 and add canonically upward planar quasi-4-trees and upward planar quasi-4-trees to it.

We start by observing that canonically upward planar quasi-4-trees are a proper subset of upward planar quasi-4-trees. For example, the graph depicted in Figure 5.7 is a planar quasi-4-tree that has an upward planar embedding but has no upward canonical embedding with the given orientation. However, the graph is a subgraph of a canonically upward planar quasi-4-tree. This leads us to the following open question.

Question 5.4: *Is there an upward planar quasi-*4*-tree that is not a subgraph of a canonically upward planar quasi-*4*-tree?*



(a) The unique combinatorial upward planar embedding of G in black.

(b) A canonical embedding of G that is not upward.

Figure 5.7: A planar quasi-4-tree G that is upward planar but not canonically upward planar in black. Adding the red edge yields a canonically upward planar quasi-4-tree.

We already discussed in Section 2.4 that there are graphs with treewidth 2 that are not a subgraph of an upward planar 2-tree as well as upward planar 2-trees that are not a subgraph of an upward planar 3-tree. The two examples we gave for such graphs are depicted in Figures 2.9 and 2.10. However, both of these graphs are subgraphs of canonically upward planar quasi-4-trees. Canonically upward planar quasi-4-trees that are supergraphs of the

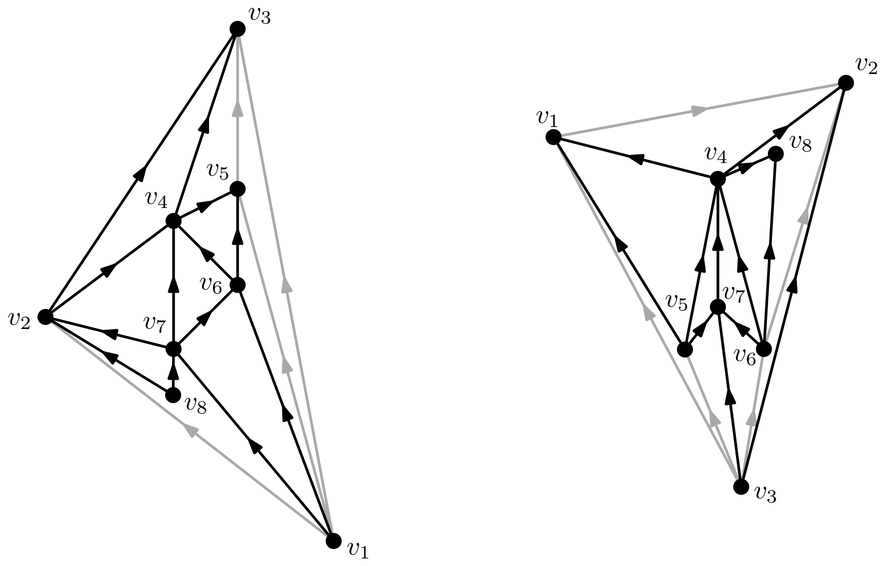


Figure 5.8: On the left is the graph depicted in Figure 2.9 in black. On the right is the graph depicted in Figure 2.10 in black. In both cases adding the gray edges results in a canonically nice upward planar quasi-4-tree with construction sequence $(v_1, \dots, v_8)$.

graphs depicted in Figures 2.9 and 2.10 are depicted in Figure 5.8. It is an open question whether there is an upward planar graph with treewidth at most 4 that is not a subgraph of a (canonically) upward planar quasi-4-tree. This leads us to the following open questions and to the Venn diagram depicted in Figure 5.9.

Question 5.5: *Is there an upward planar* 2*-tree that is not a subgraph of a (canonically) upward planar quasi-*4*-tree?*

Question 5.6: *Is there an upward planar graph with treewidth* 2, 3, *or* 4 *that is not a subgraph of a (canonically) upward planar quasi-*4*-tree?*

***A*-constructible and *A*-free.** As it proves to be rather difficult to bound the stack number of all nice canonically upward planar quasi-4-trees, we look at several subclasses of nice canonically planar quasi-4-trees instead. We believe that insights for those subclasses can prove useful in the pursuit of a bound for the entire graph class. We obtain these subclasses by restricting the allowed vertex insertions. We define these restricted graph classes in the following way.

Definition 5.7: *A nice upward planar quasi-*4*-tree is A-free for a set of insertion rules A if there exists a nice construction sequence of G that does not contain any rule in A. Conversely, G is A-constructible if there exists a nice construction sequence of G that contains only rules in A.*

We look at two different ways of doing this in the following two sections. In Section 5.2, we look at $\{(Q2.2)^-, (Q3.4)^-, (Q4.4)^-\}$-free nice canonically upward planar quasi-4-trees. In Section 5.3, we look at $\{(Q3.1)^-, (Q3.2)^-, (Q3.3)^-, (Q4.1)^-, (Q4.2)^-, (Q4.3)^-\}$-constructible nice upward planar quasi-4-trees.

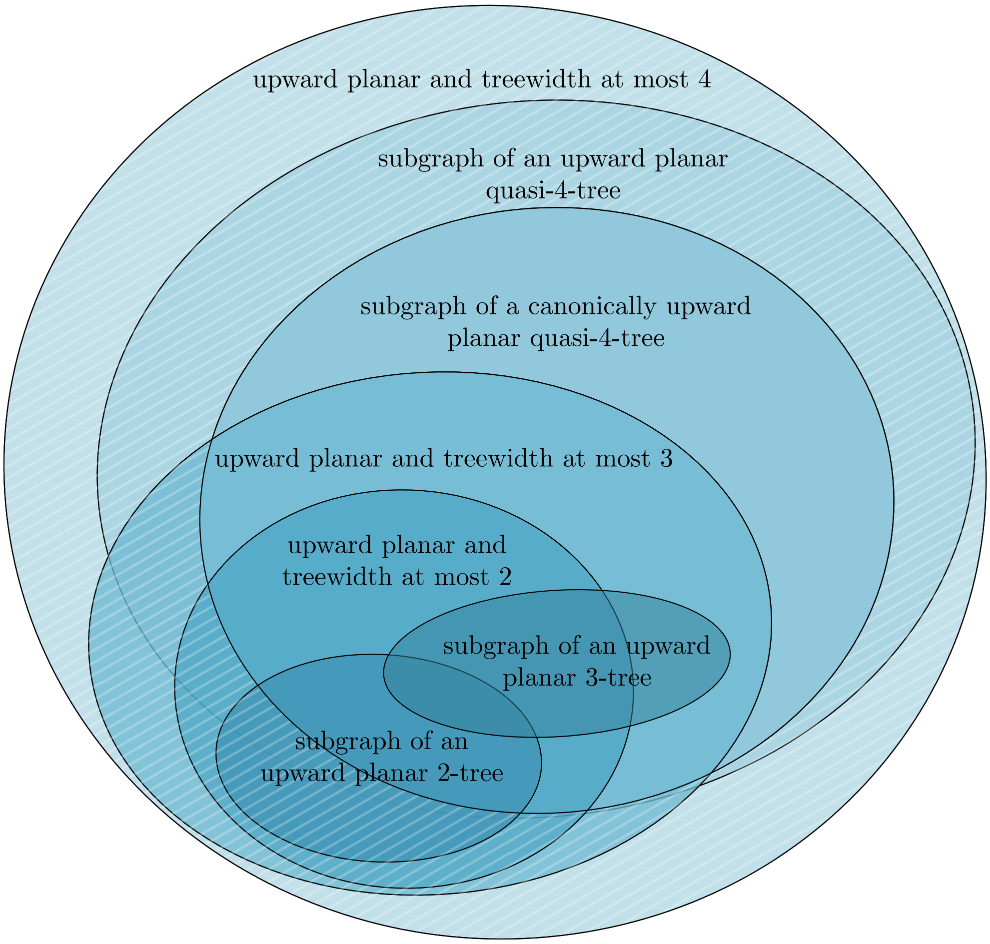


Figure 5.9: A Venn diagram relating upward planar graphs with treewidth 2, 3, and 4, upward planar 2-trees, upward planar 3-trees, canonically upward planar quasi-4-trees, and upward planar quasi-4-trees. The hatched areas may be empty.

5.2 Nice Canonically Upward Planar Quasi-4-Trees With Bounded Depth

In this section, we examine the stack number of nice canonically upward planar quasi-4-trees with bounded *depth*. For that, we start by defining the depth of a planar quasi-k-tree. We do this inductively. To do this we have to make a small observation first.

Observation 5.8: *Let G be a nice planar quasi-k-tree with a nice construction sequence $\sigma := (v_1, \dots, v_n)$ and an outer face that contains exactly the vertices $v_1, \dots, v_l$ with $l \leq k$. Furthermore, let $f_1, \dots, f_m$ be the faces in $G[\{v_1, \dots, v_{l+1}\}]$. Then, the graphs $G_1, \dots, G_m$ that are bounded by the cycles bounding $f_1, \dots, f_m$ are nice planar quasi-k-trees (see Figure 5.10).*

Intuitively, one can think of the depth of a planar quasi-k-tree as the number of steps of its construction if we allow to insert a vertex into each face in each step of the construction. This differs from our usual definition where we place a vertex in exactly one face in each step of the construction. Formally, we define the depth of a planar quasi-k-tree in the following way.

Definition 5.9: *Let G be a nice planar quasi-k-tree with a nice construction sequence $\sigma := (v_1, \dots, v_n)$ and an outer face that contains exactly the vertices $v_1, \dots, v_l$ with $l \leq k$. If $l = n$, the depth $d(\sigma)$ of σ is 1. Otherwise, let $f_1, \dots, f_m$ be the faces in $G[\{v_1, \dots, v_{l+1}\}]$ and $G_1, \dots, G_m$ be the graphs bounded by the cycles bounding $f_1, \dots, f_m$ respectively. Furthermore, let $\sigma_1, \dots, \sigma_m$ be nice construction sequences of $G_1, \dots, G_m$ respectively. Then, the depth of σ is defined as $d(\sigma) = \max\{d(\sigma_i) : 1 \leq i \leq m\} + 1$. The depth $d(G)$ of G is the minimum over the depths of all nice construction sequences of G.*

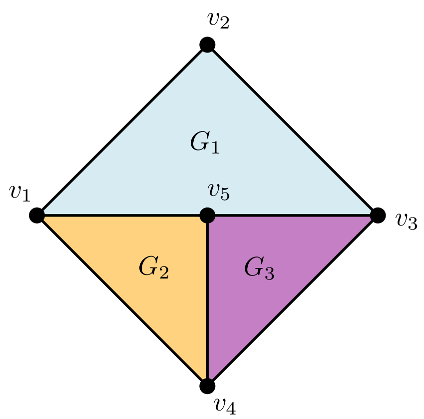


Figure 5.10: A nice planar quasi-k-tree G. It is the union of the graphs G_1, G_2 and G_3. The depth $d(G)$ of G is $\max\{d(G_1), d(G_2), d(G_3)\} + 1$.

Now that we have defined the *depth* of a planar quasi-k-tree, we go back to nice canonically upward planar quasi-4-trees. We aim to show a bound for a subclass of nice canonically upward planar quasi-4-trees that is dependent on the depth of the graph. To do this, we limit the graph class by disallowing three possible ways of inserting a vertex. Namely, we look at nice canonically upward planar quasi-k-trees with a $\{(Q2.2)^-, (Q3.4)^-, (Q4.4)^-\}$-free construction sequence. Recall that for an insertion rule R, the rule R^- is the same insertion rule, but without any optional edges added. On the other hand, R^+ is the same insertion rule, but

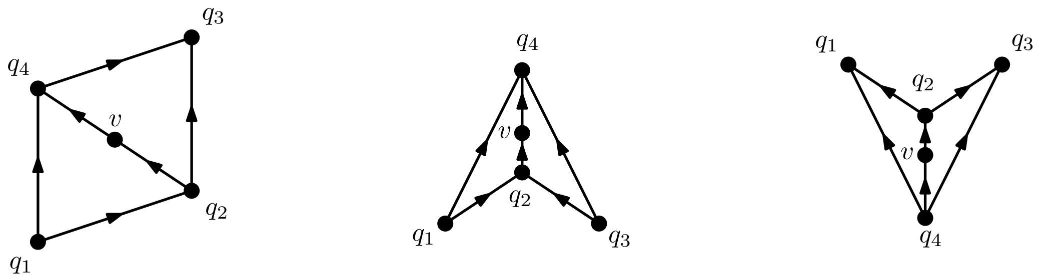


Figure 5.11: From left to right the insertion rules $(Q2.2)^-$, $(Q3.4)^-$, and $(Q4.4)^-$.

with at least one optional edge added. Thus, a $\{(Q2.2)^-, (Q3.4)^-, (Q4.4)^-\}$-free construction sequence can contain any insertion rule that is entirely different from $(Q2.2)$, $(Q3.4)$, and $(Q4.4)$ as well as the rules $(Q2.2)^+$, $(Q3.4)^+$, and $(Q4.4)^+$ (and no other rules). The three forbidden rules $(Q2.2)^-$, $(Q3.4)^-$, and $(Q4.4)^-$ are depicted in Figure 5.11.

In the following proof, we combine multiple vertex orderings. For that, we introduce the following notation:

- Let $\phi := (v_1, \ldots, v_n)$ be a vertex ordering of a graph bounded by a triangle (v_1, v_i, v_n) with $1 < i < n$. Then, we define $L_\phi := (v_2, \ldots, v_{i-1})$ and $R_\phi := (v_{i+1}, \ldots, v_{n-1})$.
- Let $\phi := (v_1, \ldots, v_n)$ be a vertex ordering of a graph bounded by a quadrangle $(v_1, v_i, v_j v_n)$ with $1 < i < j < n$. Then, we define $L_\phi := (v_2, \ldots, v_{i-1})$, $M_\phi := (v_{i+1}, \ldots, v_{j-1})$, and $R_\phi := (v_{j+1}, \ldots, v_{n-1})$.

Theorem 1.6: *Let G be a $\{(Q2.2)^-, (Q3.4)^-, (Q4.4)^-\}$-free nice canonically upward planar quasi-4-tree. Then, it holds that $\mathrm{tn}(G) \leq 5d(G)$, with $d(G)$ being the depth of G.*

Proof. We show the following stronger statement by induction on the depth $d(G)$ of G.

Claim: *Let G be a nice canonically upward planar quasi-4-tree with a $\{(Q2.2)^-, (Q3.4)^-, (Q4.4)^-\}$-free construction sequence σ with a fixed canonical upward planar embedding. Let v_t be the vertex that has the highest y-coordinate of all vertices in G and v_b be the vertex that has the lowest y-coordinate of all vertices in G. Then, there is a topological vertex ordering ϕ of G such that*

(i) $\mathrm{tn}_\phi(G) \leq 5d(G)$.

(ii) $v_b \leq_\phi v \leq_\phi v_t$ *for all vertices* $v \in V(G)$.

(iii) If G is bounded by a quadrangle as depicted in Figure 5.5, it holds that $q_1 <_\phi q_2 <_\phi v$ for all $v \in V(G) - \{q_1, q_2\}$.

(iv) If G is bounded by a quadrangle as depicted in Figure 5.6, it holds that $v <_\phi q_2 <_\phi q_1$ for all $v \in V(G) - \{q_1, q_2\}$.

For $d = 1$, the statement holds as $|V(G)| \leq 4$ and thus $\mathrm{tn}(G) \leq 2$.

Now, let G be a $\{(Q2.2)^-, (Q3.4)^-, (Q4.4)^-\}$-free nice upward planar quasi-4-tree with a $\{(Q2.2)^-, (Q3.4)^-, (Q4.4)^-\}$-free construction sequence $\sigma := (v_1, \ldots, v_n)$ with depth $d(G) \in \mathbb{N}$ and assume that the claim holds for all $\{(Q2.2)^-, (Q3.4)^-, (Q4.4)^-\}$-free nice upward planar quasi-4-trees G' with depth $d(G') < d(G)$. We do a case distinction based on the first insertion rule in σ, i. e. the first vertex in σ that is not incident to the outer face of G. For each possible insertion rule, we assume that the vertices incident to the outer face of G are named as in the figure depicting the respective rule to the right of the respective paragraph. For each rule, we look at two or three subgraphs G_1 and G_2, or G_1, G_2, and G_3 of G that have smaller depth and then combine the topological orderings of these subgraphs to obtain a topological ordering of G. As G_1, G_2, and G_3 have less depth than G, there exist orderings ϕ_1, ϕ_2, and ϕ_3 for G_1, G_2, and G_3, respectively, that fulfill the properties of the claim. For every rule, there are optional edges that may or may not be added. The orderings we construct respect the restraints these optional edges might put onto our topological ordering. However, if such an edge exists, this divides a quadrangle into two triangles. This makes it easier to combine the vertex orderings. Thus, we assume without loss of generality, that no optional edges exist, but that the constraints on our topological ordering hail from directed paths (that are not a single edge).

All orderings ϕ we construct will clearly fulfill Properties (ii) to (iv). Furthermore, for the subgraphs $G_1, \ldots, G_i$ with $i \in \{2, 3\}$, that we define in each case, the following property holds.

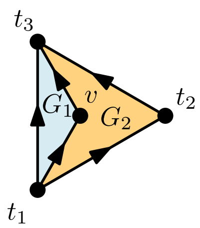
(v) Let ϕ' be the ordering ϕ without the vertices incident to the outer face of G (t_1, t_2, t_3 if G is bounded by a triangle and q_1, q_2, q_3, q_4 if G is bounded by a quadrangle) and without the first vertex that is inserted v. Then, for at most one graph G_i with $i \in \{1, 2, 3\}$, the vertices of G_i are not continuous in ϕ'.

Let ϕ be a vertex ordering fulfilling Properties (ii) to (v) and let ϕ' be defined as in Property (v). We claim that ϕ fulfills Property (i). By Property (v), we have that the vertices in all but one subgraph G_1, G_2, and G_3 are continuous in ϕ'. Thus, every twist in ϕ may only contain vertices in at most one subgraph G_1, G_2, or G_3 plus potentially the vertices incident to the outer face of G and v. As the number of vertices incident to the outer face is at most 4, i. e. 5 together with v, we get that

$$\begin{aligned}
\mathrm{tn}_\phi(G) &\le \max\{\mathrm{tn}_{\phi_1}(G_1), \mathrm{tn}_{\phi_2}(G_2), \mathrm{tn}_{\phi_3}(G_3)\} + 5 \\
&\le 5\max\{d(G_1), d(G_2), d(G_3)\} + 5 \\
&\le 5(d(G) - 1) + 5 \\
&= 5d(G)
\end{aligned}$$

Thus, Property (i) holds for every ordering fulfilling Properties (ii) to (v). It remains to construct a vertex ordering for every insertion rule. The rest of the proof is dedicated to this.

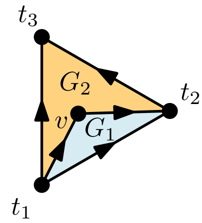
(T1) Let $G_1 := \mathrm{int}_G((t_1, v, t_3))$ and $G_2 := \mathrm{int}_G((t_1, v, t_3, t_2))$. We assume without loss of generality that there is a directed path from t_2 to v (if there is a directed path from v to t_2, it is symmetrical). Let $\phi_1 := (t_1, \ldots, v, \ldots, t_3)$ and $\phi_2 := (t_1, \ldots, t_2, \ldots, v, \ldots, t_3)$. Then, we construct the following ordering:



$$(t_1, L_{\phi_2}, t_2, M_{\phi_2}, L_{\phi_1}, v, R_{\phi_1}, R_{\phi_2}, t_3)$$

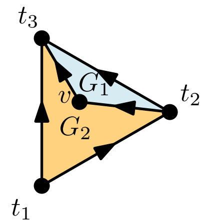
(T2) Let $G_1 := \text{int}_G((t_1, v, t_2))$ and $G_2 := \text{int}_G((t_1, v, t_2, t_3))$. We assume without loss of generality that there is a directed path from v to t_3. Let $\phi_1 := (t_1, \dots, v, \dots, t_2)$ and $\phi_2 := (t_1, \dots, v, \dots, t_2, \dots, t_3)$. Then, we construct the following ordering:

$$(t_1, L_{\phi_2}, L_{\phi_1}, v, R_{\phi_1}, M_{\phi_2}, t_2, R_{\phi_2}, t_3)$$



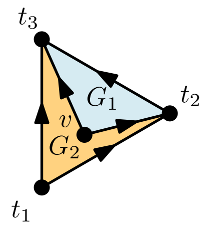
(T3) Let $G_1 := \text{int}_G((t_2, v, t_3))$ and $G_2 := \text{int}_G((t_1, t_2, v, t_3))$. We assume without loss of generality that there is a directed path from t_1 to v. Let $\phi_1 := (t_2, \dots, v, \dots, t_3)$ and $\phi_2 := (t_1, \dots, t_2, \dots, v, \dots, t_3)$. Then, we construct the following ordering:

$$(t_1, L_{\phi_2}, t_2, M_{\phi_2}, L_{\phi_1}, v, R_{\phi_1}, R_{\phi_2}, t_3)$$



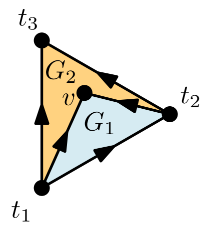
(T4) Let $G_1 := \text{int}_G((v, t_2, t_3))$ and $G_2 := \text{int}_G((t_1, t_2, v, t_3))$. We assume without loss of generality that there is a directed path from t_1 to v. Let $\phi_1 := (v, \dots, t_2, \dots, t_3)$ and $\phi_2 := (t_1, \dots, v, \dots, t_2, \dots, t_3)$. Then, we construct the following ordering:

$$(t_1, L_{\phi_2}, v, M_{\phi_2}, L_{\phi_1}, t_2, R_{\phi_1}, R_{\phi_2}, t_3)$$



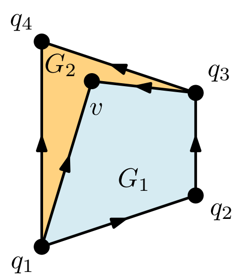
(T5) Let $G_1 := \text{int}_G((t_1, t_2, v))$ and $G_2 := \text{int}_G((t_1, v, t_2, t_3))$. We assume without loss of generality that there is a directed path from v to t_3. Let $\phi_1 := (t_1, \dots, t_2, \dots, v)$ and $\phi_2 := (t_1, \dots, t_2, \dots, v, \dots, t_3)$. Then, we construct the following ordering:

$$(t_1, L_{\phi_2}, L_{\phi_1}, t_2, R_{\phi_1}, M_{\phi_2}, v, R_{\phi_2}, t_3)$$

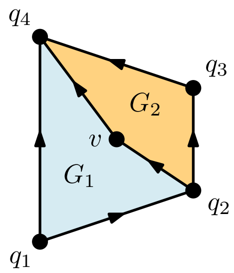


(Q1.1) Let $G_1 := \text{int}_G((q_1, q_2, q_3, v))$ and $G_2 := \text{int}_G((q_1, v, q_3, q_4))$. We assume without loss of generality that there is a directed path from v to q_4. Let $\phi_1 := (q_1, \dots, q_2, \dots, q_3, \dots, v)$ and $\phi_2 := (q_1, \dots, q_3, \dots, v, \dots, q_4)$. Then, we construct the following ordering:

$$(q_1, L_{\phi_2}, L_{\phi_1}, q_2, M_{\phi_1}, q_3, R_{\phi_1}, M_{\phi_2}, v, R_{\phi_2}, q_4)$$



(Q1.2) Symmetrical to (Q1.1).

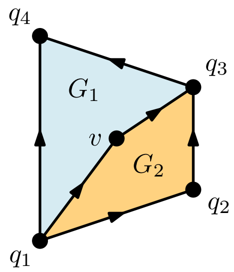


(Q1.3) Let $G_1 \coloneqq \mathrm{int}_G((q_1, q_2, v, q_4))$ and $G_2 \coloneqq \mathrm{int}_G((q_2, v, q_4, q_3))$. Let $\phi_1 \coloneqq (q_1, \dots, q_2, \dots, v, \dots, q_4)$ and ϕ_2. Depending on whether there is a directed path from v to q_3 or from q_3 to v, we have either $\phi_2 = (q_2, \dots, q_3, \dots, v, \dots, q_4)$ or $\phi_2 = (q_2, \dots, v, \dots, q_3, \dots, q_4)$. In the first case, we construct the following ordering:

$$(q_1, L_{\phi_1}, q_2, M_{\phi_1}, L_{\phi_2}, q_3, M_{\phi_2}, v, R_{\phi_2}, R_{\phi_1}, q_4)$$

In the second case, we construct the following ordering:

$$(q_1, L_{\phi_1}, q_2, M_{\phi_1}, L_{\phi_2}, v, M_{\phi_2}, q_3, R_{\phi_2}, R_{\phi_1}, q_4)$$

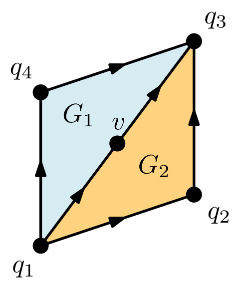


(Q1.4) Let $G_1 \coloneqq \mathrm{int}_G((q_1, v, q_3, q_4))$ and $G_2 \coloneqq \mathrm{int}_G((q_1, v, q_3, q_2))$. Let $\phi_1 \coloneqq (q_1, \dots, v, \dots, q_3, \dots, q_4)$ and depending on whether there is a directed path from v to q_2 or from q_2 to v, we have either $\phi_2 = (q_1, \dots, v, \dots, q_2, \dots, q_3)$ or $\phi_2 = (q_1, \dots, q_2, \dots, v, \dots, q_3)$. In the first case, we construct the following ordering:

$$(q_1, L_{\phi_1}, L_{\phi_2}, v, M_{\phi_2}, q_2, R_{\phi_2}, M_{\phi_1}, q_3, R_{\phi_1}, q_4)$$

In the second case, we construct the following ordering:

$$(q_1, L_{\phi_1}, L_{\phi_2}, q_2, M_{\phi_2}, v, R_{\phi_2}, M_{\phi_1}, q_3, R_{\phi_1}, q_4)$$

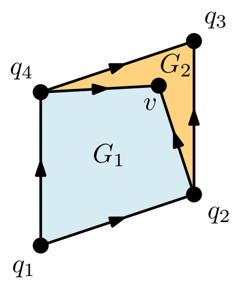


(Q2.1) Let $G_1 \coloneqq \mathrm{int}_G((q_1, v, q_3, q_4))$ and $G_2 \coloneqq \mathrm{int}_G((q_1, v, q_3, q_2))$. We assume without loss of generality that there is a directed path from v to q_4 (if there is a directed path from q_4 to v it is symmetrical). Let $\phi_1 \coloneqq (q_1, \dots, v, \dots, q_4, \dots, q_3)$ and depending on whether there is a directed path from v to q_2 or from q_2 to v, we have either $\phi_2 = (q_1, \dots, v, \dots, q_2, \dots, q_3)$ or $\phi_2 = (q_1, \dots, q_2, \dots, v, \dots, q_3)$. In the first case, we know that there is no directed path from q_2 to q_4 and no directed path from q_4 to q_2. Thus, we may choose their order in any topological ordering. Then, we construct the following ordering:

$$(q_1, L_{\phi_1}, L_{\phi_2}, v, M_{\phi_2}, q_2, R_{\phi_2}, M_{\phi_1}, q_4, R_{\phi_1}, q_3)$$

In the second case, we construct the following ordering:

$$(q_1, L_{\phi_1}, L_{\phi_2}, q_2, M_{\phi_2}, v, R_{\phi_2}, M_{\phi_1}, q_4, R_{\phi_1}, q_3)$$



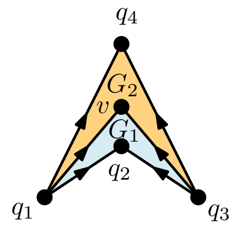
(Q2.3) Let $G_1 \coloneqq \mathrm{int}_G((q_1, q_2, v, q_4))$ and $G_2 \coloneqq \mathrm{int}_G((q_2, q_3, q_4, v))$. We assume without loss of generality that there is no directed path from q_4 to q_2. Let $\phi_1 \coloneqq (q_1, \dots, q_2, \dots, q_4, \dots, v)$ and $\phi_2 \coloneqq (q_2, q_4, \dots, v, \dots, q_3)$. Then, we construct the following ordering:

$$(q_1, L_{\phi_1}, q_2, M_{\phi_1}, q_4, R_{\phi_1}, M_{\phi_2}, v, R_{\phi_2}, q_3)$$

Note that L_{ϕ_2} is empty.

(Q2.4) Symmetrical to (Q2.3).

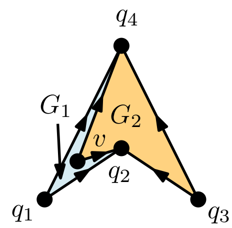
(Q3.1) Let $G_1 := \mathrm{int}_G((q_1, q_2, q_3, v))$ and $G_2 := \mathrm{int}_G((q_1, v, q_3, q_4))$. We assume without loss of generality that q_1 has a smaller y-coordinate than q_3 and that there are directed paths from q_2 to v and from v to q_4. Let $\phi_1 := (q_1, q_3, \dots, q_2, \dots, v)$ and $\phi_2 := (q_1, q_3, \dots, v, \dots, q_4)$. Then, we construct the following ordering:



$$(q_1, q_3, M_{\phi_2}, M_{\phi_1}, q_2, R_{\phi_1}, v, R_{\phi_2}, q_4)$$

Note that L_{ϕ_1} and L_{ϕ_2} are empty.

(Q3.2) Let $G_1 := \mathrm{int}_G((q_1, q_2, v, q_4))$ and $G_2 := \mathrm{int}_G((v, q_2, q_3, q_4))$. We assume without loss of generality that q_1 is lower than q_3, q_3 is lower than v and that there are directed paths from q_2 to v and from v to q_4. Let $\phi_1 := (q_1, \dots, v, \dots, q_2, q_4)$ and $\phi_2 := (q_3, v, \dots, q_2, \dots, q_4)$. Then, we construct the following ordering:



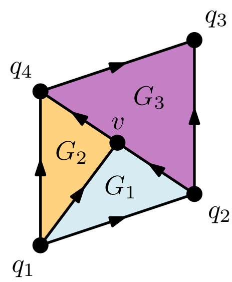
$$(q_1, q_3, L_{\phi_1}, v, M_{\phi_1}, M_{\phi_2}, q_2, R_{\phi_2}, q_4)$$

Note that R_{ϕ_1} and L_{ϕ_2} are empty.

(Q3.3) Symmetrical to (Q3.2).

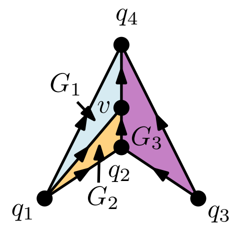
(Q4.1) to (Q4.3) Symmetrical to (Q3.1) to (Q3.3).

(Q2.2)⁺ We assume without loss of generality that $q_1 v$ is an edge (the case that $q_1 v$ is not an edge but $v q_3$ is an edge is symmetrical). Let $G_1 := \mathrm{int}_G((q_1, q_2, v))$, $G_2 := \mathrm{int}_G((q_1, v, q_4))$, and $G_3 := \mathrm{int}_G((q_2, q_3, q_4, v))$. Let $\phi_1 := (q_1, \dots, q_2, \dots, v)$, $\phi_2 := (q_1, \dots, v, \dots, q_4)$, and $\phi_3 := (q_2, \dots, v, \dots, q_4, \dots, q_3)$. Then, we construct the following ordering:



$$(q_1, L_{\phi_1}, q_2, R_{\phi_1}, L_{\phi_3}, L_{\phi_2}, v, R_{\phi_2}, M_{\phi_3}, q_4, R_{\phi_3}, q_3)$$

(Q3.4)⁺ We assume without loss of generality that $q_1 v$ is an edge (the case that $q_1 v$ is not an edge but $v q_3$ is an edge is symmetrical) and that q_1 is lower than q_3. Let $G_1 := \mathrm{int}_G((q_1, q_2, v))$, $G_2 := \mathrm{int}_G((q_1, v, q_4))$, and $G_3 := \mathrm{int}_G((q_2, q_3, q_4, v))$. Let $\phi_1 := (q_1, \dots, q_2, \dots, v)$, $\phi_2 := (q_1, \dots, v, \dots, q_4)$, and $\phi_3 := (q_3, \dots, q_2, \dots, v, \dots, q_4)$. Then, we construct the following ordering:



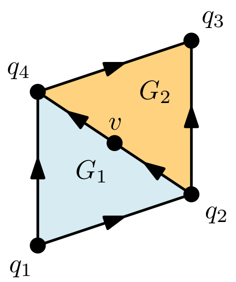
$$(q_1, q_3, L_{\phi_3}, L_{\phi_1}, q_2, R_{\phi_1}, M_{\phi_3}, L_{\phi_2}, v, R_{\phi_2}, R_{\phi_3}, q_4)$$

(Q4.4)⁺ Symmetrical to (Q3.4)⁺.

We have constructed a vertex ordering for every allowed insertion rule. Furthermore, every vertex ordering that we constructed clearly fulfills Properties (ii) to (v). By the argument from before, they also fulfill Property (i). This concludes the proof of the claim and therefore the proof of the theorem. □

Unfortunately, the proof of Theorem 1.6 cannot be easily generalized for all nice canonically upward planar 4-trees. To see this, we look at the insertion rule $(Q2.2)^-$ and the problems that arise when using the invariants used in the proof of Theorem 1.6.

(Q2.2)⁻ Let $G_1 := \text{int}_G((q_1, q_2, v, q_4))$ and $G_2 := \text{int}_G((q_2, q_3, q_4, v))$. Furthermore, let $\phi_1 := (q_1, \dots, q_2, \dots, v, \dots, q_4)$ and $\phi_2 := (q_2, \dots, v, \dots, q_4, \dots, q_3)$ of G_1 and G_2, respectively, such that $\text{tn}_{\phi_1}(G_1) \leq 5(d(G) - 1)$ and $\text{tn}_{\phi_2}(G_2) \leq 5(d(G) - 1)$. If we combine these orderings in the way that we did for the other rules, i. e. have each block $L_{\phi_1}, M_{\phi_1}, R_{\phi_1,}, L_{\phi_2}, M_{\phi_2}, R_{\phi_2}$ continuous in our constructed ordering, we get the ordering



$$(q_1, L_{\phi_1}, q_2, X_L, v, X_R, q_4, R_{\phi_2}, q_3),$$

with $X_L \in \{(M_{\phi_1}, L_{\phi_2}), (L_{\phi_2}, M_{\phi_1})\}$ and $X_R \in \{(M_{\phi_2}, R_{\phi_1}), (R_{\phi_1}, M_{\phi_2})\}$.

No matter how we choose X_L and X_R, neither the vertices of G_1 nor the vertices of G_2 are continuous in the ordering. Thus, there can be a twist that contains edges from G_1 and edges from G_2. Therefore, the argument from the proof of Theorem 1.6 does not hold. Similarly, for the rules (Q4.4)⁻ and (Q3.4)⁻, for all possible orderings such that each block $L_{\phi_1}, M_{\phi_1}, R_{\phi_1,}, L_{\phi_2}, M_{\phi_2}, R_{\phi_2}$ is continuous, neither the vertices of G_1 nor the vertices of G_2 are continuous in the ordering. Thus, in order to show the statement for graphs with construction sequences that contain at least one of these rules, there are most likely different invariants needed.

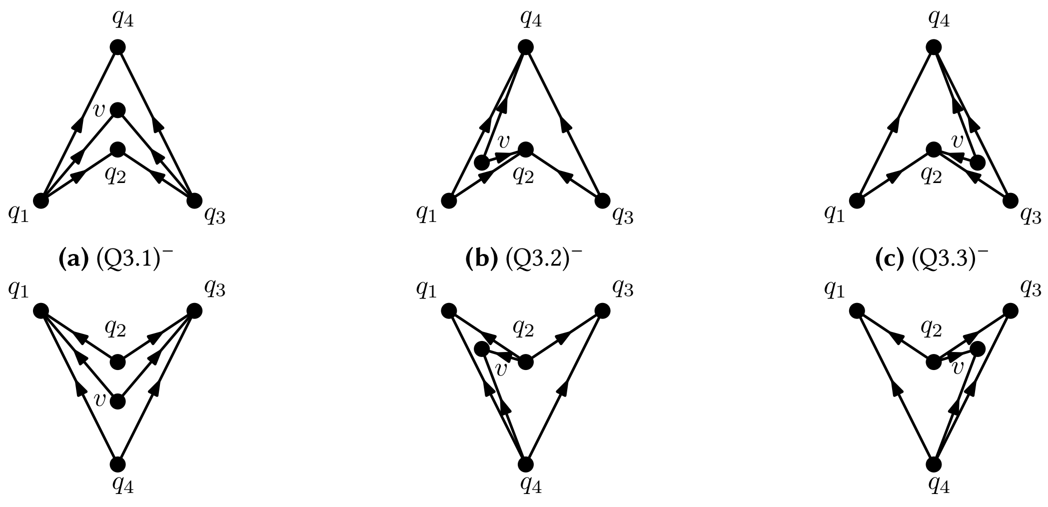
5.3 Quadrangulations With Only Sources and Sinks

In this section, we look at a subclass of upward planar quasi-4-trees that might be a good starting point for trying to bound the stack number on all upward planar quasi-4-trees. The graph class we want to look at is the class of all {(Q3.1)⁻, (Q3.2)⁻, (Q3.3)⁻, (Q4.1)⁻, (Q4.2)⁻, (Q4.3)⁻}-constructible nice upward planar quasi-4-trees. The allowed insertion rules are depicted in Figure 5.12. Clearly, to construct any graph with more than four vertices, we have to start with a quadrangle that is compliant with these insertion rules. This means that we have to start with one of the two quadrangles depicted in Figure 5.13.

We allow exactly the rules $(Q3.1)^-$, $(Q3.2)^-$, $(Q3.3)^-$, $(Q4.1)^-$, $(Q4.2)^-$, and $(Q4.3)^-$ because they replicate only the quadrangles of the base case. This means that for every face f of a graph that is constructed using only these rules, we can insert a vertex into f.

If we disregard the orientation of the edges and look at the underlying undirected graphs, this graph class is exactly the class of all 2-degenerate quadrangulations. For 2-degenerate quadrangulations, Förster, Kaufmann, Merker, Pupyrev, and Raftopoulou [51] show the following theorem.

Theorem 5.10 (Förster, Kaufmann, Merker, Pupyrev, and Raftopoulou, 2023 [51]): *Every 2-degenerate quadrangulation admits a 5-queue layout.*



(a) (Q3.1)$^-$ **(b)** (Q3.2)$^-$ **(c)** (Q3.3)$^-$

(d) (Q4.1)$^-$ **(e)** (Q4.2)$^-$ **(f)** (Q4.3)$^-$

Figure 5.12: All possible ways to insert a vertex v in a quadrangular face (q_1, q_2, q_3, q_4) of a nice upward planar quasi-4-tree G with only global sources and global sinks.

Figure 5.13: The two base cases for constructing a nice upward planar quasi-4-tree G with only global sources and global sinks.

To use their result on the queue number for a bound on the stack number, we can use the following result by Dujmović, Pór, and Wood [42] (something similar was shown before by Pemmaraju [76]).

Theorem 5.11 (Dujmović, Pór, and Wood, 2004 [42])**:** *Let G be a bipartite graph that admits a k-queue layout. Let $V(G) = A\dot{\cup}B$ such that A and B are independent sets. Then, there exists a vertex ordering σ of G such that $\mathrm{sn}_\sigma(G) \leq 2k$ and $a <_\sigma b$ for all $a \in A$ and $b \in B$.*

Note that Dujmović, Pór, and Wood do not explicitly state this theorem. However, it follows directly from Lemma 13 and Lemma 18 in [42]. Using Theorems 5.10 and 5.11, we can show the following bound for the stack number of {(Q3.1)$^-$, (Q3.2)$^-$, (Q3.3)$^-$, (Q4.1)$^-$, (Q4.2)$^-$, (Q4.3)$^-$}-constructible graphs.

Theorem 5.12: *Let G be a {(Q3.1)$^-$, (Q3.2)$^-$, (Q3.3)$^-$, (Q4.1)$^-$, (Q4.2)$^-$, (Q4.3)$^-$}-constructible graph. Then, it holds that $\mathrm{sn}(G) \leq 10$.*

Proof. First, observe that every vertex in G is either a sink or a source, i. e. it has either only incoming or only outgoing edges. Thus, the graph G is bipartite. Let G' be the underlying undirected graph of G. Furthermore, let A be the set of all sources in G and B the set of all sinks in G. By Theorem 5.10, G' admits a 5-queue layout. Thus, by Theorem 5.11, there exists a vertex ordering σ of G' such that $\mathrm{sn}_\sigma(G') \leq 10$ and $a <_\sigma b$ for all $a \in A$ and $b \in B$. Therefore, σ is a topological ordering of G, as every source is to the left of every sink, and we have that $\mathrm{sn}(G) \leq \mathrm{sn}_\sigma(G) \leq 10$. This concludes the proof. □

While this result gives us a constant bound on the stack number, it is entirely unclear how this approach could be used for general upward planar quasi-4-trees. We believe that it is necessary to construct vertex orderings that are more closely related to the canonical embedding to properly use the construction sequence of an upward planar quasi-4-tree. In particular, it would be interesting to find a way to embed upward planar quasi-4-trees such that there is a constant $c \in \mathbb{N}$ such that there exist no c edges $u_1v_1, \dots, u_cv_c$ with y-coordinates $y_{u_1}, \dots, y_{u_c}$ and $y_{v_1}, \dots, y_{v_c}$, respectively, with $y_{u_1} < \cdots < y_{u_c} < y_{v_1} < \cdots < y_{v_c}$. If we can find such an embedding, we can order the vertices along the y-axis to obtain a stack layout using a constant number of stacks, as the twist number of this ordering is bounded by c. Note that for the vertex orderings σ in both existing proofs for upward planar 3-trees by Frati, Fulek, and Ruiz-Vargas [52] and by Nöllenburg and Pupyrev [74], as well as our proof for upward planar 2-trees in Chapter 3, we can find an upward planar embedding of the respective graph such that σ is the result of ordering the vertices along the y-axis. Thus, it seems reasonable for something similar to be possible for upward planar quasi-4-trees.

However, we were unable to find such an embedding. We conclude this section by showing a problem that arises when trying to obtain such an embedding in a particular way. Namely, we look at what happens if we place the vertex v in the application of (Q3.2) directly above q_1. That is, we place v such that there is no vertex v' with a y-coordinate larger than q_1 and less than v at the time of the insertion of v. For the definition of the vertices, refer to Figure 5.12.

Theorem 5.13: *For every $k \in \mathbb{N}$, there is a {(Q3.1)$^-$, (Q3.2)$^-$}-constructible nice upward planar quasi-4-tree G with a construction sequence σ of G such that placing every vertex v that is inserted using (Q3.2)$^-$ immediately above q_1 results in a corresponding vertex ordering along the y-axis that contains a k-twist for every canonical embedding of G abiding this rule.*

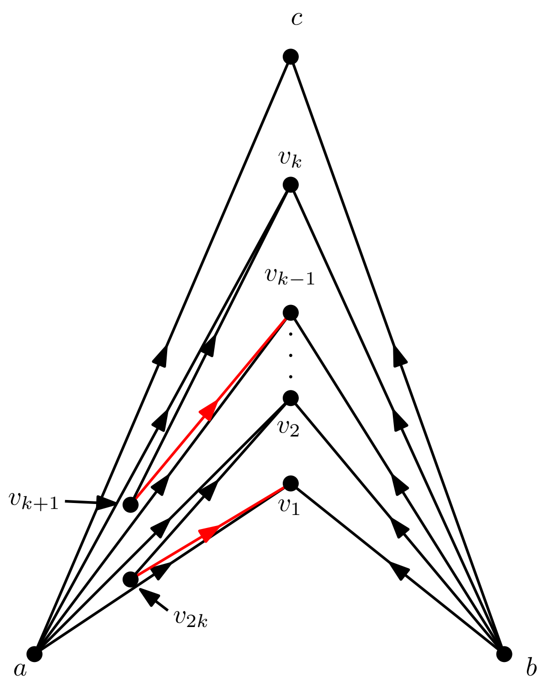


Figure 5.14: An upward planar quasi-4-tree that is $\{(Q3.1), (Q3.2)\}$-constructible. If the vertices that are inserted using (Q3.2)$^-$ are placed directly above a, there is a k-twist consisting of the edges $v_{2k}v_1, v_{2k-1}v_2, \ldots, v_{k+1}v_k$ in the corresponding ordering of the vertices along the y-axis $(a, b, v_{2k}, \ldots, v_{k+1}, v_1, \ldots, v_k, c)$ for the construction sequence $(a, b, c, v_1, \ldots, v_{2k})$.

Proof. We show the statement by constructing such a graph G. Our goal is to construct the graph G depicted in Figure 5.14. We start with the outer quadrangle (a, v_1, b, c). Then, we apply the insertion rule (Q3.1)$^-$ $k-1$ times to insert the vertices $v_2, \ldots, v_k$. Note that all canonical embeddings that result from this have a total ordering on the y-coordinates of $v_1, \ldots, v_n$. Thus, we can assume without loss of generality that $v_1 < \cdots < v_n$ in terms of their y-coordinate. This means we have the following quadrangles that bound faces: (a, v_i, b, v_{i+1}) for all $1 \leq i < k$. Finally, we apply the insertion rule (Q3.2)$^-$ k times to insert vertices $v_{k+1}, \ldots, v_{2k}$, with v_{k+i} being inserted in the quadrangle $(a, v_{k-i+1}, b, v_{k-i+2})$ for all $1 \leq i \leq k$. Thus, v_{k+i} is adjacent to v_{k-i+1} and v_{k-i+2}. Then, the construction sequence $(a, b, c, v_1, \ldots, v_{2k})$ results in the graph depicted in Figure 5.14. Furthermore, since $v_{2k} < \cdots < v_{k+1} < v_1$ in terms of their y-coordinate, we have that the corresponding vertex orderings along the y-axis of all canonical embeddings of G contain a k-twist that consists of the edges $v_{2k}v_1, v_{2k-1}v_2, \ldots, v_{k+1}v_k$. This concludes the proof. □

Note that the proof only gives a specific construction sequence of the graph that results in a canonical embedding with large twists. Indeed, there are construction sequences for the graph depicted in Figure 5.14 that do not result in large twists. For example, the construction sequence $(a, b, c, v_1 \ldots, v_k, v_{2k}, \ldots, v_{k+1})$ results in twists of constant size. This leads us to the following open question.

Question 5.14: *Is there an upward planar quasi-4-tree G for each $c \in \mathbb{N}$ such that for all canonical embeddings of G, the ordering of $V(G)$ along the y-axis contains a twist of size at least c?*

6 Conclusion

In this thesis, we looked at how we can construct planar graphs with bounded treewidth and how we can find upper bounds on their stack number. Regarding the stack number, the main contribution of this thesis is the constant upper bound on the stack number of upward planar 2-trees in Chapter 3. Together with the constant upper bound on upward planar 3-trees [52, 74] and the constant upper bound on directed trees [60], this gives us a constant bound on all upward orientations of planar edge maximal graphs with treewidth at most 3. This gives us two possible directions of how to improve the state of the art further. The first is to look at general upward planar graphs with treewidth at most 3.

Question 6.1: *Is there a constant* $c \in \mathbb{N}$ *such that* $\mathrm{sn}(G) \leq c$ *for all upward planar graphs with treewidth at most* 3*?*

Note that this is also an open question for upward planar graphs with treewidth at most 2. The second possible direction, is to look at upward planar graphs with treewidth 4. We believe that the progress on upward planar graphs with treewidth 4 is largely inhibited by the lack of a proper characterization of these graphs. In this thesis we contribute to a better understanding of planar graphs with treewidth 4 by showing that every planar graph with treewidth at most 4 is a subgraph of a nice planar quasi-4-tree. This confirms a conjecture by Förster [50]. Furthermore, we generalize planar quasi-4-trees to planar quasi-k-trees and show that every planar graph with treewidth at most k is a subgraph of a planar quasi-k-tree. However, the following question remains open.

Question 4.14: *Is there a planar quasi-k-tree that is not a subgraph of a nice planar quasi-k-tree?*

To use planar quasi-4-trees for improvements on the stack number, we need to look at upward orientations of planar quasi-4-trees. We do this in Chapter 5. Unlike the undirected case, we do not know much about the relation of upward planar quasi-4-trees to other classes of upward planar graphs. In particular, we do not know whether subgraphs of canonically upward planar quasi-4-trees and subgraphs of upward planar quasi-4-trees are the same graph class.

Question 5.4: *Is there an upward planar quasi-*4*-tree that is not a subgraph of a canonically upward planar quasi-*4*-tree?*

While we do know that upward planar 3-trees are a subset of canonically upward planar quasi-4-trees, we do not know whether every upward planar 2-tree is a subgraph of a (canonically) upward planar quasi-4-tree. This leads us o the following question.

Question 5.5: *Is there an upward planar* 2*-tree that is not a subgraph of a (canonically) upward planar quasi-*4*-tree?*

Even if there is an upward planar 2-tree, that is not a subgraph of a (canonically) upward planar quasi-4-tree, it would be of interest, if there is some $k \in \mathbb{N}$ such that every upward planar 2-tree is a subgraph of a (canonically) upward planar quasi-k-tree. One way of tackling this problem is to look whether we can triangulate upward planar 2-trees in way that keeps the treewidth *small*. In particular, we can ask the following question.

Question 6.2: *Is there a $c \in \mathbb{N}$ such that for all upward planar 2-trees G, there is an upward triangulation of G with treewidth at most c?*

If we can find such a constant c, this would give us that every upward planar 2-tree is a subgraph of a canonically upward planar quasi-c-tree. More generally, we can also ask the following question.

Question 6.3: *Is there a function f such that every upward planar graph with treewidth k is a subgraph of an upward triangulation with treewidth at most $f(k)$?*

If such a function exists, this would imply that every upward planar graph with treewidth at most k is a subgraph of a canonically upward planar quasi-$f(k)$-tree. Thus, it would be sufficient to bound the stack number of canonically upward planar quasi-$f(k)$-trees, to bound the stack number of all upward planar graphs with treewidth at most k.

Even if such a function does not exist, it is of interest, whether we can bound the stack number of (canonically) upward planar quasi-k-trees, as this would give us a bound on upward orientations of planar edge maximal graphs with treewidth k. It would be of particular interest, whether there is a bound on the stack number of (canonically) upward planar quasi-k-trees that is only dependent on k. Therefore, we ask the following question.

Question 6.4: *Is there a function f such that $\mathrm{sn}(G) \leq f(k)$ for all (canonically) upward planar quasi-k-trees G?*

As every planar graph has treewidth in $\mathcal{O}(\sqrt{n})$, this would yield a bound of $\mathcal{O}(f(\sqrt{n}))$ on the stack number of upward planar graphs. As the best currently known upper bound is $\mathcal{O}(n^{2/3} \log^{2/3}(n))$ due to Jungeblut, Merker, and Ueckerdt [65], this would be an improvement on the state of the art if $f \in o(k^{4/3} \log^{2/3}(k))$.

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