Master's Thesis

Ordered Covering Numbers

Laura Merker

May 28, 2020 – November 27, 2020

At the Department of Informatics
Institute of Theoretical Informatics
Karlsruhe Institute of Technology

Reviewers:	PD Dr. Torsten Ueckerdt
	Prof. Dr. Peter Sanders
Advisors:	PD Dr. Torsten Ueckerdt
	Paul Jungeblut

Statement of Authorship

I hereby declare that this document has been composed by myself and describes my own work, unless otherwise acknowledged in the text. I declare that I have observed the *Satzung des KIT zur Sicherung guter wissenschaftlicher Praxis*, as amended.

Ich versichere wahrheitsgemäß, die Arbeit selbstständig verfasst, alle benutzten Hilfsmittel vollständig und genau angegeben und alles kenntlich gemacht zu haben, was aus Arbeiten anderer unverändert oder mit Abänderungen entnommen wurde, sowie die Satzung des KIT zur Sicherung guter wissenschaftlicher Praxis in der jeweils gültigen Fassung beachtet zu haben.

Karlsruhe, November 27, 2020

Laura Merker

Abstract

A *page* (*queue*) with respect to a vertex ordering of a graph is a set of edges such that no two edges cross (nest), i.e. have their endpoints ordered in an ABAB-pattern (ABBA-pattern). A *union page* (*union queue*) is a vertex-disjoint union of pages (queues). The *page number* (*queue number*, *union page number*, *union queue number*) of a graph is the smallest k such that there is a vertex ordering and a partition of the edges into k pages (queues, union pages, union queues). The *local page number* (*local queue number*) is the smallest k for which there is a vertex ordering and a partition into pages (queues) such that each vertex has incident edges in at most k pages (queues). For directed acyclic graphs, we additionally require all edges to point into the same direction with respect to the vertex ordering, i.e. from a smaller vertex to a larger vertex. The *track number* of a graph is the smallest k such that there is a partition of the vertices into independent sets, each having a vertex ordering, such that any two edges with endpoints into the same two independent sets do not cross.

We show for a complete graph on n vertices that the local page number is $n/3 + \Theta(1)$, that the union page number is upper-bounded by $4n/9 + \Theta(1)$, and that both the local queue number and the union queue number are $(1 - 1/\sqrt{2})n + \Theta(1)$. In addition, we show that there is a graph with treewidth 2 and track number at least 7 and that there is a poset whose cover graph has page number at least 5.

Eine *Page* (*Queue*) ist bezüglich einer Knotenordnung eines Graphen definiert als eine Menge von Kanten, in der sich je zwei Kanten nicht kreuzen (nicht verschachtelt sind), d.h. die Endpunkte sind nicht in der Reihenfolge ABAB (ABBA) angeordnet. Eine *Union Page* (*Union Queue*) ist eine Vereinigung aus paarweise knotendisjunkten Pages (Queues). Die *Page Number* (*Queue Number*, *Union Page Number*, *Union Queue Number*) ist definiert als das kleinste k, für das es eine Knotenordnung gibt, die eine Partitionierung der Kanten in k Pages (Queues, Union Pages, Union Queues) ermöglicht. Die *Local Page Number* (*Local Queue Number*) ist das kleinste k, für das es eine Knotenordnung gibt, sodass die Kanten so in Pages (Queues) partitioniert werden können, dass jeder Knoten in höchstens k der Pages (Queues) inzidente Kanten hat. Für gerichtete azyklische Graphen fordern wir zusätzlich, dass alle Kanten bezüglich der Knotenordnung in die gleiche Richtung zeigen, d.h. von einem kleineren zu einem größeren Knoten. Die *Track Number* eines Graphen ist das kleinste k, für das es eine Knotenpartitionierung in unabhängige Mengen mit jeweils einer Knotenordnung gibt, sodass sich je zwei Kanten mit Endpunkten in den gleichen zwei Mengen nicht kreuzen.

Wir zeigen, dass die Local Page Number vollständiger Graphen mit n Knoten $n/3+\Theta(1)$ ist, dass die Union Page Number höchstens $4n/9 + \Theta(1)$ und dass sowohl die Local Queue Number als auch die Union Queue Number gleich $(1 - 1/\sqrt{2})n + \Theta(1)$ sind. Wir zeigen außerdem die Existenz eines Graphen mit Baumweite 2 und Track Number 7 und eines Posets, dessen Cover Graph mindestens Page Number 5 hat.

Contents

1 Introduction

Ordered covering numbers have been investigated intensively over the past decades with a focus on page numbers and queue numbers. Given a graph, we aim to find a vertex ordering and a partition of the edges into ordered subgraphs that satisfy certain conditions. For instance, any two edges may not cross, i.e. have alternating endpoints, (page number) or may not nest (queue number). Before introducing the main concepts more detailed, we give a brief overview of open problems and recent breakthroughs that motivate our research in this direction.

The page number was introduced by Bernhart and Kainen [10] in 1979 and the queue number by Heath and Rosenberg [32] thirteen years later. Recent results include the existence of a planar graph with page number 4 [9, 52], a constant upper bound on the queue number of planar graphs [18], and a linear lower bound and a single exponential upper bound on the queue number of graphs with treewidth k [48]. Building up on this, we introduced and investigated two new graph parameters: the local page number and the local queue number [38, 39]. Compared to the classical (also called global) variants, the investigation of local ordered covering numbers leads to stronger lower bounds and weaker upper bounds. The latter offers a way to support conjectured upper bounds on classical ordered covering numbers. In addition, there is a third type of covering numbers, the union ordered covering number, which can be located between its local and global counterparts. In this thesis, we take up this direction by investigating local and union covering numbers of complete graphs.

We then consider track numbers, which are tied to queue numbers and were used to prove the first constant upper bound on the queue number of graphs with constant treewidth [16, 19]. We improve on the previously best known lower bound on the maximum track number of graphs with treewidth 2, which is motivated by Di Giacomo et al. [16] who asked for improved bounds on the maximum track number for this graph class.

The notions of ordered covering numbers are naturally transferred to directed acyclic graphs. Here, we have the additional constraint that the vertex ordering is a topological ordering of the graph. Nowakowski and Parker [43] initially asked in 1989 whether cover graphs of planar posets have constant page number. The same question occurred for the larger class of upward planar graphs [15, 24]. Both questions are still open. For the upper bound, there is not even a sublinear bound known. We improve the best known lower bound on the page number both for upward planar graphs and planar posets.

1.1 Outline

We first introduce the main concepts on which the following chapters are based. This includes a discussion of global, union, and local covering numbers in Section 1.2. In Section 1.3, we consider some common ordered covering numbers and related graph parameters. In particular, we define page numbers, queue numbers, and track numbers. We conclude the introduction by locating this thesis in the state of the art and by pointing out the main results we obtain. Definitions that are specific to the respective chapter and more detailed surveys are given at the beginning of each chapter.

In the second chapter, we investigate local and union ordered covering numbers of complete graphs. We start with page numbers and then continue with queue numbers. In contrast to Chapter 2, where we consider dense graphs, we investigate two important sparse graph classes in the last two chapters. In Chapter 3, we first survey different variants of track layouts and then construct a graph with treewidth 2 and track number 7. Finally, we turn to directed graphs and investigate the page number of upward planar graphs and of cover graphs of planar posets in Chapter 4. This leads to the construction of a planar poset, and in particular an upward planar graph, that requires at least five pages. We conclude with open questions to encourage further research in Chapter 5.

1.2 Covering Numbers

We discuss global, union, and local covering numbers based on the covering number framework introduced by Knauer and Ueckerdt [36]. Consider a class of graphs $\mathcal{G}$, called *guest class*, and an input graph H. We say the graph H is *covered* by some covering graphs $G_1, \ldots, G_t \in \mathcal{G}$ if G_i is a subgraph of H for each i and every edge of H is contained in some covering graph, i.e. if $G_1 \cup \cdots \cup G_t = H$. The set of covering graphs is called an *injective $\mathcal{G}$-cover* of H. Note that vertices and edges in H may be covered by multiple covering graphs.

We start with the most natural number, the *global covering number* $\operatorname{cn}_g^{\mathcal{G}}(H)$, which is defined as the minimum number of covering graphs needed to cover a graph H, i.e. the minimum size if an injective $\mathcal{G}$-cover of H. Many well-known graph parameters are global covering numbers. For instance, the thickness and outerthickness are the global covering number for the guest class of all planar graphs, respectively all outerplanar graphs [27, 41].

In contrast to the global covering number, which allows only graphs from the guest class $\mathcal{G}$ as covering graphs, we may also use disjoint unions of graphs in $\mathcal{G}$ for the *union covering number* $\operatorname{cn}_u^{\mathcal{G}}(H)$. For a graph class $\mathcal{G}$, let $\mathcal{G}_u$ denote the class of graphs consisting of all graphs in $\mathcal{G}$ and all finite vertex-disjoint unions of graphs in $\mathcal{G}$. We then define the union covering number by $\operatorname{cn}_u^{\mathcal{G}}(H) = \operatorname{cn}_g^{\mathcal{G}_u}(H)$. All kinds of arboricities are union covering numbers, where the guest classes contain, for instance, all trees, paths, caterpillars, or stars [2, 3, 26, 42].

For the *local covering number* $\operatorname{cn}_\ell^{\mathcal{G}}(H)$, we use $\mathcal{G}$-covers of arbitrary size and minimize the number of covering graphs at every vertex. We say a $\mathcal{G}$-cover for a graph H is *ℓ-local*

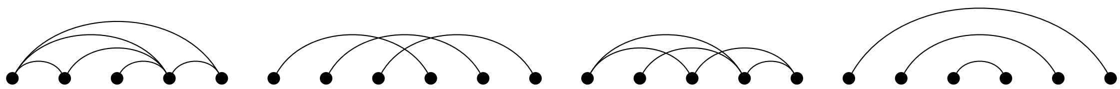


Figure 1.1: Left to right: 1-page book embedding, 3-twist, 1-queue layout, 3-rainbow

if every vertex is contained in at most ℓ covering graphs. Now, the *local covering number* of a graph H with guest class $\mathcal{G}$ is defined as the smallest ℓ such that there is an ℓ-local injective $\mathcal{G}$-cover of H. The local covering number was considered for the guest classes of complete bipartite graphs, complete graphs, and different kinds of trees [36, 40].

To conclude this section, we compare the three covering numbers presented above. Note that every $\mathcal{G}$-cover is also a $\mathcal{G}_u$-cover. In addition, every $\mathcal{G}_u$-cover of size k yields a $\mathcal{G}$-cover (of possibly larger size) mapping at most k vertices from the covering graphs to each vertex of the input graph by using additional covering graphs for each connected component. Hence, for every guest class $\mathcal{G}$ and every input graph H, we have $\mathrm{cn}_g^{\mathcal{G}}(H) \geqslant \mathrm{cn}_u^{\mathcal{G}}(H) \geqslant \mathrm{cn}_\ell^{\mathcal{G}}(H)$.

1.3 Book Embeddings, Queue Layouts, and Track Layouts

We continue with three well-known graph parameters – the page number, the queue number, and the track number. The first two are covering numbers, where we have additional constraints on the vertex ordering of the covering graphs. Based on the notions introduced in the previous section, we define global, union, and local variants of page numbers and queue numbers.

Consider a graph G with a linear ordering $\prec$ of its vertex set. The sets $V(G)$ and $E(G)$ denote the vertex set, respectively edge set, of G. For subsets $X, Y \subseteq V(G)$, we write $X \prec Y$ and say X is *to the left* of Y and Y is *to the right* of X if $x \prec y$ for all vertices $x \in X, y \in Y$. If the sets consist only of a single vertex, we use x instead of $\{x\}$.

Given a linear ordering $\prec$ of the vertices of a graph G, we say two edges $uv, xy \in E(G)$ *cross* if if $u \prec x \prec v \prec y$ or $x \prec u \prec y \prec v$, and they *nest* if $u \prec x \prec y \prec v$ or $x \prec u \prec v \prec y$. A set of k pairwise crossing edges is called a *k-twist*, a set of pairwise nesting edges a *k-rainbow*. A *page* is a set of edges with no two crossing edges. Similarly, a *queue* is an edge set in which no two edges nest. See Figure 1.1 for a page, a queue, and the respective forbidden ordered graphs. For a positive integer k, a *k-page book embedding*, respectively a *k-queue layout*, of G consists of a vertex ordering $\prec$ and a partition of the edges of G into k pages, respectively k queues. If k is not important, then we simply write book embedding and queue layout. The vertex ordering is also called the *spine ordering*. Finally, the *page number* $\mathrm{pn}(G)$ (also known as *stack number* or *book thickness*) is the smallest k such that there is a k-page book embedding for G, whereas the *queue number* $\mathrm{qn}(G)$ of a graph G is the smallest k such that there is a k-queue layout for G. Both concepts are called *ordered covering numbers* as a partition of edges can also be considered as covering the graph with pages or queues, respectively.

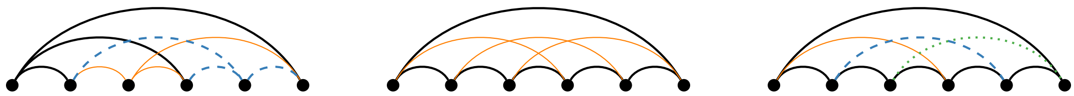


Figure 1.2: A 3-page book embedding, a 2-union book embedding, and a 2-local book embedding of $K_{3,3}$

We now define union and local variants of the parameters defined above. A vertex-disjoint union of pages, respectively queues, with a common vertex ordering is called a *union page*, respectively a *union queue*. A vertex ordering together with a partition of the edges is called a *k-union book embedding* if the parts are union pages and is called a *k-union queue layout* if the parts are queues. The *union page number* and *union queue number* are then defined as the smallest k for which there is a k-union book embedding, respectively a k-union queue layout. For local covering numbers, we allow partitions of arbitrary size but minimize the number of parts at every vertex. More formally, a *k-local* book embedding or queue layout is one in which every vertex has incident edges in at most k pages or queues, respectively. The *local page number* $\mathrm{pn}_\ell(G)$ is the smallest k allowing for a k-local book embedding. Similarly, the *local queue number* $\mathrm{qn}_\ell(G)$ is the smallest k for which there is a k-local queue layout for G.

To clearly distinguish between local and union ordered covering numbers and their classical versions, we also refer to the latter as *global page number* and *global queue number*. Figure 1.2 shows three book embeddings for $K_{3,3}$. The first is a 3-page book embedding. Note that there are vertices taking part in all three pages. As $K_{3,3}$ is not planar, three pages are best-possible. In contrast, two union pages suffice as we allow crossing edges if they belong to distinct connected components. The third book embedding reduces the complexity of each page even further by introducing more pages, while ensuring that every vertex takes part in at most two pages.

All notions defined above can be transferred to directed acyclic graphs. Here, we additionally require the spine ordering to be a topological ordering of the graph. That is, all edges point into the same direction.

We conclude with the definition of track layouts, which are closely related to queue layouts. Instead of a single vertex ordering, we first partition the vertex set into independent sets, called *tracks*, each having a linear vertex ordering. Let V and W denote two tracks with orderings $\prec_V$ and $\prec_W$. We say that two edges vw and $v'w'$ *cross* if $v \prec_V v'$ and $w' \prec_W w$ or $v' \prec_V v$ and $w \prec_W w'$, where $v, v' \in V$ and $w, w' \in W$. A *track layout* is a partition of the vertex set into tracks such that no two edges cross. The *track number* is then defined as the smallest k such that there is a track layout consisting of k tracks. The track number is related to the queue number in the sense that the two parameters are tied [20].

1.4 Related Work and Motivation

In this section, we give a brief overview on notions, results, and open questions that are related to the problems discussed in this thesis. Detailed surveys presenting the state of the art can be found at the beginning of each chapter.

Local and union ordered covering numbers unify the well-known notions of ordered covering numbers and local and union covering numbers. The investigation of local and union variants of pages numbers and queue numbers focuses on planar graphs and graphs with bounded treewidth. We summarize some known results, which we presented in [38, 39] in detail.

First, we have $\mathrm{pn}_\ell(G) \leqslant \mathrm{pn}_\mathrm{u}(G) \leqslant \mathrm{pn}(G)$ and $\mathrm{qn}_\ell(G) \leqslant \mathrm{qn}_\mathrm{u}(G) \leqslant \mathrm{qn}(G)$, where the gap between the union and global variants can be arbitrarily large. In contrast, the local page number, the local queue number, the union page number, and the union queue number are all tied to the maximum average degree. For planar graphs, there are examples known with $\mathrm{pn}_\ell(G) \geqslant 3$, respectively $\mathrm{qn}_\ell(G) \geqslant 3$. On the other hand, the local page number, the union page number, and the local queue number are at most 4, while the union queue number is upper-bounded by 5 for planar graphs. The local page number and the local queue number are both linear in the treewidth. Especially for queue layouts, these results contrast with the large gaps between the best known lower and upper bounds on the global queue number. Dujmović et al. [18] recently proved the first constant upper bound, which is 49, for the queue number of planar graphs, while the best known lower bound is 4 [4]. Considering the treewidth, Wiechert [48] showed a linear lower and an exponential upper bound on the queue number. We also remark that the proofs of the upper bounds for the local and union variants described above are straight-forward, whereas the proofs of the global upper bounds by Dujmović et al. [18], Wiechert [48], and Yannakakis [51] are very involved. We hope that the investigation of local and union ordered covering numbers gives new inside in how to find easier proofs or improved bounds for some of the mentioned problems.

Beside page numbers and queue numbers, there is another ordered covering number that is investigated in the literature. Instead of forbidding crossing or nesting edges, two edges may not be in the same part of an *arch layout* if both endpoints of one edge are to the left of both endpoints of the other. In contrast to the page number and the queue number, the arch number is tied to the chromatic number [21].

Concerning track numbers, one of the main open problems is to find the maximum track number of graphs with treewidth k between the quadratic lower bound and the exponential upper bound [16, 19, 20, 48]. Motivated by applications of drawing series-parallel graphs in three dimensions, Di Giacomo et al. [16] highlighted the special case $k = 2$ as an open problem. We remark that there is also a covering number related to track layouts. Similar to book embeddings and queue layouts, multicolor track layouts partition the edges into color classes such that no two edges of the same color cross with respect to a common partition of the vertices into tracks. See Section 3.2 for an extended survey.

Like in the unordered case, the investigation of ordered covering numbers for directed acyclic graphs focuses on book embeddings and queue layouts, initiated by Heath et al. [31]. Heath and Pemmaraju [30] restricted the considered graph class to cover graphs of posets and proved a lower bound of $\Omega(\sqrt{n})$ on the maximum queue number of n-element planar posets. In contrast, the question whether planar posets have bounded page number, first asked by Nowakowski and Parker [43], is still open. However, Heath and Pemmaraju [30] showed that there are posets whose cover graph is planar (but not upward planar) and has page number $\Theta(n)$.

Our research is mainly motivated by the open problems discussed above. Studying ordered graphs and covering numbers is additionally motivated by applications in very-large-scale integration (VLSI) circuit design and bioinformatics [1, 13, 34, 47]. Covers also appear in network design [45], while queue layouts find application in parallel multiplications of sparse matrices [28] and both queue layouts and track layouts in 3-dimensional graph drawing [50].

1.5 Results

In this thesis, we contribute to the investigation of ordered covering numbers by improving bounds on track numbers and page numbers. In addition, we continue the research on local and union ordered covering numbers by initiating the study of these covering numbers for complete graphs.

For a complete graph on n vertices, we show that the local page number is $n/3 + \Theta(1)$ (Theorems 2.1 and 2.2). This is also the best lower bound we obtain for the union page number of K_n. For the upper bound, we show that $\mathrm{qn}_\mathrm{u}(K_n) \leqslant 4n/9 + \Theta(1)$ (Theorem 2.3). In addition, we consider queue layouts and prove both the local queue number and the union queue number of complete graphs to be $(1 - 1/\sqrt{2})n + \Theta(1)$ (Theorems 2.4 and 2.6).

In the third chapter, we investigate the track number of graphs with treewidth 2. Di Giacomo et al. [16] showed an upper bound of 15 and asked for improved lower and upper bounds. We contribute to this problem by constructing a graph with treewidth 2 and track number at least 7 (Theorem 3.4).

Finally, we consider book embeddings of upward planar graphs, i.e. directed acyclic graphs that can be drawn crossing-free in the plane such that all edges monotonically increase in y-direction. We particularly consider cover graphs of planar posets, which is an important subclass of upward planar graphs. The main question concerning book embeddings of these two subclasses is whether they have bounded page number. This question was first posed by Nowakowski and Parker [43] for planar posets and was then studied and raised again in different papers [e.g. 15, 24, 30]. We continue the investigation of this problem by providing necessary conditions for upward planar graphs to have bounded page number. Prior to this this thesis, the best known lower bound on the page number of both graph classes was 4, shown by Hung [33]. However, we show that their graph admits a book embedding without 4-twist (Remark 4.8). We present a small upward planar graph that has a 4-twist in every book embedding (Proposition 4.9). Our main contribution is the construction of a planar poset that has a 5-twist in every book embedding and in particular requires five pages (Theorem 4.13).

2 Complete Graphs

This chapter is joint work with Stefan Felsner, Torsten Ueckerdt, and Pavel Valtr.

We start with the investigation of one of the most fundamental graphs classes – the complete graphs. The page number, respectively queue number, of complete graphs is $\lceil n/2 \rceil$ [10], respectively $\lfloor n/2 \rfloor$ [32]. For both upper bounds, there are straight-forward constructions that are illustrated in Figures 2.1 and 2.2.

We improve the upper bounds for the respective local and union ordered covering numbers. The trivial lower bounds due to the density of complete graphs are $(n-1)/4$ both for the local page number and the local queue number [38, 39]. In both cases, we find improved bounds matching the respective upper bounds up to an additive constant.

2.1 Local and Union Page Numbers

In this section, we discuss book embeddings of complete graphs. We construct $(n/3+\Theta(1))$-local book embeddings for K_n and show that this bound is asymptotically tight. As the union page number is lower-bounded by the local page number and upper-bounded by the (global) page number, we immediately have $n/3 + \Theta(1) = \mathrm{pn}_\ell(K_n) \leqslant \mathrm{pn}_\mathrm{u}(K_n) \leqslant \mathrm{pn}(K_n) = \lceil n/2 \rceil$. We improve on the upper bound by constructing a book embedding consisting of $4/9 + \Theta(1)$ union pages. However, the leading coefficient of the union page number of K_n remains open.

Due to the symmetries of K_n we may consider an arbitrary spine ordering. Moreover, it is convenient to think of the spine as being circularly closed. The placement of the vertices together with straight-line edges yields a convex drawing of K_n. A page assignment is a partition of the edges into non-crossing subsets, i.e. into outerplanar subdrawings of the convex drawing of K_n. We denote the vertices of K_n by $v_1, \ldots, v_n$ and assume they occur on the spine is this ordering.

First, we analyze the outerplanar subgraphs on each page of a book embedding and thereby show a lower bound on the local page number of complete graphs. The proof also gives insight into how book embeddings for a matching upper bound on the local or union page number should look like. This bound is also the best lower bound we obtain for the union page number.

Theorem 2.1. *For any n, we have $\mathrm{pn}_\ell(K_n) > \frac{1}{3}n - 1$.*

Proof. Let $\mathcal{P}$ be a page assignment of K_n which minimizes the local page number. We assume that $\mathrm{pn}_\ell(\mathcal{P}) \leqslant n/3$, otherwise we are done. Let k denote the average number of vertex-page incidences over all vertices, i.e. $k = \frac{1}{n}\sum_{P\in\mathcal{P}} |V_P|$. We shall show that

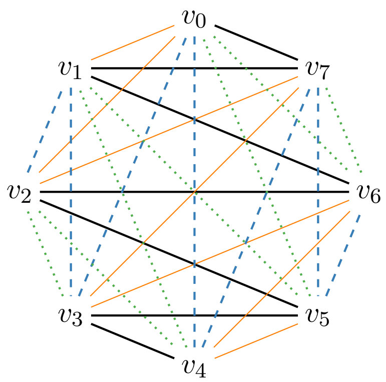


Figure 2.1: 4-page book embedding of K_8. For K_{2n}, we have pages $P_0, \ldots, P_{n-1}$, where P_k contains the edge $v_{i+k}v_{j+k}$ if $i + j \in \{n-1, n\}$. All indices are taken mod n.

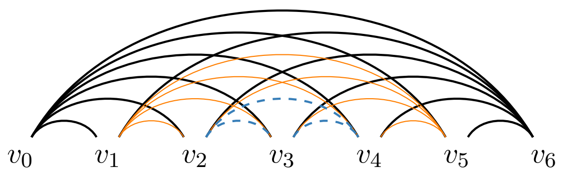


Figure 2.2: 3-queue layout of K_7. For K_{2n+1}, we have queues $Q_0, \ldots, Q_{n-1}$, where Q_k contains all edges v_kv_i and v_iv_{2n-k} with $k < i \leqslant 2n - k$.

$k > n/3 - 1$, which in particular proves that $\mathrm{pn}_\ell(\mathcal{P}) > n/3 - 1$. Later we use that $k \leqslant \mathrm{pn}_\ell(\mathcal{P}) \leqslant n/3$, i.e.

$$k \leqslant \frac{n}{3}. \tag{2.1}$$

We assume that every edge of K_n belongs to exactly one page of $\mathcal{P}$. Now for each page $P \in \mathcal{P}$ we consider an outerplanar graph O_P consisting of all edges of P and their incident vertices V_P together with the edges of the convex hull C_P of V_P. For each page $P \in \mathcal{P}$ we color the edges of O_P:

- The *black edges* are edges of C_P belonging to P.
- The *red edges* are edges of C_P which do not belong to P.
- The *green edges* are inner edges of O_P which belong to P.

Observe that every edge e of K_n has a color in $\{\text{black}, \text{green}\}$ for exactly one page, while e may be red for any number of pages.

For a vertex v and a page P containing v, let the *forward edge* $\mathrm{fwd}_\mathrm{P}(v)$ at v be the edge of C_P which leaves v in clockwise direction. Let r_v denote the number of pages for which the forward edge of v is red, and let b_v denote the number of pages for which the forward edge of v is black. As v has exactly one forward edge on each page, v is incident to exactly $r_v + b_v$ pages in $\mathcal{P}$. Hence, denoting $R = \sum_v r_v$ and $B = \sum_v b_v$, we have

$$k\,n = \sum_{P\in\mathcal{P}} |V_P| = \sum_v (r_v + b_v) = R + B. \tag{2.2}$$

Now, for each page P and each edge $e = uv$ of C_P with u clockwise followed by v, let $\mathrm{len}(e)$ denote the distance along K_n when going clockwise from u to v. That is, for $u = v_i$ and $v = v_j$, we define $\mathrm{len}(e) = j - i$ if $i \leqslant j$ and we define $\mathrm{len}(e) = j - i + n$ if $j < i$. Since C_P is a cycle, we have $\sum_{e\in C_P} \mathrm{len}(e) = n$. Thus,

$$\begin{aligned}
|\mathcal{P}| \cdot n &= \sum_{P\in\mathcal{P}} \Big(\sum_{e\in C_P} \mathrm{len}(e)\Big) = \sum_{P\in\mathcal{P}} \Big(\sum_{v\in V_P} \mathrm{len}(\mathrm{fwd}_\mathrm{P}(v))\Big) \\
&= \sum_v \Big(\sum_{P: v\in V_P} \mathrm{len}(\mathrm{fwd}_\mathrm{P}(v))\Big) \overset{(\diamond)}{\geqslant} \sum_v \Big(\sum_{\ell=1}^{b_v} \ell \Big) \geqslant \sum_v \frac{b_v^2}{2} \overset{(*)}{\geqslant} \frac{1}{2n}\Big(\sum_v b_v\Big)^2 \\
&= \frac{1}{2n}B^2 \overset{(2.2)}{=} \frac{1}{2n}(kn - R)^2 \geqslant \frac{1}{2n}(k^2n^2 - 2knR) = \frac{k^2}{2}n - kR \overset{(2.1)}{\geqslant} \frac{k^2}{2}n - \frac{n}{3}R.
\end{aligned}$$

For $(\diamond)$ ignore red forward edges at v and use that the black forward edges at v are pairwise distinct and for $(*)$ use the Cauchy-Schwarz inequality with the vectors $(b_{v_1}, \dots, b_{v_n})$ and $(1, \dots, 1)$.

Dividing both sides of the above by n we get

$$|\mathcal{P}| \geqslant \frac{k^2}{2} - \frac{R}{3}. \tag{2.3}$$

Now, consider the green edges in K_n. Since O_P is outerplanar and the green edges of O_P are the inner edges, there are at most $|V_P| - 3$ green edges on page P. Therefore,

$$\#\text{green edges} \leqslant \sum_{P \in \mathcal{P}} (|V(P)| - 3) = kn - 3|\mathcal{P}| \overset{(2.3)}{\leqslant} kn - \frac{3k^2}{2} + R. \tag{2.4}$$

On the other hand, we have

$$\#\text{green edges} = |E(K_n)| - \#\text{black edges} = \binom{n}{2} - B \overset{(2.2)}{=} \binom{n}{2} - kn + R. \tag{2.5}$$

Combining Equations (2.4) and (2.5) we conclude:

$$\begin{aligned}
& kn - \frac{3k^2}{2} + R \geqslant \binom{n}{2} - kn + R \\
\iff \quad & 0 \geqslant \frac{3k^2}{2} - 2kn + \binom{n}{2} \\
\iff \quad & 0 \geqslant k^2 - \frac{4n}{3}k + \frac{n(n-1)}{3} \\
\implies \quad & k \geqslant \frac{2n}{3} - \sqrt{\left(\frac{2n}{3}\right)^2 - \frac{n(n-1)}{3}} \\
\implies \quad & k \geqslant \frac{2n}{3} - \sqrt{\frac{n^2}{9} + \frac{n}{3}} > \frac{2n}{3} - \sqrt{\frac{(n+3)^2}{9}} = \frac{n}{3} - 1
\end{aligned}$$

Thus, we have $k > \frac{n}{3} - 1$, as desired. □

Note that for the lower bound to be tight there must be no red edges, that is each page contains the edges of the convex hull of its vertices. This is the case in the construction for the upper bound in Theorem 2.2, which is given next.

Theorem 2.2. *For any n, the local page number of K_n satisfies* $\mathrm{pn}_\ell(K_n) \leqslant \frac{1}{3}n + 4$.

Proof. We show that if $n = 18k - 3$ for some positive integer k, then we have $\mathrm{pn}_\ell(K_n) \leqslant \frac{1}{3}n = 6k - 1$.

For n of the form $n = 18k - 3 + i$ with $i < 18$ we get a page assignment with locality $6k - 1 + i$ by adding stars of the i additional vertices on an extra page each. A second option is to use a page assignment of $K_{18(k+1)-3}$ and remove $18 - i$ vertices, which yields a page assignment with locality $6(k+1) - 1 = 6k + 5$. By taking the better of these two choices we achieve a locality of at most $n/3 + 4$, as desired.

From now on, we assume that $n = 18k - 3$, i.e. $k = (n+3)/18$. We define the length of an edge as the shorter distance between its two endpoints along the cyclic ordering. The length of edge e is denoted by $\mathrm{len}(e)$. As $n = 18k - 3$ is odd, there are exactly $(n-1)/2 = 9k - 2$ different lengths, each realized by exactly n edges. For each vertex v of K_n, we define a set of k pages, each containing v and together covering exactly one edge of each possible length. These pages are denoted by $O_v(t)$, where $t = 0, \ldots, k-1$. The

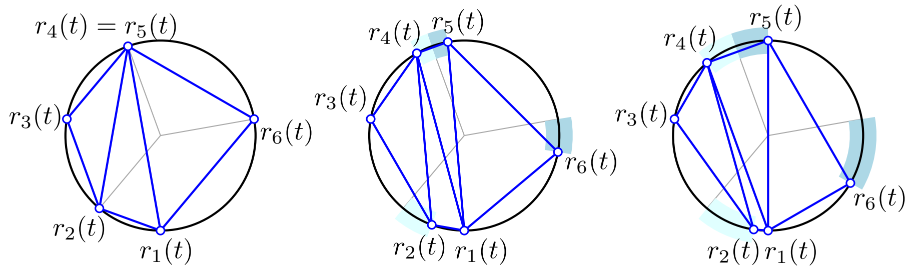


Figure 2.3: Illustrating the outerplanar graph $O(t)$ for $t = 0$ (left), $t \approx k/2$ (middle) and $t = k - 1$ (right).

page $O_v(0)$ contains five vertices and seven edges, while for $t > 0$ the page $O_v(t)$ has six vertices and nine edges. In total this makes the needed $9(k-1) + 7 = 9k - 2 = (n-1)/2$ edge lengths.

Recall that the vertices of K_n are $v_1, \ldots, v_n$ in this cyclic order. All indices are taken modulo n. Below, we describe the pages corresponding to $v_1 = v_{n+1}$. For ease of notation we let $O(t) = O_{v_1}(t)$. For any $t = 0, \ldots, k-1$, the vertices $r_1(t), \ldots, r_6(t)$ of $O(t)$ are the following:

$$\begin{array}{lll} r_1(t) = v_1 = v_{18k-2} & r_2(t) = v_{2k-2t} & r_3(t) = v_{5k+1} \\ r_4(t) = v_{8k-t} & r_5(t) = v_{8k+t} & r_6(t) = v_{13k+2t} \end{array}$$

We refer to Figure 2.3 for an illustration. Note that for $t = 0$, we have $r_4(t) = r_5(t)$ and for all $t \leqslant k$ the vertices $r_1(t), \ldots, r_6(t)$ appear in the order of their indices in the cyclic ordering of K_n. The edges of $O(t)$ are the cycle edges $e_{12}(t) = r_1(t)r_2(t)$, $e_{23}(t) = r_2(t)r_3(t)$, $e_{34}(t) = r_3(t)r_4(t)$, $e_{45}(t) = r_4(t)r_5(t)$, $e_{56}(t) = r_5(t)r_6(t)$, $e_{61}(t) = r_6(t)r_1(t)$, except for $e_{45}(0)$ which would be a loop, and the inner edges $e_{14}(t) = r_1(t)r_4(t)$, $e_{24}(t) = r_2(t)r_4(t)$, $e_{15}(t) = r_1(t)r_5(t)$. For $t = 0$, we have $e_{15}(0) = e_{14}(0)$ and only consider $e_{14}(0)$. Note that $O(t)$ is indeed outerplanar.

We claim that for every length ℓ in the interval $[1, 9k-2]$ there is an edge of length ℓ in some $O(t)$.

$$\begin{array}{rcl} \ell \text{ odd}, \ell \in [1, 2k-1] & : & \operatorname{len}(e_{12}(t)) = (2k-2t) - 1 = 2k - 2t - 1 \\ \ell \text{ even}, \ell \in [1, 2k-1] & : & \operatorname{len}(e_{45}(t)) = (8k+t) - (8k-t) = 2t \\ \ell \in [2k, 3k-1] & : & \operatorname{len}(e_{34}(t)) = (8k-t) - (5k+1) = 3k - t - 1 \\ \ell \text{ odd}, \ell \in [3k, 5k-1] & : & \operatorname{len}(e_{23}(t)) = (5k+1) - (2k-2t) = 3k + 2t + 1 \\ \ell \text{ even}, \ell \in [3k, 5k-1] & : & \operatorname{len}(e_{61}(t)) = (18k-2) - (13k+2t) = 5k - 2t - 2 \\ \ell \in [5k, 6k-1] & : & \operatorname{len}(e_{56}(t)) = (13k+2t) - (8k+t) = 5k + t \\ \ell \in [6k, 7k-1] & : & \operatorname{len}(e_{24}(t)) = (8k-t) - (2k-2t) = 6k + t \\ \ell \in [7k, 8k-1] & : & \operatorname{len}(e_{14}(t)) = (8k-t) - 1 = 8k - t - 1 \\ \ell \in [8k, 9k-2] & : & \operatorname{len}(e_{15}(t)) = (8k+t) - 1 = 8k + t - 1 \end{array}$$

For a vertex v_i and some t we obtain the page $O_{v_i}(t)$ from $O(t)$ by a rotation which maps v_1 to v_i. Hence, for each v_i we get a set $\mathcal{P}_i = \{O_{v_i}(t) \mid t = 0, \dots, k-1\}$ of pages. Let $\mathcal{P}$ denote the set of all defined pages, i.e. $\mathcal{P} = \bigcup_{i=1,\dots n} \mathcal{P}_i$. We claim that $\mathcal{P}$ covers all edges of K_n. Consider an arbitrary edge $v_a v_b$ of length ℓ for some $\ell = 1, \dots, 9k-2$. From the analysis above, we know that there is a unique t and a unique edge $e_{ij}(t) \in O(t)$ with $\operatorname{len}(e_{ij}(t)) = \ell$. There is a rotation which maps $r_i(t)$ to v_a and $r_j(t)$ to v_b or vice versa. If this rotation maps $v_1 = r_1(t)$ to v_c, then $O_{v_c}(t)$ contains the edge $v_a v_b$. Hence, $\mathcal{P}$ is a cover of the edges of K_n with outerplanar graphs.

In $\mathcal{P}_1$ there are $6(k-1)+5 = 6k-1$ vertex-page incidences Hence, the total number of vertex-page incidences in $\mathcal{P}$ is $n(6k-1) = n^2/3$. Due to symmetry, each of the n vertices is incident to exactly $n/3$ pages. This proves that $\operatorname{pn}_\ell(K_n) \leqslant n/3$ whenever n is of the form $n = 18k-3$ for some positive integer k. □

For the upper bound on the union page number, we construct union pages whose connected components are slightly sparser than the pages constructed above. In consequence, we obtain a larger upper bound.

Theorem 2.3. *For any n, the union page number of K_n satisfies* $\operatorname{pn}_\mathrm{u}(K_n) \leqslant \frac{4}{9}n + 18$.

Proof. For any $k > 0$ that is divisible by 3 and $n = 18k$, we prove that $\operatorname{pn}_\mathrm{u}(K_n) \leqslant 4n/9+4$. For any other n, we add stars or use the book embedding for $K_{n'}$, where n' is the smallest integer with $n' \geqslant n$ and $n' = 54m$ for some integer m. The better of these options gives $\operatorname{pn}_\mathrm{u}(K_n) \leqslant 4n/9 + 18$.

The vertices of K_n are denoted by $v_0, \dots, v_{n-1}$ and lie on the circularly closed spine in this ordering. All indices are taken modulo n. Let the *length* $\operatorname{len}(vw)$ of an edge vw denote the shorter distance between v and w along the cyclic vertex ordering. We have $n/2 = 9k$ different lengths, where length $9k$ is realized by $n/2$ edges and all other lengths by n edges. We first define $n/3$ union pages $\mathcal{P}_1, \dots, \mathcal{P}_{n/3}$ that cover 7/9 of all possible lengths and then cover the remaining edges with unions of stars. The union pages $\mathcal{P}_1, \dots, \mathcal{P}_{n/3}$ consist of graphs $G(i,t)$ and $H(i,t)$ defined below. We refer to Figure 2.4 for an illustration. For $t = 0, \dots, k-1$, we define vertices

$$r_1(t) = v_{1+t} \qquad r_2(t) = v_{8k+1-t} \qquad r_3(t) = v_{9k+1+2t} \qquad r_4(t) = v_{12k-t}$$
$$r_5(t) = v_{17k-2t} \qquad s_1(t) = v_{3k} \qquad s_2(t) = v_{8k+1+t}.$$

For $t = 0, \dots, k-1$, the graph $G(1,t)$ is then defined to consist of the vertices $r_1(t), \dots, r_5(t)$ and the edges $e_{12}(t) = r_1(t)r_2(t)$, $e_{13}(t) = r_1(t)r_3(t)$, $e_{14}(t) = r_1(t)r_4(t)$, $e_{15}(t) = r_1(t)r_5(t)$, $e_{23}(t) = r_2(t)r_3(t)$, $e_{34}(t) = r_3(t)r_4(t)$, and $e_{45}(t) = r_4(t)r_5(t)$. The graph $H(1,t)$ consists of a single edge $s_1(t)s_2(t)$ for $t = 0, \dots, k-1$. For $i = 1, \dots, n$, we obtain $G(i,t)$ by rotating $G(1,t)$, i.e. we use the vertices $r'_h = v_j + i - 1$ instead of $r_h = v_j$ for $h = 1, \dots, 5$. Similarly, $H(i,t)$ is obtained from $H(1,t)$ by rotation.

We claim that for $i = 1, \dots, n$ and $t = 0, \dots, k-1$, the graphs $G(i,t)$ and $H(i,t)$ cover all edges of length ℓ with $\ell = k, \dots, 3k$ and $\ell = 4k+1, \dots, 9k$. We only consider the graphs for $i = 1$ and observe that each length is covered at least once. By symmetry,

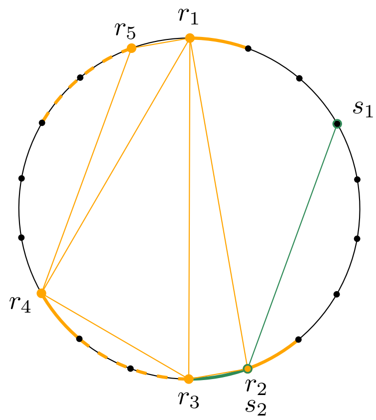


Figure 2.4: The graphs $G(1,0)$ (orange) and $H(1,0)$ (green). The orange, respectively green, arcs indicate the movement of the vertices for $t = 0, \ldots, k-1$.

all edges of the claimed lengths are covered. Recall that k is divisible by 3. An edge of length ℓ can be found in $G(1,t)$ or $H(1,t)$ as follows:

$$\begin{array}{rlrcl}
\ell \equiv 0 \mod 3, & \ell \in [k, 3k] & : & \operatorname{len}(e_{23}(t)) & = & k + 3t \\
\ell \equiv 1 \mod 3, & \ell \in [k, 3k] & : & \operatorname{len}(e_{15}(t)) & = & k + 1 + 3t \\
\ell \equiv 2 \mod 3, & \ell \in [k, 3k] & : & \operatorname{len}(e_{34}(t)) & = & 3k - 1 - 3t \\
 & \ell \in [4k+1, 5k] & : & \operatorname{len}(e_{45}(t)) & = & 5k - t \\
 & \ell \in [5k+1, 6k] & : & \operatorname{len}(s_1(t)s_2(t)) & = & 5k + 1 + t \\
\ell \text{ even}, & \ell \in [6k+1, 8k] & : & \operatorname{len}(e_{12}(t)) & = & 8k - 2t \\
\ell \text{ odd}, & \ell \in [6k+1, 8k] & : & \operatorname{len}(e_{14}(t)) & = & 6k + 1 + 2t \\
 & \ell \in [8k+1, 9k] & : & \operatorname{len}(e_{13}(t)) & = & 9k - t
\end{array}$$

We now use the given graphs to define the union pages $\mathcal{P}_1, \ldots, \mathcal{P}_{n/3}$. For $i = 1, \ldots, n/3$, we define $\mathcal{P}_i$ as the union of $G(i,t)$, $G(i+n/3,t)$, and $G(i+2n/3,t)$ for $t = 0, \ldots, k-1$ and $H(i,t)$, $H(i+n/3,t)$, and $H(i+2n/3,t)$ for $t = 1, \ldots, k-1$. To observe that $\mathcal{P}_1, \ldots, \mathcal{P}_{n/3}$ are indeed union pages, we list for each vertex $v_1, \ldots, v_n = v_0$ by which vertices of $\mathcal{P}_1$ it is hit. For ease of presentation, we assume that k is even. For odd k, swap *odd* and *even* in the listing below.

$$\begin{array}{llllll}
v_1, \ldots, v_k, & v_{6k+1}, \ldots, v_{7k}, & v_{12k+1}, \ldots, v_{13k} & : & r_1(t) & \\
v_{k+2}, \ldots, v_{2k+1}, & v_{7k+2}, \ldots, v_{8k+1}, & v_{13k+2}, \ldots, v_{14k+1} & : & r_2(t) & \\
v_{2k+2}, \ldots, v_{3k}, & v_{8k+2}, \ldots, v_{9k}, & v_{14k+2}, \ldots, v_{15k} & : & s_2(t) & t \geqslant 1 \\
v_{3k}, & v_{9k}, & v_{15k} & : & s_1(t) & t \geqslant 1 \\
v_{3k+1}, \ldots, v_{5k}, & v_{9k+1}, \ldots, v_{11k}, & v_{15k+1}, \ldots, v_{17k} & : & r_3(t) & \text{odd indices} \\
v_{3k+1}, \ldots, v_{5k}, & v_{9k+1}, \ldots, v_{11k}, & v_{15k+1}, \ldots, v_{17k} & : & r_5(t) & \text{even indices} \\
v_{5k+1}, \ldots, v_{6k}, & v_{11k+1}, \ldots, v_{12k}, & v_{17k+1}, \ldots, v_{18k} & : & r_4(t) &
\end{array}$$

The vertices v_{3k}, v_{9k}, and v_{15k} are hit by $s_1(t)$ for all t and by $s_2(k-1)$, whereas all other vertices are hit for at most one t. In particular, each vertex that is contained in some

$G(j,t)$ is not contained in any other component of $\mathcal{P}_1$. In contrast, the union of the graphs $H(j,t)$, $j = i, i+n/3, i+2n/3$ and $t = 1,\dots,k-1$, forms a single connected component whose edges do not cross. Hence, each connected component of $\mathcal{P}_1$ is crossing-free and by symmetry $\mathcal{P}_1,\dots,\mathcal{P}_{n/3}$ are union pages. None of the graphs $H(i,0)$, for $i = 1,\dots,n$, is contained in the pages $\mathcal{P}_1,\dots,\mathcal{P}_{n/3}$, i.e. the edges of length $5k+1$ are left to cover. We cover these edges with two additional union pages, each containing a perfect matching.

Finally, we define union pages consisting of disjoint unions of stars to cover the remaining edge lengths $1,\dots,k-1$ and $3k+1,\dots,4k$. For this, we define stars S_i consisting of the edges $v_iv_{i+1},\dots,v_iv_{i+k-1}$ and T_i consisting of the edges $v_iv_{i+3k+1},\dots,v_iv_{i+4k-1}$ for $i = 1,\dots,n$. For $i = 1,\dots,k$, the union page $\mathcal{S}_i$, respectively $\mathcal{T}_i$, is defined as the union of S_{i+jk}, respectively T_{i+jk}, where $j = 0,\dots,17$. As each union page is the disjoint union of stars, each connected component is crossing-free. The union pages $\mathcal{S}_1,\dots,\mathcal{S}_k$ and $\mathcal{T}_1,\dots,\mathcal{T}_k$ cover all remaining edge lengths except for the length $4k$. Again, we use two additional union pages containing a perfect matching each. Summing up, we have $n/3+2+2k+2 = 4n/9+4$ union pages. □

Comparing the presented construction with the lower bound of Theorem 2.1, we remark that we have $n^2/18+\Theta(n)$ connected components and $\Theta(n^2)$ red edges due to the stars. To obtain an upper bound of $n/3$, however, we need exactly $n^2/18$ connected components that are partitioned into $n/3$ union pages. In this case, each union page uses all vertices and each connected component is a maximal outerplanar graph, i.e. there are no red edges. It remains open whether such a book embedding exists. Note that the book embedding constructed for Theorem 2.2 consists of $n^2/18$ pages and has no red edges. Partitioning these pages into union pages containing $n/6$ outerplanar graphs each thus would suffice to prove an upper bound of $n/3+\Theta(1)$ for the union page number of K_n.

2.2 Local and Union Queue Numbers

In this section, we investigate queue layouts of complete graphs. We first give a lower bound on the local queue number and then show that it is tight up to a constant additive term both for the local queue number and the union queue number.

Theorem 2.4. *For any n, we have* $\mathrm{qn}_\ell(K_n) > (1-\frac{1}{\sqrt{2}})(n-\frac{1}{2})$.

Proof. Consider a k-local queue layout $\mathcal{Q}$ of K_n. Without loss of generality, each edge is contained in exactly one queue. Moreover, the vertices are ordered $v_1 \prec \cdots \prec v_n$, and the length of an edge v_iv_j is defined as $|i-j|$. For any edge $e = v_iv_j$ with $i<j$, consider the queue $Q \in \mathcal{Q}$ containing e. We call e *left-longest* if there is no edge in Q that is longer than e and has the same right endpoint as e, i.e. Q contains no edge $v_{i'}v_j$ with $i' < i$. Similarly, we call $e = v_iv_j \in Q$ *right-shortest* if there is no edge in Q that is shorter than e and has the same left endpoint as e, i.e. Q contains no edge $v_iv_{j'}$ with $i < j' < j$. We observe that

(i) every edge of K_n is left-longest or right-shortest (or both).

In fact, if $v_iv_j \in Q$ is of neither type, then Q would contain two nesting edges $v_{i'}v_j$ and $v_iv_{j'}$ with $i' < i < j' < j$, and hence Q would not be a queue.

For each vertex v_i let ℓ_i, respectively r_i, denote the number of left-longest edges whose right endpoint is v_i, respectively the number of right-shortest edges whose left endpoint is v_i. That is,

$$\ell_i = \#\{v_a \in V(K_n) \mid a < i \text{ and } v_av_i \text{ left-longest}\} \text{ and}$$
$$r_i = \#\{v_b \in V(K_n) \mid i < b \text{ and } v_iv_b \text{ right-shortest}\}.$$

Further, let b_i denote the number of queues in $\mathcal{Q}$ with at least one edge whose right endpoint is v_i *and* at least one edge whose left endpoint is v_i. That is,

$$b_i = \#\{Q \in \mathcal{Q} \mid \exists\, a, b \text{ with } a < i < b \text{ and } v_av_i, v_iv_b \in Q\}.$$

We can then write the number of queues in $\mathcal{Q}$ containing the vertex v_i in terms of ℓ_i, r_i and b_i. If $Q \in \mathcal{Q}$ contains an edge incident to v_i, then it contains a left-longest or a right-shortest or both, i.e. the contribution of Q to $\ell_i + r_i - b_i$ is exactly one. We conclude that

(ii) vertex v_i has incident edges in exactly $\ell_i + r_i - b_i$ queues in $\mathcal{Q}$.

As every vertex is in at most k queues, we have $b_i \leqslant k$ for $i = 1, \ldots, n$. In addition, every vertex v_i is the right endpoint of at most $i - 1$ edges and thus $b_i \leqslant i - 1$. Similarly, v_i is the left endpoint of at most $n - i$ edges and thus $b_i \leqslant n - i$. Together, we know that

(iii) for every vertex v_i we have $b_i \leqslant \min\{i - 1, n - i, k\}$.

Using the above we calculate

$$\begin{aligned} kn &\geqslant \sum_{i=1}^{n} \#\{Q \in \mathcal{Q} \mid v_i \in V(Q)\} \overset{(ii)}{=} \sum_{i=1}^{n} (\ell_i + r_i - b_i) \\ &\overset{(i)}{\geqslant} |E(K_n)| - \sum_{i=1}^{n} b_i \overset{(iii)}{\geqslant} \binom{n}{2} - \sum_{i=1}^{k}(i-1) - \sum_{i=n-k+1}^{n}(n-i) - (n-2k)k \\ &= \binom{n}{2} - 2\binom{k}{2} - (n-2k)k, \end{aligned}$$

which can be solved for the desired $k \geqslant (1 - 1/\sqrt{2})(n - 1/2)$ as follows.

$$\begin{aligned} & kn \geqslant \binom{n}{2} - 2\binom{k}{2} - (n-2k)k \\ \iff \quad & 0 \geqslant k^2 + (1-2n)k + \binom{n}{2} \\ \implies \quad & k \geqslant n - \frac{1}{2} - \sqrt{\left(n - \frac{1}{2}\right)^2 - \binom{n}{2}} \\ \implies \quad & k > n - \frac{1}{2} - \sqrt{\left(n - \frac{1}{2}\right)^2 - \frac{1}{2}\left(n - \frac{1}{2}\right)^2} = \left(1 - \frac{1}{\sqrt{2}}\right)\left(n - \frac{1}{2}\right) \qquad \square \end{aligned}$$

We conclude this chapter by proving a matching upper bound both for the local queue number and the union queue number. We first make an observation that might be of independent interest. We follow the proof of the same statement for book embeddings in [38]. However, we stress that the vertex ordering is arbitrary as we make use of this in the proof of Lemma 2.7. The maximum average degree $\mathrm{mad}(G)$ of a graph G is defined by $\mathrm{mad}(G) = \max\{2\,|E(H)|\,/\,|V(H)| : H \subseteq G, H \neq \emptyset\}$.

Proposition 2.5. *Every graph G with maximum average degree $\mathrm{mad}(G)$ admits a $(\mathrm{mad}(G)+2)$-union queue layout with any vertex ordering.*

Proof. Nash-Williams [42] proved that every graph can be partitioned into $\mathrm{mad}(G)/2+1$ forests. Each forest, in turn, can be partitioned into two star forests [5]. Choosing an arbitrary vertex ordering, each of the $\mathrm{mad}(G)+2$ star forests is a union queue as the edges of a star cannot nest. □

Using this proposition, we set out to construct a queue layout whose queues can be merged into few union queues. .

Theorem 2.6. *For any $n \geqslant 0$, we have*

$$\mathrm{qn}_\ell(K_n) \leqslant \left\lceil 1 - \frac{1}{\sqrt{2}} \right\rceil n + 11 \text{ and } \mathrm{qn}_\mathrm{u}(K_n) \leqslant \left\lceil 1 - \frac{1}{\sqrt{2}} \right\rceil n + 42.$$

We prove that whenever $k \geqslant (1 - 1/\sqrt{2})(n+1)$, there is a $(k+11)$-local queue layout and a $(k+42)$-union queue layout of K_{n+1}. Let $v_1 \prec \cdots \prec v_{n+1}$ be a fixed vertex ordering of K_{n+1}. For ease of presentation, we model the edge set of K_{n+1} as a point set T_n in $\mathbb{Z}^2$ with triangular shape defined by

$$T_n = \{(x, y) \in \mathbb{Z}^2 \mid x + y \leqslant n + 1;\ x \geqslant 1;\ y \geqslant 1\}.$$

Element (x, y) of T_n corresponds to edge $v_{n+2-y}v_x$ in K_{n+1} and conversely edge v_iv_j in K_{n+1} with $i > j$ corresponds to element $(j, n+2-i)$ in T_n. Two edges v_iv_j with $i > j$ and $v_{i'}v_{j'}$ with $i' > j'$ nest if and only if the corresponding elements $(j, n+2-i)$ and $(j', n+2-i')$ in T_n are comparable in the strict dominance order of $\mathbb{Z}^2$ (i.e. coordinate-wise strict inequalities of points). Hence, an edge set $Q \subseteq E(K_{n+1})$ forms a queue if and only if the corresponding points in T_n form a weakly monotonically decreasing chain, see Figure 2.5.

A vertex v_i of K_{n+1} corresponds to column i and row $n+2-i$ in T_n. We call the union of row i and column $n+2-i$ the *hook* of vertex v_i. If H is the hook of vertex v_i and Q is a queue corresponding to chain $C \subseteq T_n$, then vertex v_i is contained in queue Q if and only if $H \cap C \neq \emptyset$. For our construction of a $(k+11)$-local queue assignment of K_{n+1}, we use the equivalent model of covering the triangular point set T_n with monotone chains such that no hook intersects more than $k+11$ chains.

Analogously to union queues, we call a subset $S \subseteq T_n$ a *union chain* if there is a partition of S into weakly monotonically decreasing chains $C_1, \ldots, C_m$ such that each hook intersects at most one of the chains $C_1, \ldots, C_m$. To prove Theorem 2.6, we partition T_n into $k+42$ union chains and therefore get a $(k+42)$-union queue layout for K_{n+1}.

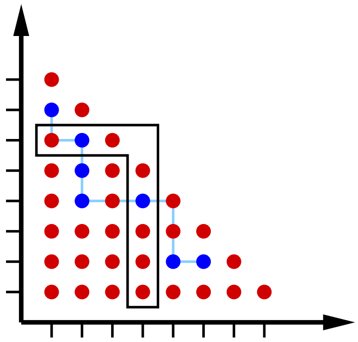


Figure 2.5: An example showing the triangle T_8 corresponding to K_9 and the hook of vertex 4. The blue entries represent a queue with 7 edges, the light blue zig-zag emphasizes that the blue edges are non-nesting.

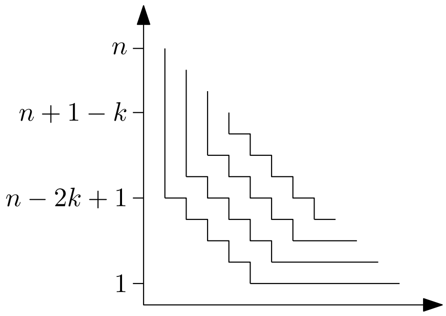


Figure 2.6: Chains $L_1, \ldots, L_k$. The bottom left triangle T_{n-2k} is covered by vertical and horizontal chains as shown in Figure 2.7.

Lemma 2.7. *For any integer $n \geqslant 0$ and any integer $k \geqslant (1 - 1/\sqrt{2})(n+1)$, the points of T_n can be partitioned into $k + 42$ union chains. In addition, the points of T_n can be partitioned into chains such that each hook intersects at most $k + 11$ chains.*

Proof. First, we define weakly monotonically decreasing chains that cover T_n such that no hook intersects more than $k + 11$ chains. We then partition the constructed chains into sets of chains that form the basis for our union chains. We assume that n is even and that k is the smallest even integer with $k \geqslant (1 - 1/\sqrt{2})(n+1)$. To compensate for this assumption, we construct chains such that each hook intersects at most $k + 9$ chains and a partition of T_n into $k + 40$ union chains. We need $n \geqslant 3k$ for the construction, which is the case for $n \geqslant 42$. For smaller n, however, we have $n/2 \leqslant k + 9$, so an $(n/2)$-queue layout of K_{n+1} gives the desired partition of K_{n+1}, respectively T_n, into queues, respectively chains.

We start by defining a family $\mathcal{L}$ of k chains $L_1, \ldots, L_k$, illustrated in Figure 2.6. Chain L_i is composed of three blocks. The first block consists of the $2(k - i + 1)$ topmost elements in column i of T_n. The second block starts at the lowest element of the first block, continues with a right and down alternation for $2(n - 2(k - 1))$ steps, and ends in row i. The last block consists of the $2(k-i+1)$ rightmost elements in row i. Formally, for $i = 1, \dots, k$ we set

$$\begin{aligned} L_i = &\{(i,y) \in T_n \mid n+1-i \geqslant y \geqslant n-2k+i\} \\ &\cup \{(x,y) \in T_n \mid x,y \geqslant i \text{ and } n-2(k-i) \leqslant x+y \leqslant n-2(k-i)+1\} \\ &\cup \{(x,i) \in T_n \mid n-2k+i \leqslant x \leqslant n+1-i\}. \end{aligned}$$

The chains of $\mathcal{L}$ cover all points of T_n except for the bottom left triangle T_{n-2k}. The remaining points are now covered by chains containing only points of a single column or row. We refer to these chains as *vertical* and *horizontal* chains, respectively. Note that vertical and horizontal chains correspond to stars in K_{n+1}. We only define families $\mathcal{A}, \dots, \mathcal{G}$ containing vertical chains, the horizontal chains $\mathcal{A}', \dots, \mathcal{G}'$ are then defined symmetrically, i.e. $\mathcal{M}' = \{(y,x) \in T_n \mid (x,y) \in \mathcal{M}\}$ for $\mathcal{M} = \mathcal{A}, \dots, \mathcal{G}$. The resulting layout of T_{n-2k} is illustrated in Figure 2.7.

The families $\mathcal{A}, \dots, \mathcal{D}$ cover the bottom left square $S_{n/2-k} = \{(x,y) \in T_n \mid 1 \leqslant x, y \leqslant n/2-k\}$. For this, let $\mathcal{A}$ consist of chains $A_1, \dots, A_{n-3k}$ and $\hat{A}_1, \dots, \hat{A}_{n-3k}$ with $A_i \cup \hat{A}_i$ having size $(n-3k)/2$ or $(n-3k)/2+1$ for $i = 1, \dots, n-3k$. The chains A_i consist of points in column i starting with the bottommost point in row i. Chains A_i with points above $y = n-3k$ continue as $\hat{A}_i$ from the bottom. Note that $\hat{A}_i$ is empty for $i \leqslant (n-3k)/2$. For $i = 1, \dots, n-3k$ we define

$$\begin{aligned} A_i &= \{(i,y) \in T_n \mid i \leqslant y < (n-3k)/2+i \text{ and } y \leqslant n-3k\} \text{ and} \\ \hat{A}_i &= \{(i,y) \in T_n \mid 1 \leqslant y \leqslant i-(n-3k)/2\}. \end{aligned}$$

The defined chains together with the chains in $\mathcal{A}'$ cover all points whose x- and y-coordinates are at most $n-3k$. Some points, however, are covered twice. We may choose any of the two covering chains for the respective points to obtain a partition.

Family $\mathcal{B}$ is located to the right of $\mathcal{A}$ and consists of chains $B_1, \dots, B_{(4k-n)/2}$ and $\hat{B}_1, \dots, \hat{B}_{(4k-n)/2}$. Above $\mathcal{A}$, we have chains $C_1, \dots, C_{n-3k}$ and $\hat{C}_1, \dots, \hat{C}_{n-3k}$ forming family $\mathcal{C}$. Considering the y-coordinates of the bottommost points of the chains B_i in $\mathcal{B}$, we have a slope $s = \frac{n-3k}{(4k-n)/2}$ ($= \sqrt{2}$ for $k = 1-1/\sqrt{2}$). Symmetrically, the slope is $1/s$ in $\mathcal{C}$. For $i = 1, \dots, (4k-n)/2$ and $j = 1, \dots, n-3k$ we define

$$\begin{aligned} B_i &= \{(n-3k+i,y) \in T_n \mid \lfloor si \rfloor \leqslant y \leqslant \lceil si \rceil + (3n-10k)/2 \text{ and } y \leqslant n-3k\}, \\ \hat{B}_i &= \{(n-3k+i,y) \in T_n \mid 1 \leqslant y \leqslant \lceil si \rceil + (3n-10k)/2 - (n-3k)\}, \\ C_j &= \{(j,y) \in T_n \mid n-3k+\lfloor j/s \rfloor \leqslant y \leqslant 3(n-3k)/2 + \lceil j/s \rceil \text{ and} \\ &\qquad n-3k < y \leqslant n-2k\}, \text{ and} \\ \hat{C}_j &= \{(j,y) \in T_n \mid n-3k < y \leqslant 3(n-3k)/2 + \lceil j/s \rceil - (n-2k)\}. \end{aligned}$$

Note that $\hat{B}_i$ and $\hat{C}_j$ are empty for $i \leqslant (n-3k)/2$ and $j \leqslant (3n-10k)/2$. The chains in $\mathcal{B} \cup \mathcal{C}'$, and symmetrically $\mathcal{B}' \cup \mathcal{C}$, form a rectangle containing $(n-3k)(4k-n)/2$ points. Again, we choose any chain for points that are covered by multiple chains.

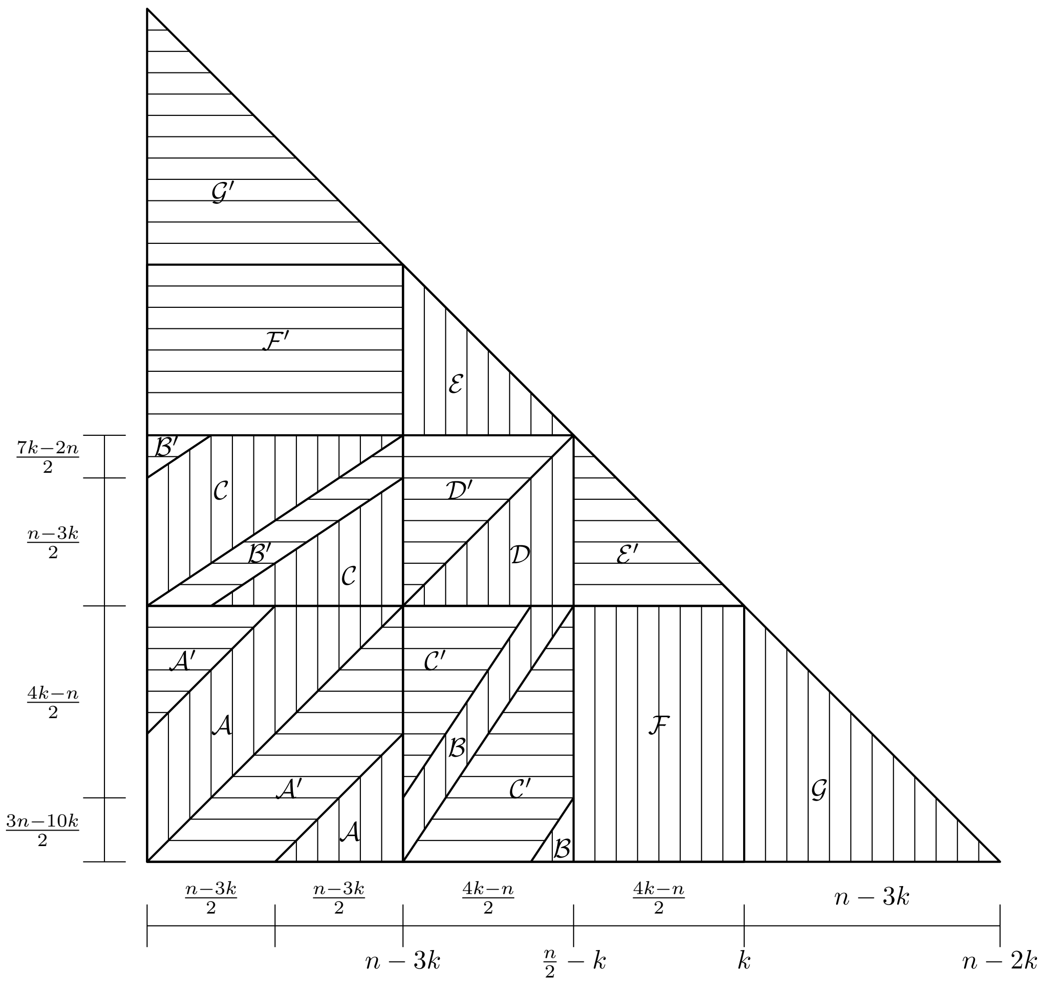


Figure 2.7: Triangle T_{n-2k} is covered by families $\mathcal{A}, \ldots, \mathcal{G}$ and $\mathcal{A}', \ldots, \mathcal{G}'$ of vertical, respectively horizontal, chains

Family $\mathcal{D}$, together with the corresponding horizontal chains in $\mathcal{D}'$, accomplishes the square $S_{n/2-k}$. For this, we have $(4k-n)/2$ chains $D_1, \dots, D_{(4k-n)/2}$, where D_i has size i. For $i = 1, \dots, (4k-n)/2$ let

$$D_i = \{(n-3k+i, y) \in T_n \mid n-3k < y \leqslant n-3k+i\}.$$

Finally, we cover to remaining two triangles with three families $\mathcal{E}, \mathcal{F}$, and $\mathcal{G}$ of vertical chains. Families $\mathcal{E}$ and $\mathcal{G}$ consist of chains $E_1, \dots, E_{(4k-n)/2}$, respectively $G_1, \dots, G_{n-3k}$, each filling a triangle, whereas the chains $F_1, \dots, F_{(4k-n)/2} \in \mathcal{F}$ all have size $n-3k$ and cover a rectangle. For $i = 1, \dots, (4k-n)/2$ and $j = 1, \dots, n-3k$ we define

$$\begin{aligned} E_i &= \{(n-3k+i, y) \in T_n \mid n/2-k < y \leqslant k-(i-1)\}, \\ F_i &= \{(n/2-k+i, y) \in T_n \mid 1 \leqslant y \leqslant n-3k\}, \text{ and} \\ G_j &= \{(k+j, y) \in T_n \mid 1 \leqslant y \leqslant n-3k-(j-1)\}. \end{aligned}$$

The chains in $\mathcal{E}'$, $\mathcal{F}$, and $\mathcal{G}$ together cover the all points (x, y) of T_{n-2k} with $x > n-3k$, and symmetrically $\mathcal{E}$, $\mathcal{F}'$, and $\mathcal{G}'$ cover the triangle above $S_{n/2-k}$.

Next we show that each hook intersects at most $k+9$ chains. Recall that a hook of a vertex is the union of the row and the column representing that vertex. We first count for each row $y = 1, \dots, n$ and each column $x = 1, \dots, n$ the number of intersecting chains and then add up the results to obtain the number of intersecting chains for each hook. We start by counting the vertical chains, which intersect rows $1, \dots, k$. Note that no vertical chain intersects any row above $y = k$. If $n = 3k$, then $\mathcal{B}$ and $\mathcal{C}$ are empty, so we may assume $s > 0$ when counting these chains.

$\mathcal{A}$: For $y = 1, \dots, (n-3k)/2$, row y intersects chains $A_1, \dots, A_y$ and $\hat{A}_{(n-3k)/2+y}, \dots, \hat{A}_{n-3k}$, which sums up to $(n-3k)/2+1$ chains. The chains $\hat{A}_1, \dots, \hat{A}_{n-3k}$ do not contain any points above row $(n-3k)/2$ and thus row $y = (n-3k)/2+1, \dots, n-3k$ intersects exactly $A_{y-(n-3k)/2+1}, \dots A_y$, i.e. $(n-3k)/2$ chains.

$\mathcal{B}$: For the upcoming calculation, note that $(4k-n)^2 \geqslant 2(n-3k)^2$ for $k \geqslant (1-1/\sqrt{2})n$. For row $y = 1, \dots, (3n-10k)/2$, we have chains B_i for $i \leqslant (y+1)/s$ and $\hat{B}_i$ for $(4k-n)/2 \geqslant i > \frac{y-1-(n-3k)-(3n-10k)/2}{s} = \frac{y-1+(4k-n)/2}{s}$. This sums up to

$$\frac{2}{s} - \frac{(4k-n)^2}{2(n-3k)^2}\frac{n-3k}{2} + \frac{(4k-n)}{2} \leqslant \frac{(4k-n)}{2} - \frac{n-3k}{2} + \frac{2}{s} \overset{(*)}{<} \frac{7k-2n}{2} + 4$$

chains. For $y = (3n-10k)/2+1, \dots, n-3k$, each row intersects at most

$$\begin{aligned} \frac{y+1}{s} - \frac{y-(3n-10k)/2-1}{s} &= \frac{3n-10k}{2s} + \frac{2}{s} = (n-3k-\frac{4k-n}{2})/s + \frac{2}{s} \\ &= \frac{4k-n}{2} - \frac{(4k-n)^2}{2(n-3k)^2}\frac{n-3k}{2} + \frac{2}{s} \leqslant \frac{4k-n}{2} - \frac{n-3k}{2} + \frac{2}{s} \overset{(*)}{<} \frac{7k-2n}{2} + 4 \end{aligned}$$

chains. For $(*)$, we remark that $2/s < 4$ holds for all $n \geqslant 104$. For smaller n, recall that there are only $(4k-n)/2$ columns containing vertical chains in $\mathcal{B}$. That is, if $(7k-2n)/2+4 > (4k-n)/2$, then $(7k-2n)/2+4$ certainly upper-bounds the number of chains in $\mathcal{B}$ that intersect any row. However, if $n < 104$ and $(7k-2n)/2+4 < (4k-n)/2$, then we also have $2/s < 4$.

$\mathcal{C}$: For $y = 1, \ldots, (n-3k)/2$, chain C_i intersects row $n-3k+y$ only if $i \leqslant (y+1)s$ and $\hat{C}_i$ intersects row $n-3k+y$ only if $n-3k \geqslant i > (y-1-(7k-2n)/2)$. So we have at most

$$(y+1)s + n - 3k - (y-1+(7k-2n)/2)s$$
$$= 2s + \frac{2(n-3k)^2}{(4k-n)^2}\frac{4k-n}{2} \leqslant 2s + \frac{4k-n}{2} \leqslant \frac{4k-n}{2} + 3$$

chains. For $y = (n-3k)/2+1, \ldots, (4k-n)/2$, we have C_i in row $n-3k+y$ for $(y-1-(n-3k)/2)s < i < (y+1)s$, which upper-bounds the number of chains by

$$((n-3k)+2)s = \frac{2(n-3k)^2}{(4k-n)^2}\frac{4k-n}{2} + 2s \leqslant \frac{4k-n}{2} + 3.$$

$\mathcal{D}$: The vertical chains in $\mathcal{D}$ intersect the rows $n-3k+1, \ldots, n/2-k$. Precisely, for $y = 1, \ldots, (4k-n)/2$, row $n-3k+y$ intersects the chains $D_y, \ldots, D_{(4k-n)/2}$, i.e. $(4k-n)/2-(y-1)$ chains.

$\mathcal{E}$: Similarly, the vertical chains in $\mathcal{E}$ intersect the rows $n/2-k+1, \ldots, k$. For $y = 1, \ldots, (4k-n)/2$, row $n/2-k+y$ intersects the chains $E_1, \ldots, E_{(4k-n)/2-(y-1)}$, i.e. $(4k-n)/2-(y-1)$ chains.

$\mathcal{F}$: All chains of $\mathcal{F}$ intersect the rows $1, \ldots, n-3k$ exactly once, that is we have $(4k-n)/2$ additional vertical chains in these rows.

$\mathcal{G}$: Row $y = 1, \ldots, n-3k$ intersects the chains $G_1, \ldots, G_{n-3k-(y-1)}$, i.e. $n-3k-(y-1)$ chains.

Summing up the number of vertical chains intersecting the rows $y = 1, \ldots, n-3k$, we get at most $k+5-y$ chains from $\mathcal{A}, \mathcal{B}, \mathcal{F}$, and $\mathcal{G}$. Additionally we have at most four horizontal chains in $\mathcal{A}'$ and $\mathcal{C}'$. Rows $y = n-3k+1, \ldots, n/2-k$ intersect vertical chains in $\mathcal{C}$ and $\mathcal{D}$, which sums up to $k-y+4$ chains. There are at most four additional horizontal chains in $\mathcal{B}', \mathcal{D}'$, and $\mathcal{E}'$ intersecting these rows. Finally, rows $y = n/2-k+1, \ldots, k$ intersect only vertical chains in $\mathcal{E}$ and horizontal chains in $\mathcal{F}'$, summing up to $k-y+2$ chains. Together, we have at most $k-y+9$ vertical and horizontal chains in row $y = 1, \ldots, k$ and only one horizontal chain from $\mathcal{G}'$ in row $k+1, \ldots, n-2k$. Symmetrically, there are at most $k-x+9$ vertical and horizontal chains intersecting column $x = 1, \ldots, k$ and one vertical chain in each column $k+1, \ldots, n-2k$.

Now, we count the number of chains intersecting the hook H_x of vertex v_x for $x = 1, \ldots, n+1$. Recall that hook H_x corresponds to column x and row $n+2-x$. For $x = 1, \ldots, k$, hook H_x intersects $L_1, \ldots, L_x$ and $k-x+9$ vertical and horizontal chains. Note that row $n+2-x$ is above row $n-2k$ and thus does not intersect any vertical or horizontal chains in T_{n-2k}. Hooks $H_{k+1}, \dots, H_{n/2+1}$ intersect all k chains in $\mathcal{L}$ and at most one vertical chain from $\mathcal{G}$. By symmetry all hooks intersect at most $k+9$ chains.

The chains defined above induce a $(k+9)$-local queue layout. To obtain a $(k+40)$-union queue layout, we partition the set of all chains into sets of chains $\mathcal{S}_1, \dots, \mathcal{S}_{k+6}$ with $L_i \in \mathcal{S}_i$ for $i = 1, \dots, k$.

We first introduce some notions that allow us to transform a set of chains into a union chain and outline the rest of the proof. Consider some set $\mathcal{S} = \{S_1, \dots, S_m\}$ of chains. Note that $\bigcup_{i \in [m]} S_i$ is not necessarily a union chain as two chains may intersect the same hook. We call the set of all hooks that intersect at least two chains of $\mathcal{S}$ the *common hooks* of $\mathcal{S}$. If $\mathcal{S}$ has no common hooks, then $\mathcal{S}$ is already a union chain. Otherwise, we assign each of the common hooks to at most one chain and remove points from the chains of $\mathcal{S}$ until each common hook intersects only the chain to which it is assigned. In particular, each hook then intersects at most one chain and thus the resulting chains form a union chain. Note that a hook that does not intersect the chain to which it is assigned yields a vertex that has no incident edges in the corresponding queue of K_{n+1}. Consider an assignment of common hooks to chains of $\mathcal{S}$. A point that is contained in some chain $S \in \mathcal{S}$ and in some common hook that is not assigned to S is called a *bad point*. Removing all bad points yields chains that together form a union chain.

We now aim to define $\mathcal{S}_1, \dots, \mathcal{S}_{k+6}$ such that the resulting bad points can be covered by a constant number of union chains. For this, we associate each vertical (horizontal) chain C with the interval $I_C \subseteq [k]$ that consists of the y-coordinates (x-coordinates) of the points that are contained in C. We say two vertical (horizontal) chains *overlap* if the corresponding intervals are not disjoint.

Consider the set $\mathcal{S}_i$ for some $i = 1, \dots, k+6$. We add vertical and horizontal chains to $\mathcal{S}_i$ such that

(i) chains in $\mathcal{S}_i$ do not overlap,

(ii) the y-coordinates (x-coordinates) of all points in vertical (horizontal) chains in $\mathcal{S}_i$ are smaller than i, and

(iii) there is no vertical (horizontal) chain in $\mathcal{S}_i$ in column (row) i.

We first assume that Conditions (i) to (iii) hold and show that they can indeed be satisfied at the end of the proof. We merge vertical (horizontal) chains of $\mathcal{S}_i$ that are in the same column (row) into a single chain. Next, we assign common hooks to chains and use the three conditions to show that 34 union chains suffice to cover all bad points. For the analysis of bad points, we concentrate on vertical chains. The result for horizontal chains follows symmetrically.

Consider a vertical chain $C \in \mathcal{S}_i$ in column x and let H_C denote the hook that contains C. Recall that vertical chains of $\mathcal{S}_i$ that are in the same column are merged into a single chain. If H_C is a common hook, we assign it to C. Note that each hook either contains vertical chains or horizontal chains, and thus no hook is assigned to multiple chains. See Figure 2.8 for an illustration of bad points in $\mathcal{S}_i$. Assigning H_C to C implies that all points of H_C that are contained in some other chain of $\mathcal{S}_i$ are bad points. We say that

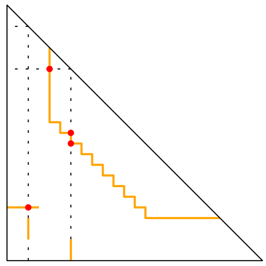


Figure 2.8: A set $\mathcal{S}_i$ with two vertical chains and their hooks (dashed). Bad points are marked red.

these bad points are *caused by* C. We now analyze how many bad points are caused by C. If $x > i$, then H_C intersects L_i in at most four points, i.e. we have at most four bad points in $H_C \cap L_i$. In this case, no horizontal chain in $\mathcal{S}_i$ intersects H_C by Condition (ii). Otherwise we have $x < i$ (due to Condition (iii)), and Condition (i) ensures that H_C intersects at most one horizontal chain $C' \in \mathcal{S}_i$. We thus have at most one bad point in $H_C \cap C'$. In either case, C causes at most four bad points.

It is left to show that each bad point is caused by some vertical or horizontal chain. For this, we keep $\mathcal{S}_i$ fixed and consider a hook H whose column x intersects the triangle T_{n-2k}, i.e. with $x \leqslant n - 2k$. We show that each bad point in H is caused by a vertical chain. The result for hooks whose row intersect T_{n-2k} follows symmetrically with horizontal chains. Hooks that do not intersect T_{n-2k} contain no bad points since L_i is the only chain in $\mathcal{S}_i$ that intersects such a hook. Note that the row of H is above T_{n-2k} and thus does not intersect any vertical or horizontal chains. If $x < i$, then H does not intersect L_i. It intersects at most one horizontal chain due to Condition (i). That is, if H contains a bad point, then it also contains a vertical chain causing this bad point. If $x = i$, then H intersects neither vertical nor horizontal chains by Conditions (ii) and (iii). Thus, L_i is the only intersected chain in $\mathcal{S}_i$ and there are no bad points. If $x > i$, then H intersects no horizontal chains in $\mathcal{S}_i$ by Condition (ii). Hence, if there are bad points, then they are caused by a vertical chain in column x.

We are now ready to cover the bad points by a constant number of union chains. Let $G \subseteq K_{n+1}$ denote the graph that is induced by all bad points. For a vertical or horizontal chain $C \subseteq T_{n-2k}$, let $v_C \in V(K_{n+1})$ denote the vertex that is represented by hook H_C. We orient the edges of G such that every edge whose corresponding bad point is caused by chain C is oriented away from v_C. Each hook contains at most four vertical or horizontal chains (see Figure 2.7 and recall that each hook either contains vertical chains or horizontal chains). In addition, each chain causes at most four bad points. Thus, we have a 16-orientation of G, i.e. the out-degree of every vertex of G is at most 16. By Proposition 2.5, the graph G can be covered with $\text{mad}(G) + 2 = 2 \cdot 16 + 2 = 34$ union queues using an arbitrary vertex ordering. Hence, the bad points can be covered by 34 union chains.

Finally, we show how to partition the vertical chains of the presented $(k+9)$-local layout such that Conditions (i) to (iii) are satisfied. Let H denote the interval graph that is given by the intervals that correspond to vertical chains, i.e. $V(H) = \{I_C \subseteq [k] \mid C \in \mathcal{A} \cup \cdots \cup \mathcal{G}\}$ and there is an edge between two vertices if and only if the intervals are not disjoint. A clique of m vertices in H corresponds to a row that intersects m vertical chains. Recall that every row $y = 1, \ldots, k$ intersects at most $k - y + 5$ vertical chains. In particular, the clique number of H is at most $k + 4$. Note that any proper $(k + 6)$-coloring and an arbitrary mapping between color classes and the sets of chains $\mathcal{S}_1, \ldots, \mathcal{S}_{k+6}$ satisfies Condition (i).

To satisfy Conditions (ii) and (iii), we define an ordering on the vertices of H by decreasing topmost points of the intervals, i.e. $[a, b] \prec [a', b']$ if and only if $b > b'$ or $b = b'$ and $a > a'$. We color the vertices of H greedily with $k + 5$ colors $k + 6, \ldots, 2$. That is, for an interval in column x, we choose the largest color that is not used by any smaller neighbor and that does not equal x. We then define $\mathcal{S}_i$ to contain the vertical chains whose intervals have color i. Since H is an interval graph, the set consisting of a vertex $[a, b] \in V(H)$ and its smaller neighbors induces a clique that corresponds to row b. The vertex $[a, b]$ thus has at most $k - b + 4$ smaller neighbors. There are at least two colors left that are larger than b and that are not already used by a smaller neighbor. Choosing one that does not equal x satisfies Conditions (ii) and (iii).

By symmetry, we color the horizontal chains with colors $k + 6, \ldots, 2$ such that Conditions (i) to (iii) hold. We now have a partition of all chains into $k + 6$ sets of chains $\mathcal{S}_1, \ldots, \mathcal{S}_{k+6}$, each forming a union chain when bad points are removed. Together with the 34 union chains for bad points, we get a partition of T_n into $k + 6 + 34 = k + 40$ union chains. □

3 Track Layouts of Graphs with Bounded Treewidth

We first give the necessary definitions for this chapter and then survey the state of the art on track numbers of different graphs classes. We also discuss the relation between track layouts and queue layouts more detailed. The main contribution of this chapter is an improved lower bound on the maximum track number of graphs with treewidth 2.

3.1 Definitions

We start by defining the graph classes discussed in this chapter and then give definitions of different variants of track layouts and related concepts. The graph classes in this chapter mainly include k-trees (defined in Section 3.1.1). We further consider some subgraphs of planar graphs, in particular leveled planar graphs and weakly leveled planar graphs. A graph is called *weakly leveled planar* if its vertex set can be partitioned into levels such that there is a plane straight-line drawing with x-coordinates indicated by the levels and the edges only connect vertices in the same or in consecutive levels. If there is such a drawing without edges between vertices in the same level, the graph is called *leveled planar*. Recall that the arched leveled planar graphs are the 1-queue graphs (see Section 1.3). We remark that the 2-trees considered in Section 3.3 are also planar graphs.

3.1.1 Tree Decompositions

A *k-tree* is a $(k+1)$-clique or is obtained from a smaller k-tree by choosing a clique C of size k and adding a new vertex u which is adjacent to all vertices of C. Fixing an arbitrary construction ordering, the vertex u is called a *child* of C, and C is called the *parent clique* of u. We also say u is a child of each vertex of C. For a parent vertex v of some vertex in C, we say u is a *grandchild* of v and v is a *grandparent* of u. If there is a path $(u = v_1, \dots, v_n = w)$ between two vertices u and w such that v_i is a child of v_{i+1} for $i = 1, \dots, n$, then we call u a descendant of w and w an ancestor of u. Two vertices having the same parent clique are called *twins*.

A *tree decomposition* of a graph G is a tree T whose vertex set consists of subsets of $V(G)$ such that

(i) each vertex of G is contained in some vertex of T, i.e. $\bigcup_{X \in V(T)} X = V(G)$,

(ii) for each vertex $v \in V(G)$, the vertices of T containing v induce a subtree in T, and

(iii) for each edge $vw \in E(G)$, there is a vertex $X \in V(T)$ with $v, w \in X$.

The vertices of T are called *bags*. The *width* of a tree decomposition is defined as the size of a largest vertex of T minus 1. The minimum width over all tree decompositions of a graph G is called the *treewidth* of G. It is well known that the maximal graphs with treewidth k are exactly the k-trees.

A *path decomposition* is a tree decomposition that is a path, and the minimum width over all path decompositions of a graph G is called the *pathwidth* of G. A *layered path decomposition* is a path decomposition together with a layering, i.e. a partition of the vertices of G into a sequence of layers $L_1, \ldots, L_t$ such that the endpoints of each edge are in the same layer or in two consecutive layers. The *layered pathwidth* is defined as the smallest k such that there is a layered path decomposition with at most k vertices in the intersection of any bag of the decomposition with any layer.

3.1.2 Track Layouts

A *t-track assignment* of a graph G is a partition of the vertex set $V(G)$ into t independent sets, called *tracks*, each having a linear vertex ordering. Fixing an arbitrary ordering of the tracks and denoting them by $V_1, \ldots, V_t$, we define the *length* of an edge vw with $v \in V_i$ and $w \in W_j$ by $|i - j|$. The *span* then denotes the maximum length among all edges.

Consider two tracks V and W with orderings $\prec_V$ and $\prec_W$. Two edges vw and $v'w'$ with $v, v' \in V$ and $w, w' \in W$ *cross* if $v \prec_V v'$ and $w' \prec_W w$ or $v' \prec_V v$ and $w \prec_W w'$. In this case, we also say that vw and $v'w'$ form an *X-crossing.* A set of c edges that pairwise form an X-crossing is called a c-crossing tuple. If a t-track assignment has no X-crossings, then it is called a t-track layout. Now, the *track number* $\mathrm{tn}(G)$ of a graph G is defined as the smallest t such that there is a t-track layout for G.

More generally, a *(c, t)-track layout* consists of a t-track assignment and a c-edge coloring such that there is no monochromatic X-crossing. The minimum t such that a graph G admits a (c, t)-track layout is denoted by $\mathrm{tn}_c(G)$. That is, $(1, t)$-track layouts are t-track layouts and we have $\mathrm{tn}_1(G) = \mathrm{tn}(G)$ for every graph G. On the other hand, the *t-track thickness* $\Theta_t(G)$ of a graph G denotes the smallest c such that there is a (c, t)-track layout.

Dujmović et al. [19] and Di Giacomo et al. [16] used the notion of nice orderings of cliques to deal with track layouts of graphs with bounded treewidth. Recently, Pupyrev [44] used this concept to improve upper bounds on the track number of planar graphs and subclasses thereof. We say a clique C *covers* the set of tracks containing a vertex of C. Consider a track assignment of a set $\mathcal{C}$ of cliques. We denote the tracks by $V_1, \ldots, V_t$ and its orderings by $\prec_1, \ldots, \prec_t$. We call a linear ordering $\prec_{\mathcal{C}}$ of $\mathcal{C}$ *nice* if for each two cliques $C, C' \in \mathcal{C}$ that have vertices $v \in V(C)$ and $v' \in V(C')$ with $v \prec_i v'$ in the same track V_i, we have $C \prec_{\mathcal{C}} C'$. That is, if there is one track in which a vertex of C is to the left of a vertex of C', then we have $v \prec_i v'$ in all tracks that contain a vertex v of C and a vertex v' of C'. In this case, we also say that $\mathcal{C}$ is *nicely ordered* by the track assignment. Finally, we call a track layout of a graph G *ℓ-clique-colorable* if the set of maximal cliques of G can be partitioned into at most ℓ nicely ordered subsets of cliques.

3.2 State of the Art

We first survey track numbers of several graph classes and then consider multicolor track layouts. We conclude the section by relating track layouts and queue layouts.

Dujmović et al. [20] characterized 2-track graphs as forests of caterpillars. Bannister et al. [7] showed that the bipartite graphs with track number at most 3 are exactly the leveled planar graphs, which are exactly the graphs with layered pathwidth 1. As recognizing leveled planar graphs is NP-complete [32], it is also NP-complete to test whether a (bipartite) graph has track number at most 3. The maximum track number of the related classes of weakly leveled planar graphs, respectively arched leveled planar graphs, is 6 [7, 44], respectively 4 [17, 20]. Continuing with planar graphs, Pupyrev [44] proved that the track number of planar graphs is at most 225. The best lower bound for the maximum track number of planar graphs is given by a 3-tree with track number 8, while the best known upper bound on the track number of planar 3-trees is 25 [44]. For the track number of outerplanar graphs, Dujmović et al. [20] gave an upper bound of 5 that is tight due to Pupyrev [44]. Felsner et al. [23] showed that the maximum track number of trees is 3. Turning to cliques, Dujmović et al. [19] showed that a set of cliques that all cover the same set of tracks is nicely ordered by any track layout. This implies that caterpillars admit 1-clique-colorable 2-track layouts. In addition, Pupyrev [44] proved that every tree admits a 2-clique-colorable 3-track layout and that every outerplanar graph admits a 2-clique-colorable 5-track layout. For 2-trees and 3-trees, Di Giacomo et al. [16] presented track layouts with at most 15, respectively 5415, tracks. For $k \geqslant 4$, Wiechert's [48] bound of $(k+1)(2^{k+1}-2)^k$ is the best known upper bound on the track number of k-trees. On the other hand, Dujmović et al. [19] constructed a k-tree with track number $(k+1)(k+2)/2$ for every $k > 0$.

We continue with multicolor track layouts and their relation to the track number and the queue number. Consider a graph G with t-track assignment $\mathcal{T}$. Dujmović et al. [20] showed that there is a (c, t)-track layout using $\mathcal{T}$ as track assignment if and only if $\mathcal{T}$ contains no two tracks inducing a $(c+1)$-crossing tuple. Note that coloring the edges between two tracks such that there is no monochromatic X-crossing is equivalent to finding a proper coloring of a permutation graph.

Dujmović et al. [20] gave a general lower bound on the track number in terms of the number of edges. They showed that for any graph on n vertices and $m \geqslant 1$ edges, we have $\mathrm{tn}_c(G) \geqslant (m+c)/(cn)+1$. For upper bounds, the span of a track layout turns out to be more important than the actual number of tracks. In particular, Dujmović et al. [20] showed that for every graph G admitting a (c, t)-track layout with maximum span s, we have $\mathrm{tn}_c(G) \leqslant 2s+1$ and $\mathrm{tn}_{2c}(G) \leqslant s+1$. It is worth mentioning that the structure of the original layout is preserved when reducing the number of tracks. In fact, the track layout is *wrapped*, that is track $V_{i+(k+1)m}$ is appended to track V_{i+km} for $m = 2s+1$, respectively $m = s+1$, for each $i = 1, \dots, m$, and $k \geqslant 0$.

In addition, Dujmović et al. [20] provided means for a trade-off between the number of colors and the number of tracks. To reduce the number of tracks, they showed that every (c, t)-track layout having maximum span s and a proper vertex coloring with t' colors admits a $(2sc, t')$-track layout such that the tracks correspond to the color classes. On the other hand, every (c,t)-track graph G has track number at most $t \cdot 4^{\binom{c}{2}(t-1)}$. If all cycles of G contain vertices in at least three tracks, then we have $\operatorname{tn}(G) \leqslant tc^{t-1}$.

Comparing track number and queue number, Dujmović et al. [20] showed that these two graph parameters are tied. They proved that every (c,t)-track graph G with maximum span s admits a cs-queue layout. In particular, we have $\operatorname{qn}(G) \leqslant c(t-1)$. Bounding the track number in terms of the queue number q of a graph G, they showed $\operatorname{tn}(G) \leqslant 4q \cdot 4^{q(2q-1)(4q-1)}$. If G admits a proper vertex coloring with t colors, then it also admits a $(2q,t)$-track layout such that the tracks correspond to the color classes. As the chromatic number is at most four times the queue number [21], G admits a $(2q,4q)$-track layout. They conclude that we have $\operatorname{tn}(G) \leqslant t(2q)^{t-1}$ if G admits an acyclic t-coloring, that is a proper t-coloring that does not result in bichromatic cycles.

3.3 Lower Bounds

In this section, we investigate track layouts of graphs with bounded treewidth. As the track number is a monotone graph parameter, it suffices to consider maximal graphs with treewidth k, i.e. k-trees, instead of arbitrary graphs with treewidth k.

Di Giacomo et al. [16] asked for improved upper and lower bounds on the maximum track number of 2-trees. Prior to this thesis, their upper bound of 15 and the lower bound of 6 due to Dujmović et al. [19] are the best known bounds. We first present (a slight variation of) the proof given by Dujmović et al. and then improve the lower bound.

Recall that a vertex may not be assigned to the same track as any of its parents as tracks are independent sets. The following lemma basically forbids to assign vertices to the same track as any of its grandparents.

Lemma 3.1. *For any $k,t \geqslant 0$ and any k-tree G, there is a k-tree G' such that for every t-track layout, G' contains G as a subgraph such that no vertex of G is assigned to the same track as any of its grandparents.*

Proof. Let n denote the number of vertices of G and let $m = 2(t-k)+1$. We fix an arbitrary construction ordering of G and denote the vertices of G by $1,\dots,n$ according to this ordering. We now construct G' and thereby assign a label between 1 and n to each vertex. We start with a k-clique whose vertices get labels $1,\dots,k$. For each $i = k+1,\dots,n$ in increasing order, we proceed as follows. Let $p_1,\dots,p_k$ denote the parent vertices of vertex i in G. To each k-clique in G' whose vertices have labels $p_1,\dots,p_k$, we add m vertices if vertex i has children in G and only a single vertex otherwise. In either case, all added children get the label i.

Note that there are many copies of G in G', where the label of a vertex in G' indicates to which vertex of G it corresponds. We fix an arbitrary t-track layout of G' and find a copy of G such that no vertex is assigned to the same track as any of its grandparents. We start with the k vertices in G' that have labels $1,\dots,k$. For $i = k+1,\dots,n$ in increasing order, we shall find a vertex with label i that has no children in the same track as any of its parent vertices. We only consider vertices whose parents have been

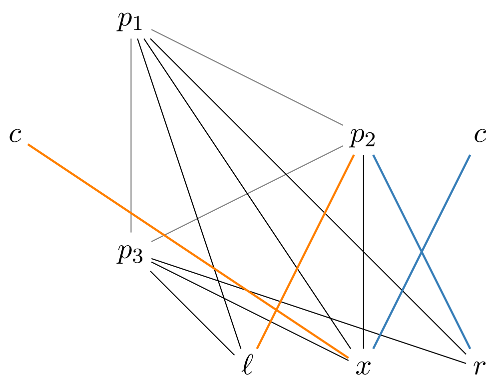


Figure 3.1: A 3-clique with three children ℓ, x, and r in the same track. If a child c of x is placed in the same track as one of the parent vertices p_1, p_2, or p_3 of x, then there is an X-crossing.

chosen for the copy of G. If i has no children in G, then there is only one such vertex. We choose it and are done for vertex i as there are no children to take care of.

If i has children, then there are $m = 2(t-k)+1$ twins with label i in G' whose parents have been chosen for G. As vertices are not placed in the same track as their parents, there are $t-k$ tracks left for these twins. Thus, there is a track containing three of them. We denote these vertices by ℓ, x, and r. Without loss of generality, we have $\ell \prec x \prec r$ in this track. We choose x for the copy of G.

Suppose that x has a child c in the same track as some parent vertex p of x as shown in Figure 3.1. If c is to the left of p, then $p\ell$ and xc cross. Symmetrically, pr and xc cross if c is to the right of p. We conclude that no child of x is in the same track as any of its parents. Continuing with the subgraph of G' containing only descendants of the chosen vertices, we find the desired copy of G. □

As the constructed graph G' again is a k-tree, we get the following corollary.

Corollary 3.2. *Let $k, t \geqslant 0$. If every k-tree admits a t-track layout, then every k-tree admits a t-track layout such that no vertex is placed in the same track as any of its grandparents.*

Next, we prove a lower bound on the track number of k-trees. Our proof differs from that given by Dujmović et al. [19] in that we make use of Corollary 3.2 and thus can present a smaller k-tree that requires $(k+1)(k+2)/2$ tracks under the condition that no vertex is assigned to the same track as any of its grandchildren. In contrast to Dujmović et al., we do not use the arguments of Lemma 3.1 until the end of the proof, which increases the size of the constructed graph. However, our graph contains their graph as a subgraph.

Theorem 3.3 (Dujmović et al. [19]). *For every $k \geqslant 0$, there is k-tree with track number at least $(k+1)(k+2)/2$.*

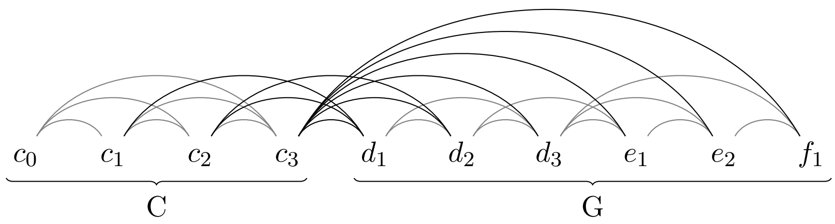


Figure 3.2: 3-Tree that needs 10 tracks if no vertex is assigned to the same track as any of its grandparents. The vertices e_1, e_2, and f_1 belong to the cliques that were added in previous induction steps.

Proof. We proceed by induction on k showing that there is a k-tree that requires $(k+1)(k+2)/2$ tracks if no vertex is assigned to the same track as any of its grandparents. A single vertex is a 0-tree and needs one track. Let $k > 0$ and assume that there is a k-tree G needing $k(k+1)/2$ tracks to avoid vertices being in the same track as their grandparents. We construct a k-tree by starting with a $(k+1)$-clique C whose vertices we denote by $c_0, \ldots, c_k$. Now take a copy of G and denote the vertices of an arbitrary k-clique by $d_1, \ldots, d_k$. Connecting d_i to the vertices $c_i, \ldots, c_k$ for each $i = 1, \ldots, k$ and additionally connecting each vertex of G to c_k yields a k-tree. See Figure 3.2 for an illustration.

Note that all vertices of C are parents or grandparents to each vertex of G. Hence, we need $k+1$ tracks for the clique C and by induction additionally $k(k+1)/2$ tracks for G, i.e. $(k+1)(k+2)/2$ tracks in total. By Corollary 3.2, there is a k-tree with track number at least $(k+1)(k+2)/2$. □

Note that the proof by Dujmović et al. [19] above gives a lower bound of 6 on the maximum track number of 2-trees. Based on the presented construction, we improve this bound.

Theorem 3.4. *There is a* 2*-tree with track number at least* 7*.*

Proof. We construct a 2-tree and suppose there is a 6-track layout such that no vertex is in the same track as any of its parents or grandparents. We show that such a track layout does not exist and conclude with Corollary 3.2 that there is a 2-tree with track number at least 7.

We start with a 2-clique consisting of the vertices a and b. For ease of presentation, we introduce vertices a_i and b_i with $a_i = a$ and $b_i = b$ for $i = 1, \ldots, 49$. For each $i = 1, \ldots, 49$, we add vertices

$$c_i \text{ with parents } a_i \text{ and } b_i,$$
$$d_i \text{ with parents } b_i \text{ and } c_i,$$
$$e_i \text{ with parents } c_i \text{ and } d_i,$$

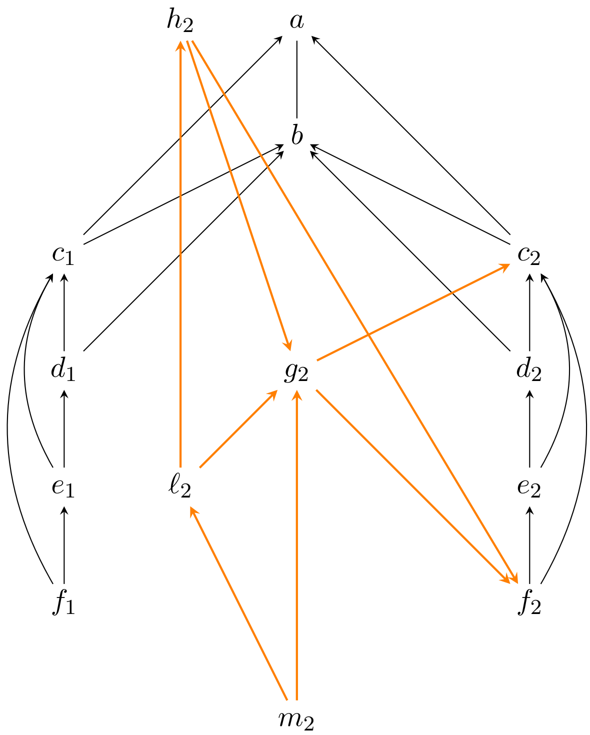


Figure 3.3: Parts of the 2-tree with track number at least 7. The edges are oriented toward their parent vertices.

$$
\begin{aligned}
f_i &\text{ with parents } c_i \text{ and } e_i,\\
g_i &\text{ with parents } c_i \text{ and } f_i,\\
h_i &\text{ with parents } g_i \text{ and } f_i,\\
\ell_i &\text{ with parents } g_i \text{ and } h_i, \text{ and}\\
m_i &\text{ with parents } g_i \text{ and } \ell_i.
\end{aligned}
$$

See Figure 3.3 for an illustration. Let G_i denote the subgraph induced by the vertices $a_i, \ldots, f_i$. Note that each G_i is isomorphic to the 2-tree constructed in Theorem 3.3. We observe that each vertex of G_i is a child or grandchild of all formerly introduced vertices with the same index. Thus, no two vertices of G_i are assigned to the same track of any track layout.

Fix an arbitrary 6-track layout. We continue by analyzing the track assignment of multiple G_i's. The vertices a and b are assigned to two distinct tracks which we denote by T_a and T_b, respectively. For each i, there are four tracks left for the vertices c_i, d_i, e_i, and f_i. As there are 24 possible permutations of these four vertices, there are three indices i, j, and k such that x_i, x_j, and x_k are in the same track for each $x = c, d, e, f$. We denote these tracks by T_c, T_d, T_e, and T_f, respectively. Without loss of generality, we have $i = 1$, $j = 2$, and $k = 3$ and $c_1 \prec c_2 \prec c_3$ in track T_c.

Next we observe that the ordering of track T_c is preserved in the tracks T_d, T_e, and T_f. Recall that d_i, e_i, and f_i are adjacent to c_i for $i = 1, 2, 3$. If we have $x_j \prec x_i$ for some $x = d, e, f$ and $1 \leqslant i < j \leqslant 3$, then the edges $c_i x_i$ and $c_j x_j$ cross. Hence, we have $x_1 \prec x_2 \prec x_3$.

Finally, we consider the vertices g_2, h_2, ℓ_2, and m_2. For the first three vertices, we determine a unique track that contains neither parents nor grandparents and in which they can be placed without creating an X-crossing. The last vertex, however, fits in none of the six tracks. The subgraphs G_1 and G_2 and the four additional descendants of G_2 are shown in Figure 3.3.

Vertex g_2 has parents in tracks T_c and T_f and grandparents in T_a, T_b, and T_e. Thus, we have $g_2 \in T_d$. Note that the G_1 and G_3 are symmetric with respect to G_2. Without loss of generality, g_2 is to the left of d_2. On the other hand, g_2 is to the right of d_1 as otherwise $c_2 g_2$ and $c_1 d_1$ cross. Next, the vertex h_2 has parents in the tracks T_d and T_f and grandparents in T_c and T_e. If h_2 is in track T_b, then the edge $g_2 h_2$ forms an X-crossing either with bd_1 or with bd_2. We conclude that h_2 is in track T_a. The vertex ℓ_2 is in track T_e as it has parents or grandparents in the tracks T_a, T_c, T_d, and T_f and creates an X-crossing with bd_1 or bd_2 if placed in track T_b. Finally, the vertex m_2 has parents in the tracks T_d and T_e and grandparents in T_a, T_c, and T_f. Again, track T_b is not possible as $g_2 m_2$ would cross bd_1 or bd_2. We find that there is no track to place m_2 and thus we need a seventh track. □

We remark that Dujmović et al. [19] constructed their lower bound on the maximum track number of k-trees inductively. Starting the induction with our 2-tree from Theorem 3.4 slightly improves the lower bound for k-trees with $k \geqslant 2$.

Corollary 3.5. *For every $k \geqslant 2$, there is k-tree G with* $\operatorname{tn}(G) \geqslant (k+1)(k+2)/2 + 1$.

4 Book Embeddings of Upward Planar Graphs

In this chapter we discuss book embeddings of upward planar graphs. A directed graph is called *upward planar* if there is a plane drawing such that all edges are y-monotone. In particular, upward planar graphs are acyclic. Heath et al. [31] initiated the study of book embeddings of directed acyclic graphs, where the spine ordering respects the orientation of the edges. Building up on their work, Binucci et al. [11], Di Giacomo et al. [15], and Frati et al. [24] particularly highlighted the question whether upward planar graphs have bounded page number and, if not, a characterization of upward planar graphs with constant page number, as open problems. We first give some definitions used in this chapter and summarize the state of the art. Subsequently, we investigate book embeddings of upward planar graphs without large twists. Section 4.3 establishes necessary conditions that upward planar graphs need to satisfy to admit book embeddings with few pages. This prepares the construction of a planar poset, and in particular an upward planar graph, that requires at least five pages. Our poset with page number at least 5 improves on the previously best known lower bounds on the page number of planar posets and of upward planar graphs, which were proved by Hung [33] in 1993.

4.1 Definitions

A directed graph G consists of a vertex set and an edge set containing ordered pairs of vertices. As in the undirected case, we denote the vertex set by $V(G)$ and the edge set by $E(G)$. For a directed graph, vw denotes the directed edge from v to w, that is the pair (v, w). Replacing each directed edge vw by an undirected edge with the same endpoints results in a graph that is called the *underlying undirected graph.* Note that the underlying undirected graph still has an edge *set*, that is we do not have multi-edges, even if there are directed edges in both directions between two vertices. In the case of a directed graph, a *cycle* refers to a subgraph consisting of pairwise distinct vertices $v_1, \dots, v_n$, for some $n \geqslant 1$, and edges $v_n v_1$ and $v_i v_{i+1}$ for $i = 1, \dots, n-1$. A directed graph is called *acyclic* if it contains no cycle. Note that the underlying undirected graph of a directed acyclic graph may have cycles. Every directed acyclic graph admits a *topological ordering*, that is a linear vertex ordering $\prec$ such that for every edge vw, we have $v \prec w$. Considering a drawing of a directed graph, an edge e is called *upward* if the curve representing e increases strictly monotonically in vertical direction. A drawing is called *upward plane* if all edges are upward and no two edges intersect except maybe at a common endpoint. A graph is called *upward planar* if it admits an upward plane drawing.

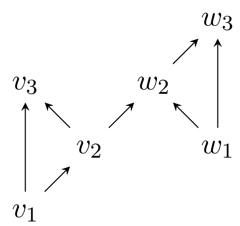
A *book embedding* of a directed acyclic graph G consists of a topological vertex ordering and a partition of the edges into *pages*, where no two edges vw and xy in the same page cross, i.e. have alternating endpoints. That is, we have an additional constraint on the vertex ordering compared to book embeddings of undirected graphs. In this chapter, all our graphs are directed and the term *book embedding* refers to a book embedding of a directed graph as defined above.

Next, we give some definitions that refer to the relation between two vertices and identify some special vertices. Consider an upward planar graph. We say a vertex w is *reachable* from some vertex v and v *reaches* w if there is a path from v to w. The *down-set* of a vertex v is defined as the set of all vertices that reach v. Similarly, the *up-set* is defined to contain all vertices that are reachable from v. In particular, every vertex is contained in its own up-set and down-set. We say two vertices v and w are *comparable* if v reaches w or w reaches v. Otherwise they are called *incomparable*. A vertex that has no incoming edges is called a *source* and a vertex that has no outgoing edges is called a *sink*. If a graph has a single source s, a single sink t, and an edge st, it is called an *st-graph*.

Having the basic definitions, we define two subclasses of upward planar graphs whose page number is investigated in the literature. First, *series-parallel digraphs* (also called two-terminal series-parallel digraphs [15]) are defined as follows. A directed edge is a series-parallel digraph. Consider two series-parallel digraphs G and G' with sources s, respectively s', and sinks t, respectively t'. Identifying the sink t of G with the source s' of G' yields a series-parallel digraph. Identifying the sources s and s' and the sinks t and t' also results in a series-parallel digraph. Note that series-parallel digraphs have a single source and a single sink. We remark that the underlying undirected graphs of maximal series-parallel digraphs are exactly the 2-trees, that is the maximal graphs with treewidth 2.

We conclude this section by discussing another important subclass of upward planar graphs. A *partially ordered set (poset)* is a set X together with a relation $\leqslant$ that is reflexive, transitive, and antisymmetric. If $x \leqslant y$ and x and y are distinct, we write $x < y$. We call two elements x and y of X *incomparable* if neither $x \leqslant y$ nor $y \geqslant x$ holds. We denote this by $x \parallel y$. Otherwise, x and y are called *comparable*. The set of elements x with $x \leqslant y$ is called the *down-set of y* and the set of elements z with $y \leqslant z$ is called the *up-set of y*. If the intersection of the up-sets of two elements x and y has a least element, we call it the *join* of x and y. If the intersection of the down-sets has a largest element, we call it the *meet* of x and y.

A relation $x < y$ is called a *cover relation* if there is no z with $x < z < y$. The *cover graph* of a poset $(X, \leqslant)$ is defined as the directed graph with X as vertex set and an edge xy if and only if $x < y$ is a cover relation. Note that the definitions of comparable, up-set, and down-set coincide for a poset and its cover graph. A poset is called *planar* if its cover graph is upward planar. An upward drawing of a cover graph is called a *Hasse diagram*. We define a *book embedding*, respectively the *page number*, of a poset as a book embedding, respectively the page number, of its cover graph.



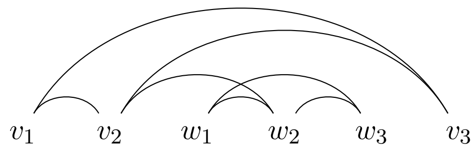


Figure 4.1: Outerplanar directed graph with page number 2 (left) and a 2-page book embedding of the same graph (right)

4.2 State of the Art

Asymptotically, there are no nontrivial bounds known for the page number of upward planar graphs. That is, the page number is in $\Omega(1)$ and $\mathcal{O}(n)$. The best known lower bound is 4, provided by Hung [33] with a poset and by Bekos et al. [9] and Yannakakis [52] with an undirected planar graph with page number 4. We improve on this in Section 4.4.1. Upper bounds are known for some subclasses of upward planar graphs. If the underlying undirected graph of some directed graph G is a tree, then G has page number 1 [31]. Di Giacomo et al. [15] showed that every series-parallel digraph admits a 2-page book embedding. Directed acyclic graphs whose underlying undirected graph is a planar 3-tree also have constant page number [24].

Frati et al. [24] gave improved upper bounds under certain conditions. In particular, if every 4-connected component of any n-vertex upward planar triangulation G admits a vertex ordering with maximum twist size $f(n)$, then G admits a vertex ordering with maximum twist size $\mathcal{O}(f(n))$, where $f(n) \in \Omega(1)$ and $f(n) \in \mathcal{O}(n)$. Using results on the χ-boundedness of circle graphs by Černý [12] and Kostochka and Kratochvíl [37], they conclude that every upward planar triangulation G whose 4-connected components have page number at most k, satisfies $\text{pn}(G) \leqslant \min\{\mathcal{O}(k \log n), \mathcal{O}(2^k)\}$. However, Davies and McCarty [14] later bounded the chromatic number of circle graphs with maximum clique size ω by $7\omega^2$, which improves the upper bound to $\text{pn}(G) \leqslant \min\{\mathcal{O}(k \log n), \mathcal{O}(k^2)\}$. We remark that Bekos et al. [9] and Yannakakis [52] constructed their undirected planar graphs with page number 4 by repeatedly inserting small graphs into triangular faces, which yields 4-connected components that require only few pages. That is, orienting these graphs is not a promising approach for constructing upward planar graphs with large page number.

Consider a subclass of upward planar graphs whose n-vertex graphs have a maximum path of length of $o(n/\log(n))$. Frati et al. [24] showed that such a graph class has page number $o(n)$. In addition, they showed that every upward planar triangulation has page number $o(n)$ if and only if every upward planar triangulation with maximum degree $\mathcal{O}(\sqrt{n})$ has page number $o(n)$.

Concerning the computational complexity, Heath and Pemmaraju [29] presented a linear time algorithm for recognizing directed graphs with page number 1. Note that there are outerplanar directed graphs that do not admit a 1-page book embedding, see for

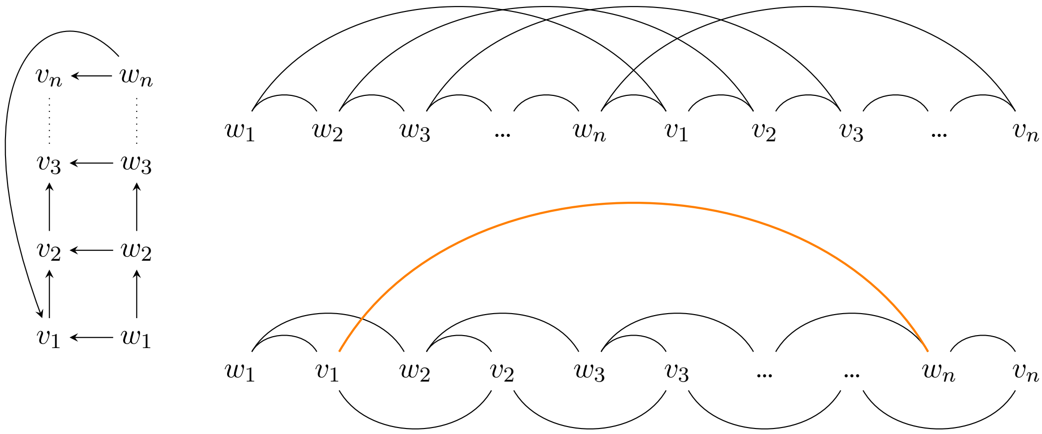


Figure 4.2: A family of planar directed acyclic (but not upward planar) graphs with $2n$ vertices and page number n (left). The vertex ordering is uniquely determined by the directed path $(w_1, \ldots, w_n, v_1, \ldots v_n)$, which results in an n-twist (top). The underlying undirected graph admits a 3-page book embedding (bottom).

instance Figure 4.1. On the other hand, Binucci et al. [11] showed that deciding whether a (not necessarily planar) *st*-graph admits a k-page book embedding is NP-complete for each $k \geqslant 3$. However, it is open whether 2-page book embeddings can be found efficiently for directed acyclic graphs. Note that every directed acyclic graph with page number at most 2 is upward planar. In comparison, deciding whether an undirected (planar) graph has page number 2 is NP-complete [10, 49].

For undirected planar graphs, Yannakakis [51] established an upper bound of 4. That is, if upward planar graphs do not have constant page number, then the page numbers of upward planar graphs are not tied to the page numbers of their underlying undirected graphs. We remark that this is the case if we consider planar directed acyclic graphs instead of upward planar graphs, as shown in Figure 4.2.

4.3 Topological Orderings of Upward Planar Graphs

We now investigate book embeddings of upward planar graphs that have no large twists. First, we observe that there are upward planar graphs that require stronger constraints on the vertex ordering than implied by the orientation of the edges. That is, there are upward planar graphs such that not every topological ordering admits a book embedding with bounded maximum twist size. We identify structures that can lead to large twists and prevent these twists by adding additional edges and choosing a topological ordering of the augmented graph. At the end of this section, we show that if the augmented graph is acyclic, then there is a plane drawing of the original graph G such that all edges of G and all added edges are upward.

Consider two vertices v and w. For $k \geqslant 2$, a *k-flag from v to w* consists of k pairwise

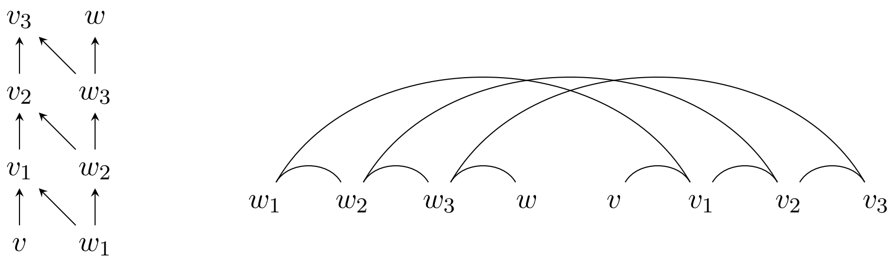


Figure 4.3: A 3-flag and a book embedding with $w \prec v$

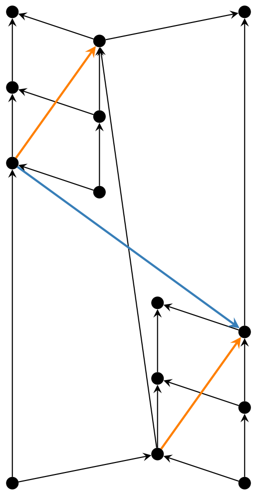


Figure 4.4: A graph (black) with multiple 3-flags. The orange edges belong to G_3^+, the blue edge belongs to G_3^* but not to G_3^+.

comparable vertices in the up-set of v, denoted by $v_1 \prec \cdots \prec v_k$, k pairwise comparable vertices in the down-set of w, denoted by $w_1 \prec \cdots \prec w_k$, and edges $w_i v_i$ for each $i = 1, \ldots, k$. The edges $w_i v_i$ are called *flag edges*. Recall that every vertex is contained in its up-set and down-set and thus $v = v_1$ or $w_k = w$ is possible. If k is not important, we simply say *flag*. Figure 4.3 shows a 3-flag. Observe that v and w are not necessarily comparable. However, we show that v needs to be to the left of w to avoid a k-twist.

Lemma 4.1. *Every book embedding of a k-flag from v to w in which w is to the left of v has a k-twist.*

Proof. As v_1 is in the up-set of v and w_k is in the down-set of w, we have $v \prec v_1$ and $w_k \prec w$. Assuming $w \prec v$, we obtain $w_1 \prec \cdots \prec w_k \preccurlyeq w \prec v \preccurlyeq v_1 \prec \cdots \prec w_k$. Hence, the flag edges form a k-twist. See Figure 4.3 for an illustration. □

Given an upward planar graph G, we augment it to a directed graph G_k^+ by adding the edge vw for each two vertices v and w for which there is a k-flag from v to w, where $v = w$ is allowed. This may result in multi-edges if the edge vw already exists in G.

However, we add at most one new edge vw, even if there are multiple flags from v to w. We now say that two vertices v and w are comparable if they are comparable in G_k^+. If there is a directed path consisting of edges of G between two vertices, then we say they are *comparable in* G. This way, we possibly obtain new flags in G whose vertices are only comparable in G_k^+ as shown in Figure 4.4. Note that flag edges still need to be edges of the original graph G. We continue adding edges for each k-flag of G until no new flags are created. The resulting graph is denoted by G_k^*. We say that some vertex v is in the up-set, respectively down-set, of some vertex w if there is a path from w to v, respectively from v to w, in G_k^*.

Lemma 4.1 shows that a spine ordering that is not a topological ordering of G_{k+1}^*, yields a $(k+1)$-twist. In particular, G_{k+1}^* being acyclic is a necessary condition for G admitting a k-page book embedding.

Lemma 4.2. *Every k-page book embedding of any upward planar graph G uses a topological ordering of G_{k+1}^* as spine ordering.*

However, this condition is not sufficient for every k. Even stronger, we find that choosing a topological ordering of G_k^* for some small k can lead to arbitrarily large twists even if the graph admits a book embedding with few pages. The following example shows that k needs to be chosen carefully to obtain a book embedding with small maximum twist size. Note that the ideas presented here also find application in Proposition 4.11 in the next section, where we prove a similar statement for a larger k.

Example 4.3. *For every $p \geqslant 0$, there is an upward planar graph G such that no topological ordering of G_2^* admits a p-page book embedding.*

Proof. Let $r = p^3 + 1$. We define a graph G on $5r$ vertices that are partitioned into five levels and illustrated in Figure 4.5. For each $i = 1, \dots, r$, we have vertices a_i, b_i, c_i, d_i, and e_i that belong to levels $A, \dots, E$, respectively. They are connected by two paths $(a_i, b_i, c_i, d_i, e_i)$ for $i = 1, \dots, r$ and $(a_i, b_{i+1}, c_i, d_{i+1}, e_i)$ for $i = 1, \dots, k-1$. In addition, there are edges a_ic_i, b_id_i, and c_ie_i connecting non-consecutive levels.

Consider a topological ordering $\prec$ of G_2^*. We show that all vertices of level B are placed to the left of all vertices of level C, whereas all vertices of level D are to the right of all vertices of level C. That is, we prove $B \prec C \prec D$. For this, we show that there are edges b_ic_j and c_id_j in G_2^* if $|i-j| \leqslant t$ by induction on t. It follows that $b_i \prec c_j \prec d_i$.

For $t = 0$, we have edges b_ic_i and c_id_i in G, which settles the base case. For $t > 0$, we now find a 2-flag from b_i to c_{i+t}, see Figure 4.6. The edges $b_{i+t}c_i$, c_id_{i+t}, and $c_{i+t}d_i$ exist by symmetry. By induction, there is an edge b_ic_{i+t-1} in G_2^* and thus c_{i+t-1} is in the up-set of b_i. In addition, there is an edge $c_{i+t-1}d_{i+t} \in E(G)$. In the down-set of c_{i+t}, we have the comparable vertices a_{i+t-1} and b_{i+t}. As $a_{i+t-1}c_{i+t-1}$ and $b_{i+t}d_{i+t}$ are edges in G, we have a 2-flag from b_i to c_{i+t}.

Having $B \prec C \prec D$, we find a $(p+1)$-twist. We consider the r triangles induced by the vertices b_i, c_i, and d_i for $i = 1, \dots, r$. As we only consider these triangles, we may assume, without loss of generality, that $b_1 \prec \cdots \prec b_r$. We call a sequence of vertices *increasing*, respectively *decreasing*, if their indices are increasing, respectively decreasing, in the

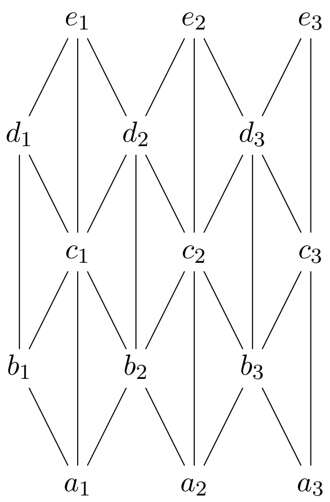


Figure 4.5: Upward planar drawing of the graph constructed in Example 4.3 with $r = 3$

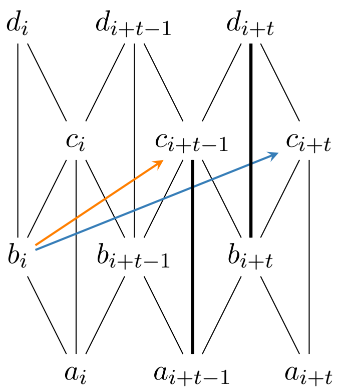


Figure 4.6: Partly augmented graph, where $t = 2$. The orange edge belongs to G_2^+ and is part of the 2-flag that creates the blue edge. The flag edges for the blue edge are drawn thick.

vertex ordering. If there is an increasing sequence of $p+1$ (not necessarily consecutive) vertices in C, then we have a $(p+1)$-twist between these vertices and their neighbors in B. Hence, there are at most p increasing vertices and by Erdős and Szekeres [22] we have a decreasing sequence of p^2+1 vertices $v_1, \dots, v_{p^2+1}$ in C. Consider the neighbors of $v_1, \dots, v_{p^2+1}$ in D. Again by Erdős and Szekeres [22], we have an increasing or a decreasing sequence of size $p+1$. In the first case, we have a $(p+1)$-twist consisting of edges between B and D. In the second case, we have $(p+1)$-twist consisting of edges between C and D. In either case, the constructed graph does not admit a p-page book embedding. □

The graph constructed in Example 4.3 is a subgraph of a triangulated grid considered by Frati et al. [24], who showed that there is a book embedding with constant maximum twist size. Following their approach, we construct a 6-page book embedding for arbitrary $r > 0$. The chosen vertex ordering is a topological ordering of G_7^* but not of G_2^*. We define the vertex ordering as follows. The vertices of the levels A and B are laid out alternatingly, that is $a_1 \prec b_1 \prec a_2 \prec b_2 \prec \cdots \prec a_r \prec b_r$. To the right of b_r, we have c_r. We then continue with $c_{r-1} \prec d_r \prec c_{r-2} \prec d_{r-1} \prec \cdots \prec c_2 \prec d_3 \prec c_1 \prec d_2$. Finally, we have $d_1 \prec e_1 \prec e_2 \prec \cdots \prec e_r$. The book embedding is illustrated in Figure 4.7. All edges are oriented from left to right since we have $A \cup B \prec C \cup D \prec E$ and for each $i = 1, \dots, r$, we have $a_i \prec \{b_i, b_{i+1}\}$ and $c_i \prec \{d_i, d_{i+1}\}$. Observe that the triangles considered in the proof of Example 4.3 are not laid out as required by a topological ordering of G_2^*. In particular, we do not have $C \prec D$.

Two pages suffice for the edges between A and B and between C and D. Note that the vertices in C and the vertices in D are laid out in reverse order, that is $c_j \prec c_i$ and $d_j \prec d_i$ for $i < j$. Because of that, the edges $a_i c_i$ and $b_i c_i$, $i = 1, \dots, r$, can be embedded in a single page. For the same reason, the edges $b_i d_i$, $i = 1, \dots, r$, and $b_{i+1} c_i$, $i = 1, \dots, r-1$, do not cross. The remaining two pages are used for the edges between C, respectively D, and E.

Recall that an acyclic G_{k+1}^* is a necessary condition for G admitting a k-page book embedding. However, for $k \leqslant 4$ there are graphs G such that G_k^* has loops or larger cycles as shown in Figures 4.8 and 4.9.

To conclude this section, we consider drawings of upward planar graphs whose flags do not create cycles. Consider an upward planar graph G such that G_k^* is acyclic for some k and let $\prec$ denote a topological ordering of G_k^*. As G is a subgraph of G_k^*, the ordering $\prec$ is also a topological ordering of G. Using a result by Giordano et al. [25] on a variant of book embeddings, we conclude that G admits an upward plane drawing such that adding the edges of G_k^* keeps the drawing upward. As every upward planar graph is a subgraph of a maximal planar st-graph [35], it suffices to consider the latter.

An *upward topological book embedding* of a directed graph G is an upward plane drawing such that the vertices of G lie on an oriented line in y-direction, called the *spine*. The ordering in which the vertices occur on the spine is called the *spine ordering*. Note that the spine ordering is a topological ordering of G. The spine divides the plane into two half-planes that correspond to the pages of a 2-page book embedding. However, edges may cross the spine in an upward topological book embedding. Given a topological

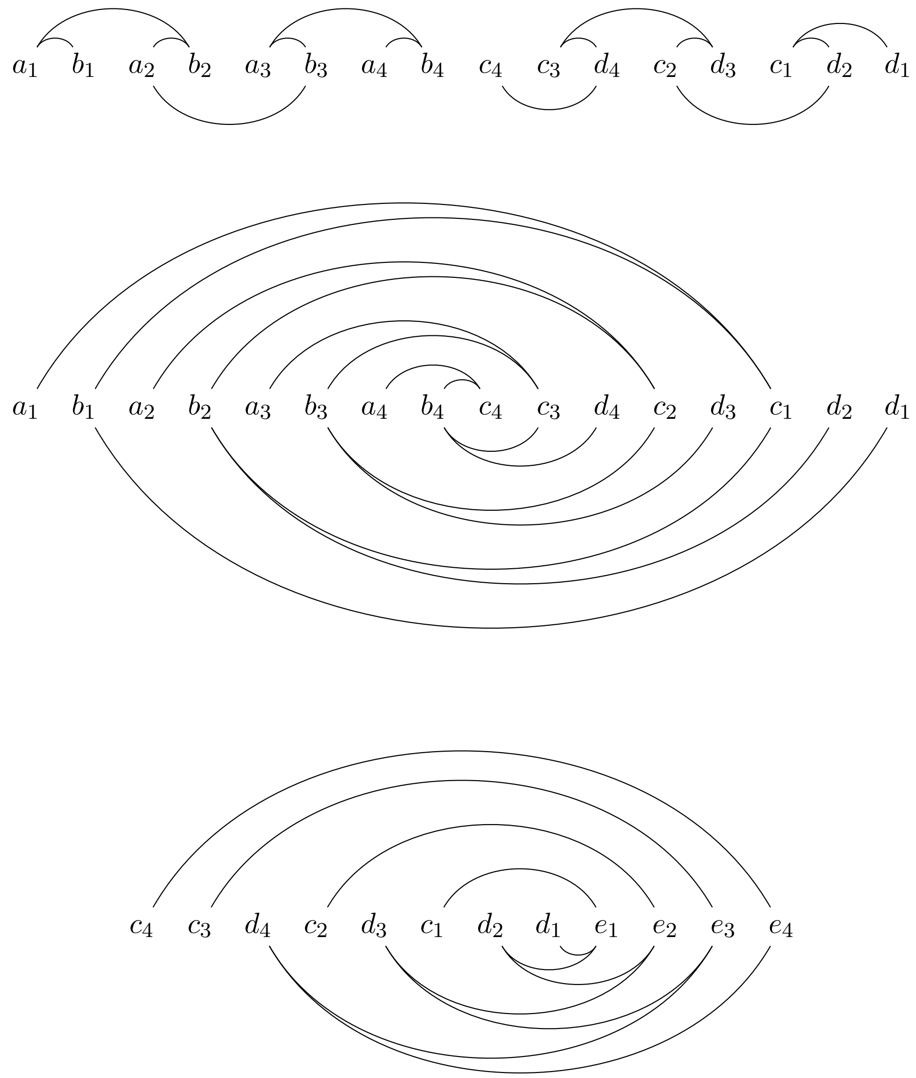


Figure 4.7: 6-page book embedding for the graph constructed in Example 4.3 with $r = 4$

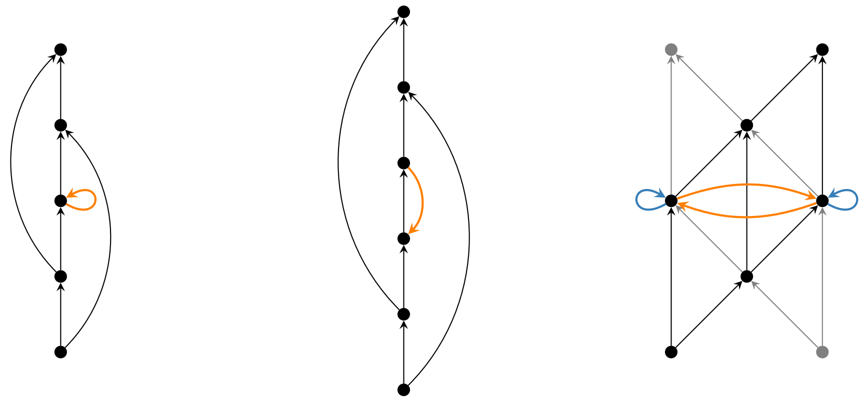


Figure 4.8: A graph G such that G_2^+ has a loop (left), G_2^+ has a 2-cycle (middle), respectively G_3^* has a 2-cycle and loops (right). The orange edges belong to G_2^+, respectively G_3^+, but not to G. The blue loops belong to G_3^* but not to G_3^+. The vertices that form the flag creating the orange left-to-right edge and the flag creating the left loop are highlighted in black.

ordering $\prec$, an upward topological book embedding is called $\prec$-constrained if $\prec$ is used as spine ordering.

Theorem 4.4 (Giordano et al. [25]). *For every maximal planar st-graph G and every topological vertex ordering $\prec$ of G, there is a $\prec$-constrained upward topological book embedding.*

Note that a $\prec$-constrained upward topological book embedding is, in particular, an upward plane drawing such that the vertices are ordered in y-direction according to $\prec$. Choosing a topological vertex ordering of G_k^*, we get the following corollary.

Corollary 4.5. *Let $k \geqslant 2$ and let G be a maximal upward planar st-graph. If G_k^* is acyclic, then there is an upward drawing of G_k^* that contains a plane drawing of G.*

4.4 Lower Bounds

The previously best known lower bound on the maximum page number of upward planar graphs is provided by Hung [33] with a planar poset requiring four pages. However, we show that this poset admits a book embedding without 4-twists. In this section, we present a small upward planar graph that has a 4-twist in every book embedding. Building up on the previous section, we then construct a planar poset that has a 5-twist in every book embedding and, in particular, requires five pages.

Hung's [33] example for a planar poset with page number at least 4 is a 48-element poset (see also [6]). We define a family of posets that contains their construction. For $i > 1$, we define a poset P_i as follows, see Figure 4.10. We start with i elements $c_1, \dots, c_i$ having a common meet a and a common join e. We then add elements $b_{j,j+1}$ and $d_{j,j+1}$

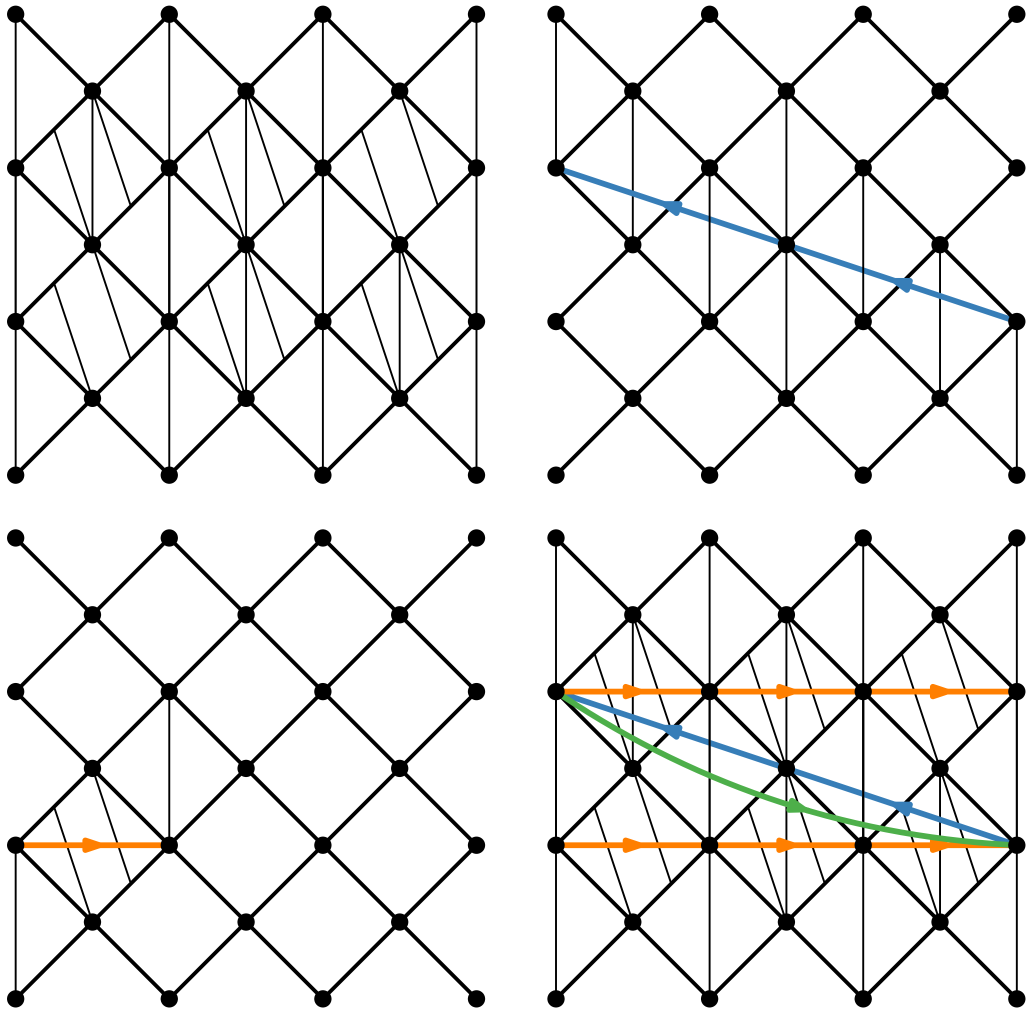


Figure 4.9: An upward planar graph whose augmented graph G_4^* contains a cycle (top left). The upward edges (blue) of the cycle are created using four vertical flag edges each (top right). The horizontal edges of G_4^* (orange) are created using the thin edges of G (bottom left) and then connect the vertical flag edges for the downward edge (green) of the cycle (bottom right).

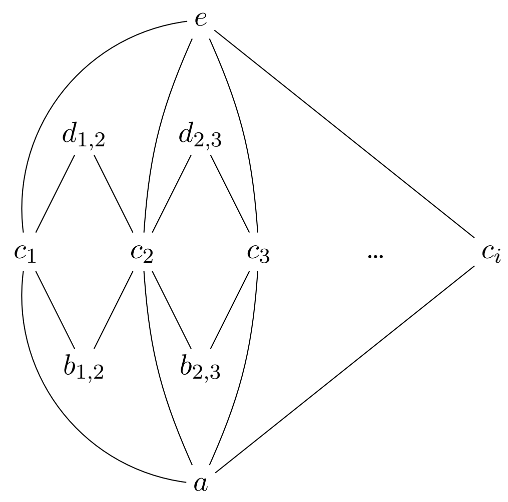


Figure 4.10: Poset P_i

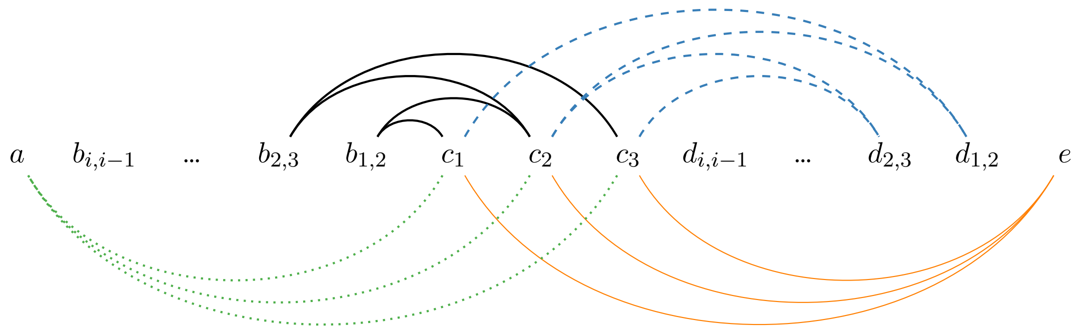


Figure 4.11: 4-page book embedding for P_i

with four cover relations $b_{j,j+1} \leqslant c_j, c_{j+1} \leqslant d_{j,j+1}$ for each $j = 1, \dots i-1$. The poset P_3 has page number 3 [43] and Hung [33] showed that P_{16} requires four pages.

Theorem 4.6 (Hung [33])**.** *The page number of P_{16} is at least 4.*

Using an online framework for computing linear layouts by Bekos et al. [8], we obtain a 3-page book embedding of P_7 but already P_8 requires four pages. On the other hand, the upper bound of 4 holds not only for P_{16} but for all P_i.

Remark 4.7. *For each $i > 1$, there is a 4-page book embedding for P_i.*

Proof. We obtain a 4-page book embedding by choosing the vertex ordering $a \prec b_{i-1,i} \prec \cdots \prec b_{1,2} \prec c_1 \prec \cdots \prec c_i \prec d_{i-1,i} \prec \dots d_{1,2} \prec e$ as shown in Figure 4.11. We use one page for the edges $b_{j,j+1}c_j$ and $b_{j,j+1}c_{j+1}$ and another page for the edges $c_jd_{j,j+1}$ and

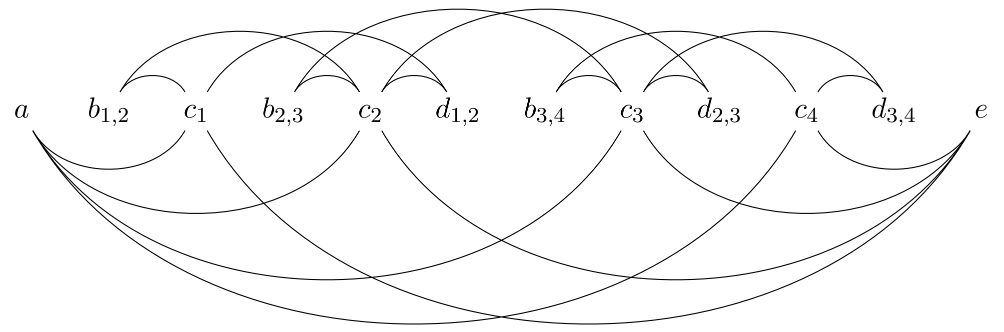


Figure 4.12: A topological ordering of the cover graph of P_4 that does not create a 4-twist

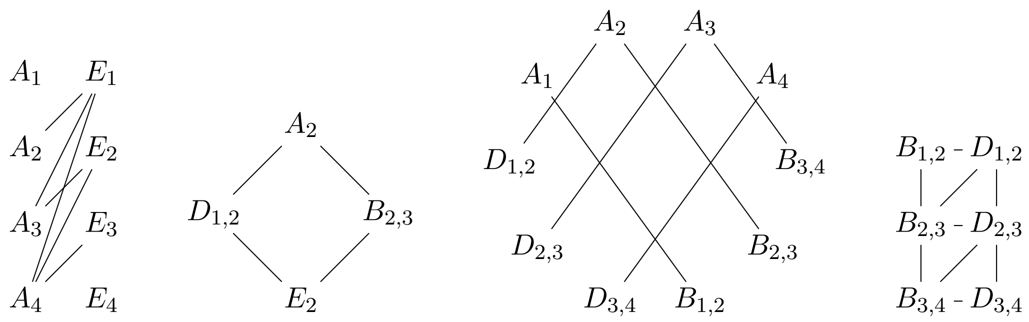


Figure 4.13: Parts of the conflict graph H_4

$c_{j+1}d_{j,j+1}$ for $j = 1, \ldots, i-1$. In both pages, the edges are pairwise incident or nesting. The other two pages each contain a star with all edges incident to a, respectively e. Since edges of a star cannot cross, both pages are crossing-free. $\square$

Despite requiring four pages in any book embedding, all of the constructed posets admit a book embedding without 4-twist.

Remark 4.8. *For each $i > 1$, there is a book embedding for P_i without 4-twist.*

Proof. Let G_i denote the cover graph of P_i. We consider the vertex ordering $a \prec c_1 \prec \cdots \prec c_i \prec e$, where $b_{j,j+1}$ is inserted directly to the left of c_j and $d_{j,j+1}$ directly to the right of c_{j+1} for $j = 1, \ldots, i-1$. See Figure 4.12 for an illustration. Note that the b-vertices and d-vertices are inserted such that their incident edges are oriented to the right. The edges incident to a, respectively e, are also oriented to the right as a is the leftmost vertex and e is the rightmost vertex.

For ease of presentation, we consider the conflict graph H_i of G_i with respect to $\prec$. That is, the vertex set of H_i consists of the edges of the cover graph G_i and two vertices of H_i are adjacent if the respective edges cross. Note that a clique in H_i corresponds to a twist in G_i of the same size. We call the vertices of H_i *nodes* and the edges *conflicts* to avoid ambiguities. Note that the edges $b_{j,j+1}c_j$ and $c_{j+1}d_{j,j+1}$ do not cross any other edge as their endpoints are consecutive in the vertex ordering. The respective nodes are isolated in H_i and are omitted from now on. Observe that each edge of the cover graph G_i is incident to exactly one c-vertex. We identify the edges by its other endpoint and denote ac_j by A_j, c_je by E_j, $b_{j,j+1}c_{j+1}$ by $B_{j,j+1}$, and $c_jd_{j,j+1}$ by $D_{j,j+1}$.

We now show that H_i does not contain a 4-clique. See Figure 4.13 for the described parts of H_4. First, observe that $\{A_j : j = 1, \dots, i\}$ and $\{E_j : j = 1, \dots, i\}$ are independent sets in H_i since they form stars in G_i. The nodes A_j and E_ℓ have a conflict if and only if $j > \ell$. In addition, A_j and E_j have conflicts with B-nodes and D-nodes if and only if c_j is between the endpoints of the respective B- or D-edge in the vertex ordering of G_i. That is, $A_jB_{\ell,\ell+1}$, $A_jD_{\ell-1,\ell}$, $B_{\ell,\ell+1}E_j$, and $D_{\ell-1,\ell}E_j$ are conflicts if and only if $j = \ell$. In particular, each B-node and each D-node has conflicts with A_j and E_j for exactly one j. Recall that A_jE_j is not a conflict. It follows that every triangle of the conflict graph H_i contains at most one of $A_1, \dots, A_i, E_1, \dots, E_i$. However, each node $X \in \{A_j, E_j : i = 1, \dots, i\}$ has degree 2 in the subgraph of H_i induced by X and all B-nodes and D-nodes and therefore is not contained in any 4-clique.

Hence, a possible 4-clique consists only of B-nodes and D-nodes. We observe that the nodes $B_{1,2}, \dots, B_{i-1,i}$ form an induced path. To see this, note that $b_{j,j+1} \prec c_j \prec c_{j+1} \prec b_{j+1,j+2}$ implies a crossing of the edges $b_{j,j+1}c_{j+1}$ and $c_jb_{j+1,j+2}$. On the other hand, we have $B_{j,j+1} \prec B_{\ell,\ell+1}$ for $j + 1 < \ell$, that is these two nodes have no conflict. Analogously, the nodes $D_{1,2}, \dots, D_{i-1,i}$ form an induced path. Finally, we consider conflicts between a B-node and a D-node. The nodes $B_{j,j+1}$ (edge $b_{j,j+1}c_{j+1}$) and $D_{j,j+1}$ (edge $c_jd_{j,j+1}$) have a conflict as we have $b_{j,j+1} \prec c_j \prec c_{j+1} \prec d_{j,j+1}$. In addition, we have conflicts $D_{j-1,j}B_{j,j+1}$ due to the edges $c_{j-1}d_{j-1,j}$ and $b_{j,j+1}c_{j+1}$ with $c_{j-1} \prec b_{j,j+1} \prec c_j \prec d_{j-1,j} \prec c_{j+1}$. Note that there is no conflict between $B_{j-1,j}D_{j,j+1}$ as both edges are incident to c_j. To conclude, we observe that we have $B_{j,j+1} \prec D_{\ell,\ell+1}$ and $D_{j,j+1} \prec B_{\ell,\ell+1}$ for $j + 1 < \ell$. Thus, there are no conflicts between these nodes. In particular, the largest clique induced by B-nodes and D-nodes consists of three nodes. It follows that the conflict graph H_i has no 4-clique and therefore the chosen vertex ordering does not yield a 4-twist. □

Next, we discuss graphs that have 4-twists in every book embedding. We start with a small upward planar graph and then construct a planar poset.

Proposition 4.9. *There is a* 13*-vertex upward planar graph that does not admit a book embedding without* 4*-twist.*

Proof. We construct the graph starting with two paths (b_1, b_2, b_3, b_4) and (d_1, d_2, d_3, d_4) that are joined by the edges b_id_i for $i = 1, 2, 3, 4$. In each of the inner faces, we add a vertex and connect it to all vertices incident to that face. That is, we add three vertices c_1, c_2, c_3 and the edges b_ic_i, $b_{i+1}c_i$, c_id_i, and c_id_{i+1} for $i = 1, 2, 3$. Finally, we add a source a that is incident to all b_i and a sink e that is incident to all d_i. Figure 4.14 shows an upward planar embedding.

We now analyze book embeddings of the constructed graph and show that every book embedding has a 4-twist. Consider the edges b_id_i for $i = 1, 2, 3, 4$ and assume they do

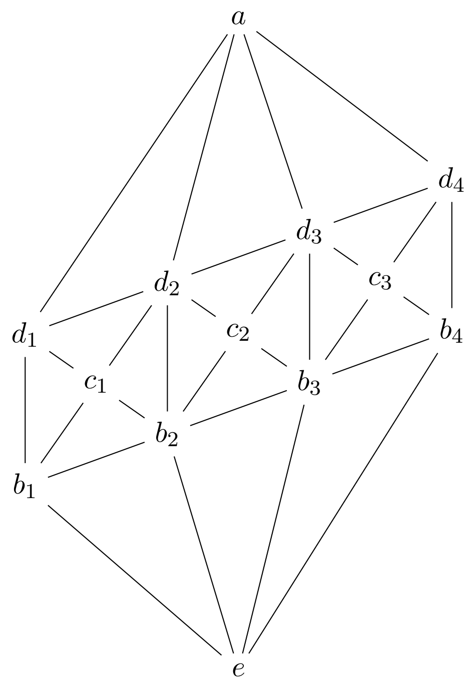
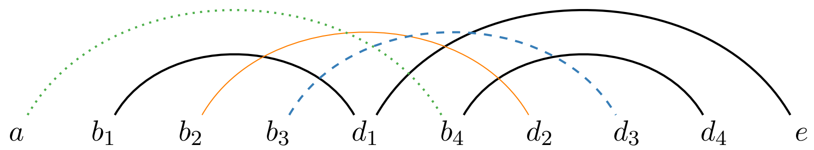
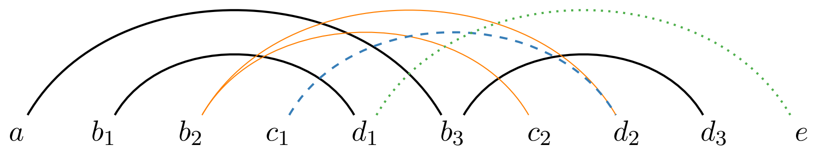


Figure 4.14: Upward planar graph that does not admit a book embedding without 4-twist

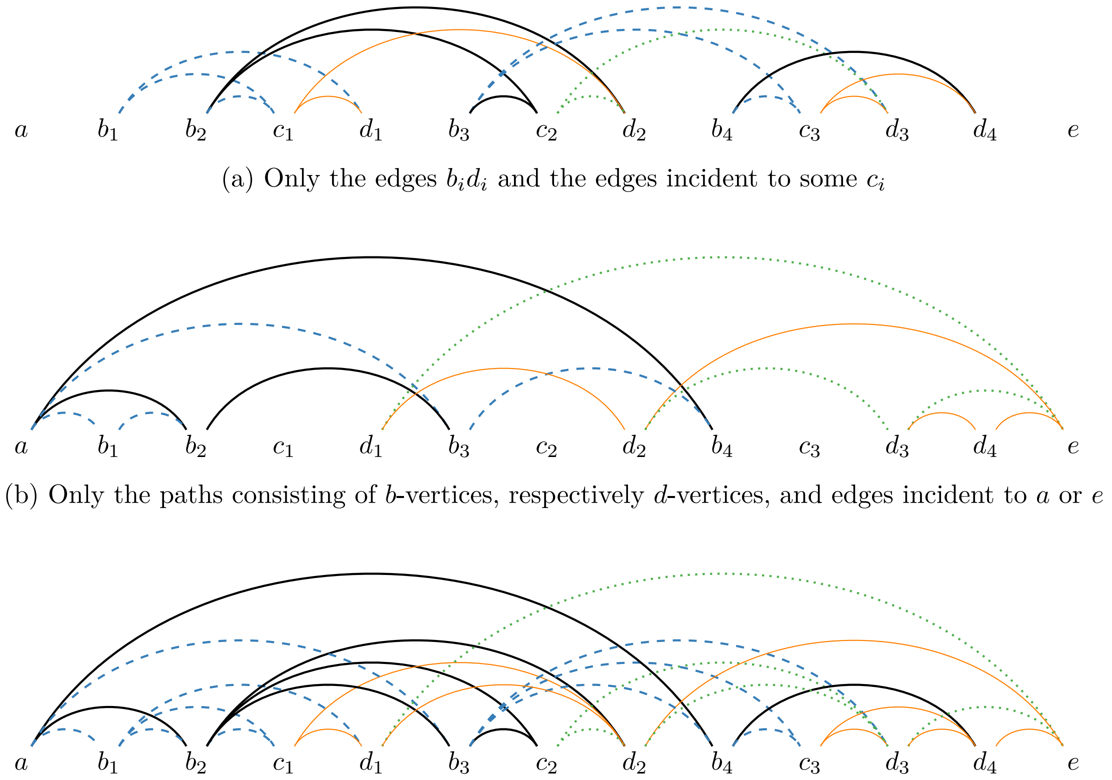


(a) The edges b_1d_1, b_2d_2, b_3, d_3 and b_2d_2, b_3d_3, b_4d_4 each form a 3-twist.



(b) The edges b_1d_1, b_2d_2, and b_3, d_3 do not form a 3-twist.

Figure 4.15: 4-twists in book embeddings of the upward planar graph constructed for Proposition 4.9



(a) Only the edges $b_i d_i$ and the edges incident to some c_i

(b) Only the paths consisting of b-vertices, respectively d-vertices, and edges incident to a or e

(c) The complete book embedding

Figure 4.16: 4-page book embedding of the graph constructed for Proposition 4.9

not form a 4-twist. We observe that d_i is reachable from b_{i+1} for $i = 1, 2, 3$. Thus, we have $b_i \prec b_{i+1} \prec d_i \prec d_{i+1}$, and the edges $b_i d_i$ and $b_{i+1} d_{i+1}$ cross. Hence, the maximal twists formed by the four considered edges $b_i d_i$ are 2-twists or 3-twists. We distinguish whether all of these maximal twists are 3-twists or whether there is some 2-twist. In both cases we have $a \prec b_1$ and $d_4 \prec e$ as the respective edges exist.

The first case is shown in Figure 4.15a. The first three edges (i.e. $b_i d_i$ for $i = 1, 2, 3$) and the last three edges (i.e. $b_i d_i$ for $i = 2, 3, 4$) each form a 3-twist. That is, we have $b_1 \prec b_2 \prec b_3 \prec d_1 \prec b_4 \prec d_2 \prec d_3 \prec d_4$. It follows that the edges ab_4, $b_2 d_2$, $b_3 d_3$, and $b_1 e$ form 4-twist.

Now, assume that the three edges $b_i d_i$ do not form a 3-twist for $i = 1, 2, 3$ or for $i = 2, 3, 4$. Without loss of generality, this is the case for $i = 1, 2, 3$ as shown in Figure 4.15b. Hence, we have $b_1 \prec b_2 \prec d_1 \prec b_3 \prec d_2 \prec d_3$. Inserting a, c_1, c_2, and e, we get $a \prec b_2 \prec c_1 \prec d_1 \prec b_3 \prec c_2 \prec d_2 \prec e$, which implies a 4-twist consisting of the edges ab_3, $b_2 c_2$, $c_1 d_2$, and $d_1 e$. We conclude that there is a 4-twist in all cases. □

Note that the graph constructed for Proposition 4.9 has a 4-page book embedding,

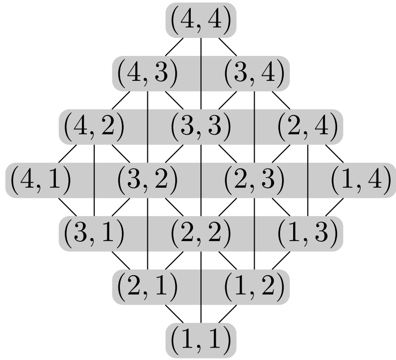


Figure 4.17: A 4×4 upward grid. Consider the vertex $(2, 2)$. The first left upper vertex is $(3, 2)$, the second left upper vertex is $(4, 1)$, the first right upper vertex is $(2, 3)$, and the second right upper vertex is $(1, 4)$.

which is shown in Figure 4.16. We also remark that the graph is upward planar but not the cover graph of a poset.

4.4.1 5-Twists

Next, we use the concept of k-flags introduced in Section 4.3 to first construct an upward planar graph that has a 5-twist in every book embedding. We then find this graph in the augmented graph G_5^* of the cover graph G of a planar poset.

Recall that for an upward planar graph G, the graph G_5^* denotes the graph that is augmented with all edges vw for which there is a 5-flag from v to w (including the flags created after partly augmenting the graph). By Lemma 4.2, the spine ordering of any book embedding of an upward planar graph G that does not have a 5-twist is a topological ordering of G_5^*. The following lemmas build on arguments used in the proof of Example 4.3 and the graph we now construct contains that of Example 4.3 as a subgraph.

For any $n > 0$, we define an $n \times n$ *upward grid* Grid_n as follows (see Figure 4.17). The vertex set of Grid_n consists of vertices (ℓ, r) for $1 \leqslant \ell, r \leqslant n$. The vertices are partitioned into *levels*, where level L_h contains the vertices (ℓ, r) with $\ell + r = h$. The edge set of Grid_n consists of three subsets. There are *left edges* containing the edges $(\ell, r)(\ell + 1, r)$ for each $r = 1, \dots, n$ and $\ell = 1, \dots, n - 1$. Symmetrically, the edges $(\ell, r)(\ell, r + 1)$ for $\ell = 1, \dots, n$ and $r = 1, \dots, n - 1$ are called *right edges*. Finally, we have edges $(\ell, r)(\ell + 1, r + 1)$ for $1 \leqslant \ell, r \leqslant n - 1$ and call them *vertical edges*.

Consider a vertex $v = (\ell_v, r_v)$ in some level L_h of an upward grid. A vertex $w = (\ell_w, r_w)$ in level L_{h+1} is called an *i-th left (right) upper vertex of* v if $\ell_w = \ell_v + i$ ($r_w = r_v + i$). A vertex that is an i-th left upper vertex or an i-th right upper vertex of v is also called an *i-th upper vertex of* v. Note that every vertex in L_{h+1} is an i-th upper vertex of v for some $i > 0$.

Based on an $n \times n$ upward grid, we define an $n \times n$ N-grid, which we denote by N_n, where n is an integer. We then show that an $n \times n$ N-grid has a 5-twist in every book embedding for large n. An $n \times n$ N-grid contains an $n \times n$ upward grid as a subgraph and

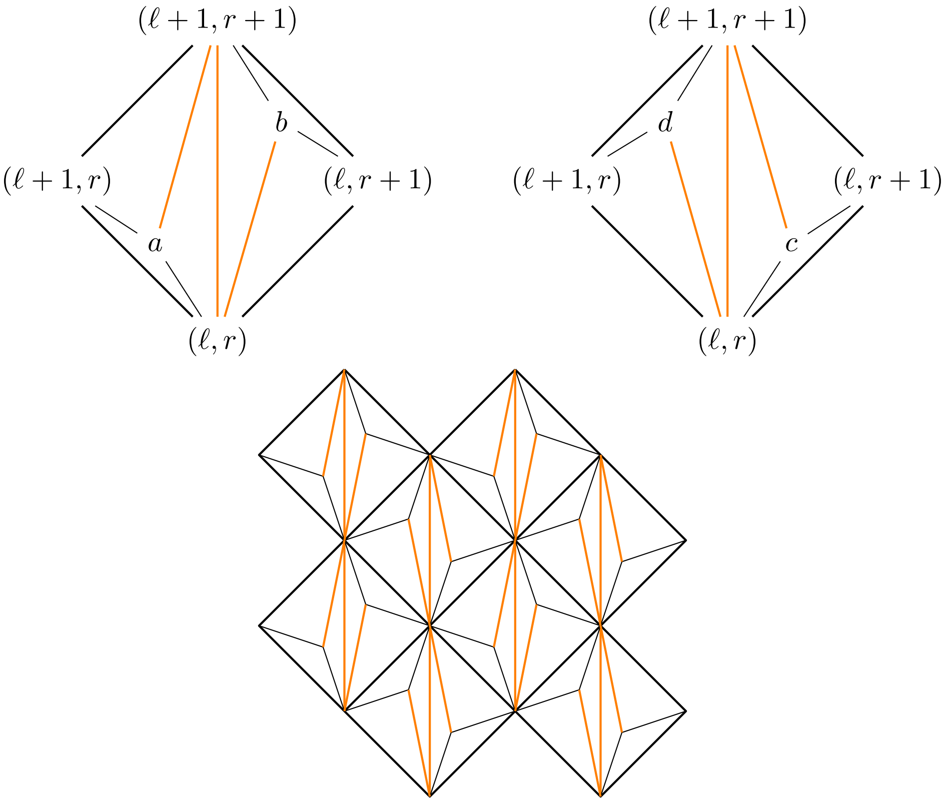


Figure 4.18: Parts of an N-grid with N-vertices $a = a_{\ell,r}$, $b = b_{\ell,r}$, $c = c_{\ell,r}$, and $d = d_{\ell,r}$. The N-edges are highlighted orange.

an additional vertex in each face of Grid_n. The additional vertices are called *N-vertices*, whereas the vertices that belong to Grid_n are called *grid vertices*. See Figure 4.18 for an illustration. Consider two triangles in Grid_n that share a vertical edge. That is, they consist of vertices (ℓ, r), $(\ell+1, r)$, $(\ell, r+1)$, and $(\ell+1, r+1)$ as shown in Figure 4.18. If $\ell - r$ is even, then we insert a vertex $a_{\ell,r}$ into the left triangle and add edges $(\ell, r)a_{\ell,r}$, $a_{\ell,r}(\ell+1, r)$, and $a_{\ell,r}(\ell+1, r+1)$. In addition, we insert a vertex $b_{\ell,r}$ together with the edges $(\ell, r)b_{\ell,r}$, $(\ell, r+1)b_{\ell,r}$, and $b_{\ell,r}(\ell+1, r+1)$ into the right triangle in this case. If $\ell - r$ is odd, then we insert vertices $c_{\ell,r}$ and $d_{\ell,r}$ into the right, respectively left, triangle and add edges $(\ell, r)c_{\ell,r}$, $c_{\ell,r}(\ell, r+1)$, $c_{\ell,r}(\ell+1, r+1)$, $(\ell, r)d_{\ell,r}$, $(\ell+1, r)d_{\ell,r}$, and $d_{\ell,r}(\ell+1, r+1)$. We call an edge that is vertical or of the form $a_{\ell,r}(\ell+1, r+1)$, $(\ell, r)b_{\ell,r}$, $c_{\ell,r}(\ell+1, r+1)$, or $(\ell, r)d_{\ell,r}$ an *N-edge* (note that they form N-shapes, compare Figure 4.18). We keep the definitions of levels and upper vertices, where the N-vertices do not belong to any level.

The rest of this chapter is devoted to proving that a sufficiently large N-grid yields a 5-twist with every vertex ordering and then to construct a poset whose augmented cover graph contains an N-grid. For this, we consider the graph $N^*_{n,5}$ that results from augmenting N_n via 5-flags. Recall that every vertex ordering that is not a topological ordering of $N^*_{n,5}$ yields a 5-twist. Hence, we only need consider topological orderings of $N^*_{n,5}$. To find an N-grid in the poset that we construct at the end of this chapter, we make use of edges that are created by 5-flags and do not belong to the cover graph. However, flag edges have to be edges of the original graph, which is why we need to keep track of which edges are used as flag edges. The next lemma separates the levels of an N-grid given that the vertex ordering is a topological ordering of $N^*_{n,5}$.

Lemma 4.10. *For every $n > 0$, there is an $n' \geqslant n$ such that every book embedding $(\mathcal{P}, \prec)$ of $N_{n'}$ that uses a topological ordering of $N^*_{n',5}$ as spine ordering, contains a copy of N_n such that the levels of N_n are separated by $\prec$. That is, we have $L_2 \prec \cdots \prec L_{2n}$ for the levels L_h, $h = 2, \ldots, 2n$, of N_n. Moreover, the statement still holds if we replace all edges except for the N-edges by edges in $N^*_{n',5}$, that is only N-edges may be used as flag edges.*

Proof. We show by induction on i that for any $i > 0$ and any $n > 0$, there is an $n' \geqslant n$ such that $N^*_{n',5}$ contains a copy of N_n, where each grid vertex of N_n has edges in $N^*_{n',5}$ to all its i-th upper vertices. We thereby only use N-edges as flag edges to augment $N_{n'}$ to $N^*_{n',5}$. Note that we quantify n in each induction step. We then conclude for $n = i$ that in every book embedding of N_n whose spine ordering respects the orientation of the additional edges of $N^*_{n',5}$, all vertices of level L_h are to the left of all vertices of the subsequent level L_{h+1} for $h = 2, \ldots, 2n-1$.

Observe that each grid vertex is adjacent to its first left upper vertex via a left edge and to its first right upper vertex via a right edge, which settles the base case $i = 1$. Let $i > 1$ and assume that we find an N_{n+2} such that every vertex v in N_{n+2} has edges to all j-th upper vertices for $0 < j < i$. We first drop all grid vertices on the outer face of N_{n+2} then remove all N-vertices that are now on the outer face. This yields an $n \times n$ N-grid N_n. Note that for every grid vertex in N_n the incoming vertical edge and the outgoing vertical edge is an edge of N_{n+2}. We next find a 5-flag from each grid vertex of N_n to its i-th upper vertices in N_n.

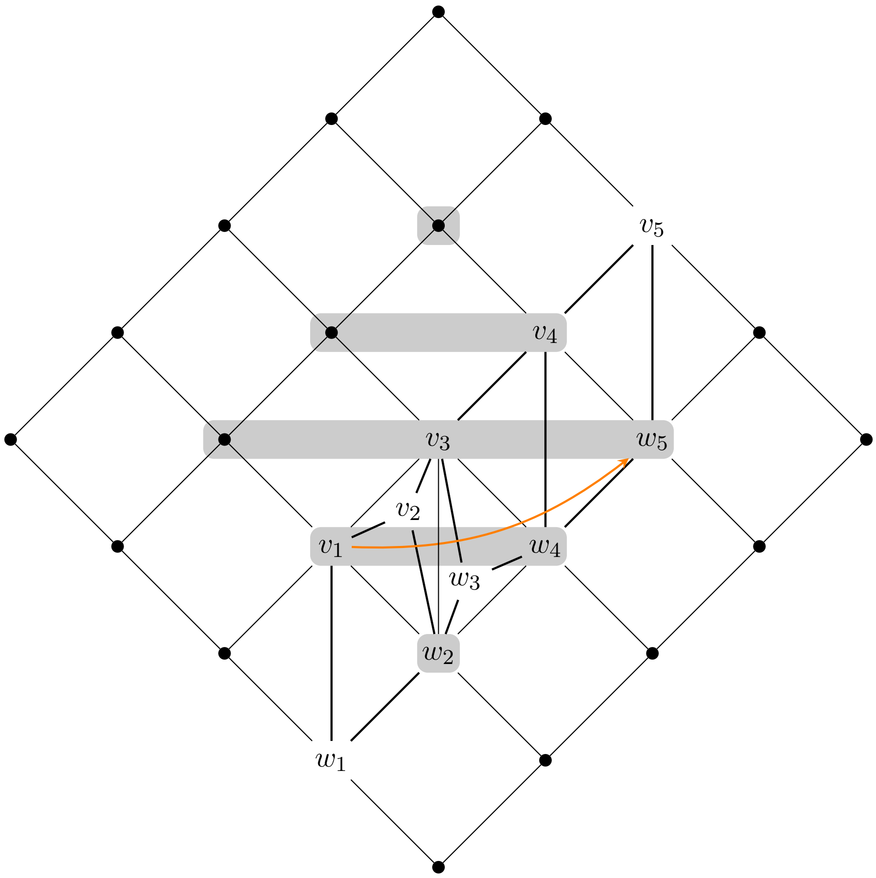


Figure 4.19: 5-Flag from $v = v_1$ to $w = w_5$, where w is the second right upper vertex of v

Consider a grid vertex $v = (\ell_v, r_v)$ of N_n. Without loss of generality, we assume that $\ell_v - r_v$ is even. Swap left and right otherwise. Let $w = (\ell_w, r_w) \in E(N_n)$ denote the i-th right upper vertex of v. By definition of an i-th right upper vertex, we have $r_w = r_v + i$. As the two vertices are in consecutive levels, we have $\ell_v + r_v = h$ and $\ell_w + r_w = h + 1$, where L_h is the level of (ℓ_v, r_v). It follows that $\ell_w = \ell_v - i + 1$.

Now, consider the vertices

$$\begin{aligned} w_1 &= (\ell_v - 1, r_v - 1), \\ w_2 &= (\ell_v - 1, r_v), \\ w_3 &= c_{\ell_v - 1, r_v}, \\ w_4 &= (\ell_v - 1, r_v + 1), \text{ and} \\ w_5 &= (\ell_v - i + 1, r_v + i) = w. \end{aligned}$$

See Figures 4.19 and 4.20 for an illustration. These five vertices form the lower part of the desired 5-flag. Note that w_1 is connected to v by a vertical edge and thus is a vertex of N_{n+2}. We observe that $w_1, \ldots, w_5$ are pairwise comparable. The first four vertices

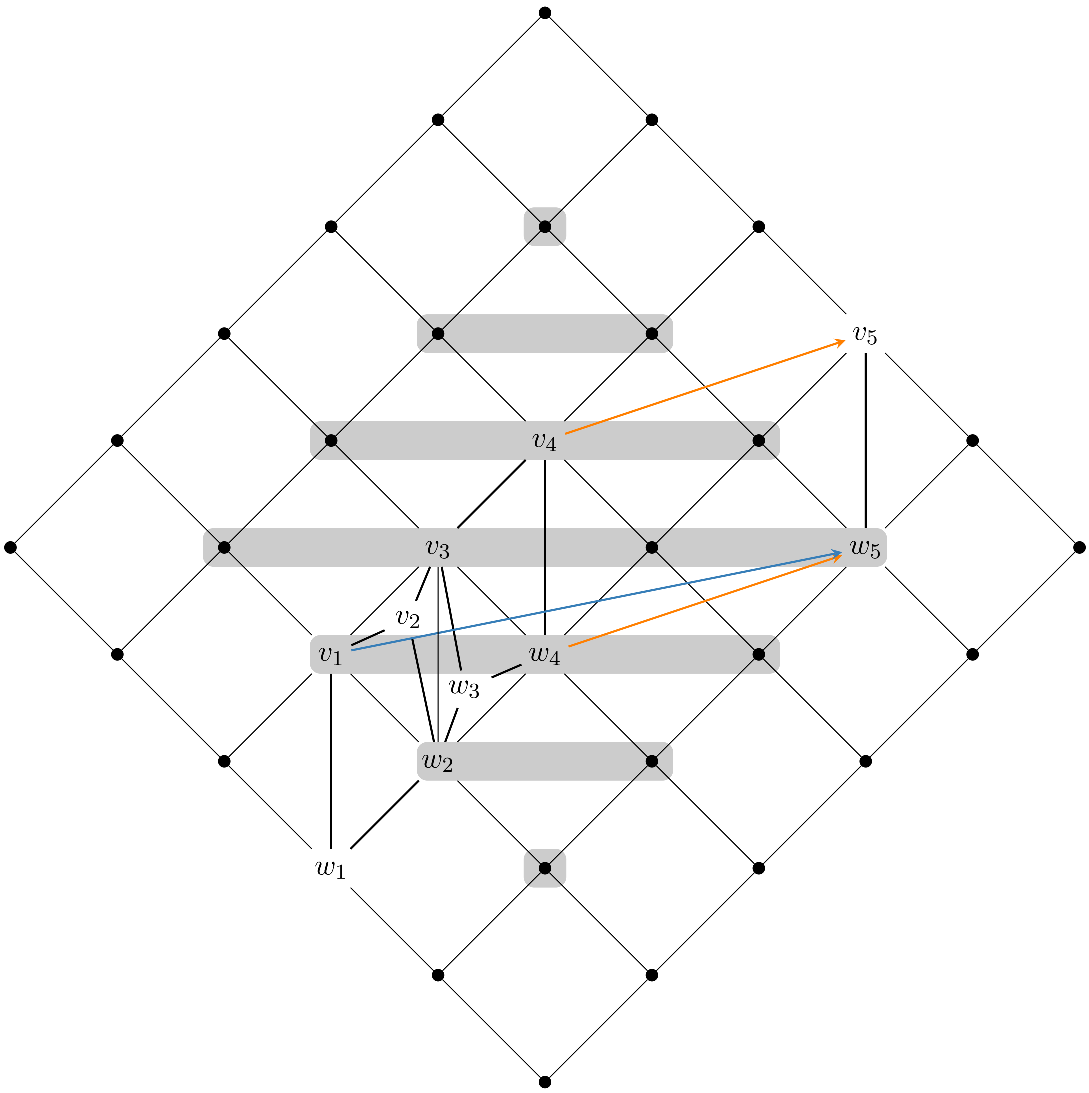


Figure 4.20: 5-Flag from $v = v_1$ to $w = w_5$, where w is the third right upper vertex of v. The orange edges exist by induction.

form a path in N_{n+2}. The vertices w_4 and w_5 are incomparable in N_{n+2} (unless $i = 2$) but there is an edge $w_4w_5 \in E(N^*_{n+2,5})$ by induction. To see this, observe that w_4 and w_5 are in consecutive levels as $(\ell_v - i + 1 + r_v + i) - (\ell_v - 1 + r_v + 1) = 1$. The r-coordinates of the two vertices differ exactly by $i - 1$ and thus w_5 is an $(i-1)$-st right upper vertex of w_4.

Next, consider the vertices

$$\begin{aligned}
v_1 &= (\ell_v, r_v) = v,\\
v_2 &= d_{\ell_v - 1, r_v},\\
v_3 &= (\ell_v, r_v + 1),\\
v_4 &= (\ell_v, r_v + 2), \text{ and}\\
v_5 &= (\ell_w + 1, r_w + 1) = (\ell_v - i + 2, r_v + i + 1).
\end{aligned}$$

These five vertices serve as the upper part of the 5-flag from v to w. Again, we find that there is a path connecting the five vertices in $N^*_{n+2,5}$. First, the edges v_1v_2 and v_2v_3 exist by construction of an N-grid. The edge v_3v_4 is a right edge in Grid_{n+2}. We obtain the remaining edge v_4v_5 by induction as v_5 is an $(i-1)$-st right upper vertex of v_4. Finally, we find the desired flag by observing that w_1v_1, w_4v_4, and w_5v_5 are vertical edges (and in particular N-edges), while w_2v_2 and w_3v_3 are non-vertical N-edges.

The proof for the i-th left upper vertex works symmetrically, except for that we first use edges we obtain be induction and then use N-vertices to find the remaining four flag edges. Let $w = (\ell_w, r_w) = (\ell_v + i, r_v - i + 1) \in E(N_n)$ denote the i-th left upper vertex of v. We find a 5-flag from v to w using the vertices

$$\begin{aligned}
w_1 &= (\ell_w - i - 1, r_w + i - 2) = (\ell_v - 1, r_v - 1),\\
w_2 &= (\ell_w - 2, r_w),\\
w_3 &= (\ell_w - 1, r_w),\\
w_4 &= a_{\ell_w - 1, r_w}, \text{ and}\\
w_5 &= (\ell_w, r_w) = w
\end{aligned}$$

for the lower part, while the upper part is formed by the vertices

$$\begin{aligned}
v_1 &= (\ell_w - i, r_w + i - 1) = v,\\
v_2 &= (\ell_w - 1, r_w + 1),\\
v_3 &= b_{\ell_w - 1, r_w},\\
v_4 &= (\ell_w, r_w + 1), \text{ and}\\
v_5 &= (\ell_w + 1, r_w + 1).
\end{aligned}$$

See Figures 4.21 and 4.22 for an illustration. Note that the coordinates of w_3 have an even difference as $(\ell_w - 1) - r_w = (\ell_v + i - 1) - (r_v - i + 1) = \ell_v - r_v + 2i - 2$, which means that we indeed have the claimed a- and b-vertices. The edges w_1w_2 and v_1v_2 exist by induction, the other vertices are connected by two paths in N_{n+2}. Again, the flag edges w_iv_i are N-edges for $i = 1, \ldots, 5$, which completes the 5-flag from v to w. □

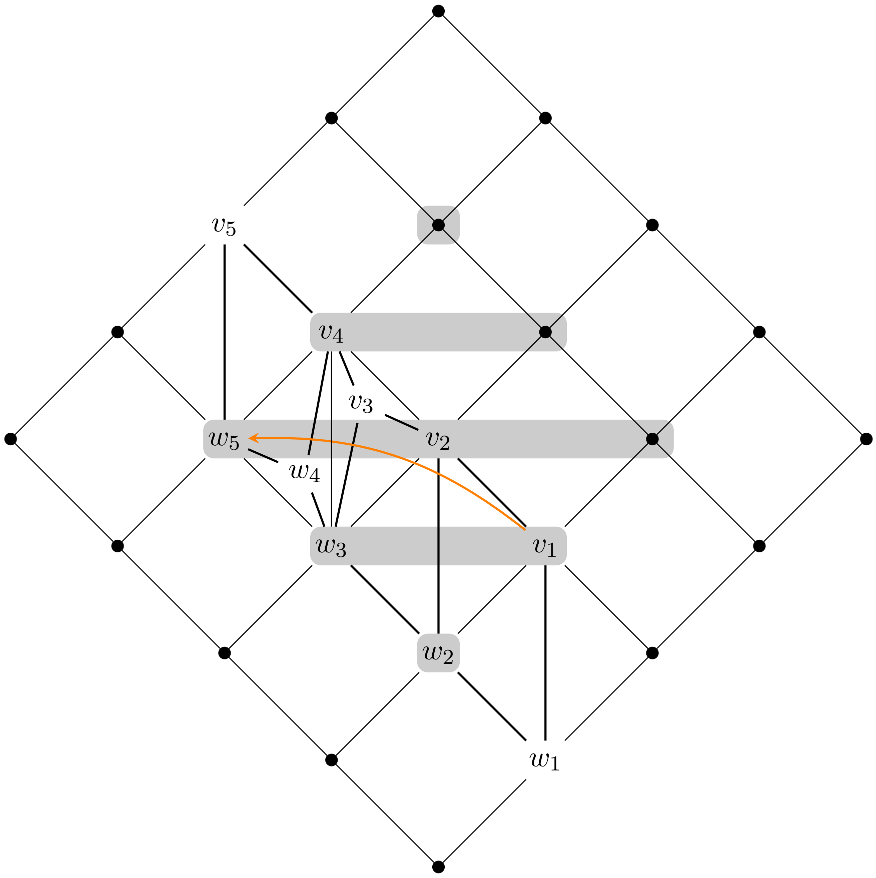


Figure 4.21: 5-Flag from $v = v_1$ to $w = w_5$, where w is the second left upper vertex of v

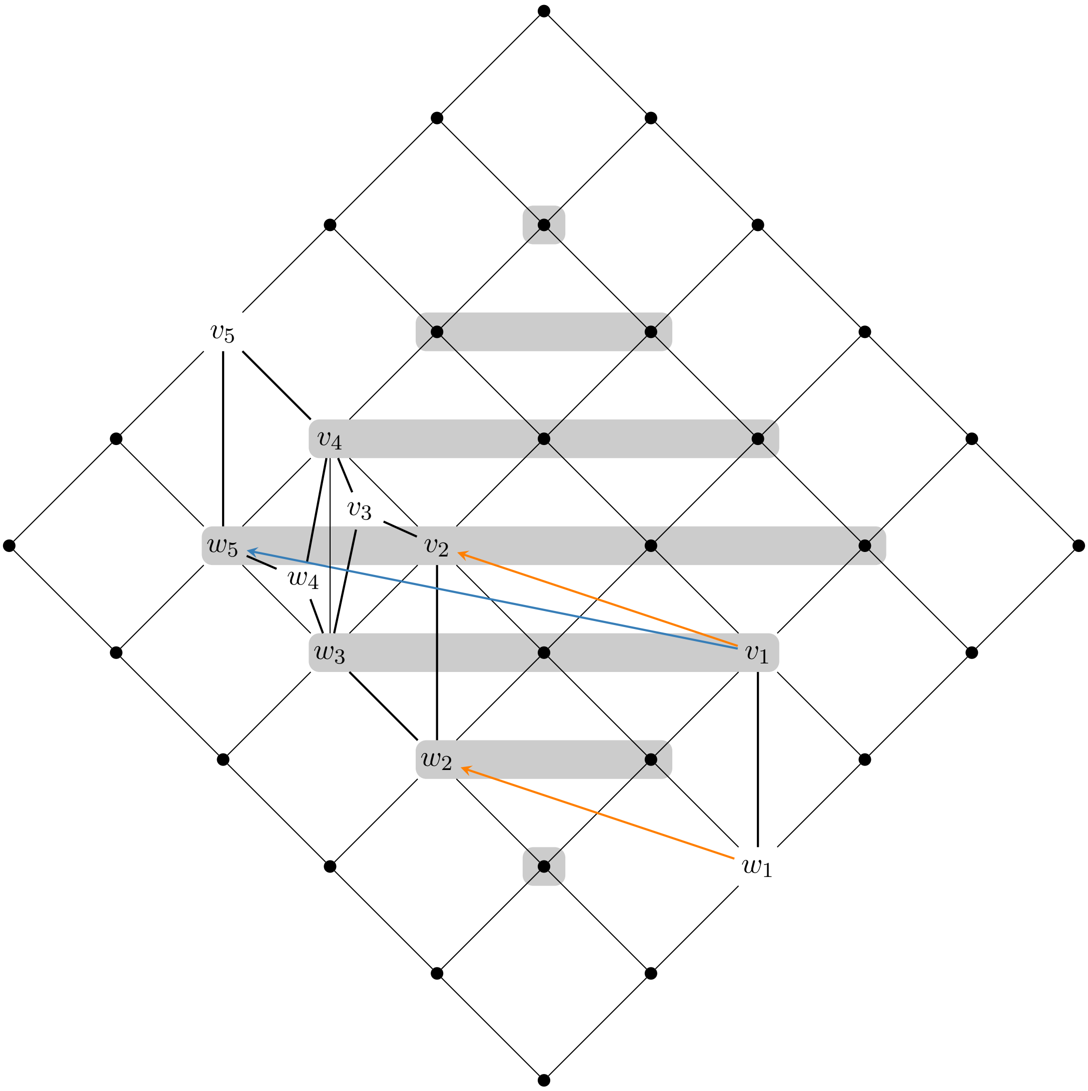


Figure 4.22: 5-Flag from $v = v_1$ to $w = w_5$, where w is the third left upper vertex of v. The orange edges exist by induction.

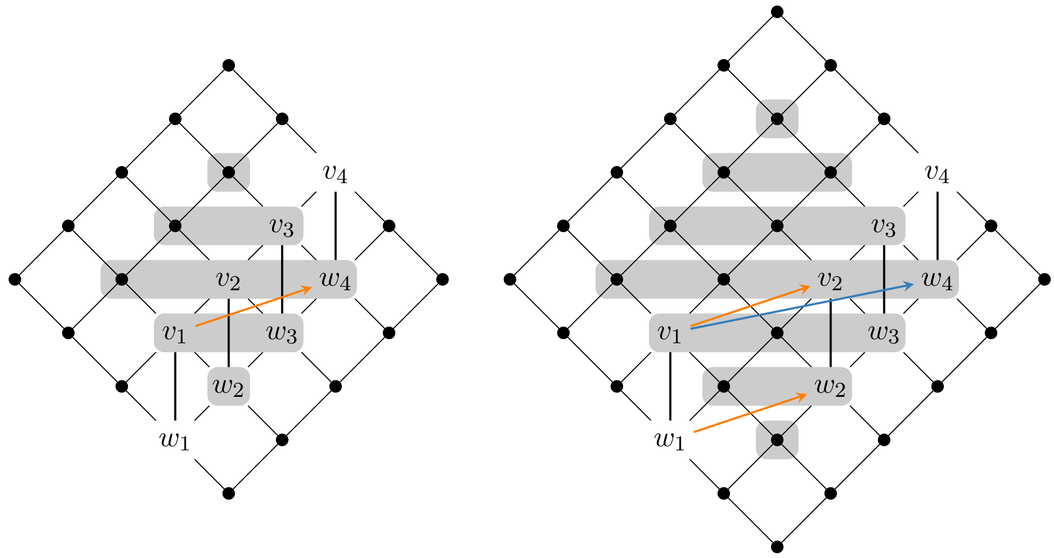


Figure 4.23: 4-flag from $v = v_1$ to $w = w_4$, where w is the second right upper vertex (left), respectively the third right upper vertex of v (right)

We remark that the same argumentation as for N_n in the proof of Lemma 4.10 works for Grid_n if we use 4-flags instead of 5-flags. For this, the two non-vertical N-edges in each 5-flag are replaced by a vertical edge as shown in Figure 4.23. Also note that the graph shown in Figure 4.9 whose augmented graph G_4^* contains a cycle is a subgraph of an N-grid.

We next use the separated levels of an N-grid the find large twists. Note that the following proposition strengthens Example 4.3 in that it uses 5-flags instead of 2-flags. Also recall that if the maximum page number among upward planar graphs is bounded by some constant c, then every optimal book embedding of an upward planar graph uses a topological ordering of G_k^* as spine ordering, for some $k \leqslant c+1$. We show that such a k, if it exists, is at least 6.

Proposition 4.11. *For every $p \geqslant 0$, there is an upward planar graph G such that no topological ordering of G_5^* admits a p-page book embedding.*

Proof. Following the ideas of Example 4.3, we find $r = p^3 + 1$ triangles whose vertices are ordered by the levels of an $n \times n$ N-grid, where $n = r + 1$. For this, consider the levels L_n, L_{n+1}, and L_{n+2} of N_n. See Figure 4.24 for an example. Observe that each of these levels has at least r vertices. For $i = 1, \ldots, r$, we define a triangle T_i consisting of the vertices $x_i = (n-i, i) \in L_n$, $y_i = (n-i, i+1) \in L_{n+1}$, and $z_i = (n-i+1, i+1) \in L_{n+2}$. We fix a book embedding whose spine ordering $\prec$ is a topological ordering of G_5^*. By Lemma 4.10, we may assume that $L_n \prec L_{n+1} \prec L_{n+2}$.

We now define an ordering $\prec_T$ on the triangles and use it to find a $(p+1)$-twist. We define $T_i \prec_T T_j$ if and only if $x_i \prec x_j$. We then say a sequence of vertices $y_{i_1}, \ldots, y_{i_s}$ is *increasing* if their ordering corresponds to $\prec_T$, that is if $y_i \prec y_j$ if and only if $T_i \prec_T T_j$

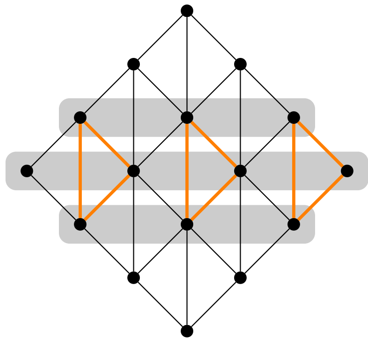


Figure 4.24: Three triangles in N_4 with vertices in levels L_4, L_5, L_6 (bottom to top)

for every $i, j = i_1, \ldots, i_s$. Similarly, a sequence of vertices $y_{i_1}, \ldots, y_{i_s}$ is called *decreasing* if their reverse ordering corresponds to $\prec_T$, that is if $y_j \prec y_i$ if and only if $T_i \prec_T T_j$ for every $i, j = i_1, \ldots, i_s$. Increasing and decreasing sequences of vertices in level L_{n+2} are defined analogously.

We now only consider the subgraph of N_n that is induced by the triangles $T_1, \ldots, T_r$. That is, a neighbor of a vertex v refers to a vertex in the same triangle as v. If there is an increasing sequence of $p+1$ (not necessarily consecutive) vertices in L_{n+1}, then we have a $(p+1)$-twist between these vertices and their neighbors in L_n. Hence, there are at most p increasing vertices and by Erdős and Szekeres [22] we have a decreasing sequence of p^2+1 vertices $v_1, \ldots, v_{p^2+1}$ in L_{n+1}. Consider the neighbors of $v_1, \ldots, v_{p^2+1}$ in L_{n+2}. Again by Erdős and Szekeres [22], we have an increasing or a decreasing sequence of size $p+1$. In the first case, we have a $(p+1)$-twist consisting of edges between L_n and L_{n+2}. In the second case, we have $(p+1)$-twist consisting of edges between L_{n+1} and L_{n+2}. In either case, the constructed graph does not admit a p-page book embedding. □

The next lemma again restricts which edges we use as flag edges. Edges that are not used as flag edges can then be replaced by flags to obtain a poset with page number at least 5.

Lemma 4.12. *There is an n such that every book embedding of an $n \times n$ N-grid has a 5-twist that uses only N-edges and right edges as flag edges.*

Proof. Similar to the proof of Proposition 4.11, we find many copies of a small graph in an N-grid whose levels are separated in the spine ordering. Recall that Lemma 4.10 separates the levels of an N-grid using only N-edges as flag edges. Here, the small subgraphs consist only of vertical edges and right edges. We then find a 5-twist using multiple of these copies.

We choose a large n with $n \equiv 1 \mod 3$ in the course of the proof and first define pairwise disjoint subgraphs $G_1, \ldots, G_{(n-1)/3}$ whose vertices are contained in six levels of N_n, see Figure 4.25. For $i = 1, \ldots, (n-1)/3$, the subgraph G_i consists of the vertices

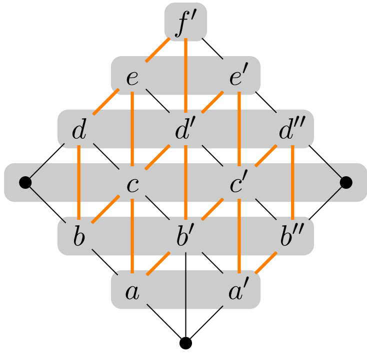


Figure 4.25: Subgraph G_1 in an 4×4 N-grid

$$
\begin{array}{rcll}
a_i &=& (n-2-3(i-1), & 1+3(i-1)) \in L_{n-1}, \\
a'_i &=& (n-3-3(i-1), & 2+3(i-1)) \in L_{n-1}, \\
b_i &=& (n-1-3(i-1), & 1+3(i-1)) \in L_{n}, \\
b'_i &=& (n-2-3(i-1), & 2+3(i-1)) \in L_{n}, \\
b''_i &=& (n-3-3(i-1), & 3+3(i-1)) \in L_{n}, \\
c_i &=& (n-1-3(i-1), & 2+3(i-1)) \in L_{n+1}, \\
c'_i &=& (n-2-3(i-1), & 3+3(i-1)) \in L_{n+1}, \\
d_i &=& (n \quad\; -3(i-1), & 2+3(i-1)) \in L_{n+2}, \\
d'_i &=& (n-1-3(i-1), & 3+3(i-1)) \in L_{n+2}, \\
d''_i &=& (n-2-3(i-1), & 4+3(i-1)) \in L_{n+2}, \\
e_i &=& (n \quad\; -3(i-1), & 3+3(i-1)) \in L_{n+3}, \\
e'_i &=& (n-1-3(i-1), & 4+3(i-1)) \in L_{n+3}, \text{ and} \\
f'_i &=& (n \quad\; -3(i-1), & 4+3(i-1)) \in L_{n+4}.
\end{array}
$$

The edges of G_i are the vertical and right edges with both endpoints in G_i. We call the graphs $G_1, \dots, G_{(n-1)/3}$ *copies*.

Consider a book embedding that does not have a 5-twist consisting of N-edges. By Lemma 4.10, we may assume that the levels of N_n are separated, that is we have $L_h \prec L_{h+1}$ for $h = 2, \dots, 2n-1$. We aim to find a 5-twist consisting of edges of some copies. As the copies contain only vertical edges and right edges, this gives a 5-twist consisting of vertical and right edges.

Let $X = \{a, a', b, b', b'', c, c', d, d', d'', e, e', f'\}$ denote a set of thirteen elements representing the respective vertices in some copy G_i. Two corresponding vertices x_i and x_j in different copies are called *twin vertices* and two edges $x_i y_i \in E(G_i)$ and $x_j y_j \in E(G_j)$ are called *twin edges* for $x, y \in X$. Now, consider two copies G_i and G_j with $i < j$. We define a vector v^{ij}, indexed by X, that indicates the order of each pair of corresponding vertices of G_i and G_j. That is, for $x \in X$ we define

$$
v_x^{ij} = \begin{cases} \text{forward} & \text{if } x_i \prec x_j \\ \text{backward} & \text{if } x_i \succ x_j. \end{cases}
$$

We call v^{ij} the *interleaving vector* of G_i and G_j.

Next, we argue that for large enough n, there is an arbitrary number of copies satisfying the following conditions.

(i) The interleaving vector is the same for each pair of copies.

(ii) Each two twin edges are nesting.

(iii) Twin vertices of each two copies are laid out in the same order. That is, if x_i and y_i are vertices in some copy and x_j and y_j are their twin vertices in another copy, then we have $x_i \prec y_i$ if and only if $x_j \prec y_j$.

For Condition (i), we consider the complete graph K having the copies $G_1, \dots, G_{(n-1)/3}$ as vertex set. We color each edge of K with the interleaving vector of its endpoints. Note that we have at most $2^{|X|} = 2^{13}$ colors. By the multi color Ramsey theorem [46], for each k there is an n such that every edge-coloring of $K_{(n-1)/3}$ with 2^{13} colors contains a monochromatic complete graph on k vertices. This monochromatic complete graph then corresponds to k copies having pairwise the same interleaving vector.

To justify the second condition, note that all twin edges have endpoints in two distinct levels. Hence, any two twin edges are crossing or nesting. We observe that twin edges of k copies with pairwise the same interleaving vector form a k-twist or a k-rainbow. We choose n large enough such that there are at least five copies having pairwise the same interleaving vector. If we have crossing twin edges, then we have a 5-twist and are done. Hence, the twin edges nest.

For Condition (iii), note that there are only three levels containing two vertices of the same copy and only two levels containing three vertices. Thus, there are at most $2^3 \cdot 6^2 = 288$ possibilities to lay out each copy. We choose $288(k-1)+1$ copies that satisfy Conditions (i) and (ii). This yields k copies whose vertices are laid out in the same order and thus satisfy all three conditions, for any k.

From now on, we assume that all considered copies satisfy Conditions (i) to (iii). We continue the proof by finding configurations that imply twists, given that we have at least five copies satisfying the first three conditions. We write G for an arbitrary copy G_i. That is, if we say that G contains a certain subgraph, then we mean that each copy contains this subgraph. The following configurations are shown in Figure 4.26.

(iv) Consider six vertices $u \prec v \prec w \prec x \prec y \prec z$ in G together with the edges uw, vw, wy, and wz forming a star and the edge vx. If such a configuration exists, then there is a 5-twist.

(v) Symmetrically, consider six vertices $u \prec v \prec w \prec x \prec y \prec z$ in G together with the edges ux, vx, xy, and xz forming a star and the edge wy. If such a configuration exists, then there is a 5-twist.

To find the claimed 5-twists, we take five copies $G_1, \dots, G_5$. We consider Configuration (iv). The 5-twist in Configuration (v) exists by symmetry. Without loss of generality, we assume $w_1 \prec \dots \prec w_5$. Since the edges $v_1w_1, \dots, v_5w_5$ nest by Condition (ii), we conclude $v_5 \prec \dots \prec v_1$. Analogously, the edges $v_1x_1, \dots, v_5x_5$ imply $x_1 \prec \dots \prec x_5$.

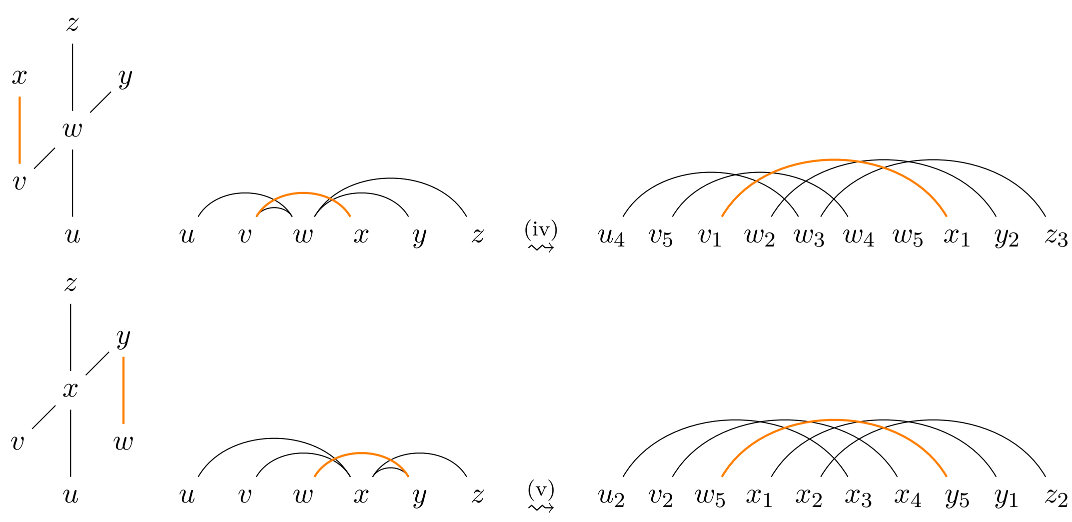


Figure 4.26: Configurations that imply a 5-twist

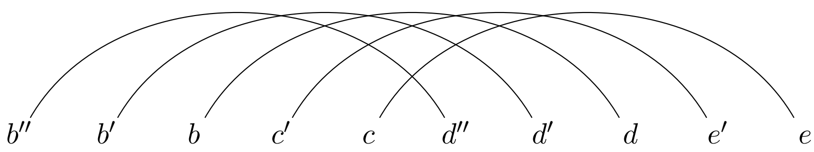


Figure 4.27: 5-twist in N_n. The vertex ordering inside each level is established by Configurations (iv) and (v).

As we have $x \prec y$ in Configuration (iv), we obtain $x_1 \prec x_2 \prec y_2$. Together, this gives $u_4 \prec v_5 \prec v_1 \prec w_2 \prec w_3 \prec w_4 \prec w_5 \prec x_1 \prec y_2 \prec z_3$ and a 5-twist consisting of the edges $u_4w_4, v_5w_5, v_1x_1, w_2y_2$, and w_3z_3.

We now set out to identify the vertex ordering of some copy. Recall that all copies have the same vertex ordering by Condition (iii). We omit the indices indicating the copy therefore. As the graph is laid out level-wise, we only need to consider pairs of vertices that are in the same level. We first find the ordering of the vertices b and b'. Suppose we have $b \prec b'$. Then we have Configuration (v) consisting of the vertices a, b, b', c, d', and e, which implies a 5-twist. Analogously, we obtain $b'' \prec b'$. Symmetrically, we get Configuration (v) consisting of the vertices a, b, c, d, d', and e if $d \prec d'$. Again, we also have $d'' \prec d'$. The same configurations show up one level higher. That is, the vertices b', c, c', d', e', and f' form Configuration (v) if $c \prec c'$. If $e \prec e'$, then the vertices b', c, d', e, e', and f' form Configuration (iv). Hence, we need $c' \prec c$ and $e' \prec e$ to avoid a 5-twist.

However, we now have $b'' \prec b' \prec b \prec c' \prec c \prec d'' \prec d' \prec d \prec e' \prec e$ (see Figure 4.27), which implies a 5-twist consisting of the edges $b''d'', b'd', bd, c'e'$, and ce. □

Note that increasing the size of the copies yields twists of arbitrary size, given that we

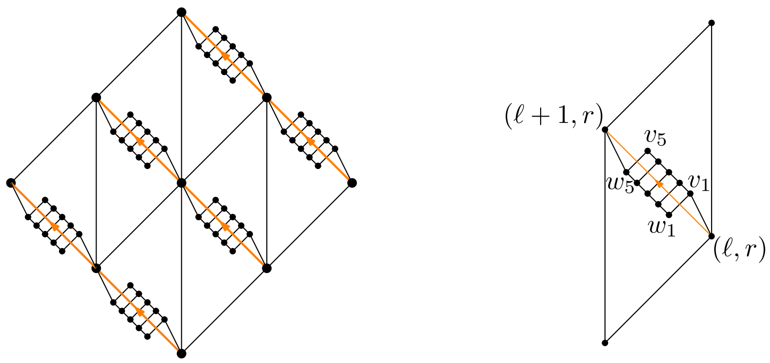


Figure 4.28: Replacing left edges by 5-flags

use a topological ordering of $N^*_{n,5}$, similar to Proposition 4.11 where we find arbitrarily many triangles. However, if we choose a spine ordering that is a topological ordering of $N^*_{n,6}$ but not of $N^*_{n,5}$, we loose the separated levels that are guaranteed by Lemma 4.10.

We are now ready to construct a poset based on an N-grid.

Theorem 4.13. *There is a planar poset that does not admit a vertex ordering without 5-twist.*

Proof. We choose n such that every book embedding of N_n has a 5-twist using only N-edges and right edges as flag edges by Lemma 4.12. We construct a poset $P = (X, <)$ with cover graph G such that G_5^+ contains N_n as a subgraph, where all N-edges and right edges are edges of G, while the other edges are created by 5-flags. Lemma 4.12 then shows that G does not admit a book embedding without 5-twist.

We start with the vertex set of Grid_n, that is $\{(\ell, r) \in \mathbb{Z}^2 \colon 1 \leqslant \ell, r \leqslant n\} \subseteq X$. Between these elements, we have cover relations $(\ell, r) < (\ell+1, r+1)$ for $1 \leqslant \ell, r < n$ corresponding to vertical edges and $(\ell, r) < (\ell, r+1)$ for $\ell = 1, \dots, n$ and $r = 1, \dots, n-1$ corresponding to right edges.

Next, we add 5-flags that result in the missing left edges as shown in Figure 4.28. For this, consider two elements (ℓ, r) and $(\ell+1, r)$ in X, where $\ell = 1, \dots, n-1$ and $r = 1, \dots, n$. We add five elements $v_1^{\ell,r}, \dots, v_5^{\ell,r}$ with cover relations $(\ell, r) < v_1^{\ell,r} < \dots < v_5^{\ell,r}$. Similarly, we add another five elements $w_1^{\ell+1,r}, \dots, w_5^{\ell+1,r}$ with cover relations $w_1^{\ell+1,r} < \dots < w_5^{\ell+1,r} < (\ell+1, r)$. Finally, we introduce cover relations $w_i^{\ell+1,r} v_i^{\ell,r}$ for $i = 1, \dots, 5$. Note that (ℓ, r) and $(\ell+1, r)$ together with the ten new vertices form a 5-flag from (ℓ, r) to $(\ell+1, r)$ in G. That is, all left edges of Grid_n occur in G_5^+ and we conclude that $\text{Grid}_n \subseteq G_5^+$.

We continue by adding the N-vertices to G as shown in Figure 4.29. Consider an element $(\ell, r) \in X$ with $\ell, r < n$. For ℓ and r with even difference, we add elements $a_{\ell,r}$ and $b_{\ell,r}$ with cover relations $a_{\ell,r}(\ell+1, r+1)$ and $(\ell, r)b_{\ell,r}$. In addition, we introduce new elements forming 5-flags from (ℓ, r) to $a_{\ell,r}$, from $a_{\ell,r}$ to $(\ell+1, r)$, from $(\ell, r+1)$ to $b_{\ell,r}$, and from $b_{\ell,r}$ to $(\ell+1, r+1)$ as above. If $\ell - r$ is odd, then we insert elements $c_{\ell,r}$ and $d_{\ell,r}$. We add edges cover relations $c_{\ell,r}(\ell+1, r+1)$ and $(\ell, r)d_{\ell,r}$ and 5-flags from (ℓ, r) to $c_{\ell,r}$, from $c_{\ell,r}$ to $(\ell, r+1)$, from $(\ell+1, r)$ to $d_{\ell,r}$, and from $d_{\ell,r}$ to $(\ell+1, r+1)$.

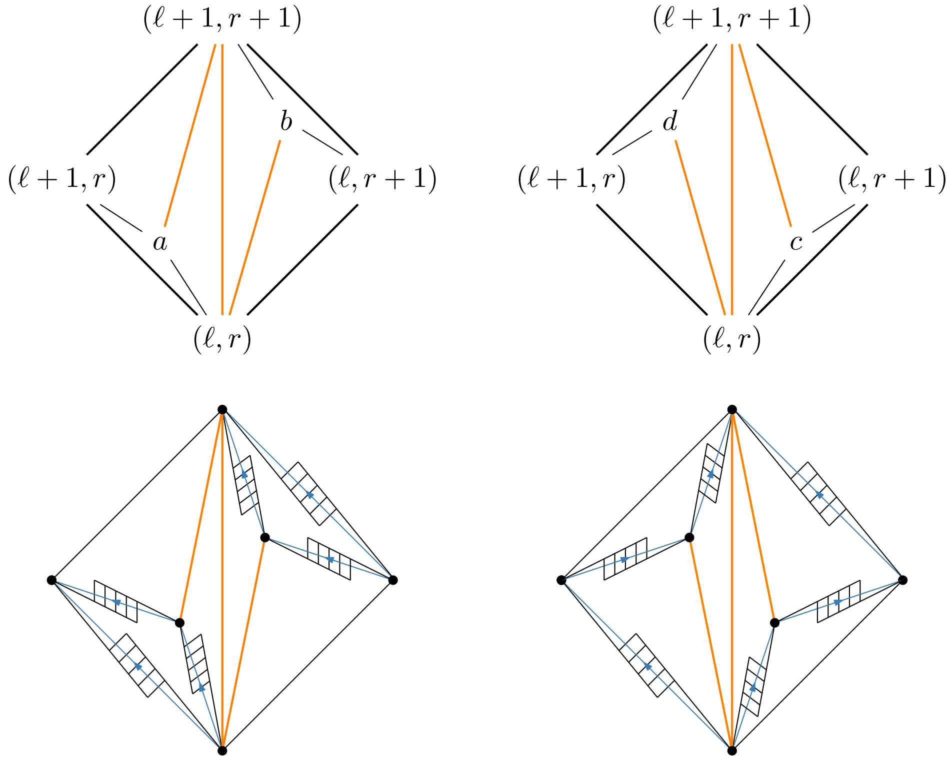


Figure 4.29: N-vertices and 5-flags in the cover graph of the poset constructed for Theorem 4.13 (bottom) in comparison with the N-edges of N_n, where $a = a_{\ell,r}$, $b = b_{\ell,r}$, $c = c_{\ell,r}$, and $d = d_{\ell,r}$ (top)

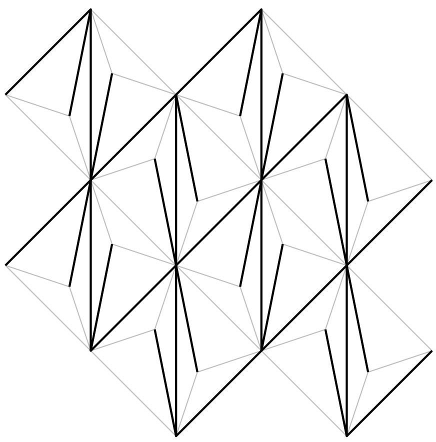


Figure 4.30: Cover relations between elements that are represented by vertices of N_n (black). The gray edges connect elements that share an edge in the augmented cover graph G_5^+ but are incomparable in P.

We observe that each added flag is the cover graph of a planar poset. Also note that adding the 5-flags keeps the cover graph planar and does not create new relations between the vertices of N_n. That is, the cover relations between these elements correspond to the N-edges and right edges in the cover graph. See Figure 4.30 for an illustration. We conclude that P is indeed the cover graph of a poset. □

Recall that a k-twist in every book embedding is stronger than having page number at least k as circle graphs are not perfect. Nevertheless, we stress the following corollary of Theorem 4.13.

Corollary 4.14. *There is a planar poset which requires at least five pages in every book embedding.*

5 Conclusions

In this thesis, we investigated local and union ordered covering numbers of complete graphs, track layouts of 2-trees, and book embeddings of upward planar graphs and planar posets. In particular, we showed bounds for the local page number, the local queue number, and the union queue number of complete graphs that are tight up to a constant additive term. However, there remains a gap between the lower bound of $n/3 + \Theta(1)$ and the upper bound of $4n/9 + \Theta(1)$ on the union page number of K_n.

Question 5.1. *What is the union page number of complete graphs?*

It also remains open to find the exact maximum track number of 2-trees between our lower bound of 7 and the upper bound of 15 by Di Giacomo et al. [16]. For arbitrary k, the lower bound on the maximum track number of k-trees is quadratic [19], while the upper bound is $2^{\mathcal{O}(k^2)}$ [48]. It would be interesting to see whether our approach for 2-trees also works for larger k, possibly resulting in a cubic lower bound.

Question 5.2. *What is the maximum track number of graphs with treewidth k, especially for $k = 2$?*

Finally, we augmented upward planar graphs with further edges to find possible vertex orderings for book embeddings with small maximum twist size. In particular, we showed that for any upward planar graph G, the augmented graph G^*_{k+1} is acyclic if the page number of G is at most k. For small k up to 4, however, we presented upward planar graphs whose augmented graphs contain cycles. We do not know whether such cycles exist for larger k and whether it helps to forbid transitive edges.

Question 5.3. *Is there a k such that for every upward planar graph G, the augmented graph G^*_k is acyclic?*

Question 5.4. *Is there a k such that for every planar poset with cover graph G, the augmented graph G^*_k is acyclic?*

Note that acyclicity is necessary but not sufficient to avoid k-twists. For instance, an $n \times n$ N-grid is acyclic when augmented by 5-flags but does not admit a vertex ordering without 5-twist. However, using 4-flags instead does create cycles (see Figure 4.9). We propose to investigate how large this gap can be.

Question 5.5. *Is there a function f such that every upward planar graph (cover graph of a planar poset) G with acyclic G^*_k satisfies* $\mathrm{pn}(G) \leqslant f(k)$*?*

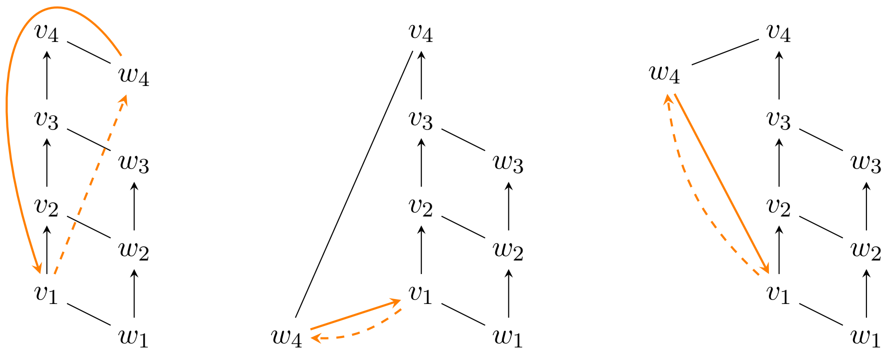


Figure 5.1: Assume there is a path (orange) from w_4 to v_1 in the augmented graph. We insert the edge w_4w_3 to avoid a 4-flag from v_1 to w_4 as this would result in the orange dashed edge closing a cycle. The left graph is an illustration of the setting. However, it is not clear whether the orange path can be realized in this embedding. In contrast, the paths in the middle and right graph can be created by an edge and by a flag from w_4 to v_1, respectively.

Given a fixed vertex ordering, recall that the page number is bounded by the maximum twist size [14]. That is, we may as well replace page number by maximum twist size in this question. A negative answer in particular would mean that the page number of upward planar graphs, respectively planar posets, is unbounded, answering longstanding open problems [15, 24, 30, 43]. On the other hand, an affirmative answer would give new hope to find upper bounds on the page number of upward planar graphs and planar posets. In the particular case of a constant upper bound, it would suffice to answer Questions 5.3 and 5.4 affirmatively. That is, the condition that there is some k such that G_k^* is acyclic for every upward planar graph G would be not only necessary but also sufficient.

Although k-flags that are laid out in the wrong order are a natural way to cause large twists, there is no obvious reason why this should be the only way. We are thus interested in more constraints on the vertex ordering. For instance, assume we try to avoid k-twists and there is a graph in which adding some edge vw yields a k-flag that closes a cycle, where vw is not used as flag edge (see Figure 5.1). Then we certainly may insert the edge wv, i.e. enforce $w \prec v$, as a cycle results in a k-twist. Note that we use the same argument for Configurations (iv) and (v) in the proof of Lemma 4.12, see Figure 5.2.

In addition, we can translate Conditions (i) to (iii) of the proof of Lemma 4.12 into a setting in which we only consider copies instead of the whole graph but in return have stronger constraints how edges inside and between these copies are directed, as indicated in Figure 5.2. For instance, if we have sufficiently many copies, we may fix an ordering of some copies and then know that edges between a set of twin vertices are all forward or all backward. Note that the order of the copies does not necessarily correspond to the left-to-right order in an upward planar embedding. However, finding flags does not rely

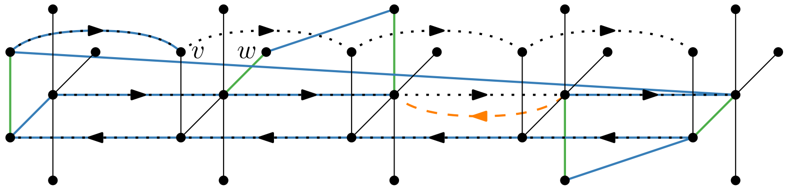


Figure 5.2: Five copies of a subgraph considered in Configuration (iv) in the proof of Lemma 4.12. The vertices are ordered level-wise, i.e. we already have edges between each two vertices that are not in the same level (omitted here except for three). The dotted edges are justified by Conditions (i) to (iii). If there is an edge from v to w, then we have a 5-flag (blue with green flag edges) that creates the orange dashed edge closing a cycle. We hence insert the edge wv.

on the embedding. Following the ideas of Lemma 4.12, we find a cycle in such a stronger augmentation of an N-grid, which is another (but very similar) proof of that lemma. We conclude by asking for stronger ways to augment an upward planar graph such that a vertex ordering that does not result in k-twists needs to be a topological ordering of the augmented graph. Questions 5.3 to 5.5 are naturally transferred to such an augmented graph.

Bibliography

[1] A. Aggarwal, M. Klawe, and P. Shor. "Multilayer grid embeddings for VLSI." In: *Algorithmica* 6.1 (June 1991), pp. 129–151. ISSN: 1432-0541. DOI: 10.1007/BF01759038.

[2] J. Akiyama, G. Exoo, and F. Harary. "Covering and packing in graphs IV: Linear arboricity." In: *Networks* 11.1 (1981), pp. 69–72. DOI: 10.1002/net.3230110108.

[3] J. Akiyama and M. Kano. "Path factors of a graph." In: *Graph theory and its Applications* (1984), pp. 11–22.

[4] J. M. Alam, M. A. Bekos, M. Gronemann, M. Kaufmann, and S. Pupyrev. "Queue Layouts of Planar 3-Trees." In: *Algorithmica* (2020), pp. 1–22. DOI: 10.1007/s00453-020-00697-4.

[5] I. Algor and N. Alon. "The star arboricity of graphs." In: *Discrete Mathematics* 75.1 (1989), pp. 11–22. ISSN: 0012-365X. DOI: 10.1016/0012-365X(89)90073-3.

[6] M. Alzohairi. "The pagenumber of ordered sets." PhD thesis. 1997. DOI: 10.20381/ruor-7833.

[7] M. J. Bannister, W. E. Devanny, V. Dujmović, D. Eppstein, and D. R. Wood. "Track Layouts, Layered Path Decompositions, and Leveled Planarity." In: *Algorithmica* 81.4 (Apr. 2019), pp. 1561–1583. ISSN: 1432-0541. DOI: 10.1007/s00453-018-0487-5.

[8] M. A. Bekos, M. Haug, M. Kaufmann, and J. Männecke. *An Online Framework to Interact and Efficiently Compute Linear Layouts of Graphs.* source code available at https://github.com/linear-layouts/SAT. 2020. arXiv: 2003.09642. URL: http://algo.inf.uni-tuebingen.de/linearlayouts/.

[9] M. A. Bekos, M. Kaufmann, F. Klute, S. Pupyrev, C. Raftopoulou, and T. Ueckerdt. *Four Pages Are Indeed Necessary for Planar Graphs.* 2020. arXiv: 2004.07630.

[10] F. Bernhart and P. C. Kainen. "The book thickness of a graph." In: *Journal of Combinatorial Theory, Series B* 27.3 (1979), pp. 320–331. ISSN: 0095-8956. DOI: 10.1016/0095-8956(79)90021-2.

[11] C. Binucci, G. D. Lozzo, E. D. Giacomo, W. Didimo, T. Mchedlidze, and M. Patrignani. "Upward Book Embeddings of st-Graphs." In: *35th International Symposium on Computational Geometry (SoCG 2019)*. Ed. by G. Barequet and Y. Wang. Vol. 129. Leibniz International Proceedings in Informatics (LIPIcs). Dagstuhl, Germany: Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2019, 13:1–13:22. ISBN: 978-3-95977-104-7. DOI: 10.4230/LIPIcs.SoCG.2019.13.

[12] J. Černý. "Coloring circle graphs." In: *Electronic Notes in Discrete Mathematics* 29 (2007). European Conference on Combinatorics, Graph Theory and Applications, pp. 457–461. ISSN: 1571-0653. DOI: 10.1016/j.endm.2007.07.072.

[13] P. Clote, S. Dobrev, I. Dotu, E. Kranakis, D. Krizanc, and J. Urrutia. "On the page number of RNA secondary structures with pseudoknots." In: *Journal of Mathematical Biology* 65.6 (Dec. 2012), pp. 1337–1357. ISSN: 1432-1416. DOI: 10.1007/s00285-011-0493-6.

[14] J. Davies and R. McCarty. *Circle graphs are quadratically χ-bounded.* 2019. arXiv: 1905.11578.

[15] E. Di Giacomo, W. Didimo, G. Liotta, and S. K. Wismath. "Book embeddability of series–parallel digraphs." In: *Algorithmica* 45.4 (2006), pp. 531–547. DOI: 10.1007/s00453-005-1185-7.

[16] E. Di Giacomo, G. Liotta, and H. Meijer. "Computing straight-line 3D grid drawings of graphs in linear volume." In: *Computational Geometry* 32.1 (2005), pp. 26–58. ISSN: 0925-7721. DOI: 10.1016/j.comgeo.2004.11.003.

[17] E. Di Giacomo and H. Meijer. "Track Drawings of Graphs with Constant Queue Number." In: *Graph Drawing.* Ed. by G. Liotta. Berlin, Heidelberg: Springer Berlin Heidelberg, 2004, pp. 214–225. ISBN: 978-3-540-24595-7. DOI: 10.1007/978-3-540-24595-7_20.

[18] V. Dujmović, G. Joret, P. Micek, P. Morin, T. Ueckerdt, and D. R. Wood. "Planar Graphs Have Bounded Queue-Number." In: *J. ACM* 67.4 (Aug. 2020). ISSN: 0004-5411. DOI: 10.1145/3385731.

[19] V. Dujmović, P. Morin, and D. R. Wood. "Layout of Graphs with Bounded Tree-Width." In: *SIAM Journal on Computing* 34.3 (2005), pp. 553–579. DOI: 10.1137/S0097539702416141.

[20] V. Dujmović, A. Pór, and D. R. Wood. "Track Layouts of Graphs." In: *Discrete Mathematics & Theoretical Computer Science* Vol. 6 no. 2 (Jan. 2004). URL: https://dmtcs.episciences.org/315.

[21] V. Dujmović and D. R. Wood. "On Linear Layouts of Graphs." In: *Discrete Mathematics and Theoretical Computer Science, DMTCS* 6.2 (2004). hal-00959012, pp. 339–358. URL: http://dmtcs.episciences.org/317.

[22] P. Erdős and G. Szekeres. "A combinatorial problem in geometry." In: *Compositio Mathematica* 2 (1935), pp. 463–470. URL: http://www.numdam.org/item/CM_1935__2__463_0.

[23] S. Felsner, G. Liotta, and S. Wismath. "Straight-Line Drawings on Restricted Integer Grids in Two and Three Dimensions." In: *Graph Algorithms and Applications 4*. 2006, pp. 363–398. DOI: 10.1142/9789812773296_0017.

[24] F. Frati, R. Fulek, and A. J. Ruiz-Vargas. "On the Page Number of Upward Planar Directed Acyclic Graphs." In: *Journal of Graph Algorithms and Applications* 17.3 (2013), pp. 221–244. DOI: 10.7155/jgaa.00292.

[25] F. Giordano, G. Liotta, and S. H. Whitesides. "Embeddability Problems for Upward Planar Digraphs." In: *Graph Drawing*. Ed. by I. G. Tollis and M. Patrignani. Berlin, Heidelberg: Springer Berlin Heidelberg, 2009, pp. 242–253. ISBN: 978-3-642-00219-9. DOI: 10.1007/978-3-642-00219-9_23.

[26] D. Gonçalves. "Caterpillar arboricity of planar graphs." In: *Discrete Mathematics* 307.16 (2007). EuroComb '03 - Graphs and Algorithms, pp. 2112–2121. ISSN: 0012-365X. DOI: 10.1016/j.disc.2005.12.055.

[27] R. K. Guy and R. J. Nowakowski. "The outerthickness & outercoarseness of graphs I. The complete graph & the n-cube." In: *Topics in Combinatorics and Graph Theory: Essays in Honour of Gerhard Ringel*. Ed. by R. Bodendiek and R. Henn. Heidelberg: Physica-Verlag HD, 1990, pp. 297–310. ISBN: 978-3-642-46908-4. DOI: 10.1007/978-3-642-46908-4_34.

[28] L. S. Heath, C. J. Ribbens, and S. V. Pemmaraju. "Processor-efficient sparse matrix-vector multiplication." In: *Computers and Mathematics with Applications* 48.3 (2004), pp. 589–608. ISSN: 0898-1221. DOI: 10.1016/j.camwa.2003.06.009.

[29] L. S. Heath and S. V. Pemmaraju. "Stack and Queue Layouts of Directed Acyclic Graphs: Part II." In: *SIAM Journal on Computing* 28.5 (1999), pp. 1588–1626. DOI: 10.1137/S0097539795291550.

[30] L. S. Heath and S. V. Pemmaraju. "Stack and Queue Layouts of Posets." In: *SIAM Journal on Discrete Mathematics* 10.4 (1997), pp. 599–625. DOI: 10.1137/S0895480193252380.

[31] L. S. Heath, S. V. Pemmaraju, and A. N. Trenk. "Stack and Queue Layouts of Directed Acyclic Graphs: Part I." In: *SIAM J. Comput.* 28.4 (1999), pp. 1510–1539. DOI: 10.1137/S0097539795280287.

[32] L. S. Heath and A. Rosenberg. "Laying Out Graphs Using Queues." In: *SIAM Journal on Computing* 21.5 (1992), pp. 927–958. DOI: 10.1137/0221055.

[33] L. Hung. "A planar poset which requires 4 pages." In: *Ars Combinatoria* 35 (1993), pp. 291–302.

[34] D. Joseph, J. Meidanis, and P. Tiwari. "Determining DNA sequence similarity using maximum independent set algorithms for interval graphs." In: *Algorithm Theory — SWAT '92*. Ed. by O. Nurmi and E. Ukkonen. Berlin, Heidelberg: Springer Berlin Heidelberg, 1992, pp. 326–337. ISBN: 978-3-540-47275-9. DOI: 10.1007/3-540-55706-7_29.

[35] D. Kelly. "Fundamentals of planar ordered sets." In: *Discrete Mathematics* 63.2 (1987), pp. 197–216. ISSN: 0012-365X. DOI: 10.1016/0012-365X(87)90008-2.

[36] K. Knauer and T. Ueckerdt. "Three ways to cover a graph." In: *Discrete Mathematics* 339.2 (2016), pp. 745–758. ISSN: 0012-365X. DOI: 10.1016/j.disc.2015.10.023.

[37] A. Kostochka and J. Kratochvíl. "Covering and coloring polygon-circle graphs." In: *Discrete Mathematics* 163.1 (1997), pp. 299–305. ISSN: 0012-365X. DOI: 10.1016/S0012-365X(96)00344-5.

[38] L. Merker and T. Ueckerdt. "Local and Union Page Numbers." In: *Graph Drawing and Network Visualization.* Ed. by D. Archambault and C. D. Tóth. Cham: Springer International Publishing, 2019, pp. 447–459. ISBN: 978-3-030-35802-0. DOI: 10.1007/978-3-030-35802-0_34.

[39] L. Merker and T. Ueckerdt. *The Local Queue Number of Graphs with Bounded Treewidth.* Appears in the Proceedings of the 28th International Symposium on Graph Drawing and Network Visualization (GD 2020). 2020. arXiv: 2008.05392.

[40] F. Moazami. "Biclique Cover and Local Clique Cover of Graphs." In: *Bulletin of the Iranian Mathematical Society* 44 (2018), pp. 225–235. DOI: 10.1007/s41980-018-0016-y.

[41] P. Mutzel, T. Odenthal, and M. Scharbrodt. "The Thickness of Graphs: A Survey." In: *Graphs and Combinatorics* 14.1 (Mar. 1998), pp. 59–73. ISSN: 1435-5914. DOI: 10.1007/PL00007219.

[42] C. S. A. Nash-Williams. "Decomposition of Finite Graphs Into Forests." In: *Journal of the London Mathematical Society* s1-39.1 (1964), pp. 12–12. DOI: 10.1112/jlms/s1-39.1.12.

[43] R. Nowakowski and A. Parker. "Ordered sets, pagenumbers and planarity." In: *Order* 6 (1989), pp. 209–218. DOI: 10.1007/BF00563521.

[44] S. Pupyrev. "Improved Bounds for Track Numbers of Planar Graphs." In: *Journal of Graph Algorithms and Applications* 24.3 (2020), pp. 323–341. DOI: 10.7155/jgaa.00536.

[45] S. Ramanathan and E. L. Lloyd. "Scheduling Algorithms for Multi-hop Radio Networks." In: *SIGCOMM Comput. Commun. Rev.* 22.4 (Oct. 1992), pp. 211–222. ISSN: 0146-4833. DOI: 10.1145/144191.144283.

[46] F. P. Ramsey. "On a Problem of Formal Logic." In: *Proceedings of the London Mathematical Society* s2-30.1 (1930), pp. 264–286. DOI: 10.1112/plms/s2-30.1.264.

[47] A. Rosenberg. "The Diogenes Approach to Testable Fault-Tolerant Arrays of Processors." In: *IEEE Transactions on Computers* C-32.10 (Oct. 1983), pp. 902–910. ISSN: 0018-9340. DOI: 10.1109/TC.1983.1676134.

[48] V. Wiechert. "On the Queue-Number of Graphs with Bounded Tree-Width." In: *Electronic Journal of Combinatorics* 24.1 (2017), P1.65. URL: http://www.combinatorics.org/ojs/index.php/eljc/article/view/v24i1p65.

[49] A. Wigderson. *The Complexity of the Hamiltonian Circuit Problem for Maximal Planar Graphs.* Tech. rep. Tech. Rep. EECS 198, Princeton University, USA, 1982. URL: https://www.math.ias.edu/avi/node/820.

[50] D. R. Wood. "Queue Layouts, Tree-Width, and Three-Dimensional Graph Drawing." In: *FST TCS 2002: Foundations of Software Technology and Theoretical Computer Science*. Ed. by M. Agrawal and A. Seth. Berlin, Heidelberg: Springer Berlin Heidelberg, 2002, pp. 348–359. ISBN: 978-3-540-36206-7. DOI: 10.1007/3-540-36206-1_31.

[51] M. Yannakakis. "Embedding Planar Graphs in Four Pages." In: *J. Comput. Syst. Sci.* 38.1 (1989), pp. 36–67. DOI: 10.1016/0022-0000(89)90032-9.

[52] M. Yannakakis. "Planar Graphs That Need Four Pages." In: *Journal of Combinatorial Theory, Series B* 145 (2020), pp. 241–263. ISSN: 0095-8956. DOI: 10.1016/j.jctb.2020.05.008.