Karlsruhe Institute of Technology

Coloring Hypergraphs Induced by Hanging Rectangles

Master Thesis of

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Statement of Authorship

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Karlsruhe, October 28, 2022

Abstract

In this work, we study m-uniform hypergraphs captured by different geometric range families. For the range family $\mathcal{R}_R$ of hanging axis-aligned rectangles, the hyperedges of a capturing hypergraph all consist of axis-aligned rectangles which contain exactly m points and whose upper left corner intersects the first angle bisector. We want to properly k-color this hypergraph, meaning coloring the point set with k colors such that every hyperedge consists of points with at least two different colors. Thereby, we try to specify the smallest number of colors k, such that any m-uniform hypergraph admits a proper k-coloring. We address the problem of hanging rectangles by investigating the range families of hanging bottomless rectangles, hanging right-open rectangles and their union. For the latter, we show that $k \geq 4$ for $m = 2$, $k \leq 4$ for $m \geq 3$ and for $m \geq 7$, any hypergraph captured by this range family is proper 3-colorable. Additionally, we prove that for $m = 3$ any hypergraph captured by hanging bottomless rectangles is proper 2-colorable. This is an improvement compared to *non-hanging* bottomless rectangles, which are only proper 3-colorable for $m = 3$ [Kes12]. We further give a lower bound of $k > 2$ for the union of non-hanging bottomless rectangles and hanging right-open rectangles for $m \geq 2$. This result is achieved with the help of a semi-online approach for hanging bottomless and hanging right-open rectangles. The approach is then transferred into an offline setting for non-hanging bottomless and hanging right-open rectangles. It is similar to the offline approach of Chekan and Ueckerdt [CU21] for bottomless rectangles and horizontal strips. For hanging rectangles, the lower bound of the proper colorability for $m = 2$ is four.

Deutsche Zusammenfassung

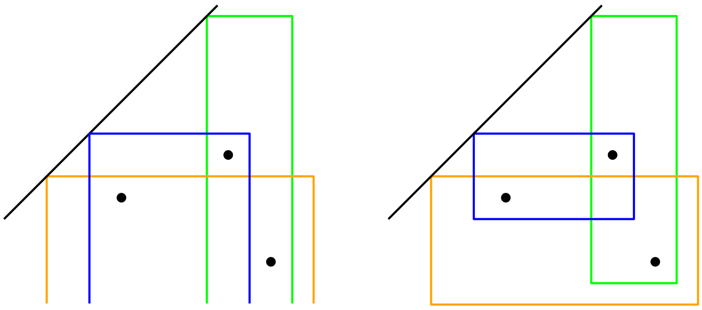
In dieser Arbeit untersuchen wir m-uniforme Hypergraphen, die von verschiedenen Familien von Bereichen erfasst werden. Für die Familie von Bereichen $\mathcal{R}_R$ von hängenden achsenparallelen Rechtecken bestehen alle Hyperkanten eines Hypergraphen, der durch diese Familie erfasst wird, aus achsenparallelen Rechtecken, die exakt m Punkte beinhalten und deren obere linke Ecke die erste Winkelhalbierende schneidet. Wir wollen diesen Hypergraphen zulässig k-färben, was bedeutet, dass die Punktmenge mit k Farben gefärbt wird, sodass jede Hyperkante aus Punkten mit mindestens zwei verschiedenen Farben besteht. Dabei versuchen wir die kleinste Anzahl an Farben k zu spezifizieren, sodass jeder m-uniforme Hypergraph eine zulässige k-Färbung besitzt. Wir gehen das Problem der hängenden Rechtecke an, indem wir die Familie von Bereichen von hängenden bodenlosen Rechtecken, hängenden rechts-offenen Rechtecken und deren Vereinigung untersuchen. Für letztere zeigen wir, dass $k \geq 4$ für $m = 2$, dass $k \leq 4$ für $m \geq 3$ und für $m \geq 7$ gilt, dass jeder Hypergraph, der durch diese Familie von Bereichen erfasst wird, zulässig 3-färbbar ist. Zusätzlich beweisen wir, dass für $m = 3$ jeder Hypergraph, der durch hängende bodenlose Rechtecke erfasst wird, zulässig 2-färbbar ist. Das ist eine Verbesserung verglichen mit *nicht-hängenden* bodenlosen Rechtecken, die nur zulässig 3-färbbar sind für $m = 3$ [Kes12]. Weiter geben wir eine untere Schranke von $k > 2$ für die Vereinigung von nicht-hängenden bodenlosen Rechtecken und hängenden rechts-offenen Rechtecken an. Dieses Ergebnis ist mit der Hilfe eines Semi-online-Ansatzes für hängende bodenlose und hängende rechts-offene Rechtecke erzielt worden. Der Ansatz ist dann in ein Offline-Szenario für nicht-hängende bodenlose und hängende rechts-offene Rechtecke überführt worden. Er ist ähnlich wie der Offline-Ansatz von Chekan und Ueckerdt [CU21] für bodenlose Rechtecke und horizontale Streifen. Für hängende Rechtecke ist die untere Schranke für die zulässige Färbbarkeit für $m = 2$ vier.

Contents

1. Introduction

In this work we will focus on *range-capturing* hypergraphs $\mathcal{H}(V, \mathcal{R}, m)$ given by a set of points $V \subset \mathbb{R}^2$, a *range family* $\mathcal{R}$ and a constant m which specifies the number of points of each hyperedge. We focus on *geometric* range families. A range family is a set of ranges, whereby a range is defined as a set of points $R \subset \mathbb{R}^2$. A point set $X \subseteq V$ defines a hyperedge of $\mathcal{H}(V, \mathcal{R}, m)$, if and only if there exists a range $R \in \mathcal{R}$ that captures X, i.e., $R \cap V = X$.

A k-coloring of a hypergraph is a coloring of the point set V with $k \in \mathbb{N}$ distinct colors. We say that a hypergraph is *polychromatic k-colorable*, if every hyperedge contains all k available colors. A *proper k-coloring* of a hypergraph is a coloring with k colors, whereby each edge contains at least two of them.



a) b)

Figure 1.1: Example of hypergraphs captured by a) hanging axis-aligned bottomless rectangles and b) hanging axis-aligned rectangles.

We will focus on the range family $\mathcal{R}_R$ of hanging axis-aligned rectangles. A range $R \in \mathcal{R}_R$ is defined as the point set $\{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, c \leq y \leq a\}$ for some constants $a, b, c \in \mathbb{R}$. According to the definition of a range $R \in \mathcal{R}_R$, we say that an axis-aligned rectangle is *hanging*, if its upper left corner intersects the first angle bisector. An example for a hypergraph captured by hanging rectangles is shown in Figure 1.1.

We want to clearly separate hanging rectangles from pierced rectangles. For a hanging rectangle, only its upper left corner intersects the first angle bisector. In contrast, we describe the range family $\mathcal{R}_P$ describing pierced rectangles as

$$\mathcal{R}_P = \{\{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, c \leq y \leq d\} \mid a, b, c, d \in \mathbb{R} \wedge \exists e \in \mathbb{R} : a \leq e \leq b, c \leq e \leq d\} \; .$$

Thus, for each pierced rectangle, there exists a point (e, e), i.e. a point which is part of the first angle bisector, that is also part of the pierced rectangle. Note that e can differ for each rectangle. These pierced rectangles form a superset of the hanging rectangles. The latter are pierced rectangles that intersect the first angle bisector in exactly one point: their upper left corner. In this work, we will only consider hanging rectangles.

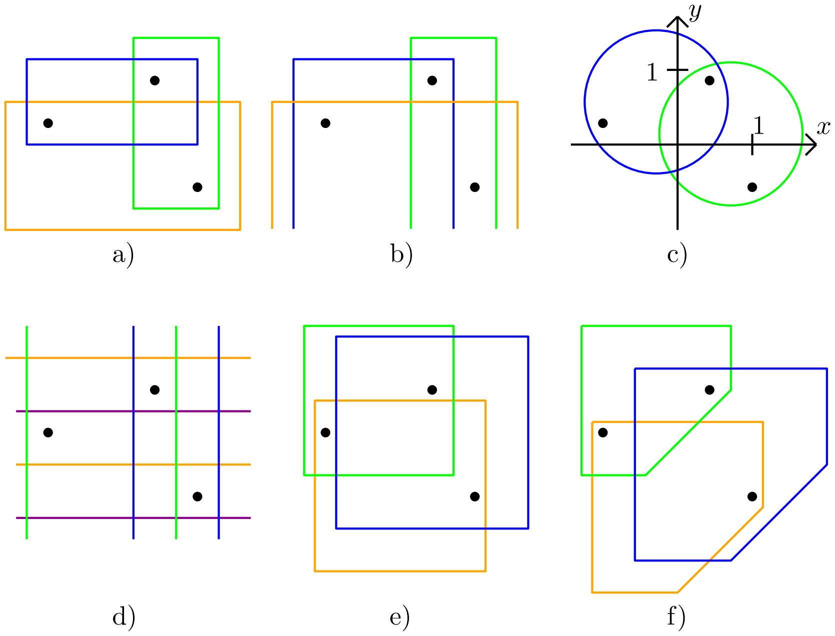


Figure 1.2: Examples of hypergraphs captured by different range families: a) axis-aligned rectangles, b) axis-aligned bottomless rectangles, c) stabbed translate unit disks whereby all edges must contain the origin, d) axis-aligned stripes, e) homothets of axis-aligned squares and f) homothets of convex polygons.

We will now look at several results in the literature about the colorability of other geometric range families. A few examples of hypergraphs captured by different range families are shown in Figure 1.1 and Figure 1.2. When studying proper colorings of hypergraphs captured by a range family $\mathcal{R}$, we often search for the smallest number of colors χ_m for which there exists a constant m such that every hypergraph $\mathcal{H}(V, \mathcal{R})$ having at least m points in every hyperedge admits a proper χ_m-coloring. For uniform hypergraphs, the question asked for these proper-coloring problems is the following:

Question 1.1. *Let $\mathcal{H}(V, \mathcal{R}, m)$ be an m-uniform hypergraph captured by the range family $\mathcal{R}$. For a sufficiently large m, what is the smallest number of colors $\chi_m = \chi_{\mathcal{R}}$ such that $\mathcal{H}$ admits a proper χ_m-coloring?*

There already exist lots of studies on this question for many different range families. A few examples are presented in the following.

1. For axis-aligned rectangles, $\chi_m = \infty$ [CPST09].

2. For axis-aligned bottomless rectangles, $\chi_m = 2$. An axis-aligned bottomless rectangle is a rectangle whose lower side lies at $y = -\infty$, i.e., below all points of the hypergraph. More precisely, $\chi_4 = 2$, whereas for $2 \le m \le 3$, we need at least three colors, to admit a proper coloring of an m-uniform hypergraph captured by axis-aligned bottomless rectangles [Kes12].

3. For unit disks, $\chi_m = 3$ [PP16, DP21b]. A unit disk is a circle with radius $r = 1$.

4. For stabbed unit disks, $\chi_m = 2$. A stabbed unit disk is a unit disk that contains one fixed point, e.g. the origin. This proper 2-coloring can already be found for $m \geq 9$ [DP20].
5. For disks with arbitrary sizes, thus homothets of disks, $\chi_m = 4$ [DP20].
6. For stabbed homothets of disks, $\chi_m = 3$ [AKP20, DP20].
7. For axis-aligned strips in two dimensions, $\chi_m = 2$ [ACC+11]. An axis-aligned strip in $\mathbb{R}^2$ is described by the point set $\{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b\}$ or $\{(x, y) \in \mathbb{R}^2 \mid a \leq y \leq b\}$ for $a, b \in \mathbb{R}$.
8. For homothets of axis-aligned squares and homothets of a fixed parallelogram, $\chi_m = 2$ [AKV17].
9. For translates of squares, $\chi_m = 2$ [Pac86].
10. For homothets of triangles, $\chi_m = 2$ [KP15].
11. For homothets of an arbitrary convex polygon, $2 \leq \chi_m \leq 3$ [KP19].
12. For translates of an arbitrary convex polygon, $\chi_m = 2$ [PT10b].
13. For half-planes, $\chi_m = 2$ [SY10, Kes12].
14. For quadrants, $\chi_m = 2$ [KP15].

A translate of a geometric figure is a translation of the figure in another position in the plane, whereby the size of the figure remains the same. A homothet of a figure is a translation and a scaling, whereby all relations of the figure stay the same.

When looking at polychromatic colorings, we search for the smallest number $m = m_k$, such that every m-uniform hypergraph is polychromatic k-colorable. Note that for $m = 2$, every hypergraph $\mathcal{H}(V, \mathcal{R}, m)$ is polychromatic k-colorable if and only if it is proper k-colorable. Observe also that for any m, if the hypergraph $\mathcal{H}(V, \mathcal{R}, m)$ admits a polychromatic k-coloring, $\mathcal{H}$ also admits a proper k-coloring. The question that is studied for polychromatic colorings of hypergraphs is the following:

Question 1.2. *Let $\mathcal{H}(V, \mathcal{R}, m)$ be an m-uniform hypergraph captured by the range family $\mathcal{R}$. What is the smallest number $m = m_k$ for the uniformity of $\mathcal{H}$, such that $\mathcal{H}$ admits a polychromatic k-coloring?*

In the following we will show some of the known results. Note that if the hypergraph $\mathcal{H}(V, \mathcal{E}, m)$ admits no polychromatic k-coloring for $k \geq 2$ and any m, it especially admits no polychromatic 2-coloring. We therefore write for the corresponding range families that $m_2 = \infty$ holds. Note also that it always holds that $m_k \geq k$.

1. For axis-aligned rectangles, $m_2 = \infty$ [CPST09]. We will repeat the proof for this result in Section 1.3.
2. For axis-aligned bottomless rectangles, it is shown that $m_2 = 4$ [Kes12] and more generally, $1.677k - 2.5 < m_k \leq 3k - 2$ [ACC+13].
3. For unit disks, $m_2 = \infty$ [PP16].
4. For stabbed unit disks, $m_k \leq 8k - 7$ [DP20].
5. For axis-aligned strips in two dimensions, thus horizontal and vertical strips, $m_k \leq 2k - 1$. For axis-aligned strips in d dimensions, it holds that

$$2 \cdot \left\lceil \frac{(2d-1)k}{2d} \right\rceil - 1 \leq m_k \leq k(4 \ln k + \ln d)$$

[ACC+11]. It was recently proven by Planken [Pla22], that it even holds that $m_k \in \mathcal{O}(k)$ for axis-aligned strips in d dimensions.

6. For homothets of axis-aligned squares and homothets of a fixed parallelogram, $m_2 \leq 215$ and $m_k \in \mathcal{O}(k^{8.75})$ [AKV17].

7. For translates of squares, $m_k \in \mathcal{O}(k)$ [ACC+10].

8. For homothets of triangles, $5 \leq m_2 \leq 9$ and $m_k \in \mathcal{O}(k^{4.09})$ [KP15].

9. For translates of an arbitrary convex polygon, $m_k \in \mathcal{O}(k)$ [GV09].

10. For half-planes, $m_2 = 3$ [Kes12] and $m_k = 2k - 1$ [SY10].

11. For quadrants, $m_k = k$ [KP15].

For an up-to-date overview of other geometric hypergraphs and their properties regarding colorability, we refer to [MTA].

1.1 Union of Range Families

We now take a look at the union of different range families and their polychromatic colorability properties. Chekan [Che21] and Chekan and Ueckerdt [CU21] investigated the union of several range families. For example, they obtained the following results:

1. For the union of quadrants with two adjacent directions, $m_k \leq 2k - 1$ [Che21].

2. For the union of quadrants with two non-adjacent directions, $m_k \leq 3k - 3$ [Che21].

3. For the union of quadrants with three directions and for the union of quadrants in all four directions, $m_k \leq 4k - 3$ [Che21].

4. For the union of bottomless rectangles and quadrants in all four directions, $m_k \leq 5k-2$ [Che21].

5. For axis-aligned strips and quadrants in all four directions, $m_k \leq 10k - 1$ [CU21].

6. For the union of bottomless (or topless) rectangles and horizontal strips, $m_2 = \infty$ [CU21].

7. For the union of south-west or north-east quadrants or bottomless or topless rectangles and diagonal strips, $m_2 = \infty$ [CU21, Che21].

8. For the union of north-west and south-east quadrants and axis-aligned and diagonal strips, $m_k \leq \lceil 4k \ln k + k \ln 3 \rceil + 4k$ [CU21].

9. For the union of topless and bottomless rectangles, it holds that $m_k \in \mathcal{O}(k^{8.75})$ [CU21]. They follow this from the result that $m_k \in \mathcal{O}(k^{8.75})$ for axis-aligned squares by [AKV17]. We will repeat the reduction of topless and bottomless rectangles to axis-aligned squares in Section 1.5.

We will partition our problem of hanging rectangles into the union of two range families in the upcoming chapters: the range families of hanging bottomless rectangles and of hanging right-open rectangles. These range families are formally introduced later. We will see that a hypergraph captured by the union of them forms a subset of a hypergraph with the same uniformity and set of points captured by hanging rectangles.

1.2 Dual hypergraphs

For a hypergraph $\mathcal{H}(V, \mathcal{R})$ captured by the range family $\mathcal{R}$, there exists a *dual hypergraph* $\mathcal{H}^*(\mathcal{R}', V)$. Its vertices consist of a finite set of ranges $\mathcal{R}' \subseteq \mathcal{R}$. For $v \in V$, a hyperedge of $\mathcal{H}^*$ is the set of ranges covering v. We refer to $\mathcal{H}^*(\mathcal{R}', V, m)$ for $m \in \mathbb{N}$ as the dual hypergraph whose hyperedges all have *at least* size m. The polychromatic coloring problem concerning these dual hypergraphs $\mathcal{H}^*(\mathcal{R}', V, m)$ is therefore simply defined as follows: We are searching for a k-coloring $c : \mathcal{R}' \to \{1, \ldots, k\}$ of the ranges such that every point $v \in V$ that is covered by at least m different ranges is covered by at least one range of each color. The main question studied for polychromatic colorings of those dual hypergraphs is the following:

Question 1.3. *For $\mathcal{R}' \subseteq \mathcal{R}$, let $\mathcal{H}^*(\mathcal{R}', V, m^*)$ be a dual hypergraph captured by the range family $\mathcal{R}$. What is the smallest number $m^* = m_k^*$ for the uniformity of $\mathcal{H}^*$, such that $\mathcal{H}^*$ admits a polychromatic k-coloring?*

Again, we will show some of the already known results in the literature:

1. For axis-aligned rectangles, $m_k^* = \infty$ [PT10a].
2. For axis-aligned bottomless rectangles, $m_2^* = 3$ [Kes12] and $m_k^* \in \mathcal{O}(k^{5.09})$ [KP15].
3. For axis-aligned strips, $m_2^* = 3$ and $k \leq m_k^* \leq 2k - 1$ in the plane. For axis-aligned strips in d dimensions, it holds that

$$\left\lfloor \frac{k}{2} \right\rfloor d + 1 \leq m_k^* \leq d(k-1) + 1$$

   [ACC+11]. We will repeat the proof of this result in Section 1.4.
4. For translates of squares, $m_k^* \in \mathcal{O}(k)$ [ACC+10].
5. For homothets of triangles, $5 \leq m_2^* \leq 9$ and $m_k^* \in \mathcal{O}(k^{5.09})$ [KP15].
6. For translates of an arbitrary convex polygon, $m_k^* \in \mathcal{O}(k)$ [GV09].
7. For half-planes, $m_2^* = 3$ [Ful10] and $2k - 1 \leq m_k^* \leq 3k - 2$ [SY10].
8. For hanging arrangements, $m_k^* \leq 2k - 1$ [CKM+20]. A hanging arrangement is a configuration that is attached to a line of slope one.

Another coloring problem for hypergraphs is called *conflict-free coloring* and was introduced by Even et al. [ELRS03]. For a hypergraph $\mathcal{H}(V, \mathcal{E})$, a conflict-free coloring is a coloring of the points in V with as few colors as possible such that every hyperedge $e \in \mathcal{E}$ contains a point that has a unique color among all points in e. This means that we search for a coloring $c : V \to \{1, \ldots, k\}$ with k as small as possible, such that for every hyperedge $e \in \mathcal{E}$ a point $v \in V$ exists that is contained in e and for which it holds that for all other points $w \in e$ with $w \neq v$, the coloring of w is different to that of v, i.e., $\forall w \in e, w \neq v : c(w) \neq c(v)$.

We can extend this coloring problem to a dual hypergraph $\mathcal{H}^*(\mathcal{E}, V)$. Thereby, we search for a coloring $c : \mathcal{E} \to \{1, \ldots, k\}$ of the hyperedges with k colors, such that every node $v \in V$ is covered by a hyperedge $e \in \mathcal{E}$ with unique color among all hyperedges covering v, i.e.,

$$\forall v \in V : \exists e \in \mathcal{E}, v \in e : \forall e' \in \mathcal{E}, v \in e', e' \neq e : c(e') \neq c(e) \; .$$

Again, the goal is to use as few colors as possible.

1.3 Axis-aligned Rectangles

We will now summarize the proof that $m_k = \infty$ for $k \geq 2$ holds for axis-aligned (or axis-parallel) rectangles by Chen et al. [CPST09] (and Pach and Tardos [PT10a]). Recall that $m_k = \infty$ means that no matter the uniformity m_k of a hypergraph $\mathcal{H}$, there never exists a coloring of the point set V such that $\mathcal{H}(V, \mathcal{E}, m_k)$ admits a polychromatic k-coloring.

Theorem 1.4 (Chen et al. [CPST09]). *For any positive integers k and m, there is a finite point set in the plane with the property that no matter how we color its elements with k colors, there always exists an axis-parallel rectangle containing at least m points, all of which have the same color.*

Proof. Chen et al. [CPST09] remark that for a hypergraph $\mathcal{H}(V, \mathcal{R})$ captured by the range family $\mathcal{R}$ of axis-aligned rectangles and its dual hypergraph $\mathcal{H}^*(\mathcal{R}', V)$ with $\mathcal{R}' \subseteq \mathcal{R}$, the property of Theorem 1.4 for $\mathcal{H}$ is equivalent to $\mathcal{H}^*$ having the following property, proved by Pach and Tardos [PT10a] (for $m \geq r$):

Claim 1.5 (Pach and Tardos [PT10a]). *Let $k, r \geq 2$ be fixed. There exists a family of $k(2r)^{2kr}$ axis-parallel rectangles in the plane such that for any coloring of these rectangles with k colors, one can find a point covered by exactly r rectangles, all of which have the same color.*

Proof of Claim 1.5. To proof this Claim, Pach and Tardos [PT10a] construct a hypergraph $\mathcal{H}(V, \mathcal{R})$ and its dual hypergraph $\mathcal{H}^*(\mathcal{R}', V)$ that fulfills the property of Claim 1.5. Thereby V is a set of points in $(0,1)^2$ and $\mathcal{R}' \subseteq \mathcal{R}$. In the following, we will repeat their construction.

For two arbitrary but fixed integers $c \geq 2$ and $k \geq 0$, $[c] = \{0, \ldots, c-1\}$ describes the base of the number system that describes the positions of the corners of the rectangles. They denote with $[c]^k$ the set of strings over the possible digits $[c]$ having a length of k. For $x \in [c]^k$, the j-th digit of x for $1 \leq j \leq k$ is x_j. For any x we obtain $\bar{x}$, i.e. the *value* of x, in the following way:

$$\bar{x} = \sum_{j=1}^{k} \frac{x_j}{c^j} \ .$$

The empty string is denoted by ε and it holds that $\bar{\varepsilon} = 0$. Informally speaking, we interpret x as the digits after the decimal point of the c-ary number $0.x$.

For two integers $c \geq 2$ and $d \geq 1$ and any $0 \leq k \leq d$, they define an open (that is, the points that lie on the four sides of the rectangle are not included) axis-parallel rectangle $R^k_{u,v}$ with $u \in [c]^k$ and $v \in [c]^{d-k}$ as follows:

$$R^k_{u,v} \coloneqq (\bar{u}, \bar{u} + c^{-k}) \times (\bar{v}, \bar{v} + c^{k-d}) \ .$$

Thus, the rectangles are defined by a cartesian product of two intervals. As for the definition of $\bar{x}$, all rectangles lie inside the unit square $(0,1)^2$.

The sets of rectangles $\mathcal{R}'$ that fulfill the property of Claim 1.5 are then defined as

$$\mathcal{R}' = \mathcal{R}'(c,d) = \{R^k_{u,v} \mid 0 < k < d, u \in [c]^k, v \in [c]^{d-k}, u_k = v_{d-k}\}$$
$$\cup \{R^0_{\varepsilon,v} \mid v \in [c]^d, v_d = 0\} \ \cup \ \{R^d_{u,\varepsilon} \mid u \in [c]^d, u_d = 0\} \ .$$

An example for the set of rectangles $\mathcal{R}'(2,3)$ is shown in Figure 1.3. It holds that the number of rectangles $|\mathcal{R}'| = (d+1)c^{d-1}$. They now define the dual hypergraph $\mathcal{H}^*$ as $\mathcal{H}^*(c,d) = \mathcal{H}^*(\mathcal{R}'(c,d), \mathbb{R}^2)$. Additionally, they show that the following property, which we will not prove here, holds for this construction:

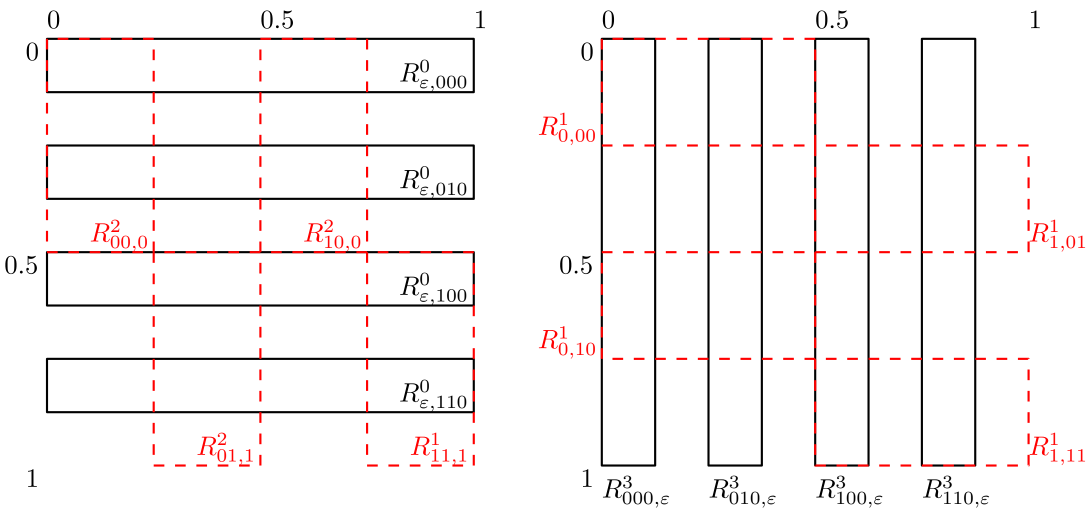


Figure 1.3: The set of rectangles $\mathcal{R}'(2,3)$. For better readability, the rectangles are drawn into two different $(0,1)^2$ unit squares.

Claim 1.6 (Pach and Tardos [PT10a]). *Let $d \geq 1$, $2 \leq r < c$, and let $\mathcal{H}^* = \mathcal{H}^*(c,d)$ denote the hypergraph defined above. If a subset $I \subseteq \mathcal{R}'(c,d)$ contains no hyperedge of $\mathcal{H}^*$ of size r, then we have*

$$|I| \leq \frac{c^{d-1}}{\frac{1}{r-1} - \frac{1}{c-1}} .$$

We now set $c = 2r$ and $d = 2kr - 1$ and k-color the rectangles of $\mathcal{R}'(c,d) = \mathcal{R}'(2r, 2kr-1)$. Thereby, the color that appears most often appears at least

$$\frac{(d+1)c^{d-1}}{k} = \frac{(2kr-1+1)(2r)^{(2kr-1)-1}}{k} = 2r \cdot (2r)^{(2kr-1)-1} = (2r)^{2kr-1} = c^d$$

times. This number of appearances is greater than the bound for the subset I in Claim 1.6. Thus, there is a hyperedge of size r in the dual hypergraph $\mathcal{H}^*(\mathcal{R}'(c,d), V)$ that is monochromatic. $\square$

From the dual hypergraph $\mathcal{H}^*(\mathcal{R}', V)$ we can determine the primal hypergraph $\mathcal{H}(V, \mathcal{R})$ that then satisfies the desired condition that no matter how the points in V are colored with k colors, there exists a monochromatic axis-parallel rectangle of size at least m. Thus, hypergraphs of any uniformity captured by axis-aligned rectangles are not polychromatic k-colorable for any $k \in \mathbb{N}$. $\square$

1.4 Axis-aligned Strips in d Dimensions

We already know that for axis-aligned strips in d dimensions, it holds that

$$\left\lfloor \frac{k}{2} \right\rfloor d + 1 \leq m_k^* \leq d(k-1) + 1 \tag{1.1}$$

[ACC+11]. In the following, we will summarize the proofs by Aloupis et al. [ACC+11] for the lower and the upper bound and also explain the bounds for axis-aligned strips in the plane. For $d = 1$, they refer to the strips as intervals. Note that as we talk about a property of a dual hypergraph captured by strips in d dimensions, we color the strips.

Lemma 1.7 ([ACC$^+$11])**.** *Let $\mathcal{I}$ be a finite set of intervals. Then for every k, $\mathcal{I}$ can be k-colored so that every point that is contained in at least k strips is polychromatic, while any point covered by fewer than k intervals will be covered by distinct colors.*

Proof. For an interval I, let $r(I)$ ($l(I)$) denote the right (left) end point of the interval I. The proof works inductively via the size of the set of intervals $\mathcal{I}$. We order the intervals $\mathcal{I} = \{I, I_1, \ldots, I_{|\mathcal{I}|-1}\}$ according to their right end point, i.e., $r(I) \leq r(I_1) \leq \cdots \leq r(I_{|\mathcal{I}|-1})$. We now want to color the interval I with the leftmost right end point. For the other intervals in $\mathcal{I} \setminus \{I\}$, we know by induction, that they can be k-colored such that the desired properties hold. Let now $I'_1, \ldots, I'_{k-1}$ be the $k-1$ intervals intersecting I with the leftmost starting points. We color the interval I with a color $c \in \{1, \ldots, k\} \setminus \{c(I'_1), \ldots, c(I'_{k-1})\}$. Therefore, the color c of I is distinct to the colors of the $k-1$ leftmost intersecting intervals. This produces a valid k-coloring with the desired properties. $\square$

Theorem 1.8 ([ACC$^+$11])**.** *For any d and k, one can k-color any set of axis-aligned strips in $\mathbb{R}^d$ so that every point that is contained in at least $d(k-1)+1$ strips is polychromatic. That is,*

$$m_k^* \leq d(k-1)+1 \ .$$

Proof. The strips parallel to any axis $x_1, \ldots, x_d$ are colored separately as described in Lemma 1.7. Let s be a point contained in a set of $d(k-1)+1$ strips $H(s)$. The strips in $H(s)$ must each be parallel to an axis $x_1, \ldots, x_d$. By the pigeonhole principle, as there are only d axes, there are at least

$$\left\lceil \frac{d(k-1)+1}{d} \right\rceil = \left\lceil k - 1 + \frac{1}{d} \right\rceil = k$$

strips, that are parallel to the same axis. Thus, these at least k strips fulfill the property of Lemma 1.7 and therefore s is polychromatic. $\square$

Theorem 1.9 ([ACC$^+$11])**.** *For any fixed dimension d and integer k, it holds that*

$$m_k^* \geq \left\lfloor \frac{k}{2} \right\rfloor d + 1 \ .$$

Proof. We define $2d$ strips $\{s_i\}_{\{1,\ldots,2d\}}$ as follows: For $x_1, \ldots, x_d$ being the coordinates of the d dimension, we define the strips s_{2i} as $0 < x_i < 2$ and s_{2i+1} as $1 < x_i < 3$. For those strips it holds that there always exists a point that is covered by any subset of these $2d$ strips.

Each of these $2d$ strips is now replaced by $\lfloor k/2 \rfloor$ strips $\{s_{i,j}\}_{j \in \{1,\ldots,\lfloor k/2 \rfloor\}}$ that all overlap but have slightly different ends. We call each of the initial $2d$ strips a cluster containing $\lfloor k/2 \rfloor$ strips. For a point p it holds that $p \in s_{i,j}$ if and only if $p \in s_i$. As we color the strips and each cluster contains $\lfloor k/2 \rfloor$ strips, each cluster is missing $k - \lfloor k/2 \rfloor = \lceil k/2 \rceil$ colors. Thus, by the pigeonhole principle, we find that there is one color that is missing in at least

$$\frac{\left\lceil \frac{k}{2} \right\rceil 2d}{k} \geq d$$

clusters. For the set of clusters I that are all missing the same color, there is a point p that is contained in exactly these clusters. This point p then is covered by at least d clusters and thus by at least $\lfloor k/2 \rfloor d$ strips. But it is not polychromatic. Therefore, $m_k^* \geq \lfloor k/2 \rfloor d + 1$. $\square$

As we proved the lower and upper bound for m_k^* for d dimensions, we can determine the bounds for $d = 2$. Note that it must hold that $m_k^* \geq k$, as otherwise a point p cannot be covered by k strips of distinct colors as it is not even covered by k strips. For an odd number of colors $k = 2a + 1$ for $a \in \mathbb{N}_0$, we get

$$\left\lfloor \frac{k}{2} \right\rfloor 2 + 1 = \left\lfloor \frac{2a+1}{2} \right\rfloor 2 + 1 = 2a + 1 = k \leq m_k^* \leq 2k - 1 = 2(k-1) + 1 \ .$$

For $k = 2a$, the lower bound is as well equal to $2a + 1$, which is equal to $k + 1$ this time. We need to take the lower bound that is less restrictive, thus $k \leq m_k^*$ for $d = 2$. If we insert the values for two colors, thus $k = 2$, and two dimension into Equation 1.1, it holds that

$$\left\lfloor \frac{2}{2} \right\rfloor 2 + 1 = 3 \leq m_2^* \leq 3 = 2(2-1) + 1$$

and therefore $m_2^* = 3$.

1.5 Union of Topless and Bottomless Rectangles

We will now repeat the reduction of hypergraphs captured by the union of topless and bottomless rectangles to hypergraphs captured by axis-aligned squares by Chekan and Ueckerdt [CU21]. This proves the following Lemma:

Lemma 1.10 ([CU21])**.** *Let $\mathcal{H}(V, \mathcal{E})$ be an m-uniform hypergraph captured by the union of all topless and all bottomless rectangles. For $\mathcal{H}$, it holds that $m_k \in \mathcal{O}(k^{8.75})$.*

The idea of the proof is to shift the aspect ratios of the topless and bottomless rectangles so that the rectangles become axis-aligned squares. Then, we can apply the result of Ackerman et al. [AKV17] that for axis-aligned squares it holds that $m_k \in \mathcal{O}(k^{8.75})$.

The range families of topless rectangles $\mathcal{T}$ and bottomless rectangles $\mathcal{B}$ are defined as

- $\mathcal{T} = \{\{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, y \geq c\} \mid a, b, c \in \mathbb{R}\}$.
- $\mathcal{B} = \{\{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, y \leq c\} \mid a, b, c \in \mathbb{R}\}$.

The construction works as follows: Let $\mathcal{H}(V, \mathcal{T} \cup \mathcal{B}, m)$ be an m-uniform hypergraph captured by the union of topless and bottomless rectangles. Let $\mathcal{E}_\mathcal{T}$ and $\mathcal{E}_\mathcal{B}$ be the topless and bottomless rectangle hyperedges of $\mathcal{H}$, respectively.

First of all, we close the rectangles in such a way, that the upper side of $e_\mathcal{T} \in \mathcal{E}_\mathcal{T}$ lies above the topmost point of V, and the lower side of $e_\mathcal{B} \in \mathcal{E}_\mathcal{B}$ lies below all points. Thus, all rectangles are bounded. Now, we stretch the x-axis, such that for every hyperedge e, the width of e becomes greater than its height. Thereby, the relative positions of the points are maintained. As there are still no points above or below any bounded topless or bounded bottomless rectangle, respectively, we prolong the left and right sides of each bounded topless hyperedge to the top and for each bounded bottomless hyperedge to the bottom, such that each hyperedge becomes a square. Note that the uniformity of each hyperedge remains equal. Now we can apply the proof that homothets of axis-aligned squares are polychromatic k-colorable for $m \in \mathcal{O}(k^{8.75})$ by Ackerman et al. [AKV17].

1.6 Outline

We structure the thesis as follows: In Chapter 2, we formally define the notations and terms that we use in the rest of the work. We also make some simple, but important observations about the coloring terms. In Chapter 3, we simplify our studied problem:

Instead of hanging rectangles, we consider hanging bottomless rectangles and hanging right-open rectangles, separately. These two range families each form a subset of the range family of hanging rectangles. We introduce an algorithm for the proper 2-colorability of these two range families. Further, in Chapter 4, we investigate the union of those two range families. Therefore, we play two different presenter-painter-games which we then convert into one joint game. These three considered games are all semi-online and work for a coloring with two colors. We then generalize the game for the union of hanging bottomless and hanging right-open rectangles to any number of colors in Chapter 5. Thereby, we lose the bottomless property of the hanging bottomless rectangles but instead we can discuss hanging topless and hanging right-open rectangles. We also transfer the game for two colors into an offline game for bottomless and hanging right-open rectangles in Section 5.1. In Section 5.1.2, we discuss the challenges that occur when trying to transfer the semi-online game for any number of colors into an offline game. Finally, in Chapter 6, we summarize our elaborated results and name a few remaining questions.

2. Preliminaries

Let V be a **finite** set of points in $\mathbb{R}^2$. For a hypergraph $\mathcal{H}(V, \mathcal{E})$, we denote the number of points contained in a hyperedge $e \in \mathcal{E}$ as $|e|$. A *k-coloring* of $\mathcal{H}$ is a coloring of V with k distinct colors defined by a mapping $c : V \to \{1, 2, \ldots k\}$. We denote as $c(e)$ the set of colors used to color the points contained in the hyperedge e.

A *polychromatic* k-coloring of $\mathcal{H}$ is defined as a k-coloring whereby each edge contains all k colors, i.e.,

$$\forall e \in \mathcal{E} : c(e) = \{1, 2, \ldots k\} \ .$$

A *proper* k-coloring of $\mathcal{H}$ is a k-coloring with the additional condition, that every edge that contains at least two points also contains at least two different colors, i.e.,

$$\forall e \in \mathcal{E} : |e| \geq 2 \Rightarrow |c(e)| \geq 2 \ .$$

If a hypergraph $\mathcal{H}(V, \mathcal{E})$ has no proper k-coloring, there exits a *monochromatic* hyperedge $e \in \mathcal{E}$ that only contains one color, i.e., $|c(e)| = 1$.

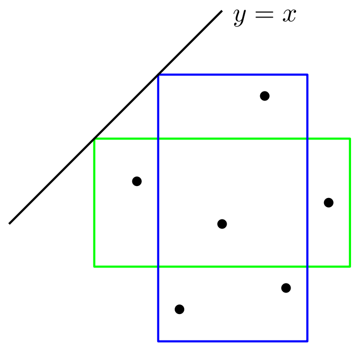


Figure 2.1: Example of hyperedges described by hanging axis-aligned rectangles. For simplification, we do not draw the axes. The black line describes the first angle bisector.

A *range* is defined as a set of points $R \subset \mathbb{R}^2$. A *range family* $\mathcal{R}$ is a set of ranges. For a range family $\mathcal{R}$ we denote $\mathcal{H}(V, \mathcal{R})$ as a *range-capturing hypergraph*. A point set $X \subseteq V$ defines a hyperedge of $\mathcal{H}(V, \mathcal{R})$ if and only if there exists a range $R \in \mathcal{R}$ for which holds that $R \cap V = X$. In this work, we study the proper k-colorability of hypergraphs captured by the range family $\mathcal{R}_R$ of hanging axis-aligned rectangles. The range family $\mathcal{R}_R$ is defined as follows:

$$\mathcal{R}_R = \{\{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, c \leq y \leq a\} \mid a, b, c \in \mathbb{R}\}\,.$$

It consists of a set of ranges each describing an axis-aligned rectangle whose upper left corner is attached to the first angle bisector defined by the line $y = x$. An example can be seen in Figure 2.1. Note that in the upcoming chapters, we do not always specify the exact coordinates of the points. We always assume that the relevant points of the graph are situated in an appropriate manner below the first angle bisector. Note also that it is not necessary that the rectangles are attached to the first angle bisector. Any other line with a slope of one is sufficient. But for simplicity, we always refer to the line as the first angle bisector.

In this work, we only consider *m-uniform* hypergraphs $\mathcal{H}(V, \mathcal{R}, m)$. This means that every hyperedge e in the set of hyperedges $\mathcal{E}$ of $\mathcal{H}$ contains exactly m points, i.e.,

$$\forall e \in \mathcal{E} : |e| = m\,.$$

Instead of $\mathcal{H}(V, \mathcal{R}, m)$ for an m-uniform hypergraph captured by the range family $\mathcal{R}$, we also refer to the same hypergraph $\mathcal{H}$ as $\mathcal{H}(V, \mathcal{E}, m)$, whereby $\mathcal{E}$ describes the set of hyperedges of $\mathcal{H}$. For every $e \in \mathcal{E}$ it holds that there exists a range $R \in \mathcal{R}$ with $R \cap V = e$.

Unless explicitly stated, we always consider *axis-aligned* rectangles in the following.

We further demand that all points in V have pairwise different x- and y-coordinates. Thus, for a pair of points $u = (x_u, y_u) \in V$ and $v = (x_v, y_v) \in V$, it holds that $x_u \neq x_v$ and $y_u \neq y_v$. Additionally, we say that the upper left corners of the hanging rectangles differ from each other. Therefore, we are able to order the hyperedges and all points along their attachment points or coordinates, respectively.

We will study proper k-colorings of m-uniform hypergraphs induced by all hanging rectangles. Thereby we will concentrate on smaller numbers of k, starting with $k = 3$. In the following, we will prove some important properties of polychromatic and proper colorings and will show an example for the not proper 3-colorability of 2-uniform hypergraphs captured by hanging rectangles.

Lemma 2.1. *Let $\mathcal{H} = (V, \mathcal{E}, m)$ be an m-uniform proper k-colorable hypergraph for a fixed $k \geq 2$. Then $\mathcal{H}$ is also proper $(k + 1)$-colorable.*

Proof. Let $\mathcal{H} = (V, \mathcal{E}, m)$ be an m-uniform proper k-colorable hypergraph for a fixed $k \geq 2$. The coloring of the point set of $\mathcal{H}$ is a mapping $c_k : V \to \{1, 2, \ldots, k\}$. Since the coloring is proper, we know that every edge with at least two points contains at least two of the colors $1, 2, \ldots, k$. We then define the coloring $c_{k+1} : V \to \{1, 2, \ldots, k, k + 1\}$ for any $v \in V$ as $c_{k+1}(v) = c_k(v)$, simply not using the $(k + 1)$-th color. As all points have the same colors no matter if colored using the coloring c_k or c_{k+1}, there are still at least two different colors in every hyperedge containing at least two points. Therefore $\mathcal{H}$ is also proper $(k + 1)$-colorable. $\square$

We now name a few additional elementary properties of polychromatic and proper k-colorings. For an arbitrary hypergraph $\mathcal{H}(V, \mathcal{R})$ it holds that

- if $\mathcal{H}$ admits a polychromatic k-coloring, then $\mathcal{H}$ also admits a polychromatic $(k-1)$-coloring. We define the $(k-1)$-coloring $c_{k-1} : V \to \{1, \ldots, k-1\}$ with the help of the k-coloring $c_k : V \to \{1, \ldots, k\}$ for every point $v \in V$ as follows:

$$c_{k-1}(v) = \begin{cases} c_k(v) & \text{if } c_k(v) \in \{1, \ldots, k-1\} \\ k-1 & \text{if } c_k(v) = k \end{cases} .$$

  In the polychromatic k-coloring c_k, each hyperedge contains the colors $\{1, \ldots, k\}$ and hence the colors $\{1, \ldots, k-1\}$. If we simply recolor the points with color k to color $k-1$, each hyperedge still contains the colors $\{1, \ldots, k-1\}$ and thus we have a polychromatic $(k-1)$-coloring c_{k-1}.

  It further holds that if $\mathcal{H}$ is not polychromatic $(k-1)$-colorable, $\mathcal{H}$ also admits no polychromatic k-coloring.

- if $\mathcal{H}$ admits a polychromatic k-coloring, then $\mathcal{H}$ is also properly k-colorable. As each hyperedge e contains all k colors and $k \geq 2$, e also contains at least two colors. Note especially that for $k = 2$ colors, it even holds that $\mathcal{H}$ admits a polychromatic k-coloring if and only if $\mathcal{H}$ admits a proper k-coloring.

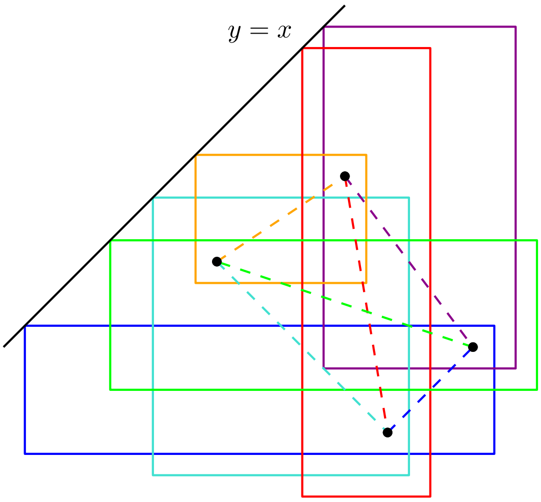


Figure 2.2: The graph K_4 represented using hanging rectangles. The dashed lines represent the edges of K_4. Each edge is represented in the hypergraph captured by hanging rectangles by the rectangle having the same color.

Theorem 2.2. *For $m = 2$, there exits a hypergraph captured by the range family $\mathcal{R}_R$ that is not proper 3-colorable.*

Proof. For $m = 2$, an m-uniform hypergraph corresponds to a graph having exactly two nodes incident to each edge. It is known that the clique number $\omega(G)$ of a graph G is less than or equal to the chromatic number $\chi(G)$, i.e.,

$$\omega(G) \leq \chi(G) .$$

That means that the size of the largest clique in a graph is a lower bound on the number of colors needed to color the graph such that adjacent nodes have different colors. Finding such a coloring is equivalent to finding a proper k-coloring of a 2-uniform hypergraph.

The graph K_4 is the graph consisting of a clique of size four. This graph needs at least four colors to be colored. Figure 2.2 now maps K_4 into the setting of a hypergraph captured by hanging rectangles.

Hence, this graph is an example of a 2-uniform hypergraph, that is not proper 3-colorable, as using only three colors would result in a monochromatic edge. □

The simplest next approach is to try to extend K_4 to K_5. If this is possible, K_5 is a counterexample for the proper 4-colorability of 2-uniform hypergraphs captured by hanging rectangles. Unfortunately, we cannot simply extend the embedded K_4 to a K_5.

Observation 2.3. *It is not possible to extend the embedded K_4 to an embedded K_5 that is still captured by hanging rectangles.*

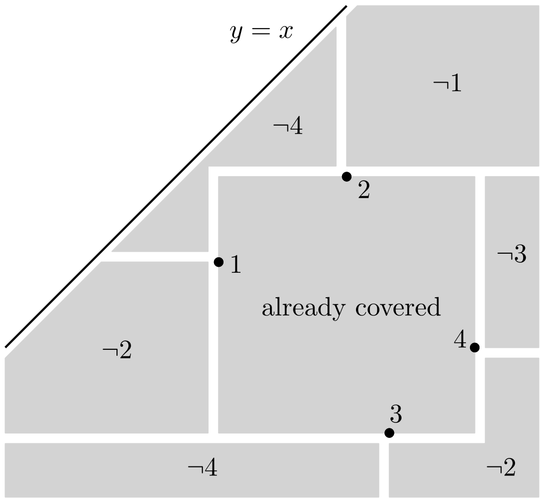


Figure 2.3: Illustration of possible areas relative to the points of K_4, where the fifth point can be located. The area between the four other points is already completely covered by other hyperedges. The figure shows which point contradicts the extension to K_5. Thereby, $\neg x$ denotes that x and the fifth point cannot be captured by a hanging rectangle of size two, if the latter is positioned in the respective region. For example, if the fifth point is located in the upper right of the second point, we cannot form a hyperedge covering point 5 and point 1, without containing point 2 as well.

It is clear that the new, fifth point, needs to be situated outside of the other hyperedges. It can therefore not be situated inside the axis-aligned rectangle having all four existing points on one end, respectively. If we situate the fifth point on any other position outside of this rectangle, there is always one point, that cannot be part of a hanging rectangle of size two together with point 5. An illustration of possible areas where we can situate the fifth point and the specification, with which point it cannot form a hanging rectangle can be seen in Figure 2.3. Thus, we cannot easily extend K_4 to K_5 and can therefore not assure that we need at least five colors to color any 2-uniform hypergraph captured by hanging rectangles. Analogously, we nevertheless cannot assure that four colors are sufficient to color any of those hypergraphs, as it may be still possible to embed K_5 with hanging rectangles.

3. Hanging Bottomless and Right-Open Rectangles

In this chapter, we investigate the range families of hanging bottomless rectangles, $\mathcal{R}_{BL}$, and hanging right-open rectangles, $\mathcal{R}_{RO}$. We will prove that hypergraphs $\mathcal{H}(V, \mathcal{R}_{BL}, m)$ and $\mathcal{H}(V, \mathcal{R}_{RO}, m)$ captured by one of those range families form a subset of a hypergraph $\mathcal{H}'(V, \mathcal{R}_R, m)$ over the same set of points captured by hanging rectangles. The range families $\mathcal{R}_{BL}$ and $\mathcal{R}_{RO}$ are defined as follows:

- $\mathcal{R}_{BL} = \{\{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, y \leq a\} \mid a, b \in \mathbb{R}\}$ and
- $\mathcal{R}_{RO} = \{\{(x, y) \in \mathbb{R}^2 \mid a \leq x, b \leq y \leq a\} \mid a, b \in \mathbb{R}\}$.

We say that a hypergraph $\mathcal{H}(V, \mathcal{E})$ forms a subset of a hypergraph $\mathcal{H}'(V', \mathcal{E}')$ if it holds that $V = V'$ and $\mathcal{E} \subseteq \mathcal{E}'$.

Lemma 3.1. *Any m-uniform hypergraph $\mathcal{H} = (V, \mathcal{E}, m)$ captured by the range family $\mathcal{R}_{BL}$ ($\mathcal{R}_{RO}$) forms a subset of the m-uniform hypergraph $\mathcal{H}' = (V, \mathcal{E} \cup \mathcal{E}', m)$ captured by the range family $\mathcal{R}_R$.*

Proof. Let $\mathcal{H}(V, \mathcal{E}, m)$ be an m-uniform hypergraph captured by the range family $\mathcal{R}_{BL}$. We can adapt the definition of this range family to $\mathcal{R}'_{BL} = \{\{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, e \leq y \leq a\} \mid a, b \in \mathbb{R}\}$ whereby e denotes a baseline that lies below the point $v_b = (x_b, y_b) \in V$ with the smallest y-coordinate.

The definition of $\mathcal{R}'_{BL}$ is then similar to that of $\mathcal{R}_R$, the only difference being the variable lower side of the edges in $\mathcal{R}_R$. The edges created by the range family $\mathcal{R}'_{BL}$ then correspond to the edges in $\mathcal{R}_R$ whose lower side lies in the interval $(-\infty, y_b)$. We can substitute $-\infty$ with the baseline e, as e also lies in this interval. And as v_b is the lowest point of $\mathcal{H}$, it does not matter where exactly the bottom side of the rectangle lies in this interval. Thus, any set of points of size m that is captured by a hanging bottomless rectangle is also captured by a hanging rectangle. This results in $\mathcal{H} = (V, \mathcal{E}, m)$ being a subgraph of $\mathcal{H}'$ captured by $\mathcal{R}_R$ whereby $\mathcal{H}' \coloneqq (V, \mathcal{E} \cup \mathcal{E}', m)$.

Analogously we can adapt the definition of $\mathcal{R}_{RO}$ to $\mathcal{R}'_{RO} = \{\{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq e, b \leq y \leq a\} \mid a, b \in \mathbb{R}\}$ where e is a baseline right of $v_r = (x_r, y_r)$, the point with the highest x-coordinate. The hyperedges of size m in $\mathcal{R}'_{RO}$ then correspond to those in $\mathcal{R}_R$ whose right side lies in $(x_r, +\infty)$ where we again can substitute $+\infty$ with e which also lies in this interval as v_r is the rightmost point of $\mathcal{H}$. Therefore, also for hanging right-open rectangles, an m-uniform hypergraph $\mathcal{H}(V, \mathcal{E}, m)$ captured by $\mathcal{R}_{RO}$ is a subgraph of $\mathcal{H}' := (V, \mathcal{E} \cup \mathcal{E}', m)$ captured by hanging rectangles. □

Another generalization of $\mathcal{R}_{BL}$ is the range family of axis-aligned bottomless rectangles $\mathcal{B}$ defined as

$$\mathcal{B} = \{\{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, y \leq c\} \mid a, b, c \in \mathbb{R}\} .$$

Ranges $B \in \mathcal{B}$ describing bottomless rectangles do not necessarily have their upper left corner attached to the first angle bisector. If we know that all hypergraphs $\mathcal{H}(V, \mathcal{B}, m)$ are polychromatic or proper k-colorable, we can apply this information to the range family of hanging bottomless rectangles $\mathcal{R}_{BL}$, as the following holds:

Lemma 3.2. *Any m-uniform hypergraph $\mathcal{H} = (V, \mathcal{E}, m)$ captured by the range family $\mathcal{R}_{BL}$ of hanging bottomless rectangles forms a subset of the m-uniform hypergraph $\mathcal{H}' = (V, \mathcal{E} \cup \mathcal{E}', m)$ captured by the range family $\mathcal{B}$ of bottomless rectangles.*

Proof. Let $R = \{(x, y) \mid a \leq x \leq b, y \leq a\} \in \mathcal{R}_{BL}$ be a hanging bottomless rectangle defined by some constants $a, b \in \mathbb{R}$. R is also an element of $\mathcal{B} = \{\{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, y \leq c\} \mid a, b, c \in \mathbb{R}\}$, as c can take the value a. Thus, as we only consider hyperedges of size m, each set of m points that is captured by a hanging bottomless rectangle is also captured by a non-hanging bottomless rectangle of size m. Therefore, the hypergraph $\mathcal{H} = (V, \mathcal{E}, m)$ captured by the range family $\mathcal{R}_{BL}$ of hanging bottomless rectangles forms a subset of the hypergraph $\mathcal{H}' = (V, \mathcal{E} \cup \mathcal{E}', m)$ captured by the range family $\mathcal{B}$ of bottomless rectangles over the same set of points. □

This means, that every hypergraph captured by hanging bottomless rectangles is also a hypergraph captured by non-hanging bottomless rectangles. Therefore, the colorability results of bottomless rectangles apply to hanging bottomless rectangles. Note that it is especially not possible to conclude from the non-colorability of hypergraphs captured by $\mathcal{B}$ to the non-colorability of hypergraphs captured by $\mathcal{R}_{BL}$. The hanging property of the hyperedges can bring a certain advantage to the colorability, as it restricts the number of possible hyperedges in contrast to the number of hyperedges of a hypergraph over the same set of points V captured by the range family $\mathcal{B}$ without the hanging property. We refer to the image in the middle of Figure 3.2 as example where we cannot capture the second, third and fourth point in x-direction with a hanging bottomless rectangle. But it is possible to capture those points with a non-hanging bottomless rectangle.

Keszegh [Kes12] has proven that, given a finite set of points, any m-uniform hypergraph consisting of bottomless rectangles has a proper 3-coloring for $2 \leq m \leq 3$. He has further shown that for $m \geq 4$, any m-uniform hypergraph captured by $\mathcal{R}_{BL}$ has a proper 2-coloring. Note that a proper 2-coloring corresponds to a polychromatic 2-coloring.

Asinowski et al. [ACC$^+$13] studied the polychromatic colorability of bottomless rectangles: Any hypergraph consisting of bottomless rectangles that each have a size $m \geq 3k - 2$ has a polychromatic k-coloring.

Lemma 3.3 (Asinowski et al. [ACC$^+$13])**.** *Every point set $S \subset \mathbb{R}^2$ can be colored with k colors so that any bottomless rectangle containing at least $3k - 2$ points of S contains at least one point of each color.*

Proof. For completeness, we repeat the proof by Asinowski et al. [ACC$^+$13]. They show that the polychromatic coloring problem for bottomless rectangles is equivalent to the following coloring problem:

Claim 3.4 (Asinowski et al. [ACC$^+$13])**.** *Every dynamic point set without disappearing points can be k-colored in the semi-online model such that at any time, every subsequence of at least $3k-2$ consecutive points contains at least one point of each color.*

Proof of Claim 3.4. Consider the points to appear on a line one at a time. We define the set of points between two consecutive points with color c as a *gap for color c*. This also includes the points left to the first point with color c (first gap) and those right to the last one (last gap). Note that in the case that there is no point with color c, the first gap is equal to the last gap. The idea of the algorithm is to assure that all gaps have a size of at most $3k-3$.

For an empty set and for an arbitrary but fixed set of points we maintain the following invariants:

a) There are at most $3k-3$ points in each gap.

b) If there exists a point that is colored with color c, all gaps for color c, excluding the first and the last gap, have at least $k-1$ points.

By adding a new point, it is not possible to violate the invariant b) as it only assures a minimum number of points that cannot decrease when one is added. As long as invariant a) holds, we do not color any point. If there is a gap for color c, that increases to a size of $3k-2$, there is a violation to invariant a) and we need to color at least one point to make sure that invariant a) holds again. We denote the points of this gap from the left to the right as $l_1, \ldots, l_{k-1}, m_1, \ldots, m_k, r_1, \ldots, r_{k-1}$. We know that none of those points has color c, as they form a gap for that color. Due to invariant b), we also know that among the k consecutive points $m_1, \ldots, m_k$, there cannot be the same color twice, as every gap has a size of at least $k-1$. Additionally, there are at most $k-1$ colors presented within these k points. Hence, at least one of the points $m_1, \ldots, m_k$ is uncolored, say m_i. Coloring it with color c, we split the large gap into two smaller ones, both having at least $k-1$ points as we have $k-1$ points to the left and to the right of any point in $\{m_1, \ldots, m_k\}$. Thus, invariant b) holds. The new gaps have a size of at most $2(k-1) = 2k-2$ points (if $m_i = m_1$ or $m_i = m_k$) and thus invariant a) holds. After presenting all the points, the points that remained uncolored are colored arbitrarily.

To summarize, no gap has a size greater than $3k-3$. Hence, every sequence of $3k-2$ consecutive points contains each color at least once. □

For the polychromatic coloring of bottomless rectangles, we can order the points by their x-coordinates and process them vertically in increasing y-coordinate. For a point in time t, we have a dynamic point set $S_t \subseteq S$ with all points having a y-coordinate lower or equal to $y = t$. A bottomless rectangle $r = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, y \leq t\}$ in S is then represented by the interval $[a, b]$ in the subset S_t. We can then apply the algorithm described in the proof of the Claim to color the bottomless rectangles. □

The runtime of this algorithm is $\mathcal{O}(k \cdot |V| \cdot \log |V|)$. For each new point $v \in V$, its position in the list of already presented points can be found in $\mathcal{O}(\log |V|)$. We then need to check if the new point v increases the gap of any color to $3k-2$. Therefore, we need to look at the $3k-3$ points to the left and to the right of v in the list, respectively. For every color c_i, we save in a counter l^{c_i} the number of points between its first appearance on the left of v and v and in another counter r^{c_i} the number of points between v and the first appearance on the right of v. The gap size for c_i then equals to $r^{c_i} + l^{c_i} + 1$. If a gap grows too big, say for color c, we color one of the uncolored points in the middle of the gap with the corresponding color. The middle of the gap are the points $m_1, \ldots, m_k$ mentioned in the proof, thus not the first and not the last $k-1$ points of the gap. Due to the counters for color c, we know which points are identified as $l_1, \dots, l_{k-1}$, $m_1, \dots, m_k$ and $r_1, \dots, r_{k-1}$. The calculation of the gap sizes and the coloring of a point if necessary works in $\mathcal{O}(k)$.

Observation 3.5. *Results about the colorability of (hanging) bottomless rectangles can also be applied to (hanging) right-open rectangles.*

For a set of points V of a hypergraph captured by hanging right-open rectangles, we swap the x- and y-coordinates of every point in V. Thus, we get a new point set $V' = \{(y,x) \mid (x,y) \in V\}$. Additionally, instead of considering the points in increasing order of their coordinates, we consider them in decreasing order and vice versa. In this way, we can consider any hypergraph $\mathcal{H}(V,\mathcal{E})$ captured by hanging right-open rectangles as a hypergraph $\mathcal{H}'(V',\mathcal{E}')$ captured by hanging bottomless rectangles.

We can further use the Union-Lemma introduced by Damásdi and Pálvölgyi [DP21a] to apply it to the previous results to generate results about the colorability of hypergraphs captured by the union of hanging bottomless rectangles and hanging right-open rectangles.

Lemma 3.6 (Union-Lemma, Damásdi and Pálvölgyi [DP21a])**.** *Let*

$$\mathcal{H}_1 = (V,\mathcal{E}_1), \dots, \mathcal{H}_{k-1} = (V,\mathcal{E}_{k-1})$$

be hypergraphs on a common point set V. If $\mathcal{H}_1, \dots, \mathcal{H}_{k-1}$ are polychromatic k-colorable, then the hypergraph

$$\bigcup_{i=1}^{k-1} \mathcal{H}_i = (V, \bigcup_{i=1}^{k-1} \mathcal{E}_i)$$

is proper k-colorable.

Proof. For completeness, we repeat the proof of the Lemma by Damásdi and Pálvölgyi [DP21a]. We construct a proper coloring $c : V \to \{1,\dots,k\}$ for $\bigcup_{i=1}^{k-1} \mathcal{H}_i$ by using the polychromatic colorings $c_i^{poly} : V \to \{1,\dots,k\}$ of $\mathcal{H}_1, \dots, \mathcal{H}_{k-1}$. For every hyperedge $E \in \mathcal{E}_i$ and for all colors $j \in \{1,\dots k\}$, there is a point $v \in E$ that has color j. As there are only $k-1$ polychromatic colorings, the $k-1$ hypergraphs can only occupy $k-1$ different colors for a fixed $v \in E$. Thus, there is always a k-th color c available for v in the proper coloring, that is different to the colors of all polychromatic colorings. This remaining color c is not the same for all $v \in E$, as there exists a point $w \in E$ where $c_i^{poly}(w) = c$ and as a consequence, $c(w) \neq c$. Therefore, every hyperedge $E \in \mathcal{E}_i$ always contains at least two points with different colors. $\square$

This construction can be done in $\mathcal{O}(|V| \cdot (k-1))$, as we need to take a look at every point $v \in V$ and for every v, we need to find the unused color. Therefore we need to check all hypergraphs $\mathcal{H}_1, \dots, \mathcal{H}_{k-1}$, thus $k-1$ copies of v.

Using the result of Asinowski et al. [ACC+13] about polychromatic k-colorability together with the Union-Lemma by Damásdi and Pálvölgyi [DP21a] (Lemma 3.6), we can conclude the following:

Theorem 3.7. *Let $\mathcal{H} = (V,\mathcal{E},m)$ be a hypergraph captured by $\mathcal{R}_{BL} \cup \mathcal{R}_{RO}$, i.e., the union of the range families of hanging bottomless and hanging right-open rectangles. For any $k \geq 3$ and $m \geq 3k-2$, $\mathcal{H}$ is proper k-colorable.*

Proof. Let $k \geq 3$ and let $\mathcal{H}_1 = (V, \mathcal{E}_1, m)$ and $\mathcal{H}_2 = (V, \mathcal{E}_2, m)$ be hypergraphs over a common set of points V captured by hanging bottomless and hanging right-open rectangles, respectively. If $m \geq 3k-2$, we know from [ACC+13] that $\mathcal{H}_1$ is polychromatic k-colorable for any k, as Lemma 3.2 shows that we can apply the results about the colorability of hypergraphs for bottomless rectangles to hanging bottomless rectangles. Furthermore, $\mathcal{H}_2$ is polychromatic k-colorable as by Observation 3.5, we know that results about the colorability of hanging bottomless rectangles equally apply to hanging right-open rectangles. Let now $\mathcal{H}_3, \ldots, \mathcal{H}_{k-1}$ be $k-3$ copies of the hypergraph $\mathcal{H}_1$. As $\mathcal{H}_1$ is polychromatic k-coloarble, all of the copies are polychromatic k-colorable too. Applying the Union-Lemma 3.6, we can now conclude that the hypergraph

$$\mathcal{H}(V, \mathcal{E}, m) := \bigcup_{i=1}^{k-1} \mathcal{H}_i = (V, \bigcup_{i=1}^{k-1} \mathcal{E}_i, m) = (V, \mathcal{E}_1 \cup \mathcal{E}_2, m)$$

captured by $\mathcal{R}_{BL} \cup \mathcal{R}_{RO}$ is proper k-colorable. □

Corollary 3.8. *Let $\mathcal{H} = (V, \mathcal{E}, m)$ be a hypergraph captured by $\mathcal{R}_{BL} \cup \mathcal{R}_{RO}$, i.e., the union of the range families of hanging bottomless and hanging right-open rectangles. For $m \geq 7$, $\mathcal{H}$ is proper k-colorable for any $k \geq 3$.*

Proof. Let $\mathcal{H} = (V, \mathcal{E}, m)$ be a hypergraph captured by hanging bottomless and hanging right-open rectangles. Theorem 3.7 shows that if $k = 3$ and $m \geq 3k-2 = 3 \cdot 3 - 2 = 7$, $\mathcal{H}$ is proper 3-colorable. As $\mathcal{H}$ is proper 3-colorable, $\mathcal{H}$ is also proper k-colorable for every $k > 3$ according to Lemma 2.1. To summarize, $\mathcal{H} = (V, \mathcal{E}, m)$ is proper k-colorable for every $k \geq 3$ if $m \geq 7$. □

3.1 Proper 2-colorability of Hanging Bottomless and Hanging Right-Open Rectangles

It is already known that any m-uniform hypergraph captured by the range family of bottomless rectangles $\mathcal{B}$ is best possible proper 3-colorable for $2 \leq m \leq 3$ and proper 2-colorable for every $m \geq 4$ [Kes12]. This applies at least also to hypergraphs captured by the range family of hanging bottomless rectangles $\mathcal{R}_{BL}$, as $\mathcal{R}_{BL} \subseteq \mathcal{B}$ (see Lemma 3.2). We now want to examine, if we can improve these values for 2- and 3-uniform hypergraphs captured by *hanging* bottomless rectangles, by using the additional characteristics of this range family.

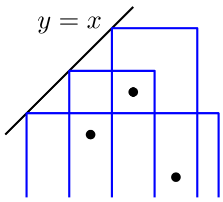


Figure 3.1: Illustration of a 2-uniform hypergraph captured by hanging bottomless rectangles. It contains three points and three hyperedges. As these points form a clique of size three, we need at least three colors to color this hypergraph properly.

Lemma 3.9. *There exists a 2-uniform hypergraph $\mathcal{H}(V, \mathcal{R}_{BL}, 2)$ captured by hanging bottomless rectangles that is not proper 2-colorable.*

Proof. We show a point set V as counterexample which is similar to that of Keszegh [Kes12] and with which it is shown that we need at least three colors to be able to color every 2-uniform hypergraph captured by bottomless rectangles.

We transfer this counterexample to *hanging* bottomless rectangles in Figure 3.1. There we can see that the three points of the hypergraph can pairwise be captured by a hanging bottomless rectangle. So these three points and the three edges can be seen as a graph $G(V, E) = K_3$ that forms a clique of size three, which results in $3 = \omega(G) \leq \chi(G)$, i.e., we need at least three colors to color this graph properly. □

Due to Lemma 3.9, we cannot improve the number of colors needed to color a 2-uniform hypergraph captured by hanging bottomless rectangles. Therefore, we can say that these 2-uniform hypergraphs are best possible proper 3-colorable, as the result of Keszegh [Kes12] applies to m-uniform hypergraphs captured by hanging bottomless rectangles, as they form a subset of m-uniform hypergraphs captured by bottomless rectangles (see Lemma 3.2). We now want to show that 3-uniform hypergraphs captured by *hanging* bottomless rectangles are proper 2-colorable, in contrast to the case of bottomless rectangles. We also show how to find such a coloring. Therefore we need to introduce a few definitions:

Definition 3.10. *Let e_1 and e_2 be hanging rectangles attached to the first angle bisector at the positions (x_1, y_1) and (x_2, y_2), respectively, with $x_1 < x_2$. We denote e_1 and e_2 as consecutive, if and only if $x_2 < x_k$ for any other hanging rectangle e_k attached at position (x_k, y_k) with $x_1 < x_k$.*

We also define a total order on the edges in the following Definition 3.11.

Definition 3.11. *Let $e_1, e_2, \ldots, e_k$ be hanging rectangles attached to the first angle bisector at the positions $(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)$, respectively. For two edges e_i and e_j, $1 \leq i, j \leq k$, e_i has a higher order than e_j if and only if $x_i > x_j$, otherwise e_i has a lower order than e_j. This relation is transitive. Therefore, these pairwise relations between all edges lead to a total order of the edges, whereby the edge e_l with the lowest x-coordinate among all edges is the edge with the smallest order overall.*

This means, that when we walk along the first angle bisector from the bottom left to the top right, the order of the attachment points corresponds to the order of the hyperedges. And therefore two hyperedges are consecutive if we find them attached directly one after the other. Furthermore, for a currently considered hyperedge e, we say that all hyperedges that are attached to the first angle bisector after e are higher ordered hyperedges than e.

Observation 3.12. *Let $\mathcal{H}$ be a hypergraph containing the hyperedges $e_i = \{p_i^1, \ldots p_i^m\}$ and $e_j = \{p_j^1, \ldots p_j^m\}$. Let e_i and e_j be consecutive hanging bottomless rectangles with m points each. Then e_i and e_j can have zero to $m-1$ points in common, i.e.,*

$$\exists x \in \{0, \ldots, m-1\} : |e_i \cap e_j| = x \; .$$

Figure 3.2 shows exemplary consecutive edges with both containing $m = 3$ points that have two, one or zero points in common. It is easy to see that the hanging characteristic of the edges together with the distance of the points determines how many points the edges have in common. This example for 3-uniform edges can be generalized to any uniformity m. Given m points that form an edge e_1. Roughly speaking, the further up right we place m additional points for a hyperedge e_2, the less points the two edges have in common. The upper side of a hanging rectangle r needs to be above the highest point contained in it. As

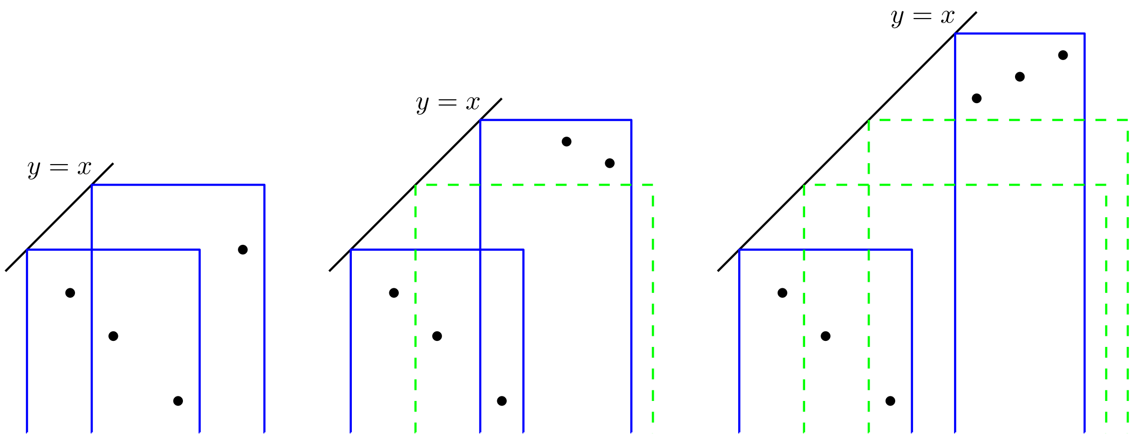


Figure 3.2: Illustration of the three possible overlapping scenarios of two consecutive edges with uniformity $m = 3$. They differ in one, two or three points. The blue lines each indicate a hanging bottomless rectangle. The dashed green lines indicate positions at which there cannot be hyperedges of this 3-uniform hypergraph, as they each capture only less than three points. On the right, the two blue edges have no points in common. This results in two different connected components.

the x-coordinate of the left side of r equals the y-coordinate of the upper side, the higher the highest point in r, the further right is r.

Let now $\mathcal{H}$ be an m-uniform hypergraph captured by the range family $\mathcal{R}_{BL}$ of hanging bottomless rectangles. We shall find a proper 2-coloring of the points of $\mathcal{H}$. Therefore we consider the hyperedges to be ordered along the first angle bisector and process them one after the other starting from the left. The two available colors c_1 and c_2 are also ordered: $c_1 < c_2$. As long as the currently processed hyperedge e does not contain two points with different colors, we color the leftmost uncolored point (in x-direction) with the smallest unused color in that edge e. If the edge contains two points with two different colors, we pass over to the next edge, leaving possible uncolored points uncolored. After having processed all the edges, we color all points that remained uncolored with the same color c_1.

Lemma 3.13. *The algorithm described above calculates a proper 2-coloring of the hypergraph $\mathcal{H}(V, \mathcal{R}_{BL}, m)$ captured by hanging bottomless rectangles with uniformity $m \geq 3$.*

Proof. We define the already considered points as *active*, and the points that did not yet appear in any edge as *inactive*. Two active points are called *consecutive* (in x-direction), if there is no other active point positioned between them (in x-direction).

If all points of a processed hyperedge are still uncolored, we color the leftmost uncolored point with color c_1 and the following uncolored point with color c_2. As $m \geq 3$, this is always possible.

When processing a hyperedge that already contains one color but still has at least one uncolored point, we are always able to color the points in the hyperedge with two different colors. Let at most $m - 1$ points be already colored with the same color, say c_1. We then color the leftmost uncolored point with c_2 and the hyperedge contains both colors. Hence, in this scenario, we never get a monochromatic hyperedge. This case only occurs during the algorithm if a hyperedge is processed, whose points are all already colored in the same color due to the processing of lower ordered hyperedges.

We do not color any point, if there already exist at least two points with different colors among the m points of the hyperedge.

Thus, a monochromatic hyperedge only occurs during the algorithm, if all m points of the hyperedge are active, consecutive and already colored in the same color, say c_1. We show now, that there cannot even be two consecutive points with the same color, which can be captured by higher ordered hyperedges to produce a monochromatic hyperedge later in the algorithm:

Claim 3.14. *Using the algorithm, there never exist two active points p and q with the same color and consecutive x-coordinates, which are both part of higher ordered hyperedges, if $m \geq 3$.*

Proof of Claim 3.14. Let $p = (x_p, y_p)$ and $q = (x_q, y_q)$ be the first two active points with consecutive x-coordinates which are colored with the same color c. Consider the situation in which the second point, say q, just got its color by processing the hyperedge $e = \{(x, y) \in \mathbb{R}^2 \mid x_e^l \leq x \leq x_e^r, y \leq x_e^l\}$ defined by its attachment point (x_e^l, x_e^l) and the coordinate of its right border x_e^r. As q is colored when processing e, it is part of e.

As q is colored with the color c, there is no other point with color c in e, due to the coloring strategy of the algorithm. This implies that p is not part of the hyperedge e which means $x_p < x_e^l$ or $x_p > x_e^r$ or if $x_e^l \leq x_p \leq x_e^r$ then $y_p > x_e^l$, i.e., p is situated above e.

If p is on the left of e, so $x_p < x_e^l$, it cannot be part of any higher ordered hyperedge. Because of the attachment of e to the first angle bisector, the points situated left of this edge are only part of a higher ordered hyperedge e', if e' is situated further left than e. But then, e' is considered before e and is therefore a lower ordered edge than e.

If p is above e, meaning $y_p > x_e^l$ regardless of p's x-coordinate, the following holds: All hyperedges that contain p have an upper side y_u and an attachment point (y_u, y_u) for which applies $y_u \geq y_p$. This results in $y_u > x_e^l$ making every hyperedge that contains p a higher ordered edge than e. Therefore these hyperedges are not yet considered and thus p is still inactive.

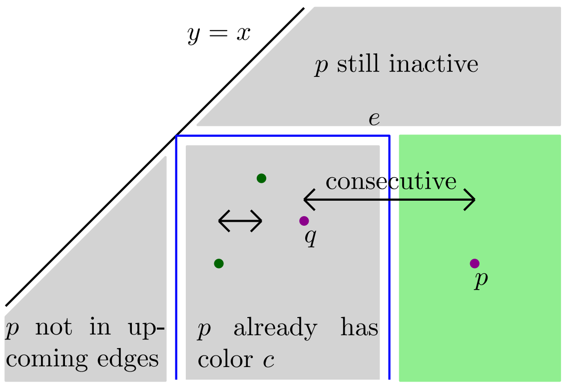


Figure 3.3: Illustration of the proof of Claim 3.14 for $m = 3$. Points with color c are represented in purple, points with color c' in dark green. The blue hanging rectangle represents the edge e containing q. In the gray areas, p cannot be situated. Instead, p is part of the green area.

Thus, the only remaining valid position for p is on the right side of q with $x_p > x_e^r$ and $y_p < x_e^l$. Additionally, p is already colored which means it is part of a lower ordered hyperedge and therefore active. As we assumed that p and q are consecutive in x-direction, and p is not part of e, this results in q being the rightmost point of e. But as q is colored, there were less than two colors represented in the other points of e. Additionally. all other points of e were already colored. For an uniformity of $m \geq 3$, this results in the situation, that the other points in e are all on the left of q and are all colored in the same color c'. If $m \geq 3$, there are at least two other points in e which are active and consecutive. As they are all colored with the same color c', this is a contradiction to p and q being the first two consecutive points with the same color. Thus, there can never exist two consecutive points having the same color that can both be part of higher ordered hyperedges if $m \geq 3$. An illustration of the proof for $m = 3$ is shown in Figure 3.3. □

To summarize, this means that, as there cannot even be two consecutive points with the same color that can be part of higher ordered hyperedges, there especially cannot be more than two consecutive points with the same properties. Thus, the algorithm does not produce monochromatic hyperedges for hypergraphs with uniformity $m \geq 3$.

As we only use the two colors c_1 and c_2 and are always able to guarantee both of them in the m points of each hyperedge, it is irrelevant how we color the still uncolored points at the end. Thus we need only two colors to avoid monochromatic hyperedges using the algorithm described above. In conclusion, hypergraphs with $m \geq 3$ points in every hyperedge captured by hanging bottomless rectangles are proper 2-colorable. □

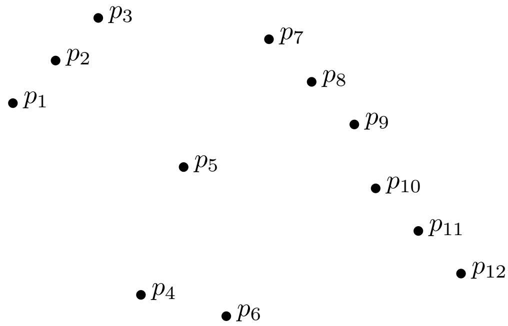


Figure 3.4: Counterexample taken from [Kes12] that shows that we need at least three colors to properly color hypergraphs captured by bottomless rectangles with uniformity $m = 3$.

For completeness, we present the counterexample for non-hanging bottomless rectangles taken from [Kes12] in Figure 3.4. It shows that not all 3-uniform hypergraphs captured by bottomless rectangles are proper 2-colorable. Therefore we try to color the hypergraph induced by the points $p_1, \ldots, p_{12}$ in Figure 3.4 with only two colors and show, that this always induces a monochromatic hyperedge. The points $p_1, \ldots, p_{12}$ are ordered according to their x-coordinate. We can see that the points p_4, p_5 and p_6 form a bottomless rectangle edge, and as we only use two colors, we know that two of those points must be colored with the same color c. Assuming p_4 and p_5 are colored with color c, there exist three edges $\{p_4, p_5, p_1\}$, $\{p_4, p_5, p_2\}$ and $\{p_4, p_5, p_3\}$ containing each two points with color c and one uncolored point. The uncolored points thus must all be colored with the second color c'. But this leads to the monochromatic hyperedge $\{p_1, p_2, p_3\}$. The same reasoning leads to a monochromatic hyperedge $\{p_7, p_8, p_9\}$ if p_5 and p_6 have color c, and a monochromatic hyperedge $\{p_{10}, p_{11}, p_{12}\}$ if p_4 and p_6 have color c. Hence, this 3-uniform hypergraph is not proper 2-colorable [Kes12].

We now move to the range family of hanging right-open rectangles instead of hanging bottomless rectangles. We can perform a similar algorithm to that for hanging bottomless rectangles to show the proper 2-colorability of hypergraphs captured by hanging right-open rectangles for $m \geq 3$: Let $\mathcal{H}$ be an m-uniform hypergraph captured by $\mathcal{R}_{RO}$. We consider the hyperedges to be ordered along the first angle bisector and process them one after the other starting with the hyperedge attached to the first angle bisector at the highest point. As above, we order the colors: $c_1 < c_2$. As long as there are not two different colors in the currently considered hyperedge e, we color the topmost uncolored point with the smallest unused color in e. If there are two points with two different colors, we pass over to the next edge. In the end, we color all still uncolored points with the same color c_1.

Lemma 3.15. *The algorithm described above calculates a proper 2-coloring of the hypergraph $\mathcal{H}(V, \mathcal{R}_{RO}, m)$ captured by hanging right-open rectangles with uniformity $m \geq 3$.*

Proof. We can perform the proof analogously to the proof of Lemma 3.13, but, as in the algorithm, instead of starting the processing of the edges on the left, we start from the right. A higher ordered hyperedge is then a hyperedge, that is attached further left to the first angle bisector. Two points are then consecutive, if they are active and there is no other active point between them in y-direction. The point p has to be below the hyperedge e and we conclude with the same contradiction, that if p and q are both active, consecutive and can be part of higher ordered hyperedges, then they were not the first two points with those characteristics.

In conclusion, m-uniform hypergraphs captured by hanging right-open rectangles are proper 2-colorable for $m \geq 3$. □

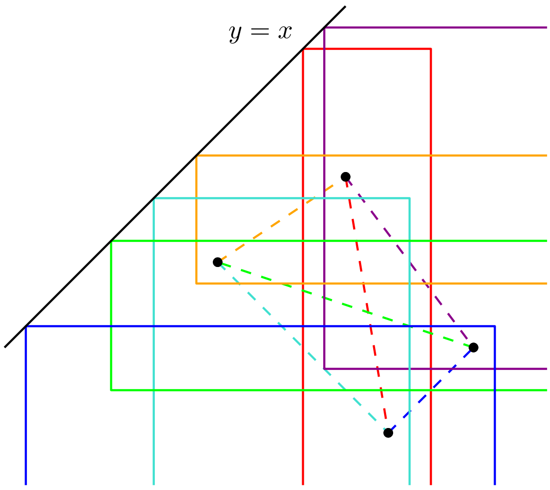


Figure 3.5: Embedding of the graph K_4 with hanging bottomless and hanging right-open rectangles.

If we now take a look at the union of hanging bottomless and hanging right-open rectangles, we can use the 2-coloring of each individual range family and construct a proper 4-coloring for the union. Hence, we can conclude the following for $m \geq 3$:

Corollary 3.16. *For $m \geq 3$, the hypergraph $\mathcal{H}(V, \mathcal{R}_{BL} \cup \mathcal{R}_{RO}, m)$ captured by the union of hanging bottomless and hanging right-open rectangles is proper 4-colorable.*

Proof. For $m \geq 7$, we already know from Corollary 3.8 that $\mathcal{H}$ is proper 3-colorable. Due to Lemma 2.1, $\mathcal{H}$ then also admits a proper 4-coloring.

For $3 \leq m \leq 6$, we create a proper 4-coloring of $\mathcal{H}$ as follows: Let $\mathcal{E}_{BL}$ and $\mathcal{E}_{RO}$ be the hyperedges described by bottomless rectangles and right-open rectangles, respectively. We color the points of these sets of hyperedges separately with the colorings $c_{BL} : V \to \{c_1, c_2\}$ and $c_{RO} : V \to \{c_1, c_2\}$ such that the underlying hypergraphs are proper 2-colorable. This is possible due to Lemma 3.13 and Lemma 3.15.

For the hypergraph $\mathcal{H}$ containing all the hyperedges $\mathcal{E} = \mathcal{E}_{BL} \cup \mathcal{E}_{RO}$, we then define a proper 4-coloring $c : V \to \{1, 2, 3, 4\}$ as follows: For any $v \in V$,

$$c(v) = \begin{cases} 1, & \text{if } c_{BL}(v) = c_1 \text{ and } c_{RO}(v) = c_1 \\ 2, & \text{if } c_{BL}(v) = c_1 \text{ and } c_{RO}(v) = c_2 \\ 3, & \text{if } c_{BL}(v) = c_2 \text{ and } c_{RO}(v) = c_1 \\ 4, & \text{if } c_{BL}(v) = c_2 \text{ and } c_{RO}(v) = c_2 \end{cases} .$$

Let $e \in \mathcal{E}$ be an arbitrary hyperedge of $\mathcal{H}$, say a bottomless rectangle. As none of the edges are monochromatic, $|c_{BL}(e)| = 2$ and $c_1, c_2 \in c_{BL}(e)$. No matter how the three points of e are colored in c_{RO}, at least two of the points, p and q, with different colors in c_{BL} also have different colors in c, as if $c_{BL}(p) \neq c_{BL}(q)$ then $c(p) \neq c(q)$ due to the definition of c. Hence, all hyperedges still contain at least two different colors and thus $\mathcal{H}$ is proper 4-colorable. □

For $3 \leq m \leq 6$, $k = 4$ colors is the lowest number of colors for which we know, that an m-uniform hypergraph captured by hanging bottomless and hanging right-open rectangles is proper k-colorable. We do not know, if these hypergraphs can be properly 3-colored or even properly 2-colored.

We observe that for $m = 2$, we can embed the graph K_4 as a 2-uniform hypergraph captured by the union of hanging bottomless and hanging right-open rectangles. See therefore Figure 3.5. Thus, as we need at least four colors to color K_4, there exists a 2-uniform hypergraph captured by the union of hanging bottomless and hanging right-open rectangles that is not proper 3-colorable. It remains open, if we can proper 4-color any hypergraph captured by the union of those range families for $m = 2$. Due to this observation, following Theorem holds:

Theorem 3.17. *For $m = 2$, there exits a hypergraph captured by the union of the range families $\mathcal{R}_{BL}$ and $\mathcal{R}_{RO}$ of hanging bottomless and hanging right-open rectangles, that is not proper 3-colorable.*

In the next two chapters, we will take a further look at the union of those two range families. We will develop a semi-online presenting strategy for the points of the hypergraph, such that no matter how the painter colors the presented points along the way with two colors, there will always arise a monochromatic hyperedge. This strategy is then transferred into an offline strategy for *non-hanging* bottomless rectangles and hanging right-open rectangles in Section 5.1.1 for $m \geq 3$ and $k = 2$. We also develop a semi-online strategy for $k > 2$ colors.

4. Union of Hanging Bottomless and Hanging Right-Open Rectangles

We now take a look at the union of two range families: The range family of hanging bottomless rectangles $\mathcal{R}_{BL}$ and the range family of hanging right-open rectangles $\mathcal{R}_{RO}$. Recall therefore their definitions:

- $\mathcal{R}_{BL} = \{\{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, y \leq a\} \mid a, b \in \mathbb{R}\}$ and
- $\mathcal{R}_{RO} = \{\{(x, y) \in \mathbb{R}^2 \mid a \leq x, b \leq y \leq a\} \mid a, b \in \mathbb{R}\}$.

Hypergraphs captured by $\mathcal{R}_{BL}$ or $\mathcal{R}_{RO}$ form a subset of a hypergraph over the same set of points captured by the range family of hanging axis-aligned rectangles R_R (see Lemma 3.1). Hence, we can obtain lower bounds of the colorability of hanging rectangles from studying $\mathcal{R}_{BL}$ and $\mathcal{R}_{RO}$. If we need k colors to properly k-color an m-uniform hypergraph captured by $\mathcal{R}_{BL}$ or $\mathcal{R}_{RO}$, then we also need at least k colors to properly color the m-uniform hypergraph over the same set of points captured by $\mathcal{R}_R$.

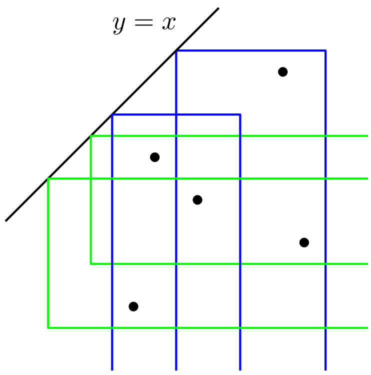


Figure 4.1: Example for the union of hypergraphs captured by hanging bottomless rectangles (blue) and hanging right-open rectangles (green) over the same set of points. If we consider the open sides of the hyperedges to be closed at a position below or right of all points of the hypergraph, those edges are part of a hypergraph captured by hanging axis-aligned rectangles.

Figure 4.1 shows an example of the union of two hypergraphs over the same set of points, once captured by $\mathcal{R}_{BL}$ and once by $\mathcal{R}_{RO}$. We already know from Lemma 3.13 and Lemma

3.15, that for a uniformity $m \geq 3$, hypergraphs captured by $\mathcal{R}_{BL}$ or $\mathcal{R}_{RO}$ separately are proper 2-colorable. Corollary 3.8 additionally says that for $m \geq 7$ and $k \geq 3$, any hypergraph $\mathcal{H}(V, \mathcal{R}_{BL} \cup \mathcal{R}_{RO}, m)$ is proper k-colorable. For $m \geq 3$, $\mathcal{H}(V, \mathcal{R}_{BL} \cup \mathcal{R}_{RO}, m)$ is proper 4-colorable, due to Corollary 3.16.

In the following, we present three different semi-online presenter-painter games for an m-uniform hypergraph $\mathcal{H}$ over a common point set V: one for hanging bottomless rectangles, one for hanging right-open rectangles and one that combines the two aforementioned games. A presenter-painter game is a game with two players, the presenter and the painter. The presenter knows the point set V of the hypergraph and presents the points $v \in V$ according to a defined strategy to the painter. The goal for the presenter is to force the painter to color the presented points so that a monochromatic hyperedge is created. The painter only knows the points that the presenter already revealed. His goal is to assure that in every hyperedge, there are at least two points with different colors. Therefore, he either colors a point or does not make a move if no coloring of a point is necessary to assure the desired property. Since the painter does not know all the points of the hypergraph from the beginning and only needs to color if otherwise the desired property no longer applies, the games are semi-online.

We are especially interested in small uniformities $m \in \{2, \ldots, 6\}$, as we already know that hypergraphs captured by the union of hanging bottomless and hanging right-open rectangles are proper k-colorable for $k \geq 3$ if $m \geq 7$. For the first two games, we present a strategy for the painter that works against any presenting strategy and thus, for $m \geq 3$, hanging bottomless and hanging right-open rectangles are proper 2-colorable, respectively. In the game for the union of the two range families, we propose a semi-online strategy for the presenter such that no matter how the painter colors the presented points with two colors, we always result in a monochromatic hyperedge. This leads to the results that for $m \geq 2$ and with our proposed semi-online presenting-strategy, the painter cannot properly 2-color the underlying hypergraph. The strategy for two colors is then transferred into an offline strategy in Section 5.1.1. Thereby, we lose the hanging property of the bottomless rectangles, which then gives us a lower bound for the proper colorability of the union of bottomless rectangles and hanging right-open rectangles for $m \geq 2$. We also transfer the semi-online strategy for hanging bottomless and hanging right-open rectangles for two colors to a semi-online strategy for hanging right-open rectangles and hanging topless rectangles for any number of colors k in Chapter 5.

In the following we first present the game for hanging bottomless rectangles, then for hanging right-open rectangles and after that the game for the union of the two range families. The painting strategy for the game for hanging bottomless rectangles is similar to the algorithm in Section 3.1.

4.1 Game for Hanging Bottomless Rectangles

The presenter-painter game for hanging bottomless rectangles works as follows: We only consider the x-coordinates of the points and map them to the same y-coordinate. Figure 4.2 on the left shows an exemplary graph and the line of points at a point in time of the game. The presenter can insert a new point at any position on that line or make the leftmost point disappear. Inserting a new point corresponds to the event that a new point gets *active* while processing the next edge. Thus, a point is called active, if it is part of a hyperedge, that has already been or is currently processed. In contrast, points that are not discovered yet are called *inactive*. This point then appears at the position defined by its x-coordinate. When we process the next hyperedge, all points v having an x-coordinate x_v lower than the x-coordinate of the upper left corner x_e of the currently processed hyperedge e, i.e. all v for which it holds that $x_v < x_e$, can no longer be part

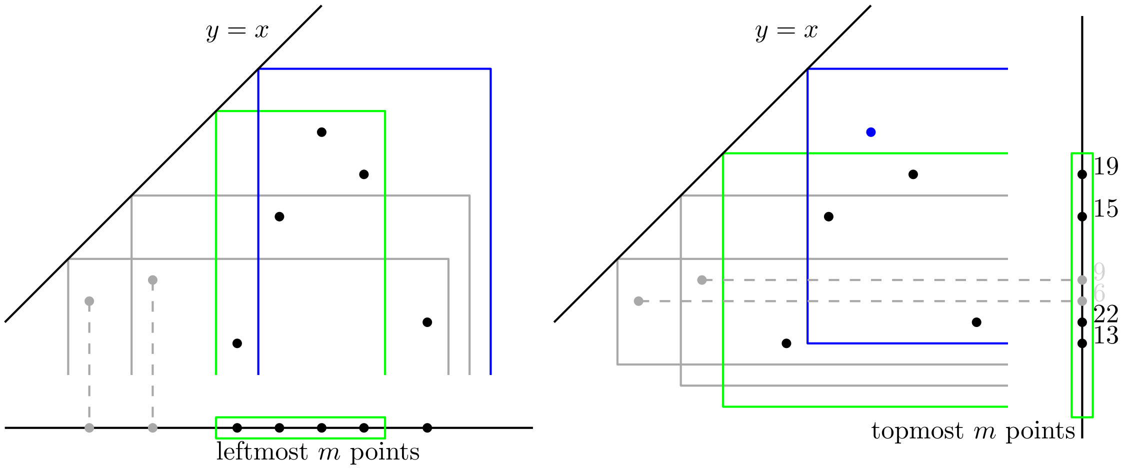


Figure 4.2: Illustration of a 4-uniform hypergraph captured by $\mathcal{R}_{BL}$ on the left and by $\mathcal{R}_{RO}$ on the right. It is shown how the points are mapped to a line for the game. Gray edges are already processed, the green edge is the currently processed edge and the blue edge is going to be processed after the current edge. The gray points already disappeared, as their x-coordinate is smaller than that of the current edge and therefore they cannot be part of an upcoming edge. The leftmost/topmost four points that have not disappeared yet represent the current edge and therefore have to be colored with at least two colors. The numbers represent the x-coordinates of the points. The blue point on the right was not yet part of any processed hyperedge.

of any upcoming hyperedge. Therefore the presenter can remove them. The painter has to ensure that among the leftmost m points are at least two different colored points. He has two colors c_1 and c_2 to achieve this goal. We define that c_1 is the smaller color. The leftmost m points are exactly the points that form the next hyperedge and therefore they must have at least two different colors.

The strategy of the painter works as follows: If among the leftmost m points there is at most one color, then color the leftmost uncolored point with the smallest color available.

Lemma 4.1. *For $m \geq 3$, there can never be two consecutive points colored with the same color among the leftmost m points.*

Proof. According to his strategy, the painter never colors an additional point of the leftmost m points with a color that is already represented in those points. Additionally, only the leftmost point can disappear. Thus two consecutive points with the same color can only occur if the painter colors the m-th point with color c and the point at the $(m+1)$-th position from the left was already colored with color c beforehand. Then, if the leftmost point disappears and the $(m+1)$-th point again becomes part of the leftmost m points, there are two consecutive points with color c within those m points.

Assume that this is the first time in the game, that two consecutive points of the same color occur within the leftmost m points. The painter then has colored the m-th point, as the other $m-1$ points already had colors before. As the painter only starts coloring if there is at most one color presented in the leftmost m points, the m-th point, before coloring, is the leftmost uncolored point and all other points have the same color. For $m \geq 3$ this is a contradiction to the assumption, that the current situation is the first time that two consecutive points among the leftmost m points are colored with the same color.

Thus, there can never be two consecutive points colored with the same color among the leftmost $m \geq 3$ points. □

Note that for $m = 2$, the coloring strategy of the painter easily leads to a monochromatic rectangle, if the presenter presents the points p_1, p_2 and p_3 with $x_{p_1} < x_{p_2} < x_{p_3}$ as follows:

- The presenter presents the points p_1 and p_3 and the painter colors p_1 with color c_1 and p_3 with color c_2 on the way. The currently considered hyperedge is therefore polychromatic.
- Now, the presenter presents p_2, which is situated between the other two points. The painter needs to colors p_2 with color c_2, as otherwise the hyperedge consisting of $\{p_1, p_2\}$ does not have two points with different colors.
- After that, the presenter removes p_1. The arising hyperedge $\{p_2, p_3\}$ is monochromatic with color c_2.

Due to Lemma 4.1, we observe that for $m \geq 3$, there are always two consecutive points in the leftmost m points that have different colors or they can be colored such that afterwards, they have two different colors. Hence, the painter wins the game if he sticks to his strategy. We can also derive the following:

Corollary 4.2. *If the painter decides to color a point, he always colors the leftmost or second most left point.*

Proof. If the painter needs to color a point further to the right than the leftmost two, all the points left of the currently colored node have the same color which is impossible according to Lemma 4.1. So in situations where there are less than two colors represented in the leftmost m points, at least the leftmost or the second most left point is still uncolored. This point or these points can then be colored with the color that is not already presented or with one color each. Hence, there is no need to color a point at another position. □

The strategy of the painter is similar to the algorithm described in Section 3.1. Points already presented by the presenter correspond to the active points of the algorithm. Additionally, the painter colors as well from the left to the right, and only if necessary. This strategy is only presented for completeness as we obtain the same results as Lemma 3.13, i.e., any hypergraph captured by hanging bottomless rectangles admits a proper 2-coloring for $m \geq 3$. We now present the game for hanging right-open rectangles, which we will use in the game for the combination of the two range familes.

4.2 Game for Hanging Right-Open Rectangles

We now look at the presenter-painter game for hanging right-open rectangles. This time, the points of the point set V are considered as having the same x-coordinate. Figure 4.2 on the right shows an exemplary situation that may occur during the game. In this game, the presenter presents new points only on the top of the line, in contrast to the last game where points could appear anywhere. Further, each point is assigned a number which indicates the point in time at which they disappear. Those numbers are unique and the points can be ordered by them. They correspond to the x-coordinates of the respective points, as these determine the point of time when the points disappear (see Figure 4.2 on the right). The presenter can remove the point with the lowest number. The painter can again color points with colors c_1 and c_2. He needs to ensure that in the top m points there are at least two different colors present. The reason for the appearance of points on the top is, that later presented points always lie above the upper side of the current edge e, otherwise they would be part of the current or a former right-open rectangle. They therefore have a greater y-coordinate than e and remain inactive until they are part of a hanging right-open rectangle for the first time. The next edge e' that contains those points then has a higher upper side than the y-coordinates of the points. As the points appear at the position of their y-coordinate on the line, new points always appear on top of the line. The number assigned to each point corresponds to their x-coordinate in the hypergraph. A point can no longer be part of any higher ordered hyperedge, if its x-coordinate is smaller than the left side of every upcoming hyperedge, as we consider the edges from the left to the right. The coordinates of the left side of a hyperedge have the same x-values as the attachment point. Thus, a point whose x-coordinate is smaller than the x-value of the attachment point of the currently considered hyperedge is removed by the presenter, independent of its y-coordinate.

As the points with the smaller numbers disappear first, we think that it is a good idea to color them first too. Thus, the painter colors the point with the smallest number whenever there are less than two different colors presented in the topmost m points. We figure out that it is very easy for the presenter to force two consecutive points with the same color. See Figure 4.3 for an example with uniformity $m = 3$.

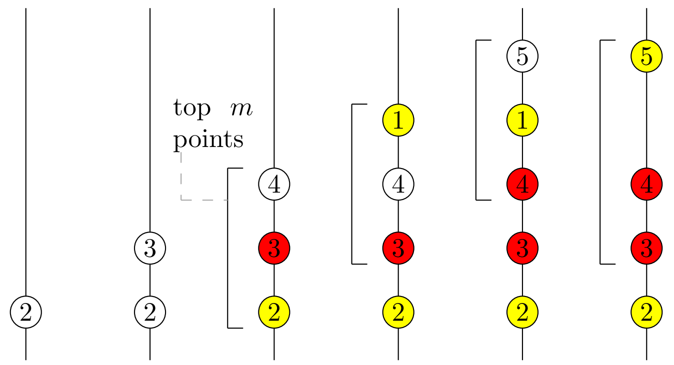


Figure 4.3: From the left to the right: Consecutive situations in the presenter-painter game for hanging right-open rectangles. The presenter first adds five points, then removes the one with the smallest number. The painter paints with the color yellow first and red second. The brackets always indicate the top three points for which the painter must ensure that two different colors are present.

We observe that two consecutive points with the same color can occur when the points that are originally situated between them disappear. Alternatively, it also happens if the lowest of the top m points is colored with the same color as the $(m+1)$-th point and then one point of the top $m-1$ disappears.

Next, we examine whether it is possible to have three consecutive points with the same color in the topmost m points. We first adjust the painter's strategy, as three consecutive, same colored points can easily occur when points disappear between them: Every time the painter colors a point v with color c, he also determines another uncolored point w next to it as a barrier for v. The barrier of v is never assigned the same color c, as long as v is active. It is important, that w has a higher number than v. For the two colors c_1 and c_2 we use the mappings $b_{c_1}, b_{c_2} : V \to V$ to describe the barrier of a point $v_1 \in V$ colored with color c_1 as $b_{c_1}(v_1) \in V$ and for a point $v_2 \in V$ colored with color c_2 as $b_{c_2}(v_2) \in V$. The idea of the barrier $b_{c_i}(v)$ with $c_i \in \{c_1, c_2\}$ is, that as it has a higher number assigned to it, it is removed later than v and therefore protects v on one side from a point v' colored with the same color as v. Hence, v and v' can never be consecutive points with the same color.

If v is to be colored with color c_i, the barrier point $b_{c_i}(v)$ must fulfill the following conditions, otherwise the painter cannot color v:

1. It has to be uncolored or colored with $c_j \neq c_i$ in the moment of assignment.
2. It must not already be a barrier $b_{c_i}(w)$ of another point w with the same color c_i.
3. The barrier $b_{c_i}(v)$ is either one of the direct neighbors of v, or all points between v and $b_{c_i}(v)$ are colored with $c_j \neq c_i$.
4. In the moment of assignment, it must lie within the topmost m points.
5. It must have assigned a higher number than the colored point v, i.e., it is removed later from the game than v.

Note that for the two colors used, the barriers of the colored points are assigned without considering the barriers of the other color. It is therefore possible, that a point is a barrier for both colors or that a point is a barrier for color c_1 and is colored with color c_2. If two points in both sides of v fulfill all these conditions, the point with the lower number is set as the barrier $b_{c_i}(v)$. The painter is not allowed to color $b_{c_i}(v)$ in future steps with color c_i, unless v is removed by the presenter. In that moment, $b_{c_i}(v)$ is no longer a barrier and can be colored or again be assigned as barrier to another colored point. Every time an additional color, say c_2, is needed, the painter colors the uncolored point v with the smallest assigned number in that color if it is not a barrier for a point w with the same color c_2, i.e. $v \neq b_{c_2}(w)$. The point v is only colored if the painter can find an uncolored or c_1-colored barrier for it. Otherwise, the painter tries to color the uncolored point with the next higher number. We later show in Lemma 4.5 that for $m \geq 3$, the painter can always find an uncolored point and a corresponding barrier as long as less than two colors are present in the topmost m points.

Observation 4.3. *Between a point colored with color c and its barrier is no other point colored with color c.*

This directly follows from the third barrier condition. If this is not the case, the barrier does not protect the point in the direction of its barrier against other points in the same color. This then leads to several consecutive points with the same color.

Lemma 4.4. *For any $m \geq 3$, there can never be three consecutive points colored with the same color among the topmost m points.*

Proof. Assume that there are three consecutive points u, v and w with the same color c in the topmost m points. Let their y-coordinates be ascending. i.e., $y_u < y_v < y_w$. Due to Observation 4.3, we know that the barrier of v cannot be above w or below u. Hence, $b_c(v)$ must be between u and v or v and w. This contradicts our assumption that u, v and w are consecutive, as $b_c(v) \neq u$ and $b_c(v) \neq w$ due to $c(b_c(v)) \neq c$. □

Lemma 4.5. *For $m \geq 3$ and $k = 2$ colors, it is always possible for the painter to guarantee that there are points with at least two different colors in the topmost m points. Furthermore, all colored points have a barrier assigned to them.*

Proof. Let $v_1, v_2, \ldots, v_m \in V$ be the topmost m points at a point in time of the presenter-painter game for hanging right-open rectangles. In the following we use the notation $v_i < v_j$ for any points v_i and v_j to denote that v_i has a smaller assigned number than v_j. Within the topmost m points, either none of the points is colored or at least one point is already colored. These points then each have a barrier point assigned to them as well.

Assume that all points $v_1, v_2, \ldots, v_m$ are still uncolored. The painter then colors the point v_i with the smallest number with color c_1 and assigns one of the points next to v_i as barrier $b_{c_1}(v_i)$ to it. If $v_i \in \{v_2, \ldots, v_{m-1}\}$, both of the neighbors of v_i fulfill the conditions for a barrier listed above, as they are uncolored, not yet a barrier for a point with color c_1, direct neighbors of v_i, have a higher assigned number than v_i as v_i has the smallest number among all points $v_1, v_2, \ldots, v_m$ (and all points have different numbers), and lie within the topmost m points. If $v_i = v_1$ then only v_2, fulfills the conditions. Analogously, if $v_i = v_m$, only v_{m-1} fulfills the conditions. Therefore it is always possible to color v_i and assign a point $b_{c_1}(v_i)$ as its barrier. To ensure two different colors, the painter needs to color a second point $v_j \in \{v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_m\}$ with color c_2. If $m = 3$, the number of uncolored points is the fewest out of all $m \geq 3$. The following argumentation for $m = 3$ will work similar for any $m > 3$: As we colored one point with color c_1, there are only two (or more, if $m > 3$) uncolored points v_j and v_k left. Let v_j be the point with the smaller number. Hence, the painter tries to color it first. Therefore v_k must fulfill the conditions of a barrier. There are two different scenarios now: As $b_{c_1}(v_i)$ is a direct neighbor of v_i, it is either the point in the middle if v_i is the first or third point, or, if v_i is the point in the middle, $b_{c_1}(v_i)$ is the first or third point.

1. Let $b_{c_1}(v_i)$ be the point in the middle. Then, the two uncolored points v_j and v_k are situated next to each other, as $v_i = v_1$ or $v_i = v_m = v_3$. The point v_k is then uncolored, a direct neighbor of v_j, lies within the topmost m points and has a higher number than v_j. Therefore the painter is able to color v_j with color c_2 and assign v_k as the barrier $b_{c_2}(v_j)$ of v_j. Thus, v_i and v_j lie among the topmost m points, are colored with two different colors and both have a barrier point.

2. Let now $b_{c_1}(v_i)$ be the first or third point and v_i is the point in the middle. It holds that $v_i < b_{c_1}(v_i) = v_j < v_k$ as when both neighbors of v_i fulfill the conditions for a barrier, the one with the smaller number is picked. Hence, v_k is again uncolored, not yet a barrier of a point with the same color c_2, lies in the topmost m points and has a higher number than v_j. It is not a direct neighbor of v_j, but all points between v_j and v_k are colored with color c_1 whereas v_j is to be colored with the color c_2. Therefore v_k fulfills all conditions and is assigned as barrier $b_{c_2}(v_j)$ to v_j.

Assume some of the points among the topmost m points are already colored with color c_1. The painter only needs to color one other point with the second color c_2. Due to Lemma 4.4, we know that for $m \geq 3$ there can never be three consecutive points with the same color among the topmost m points. Hence, for $m = 3$ not all points are colored and thus at least one uncolored point exists in the topmost m points. If $m > 3$, no three consecutive points can have color c_1 and therefore there again exists at least one uncolored point. We then color the uncolored point with color c_2. As no other c_2-colored point exists in the topmost m points, we can assign one of the uncolored or c_1-colored points as its barrier.

If there are already two colors present within the topmost $m \geq 3$ points, the painter does not need to paint another point. Note that as the points appear on the top, it is possible that either a barrier of a point or a point itself no longer lies in the topmost m points. If the barrier no longer lies there, both colors are still represented. If instead a point colored with color c_1 is pushed out of the topmost m points, we need a new point in that color. We do not free the barrier of the disappeared point, but we can use the barrier of the point with color c_2, which is uncolored, and the newly presented uncolored point as new point-barrier pair for the color c_1.

To summarize, for $m \geq 3$ the painter can always color points such that there are two different colored points within the topmost m points that all have a barrier assigned to them as well. $\square$

We saw that for $m \geq 3$, there can never be three consecutive points with the same color among the topmost m points in Lemma 4.4. Additionally, it holds that the painter can always guarantee two points with different colors in the topmost $m \geq 3$ points when coloring with $k = 2$ colors (see Lemma 4.5). Thus, as already shown in Lemma 3.15, we can find a proper 2-coloring for any hypergraph captured by hanging right-open rectangles and presented by the presenter, if we use the painting strategy using barrier points for the painter. We will now use this game as basis for the presenter-painter game for the combination of hanging bottomless and hanging right-open rectangles.

4.3 Combination of the two Games

We now want to combine the presenter-painter game for hanging bottomless rectangles and the game for hanging right-open rectangles defined in Section 4.1 and Section 4.2, respectively. Informally speaking, our goal is to study the combination of both games: We would like to know whether any hypergraph $\mathcal{H}$ captured by hanging bottomless and hanging right-open rectangles is semi-online proper 2-colorable. In Chapter 5, we extend this question to semi-online proper k-colorability.

We start with considering the presenter-painter game for hanging right-open rectangles. There, we already know that the number each point gets when it appears corresponds to its x-coordinate. Thus, the presenter always removes the points, whose x-coordinates are too small to be part of the upcoming edges. In the presenter-painter game of the previous section, we only considered hanging right-open rectangles. Now we additionally consider hanging bottomless rectangles. Therefore, we define subsets of the points where each m consecutive points induce a hanging bottomless rectangle.

Definition 4.6. *Let V be the points of the hypergraph. We order them by ascending y-coordinates. Let $V_t \subseteq V$ consist of the lowest t points $p_1, \dots, p_t$, i.e., all other points $q \notin V_t$ of the hypergraph have higher y-coordinates than p_t. As the points are ordered by ascending y-coordinates, it holds that $y_{p_1} < \dots < y_{p_t}$.*

In the hypergraph $\mathcal{H}(V_t, \mathcal{R}_{BL} \cup \mathcal{R}_{RO}, m)$, every m consecutive points in x-direction form a hanging bottomless rectangle. The x-coordinates of those points correspond to their assigned numbers. Thus, these points must use at least two different colors as otherwise, there would be a monochromatic hanging bottomless rectangle.

If we present the points in order of ascending y-coordinates, it is possible that later in the game, in a subset $V_{>t} \subseteq V$, the presenter presents a point p that lies (in x-direction) between m consecutive points $p_1, \dots, p_m$ of V_t. Hence, these m points in V_t would not form a hanging rectangle after all. But as it is also possible that no point p is presented later in the game between $p_1, \dots, p_m$, the painter must guarantee that any presented points with m consecutive numbers are not monochromatic. Only in this way, he can avoid monochromatic hanging bottomless rectangles.

Theorem 4.7. *For $m \geq 2$, there exists a semi-online strategy for the presenter, such that the corresponding m-uniform hypergraph captured by the union of hanging bottomless and hanging right-open rectangles is not semi-online proper 2-colorable.*

Proof. We only concentrate on one fixed color c and look at a presenter-painter game that handles both hypergraphs simultaneously. We show that the presenter can force the painter to color points in such a way that in the end of the game, there are points with m consecutive numbers that all have color c and that also can hang from the first angle bisector. Thus, we then have a monochromatic hanging bottomless rectangle and the hypergraph is not proper 2-colorable.

We take advantage of the fact that to properly 2-color the hypergraph, the painter must use our arbitrarily chosen color c in every hyperedge as he has only two colors available.

We play the presenter-painter game for hanging right-open rectangles whereby any m points with m consecutive numbers potentially form a hanging bottomless rectangle. Note that the presenter never removes a presented point. Thus, it cannot happen that we get two consecutive points with the same color due to the removal of a point among the topmost m points. Hence, this proof works for $m \geq 2$ as it is possible to ensure that $m \geq 2$ points have different colors and the presenter always presents exactly one point and the painter reacts directly after that. For $m = 2$, the painter then only has the possibility to color the points alternating with the two colors and thereby always knows the next color necessary, as the other color is still represented in the topmost two points. The construction described in the following is shown in Figure 4.4. Also note that we did not project the points on a vertical line but instead we present the points with their x-coordinates. For simplification, the points here are placed on the exact same x-coordinates, instead of including a shift of a small $\varepsilon > 0$.

In general, we provide a strategy for the presenter such that no matter how the painter colors the points, at some point there necessarily occurs a monochromatic hyperedge. But in this proof, we assume that whenever the topmost m points contain no point of color c, the painter colors one of them with color c. This is necessary to assure that there is always one colored point within the topmost m points. Therefore, we can assume that whenever the presenter adds a point in such a way that the point which was colored last is no longer part of the topmost m points, the painter colors exactly one other point. Any strategy of the painter which does not ensure this property will lead to a monochromatic hanging right-open rectangle immediately, as we color all points that remained uncolored with the same color in the end. We now describe the winning strategy of the presenter.

First, the presenter presents a set V_1 of m^m points $p_{1,1}, \ldots, p_{1,m^m}$ with descending x-coordinates $x_{p_{1,1}}, \ldots, x_{p_{1,m^m}}$, i.e.,

$$x_{p_{1,1}} > \cdots > x_{p_{1,m^m}}$$

starting with $p_{1,1}$. The painter colors some of the points according to his strategy along the way. We call these points in V_1 presented by the presenter with ascending y-coordinates a *sequence*. The first sequence forces the painter to color at least m^{m-1} points with color c, as he must always color a point among any m uncolored vertically consecutive points. After that, the presenter presents a second sequence V_2 with at least m^{m-1} new points, one for every point colored with color c in V_1, as follows: We refer to the points colored with color c in V_1 as the point set $V_1^c \subseteq V_1$. The x-coordinate of every point in V_2 is “almost the same” as the x-coordinate of one of the points in V_1^c, respectively. Namely, we choose a sufficiently small $\varepsilon > 0$ such that any point $p_{2,j} \in V_2$ is ε to the right of one point $p_{1,i} \in V_1^c$, i.e., $x_{p_{2,j}} = x_{p_{1,i}} + \varepsilon$, while $x_{p_{2,j}} \ll x_{p_{1,i-1}}$.

Observe that for the points of V_2, it holds again that $x_{p_{2,1}} > \cdots > x_{p_{2,|V_2|}}$ and the presenter starts with presenting $p_{2,1}$. The painter now has to color at least m^{m-2} points from V_2. Afterwards, the presenter again presents a point p_3 for each point $p_2 \in V_2^c$ such that the x-coordinate of p_3 is by a summand ε higher than the x-coordinate of p_2. This is repeated until the painter colors at least m points in the sequence V_{m-1}, and after that at least m new points are added by the presenter as a sequence V_m. By an analogous argument as before, the painter then colors at least one of the points in V_m.

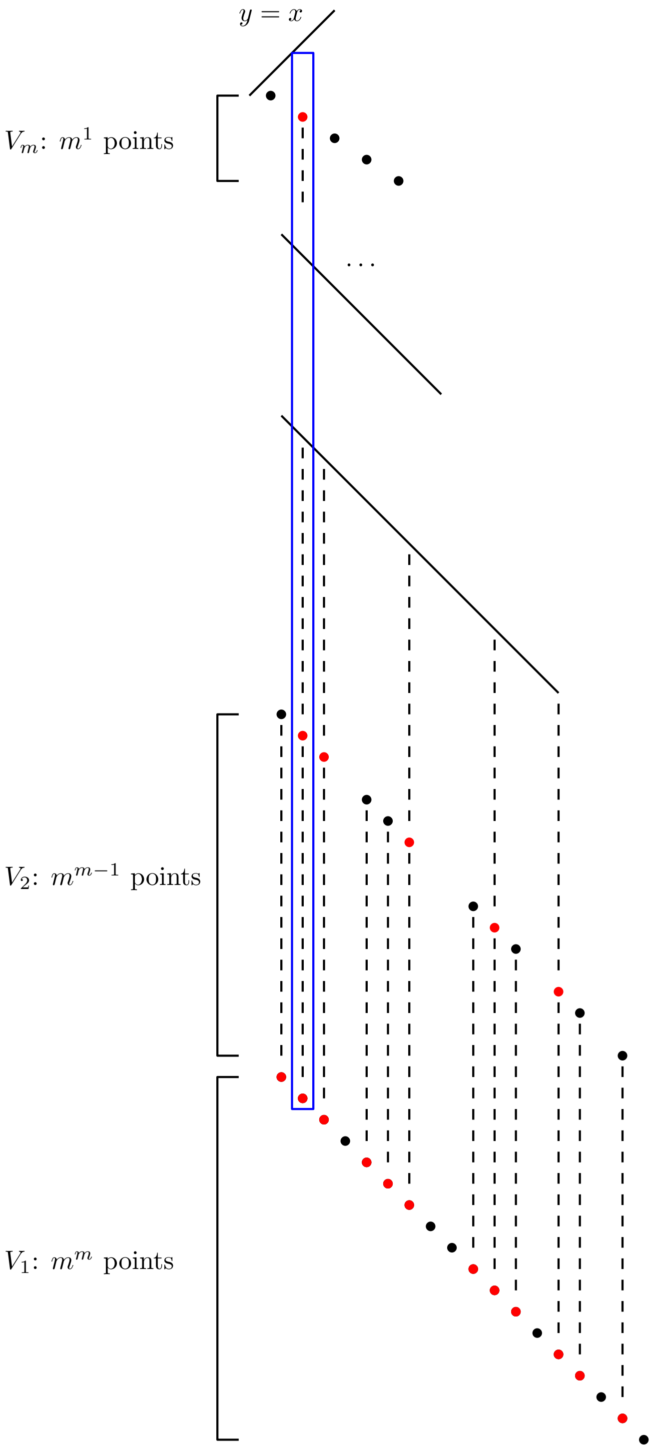


Figure 4.4: Construction for the proof of Theorem 4.7. New points are presented on the top, thus the lowest point was presented as first. Black points indicate uncolored points whereas red points indicate points colored with color c. New sequences of points with ascending y- but descending x-coordinates consist of points having "almost the same" x-coordinates than the colored points in the previous sequence but with a slight shift to the right. The blue box indicates m points colored with color c and having m consecutive x-coordinates.

Note that the painter needs to color points with both colors every topmost m points. However the presenting-strategy only reacts to one color, say c.

Now we prove that after presenting the points of $V_1, \dots, V_m$, there necessarily exists a monochromatic hyperedge: Recall that the sequence V_m contains at least one colored point, say $p_{m,i}$. By the construction of V_m, we then have a colored point $p_{m-1,j}$ with x-coordinate $x_{p_{m-1,j}} = x_{p_{m,i}} - \varepsilon$ in V_{m-1}. Similarly, we have a colored point $p_{i,k}$ with $x_{p_{i,k}} = x_{p_{i+1,l}} - \varepsilon$ in every former sequence $V_i \in \{V_{m-2}, \dots, V_1\}$. This means, starting from a colored point $p_{m,i} \in V_m^c$ as the rightmost point, that there are exactly m points with consecutive x-coordinates within the already presented numbers, one point in each of the m sequences. Therefore there is a monochromatic hanging bottomless rectangle r within the interval $i = \left[x_{p_{1,k}}, x_{p_{m,i}}\right]$, whereby $|x_{p_{m,i}} - x_{p_{1,k}}| = (m-1) \cdot \varepsilon$ and $x_{p_{m,i}} < x_{p_{1,k-1}}$.

To summarize, this means that the union of hanging bottomless rectangles and hanging right-open rectangles is not semi-online proper 2-colorable when using this presenting-strategy. $\square$

As illustrated in Figure 4.4, the total number of points needed for this strategy is at least

$$m^m + m^{m-1} + \cdots + m^1 = \sum_{i=1}^{m} m^i = \frac{m(m^m - 1)}{m-1} .$$

5. Not Semi-online Proper k-colorable

In the previous chapter, we developed a strategy for the presenter in a presenter-painter game for hanging bottomless and hanging right-open rectangles such that no matter how the painter colors the presented points with two colors, there is always a monochromatic hyperedge. The strategy of the presenter works for any uniformity $m \geq 2$ and $k = 2$ colors. The idea of this chapter is to generalize the strategy of the presenter to any number of colors k. Thereby, we consider hanging right-open and hanging (axis-aligned) topless rectangles. Note that we previously considered hanging bottomless instead of hanging topless rectangles. Moreover, we try to transfer the semi-online strategy of the previous chapter and the semi-online strategy considered in the following into offline strategies.

Figure 5.1 shows on the left an example of a hypergraph captured by topless rectangles and on the right an example of a hypergraph captured by *hanging* topless rectangles. The range family of axis-aligned topless rectangles $\mathcal{T}$ is formally defined as

$$\mathcal{T} = \{\{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, y \geq c\} \mid a, b, c \in \mathbb{R}\} .$$

For *hanging* topless rectangles, their left and right side of the hyperedge pierce the first angle bisector. It also requires that the lower side lies below the first angle bisector. Therefore, the range family of hanging topless rectangles is defined as

$$\mathcal{R}_{TL} = \{\{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, y \geq c\} \mid a, b, c \in \mathbb{R}, c < a\} .$$

We want to find for $x, a, k \in \mathbb{N}$ a semi-online presenting-strategy $S(x, a, k)$ with the property, that for $x \geq 1, m \geq a \geq 1$ and $k \geq 2$, the following holds: If the painter semi-online properly k-colors m consecutive presented points in the presenter-painter game, $S(x, a, k)$ enforces the painter to create a set of x hanging topless disjoint rectangles such that each of them contains m points and these m points are colored with the same color. Then for $x = 1$, $S(1, m, k)$ would be the semi-online winning strategy for the presenter that enforces a monochromatic hanging topless rectangles of size m while the painter colors with k colors.

We construct the strategy $S(x, a, k)$ recursively, first by the number of colors k and then by the number a of points contained in the hanging topless rectangles. Note that the monochromatic hanging topless rectangles of the recursive strategies may not all remain monochromatic due to new points of the current strategy. Therefore, the idea is, informally speaking, to enforce a sufficiently large number of monochromatic hanging topless rectangles of size $a - 1$ to ensure that when all of them grow to size a, enough of them remain monochromatic.

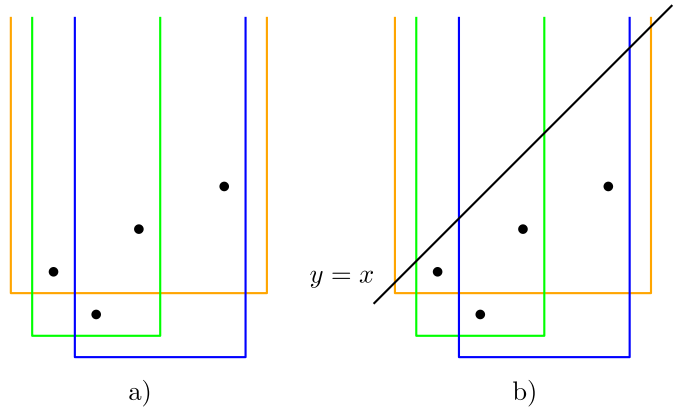


Figure 5.1: Example of hypergraphs captured by a) topless rectangles and b) hanging topless rectangles.

With $n(x, a, k)$ we denote some fixed number at least as large as the number of points used in the strategy $S(x, a, k)$. The number of points for a strategy $S(x, a, k)$ is not fixed, as the strategy is semi-online and reacts directly to the coloring of the painter. For example, it makes a huge difference if the painter most frequently uses the color the presenting-strategy is focussing on or if he mainly uses all other colors. In the first case, the presenter needs to present less points than in the second case. The presenter always stops presenting points, if no more points are necessary.

The strategy $S(x, 1, k)$ for all $x, k \in \mathbb{N}$ is defined as sequence of points with ascending y-, but descending x-coordinates. We call this sequence V_1 whereby the lower bound of $n(x, 1, k)$ is dependent on x, k and m. As the painter properly k-colors every m consecutive points of V_1, we need so many points that across the whole sequence of points, he uses one of the k colors, say c_1, at least x times. The painter has to color among every m consecutive points at least two points with different colors to not directly lose the game. Thus, for $k \cdot m$ points, at least $2k$ points are colored. Consequently, we say that for

$$n(x, 1, k) \geq k \cdot m \cdot \left\lceil \frac{x}{2} \right\rceil$$

points, at least $2k \cdot \lceil x/2 \rceil$ of them are colored and hence one color is represented at least x times. This implies that there are at least x hanging topless rectangles of size one with the same color, e.g. c_1. With $V_1^{c_1}$ we denote the set of points colored with color c_1.

The other base case for our strategy is the strategy $S(x, a, 2)$ for two colors. If $x = 1$ and $a = m$, the strategy corresponds to the strategy of Theorem 4.7, i.e., the proof that hypergraphs captured by $\mathcal{R}_{BL}$ and $\mathcal{R}_{RO}$ are not semi-online proper 2-colorable. Note that if the presenter stops presenting points in the moment where the monochromatic hanging bottomless rectangle r is enforced, we can find a hanging topless rectangle containing the exact same points as r, hence a monochromatic hanging topless rectangle. Between the left and the right side of r, there is no point above the upper side of r. That is because the point p, colored last in the topmost $a = m$ points, and all upcoming points have x-coordinates lower than the left side of r. The reason is that we present new points q_j in new sequences that correspond to a point p_i in the previous sequence, identical to the strategy of the previous chapter, always to the right of p_i, but far to the left of the previous point in the previous sequence, p_{i-1}. Thus, for $\varepsilon > 0$, $x_{q_j} = x_{p_i} + \varepsilon$ and $x_{q_j} \ll x_{p_{i-1}}$.

For $x > 1$, the strategy of the presenter is described in the proof of the following Lemma 5.1. The idea is that we can scale the strategy of Theorem 4.7 by presenting x times as many points in the first sequence V_1. This leads to a factor of x more points in every sequence V_i, for $i \in \{1, \dots, a, \dots, m\}$ and therefore x times more colored points in the sequence V_a. Note that the presenter stops presenting points after a sequences $V_1, \dots, V_a$.

Lemma 5.1. *For $m \geq a$, the strategy $S(x, a, 2)$ creates x monochromatic hanging topless rectangles of size a if the painter colors with two colors.*

Proof. Without loss of generality, we focus on one of the two colors. Note that the proof works equally for the other color. The presenter starts with presenting $x \cdot m^m$ points in the first sequence V_1. As the painter needs to color at least every m-th point, we get at least $x \cdot m^{m-1}$ points colored with the same color, say c_1, in sequence V_1. Thus, sequence V_2 has at least $x \cdot m^{m-1}$ points. After a sequences, $a \leq m$, the sequence V_a contains at least $x \cdot m^{m-(a-1)}$ points. As $m \geq a$,

$$x \cdot m^{m-(a-1)} = x \cdot m^{m-a+1} \geq x \cdot m^1 .$$

Thus, the painter colors at least x points in sequence V_a. By the construction of sequences, each colored point in V_a has "almost the same" x-coordinate as $a-1$ other points colored with the same color. Thus, we have x topless rectangles of size a that are monochromatic. □

Recursively, we define the strategy $S(x, a, k)$ for $k \geq 3$ and $a \geq 2$ given the strategies $S(x', a, k-1)$ for all x' and $S(x'', a-1, k)$ for all x'' as follows:

1. First, play the strategy $S(x_1, a-1, k)$ with $x_1 = x \cdot k \cdot n(x, a, k-1)$. This results in x_1 monochromatic topless rectangles containing $a-1$ points each. By the pigeonhole principle, at least $x_1/k = x \cdot n(x, a, k-1)$ of them have the same color, say color c_1. Let $r_1, \dots, r_{x \cdot n(x,a,k-1)}$ be those rectangles.
2. Now play x versions of $S(x, a, k-1)$. Consider for this the rightmost points $p_i, \dots, p_{i+n(x,a,k-1)-1}$ of $n(x, a, k-1)$ consecutive rectangles $r_i, \dots, r_{i+n(x,a,k-1)-1}$ with color c_1, respectively, whereby $i = y \cdot n(x, a, k-1) + 1$ for $0 \leq y \leq x-1$. For $\varepsilon > 0$, new points of one of the strategies $S(x, a, k-1)$ are then set at the positions $p_i + \varepsilon, \dots, p_{i+n(x,a,k-1)-1} + \varepsilon$. Note that we reserve one point p_j of the c_1-colored rectangle r_j for *each* point in the upcoming strategy $S(x, a, k-1)$, even if it uses several sequences. Now we proceed as follows, depending on the actions of the painter. Note that the two following cases are visualized in Figure 5.2 and Figure 5.3.
   a) If the painter does not use color c_1 in any of these versions v_j, this version already provides us the desired x rectangles, each of them containing a points, but only $k-1$ colors are used, as we already know that a strategy $S(x, a, k-1)$ provides us x monochromatic hanging topless rectangles of size a when using $k-1$ colors. This also satisfies the strategy $S(x, a, k)$, only using less colors.
   b) The painter uses color c_1 in each of the x versions at least once. In each version, the point colored with color c_1 has "almost the same" x-coordinate as the points of the $n(x, a, k-1)$ monochromatic hanging topless rectangles, as we place the new points close to them. These rectangles now contain an additional point with color c_1, thus a points in total. They are also still hanging and topless. The rectangles containing a point colored with color $c_i \neq c_1$ are no longer monochromatic. Thus, as each of the x versions produces at least one monochromatic hanging topless rectangle, we get the desired number of at least x, each containing a points colored with color c_1.

If the presenter plays the strategy $S(1, m, k)$, we get one hanging topless monochromatic rectangle with m points. The strategy is still semi-online.

Theorem 5.2. *There exists a semi-online strategy $S(1, m, k)$ for the presenter such that the union of hanging right-open rectangles and hanging topless rectangles over the same set of points V is not semi-online proper k-colorable for every $k \geq 2$.*

Proof. The presenter presents a sufficiently large sequence of points with ascending y- but descending x-coordinates as the first base case strategy $S(x, 1, k)$. As shown above, it must consist of at least $k \cdot m \cdot \lceil x/2 \rceil$ points. Along that, the painter needs to color at least two points for each m points, such that he properly k-colors the hanging right-open rectangles. Then, there are at least x rectangles of size one that are monochromatic.

The second base case strategy $S(x, a, 2)$ is described in the proof of Lemma 5.1.

It holds that for $x' \in \mathbb{N}$, $S(x', a, k-1)$ is a strategy that forces the painter to create x' hanging topless rectangles containing a points when coloring with $k-1$ colors. Moreover, it holds that for $x'' \in \mathbb{N}$, $S(x'', a-1, k)$ is a presenting-strategy that leads to x'' hanging topless rectangles of size $a-1$ when the painter uses k colors.

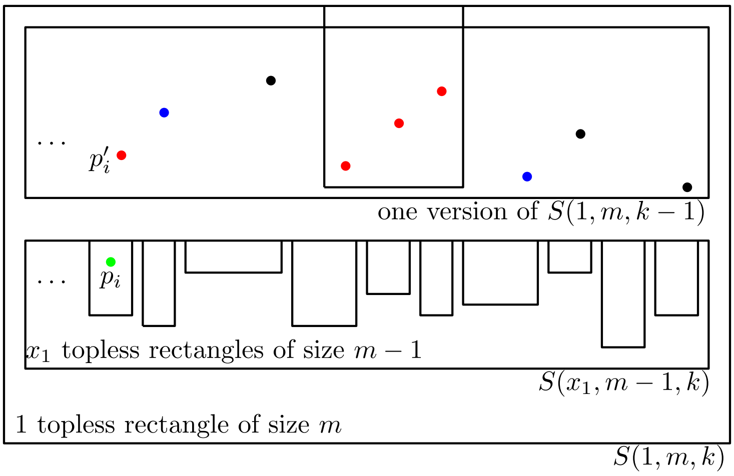


Figure 5.2: Example for the construction of $S(1, m, k)$ case a). The strategy $S(x_1, m-1, k)$ creates x_1 green monochromatic hanging topless rectangles of size $m-1$. After that, the presenter plays one version of $S(1, m, k-1)$ on the top, but the painter never reuses color green. Thus, only $k-1$ colors are used, e.g. blue and red here. But the strategy $S(1, m, k-1)$ is known to provide one monochromatic hanging topless rectangle of size m using only $k-1$ colors.

We now show that we can create the strategy $S(1, m, k)$ with the use of $S(x', m, k-1)$ and $S(x'', m-1, k)$. The construction for case a) is shown in Figure 5.2, the one for case b) in Figure 5.3. Let $x'' = x_1 = 1 \cdot k \cdot n(1, m, k-1)$. Then the strategy $S(x_1, m-1, k)$ creates us $1 \cdot k \cdot n(1, m, k-1)$ monochromatic topless rectangles of size $m-1$ using k colors. Dividing by the number of colors, we know that at least $n(1, m, k-1)$ of the rectangles have the same color, say c_1. Starting from the rightmost point p_i of the i-th (from the right) monochromatic rectangle, we create for the strategy $S(1, m, k-1)$ a new sequence of points $p_1', \ldots, p_{n(1,m,k-1)}'$ whereby for $\varepsilon > 0$, $x_{p_i}' = x_{p_i} + \varepsilon$, i.e., each point p_i' is slightly shifted to the right of the corresponding point p_i. This strategy creates one monochromatic hanging topless rectangle of size m when coloring with $k-1$ colors. Thus, if the painter does not use color c_1, he creates a monochromatic rectangle r in a second color, say c_2. If the painter colors at least one of the newly added points, e.g. p_i', with color c_1, p_i' is in any case consecutive with $m-1$ monochromatic points of the strategy $S(x'', m-1, k)$, as the rightmost point p_i in this rectangle of size $m-1$ influenced the position of p_i'. Thus, there are now m consecutive points with color c_1 that create a monochromatic hanging

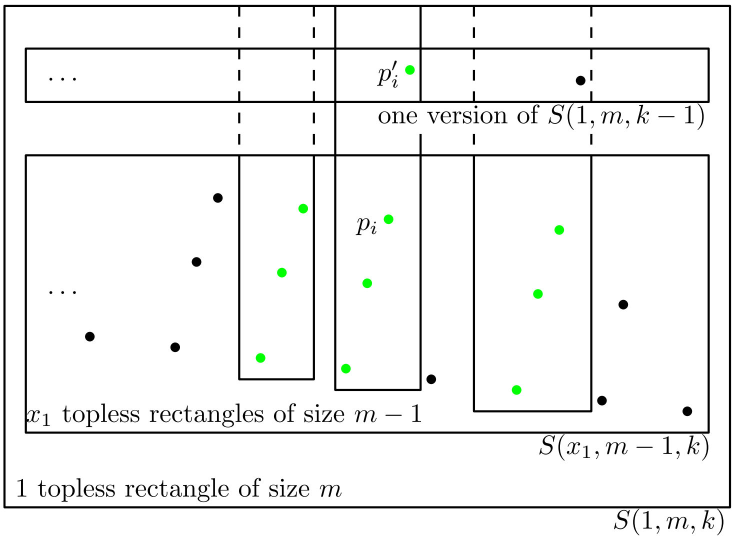


Figure 5.3: Example for the construction of $S(1, m, k)$ case b). The strategy $S(x_1, m-1, k)$ creates x_1 green monochromatic hanging topless rectangles of size $m-1$. After that, the presenter plays one version of $S(1, m, k-1)$ on the top, and as at least one point is colored with green again, there exists a monochromatic hanging topless rectangle of size m.

topless rectangle, which is exactly what the strategy $S(1, m, k)$ is supposed to do. Therefore there exists a semi-online strategy $S(1, m, k)$ that forces a monochromatic hanging topless rectangle of size m when coloring with k colors. The existence of such a strategy implies that the union of hanging right-open and hanging topless rectangles is not semi-online proper k-colorable for $k > 2$. □

5.1 Conversion into Offline Setting

We now construct an offline strategy for the presenter in the presenter-painter game for hanging right-open and hanging bottomless rectangles for $k = 2$ colors. This strategy is similar to the semi-online strategy described in the previous chapter and in the previous section. For $k > 2$, we are not able to present an offline strategy. Therefore, we describe the challenges arising during the development of such a strategy. We know no way to solve all of them together.

As these strategies are offline, all points of the hypergraph must be presented before the painter starts coloring. The painter thus knows the position of all the points from the beginning.

We want to find for $x, a, k \in \mathbb{N}$ an offline presenting-strategy $S_{\text{off}}(x, a, k)$ for $x \geq 1, a \geq 1$ and $k \geq 2$. If the painter offline properly k-colors m consecutive presented points, $S_{\text{off}}(x, a, k)$ enforces the painter to create a set of x *non-hanging* bottomless disjoint rectangles that all have a size of a points and that are all monochromatic. For $x = 1$, $S_{\text{off}}(1, m, k)$ would be the offline winning strategy for the presenter that enforces a monochromatic hanging topless rectangle of size m while the painter colors with k colors.

The idea is to start again with a starting sequence of points V_1 with ascending y- and descending x-coordinates. As the presenter does not know which of those points the painter colors with the same color, say c_1, the presenter must ensure that no matter which points are colored with color c_1, there exist points in a second sequence that are very close to the one colored with c_1 in the sequence V_1. To cover all possible subsets of points colored with color c_1, the presenter creates an individual subuniverse for every subset of V_1. Recursively, we treat the sequences V_i created in this way identically to the first one. Thus, for all sequences V_i, we again create subuniverses for all possible subset of V_i.

Definition 5.3. *Let $p = (x_p, y_p)$ and $q = (x_q, y_q)$ be two points whereby $x_p < x_q$. We define the interval (x_p, x_q) as* area *between p and q. The interval $(x_p + 0.5 \cdot (x_q - x_p), x_q)$ is called the* right area *between these two points or the* right area of *x_p.*

For the rightmost point $r = (x_r, y_r)$, the right area is defined as $(x_r + c, x_r + 2c)$ for an arbitrary $c \in \mathbb{N}$.

Lemma 5.4. *Let $p = (x_p, y_p)$ and $q = (x_q, y_q)$ be two points whereby $x_p < x_q$. There always exits a point $r = (x_r, y_r)$ that lies in the area between p and q, i.e.,*

$$\exists x_r \in \mathbb{R} : x_p < x_r < x_q \ .$$

There equally exists a point $s = (x_s, y_s)$ that lies in the right area between p and q.

Proof. Let $p = (x_p, y_p)$ and $q = (x_q, y_q)$ be two points whereby $x_p < x_q$. As $x_p, x_q \in \mathbb{R}$ and as $\mathbb{R}$ is an uncountably infinite set, there always exists a real number x_r with $x_p < x_r < x_q$. Therefore, any point $r := (x_r, y_r)$ lies in the area between p and q.

Let now $s = (x_s, y_s)$ be another point. If we define $x_s := x_p + 0.5 \cdot (x_q - x_p)$, we can analogously conclude, that we can find a point $r := (x_r, y_r)$ that lies in the area between s and q. As for the definition of x_s, the point r then lies in the right area between p and q. $\square$

We recall the base cases for our semi-online construction from the previous section and then transfer them into offline base cases:

1. The strategy $S(x, 1, k)$ is a semi-online strategy, that creates x monochromatic hanging topless rectangles of size one when coloring with k colors. Therefore we need at most $k \cdot m \cdot \lceil x/2 \rceil \leq n(x, 1, k)$ points. Then, one color appears at least x times.
2. The semi-online strategy $S(x, a, 2)$ forces x monochromatic rectangles of size a when coloring with two colors. We therefore increase the number of points presented in every sequence of the strategy described in Theorem 4.7 as described in the proof of Lemma 5.1. After the a-th sequence, there are at least x colored points that are all contained in a monochromatic hanging topless rectangle of size a, respectively.

The first strategy can be directly transferred into the offline strategy $S_{\text{off}}(x, 1, k)$. For the second strategy, in the offline setting, we do not know which of the points in the first sequence V_1^0 on level 0 will be colored with the same color. Thus, we have to create each possible subuniverse by assuming that any arbitrary subset of V_1 represents the colored points with the same color.

5.1.1 Offline-Algorithm for Two Colors

As for the semi-online strategy for two colors, we again only consider one color. But for $m \geq 2$, it is always possible to color a different point with another color. We use the following definition to describe the structure of the presented points for the strategy $S_{\text{off}}(x, a, 2)$. Note that this construction is similar to the construction of Chekan and Ueckerdt [CU21], which they use to prove that the union of bottomless rectangles and horizontal strips is not polychromatic k-colorable. In their construction, it is sufficient to show, that there exists a hypergraph captured by those two range familes, that is not polychromatic 2-colorable. Thus, their construction is also made for two colors only. During the construction, we will lose the hanging property of the bottomless rectangles. Hence, the offline strategy for two colors is for hanging right-open and non-hanging bottomless rectangles.

Definition 5.5. *In the following, we define a rooted forest F_z for $z \in \mathbb{N}$. It holds that $z = x \cdot m^m$ and $m \geq a$. For $a \geq 2$, this graph will then be extended to an a-uniform hypergraph $\mathcal{H}_z(V, \mathcal{E}, a)$.*

F_z consists of z trees. The root nodes $S = \{r_1, \ldots, r_z\}$ all have the same y-coordinate y_r and each of them a fixed x-coordinate $x_{r_1}, \ldots, x_{r_z}$. Thus, the root nodes are ordered according to their x-coordinate. The children of each (root) node will all have the same x-coordinate as their parent node.

The forest consists of a levels $0, \ldots a-1$*. The root nodes form level 0. The children of all the root nodes are on level 1, their children on level 2 and so on. Each level $j \in \{1, \ldots, a-1\}$ is divided into multiple* stages*. On level 0, there is only one unique stage. A stage is determined by a fixed y-coordinate, thus the stages are ordered along their y-coordinate.*

The child nodes of the root nodes are derived from the set S of all root nodes. Each stage on level 1 consists of a set of nodes

$$S^1 \in \binom{S}{y} \text{ for } y = \frac{1}{m}|S| ,$$

i.e., every possible subset of size $y = (1/m) \cdot |S|$. Let $S_1^j, \ldots, S_s^j$ be the stages on level j. Analogously, stage S_i^j for $1 \leq i \leq s$ induces the stages on level $j+1$ consisting of the nodes

$$S^{j+1} \in \binom{S_i^j}{y} \text{ for } y = \frac{1}{m}|S_i^j| .$$

Note that as $z = x \cdot m^m$, we can divide the number of root points $m \geq a$ times by m and every time we get a natural number, thus $y \in \mathbb{N}$. Recursively, the children of the nodes $v_1, \ldots, v_y$ of a stage S^j on level j are on different stages S_i^{j+1} on level $j+1$. Thereby, for stages $S_1^j, \ldots, S_s^j$ on level j, the s' stages $S_1^{j+1}, \ldots, S_{s'}^{j+1}$ on level $j+1$ derived by stage S_i^j on level j lie between the stages S_i^j and the next stage on level j, S_{i+1}^j. The structure of the stages and levels is exemplified in Figure 5.4. An example of the whole forest F_4 for $m = 2$ is shown in Figure 5.5.

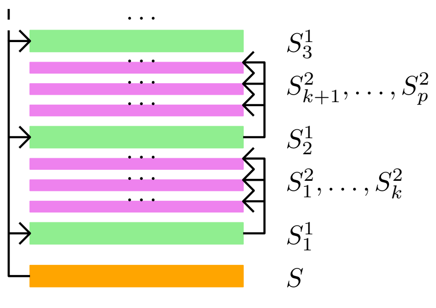


Figure 5.4: Illustration of the level and stage structure. The colored areas each describe a stage. The stages on level 0 are orange, on level 1 green and on level 2 pink. The arrows indicate the subset relations.

Using F_z, we define the a-uniform hypergraph $\mathcal{H}_z(V, \mathcal{E}, a)$ as follows:

1. *Let r_1 be the root node with the highest x-coordinate. Analogously, in every stage on level $1, \ldots, a-1$, we also consider v_1 being the node with the highest x-coordinate.*

In every stage S consisting of y nodes, we slightly increase the y-coordinate of the nodes $v_2, \dots, v_y$, such that $y_{v_1} < y_{v_2} < \dots < y_{v_y}$. This is necessary, as we do not allow two points to have the same coordinates. Note that nevertheless, we respect the stage boundaries, i.e. all points of any stage S_i^j have lower y-coordinates than those of the next stage in y-direction, S_k^{j+1}. Thus, the stages remain disjoint.

2. *Let $v_i^j, v_1^{j+1}, \dots, v_p^{j+1}$ be the points with an identical x-coordinate and increasing y coordinates, i.e., $y_{v_i^j} < y_{v_1^{j+1}} < \dots < y_{v_p^{j+1}}$. Let v_i^j on level j be the direct parent of $v_1^{j+1}, \dots, v_p^{j+1}$ on level $j+1$. We shift v_1^{j+1} in the middle of v_i^j and the point with the next higher x-coordinate in the same stage as v_i^j on level j, v_{i-1}^j. Thus, $x_{v_1^{j+1}} = x_{v_i^j} + 0.5 \cdot (x_{v_{i-1}^j} - x_{v_i^j})$. Moreover, all direct children of v_1^{j+1} lie in the right area between v_i^j and v_{i-1}^j. Analogously, we shift every v_l^{j+1} for $l \in \{2, \dots, p\}$ in the middle between v_i^j and v_{l-1}^{j+1} and their direct children lie in the right area between the latter two.*

3. *The hyperedges of $\mathcal{H}_z$ are defined as follows: Every a consecutive points in y-direction in one stage form a hyperedge. We call those hyperedges stage-hyperedges. Note that those hyperedges are not disjoint. Note further that each one is captured by a hanging right-open rectangle. Also, for every point v^{a-1} on level $a-1$, we construct a path-hyperedge where v^{a-1} is the upper right corner and the root of the tree containing v^{a-1}, called $root(v^{a-1})$, is the lower left corner. Note that this edge consists of points that have the same x-coordinate in F_z. These edges also have exactly a size of a points (see Lemma 5.6). They are captured by bottomless rectangles. We call v^{a-1} a leaf of the tree rooted in $root(v^{a-1})$. For an illustration of the path-hyperedges, see Figure 5.6.*

The different stages on each level describe the different subuniverses of the universe considered on the previous level. Note that it is not necessary to consider stage-hyperedges that contain points of different stages. We ignore if the painter colors these hyperedges monochromatically since we can still force another monochromatic hyperedge.

Lemma 5.6. *The hypergraph $\mathcal{H}_z$ is a-uniform.*

Proof. The stage-hyperedges of $\mathcal{H}_z$ each contain a points with consecutive y-coordinates. Therefore, all of them have a size of a.

It remains to show, that every path-hyperedge defined by a rectangle surrounding a leaf and its root and all points in between contains exactly a points. Let $v = (x_v, y_v)$ be the leaf of the tree situated on level $a-1$. Let $root(v)$ be the root of this tree. By construction, every direct child of $root(v)$ has its own right area, in which their direct children are situated. Furthermore, for a child v_i, its children have greater x-coordinates than v_i itself, but lower x-coordinates than the next child v_{i-1}.

Thus, only points of the path from $root(v)$ to v are in the rectangle spanned by v and $root(v)$. As we have exactly a levels and each level contains one point of the path, it contains exactly a points.

Therefore, all hyperedges consist of exactly a points and thus $\mathcal{H}_z$ is a-uniform. □

Let now $a = m$ and $x = 1$. The strategy $S_{\text{off}}(1, m, 2)$ then describes the offline version of the semi-online strategy introduced in Section 4.3. Thus, no matter how the painter colors all the presented points with two colors, we always get a monochromatic hyperedge:

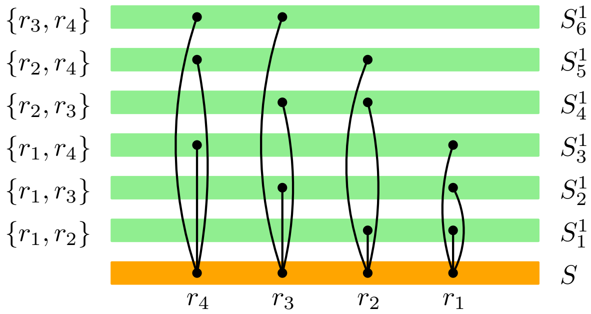


Figure 5.5: Illustration of F_4 for $m = 2$ defined in Definition 5.5 (inspired by [Che21]). The colored areas show the different stages of different levels: Stages on level 0 are orange and stages on level 1 are green. The black edges represent the edges of the forest. After shifting the points, every two consecutive points in y-direction in one stage form a stage-hyperedge. Additionally, every leaf-to-root-path forms a path-hyperedge.

Lemma 5.7. *If the painter properly 2-colors the stage-hyperedges, one of the path-hyperedges of $\mathcal{H}_z$ for $z = m^m$ is monochromatic.*

Proof. Level 0 with its unique stage S consists of $1 \cdot m^m = m^m$ points. The stages on level 1 contain m^{m-1} points. Thus, the stages on level $m-1$ consist of $m^{m-(m-1)} = m^1 = m$ points. Every m consecutive points on any stage on level $m-1$ form a stage-hyperedge. Therefore, as the painter properly 2-colors all stage-hyperedges, there is at least one point in every stage on level $m-1$, that is colored with color c_1. All point sets or stages $S_1^{m-1}, \ldots, S_{s'}^{m-1}$ on level $m-1$ form a subset of a point set $S_1^{m-2}, \ldots, S_s^{m-2}$ on level $m-2$. We call the stage S_i^j and S_k^{j-1} *related* to each other, if S_i^j contains a subset of the points of S_k^{j-1}. In the stages $S_1^{m-2}, \ldots, S_s^{m-2}$, at least $1/m$ of the points are colored with color c_1. Per definition there exists a stage on level $m-1$ that represents exactly the subset of the c_1-colored points of the related stage on level $m-2$. In this stage on level $m-1$, the point(s) with color c_1, together with the corresponding point(s) on level $m-2$, form a rectangle, which contains no other points. An example is shown in Figure 5.6. The closer a stage on level $j+1$ is to the related stage on level j, the wider is the rectangle that covers those two points. The further away a stage is, the narrower is the rectangle. As each point in a stage on level $j+1$ has its own right-area, no other points are contained in those rectangles.

For all of those monochromatic rectangles of size two, we can repeat the same argumentation for the point set on level $m-2$ and the superset on the related stage on level $m-3$. For each stage on level $m-3$, there is a subset on level $m-2$ that exactly corresponds to the colored points on the related stage on level $m-3$. Thus, we can extend the rectangle to three points with the same color. With the same reasoning for all levels, we obtain a monochromatic rectangle of size m when looking at the stage that contains the subset of the colored points of level 0. Hence, if the painter properly 2-colors the stage-hyperedges of size m, we get a monochromatic path-hyperedge of size m. This hyperedge consists of exactly one point from a stage on each level, where the points in the stage on level $j+1$ correspond to the points colored with color c_1 on level j. This rectangle is a bottomless rectangle, as there are no more points below level 0.

To summarize, the hypergraph $\mathcal{H}_z$ for $z = m^m$ contains a monochromatic bottomless rectangle and is therefore not proper 2-colorable. $\square$

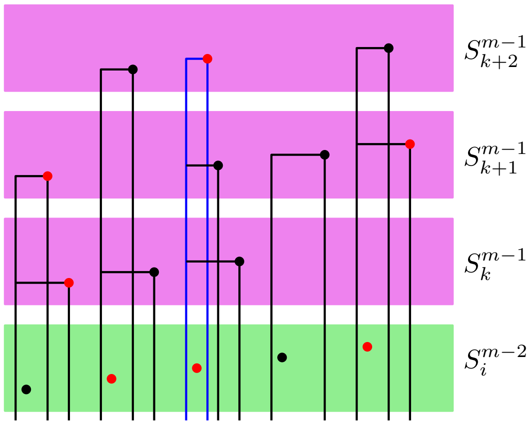


Figure 5.6: Illustration of a part of one stage S_i^{m-2} on level $m-2$ and three stages on level $m-1$, that are related to the stage S_i^{m-2}. The colored areas indicate the different stages. Every m consecutive points in y-direction form a stage-hyperedge. The path-hyperedges are indicated with black rectangles. Note that the left ends of those edges do not lie on the same x-coordinate as the points in stage S_i^{m-2}, as there are other points on lower levels, that are also part of these rectangles and have lower x-coordinates. The points of stage S_{k+2}^{m-1} are exactly the points, that are colored with red in stage S_i^{m-2}. Thus, the colored point in the stage on level $m-1$ and the corresponding point in stage S_i^{m-2} are covered by a blue rectangle, that already contains two points with the same color.

The statement of Lemma 5.7 leads to the result, that there exists an offline strategy $S_{\text{off}}(1, m, 2)$ to place points in the plane, such that the union of hanging right-open rectangles and non-hanging bottomless rectangles is not proper 2-colorable. Note that the stage-hyperedges are right-open, as they contain the complete width of the stages, and thus there is no more point on the right of any of those rectangles. They are all hanging, as they can be extended to the left as far as necessary to intersect the first angle bisector in the upper left corner. Recall also that the path-hyperedges always include the root nodes and there are no more points below them, hence they are bottomless. But not all path-hyperedges are hanging. In fact, the rectangles that are part of the stage on level $m-1$ with the highest y-coordinate can be extended to the first angle bisector. However path-hyperedges of leaf nodes that are not the sibling with the highest y-coordinate cannot be extended to the first angle bisector. This is due to siblings with higher y-coordinates being between itself and its parent in x-direction, hence these siblings would be captured in the hanging bottomless rectangle as well and the rectangles would lose their m-uniformity (and likely their monochromacity).

It is important to notice, that this result does not contradict the statement of Corollary 3.8. This Corollary applies to hypergraphs captured by $\mathcal{R}_{BL} \cup \mathcal{R}_{RO}$, the union of the range families of hanging bottomless and hanging right-open rectangles. In Lemma 5.7, as mentioned above, we prove that a hypergraph captured by hanging right-open and *non-hanging* bottomless rectangles is not proper 2-colorable for any $m \geq 2$.

5.1.2 Discussion: Offline-Algorithm for $k \geq 3$ Colors

For $k \geq 3$ colors, it becomes more difficult to transfer the strategy for the semi-online not proper k-colorability of hanging topless and hanging right-open rectangles described in Chapter 5 into an offline strategy. Additionally, in the semi-online and offline strategy for two colors, the stage-hyperedges remain hanging and right-open. But for more than two colors, the monochromatic rectangles are topless instead of bottomless even in the semi-online setting.

The challenges we face in defining the offline strategy for $k \geq 3$ colors are the following:

1. We do not know which points of the strategy $S_{\text{off}}(x, 1, k)$ the painter colors. Therefore, we again need to consider any subuniverse.
2. We do not know where the $(a-1)$-uniform monochromatic rectangles are situated to play the next strategy to extend those rectangles to monochromatic a-uniform rectangles. Thus, we again need to consider any subuniverse of possible beginnings of monochromatic rectangles. But this time, we cannot just situate the upcoming points in the right area of the already presented points. As the possible positions for the rectangles are not disjoint, placing a point p in the right area of the rightmost point of a possible rectangle r leads to p being in the middle of other possible rectangles r'. This situation is illustrated in Figure 5.7.

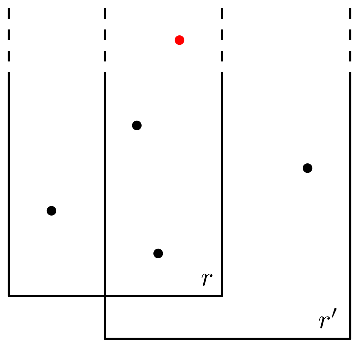


Figure 5.7: Illustration of the second challenge. The topless rectangles r and r' are both possible candidates for being monochromatic. Thus, we need to create multiple subuniverses, to respond to all coloring possibilities. An open question is, how we place the new points in regard to the already placed ones. The red point, that is placed as the rightmost point of r is only the second rightmost for the rectangle r'. This makes it difficult for us to define the different universes as pairwise disjoint, as the red point is now in any case part of r'.

If we have a fixed set of disjoint rectangles, we can play the next upcoming strategy on top of the corresponding rectangles. However, we do not know the set of rectangles in advance in the offline setting, and therefore we have to play the strategy for every possible set of rectangles. These sets are not disjoint. Hence, we have to play strategies for overlapping rectangles, whereby the point set for one set of rectangles blocks the strategy for another set of rectangles, as the sets overlap, and therefore the following strategies overlap as well. We do not know where to put the points of the upcoming strategies without conflicting with the other strategies.

Especially, if k increases, there are more and more different coloring options that require more and more subuniverses. Thus, it remains an open question whether we can transfer the semi-online strategy into an offline strategy and thus whether hanging right-open and hanging topless rectangles are proper k-colorable for $k \geq 3$ or not. In any case, it is likely that if it is possible to transfer the given strategy into an offline strategy, we will lose the hanging property of the topless rectangles. It is already lost in the offline strategy with only two colors for the bottomless rectangles. As we need to create several times multiple subuniverses, the later presented universes will most likely prevent rectangles of previous universes from extending to the first angle bisector without capturing additional points.

6. Conclusion

We investigated the question, whether the m-uniform hypergraph $\mathcal{H}(V, \mathcal{R}_R, m)$ captured by the geometric range family of hanging rectangles is proper k-colorable for $k \geq 2$ and $m \geq 2$. We first showed in Theorem 2.2 that for $m = 2$, we can embed the graph K_4, describing a clique of size four, into the setting of hanging rectangles. Hence, it holds that for $m = 2$, there exists at least one hypergraph captured by hanging rectangles which is neither proper 2-colorable nor proper 3-colorable, as we need at least four colors to color K_4. It remains unclear, if it is possible to embed K_5 as well. Thus, we have no witness that is not proper 4-colorable but can still be embedded into the setting of hanging rectangles.

Then, we investigated the two range families of hanging bottomless and hanging right-open rectangles. Hypergraphs $\mathcal{H}_{BL}(V, \mathcal{R}_{BL}, m)$ captured by the former range family form a subset of hypergraphs $\mathcal{H}_R(V, \mathcal{R}_R, m)$ over the same point set V as well as of hypergraphs $\mathcal{H}_B(V, \mathcal{B}, m)$ captured by *non-hanging* bottomless rectangles. Thus, if $\mathcal{H}_B$ or $\mathcal{H}_R$ is proper or polychromatic k-colorable for a fixed k, so is $\mathcal{H}_{BL}$. This does not hold for the not proper or not polychromatic colorability: The hanging property is more restrictive, hence it is likely that we can sometimes color these range families with less colors than the non-hanging range family. We were able to show in Lemma 3.13 that, by contrast to bottomless rectangles [Kes12], it is possible to properly 2-color any hypergraph $\mathcal{H}_{BL}$ captured by hanging bottomless rectangles with $m \geq 3$. But for $m = 2$, $\mathcal{H}_{BL}$ is not proper 2-colorable, as the counterexample for bottomless rectangles of Keszegh [Kes12] is also an counterexample for hanging bottomless rectangles.

Moreover, we can transform any hypergraph captured by (hanging) right-open rectangles to a hypergraph captured by (hanging) bottomless rectangles. Therefore, the results for (hanging) bottomless rectangles also apply for (hanging) right-open rectangles.

We then used the polychromatic colorability of bottomless rectangles by Asinowski et al. [ACC$^+$13] in combination with the Union-Lemma by Damásdi and Pálvölgyi [DP21a], to prove that for any $k \geq 3$ and for $m \geq 7$, any m-uniform hypergraph captured by hanging bottomless and hanging right-open rectangles is proper k-colorable (see Corollary 3.8).

Further, for $m \geq 2$, we developed a semi-online presenting-strategy in the presenter-painter game for the union of hanging bottomless and hanging right-open rectangles. No matter how the painter colors the presented points with two colors, the presenter can always force the painter to color a hanging bottomless rectangle of size m monochromatically. This strategy is transferred into an offline strategy in Section 5.1.1. In the process, we lose the hanging property of the bottomless rectangles. Thus, any hypergraph captured by *non-hanging* bottomless and hanging right-open rectangles is not proper 2-colorable for any $m \geq 2$.

The semi-online strategy is generalized to any number of colors k in Chapter 5. However, we were not able to transfer this strategy into an offline strategy. The challenges that arose are described in Section 5.1.2. If we were able to define an offline strategy, we would probably lose the hanging property for the bottomless rectangles again, as for two colors, but we additionally lose the bottomless property in any case.

We summarize the results of our work and some helpful additional results by other authors in Table 6.1. For the different studied range families, we specify for different uniformities m the smallest number of colors k such that we can properly k-color any m-uniform hypergraph captured by this range family. If we specify a lower or upper bound for k, we do not know an exact value.

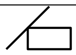
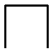
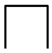
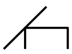
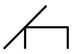
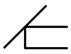
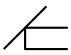
Range family	**Sketch**	**Uniformity** m	**#Colors** k	**Source**
hanging rectangles		$m = 2$	$k \geq 4$	Theorem 2.2
bottomless rectangles		$m \in \{2, 3\}$	$k = 3$	[Kes12]
bottomless rectangles		$m \geq 4$	$k = 2$	[Kes12]
hanging bottomless rectangles		$m = 2$	$k = 3$	Lemma 3.9, Lemma 3.2, [Kes12]
hanging bottomless rectangles		$m \geq 3$	$k = 2$	Lemma 3.13
hanging right-open rectangles		$m = 2$	$k = 3$	Observation 3.5
hanging right-open rectangles		$m \geq 3$	$k = 2$	Lemma 3.15, Observation 3.5
hanging bottomless and hanging right-open rectangles		$m = 2$	$k \geq 4$	Theorem 3.17
hanging bottomless and hanging right-open rectangles		$m \geq 3$	$k \leq 4$	Corollary 3.16
hanging bottomless and hanging right-open rectangles		$m \geq 7$	$k \leq 3$	Corollary 3.8
bottomless and hanging right-open rectangles		$m \geq 2$	$k > 2$	Lemma 5.7

Table 6.1: Overview of the known values for the minimum number of colors k required for different range families to properly k-color any m-uniform hypergraph captured by them. In some cases we only know lower or upper bounds for k. We also differentiate between different uniformities m. Additionally, we indicate the source of these results.

6.1 Open Questions

There are a few questions that remain open and need further investigations to be answered. In Section 5.1.2, we discussed that it is very challenging to transfer our semi-online strategy for hanging right-open and hanging topless rectangles for $k > 2$ colors described in Chapter 5 into an offline strategy. It remains open, if it is possible to transfer the strategy into an offline setting.

Question 6.1. *Is it possible to transfer our semi-online strategy for the not proper k-colorability of the union of hanging topless rectangles and hanging right-open rectangles into an offline strategy?*

It further remains an interesting question, if it is possible to properly k-color the union of hanging bottomless and hanging right-open rectangles for $m = 2$. We know due to the embedding of K_4 with hanging bottomless and hanging right-open rectangles in Section 3.1, that if it is possible, k is at least four.

Question 6.2. *What is the smallest number $k \geq 4$, such that we can properly k-color any 2-uniform hypergraph captured by the union of hanging bottomless and hanging right-open rectangles?*

For $m \geq 3$, we know due to Corollary 3.16 that any hypergraph captured by the union of those two range families is proper 4-colorable. It remains open, whether these hypergraphs are even proper 2-colorable or proper 3-colorable.

Question 6.3. *What is the smallest number $k \in \{2, 3, 4\}$, such that we can properly k-color any m-uniform hypergraph captured by the union of hanging bottomless and hanging right-open rectangles for $3 \leq m \leq 6$?*

We know that for $m \geq 7$, we can properly 3-color any m-uniform hypergraph captured by hanging bottomless and hanging right-open rectangles (see Corollary 3.8). Hence, we pose the question whether an $m' \geq 7$ exists such that all m'-uniform hypergraphs captured by $\mathcal{R}_{BL}$ and $\mathcal{R}_{RO}$ are proper 2-colorable.

Question 6.4. *Does a number $m' \geq 7$ exist such that we can properly 2-color any m'-uniform hypergraph captured by the union of hanging bottomless and hanging right-open rectangles?*

Additionally, we were not able to give a final answer on the colorability of m-uniform hypergraphs captured by hanging rectangles for $m \geq 3$:

Question 6.5. *Let $\mathcal{H}(V, \mathcal{R}_R, m)$ be an m-uniform hypergraph captured by the range family of hanging rectangles. For $m \geq 3$ and any $k \geq 2$, is $\mathcal{H}$ proper k-colorable?*

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