Tree-Independence Number and Tree-Chromatic Number of Graphs

Master's Thesis of

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I declare that I have developed and written the enclosed thesis completely by myself. I have not used any other than the aids that I have mentioned. I have marked all parts of the thesis that I have included from referenced literature, either in their original wording or paraphrasing their contents. I have followed the by-laws to implement scientific integrity at KIT.

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(Kilian Krause)

Abstract

This thesis surveys some recently introduced graph invariants, namely the tree-independence number and the tree-chromatic number of a graph. Instead of measuring the size of the bags in a tree decomposition (as the treewidth does), we measure the independence number or the chromatic number of the bags, respectively. Especially the tree-independence number is used in algorithmic graph theory quite often, because graph classes with bounded tree-independence number admit polynomial time algorithms for some hard graph problems, such as the Maximum Weight Independent Set problem. However, in this thesis we will focus on the structural properties of these graph parameters. In particular, we show the connection between graph classes with bounded tree-independence number and (tw, ω)-bounded graph classes. We are able to give another characterization of the class of perfect graphs in terms of the tree-chromatic number. Also, we are able to answer (in the negative) a question from [DMŠ24a], that asks whether the treewidth of any graph can be bounded from above by the product of its tree-independence number and its tree-chromatic number. Although this is not the case, we give lower and upper bounds for the treewidth of a graph in terms of the two graph invariants. Furthermore, we present a recently developed tool called the central bag method. This tool was invented to bound the treewidth of graph classes. We are able to generalize this method and hence we can use it do bound the tree-independence number and the tree-chromatic number of graph classes as well.

Zusammenfassung

Diese Arbeit gibt einen Überblick über einige kürzlich eingeführte Graphinvarianten, nämlich die Baumunabhängigkeitszahl und die Baumfärbungszahl eines Graphen. Anstatt die Größe der Taschen in einer Baumzerlegung zu messen (wie es die Baumweite tut), messen wir die Unabhängigkeitszahl bzw. die Färbungszahl der Taschen. Speziell die Baumunabhängigkeitszahl wird in der algorithmischen Graphentheorie oft verwendet, weil Graphenklassen mit beschränkter Baumunabhängigkeitszahl Polynomialzeitalgorithmen für schwere Probleme zulassen, wie zum Beispiel das Maximum Weight Independent Set Problem. In dieser Arbeit werden wir uns allerdings auf die strukturellen Eigenschaften der neuen Graphparameter konzentrieren. Insbesondere zeigen wir die Verbindung zwischen Graphklassen mit beschränkter Baumunabhängigkeitszahl und (tw, ω)-beschränkten Graphklassen. Wir können darüber hinaus eine weitere Charakterisierung der perfekten Graphen bezüglich der Baumfärbungszahl geben. Außerdem werden wir eine Frage aus [DMŠ24a] negativ beantworten, die danach fragt, ob die Baumweite eines Graphen von oben durch das Produkt seiner Baumunabhängigkeitszahl und seiner Baumfärbungszahl beschränkt ist. Obwohl das nicht der Fall ist, geben wir untere und obere Schranken für die Baumweite in Bezug auf die zwei neuen Graphinvarianten an. Darüber hinaus stellen wir ein kürzlich entwickeltes Tool namens Central Bag Method vor. Dieses Tool wurde entwickelt um zu zeigen, dass die Baumweite von Graphklassen beschränkt ist. Wir können die Methode verallgemeinern um nicht nur die Baumweite, sondern auch die Baumunabhängigkeitszahl und die Baumfärbungszahl von Graphklassen zu beschränken.

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1 Introduction

One of the most important graph parameters, if not the most important one, studied in both structural and algorithmic graph theory, is the treewidth of a graph. Intuitively, it measures how close a graph is to a tree. From the algorithmic point of view, treewidth is important because many problems that are **NP**-hard in general become tractable if the input graph has bounded treewidth. For example, the class of graphs of treewidth 1, namely the forests, admit linear-time algorithms for most important graph problems. On the other hand, from the structural point of view, treewidth has deep connections to the graph minor theory, as it is one of the key tools developed in this theory. Also, the structure of a tree is very simple and well-understood. There are many theorems that characterize trees in different ways. So it may be worth investigating the structure of graph classes that have small treewidth $k > 1$, too. It turns out that for small values of k, these classes have some nice structural properties as well. We provide the definition of treewidth based on tree decompositions in Chapter 2, after giving some standard graph theoretic definitions and a short overview about chordal and perfect graphs.

The treewidth measures the size of the bags in a tree decomposition. Now, since every bag itself induces a graph, we can not only measure its size, but also some other graph parameters, for example the independence number of the (subgraph induced by the) bag. Very recently, researchers started investigating this concept and called it the *tree-independence number* of a graph [DMŠ24a] (in fact, it was rediscovered by Dallard et. al. in [DMŠ24a] and independently defined by Yolov in [Yol18]). It turns out that the tree-independence number is another graph invariant with some very interesting results in both structural and algorithmic graph theory. In Chapter 3, we give an introduction to the tree-independence number and show some interesting results about this graph parameter, mostly in the area of structural graph theory.

The tree-independence number of a graph is connected to another concept in structural graph theory, the (tw, ω)*-bounded* graph classes. Intuitively, a graph class is (tw, ω)-bounded, if it has large treewidth only due to the presence of large cliques. Similarly, a graph class is (χ, ω)*-bounded* (or just χ*-bounded*), if it has large chromatic number only due to the presence of a large clique. We will formalize these notions in Chapter 4. Also, we can think of (tw, ω)-bounded and χ-bounded graph classes as generalizations of chordal and perfect graphs, respectively.

In Chapter 5, we will see that every graph class, that is bounded in terms of the tree-independence number, is also (tw, ω)-bounded. One of the most important questions recently asked in the area of tree-independence number is whether the converse holds: does every (tw, ω)-bounded graph class have bounded tree-independence number? Chudnovsky and Trotignon showed in [CT24] that this is not true by using a construction called *layered wheel*. We present their proof in Chapter 5.

Chapter 6 gives an introduction to the *tree-chromatic number* of a graph. Instead of measuring the independence number of the bags, we measure the chromatic number. This is another interesting graph invariant, although it is not yet studied as much in depth as the tree-independence number. After defining the tree-chromatic number and proving some basic properties, we are able to characterize the perfect graphs in terms of this graph parameter. The main result of this chapter is the answer to a question asked in [DMŠ24a] by Dallard, Milanič and Štorgel. They ask whether it is possible to bound the treewidth of a graph from above by the product of its tree-independence number and its tree-chromatic number. It turns out that this is not true and we construct certain graphs to prove that. Although we can not bound the treewidth from above by the product of the two graph invariants, we are able to give other upper bounds for the treewidth. In the general case, we give an exponential bound, and for some specific graph classes, we can give a polynomial bound. We also construct graphs where the treewidth is bounded from below by a function that is linear in the tree-independence number and the tree-chromatic number.

The last chapter of this thesis deals with a very powerful, recently developed tool called the *central bag method*. It is used in the literature to prove that certain graph classes have bounded treewidth. We generalize this idea in order to not only bound the treewidth of a graph class, but also the tree-independence number and the tree-chromatic number. Although the real-world applications of the central bag method are very evolved and beyond the scope of this thesis, we can apply the method on a very simple graph class, namely the outerplanar, 2-connected graphs. This does not bring us any new results, but we see an example of how the method is used.

2 Preliminaries

In this section, we introduce basic graph-theoretic definitions and notations that will be used throughout the thesis.

We start by noting that all graphs considered in this thesis are simple and finite, except explicitly stated otherwise. This means that there are no loops or parallel edges and the vertex set is finite. Given a graph G, we denote its vertex set and edge set by $V(G)$ and $E(G)$, respectively. For a vertex $v \in V(G)$, by $N_G(v)$ we denote the *neighborhood of v in G* and we set $N_G[v] := N_G(v) \cup \{v\}$, that is the *closed neighborhood of v in G*. If there can be no confusion, we omit the subscript and write $N(v)$ and $N[v]$.

We say that two vertex subsets $A, B \subseteq V(G)$ are *complete* (resp. *anticomplete*) if each pair $a \in A, b \in B$ is adjacent (resp. non-adjacent).

A *clique* in G is a subset of vertices $X \subseteq V(G)$ such that the vertices in X are pairwise adjacent. The *clique number* of G, denoted by $\omega(G)$, is defined as the size of a largest clique in G. Analogously, an *independent set* in G is a subset of vertices $X \subseteq V(G)$ such that the vertices in X are pairwise non-adjacent. The *independence number* of G, denoted by $\alpha(G)$, is defined as the size of a largest independent set in G. A mapping $\phi : V(G) \to \mathbb{N}$ is called a *coloring of G*. A coloring ϕ of a graph G is *proper*, if for any edge $uv \in E(G)$ it holds that $\phi(u) \neq \phi(v)$. The *chromatic number* of G, denoted by $\chi(G)$, is the smallest integer k such that G can be colored properly with k colors, i.e. there is a mapping $\phi : V(G) \to \{1, \ldots, k\}$ with $\phi(u) \neq \phi(v)$ if uv is an edge in G.

In this thesis, we consider different containment relations of graphs. For a subset $X \subseteq V(G)$, by $G[X]$ we denote the *induced subgraph* of G, induced by the vertices of X. Formally, we set

$$G[X] := \left(X, \binom{X}{2} \cap E(G)\right).$$

If G' is an induced subgraph of G, we also write $G' \subseteq_{\text{ind}} G$. By *subdividing an edge $e = uv \in E(G)$*, we define the operation of deleting the edge e from G and adding a vertex w and two edges uw and wv. Formally, the graph obtained from this operation is $(V(G) \cup \{w\}, (E(G) \setminus \{e\}) \cup \{uw, wv\})$. A *subdivision* (or a *topological minor*) of a graph G is a graph G' obtained from G by a sequence of edge subdivisions.

By *contracting an edge $e = uv \in E(G)$*, we define the operation of deleting u and v, adding a new vertex w and making it adjacent to all vertices that were adjacent to u and v before. All loops and parallel edges that arise by this operation are deleted afterwards. A graph H is a *minor* of a graph G, if H can be obtained from G by a sequence of vertex deletions, edge deletions and edge contractions. A graph H is an *induced minor* of a graph G, if H can be obtained from G by a sequence of vertex deletions and edge contractions.

We denote the complete graph by K_n, the empty graph (or edgeless graph) by E_n, the complete bipartite graph by $K_{m,n}$, the cycle by C_n and the path by P_n, respectively. Sometimes we refer to K_3 as a triangle.

The *complement* of a graph G is denoted by $\overline{G}$ and is obtained from G by flipping all edges and non-edges. Formally, we set

$$\overline{G} := \left(V(G), \binom{V(G)}{2} \setminus E(G)\right).$$

A *separation* of a graph G is a triple (A, C, B) such that A, C and B are disjoint, $A \cup C \cup B = V(G)$, and A is anticomplete to B. If $S = (A, C, B)$ is a separation, we set $A(S) := A$, $B(S) := B$ and $C(S) := C$. The *size* of S is $|C|$. We say that S is *balanced* if $|A| \leq 2n/3$ and $|B| \leq 2n/3$ with $n = |V(G)|$. The *separation number* $\mathrm{sn}(G)$ *of* G is the smallest integer s such that every subgraph of G has a balanced separation of size at most s.

We consider and analyse many different graph classes in this thesis, two of the most important ones, the chordal graphs and the perfect graphs, will be introduced in the following section. Another important graph class is the class of planar graphs. We say that a graph G is *planar*, if we can draw G in the plane such that no edges cross, except at their endpoints. This "definition" is far away from being mathematically precise, but it is sufficient for our purpose.

2.1 Chordal graphs and perfect graphs

Two important and well-studied graph classes are the chordal graphs and the perfect graphs. In this section we define chordal and perfect graphs and show some characterizations that we will use later.

A *hole* in a graph G is an induced cycle of length at least four. An *antihole* in G is an induced cycle of length at least four in $\overline{G}$. Holes and antiholes are called *odd* (resp. *even*) if their number of vertices is odd (resp. even).

A graph is *chordal* if it does not contain any hole. The word chordal arose from the following equivalent definition. We call a graph G *chordal*, if every cycle of G of length at least four contains a chord, i.e. an edge connecting two non-consecutive vertices of the cycle. An immediate consequence of these definitions is that every induced cycle in a chordal graph is a triangle.

A graph G is *perfect* if $\omega(G') = \chi(G')$ holds for every induced subgraph G' of G. The celebrated Strong Perfect Graph Theorem (SPGT), proven by Chudnovsky, Robertson, Seymour and Thomas in [CRST06], is a characterization of perfect graphs in terms of odd holes and odd antiholes:

Theorem 2.1: *[CRST06] A graph G is perfect if and only if G neither contains an odd hole nor an odd antihole.*

In Section 6.1, we present another characterization of perfect graphs using the SPGT. We can also use the Strong Perfect Graph Theorem to show the following result.

Corollary 2.2: *Every chordal graph is perfect.*

Proof. Let G be a chordal graph. By Theorem 2.1, it is sufficient to show that G contains no odd holes and no odd antiholes. By the definition of a chordal graph, G contains no holes and therefore no odd holes.

Now, suppose that G contains an odd antihole C. If $C = C_5$, then $\overline{G}$ contains C_5 as an induced subgraph. Observe that $\overline{C} = \overline{C_5} = C_5$, so C_5 is an induced subgraph of G, a contradiction.

Now assume that $C = C_t$ for $t > 5$, t odd. Then $\overline{G}$ contains C_t as an induced subgraph. Note that the complement $\overline{C_t}$ of C_t contains a hole of length 4, therefore G contains a hole of length 4, a contradiction. This concludes the proof. ∎

The converse of Corollary 2.2 does not hold. For two positive integers $m, n \geq 2$, the complete bipartite graph $K_{m,n}$ is perfect, since it does not contain odd holes (which is a well-known characterization of bipartite graphs). It also contains no odd antihole, since $\overline{K_{m,n}} = K_m + K_n$ is the disjoint union of two cliques. But, clearly, $K_{m,n}$ contains a C_4 as an induced subgraph, and therefore it is not chordal. This means that the chordal graphs are properly contained in the class of perfect graphs.

We now want to present some characterizations of chordal graphs that we will use later in this thesis. We begin with the following definition. Let G be a graph and let $v \in V(G)$. Then v is called *simplicial* in G, if the neighborhood of v induces a clique. One can show the following characterization of chordal graphs:

Lemma 2.3: *[KN12 | TW15] A graph G is chordal if and only if every induced subgraph G' of G contains a simplicial vertex.*

Given a family of sets $\{R_i\}_{i \in I}$, the *intersection graph* of $\{R_i\}_{i \in I}$ is the graph G with $V(G) := \{v_i \mid i \in I\}$ and there is an edge between v_i and v_j if and only if $R_i \cap R_j \neq \emptyset$. Some well-studied intersection graphs are *interval graphs* and *circular arc graphs*. Here the sets R_i are intervals on the real line and circular arcs on the unit circle, respectively.

We want to consider other families of sets. Let H be a graph and let $\{H_i\}$ be a family of subgraphs of H. We say that two subgraphs H_i and H_j intersect, if they have a vertex in common, i.e. $V(H_i) \cap V(H_j) \neq \emptyset$. That being said, we can create an intersection graph G of $\{H_i\}$. Now, the second characterization of a chordal graph is based on this idea.

Lemma 2.4: *[Gav74] A graph G is chordal if and only if there exists a tree T and a family of subtrees $\{T_v \subseteq_{ind} T \mid v \in V(G)\}$ such that $vw \in E(G)$ if and only if $V(T_v) \cap V(T_w) \neq \emptyset$.*

In other words, a graph G is chordal if and only if G is the intersection graph of subtrees of a tree.

2.2 Tree decompositions and treewidth

One of the main graph parameters studied in both structural and algorithmic graph theory is the treewidth of a graph. Roughly speaking, it measures how close a graph is to a tree. Treewidth was first defined by Bertelè and Brioschi in [BB73]. It was examined in depth by Robertson and Seymour in their famous series of papers from 1982 until 2004, where they introduced the graph minor theory, culminating in the proof of Wagner's conjecture [RS04], nowadays known as the Graph Minor Theorem or the Robertson-Seymour Theorem. The treewidth of a graph is defined via its tree decompositions.

A *tree decomposition* of a graph G is a pair $(T, \{X_t\}_{t \in V(T)})$ where T is a tree and every node $t \in V(T)$ is assigned a vertex subset $X_t \subseteq V(G)$ called a *bag* such that the following conditions are satisfied:

(i) $\bigcup_{t \in V(T)} X_t = V(G)$, i.e. every vertex $v \in V(G)$ appears in at least one bag,

(ii) for every edge $uv \in E(G)$ there exists a node $t \in V(T)$ such that $u, v \in X_t$,

(iii) for every vertex $v \in V(G)$ the induced subgraph $T_v := T[\{t \in V(T) \mid v \in X_t\}]$ of T is connected, i.e. T_v is a subtree of T.

We often refer to vertices of the tree T as *nodes* for convenience, as we already did in the definition, in order to distinguish between vertices of the original graph and vertices of the tree decomposition. We abbreviate $(T, \{X_t\}_{t \in V(T)})$ by (T, X_t) if the vertex set of the tree is clear from context.

Condition (iii) in the definition of a tree decomposition is sometimes replaced with the following, equivalent condition:

(iii)* for all nodes $i, j, k \in V(T)$, if j lies on the unique path from i to k in T, then $X_i \cap X_k \subseteq X_j$.

Given a tree decomposition $\mathcal{T} = (T, X_t)$ of a graph G, we define the *width* of $\mathcal{T}$ by $\max_{t \in V(T)} |X_t| - 1$. The *treewidth* of G, denoted by $\mathrm{tw}(G)$, is defined as the minimum possible width of a tree decomposition of G. More formally,

$$\mathrm{tw}(G) := \min_{(T, X_t)} \max_{t \in V(T)} |X_t| - 1,$$

minimizing over all tree decompositions of G.[1]

Figure 2.1 shows an example of a graph G and a possible tree decomposition of G. Since the width of this tree decomposition is 3, we can conclude that $\mathrm{tw}(G) \leq 3$, which is best possible, as we see later.

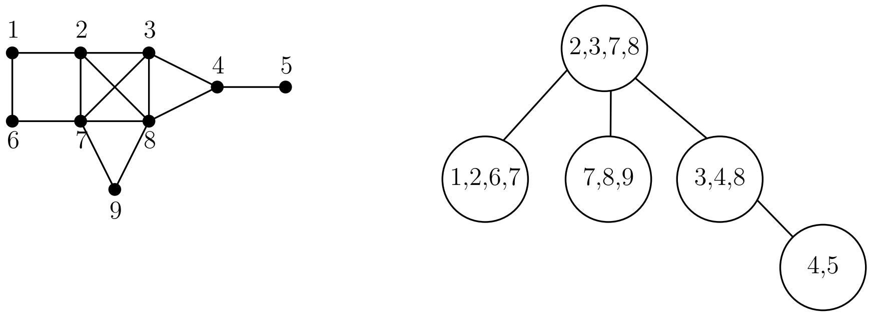


Figure 2.1: A graph G and a possible tree decomposition of G.

Now we prove some useful properties of tree decompositions which we use throughout the thesis.

Lemma 2.5: *[KN12] Let G be a graph. Then, there exists a tree decomposition $\mathcal{T} = (T, X_t)$ of G of width $\mathrm{tw}(G)$ such that $X_{t_1} \not\subseteq X_{t_2}$ for all $t_1, t_2 \in V(T), t_1 \neq t_2$.*

Proof. Let $\mathcal{T} = (T, X_t)$ be a tree decomposition of G of width $\mathrm{tw}(G)$, such that $\mathcal{T}$ is minimal with respect to the number of nodes in T. We prove that $\mathcal{T}$ satisfies the stated property.

Assume that there exist two nodes $t_1, t_2 \in V(T)$, $t_1 \neq t_2$, such that $X_{t_1} \subseteq X_{t_2}$. Let $t_3 \in V(T)$ be a node in T sitting on the unique path P between t_1 and t_2 in T. Note that $t_3 = t_1$ or $t_3 = t_2$ is possible. By condition (iii)* of the definition of a tree decomposition, we have $X_{t_1} = X_{t_1} \cap X_{t_2} \subseteq X_{t_3}$. Hence, $X_{t_1} \subseteq X_{t_3}$ for all $t_3 \in V(T) \cap V(P)$.

Consider the tree T' that is obtained from T by contracting the edge t_1t', where t' is the neighbor of t_1 in P ($t' = t_2$ is possible). After contracting the edge t_1t', we call the new node t^* and we set $X_{t^*} := X_{t'}$. Observe that we obtain a valid tree decomposition of G of width $\mathrm{tw}(G)$ and $|V(T')| < |V(T)|$, contradicting the minimality of T. ∎

Lemma 2.6: *[KN12] Let G be a graph and let C be a clique in G. Then, every tree decomposition $\mathcal{T}$ of G has a bag that contains C.*

Proof. We prove the statement by induction on $k := |C|$. For $k = 1$ and $k = 2$ the claim follows by the first two conditions of the definition of a tree decomposition.

Now let $k \geq 3$, let $u, v, w \in C$ and let $\mathcal{T} = (T, X_t)$ be a tree decomposition of G. By induction, there are bags X_a, X_b and X_c with

$$C \setminus \{u\} \subseteq X_a, \quad C \setminus \{v\} \subseteq X_b \quad \text{and} \quad C \setminus \{w\} \subseteq X_c.$$

We may assume that a, b and c are distinct nodes in T, otherwise we are done. Now consider the three unique paths $P_{a,b}, P_{b,c}$ and $P_{a,c}$ in T, that connect the nodes a and b, b and c and a and c, respectively. These paths have a node $d \notin \{a, b, c\}$ in common, otherwise there would be a cycle in T, a contradiction. By condition (iii)* in the definition of a tree decomposition we conclude that

$$C \setminus \{u, v\} \subseteq X_a \cap X_b \subseteq X_d, \quad C \setminus \{u, w\} \subseteq X_a \cap X_c \subseteq X_d \quad \text{and} \quad C \setminus \{v, w\} \subseteq X_b \cap X_c \subseteq X_d.$$

This implies $C \subseteq X_d$, which finishes the proof. ∎

[1]Indeed, the only reason for the -1 in the definition of the width is to make sure that trees have treewidth 1.

An immediate consequence of Lemma 2.6 is the following corollary.

Corollary 2.7: *For every graph G it holds that* $\mathrm{tw}(G) \geq \omega(G) - 1$. *In particular,* $\mathrm{tw}(K_n) = n - 1$.

Referring back to Figure 2.1, Lemma 2.6 is the reason why the tree decomposition is optimal: The graph G contains a clique of size 4 and by Lemma 2.6, every tree decomposition of G has a bag that contains this clique. Therefore, $\mathrm{tw}(G) \geq 3$ and hence $\mathrm{tw}(G) = 3$.

Lemma 2.8: *[DMŠ24a] Let G be a graph and let $\mathcal{T} = (T, X_t)$ be a tree decomposition of G. Then there exists a vertex $v \in V(G)$ and a node $t \in V(T)$ such that $N[v] \subseteq X_t$.*

Proof. Let G' be the graph with vertex set $V(G)$ such that two distinct vertices u and v are adjacent in G' if and only if there exists a bag X_t of $\mathcal{T}$ with $u, v \in X_t$. Note that for every vertex $v \in V(G)$ it holds that $N_G[v] \subseteq N_{G'}[v]$. Recall that for each vertex $v \in V(G')$, the set of nodes $t \in V(T)$ such that $v \in X_t$ induces a subtree T_v of T. Thus, two distinct vertices u and v of G' are adjacent if and only if the corresponding trees T_u and T_v have a node in common. This means that G' is the intersection graph of subtrees of a tree and hence, by Lemma 2.4, G' is chordal. Observe that $\mathcal{T}$ is also a tree decomposition of G'. Now since G' is chordal, it has a simplicial vertex v by Lemma 2.3, which means that $N_{G'}[v]$ forms a clique. Thus, by Lemma 2.6, there exists a node $t \in V(T)$ such that $N_{G'}[v] \subseteq X_t$. Therefore, $N_G[v] \subseteq X_t$, which concludes the proof. ∎

We already saw some characterizations of chordal graphs, that we used in the previous lemma, in Section 2.1. Here we give another one that we need in Section 3.

Lemma 2.9: *[Die17] A graph G is chordal if and only if there exists a tree decomposition $\mathcal{T}$ of G such that the bags of $\mathcal{T}$ are exactly the maximal cliques in G.*

We point out that a maximal clique is not necessarily of maximum size. A clique C in a graph G is called *maximal* if for every vertex $v \in V(G) \setminus C$, $C \cup \{v\}$ is not a clique. So, we can not add any other vertex to a maximal clique and obtain a clique that is larger in size.

Clearly, the tree decomposition in Lemma 2.9 has a bag that contains a clique of G of largest size. Together with Lemma 2.6 we conclude that the treewidth of a chordal graph G is equal to $\omega(G) - 1$.

We defined the treewidth of a graph as a measure of the bags of a tree decomposition. But we can also define the treewidth in another way, closely relating it to chordal graphs. Before we state and prove this statement, we want to note that every graph G is a subgraph of some chordal graph H. If G itself is chordal, we are done. Otherwise, take a hole in G and add a chord (an edge connecting two non-consecutive vertices of the hole) to it. Repeat this process until there are no holes left in G. Eventually, this process has to stop, since the complete graph is chordal and we can not add any edges to it. In fact, the graph obtained from the complete graph by deleting an arbitrary edge is already a chordal graph.

Theorem 2.10: *[Die17] Let G be a graph. Then*

$$\mathrm{tw}(G) = \min\{\omega(H) - 1 \mid G \subseteq H \text{ and } H \text{ is chordal}\}.$$

Proof. Let G be a graph and let H be a graph obtained from G by adding edges, such that H is chordal. By Lemma 2.6 and 2.9, there is a tree decomposition of H of width $\omega(H) - 1$. This tree decomposition is also one of G, since adding edges does not break any of the conditions (i)-(iii) in the definition of a tree decomposition. This shows $\mathrm{tw}(G) \leq \omega(H) - 1$ for every such H.

To show the converse, we construct a chordal graph H with $G \subseteq H$ such that $\omega(H) - 1 \leq \mathrm{tw}(G)$. Let $\mathcal{T} = (T, X_t)$ be a tree decomposition of G of width $\mathrm{tw}(G)$. Let H be a copy of G and for every node $t \in V(T)$ and every pair $u, v \in X_t$ we add an edge uv to H. Clearly, $\mathcal{T}$ is also a tree decomposition of H, where every bag of $\mathcal{T}$ now induces a complete graph in H. Hence, by Lemma 2.9, H is a chordal graph and by Lemma 2.6, $\omega(H) - 1$ is at most the width of $\mathcal{T}$, that is at most $\mathrm{tw}(G)$. This finishes the proof. ∎

We introduced the notion of a minor as a containment relation that is obtained by a sequence of graph operations. There exists another definition of a graph minor, which we call minor models. A *minor model of a graph H in a graph G* is a partition of $V(G)$ into sets $V_1, \dots, V_{|V(H)|}$ such that for all $1 \leq i \leq |V(H)|$, $G[V_i]$ is connected and for every edge $ij \in E(H)$ there exists an edge in G connecting the sets V_i and V_j.

The following lemma, that we state without a proof here, shows that the two definitions of a graph minor are equivalent.

Lemma 2.11: *[Die17] A graph H is a minor of a graph G if and only if there exists a minor model of H in G.*

In Figure 2.2, we see that the graph H is a minor of G by Lemma 2.11. Note that H is neither a subgraph, nor an induced subgraph of G.

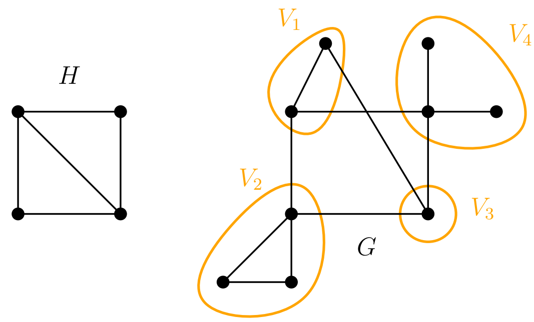


Figure 2.2: The graph H is a minor of G, since there is a minor model of H in G.

Depending on the problem at hand, we can decide which definition of a minor suits us best.

For our next results we need another definition: We say that a family of sets $\{T_i\}_{i \in I}$ satisfies the *Helly property* if for any $J \subseteq I$ the following holds: if $T_i \cap T_j \neq \emptyset$ for all $i, j \in J$, then $\bigcap_{j \in J} T_j \neq \emptyset$.

A family of closed intervals on the real line $\{[a, b] \mid a, b \in \mathbb{R}, a \leq b\}$ is one example for a family that satisfies the Helly property. On the other hand, a family of circular arcs on the unit circle does not fulfill the Helly property. The following well-known result states that a family of subtrees of a tree satisfies the Helly property.

Lemma 2.12: *Let T be a tree and let $\mathcal{F} = \{T_i\}_{i \in I}$ be a family of subtrees of T. Then $\mathcal{F}$ satisfies the Helly property.*

Proof. Assume that for all $i \neq j$ we have $V(T_i) \cap V(T_j) \neq \emptyset$. We need to show that $\bigcap_{i \in I} V(T_i) \neq \emptyset$.

We prove this by induction on $n = |V(T)|$, which is trivially true for the base case $n = 1$. So let $n \geq 2$. Observe that if there is a tree $T_i \in \mathcal{F}$ with $|V(T_i)| = 1$, we are done. So we may assume that each $T_i \in \mathcal{F}$ has at least 2 vertices. Let v be a leaf of T and let u denote its unique neighbor. Set $T' := T - \{v\}$ and $\mathcal{F}' := \{T_i - \{v\} \mid T_i \in \mathcal{F}\}$. Note that, since $|V(T_i)| \geq 2$ for each $T_i \in \mathcal{F}$, if two subtrees $T_i \neq T_j$ intersect in v, they also intersect in u. Therefore, if $T_i \in \mathcal{F}$ and $T_j \in \mathcal{F}$ intersect, for $i \neq j$, then $T_i - \{v\} \in \mathcal{F}'$ and $T_j - \{v\} \in \mathcal{F}'$ also intersect. Now $\mathcal{F}'$ is a family of subtrees of T' with $V(T_i - \{v\}) \cap V(T_j - \{v\}) \neq \emptyset$ for $i \neq j$. It follows by induction that $\bigcap_{i \in I} V(T_i - \{v\}) \neq \emptyset$. We conclude that $\bigcap_{i \in I} V(T_i) \neq \emptyset$, which completes the proof. ∎

We can now prove the following useful lemma about tree decompositions and minor models.

Lemma 2.13: *[CT24] Let G be a graph and let $\mathcal{T} = (T, X_t)$ be a tree decomposition of G. Assume that G contains a K_r as a minor and let $V_1, \dots, V_r$ be the minor model of K_r in G. Then there exists a node $s \in V(T)$ such that X_s contains at least one vertex from each V_i, $1 \leq i \leq r$.*

Proof. Recall that for each $v \in V(G)$, the subgraph $T_v \subseteq_{\text{ind}} T$, induced by the set $\{t \in V(T) \mid v \in X_t\}$, is connected, i.e. T_v is a subtree of T. Now consider the subgraph $T(V_i) \subseteq_{\text{ind}} T$, induced by the set $\{t \in V(T) \mid v \in V_i, v \in X_t\}$, i.e. $T(V_i)$ is the union of the trees T_v for each $v \in V_i$. Since T_v is connected for each $v \in V_i$, and since $G[V_i]$ is connected, it follows that $T(V_i)$ is connected, i.e. $T(V_i)$ is a subtree of T.

Since $V_1, \dots, V_r$ is a minor model of K_r in G, it follows that for each pair of distinct sets V_i, V_j, there exist vertices $u \in V_i$ and $w \in V_j$ such that $uw \in E(G)$. By condition (ii) of the definition of a tree decomposition, there is a node $t \in V(T)$ such that $u, w \in X_t$. Hence, $t \in V(T(V_i)) \cap V(T(V_j))$. Now $\{T(V_i)\}_{1 \leq i \leq r}$ is a family of subtrees of T with $V(T(V_i)) \cap V(T(V_j)) \neq \emptyset$ for $i \neq j$. By Lemma 2.12 $\{T(V_i)\}_{1 \leq i \leq r}$ satisfies the Helly property, so there exists a node $s \in V(T)$ with $s \in \bigcap_{i=1}^{r} V(T(V_i))$. We conclude that the bag X_s contains at least one vertex from each V_i, which is what we wanted to show. ∎

The following lemma is another consequence of Lemma 2.12.

Lemma 2.14: *[DMŠ24a] Let $\mathcal{T} = (T, X_t)$ be a tree decomposition of a graph G and let $S \subseteq V(G)$. If every pair of vertices in S is contained in some bag of $\mathcal{T}$, then there exists a bag X_t with $S \subseteq X_t$.*

Proof. Recall that for each vertex $v \in S$, the nodes $\{t \in V(T) \mid v \in X_t\}$ induce a subtree T_v of T. Since every pair of vertices u, v in S is contained in some bag of $\mathcal{T}$, it follows that the subtrees T_u and T_v have a node in common. So, $\{T_v \mid v \in S\}$ is a family of subtrees of a tree that pairwise intersect. By Lemma 2.12, there exists a node $t \in V(T)$ that is common to all subtrees $T_v, v \in S$. So we have $v \in X_t$ for every $v \in S$, hence $S \subseteq X_t$. ∎

We finish this section by proving that the treewidth is monotone under taking minors.

Lemma 2.15: *Let G be a graph and let H be a minor of G. Then* $\text{tw}(H) \leq \text{tw}(G)$.

Proof. Since H is a minor of G, it is obtained from G by a sequence of vertex deletions, edge deletions and edge contractions. Therefore it is sufficient to show that none of these graph operations increase the treewidth. For the rest of the proof, let $\mathcal{T}$ be a tree decomposition of G.

Assume that H is obtained from G by deleting a vertex $v \in V(G)$. We create the following tree decomposition $\mathcal{T}' = (T', X_{t'})$ of H: We set $T' := T$ and $X_{t'} := X_t \setminus \{v\}$. Observe that $\mathcal{T}'$ is a valid tree decomposition of H and the width of $\mathcal{T}'$ is at most the width of $\mathcal{T}$.

If H is obtained from G by deleting an edge $uv \in E(G)$, we set $\mathcal{T}' = \mathcal{T}$ as a tree decomposition of H, and observe that it is a valid tree decomposition for H, too.

Now let H be obtained from G by contracting an edge $uv \in E(G)$. We denote the new vertex by w. We create a tree decomposition $\mathcal{T}' = (T, X_{t'})$ of H as follows: We set $T' := T$. If a bag X_t does neither contain u nor v, we set $X_{t'} := X_t$. Otherwise, if u or v (or both) are contained in X_t, we set $X_{t'} := (X_t \setminus \{u, v\}) \cup \{w\}$. We observe that $\mathcal{T}'$ is a valid tree decomposition of H and the width of $\mathcal{T}'$ is at most the width of $\mathcal{T}$. This concludes the proof. ∎

Note that Lemma 2.15 already implies that the treewidth is monotone under taking (induced) subgraphs, since every (induced) subgraph of a graph G is also a minor of G.

3 Tree-independence number

Researchers have recently defined and studied other graph parameters defined over tree decompositions [Yol18 | DMŠ24a | Sey16 | Lim+24]. Instead of measuring the size of the bags, as the treewidth does, we now measure other graph invariants of the subgraphs induced by the bags, for example the independence number.

Let G be a graph and let $\mathcal{T} = (T, X_t)$ be a tree decomposition of G. We define the *independence number of $\mathcal{T}$*, denoted by $\alpha(\mathcal{T})$, as the maximum independence number over all bags. Formally, we set

$$\alpha(\mathcal{T}) := \max_{t \in V(T)} \alpha(G[X_t])$$

The *tree-independence number* of G, denoted by tree-$\alpha(G)$, is the minimum independence number taken over all tree decompositions of G.

We observe two simple upper bounds for the tree-independence number.

Observation 3.1: *Let G be a graph. Then*

- tree-$\alpha(G) \leq \mathrm{tw}(G) + 1$, *and*
- tree-$\alpha(G) \leq \alpha(G)$.

Proof. For the first equality, let $\mathcal{T}$ be a tree decomposition of G of width $\mathrm{tw}(G)$. Each bag of $\mathcal{T}$ has at most $\mathrm{tw}(G) + 1$ vertices, hence $\alpha(\mathcal{T}) \leq \mathrm{tw}(G) + 1$, which implies the first inequality.

For the second inequality, consider the trivial tree decomposition of G, where all vertices of G are contained in a single bag. The independence number of this bag is equal to the independence number of G, which concludes the proof. ∎

We are mostly interested in whether a graph class is bounded in terms of the tree-independence number or not. We say that a graph class $\mathcal{G}$ is *tree-α-bounded*, if there exists an integer c such that tree-$\alpha(G) \leq c$ for all $G \in \mathcal{G}$. Otherwise, we say that $\mathcal{G}$ is *tree-α-unbounded*.

Observation 3.1 has some important direct consequences. If we are given a graph class $\mathcal{G}$ that has bounded treewidth or bounded independence number, we can conclude that $\mathcal{G}$ is tree-α-bounded.

Unlike the treewidth of a graph, the tree-independence number is not monotone under taking minors. In particular, the operation of deleting an edge might increase the tree-independence number; we show that later. But, tree-α is monotone under taking induced minors.

Lemma 3.2: *[DMŠ24a] Let G' be an induced minor of a graph G. Then* tree-$\alpha(G') \leq$ tree-$\alpha(G)$.

Proof. We show that a vertex deletion does not increase the tree-independence number. Let $\mathcal{T} = (T, X_t)$ be a tree decomposition of G and let $v \in V(G)$. Let $\mathcal{T}'$ be the tree decomposition obtained from $\mathcal{T}$ by removing v from every bag. Then $\mathcal{T}'$ is a tree decomposition of $G - v$. Hence, we have $\alpha(\mathcal{T}') \leq \alpha(\mathcal{T})$ which implies tree-$\alpha(G - v) \leq$ tree-$\alpha(G)$.

Now we prove that an edge contraction does not increase the tree-independence number. Let $e = uv \in E(G)$ and G/e denote the graph obtained from G by contracting the edge e. Let w be the vertex in G/e that corresponds to the contracted edge. We construct a tree decomposition $\mathcal{T}' = (T, \{X'_t\}_{t \in V(T)})$ of G/e as follows: For each node t in T, if X_t neither contains u nor v we set $X'_t = X_t$, otherwise we set $X'_t = (X_t \setminus \{u, v\}) \cup \{w\}$. Then $\mathcal{T}'$ is a tree decomposition of G/e. Now fix a bag X'_t of $\mathcal{T}'$ and let $I \subseteq X'_t$ be an independent set in G/e. If $w \notin I$, then I is also an independent set in $G[X_t]$ and hence $|I| \leq \alpha(G[X_t])$. Otherwise, if $w \in I$, then either $(I \setminus \{w\}) \cup \{u\}$ or $(I \setminus \{w\}) \cup \{v\}$ is an independent set in $G[X_t]$. In both cases, we have $|I| \leq \alpha(G[X_t])$ and hence $|I| \leq \alpha(\mathcal{T})$, which implies $\alpha(\mathcal{T}') \leq \alpha(\mathcal{T})$ and thus tree-$\alpha(G/e) \leq$ tree-$\alpha(G)$.

An induced minor G' of G is obtained by a sequence of vertex deletions and edge contractions, which implies tree-$\alpha(G') \leq$ tree-$\alpha(G)$. ∎

Lemma 3.2 is a nice tool to bound the tree-independence number of a graph class from below, by showing that every graph in this class contains a certain graph of known tree-independence number as an induced minor.

To show that the tree-independence number is not monotone under taking minors, we need the following lemma, which is a another nice tool on its own.

Lemma 3.3: *[DMŠ24a] Let G be a graph and let G' be obtained from two copies of G by adding all possible edges between them. Then* tree-$\alpha(G') = \alpha(G)$.

Proof. We denote the two copies of G by G_1 and G_2 such that $V(G') = V(G_1) \cup V(G_2)$ and $V(G_1) \cap V(G_2) = \emptyset$. Since we add all possible edges between G_1 and G_2, every independent set in G' is completely contained in either G_1 or G_2. This implies $\alpha(G') = \alpha(G)$ and therefore tree-$\alpha(G') \leq \alpha(G)$ by Observation 3.1.

To see that tree-$\alpha(G') \geq \alpha(G)$, we consider an arbitrary tree decomposition $\mathcal{T}$ of G'. By Lemma 2.8, there exists a vertex $v \in V(G)$ and a node $t \in V(T)$ such that $N[v] \subseteq X_t$. We may assume w.l.o.g. that $v \in V(G_1)$. Then, by construction, $V(G_2) \subseteq N(v) \subseteq X_t$, which implies $\alpha(G'[X_t]) \geq \alpha(G_2) = \alpha(G)$. Thus, every tree decomposition of G' contains a bag which induces a subgraph of independence number at least $\alpha(G)$. This shows tree-$\alpha(G') \geq \alpha(G)$, and therefore tree-$\alpha(G') = \alpha(G)$, which concludes the proof. ∎

Given a graph G, we can easily compute the graph G' of Lemma 3.3 in polynomial time. Since computing $\alpha(G)$ is **NP**-hard, we conclude the following theorem.

Theorem 3.4: *[DMŠ24a] Computing the tree-independence number of a graph is* ***NP****-hard.*

Applying Lemma 3.3 to the empty graph $G := E_n$, we obtain $G' = K_{n,n}$ with tree-$\alpha(G') = \alpha(E_n) = n$, which is worth noting in a separate corollary.

Corollary 3.5: *For every positive integer n, we have* tree-$\alpha(K_{n,n}) = n$.

Here, we also see the reason why the tree-independence number is not monotone under taking minors. Starting with a complete graph K_{2n}, whose tree-independence number is obviously 1, we can delete edges in a way such that we obtain the graph $K_{n,n}$ with tree-independence number n.

By $K_n^{(1)}$ we denote the graph that is obtained from K_n by subdividing every edge exactly once.

Corollary 3.6: *For every positive integer $n \geq 2$, we have* tree-$\alpha(K_n^{(1)}) = n - 1$.

Proof. We denote the original vertices of the K_n by $v_1 \ldots, v_n$ and for each $1 \leq i < j \leq n$, we denote the subdivision vertex by $x_{i,j}$. We set $G := K_n^{(1)}$. First, we show that tree-$\alpha(G) \leq n - 1$. Let $\mathcal{T}$ be the following tree decomposition of G: we take two bags $X_1 := \{v_1, \ldots, v_{n-1}\} \cup \{x_{i,n} \mid 1 \leq i \leq n-1\}$ and $X_2 := \{v_n\} \cup \{x_{i,n} \mid 1 \leq i \leq n-1\}$ and connect them by an edge. Then, for each $1 \leq i < j \leq n-1$, we add a leaf bag $X_{i,j} := \{v_i, v_j, x_{i,j}\}$ to X_1. We see that $\mathcal{T}$ is a valid tree decomposition for G. The leaf bags $X_{i,j}$ induce a path of length 2 in G and hence, their independence number is 2. The bag X_1 induces a matching on $2(n-1)$ vertices in G and therefore, $\alpha(G[X_1]) = n - 1$. The bag X_2 induces a star on n vertices in G, thus $\alpha(G[X_2]) = n - 1$. We conclude that $\alpha(\mathcal{T}) = n - 1$, so tree-$\alpha(G) \leq n - 1$.

Now we show that tree-$\alpha(G) \geq n-1$. Let $\mathcal{T} = (T, X_t)$ be an arbitrary tree decomposition of G. If every pair of vertices in $\{v_1, \ldots, v_n\}$ is contained in some bag of $\mathcal{T}$, then by Lemma 2.14, there exists a bag X_t in $\mathcal{T}$ such that $\{v_1, \ldots, v_n\} \subseteq X_t$. Hence, $\alpha(G[X_t]) \geq n$, and we are done. So we may assume that there exist two vertices $v_i, v_j \in \{v_1, \ldots, v_n\}$ such that no bag of $\mathcal{T}$ contains both v_i and v_j. Then there exists an edge $e = t_1t_2 \in E(T)$ such that the two components T_1 and T_2 of $T - e$, with $t_1 \in T_1$ and $t_2 \in T_2$, contain only one vertex, say w.l.o.g. $v_i \in T_1$ and $v_j \in T_2$. We partition the set $\{v_1, \ldots, v_n\}$ as follows: let A be the set of vertices in $\{v_1, \ldots, v_n\}$ that are only contained in T_1, let B be the set of vertices in $\{v_1, \ldots, v_n\}$ that are only contained in T_2 and let C be the set of vertices in $\{v_1, \ldots, v_n\}$ that are contained in both T_1 and T_2. Since $v_i \in A$ and $v_j \in B$, we have $|A| \geq 1$ and $|B| \geq 1$.

We now count the number of vertices in X_{t_1} (we could also count the vertices in X_{t_2}). Clearly, every vertex $v_c \in C$ is contained in X_{t_1}. For every pair of vertices $v_a \in A$ and $v_b \in B$, their corresponding subdivision vertex $x_{a,b}$ is contained in X_{t_1}. Therefore, X_{t_1} contains at least $|C|+|A|\cdot|B| \geq |C|+|A|+|B|-1 = n-1$ vertices. The subdivision vertices are pairwise non-adjacent and the vertices in C are pairwise non-adjacent. Also, no vertex in C is adjacent to any subdivision vertex $x_{a,b}$, since $x_{a,b}$ connects two vertices $v_a \in A$ and $v_b \in B$. Thus, all of these $n-1$ (or more) vertices are pairwise non-adjacent and we conclude that $\alpha(G[X_{t_1}]) \geq n-1$. This implies $\alpha(\mathcal{T}) \geq n-1$ and so tree-$\alpha(G) \geq n-1$. ■

Corollaries 3.5 and 3.6 yield two graph classes that are not tree-α-bounded. Now, we want to show that the class of planar graphs $\mathcal{G}_{\text{planar}}$ is not tree-α-bounded either. We prove that by showing that a proper subclass of $\mathcal{G}_{\text{planar}}$ is not tree-α-bounded, and therefore, neither is $\mathcal{G}_{\text{planar}}$. We start with the following definition. The *Cartesian product* $G_1 \square G_2$ of two graphs G_1 and G_2 is defined as follows: $V(G_1 \square G_2) := V(G_1) \times V(G_2)$ and two vertices (v_1, w_1) and (v_2, w_2) are adjacent if and only if either $(v_1v_2 \in E(G_1)$ and $w_1 = w_2)$ or $(v_1 = v_2$ and $w_1w_2 \in E(G_2))$. Now, the class of square grid graphs is defined as $\mathcal{G} := \{P_k \square P_k \mid k \in \mathbb{N}\}$. Figure 3.1 shows the square grid $P_4 \square P_4$ as an example. Clearly, every square grid graph is planar and hence $\mathcal{G}$ is a proper subset of $\mathcal{G}_{\text{planar}}$.

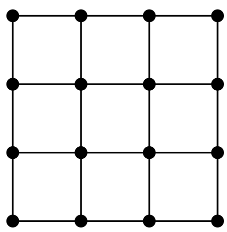


Figure 3.1: The square grid graph $P_4 \square P_4$.

The following is a well-known result on the treewidth of square grid graphs.

Theorem 3.7: *[Die17] The treewidth of the grid graph $P_k \square P_k$ is equal to k.*

The upper bound, i.e. $\text{tw}(P_k \square P_k) \leq k$, is quite easy to show. Assume that the vertices of $P_k \square P_k$ are enumerated row by row, i.e. the first vertex of the first row is labeled by 1, the last vertex of the first row is labeled by k, the first vertex of the second row is labeled by $k+1$ and so on. We create a tree decomposition with the following bags $X_i := \{i, i+1, \ldots, i+k\}$, for $1 \leq i \leq k(k-1)$. Then we connect two bags X_i and X_j if and only if $j = i+1$. Clearly, this is a valid tree decomposition of $P_k \square P_k$ and each bag has size $k+1$, which implies $\text{tw}(P_k \square P_k) \leq k$.

Using this exact tree decomposition, we obtain tree-$\alpha(P_k \square P_k) \leq \lceil \frac{k+1}{2} \rceil$, since every bag induces a path of length $k+1$.

To show that tree-$\alpha(P_k \square P_k) \geq \lceil \frac{k+1}{2} \rceil$, we need the following lemma.

Lemma 3.8: *Let G be a bipartite graph and let G' be a subgraph of G with $|V(G')| = k$. Then $\alpha(G') \geq \lceil \frac{k}{2} \rceil$.*

Proof. Observe that every subgraph of a bipartite graph is bipartite. Thus, the vertices of G' can be partitioned into two sets A and B, one of them, say A, containing at least $\lceil \frac{k}{2} \rceil$ vertices. This means that A is an independent set of size at least $\lceil \frac{k}{2} \rceil$, which proves the claim. ∎

Corollary 3.9: *The square grid graph $P_k \square P_k$ has tree-independence number at least $\lceil \frac{k+1}{2} \rceil$.*

Proof. Let $G = P_k \square P_k$. Since $\text{tw}(G) = k$ by Theorem 3.7, every tree decomposition of G contains a bag with at least $k + 1$ vertices. This bag induces a subgraph of G, and since G is bipartite, the claim follows by Lemma 3.8. ∎

This shows that the class of square grid graphs is not tree-α-bounded and therefore, $\mathcal{G}_{\text{planar}}$ is not tree-α-bounded.

Another important graph class we want to consider is the class of chordal graphs. There are many characterizations of this graph class (we saw a few of them in this thesis already) with lots of different applications. Here, we give another characterization of chordal graphs in terms of tree-independence number. It implies that the class of chordal graphs is tree-α-bounded.

Theorem 3.10: *[DMŠ24a] Let G be a graph. Then,* tree-$\alpha(G) \leq 1$ *if and only if G is chordal.*

Proof. If G is chordal, then, by Lemma 2.9, there exists a tree decomposition $\mathcal{T}$ such that the bags are exactly the inclusion maximal cliques in G. Thus, we have $\alpha(\mathcal{T}) = 1$ and tree-$\alpha(G) \leq 1$.

Now, let G be a graph with tree-$\alpha(G) \leq 1$. Then every bag of the corresponding tree decomposition $\mathcal{T} = (T, X_t)$ is a clique in G. Thus, two distinct vertices u and v of G are adjacent if and only if they belong to a same bag. Since $\mathcal{T}$ is a tree decomposition, for every vertex $u \in V(G)$ the subgraph T_u of T induced by the set $\{t \in V(T) : u \in X_t\}$ is a tree (see condition (iii) of the definition of a tree decomposition). We have

$$uv \in E(G) \Leftrightarrow \exists t \in V(T) : u, v \in X_t \Leftrightarrow V(T_u) \cap V(T_v) \neq \emptyset.$$

We conclude that G is the intersection graph of the collection of subtrees $\{T_u : u \in V(G)\}$ of T, which implies that G is chordal by Lemma 2.4. ∎

In [RTL76], Rose, Tarjan and Lueker present a linear-time algorithm for recognizing chordal graphs. By Theorem 3.10, we obtain the following complexity result for computing the tree-independence number.

Corollary 3.11: *There exists a linear-time algorithm that decides whether a graph has tree-independence number at most 1.*

In [Dal+24] it is shown that deciding whether a given graph has tree-independence number at most k, for $k \geq 4$, is a **NP**-hard problem. For $k \in \{2, 3\}$, the complexity of recognizing graphs with tree-independence number at most k is still an open problem in current research.

4 χ-bounded and (tw, ω)-bounded graph classes

In this chapter, we want to generalize the concept of chordal and perfect graphs. We start with a few simple observations.

For every graph G, the clique number $\omega(G)$ is a lower bound for the chromatic number $\chi(G)$, because every vertex in a largest clique in G must be colored with a different color. So, the presence of a large clique in a graph G is one reason for G to have a large chromatic number. But is it the only reason? Can we construct graphs, that have bounded clique number, e.g. triangle-free graphs, with arbitrary large chromatic number? The answer is yes. There are many constructions of triangle-free graphs that have arbitrary large chromatic number. See [SS20] for a survey. But, we could be interested in graph classes that have large chromatic number only due to the presence of a large clique. We define those graph classes formally now.

A graph class $\mathcal{G}$ is called (χ, ω)-*bounded*, or just χ-*bounded*, if there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that $\chi(G') \leq f(\omega(G'))$ for all $G \in \mathcal{G}$ and all induced subgraphs G' of G. We refer to f as a *binding function*.

Clearly, the class of all graphs $\mathcal{G}_{\text{all}}$ is not χ-bounded, since there exist triangle-free graphs with arbitrary large chromatic number. More specifically, even the class of triangle-free graphs $\{G \in \mathcal{G}_{\text{all}} \mid \omega(G) \leq 2\}$ is not χ-bounded.

On the other hand, the class of perfect graphs is χ-bounded. To see that, choose the identity $f = \mathrm{id}$ as the binding function. Then the statement follows immediately from the definition of a perfect graph.

By Lemma 2.6, a largest clique of a graph G is contained in some bag of every tree decomposition of G. Hence, the clique number $\omega(G)$ is a lower bound for the treewidth $\mathrm{tw}(G)$. In other words, the presence of a large clique in a graph is one reason for the graph to have large treewidth. Clearly, large cliques are not the only reason for large treewidth, as can be seen on the class of planar graphs. Now one can ask, what are the graph classes, that have large treewidth only due to the presence of a large clique. These graph classes are called (tw, ω)-bounded, which is defined as follows.

A graph class $\mathcal{G}$ is called (tw, ω)-*bounded* if there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that $\mathrm{tw}(G') \leq f(\omega(G'))$ for all $G \in \mathcal{G}$ and all induced subgraphs G' of G.

The class of planar graphs $\mathcal{G}_{\text{planar}}$ is not (tw, ω)-bounded. To see that, recall that $\omega(G) \leq 4$ for all $G \in \mathcal{G}_{\text{planar}}$ by Kuratowski's Theorem (see [Kur30]), but the treewidth of $\mathcal{G}_{\text{planar}}$ is unbounded.

Every graph class that has bounded treewidth is also (tw, ω)-bounded: If $\mathcal{G}$ is a graph class with $\mathrm{tw}(G) \leq k$ for all $G \in \mathcal{G}$, we set $f := k$ as a binding function and see that $\mathcal{G}$ is (tw, ω)-bounded. Obviously, not every (tw, ω)-bounded graph class has bounded treewidth. One example is the class of complete graphs $\{K_n \mid n \in \mathbb{N}\}$, which is (tw, ω)-bounded, but has not bounded treewidth.

The concepts of χ-bounded and (tw, ω)-bounded graph classes can also be seen as generalizations of perfect and chordal graphs, respectively. The class of perfect graphs is χ-bounded by the identity function and the class of chordal graphs is (tw, ω)-bounded by the identity function. The latter statement follows by the fact that $\mathrm{tw}(G) = \omega(G) - 1$ for every chordal graph G, see Theorem 2.10.

By Corollary 2.2, every chordal graph is perfect. Speaking in terms of graph classes, the class of chordal graphs is contained in the class of perfect graphs. We obtain a similar result for (tw, ω)-bounded graph classes and χ-bounded graph classes. Before we can prove that, we need the following well-known result.

Lemma 4.1: *Let G be a graph. Then $\chi(G) \leq \mathrm{tw}(G) + 1$.*

Proof. By Lemma 2.5, there exists a tree decomposition $\mathcal{T} = (T, X_t)$ of G of width $\mathrm{tw}(G)$ such that $X_{t_1} \nsubseteq X_{t_2}$ for all $t_1, t_2 \in V(T), t_1 \neq t_2$. Note that every bag of $\mathcal{T}$ has size at most $\mathrm{tw}(G) + 1$. We pick an arbitrary bag X_t and color the graph $G[X_t]$ properly, using at most $\mathrm{tw}(G) + 1$ colors. Once a vertex $v \in V(G)$ is colored, it keeps its color for the rest of the proof. Let t' be a neighbor of t in T and consider the corresponding bag $X_{t'}$. Since $X_{t'} \nsubseteq X_t$, there is a vertex $u \in X_{t'}$ that has not been colored yet. We consider the graph $G[X_{t'}]$ and we color it properly. Since $|X_{t'}| \leq \mathrm{tw}(G) + 1$, u has at most $\mathrm{tw}(G)$ neighbors in $G[X_{t'}]$ and hence we can color u using a color from $\{1, \ldots, \mathrm{tw}(G) + 1\}$.

We then pick another neighbor in T that has not been picked already and proceed like that, until there is no bag left. In each step, we use at most $\mathrm{tw}(G) + 1$ colors and we do not succeed this bound, since once a vertex is colored, it keeps its color for the rest of the procedure. By condition (ii) of the definition of a tree decomposition, every edge of G is contained in some bag, therefore, if two vertices u and v are in the same bag X_t, they get a different color. By condition (i) of the definition of a tree decomposition, every vertex appears in at least one bag, so we do not miss any vertex. Thus, we properly colored G with at most $\mathrm{tw}(G) + 1$ colors. ∎

Theorem 4.2: *Every (tw, ω)-bounded graph class is χ-bounded.*

Proof. Let $\mathcal{G}$ be a (tw, ω)-bounded graph class. Then there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that $\mathrm{tw}(G') \leq f(\omega(G'))$ for all $G \in \mathcal{G}$ and all induced subgraphs G' of G. By Lemma 4.1, every graph G satisfies the inequality $\chi(G) \leq \mathrm{tw}(G) + 1$. Hence, for $G \in \mathcal{G}$ and $G' \subseteq_{\mathrm{ind}} G$,

$$\chi(G') \leq \mathrm{tw}(G') + 1 \leq f(\omega(G')) + 1.$$

Now set $g := f + 1$ as a binding function for χ-boundedness. This completes the proof. ∎

The converse direction does not hold in general. A counterexample is the class of balanced complete bipartite graphs $\{K_{n,n} \mid n \in \mathbb{N}\}$. This graph class clearly is χ-bounded, since $\chi(K_{n,n}) = 2$. But it is not (tw, ω)-bounded, since $\omega(K_{n,n}) = 2$ and $\mathrm{tw}(K_{n,n}) = n$.

5 The connection between tree-α-bounded graph classes and (tw, ω)-bounded graph classes

In this chapter we want to discuss a conjecture made by Dallard, Milanič and Štorgel in [DMŠ24b]. They conjectured the following.

Conjecture 5.1: *Let $\mathcal{G}$ be a hereditary graph class. Then $\mathcal{G}$ is (tw, ω)-bounded if and only if $\mathcal{G}$ has bounded tree-independence number.*

Here, a graph class $\mathcal{G}$ is called *hereditary* if it is closed under taking induced subgraphs, i.e. if for all graphs $G \in \mathcal{G}$ and all induced subgraphs $G' \subseteq_{ind} G$, it follows that $G' \in \mathcal{G}$.

The "only if" direction of Conjecture 5.1 is quite simple. Before we can prove it, we need some definitions and facts about Ramsey Theory.

Let r and b be positive integers. Ramsey's Theorem [Ram87] tells us that there exists a positive integer $N(r,b)$ such that any edge-coloring of the complete graph $K_{N(r,b)}$ with two colors (say, red and blue) contains a red clique of size r or a blue clique of size b. The *Ramsey Number* $R(r,b)$ is defined as the least such integer $N(r,b)$. Note that the Ramsey Number is symmetric, i.e. $R(r,b) = R(b,r)$ for all $r, b \in \mathbb{N}$.

We want to discuss some simple Ramsey Numbers to get an intuition for this concept.

Lemma 5.2: *For every positive integer r it holds that $R(r,2) = r$.*

Proof. To see that $R(r,2) \geq r$ holds, consider the complete graph K_{r-1} with all edges colored in red. This graph neither contains a red clique of size r nor a blue edge.

On the other hand, consider an arbitrary red-blue-coloring of the complete graph K_r. If there is one edge that is colored in blue, we find a blue clique of size two. Otherwise, all edges are colored in red and we obtain a red K_r. This shows $R(r,2) \leq r$ and therefore $R(r,2) = r$. ∎

In view of Lemma 5.2, the first interesting Ramsey Number where neither $r = 2$ nor $b = 2$ is $R(3,3)$. We have $R(3,3) \geq 6$, since there exists an edge-coloring of K_5 that contains neither a red triangle nor a blue triangle. This coloring is shown in Figure 5.1.

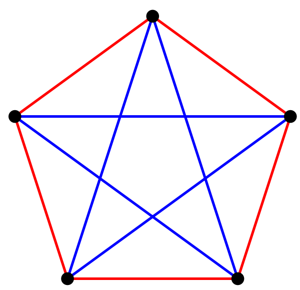


Figure 5.1: Edge-coloring of K_5 without monochromatic triangles.

To see that $R(3,3) \leq 6$, consider any red-blue edge-coloring of K_6. Let v be any vertex in this colored K_6. Since v has 5 neighbors in K_6, it follows by the pigeonhole principle that at least 3 incident edges to v are colored in the same color, say red. We denote the corresponding neighbors by v_1, v_2 and v_3. Now, if there is an edge between any pair of v_1, v_2 and v_3 that is colored in red, say w.l.o.g. the edge v_1v_2 is colored in red, we find a red triangle with vertices v, v_1 and v_2. Otherwise, if all edges between v_1, v_2 and v_3 are colored in blue, we find a blue triangle. This shows $R(3,3) \leq 6$ and hence $R(3,3) = 6$.

We want to think about Ramsey Numbers in a slightly different way. Instead of edge-colorings with two colors red and blue, we think about edges being present or not. We can imagine that red edges in an edge-coloring represent edges that are present and blue edges represent non-edges.

That being said, we can reformulate Ramsey's Theorem as follows: For every two positive integers r and b there exists a positive integer $N(r,b)$ such that every graph on $N(r,b)$ vertices contains either a clique of size r or an independent set of size b. Again, the least such integer is denoted by $R(r,b)$.

We are now able to prove the "only if" direction of Conjecture 5.1.

Lemma 5.3: *[DMŠ24a] Let k be a positive integer and let $\mathcal{G}$ be a graph class with tree-independence number at most k. Then $\mathcal{G}$ is (tw, ω)-bounded.*

Proof. Let $G \in \mathcal{G}$. By assumption, we have tree-$\alpha(G) \leq k$. Let $\mathcal{T} = (T, X_t)$ be a tree decomposition of G with independence number at most k. Then, every bag of $\mathcal{T}$ induces a subgraph of G with independence number at most k and clique number at most $\omega(G)$. Therefore, for every bag X_t of $\mathcal{T}$, we have $|X_t| \leq R(w(G)+1, k+1) - 1$ by Ramsey's Theorem. This implies $\text{tw}(G) \leq R(w(G)+1, k+1) - 2$. We set $f(x) := R(x+1, k+1)$ as a binding function, which concludes the proof. ∎

The question that remains is whether the converse direction of Conjecture 5.1 holds. Chudnovsky and Trotignon disproved this question in [CT24], which implies that Conjecture 5.1 is not true. Their construction is based on a concept called layered wheel. We will discuss this construction in Section 5.1.

Before we get there, we give a very simple construction of a graph class that is (tw, ω)-bounded but not tree-α-bounded. Let k be a positive integer. By G_k we denote the graph obtained from k independent sets, each of size k, where all possible edges between each pair of independent sets are present. Set $\mathcal{G} := \{G_k \mid k \in \mathbb{N}\}$. Clearly, $n := |V(G_k)| = k^2$ and $\omega(G_k) = k = \sqrt{n}$ and therefore $\mathcal{G}$ is (tw, ω)-bounded. On the other hand, the complete balanced bipartite graph $K_{k,k}$ is an induced minor of G_k and hence, by Lemma 3.2 and Corollary 3.5, tree-$\alpha(G_k) \geq$ tree-$\alpha(K_{k,k}) = k = \sqrt{n}$. Thus, $\mathcal{G}$ is not tree-α-bounded. Unfortunately, $\mathcal{G}$ is not hereditary, since deleting any vertex from G_k results in a graph that is not contained in $\mathcal{G}$. Therefore, it is not applicable to Conjecture 5.1.

5.1 Layered wheels

In this section we give an overview of the construction by Chudnovsky and Trotignon in [CT24] that was used to disprove Conjecture 5.1. This section is the only part of the thesis where we consider graphs with countably infinite vertex sets. Most definitions and theorems presented in this section are taken from [CT24].

For convenience, we set $\mathbb{N}^+ := \mathbb{N} \setminus \{0\}$. A function $f : \mathbb{N}^+ \to \mathbb{N}^+$ is called *slow* if $f(1) = 1, f(2) = 2, f(3) = 3$ and for every $i \in \mathbb{N}^+$, $f(i) \leq f(i+1) \leq f(i) + 1$. Note that slow functions exist, e.g. the identity $f = \text{id}$ is a slow function. By definition, slow functions are non-decreasing and thus they either tend to infinity or there exists an $i \in \mathbb{N}^+$ such that $f(i) = f(i')$ for all $i' \geq i$, i.e. slow functions eventually become constant.

If f is a slow function, we define $F(k) := \sup\{i \in \mathbb{N}^+ \mid f(i) \leq k\}$ and call F the *cumulative function of* f. If f eventually becomes a constant function, i.e. $f(i) = c$ for all large enough i and some constant c, then $F(k) = \infty$ for all $k \geq c$. If e.g. $f = \text{id}$, then $F = f$.

We make the following observations about cumulative functions and slow functions.

Observation 5.4: *[CT24] Let f be a slow function and let F be the cumulative function of f. Then $F : \mathbb{N}^+ \to \mathbb{N}^+ \cup \{\infty\}$, $F(1) = 1$, $F(2) = 2$ and $F(k+1) \geq F(k) + 1$ for all $k \in \mathbb{N}^+$.*

Proof. We see that $F : \mathbb{N}^+ \to \mathbb{N}^+ \cup \{\infty\}$, $F(1) = 1$ and $F(2) = 2$ are immediate consequences of the definitions of slow functions and their cumulative functions. It remains to prove that $F(k+1) \geq F(k)+1$ for all $k \in \mathbb{N}^+$.

If $F(k+1) = \infty$, we are done. So let $F(k+1) = j \in \mathbb{N}^+$. Then $f(j) = k+1$ and $f(j') \leq f(j)$ for all $j' < j$. We need to show that $F(k) \leq j - 1$.

If $F(k) = j$, then $f(j) = k$, a contradiction, since $f(j) = k+1$. If $F(k) = \bar{j} > j$, then $f(\bar{j}) = k$ and $f(j) \leq f(\bar{j})$. But, $f(\bar{j}) = k < k+1 = f(j)$, a contradiction. Therefore, $F(k) \leq j - 1$. ∎

Observation 5.5: *Let $F : \mathbb{N}^+ \to \mathbb{N}^+ \cup \{\infty\}$ be a function that satisfies $F(1) = 1$, $F(2) = 2$ and $F(k+1) \geq F(k) + 1$ for all $k \in \mathbb{N}^+$. Then, there exists a slow function f such that F is the cumulative function of f.*

Proof. For all $i \in \mathbb{N}^+$, we set $f(i) := \min\{k \in \mathbb{N}^+ \mid F(k) \geq i\}$. Clearly, $f : \mathbb{N}^+ \to \mathbb{N}^+$ and $f(1) = 1$, $f(2) = 2$ and $f(3) = 3$. Now we prove $f(i) \leq f(i+1) \leq f(i) + 1$.

Suppose that $f(i) > f(i+1)$. We may assume that $f(i+1) = k$ and $f(i) = k+1$. Then $F(k) = i+1$ and $F(k+1) = i$, which implies

$$i = F(k+1) \geq F(k) + 1 = i + 1 + 1 = i + 2,$$

a contradiction.

Now, suppose that $f(i+1) > f(i) + 1$. We may assume that $f(i) = k$, hence $f(i) + 1 = k+1$ and therefore $f(i+1) = k+2$. But then, $F(k) = i$ and $F(k+2) = i+1$, which implies

$$i + 1 = F(k+2) \geq F(k) + 2 = i + 2,$$

a contradiction. This finishes the proof. ∎

For every slow function f, we define a graph G called the *f-layered wheel*. Its vertex set $V(G)$ is countably infinite. Although we define G as an infinite graph, we pick a finite set $X \subseteq V(G)$ and consider the finite graph $G[X]$ in later proofs. The set $\{G[X] \mid X \subseteq V(G), X \text{ finite}\}$ forms a hereditary graph class, and we will work with that graph class instead of G directly.

We will define G inductively, but we start with some construction rules before we do that.

Rule 1 $V(G)$ is partitioned into sets L_i, $i \in \mathbb{N}^+$, called the *layers* of G.

Rule 2 For every $i \in \mathbb{N}^+$ and every vertex $v \in L_i$, there exists an integer n_v and a path $P(v) = v_1, \ldots, v_{n_v}$ such that $V(P(v)) \subseteq L_{i+1}$ and $P(v)$ contains all the neighbors of v in L_{i+1}. The vertex v_1 is adjacent to v, the paths $P(v)$, $v \in L_i$, are vertex-disjoint, $L_{i+1} = \bigcup_{v \in L_i} V(P(v))$ and if $vv' \in E(G)$, then $v_{n_v} v'_1 \in E(G)$.

It follows that a vertex $u \in L_{i+1}$ has at most one neighbor in L_i. If such a neighbor v exists, we say that v is the *parent* of u and u is a *child* of v.

Rule 3 If $v \in L_i$, then v has at most $f(i) - 1$ neighbors in $\bigcup_{1 \leq j < i} L_j$. These neighbors induce a clique that contains at most one vertex in each layer. We denote this clique by $N^{\uparrow}(v)$ and we define $N^{\uparrow}[v] := N^{\uparrow}(v) \cup \{v\}$.

Similar to the definition of a child vertex and a parent vertex, we say that a vertex $w \in L_j$ is an *ancestor* of a vertex $u \in L_i$, if w and u are adjacent and if $j < i$. Conversely, $w \in L_j$ is a *descendant* of $u \in L_i$, if w and u are adjacent and if $j > i$. So, parent and child vertices lie in adjacent layers while ancestor and descendant vertices can lie in layers that are arbitrary far away from each other.

Now we define the f-layered wheel G. We note that in [CT24], Chudnovsky and Trotignon define the layered wheel with an additional parameter ℓ that is responsible for creating "large" holes in G. We do not need this for our purposes and therefore we omit this additional parameter.

The first layer L_1 is a cycle of length 4. We assume inductively that G is defined for the first i layers and explain how to construct layer L_{i+1}. We also inductively assume that Rules 1-3 hold for the first i layers, which is trivially true for layer L_1. Observe that, by Rule 3 and the fact that f is a slow function, for every $v \in L_i$, we have $|N^{\uparrow}(v)| \leq f(i+1) - 1$.

It is sufficient to define for each $v \in L_i$ the integer n_v, and for each $v_j \in P(v)$ the set $N^{\uparrow}(v_j)$. Then L_{i+1} can be described as the cycle formed by the consecutive paths $P(v)$. So, let $v \in L_i$. We distinguish two cases:

Rule 4 $|N^{\uparrow}(v)| < f(i+1) - 1$:
Then we set $n_v := 2$, so $P(v) = v_1 v_2$ and we set $N^{\uparrow}(v_1) := N^{\uparrow}[v]$ and $N^{\uparrow}(v_2) := \emptyset$.

Rule 5 $|N^{\uparrow}(v)| = f(i+1) - 1$:
By Rule 3, we have $N^{\uparrow}(v) = \{w_1, \dots, w_{f(i+1)-1}\}$ and for all $1 \leq j \leq f(i+1) - 1$, $w_j \in L_{i_j}$ with $1 \leq i_1 < i_2 < \cdots < i_{f(i+1)-1} < i$. We set $n_v := 2(f(i+1) - 1)$, hence $P(v) = v_1, \dots, v_{2(f(i+1)-1)}$ and for all $1 \leq j \leq f(i+1) - 1$ we set $N^{\uparrow}(v_{2(j-1)+1}) := N^{\uparrow}[v] \setminus \{w_j\}$ and $N^{\uparrow}(v_{2(j-1)+2}) := \emptyset$.

The new layer L_{i+1} satisfies the Rules 1-3 and therefore we successfully described G inductively.

In Figure 5.2, we see the first three layers of the f-layered wheel G. Note that the first three layers are always the same, independent of f.

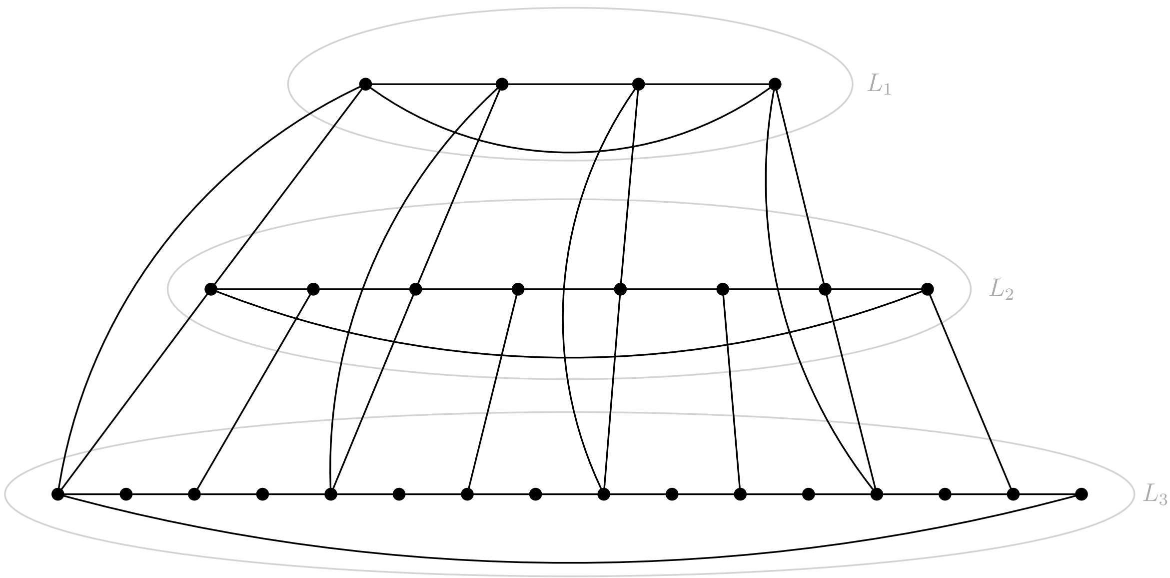


Figure 5.2: First three layers of the f-layered wheel G.

In the next section, we prove some properties of the f-layered wheel.

5.2 Properties of layered wheels

During this section, let f denote a slow function and let G be the f-layered wheel.

We start by proving that the layers of G are pairwise connected by at least one edge. Hence, for each $i \in \mathbb{N}^+$, the layers $L_1, \ldots, L_i$ form a minor model of K_i in G.

Lemma 5.6: *[CT24] For all integers $i \geq 1$ and all $i' > i$, every vertex $u \in L_i$ has at least one neighbor in $L_{i'}$.*

Proof. We prove the claim by induction on i'. In the base case, if $i' = i + 1$, the statement follows directly by Rule 2.

Now let $i' \geq i + 2$. Then $i' + 1 \geq 3$ and since f is a slow function, $f(i' + 1) \geq 3$. Let $w \in L_i$. By induction, w has a neighbor $v \in L_{i'}$. We now check that the path $P(v)$, defined in Rules 4 and 5, contains a neighbor of w. Since $P(v)$ is entirely contained in $L_{i'+1}$, we are done.

There are two cases.

- $|N^{\uparrow}(v)| < f(i + 1) - 1$:
  Rule 4 applies here and v_1 is adjacent to w, since $N^{\uparrow}(v_1) = N^{\uparrow}[v]$ by construction.
- $|N^{\uparrow}(v)| = f(i + 1) - 1$:
  Rule 5 applies here. Since $f(i' + 1) - 1 \geq 2$, there exists a $j \in \{1, \ldots, f(i' + 1) - 1\}$ such that $w_j \neq w$. Hence, the vertex $v_{2(j-1)+1} \in P(v)$ is adjacent to w because $N^{\uparrow}(v_{2(j-1)+1}) = N^{\uparrow}[v] \setminus \{w_j\}$ by construction.

This concludes the proof. ∎

Roughly speaking, the following lemma tells us, that the slow function f controls the clique number of G when adding a new layer. We then may intuitively think of F as the maximum number of layers where f is at most k (recall that F is the cumulative function of f).

Lemma 5.7: *[CT24] For all integers $i \geq 2$, we have $\omega(G[L_1 \cup \cdots \cup L_i]) = f(i)$.*

Proof. We prove by induction on i that there exists a clique C of $G[L_1 \cup \cdots \cup L_i]$ on $f(i)$ vertices such that $|C \cap L_i| = 1$. This implies $\omega(G[L_1 \cup \cdots \cup L_i]) \geq f(i)$.

We start with a trivial base case when $i = 1$.

In the induction step, we assume that such a clique C exists for some fixed $i \geq 1$ with $|V(C)| = f(i)$ and we denote the vertex in $C \cap L_i$ by v. Since f is slow, $f(i + 1) \in \{f(i), f(i) + 1\}$. We distinguish the two possible cases.

- $f(i + 1) = f(i) + 1$:
  Then $|N^{\uparrow}(v)| = f(i) - 1 = f(i + 1) - 2$, so Rule 4 applies and v has a unique child v_1 with $N^{\uparrow}(v_1) = N^{\uparrow}[v]$. Hence, $C \cup \{v_1\}$ is a clique on $f(i) + 1 = f(i + 1)$ vertices.
- $f(i + 1) = f(i)$:
  Then $|N^{\uparrow}(v)| = f(i) - 1 = f(i + 1) - 1$, so Rule 5 applies. Now any child u of v satisfies $N^{\uparrow}(u) = N^{\uparrow}[v] \setminus \{w\}$ for some $w \in N^{\uparrow}(v)$. Thus, $C \cup \{u\}$ is a clique on $f(i + 1)$ vertices.

This shows that $\omega(G[L_1 \cup \cdots \cup L_i]) \geq f(i)$.

We now prove, again by induction on i, that $\omega(G[L_1 \cup \cdots \cup L_i]) \leq f(i)$ also holds. Note that we start with $i = 2$, where the claim trivially holds. Now let $i \geq 3$ and let C be a maximum clique of $G[L_1 \cup \cdots \cup L_i]$ and let j be the maximum integer such that $C \cap L_j \neq \emptyset$. Rules 4 and 5 imply that two adjacent vertices in the layer L_j have no common neighbor in the layer $L_{j'}$ for all $j' \leq j$. Hence, $|C \cap L_j| = 1$ and let u be the vertex in $C \cap L_j$. This implies $C \subseteq N^{\uparrow}[u]$. By Rule 3, $|N^{\uparrow}(u)| \leq f(i) - 1$, so $|N^{\uparrow}[u]| \leq f(i)$ and therefore $|C| \leq f(i)$. This concludes the proof. ∎

Let $i \in \mathbb{N}^+$. An infinite path P of G is called a *vertical path starting in layer i*, if $P = p_i, p_{i+1}, p_{i+2}, \ldots$ and for all $j \geq i$, $p_j \in L_j$.

Lemma 5.8: *[CT24] Let $P = p_i, p_{i+1}, p_{i+2}, \ldots$ and $Q = q_i, q_{i+1}, q_{i+2}, \ldots$ be two vertical paths starting in layer L_i. If $p_i \neq q_i$, then $V(P) \cap V(Q) = \emptyset$.*

Proof. Assume that there exists a vertex $v \in V(P) \cap V(Q)$. Let L_j be the layer containing v such that j is minimal. Then v has two parents, contradicting Rule 2. ∎

Lemma 5.9: *[CT24] If $P = p_i, p_{i+1}, p_{i+2}, \ldots$ is a vertical path starting in layer i, then for all $j \geq i$,*

$$N^{\uparrow}[p_j] \subseteq V(P) \cup N^{\uparrow}(p_i).$$

Proof. We prove the statement by induction on j, with a trivial base case when $j = i$. So now assume that $j > i$. It follows from the Rules 4 or 5 that $N^{\uparrow}(p_j) \subseteq N^{\uparrow}[p_{j-1}]$. By induction, we have $N^{\uparrow}[p_{j-1}] \subseteq V(P) \cup N^{\uparrow}(p_i)$, which implies $N^{\uparrow}[p_j] \subseteq V(P) \cup N^{\uparrow}(p_i)$. ∎

Recall that each layer of G induces a cycle. We want to separate each layer of G into two parts in a certain way. For each Layer L_i, we fix an ordering of the vertices of L_i such that adjacent vertices differ by at most one in the ordering (we compute everything modulo $|L_i|$). Since each layer induces a cycle, we might think of this ordering as enumerating the vertices of the cycle in clockwise order.

Now, let p and q be two vertices in the same layer L_i. Note that, if $p \neq q$, there are two paths between p and q in the layer L_i, since $G[L_i]$ is a cycle. By $p\overrightarrow{L_i}q$, we denote the path from p to q in $G[L_i]$ in an ascending order, with respect to the fixed ordering. If $p = q$, then $p\overrightarrow{L_i}q$ is the path just containing the vertex p. We set $p\overleftarrow{L_i}q := \{p, q\} \cup (L_i \setminus p\overrightarrow{L_i}q)$. We observe that $p\overrightarrow{L_i}q$ and $p\overleftarrow{L_i}q$ edge-wise partition $G[L_i]$. Hence, if $p \neq q$, $(V(p\overrightarrow{L_i}q) \setminus \{p, q\}, \{p, q\}, V(p\overleftarrow{L_i}q) \setminus \{p, q\})$ is a separation of $G[L_i]$ of size 2.

Let $P = p_i, p_{i+1}, p_{i+2}, \ldots$ and $Q = q_i, q_{i+1}, q_{i+2}, \ldots$ be two vertical paths starting in the same layer L_i. We define

$$A(P, Q) := \bigcup_{u \in p_i\overrightarrow{L_i}q_i} N^{\uparrow}[u] \cup \bigcup_{j>i} p_j\overrightarrow{L_j}q_j$$

and

$$B(P, Q) := \bigcup_{1 \leq j \leq i} L_j \cup \bigcup_{j>i} p_j\overleftarrow{L_j}q_j.$$

Our next goal is to prove that $C(P, Q) := A(P, Q) \cap B(P, Q)$ separates G into parts $A^*(P, Q) := A(P, Q) \setminus B(P, Q)$ and $B^*(P, Q) := B(P, Q) \setminus A(P, Q)$. In other words, we show that $(A^*(P, Q), C(P, Q), B^*(P, Q))$ is a separation of G. We start by proving the following lemma.

Lemma 5.10: *[CT24] If P and Q are two vertical paths starting in the same layer L_i, then*

$$A(P, Q) \cup B(P, Q) = V(G)$$

and

$$C(P, Q) = A(P, Q) \cap B(P, Q) = V(P) \cup V(Q) \cup \bigcup_{u \in V(p_i\overrightarrow{L_i}q_i)} N^{\uparrow}[u].$$

Proof. First, we show that $A(P, Q) \cup B(P, Q) = V(G)$. For all $1 \leq j \leq i$, the vertices of layer L_j are contained in $B(P, Q)$ by definition. For all $j > i$, vertices in layer L_j are contained in $p_j\overrightarrow{L_j}q_j$ or $p_j\overleftarrow{L_j}q_j$. So, by definition, they are contained in $A(P, Q)$ or $B(P, Q)$. This shows $A(P, Q) \cup B(P, Q) = V(G)$.

Now we show that

$$A(P,Q) \cap B(P,Q) = V(P) \cup V(Q) \cup \bigcup_{u \in V(p_i \overrightarrow{L_i} q_i)} N^{\uparrow}[u].$$

By definition, $B(P,Q)$ contains all layers $L_1, \ldots, L_i$ and $A(P,Q)$ contains only the vertices $\bigcup_{u \in p_i \overrightarrow{L_i} q_i} N^{\uparrow}[u]$ from the layers $L_1, \ldots, L_i$. For $j > i$, the only vertices contained in both $A(P,Q)$ and $B(P,Q)$ are the vertices from the paths P and Q, since $V(p_j \overrightarrow{L_j} q_j) \cap V(p_j \overleftarrow{L_j} q_j) = \{p_j, q_j\}$. This concludes the proof. ∎

Lemma 5.11: *[CT24] Let P and Q be two vertical paths starting in the same layer L_i and let $u \in A^*(P,Q)$. Then, all descendants of u are in $A^*(P,Q)$ and all ancestors of u are in $A(P,Q)$.*

Proof. We first prove that all descendants of u are in $A^*(P,Q)$. We prove that by contradiction, i.e. we assume that there exists an edge $uw \in E(G)$ such that $u \in A^*(P,Q) \cap L_j$ and $w \in B(P,Q) \cap L_{j'}$ for some $j' > j$. We choose such a pair u, w such that $j' - j$ is minimal. Since $uw \in E(G)$, Rule 4 or Rule 5 implies that w has a parent v such that $u \in N^{\uparrow}[v]$. We have $v \neq u$ since otherwise, w is a child of u and by Rule 2, the children of u are in the interior of $p_{j+1} \overrightarrow{L_{j+1}} q_{j+1}$, and therefore in $A^*(P,Q)$. If $v \in V(P)$, Lemma 5.9 implies that $u \in V(P) \cup N^{\uparrow}(p_i)$, a contradiction to $u \in A^*(P,Q)$. Therefore, $v \notin V(P)$ and by symmetry $v \notin V(Q)$. Thus, if $v \in p_{j'-1} \overrightarrow{L_{j'-1}} q_{j'-1}$, v and w contradict the minimality of $j' - j$ and if $v \in p_{j'-1} \overleftarrow{L_{j'-1}} q_{j'-1}$, u and v contradict the minimality of $j' - j$. In either case, we end up in a contradiction, which implies that all descendants of u are in $A^*(P,Q)$.

Now we prove that all ancestors of u are in $A(P,Q)$. Again, we prove the claim by contradiction. We suppose that there exists an edge $wu \in E(G)$ such that $u \in A^*(P,Q) \cap L_j$ and $w \in B^*(P,Q) \cap L_{j'}$ for some $j' < j$. Here, we choose such a pair w, u such that $j - j'$ is minimal. Since $wu \in E(G)$, Rule 4 or Rule 5 implies that u has a parent v such that $w \in N^{\uparrow}[v]$. If $v \in V(P)$, Lemma 5.9 implies that $w \in V(P) \cup N^{\uparrow}(p_i)$, a contradiction to $w \in B^*(P,Q)$. Therefore, $v \notin V(P)$ and by symmetry $v \notin V(Q)$. If $v \in p_i \overrightarrow{L_i} q_i$, then $w \in N^{\uparrow}[v]$, a contradiction to $w \in B^*(P,Q)$. By Rule 2 it follows that $j \geq i + 2$ and v is contained in the interior of $p_{j-1} \overrightarrow{L_{j-1}} q_{j-1}$. Thus, $v \in A^*(P,Q) \cap L_{j-1}$. If $j - j' = 1$, the pair w, u contradict Rule 2 and otherwise the pair w, v contradict the minimality of $j - j'$. In either case, we end up in a contradiction, which implies that all ancestors of u are in $A(P,Q)$. This finishes the proof. ∎

Now, we are able to prove that $(A^*(P,Q), C(P,Q), B^*(P,Q))$ is a separation of G.

Lemma 5.12: *[CT24] Let P and Q be two vertical paths starting in the same layer L_i. Then $S = (A^*(P,Q), C(P,Q), B^*(P,Q))$ is a separation of G.*

Proof. We assume that S is not a separation of G. By Lemma 5.10, $A^*(P,Q) \cup C(P,Q) \cup B^*(P,Q) = V(G)$ and by definition, $A^*(P,Q)$, $C(P,Q)$ and $B^*(P,Q)$ are pairwise vertex-disjoint. Thus, since S is not a separation of G, there is an edge $uv \in E(G)$ with $u \in A^*(P,Q)$ and $v \in B^*(P,Q)$. If u and v are in the same layer L_j, then $j > i$ since $B(P,Q)$ contains all layers $L_1 \ldots, L_i$ and $u \notin B(P,Q)$. Hence, $u \in V(p_j \overrightarrow{L_j} q_j) \setminus \{p_j, q_j\}$ and $v \in V(p_j \overleftarrow{L_j} q_j) \setminus \{p_j, q_j\}$, a contradiction since u and v are adjacent but there is no edge from $V(p_j \overrightarrow{L_j} q_j) \setminus \{p_j, q_j\}$ to $V(p_j \overleftarrow{L_j} q_j) \setminus \{p_j, q_j\}$ in $G[L_j]$. So we may assume that u and v are not in the same layer. Then v is either a descendant or an ancestor of u and in either case, by Lemma 5.11, $v \in A(P,Q)$, a contradiction.

Therefore, such an edge uv cannot exist and we conclude that S is a separation of G. ∎

We want to take a short break and get an overview of where we are right now. We defined the f-layered wheel G for a slow function f and proved some basic properties, culminating in Lemma 5.12, stating that we can separate G in a certain way. Why do we want to separate G in the first place? Recall that our main goal is to find a hereditary graph class that is (tw, ω)-bounded but not tree-α-bounded. Lemma 5.12 helps us to achieve (tw, ω)-boundedness, since the treewidth of a graph can be bounded by separators in the following way.

Theorem 5.13: *[DN19] The treewidth of any graph G is at most* $15\,\mathrm{sn}(G)$.

We showed that the f-layered wheel G has a certain separation. What we need to show is that for any finite set $X \subseteq V(G)$, the graph $G[X]$ has a balanced separation of bounded size. This is our next goal.

As throughout the whole section, let f be a slow function, G be the f-layered wheel and let $X \subseteq V(G)$ be a finite set. Furthermore, let k be a positive integer such that $\omega(G[X]) \leq k$. We say that an edge $vu \in E(G)$ is *augmenting* if u is a child of v and

$$N^{\uparrow}(u) \cap X = N^{\uparrow}[v] \cap X.$$

In Figure 5.3, we see the first three layers of G where each augmenting edge is colored in red. Note that, if u is a child of a vertex v that was introduced by Rule 4, then the edge uv is augmenting by construction.

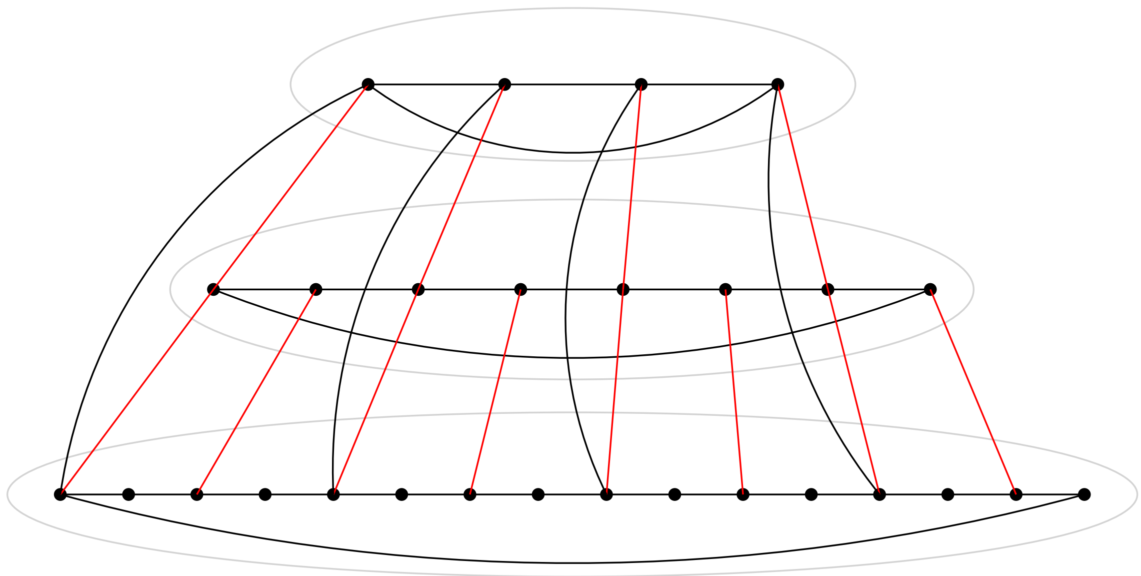


Figure 5.3: First three layers of G and its augmenting edges colored in red.

Lemma 5.14: *[CT24] If $v \in L_i$ and $f(i+1) \geq k+2$, then there exists at least one child u of v such that vu is augmenting.*

Proof. We distinguish two cases.

- $|N^{\uparrow}(v)| < f(i+1) - 1$:
  Then Rule 4 applies and so there is a child u of v with $N^{\uparrow}(u) = N^{\uparrow}[v]$, which implies $N^{\uparrow}(u) \cap X = N^{\uparrow}[v] \cap X$ trivially. Thus, vu is augmenting.

- $|N^{\uparrow}(v)| = f(i+1) - 1$:
  We have $|N^{\uparrow}(v)| = f(i+1) - 1 \geq k+1$ and since $\omega(G[X]) \leq k$, there is a vertex $w \in N^{\uparrow}(v) \setminus X$. Rule 5 implies that v has a child u such that $N^{\uparrow}(u) = N^{\uparrow}[v] \setminus \{w\}$. Now $N^{\uparrow}(u) \cap X = N^{\uparrow}[v] \cap X$ follows because $w \notin X$. This means that the edge vu is augmenting and we are done.

∎

For every $v \in V(G)$ we define $a(v)$ as follows: if v has a child u such that vu is augmenting, then we set $a(v) := u$. Otherwise, we pick any child u of v and set $a(v) := u$. We call $a(v)$ the *augmenting child of* v. Note that $a(v)$ is not necessarily unique, but it exists. Therefore, there exists a vertical path $P = v, a(v), a(a(v)), \ldots$ which we call the *augmenting path out of* v.

Lemma 5.15: *[CT24] Let $v \in V(G)$ and let P be the augmenting path out of v. Then*

$$\left(V(P) \cap \left(\bigcup_{i \geq F(k+1)} L_i\right)\right) \cap X$$

is a clique and

$$|V(P) \cap X| \leq F(k+1) + k - 1.$$

Proof. If $F(k+1) = \infty$, then $\{i \in \mathbb{N}^+ \mid i \geq F(k+1)\} = \emptyset$ and so the statement is trivially true.

Otherwise, if $F(k+1)$ is finite, then $V(P) \cap (\bigcup_{i \geq F(k+1)} L_i)$ induces an infinite vertical path $p_1, p_2, \ldots$, and for each $j \geq 1$, p_j is in a layer L_i such that $f(i+1) \geq k+2$. Thus, by Lemma 5.14, for all $j \geq 1$ there exists a child u such that $p_j u$ is augmenting. By the definition of an augmenting path, $p_j p_{j+1}$ is augmenting.

We need to prove that $\{p_1, p_2, \ldots\} \cap X$ induces a clique. We do so by proving by induction on j that $\{p_1, \ldots, p_j\} \cap X \subseteq N^{\uparrow}[p_j] \cap X$, which induces a clique by Rule 3. In the base case, if $j = 1$, the statement is true since $p_1 \in N^{\uparrow}[p_1]$. Now:

$$\begin{aligned}
\{p_1, \ldots, p_{j+1}\} \cap X &= (\{p_1, \ldots, p_j\} \cap X) \cup (\{p_{j+1}\} \cap X) & \\
&\subseteq (N^{\uparrow}[p_j] \cap X) \cup (\{p_{j+1}\} \cap X) & \text{by induction} \\
&= (N^{\uparrow}(p_{j+1}) \cap X) \cup (\{p_{j+1}\} \cap X) & \text{since } p_j p_{j+1} \text{ is augmenting} \\
&= N^{\uparrow}[p_{j+1}] \cap X &
\end{aligned}$$

Thus, $\{p_1, p_2, \ldots\} \cap X$ is a clique on at most k vertices since $\omega(G[X]) \leq k$. For the layers $L_1 \ldots, L_{F(k+1)-1}$, the path P contains at most one vertex in each layer and we conclude that $|V(P) \cap X| \leq F(k+1) + k - 1$, which finishes the proof. ∎

We call a separation $S = (A, C, B)$ of G *fair* if there exists a pair of vertical paths $P = p_i, p_{i+1}, \ldots$ and $Q = q_i, q_{i+1}, \ldots$ starting in the same layer such that

- $|V(p_i \overrightarrow{L_i} q_i)| \leq 3$,
- the paths $P - \{p_i\} = p_{i+1}, p_{i+2}, \ldots$ and $Q - \{q_i\} = q_{i+1}, q_{i+2}, \ldots$ are augmenting paths,
- $A = A^*(P, Q)$, $B = B^*(P, Q)$, $C = C(P, Q) = A(P, Q) \cap B(P, Q)$ and
- $|A(P, Q) \cap X| \geq n/3$ with $n = |X|$.

Lemma 5.16: *[CT24] There exists a fair separation in G.*

Proof. We pick two distinct, non-adjacent vertices p and q in the first layer L_1. Such vertices exist and $|V(p\overrightarrow{L_1}q)| \leq 3$ and $|V(q\overrightarrow{L_1}p)| \leq 3$, since L_1 induces a cycle of length 4. Let P and Q be the two augmenting paths out of p and q, respectively. By Lemma 5.12, $(A^*(P,Q), C(P,Q), B^*(P,Q))$ and $(A^*(Q,P), C(Q,P), B^*(Q,P))$ are separations of G.

We need to prove that $|A(P,Q) \cap X| \geq n/3$ or $|A(Q,P) \cap X| \geq n/3$. Since $p \neq q$, by Lemma 5.8, P and Q are vertex-disjoint, which implies $A(P,Q) \cup A(Q,P) = V(G)$. Thus, either $|A(P,Q) \cap X| \geq n/3$ holds or $|A(Q,P) \cap X| \geq n/3$ holds, concluding the proof. ∎

Now, we are able to prove that each induced subgraph of G has a balanced separator of bounded size.

Lemma 5.17: *[CT24] There exists a balanced separation of $G[X]$ of size at most*

$$2F(k+1) + 5k - 2.$$

Proof. The statement is trivially true for $n \leq 5$, since $k \geq 1$ and so $F(k+1) \geq 2$, so $(\emptyset, X, \emptyset)$ is a valid separation. Thus, we may assume that $n \geq 6$.

Let $S = (A, C, B)$ be a fair separation of G; it exists by Lemma 5.16. We choose S to be such that i is maximal and among all separations with maximal i, such that $|p_{i+1}\overrightarrow{L_{i+1}}q_{i+1}|$ is minimal. By Lemma 5.10, we have

$$A(P,Q) \cap B(P,Q) = V(P) \cup V(Q) \cup \bigcup_{u \in V(p_i\overrightarrow{L_i}q_i)} N^{\uparrow}[u]$$

and Lemma 5.15 implies that $|V(P - \{p_i\}) \cap X| \leq F(k+1) + k - 1$. A similar inequality clearly holds for Q. By Rule 3, we obtain for all $u \in p_i\overrightarrow{L_i}q_i$ that $|N^{\uparrow}[u]| \leq k$. Putting this all together, we see that $(A \cap X, C \cap X, B \cap X)$ has size at most $2F(k+1) + 5k - 2$, since $|V(p_i\overrightarrow{L_i}q_i)| \leq 3$.

We need to prove that $(A \cap X, C \cap X, B \cap X)$ is balanced. For the sake of contradiction, we suppose that it is not balanced. Since S is fair, we have $|A \cap X| \geq n/3$, which implies $|B \cap X| \leq 2n/3$. Therefore, $(A \cap X, C \cap X, B \cap X)$ can only be not balanced because $|A \cap X| > 2n/3$.

We assume that no internal vertex of $p_{i+1}\overrightarrow{L_{i+1}}q_{i+1}$ has a parent. Then, by Rule 2, either $p_i = q_i$ or p_iq_i is an edge in G. By Rules 4 and 5, $p_{i+1}\overrightarrow{L_{i+1}}q_{i+1}$ induces a path on 3 vertices. We set $P' := P - \{p_i\}$ and $Q' := Q - \{q_i\}$ and $A' := A^*(P', Q')$ and $B' := B^*(P', Q')$ and $C' := C(P', Q')$. We show that (A', C', B') is a fair separation, which contradicts the maximality of (A, C, B). Clearly, P' and Q' are vertical paths starting in the same layer. We already showed that $|V(p_{i+1}\overrightarrow{L_{i+1}}q_{i+1})| \leq 3$. Since S is fair, P' and Q' are augmenting paths. At most two vertices of A' are not in A and by Rules 4 and 5, at most one vertex is in $N^{\uparrow}[p_i] \setminus N^{\uparrow}(p_{i+1})$ and at most one vertex is in $N^{\uparrow}[q_i] \setminus N^{\uparrow}(q_{i+1})$. Therefore, $|A' \cap X| \geq |A \cap X| - 2 > 2n/3 - 2 = n/3 + (n-6)/3 \geq n/3$ because $n \geq 6$. This shows that (A', C', B') is a fair separation, a contradiction, since $i + 1 > i$.

So, we may assume that there is an internal vertex u of $p_{i+1}\overrightarrow{L_{i+1}}q_{i+1}$ that has a parent v. By Rule 2, $v \in p_i\overrightarrow{L_i}q_i$. Let R' be the augmenting path out of u and set $R := vuR'$. Figure 5.4 visualizes the setup. Furthermore, set $A' := A^*(P,R), A'' := A^*(R,Q), B' := B^*(P,R), B'' := B^*(R,Q), C' := C(P,R)$ and $C'' := C(R,Q)$. We show that (A', C', B') or (A'', C'', B'') is a fair separation. Clearly, P and R start in the same layer L_i. Since S is fair, $P - \{p_i\}$ is an augmenting path and R augmenting by definition. Also, since S is fair, we have $|V(p_i\overrightarrow{L_i}q_i)| \leq 3$. Furthermore, we have $A = A^*(P,Q) = A^*(P,R) \cup A^*(R,Q) = A' \cup A''$. Thus, since $|A \cap X| \geq 2n/3$, either $|A' \cap X| \geq n/3$ or $|A'' \cap X| \geq n/3$. This means that either (A', B') or (A'', B'') is fair, but that is a contradiction to the minimality of $|p_{i+1}\overrightarrow{L_{i+1}}q_{i+1}|$.

This finishes the proof. ∎

We are finally ready to proof that the induced subgraphs of the f-layered wheel form a (tw, ω)-bounded class of graphs.

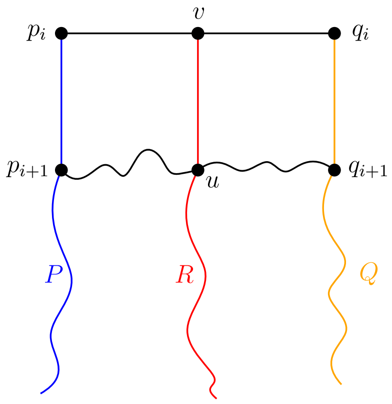


Figure 5.4: The augmenting paths P, R and Q in the proof of Lemma 5.17.

Lemma 5.18: *[CT24] Let f be an unbounded slow function and let G be the f-layered wheel. Let $\mathcal{G}$ be the hereditary graph class obtained from G by taking all finite induced subgraphs of G. Then, $\mathcal{G}$ is (tw, ω)-bounded.*

Proof. Note that, since f is unbounded, the cumulative function F does not map any $i \in \mathbb{N}^+$ to ∞, so the value of $F(i)$ is always finite. We prove that every finite induced subgraph H of G satisfies

$$\mathrm{tw}(H) \leq 15(2F(\omega(H)+1) + 5\omega(H) - 2).$$

Let H' be an induced subgraph of H and let $k := \omega(H')$. By Lemma 5.17, H' has a balanced separation of size at most $2F(k+1) + 5k - 2 \leq 2F(\omega(H)+1) + 5\omega(H) - 2$. Theorem 5.13 implies

$$\mathrm{tw}(H) \leq 15(2F(\omega(H)+1) + 5\omega(H) - 2).$$

Setting $g(\omega(H)) := 15(2F(\omega(H)+1) + 5\omega(H) - 2)$ as a binding function completes the proof. ∎

5.3 Disproving Conjecture 5.1

As a last ingredient to disprove Conjecture 5.1, we need the following lemma.

Lemma 5.19: *[CT24] Let $X \subseteq V(G)$ be finite. If X contains at most one vertex in each layer of G, then $G[X]$ is chordal.*

Proof. Let i be the maximum integer such that $X \cap L_i \neq \emptyset$. By assumption, there is a unique vertex $v \in X \cap L_i$ and it has all its neighbors in layers L_j with $j < i$. Thus, $N(v) \cap X = N^{\uparrow}(v) \cap X$ and Rule 3 implies that v is simplicial. By Lemma 2.3 and the fact that we can apply this proof to every induced subgraph of $G[X]$, it follows that $G[X]$ is chordal. ∎

We have now everything together to disprove Conjecture 5.1.

Theorem 5.20: *[CT24] Let $F : \mathbb{N}^+ \to \mathbb{N}^+$ be a super-linear function with $F(1) = 1, F(2) = 2$ and $F(k+1) \geq F(k) + 1$ for every $k \geq 1$. Then there exists a hereditary graph class $\mathcal{G}$ such that $\mathcal{G}$ is (tw, ω)-bounded but $\mathcal{G}$ contains graphs of arbitrary large tree-independence number.*

Proof. Let f be the slow function whose cumulative function is F. Such a slow function f exists by Observation 5.5. Let G be the f-layered wheel and let $\mathcal{G}$ be the class of finite induced subgraphs of G.

Clearly, $\mathcal{G}$ is hereditary and Lemma 5.18 implies that $\mathcal{G}$ is (tw, ω)-bounded.

We now show that $\mathcal{G}$ is not tree-α-bounded. So let c be a positive integer. Since F is super-linear, there exists a positive integer k such that $F(k) \geq ck$. Let $H := G[L_1 \cup \cdots \cup L_{F(k)}]$. Lemma 5.7 implies $\omega(H) = k$ and by Lemma 5.6, the layers $L_1, \ldots, L_{F(k)}$ form a minor model of a complete graph in H.

Let $\mathcal{T} = (T, X_t)$ be a tree decomposition of H. Lemma 2.13 implies that there exists a node $s \in V(T)$ such that X_s contains at least one vertex of each layer L_i, $1 \leq i \leq F(k)$. Now let Y be a subset of X_S that contains exactly one vertex in each layer. Then $|Y| = F(k)$ and $\omega(H[Y]) \leq k$. By Lemma 5.19, $H[Y]$ is chordal. Since every chordal graph is perfect (see Corollary 2.2), we have $\omega(H[Y]) = \chi(H[Y])$ and we get

$$\alpha(H[X_s]) \geq \alpha(H[Y]) \geq \frac{|Y|}{\chi(H[Y])} = \frac{F(k)}{\omega(H[Y])} \geq \frac{ck}{k} = c,$$

where the second inequality holds in any graph. This shows that every tree decomposition of H contains a bag X_s with $\alpha(H[X_s]) \geq c$, which implies tree-$\alpha(H) \geq c$. Since c is arbitrary, $\mathcal{G}$ is not tree-α-bounded.

This completes the proof. ∎

6 Tree-chromatic number

A graph parameter that was introduced in 2016 by Seymour in [Sey16], and that has not been examined a lot yet, is the tree-chromatic number of a graph. It is defined in the same way the tree-independence number is defined, replacing the independence number in definitions by the chromatic number: Given a graph G and a tree decomposition of $\mathcal{T} = (T, X_t)$ of G, we define the *chromatic number* of $\mathcal{T}$, denoted by $\chi(\mathcal{T})$, as follows:

$$\chi(\mathcal{T}) := \max_{t \in V(T)} \chi(G[X_t]).$$

The *tree-chromatic number* of G is defined as the smallest chromatic number taken over all tree decompositions of G, and it is denoted by tree-$\chi(G)$. We observe similar upper bounds for the tree-chromatic number as we did for the tree-independence number. Additionally, we give a lower bound.

Observation 6.1: *Let G be a graph. Then*

- tree-$\chi(G) \leq \text{tw}(G) + 1$, *and*
- $\omega(G) \leq$ tree-$\chi(G) \leq \chi(G)$.

Proof. The proofs of the inequalities tree-$\chi(G) \leq \text{tw}(G) + 1$ and tree-$\chi(G) \leq \chi(G)$ follow the same approach as in Observation 3.1.

To see $\omega(G) \leq$ tree-$\chi(G)$, consider an arbitrary tree decomposition $\mathcal{T}$ of G. By Lemma 2.6, each clique in G is contained in some bag of $\mathcal{T}$, in particular a largest clique. Each vertex in a clique must be colored in a different color, which implies $\omega(G) \leq$ tree-$\chi(G)$. ∎

We give a very simple characterization of graphs that have tree-chromatic number 1.

Lemma 6.2: *A graph G is edgeless if and only if* tree-$\chi(G) = 1$.

Proof. Let G be edgeless. We put every vertex of G in a different bag, and connect the bags in a way such that they form a tree. Then tree-$\chi(G) = 1$.

Now assume that G is not edgeless. Then there is an edge $uv \in E(G)$ that is contained in a bag of every tree decomposition of G. A proper coloring assigns different colors to u and v, which implies tree-$\chi(G) \geq 2$. This finishes the proof. ∎

By Lemma 6.2 we obtain the following complexity result:

Corollary 6.3: *There exists a linear-time algorithm that decides whether a graph has tree-chromatic number equal to* 1.

Unfortunately, the complexity of deciding whether tree-$\chi(G) \leq 2$ is unknown. In [HRWY21] it is conjectured that this is already **NP**-complete.

Are the two graph invariants tree-chromatic number and tree-independence number connected in some way? Especially, it would be interesting to know if one parameter is a bound for the other one. This is not the case, as we see in the following example. The class of complete graphs $\mathcal{G} := \{K_n \mid n \in \mathbb{N}\}$ is a subclass of the chordal graphs, and hence, by Theorem 3.10, $\mathcal{G}$ is tree-α bounded. But $\mathcal{G}$ is not tree-χ-bounded, since tree-$\chi(K_n) = n$ by Observation 6.1. On the other hand, the class of balanced complete bipartite graphs $\mathcal{H} := \{K_{n,n} \mid n \in \mathbb{N}\}$ is tree-χ-bounded, since $K_{n,n}$ is bipartite and therefore 2-colorable. But $\mathcal{H}$ is not tree-α-bounded, since tree-$\alpha(K_{n,n}) = n$, see Corollary 3.5.

Nevertheless, we give the following bound.

Observation 6.4: *Let G be a graph and let $\overline{G}$ be the complement of G. Then* tree-$\alpha(G) \leq$ tree-$\chi(\overline{G})$.

Proof. By Observation 3.1, we have tree-$\alpha(G) \leq \alpha(G)$. We observe that $\alpha(G) = \omega(\overline{G})$ holds in any graph G. Now, by Observation 6.1, we get $\omega(\overline{G}) \leq$ tree-$\chi(\overline{G})$, hence

$$\text{tree-}\alpha(G) \leq \alpha(G) = \omega(\overline{G}) \leq \text{tree-}\chi(\overline{G}).$$

This completes the proof. ∎

The parameter tree-χ is not monotone under taking minors. To see that, consider the graph K_n. We know that tree-$\chi(K_n) = n$. Let G be the graph obtained from K_n by subdividing every edge once. Then G is bipartite and therefore tree-$\chi(G) \leq \chi(G) = 2$. Since K_n is a minor of G, tree-χ is not monotone under taking minors. But tree-χ is monotone under taking subgraphs, since deleting vertices and edges does not increase the chromatic number of a graph.

6.1 Tree-chromatic number for odd holes and odd antiholes

We now want to find the tree-chromatic number for odd holes and their complements. We use these results to give another characterization for the class of perfect graphs.

Lemma 6.5: *For an odd integer $n \geq 5$,* tree-$\chi(C_n) = 2 = \chi(C_n) - 1$.

Proof. We denote the vertices of C_n by $v_1, \ldots, v_n$ along the cycle. We give a tree decomposition of C_n with chromatic number 2, which implies the claim. The bags are constructed in the following way: $X_{t_i} := \{v_1, v_i, v_{i+1}\}$ for $2 \leq i \leq n-1$. Now connect every bag with its natural successor bag by an edge, such that they form a path. Clearly, all conditions of the definition of a tree decomposition are satisfied and $\chi(X_{t_i}) = 2$ for every bag X_{t_i}. ∎

So, the chromatic number of an odd hole is not equal to its tree-chromatic number. The same is true for odd antiholes:

Lemma 6.6: *For an odd integer $n \geq 5$,* tree-$\chi(\overline{C_n}) = \lfloor \frac{n}{2} \rfloor = \chi(\overline{C_n}) - 1$.

Proof. As in the previous lemma, we denote the vertices of $\overline{C_n}$ by $v_1, \ldots, v_n$ (along the cycle C_n). Consider the following tree decomposition of $\overline{C_n}$ with two bags. One bag X_1 contains the vertices $V(\overline{C_n}) \setminus \{v_1\}$ and the other bag X_2 contains the vertices $V(\overline{C_n}) \setminus \{v_2\}$. Connect X_1 and X_2 by an edge.

First observe, that all conditions of a tree decomposition are satisfied. We now show that each bag can be colored with $\lfloor \frac{n}{2} \rfloor$ colors. We start with the bag X_2. Since consecutive vertices are not adjacent in $\overline{C_n}$, we can color those vertices with the same color. So, we can color v_1 and v_n with color 1, v_3 and v_4 with color 2, v_5 and v_6 with color 3 and so on. We need $\frac{|X_2|}{2} = \frac{n-1}{2} = \lfloor \frac{n}{2} \rfloor$ colors for this process, which is what we wanted to show.

We do the same procedure for bag X_1, i.e. the vertices v_2 and v_3 get color 1, v_4 and v_5 get color 2 and so on. This shows tree-$\chi(\overline{C_n}) \leq \lfloor \frac{n}{2} \rfloor$.

To see tree-$\chi(\overline{C_n}) \geq \lfloor \frac{n}{2} \rfloor$, observe that the vertices $v_2, v_4, \ldots, v_{n-1}$ form a clique in $\overline{C_n}$ of size $\lfloor \frac{n}{2} \rfloor$. Hence $\omega(\overline{C_n}) \geq \lfloor \frac{n}{2} \rfloor$, which implies tree-$\chi(\overline{C_n}) \geq \lfloor \frac{n}{2} \rfloor$ by Observation 6.1. This concludes the proof. ∎

With the previous two lemmas we are now able to give another characterization of the class of perfect graphs.

Corollary 6.7: *A graph G is perfect if and only if* tree-$\chi(G') = \chi(G')$ *for every induced subgraph G' of G.*

Proof. For the if part we assume that G is perfect. By definition, we have $\omega(G') = \chi(G')$ for every induced subgraph G' of G and by Observation 6.1 we have tree-$\chi(G') = \chi(G')$.

We prove the only if part by contraposition. So we assume that G is not perfect. By the Strong Perfect Graph Theorem (Theorem 2.1), G contains an odd hole or an odd antihole. As shown in Lemmas 6.5 and 6.6, the tree-chromatic number is not equal to the chromatic number for odd holes and odd antiholes, which implies the claim. ∎

6.2 Bounding the treewidth by the tree-chromatic number and the tree-independence number

We now want to tackle a problem introduced in [DMŠ24a]. We need some observations to pose the problem.

For every graph G, we have $|V(G)| \leq \alpha(G) \cdot \chi(G)$. To see that, observe that every color class induces an independent set, every independent set has size at most $\alpha(G)$ and we have $\chi(G)$ independent sets. Now let $\mathcal{T}$ be a tree-α-optimal tree decomposition of G, i.e. $\alpha(\mathcal{T}) =$ tree-$\alpha(G)$. Let X_t a bag of maximum size in $\mathcal{T}$. By the previous inequality, we obtain

$$\text{tw}(G) + 1 \leq |X_t| \leq \alpha(\mathcal{T}) \cdot \chi(G) = \text{tree-}\alpha(G) \cdot \chi(G).$$

Analogously, let $\mathcal{T}'$ be a tree-χ-optimal tree decomposition of G and X'_t be a bag of maximum size in $\mathcal{T}'$. Then we obtain

$$\text{tw}(G) + 1 \leq |X'_t| \leq \alpha(G) \cdot \chi(\mathcal{T}') = \alpha(G) \cdot \text{tree-}\chi(G).$$

Now the question asked in [DMŠ24a], is whether the following natural strengthening also holds in any graph G:

Question 6.8: *Does* $\text{tw}(G) + 1 \leq \text{tree-}\alpha(G) \cdot \text{tree-}\chi(G)$ *hold for every graph G?*

We give a negative answer to this question by constructing the following graph. We start with a C_5 and call the vertices $v_1, \ldots, v_5$. Then we add paths of length 2 between every pair of non-adjacent vertices v_i and v_j, with $i < j$ and call the inner vertex of that path $x_{i,j}$. Since the resulting graph is a subdivision of K_5, starting with C_5, we call the resulting graph the *C_5-subdivision of K_5*, which is shown in Figure 6.1, and denote it by $S(C_5)$. In general, given any graph G on n vertices, if we connect any pair of non-adjacent vertices in G by a path of length 2, we call the resulting graph the *G-subdivision of K_n* and denote it by $S(G)$.

Clearly, $S(C_5)$ contains K_5 as a minor, yielding $\text{tw}(S(C_5)) \geq 4$ by Lemma 2.15 and Corollary 2.7. Also, tree-$\chi(S(C_5)) \geq 2$ since $S(C_5)$ is not edgeless (see Lemma 6.2) and tree-$\alpha(S(C_5)) \geq 2$ since $S(C_5)$ is not chordal (see Theorem 3.10).

We have to show that tree-$\alpha(S(C_5)) \leq 2$ and tree-$\chi(S(C_5)) \leq 2$. We start with tree-$\alpha(S(C_5)) \leq 2$. To see that, consider the tree decomposition, consisting of one bag X containing the original C_5. For every path connecting two non-adjacent vertices of C_5, we create a leaf bag in the tree decomposition containing the two endpoints and the inner vertex. This yields a valid tree decomposition $\mathcal{T}$ with $\alpha(\mathcal{T}) = 2$. Hence, tree-$\alpha(S(C_5)) \leq 2$ and therefore tree-$\alpha(S(C_5)) = 2$.

We now prove tree-$\chi(S(C_5)) \leq 2$. We start by defining two bags $X_1 := \{v_1, v_2, v_3, v_4, x_{25}\}$ and $X_2 := \{v_1, v_3, v_4, v_5, x_{25}\}$ and add an edge between them. We then add bags $X_{13} := \{v_1, v_3, x_{1,3}\}$ and $X_{24} := \{v_2, v_4, x_{2,4}\}$ and attach them as leaf bags to the bag X_1. Also, we add bags $X_{14} := \{v_1, v_4, x_{1,4}\}$ and $X_{35} := \{v_3, v_5, x_{3,5}\}$ and attach them as leaf bags to the bag X_2. One can easily check that this is a valid tree decomposition $\mathcal{T}$ with $\chi(\mathcal{T}) = 2$, which implies tree-$\chi(S(C_5)) \leq 2$ and thus tree-$\chi(S(C_5)) = 2$.

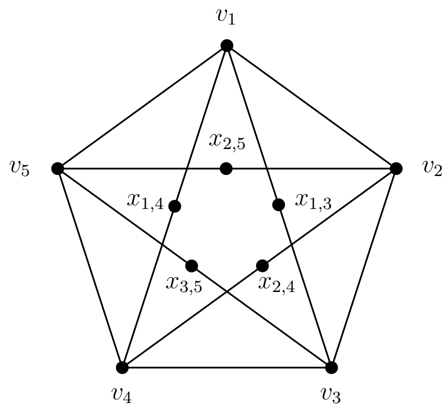


Figure 6.1: A counterexample for Question 6.8.

We obtain $\mathrm{tw}(S(C_5)) + 1 \geq 5 > 4 = \text{tree-}\alpha(S(C_5)) \cdot \text{tree-}\chi(S(C_5))$, which is a negative answer to Question 6.8.

Our construction exceeds the bound by exactly one. Now we want to know if we can construct graphs whose treewidth can be bounded from below by the product of tree-α and tree-χ and some constant $c \in \mathbb{R}$. This is the subject of the following sections.

6.3 Pushing the lower bound

The construction of the counterexample in Figure 6.1 was based on the idea of introducing paths of length 2 between any pair of non-adjacent vertices of C_5. We now study this case more generally: Given any graph G, we consider the G-subdivision of K_n, $S(G)$. Clearly, $S(G)$ is a subdivision of K_n and if $G = K_n$ then $S(G) = K_n$ and if $G = E_n$ then $S(G) = K_n^{(1)}$ (recall that $K_n^{(1)}$ is the graph obtained from K_n by subdividing every edge exactly once). We prove some useful properties of the graph $S(G)$.

Lemma 6.9: *Let G be a non-edgeless graph on $n \geq 3$ vertices. Then*

(i) $\mathrm{tw}(S(G)) = n - 1$,

(ii) $\alpha(G) - 1 \leq \text{tree-}\alpha(S(G)) \leq \alpha(G)$,

(iii) $\text{tree-}\chi(S(G)) = \text{tree-}\chi(G)$.

Proof. Since $S(G)$ is a subdivision of K_n, it contains K_n as a minor. Thus $\mathrm{tw}(S(G)) \geq n - 1$ by Lemma 2.15 and Corollary 2.7. To see that $\mathrm{tw}(S(G)) \leq n - 1$, let $\mathcal{T}$ be the following tree decomposition of $S(G)$: We put the vertices $V(G)$ in a bag X and for every subdivision vertex v, we add v and its two neighbors in a new bag X_v and add X_v as a leaf to X. This yields a valid tree decomposition of $S(G)$. The bag X has size $|V(G)| = n \geq 3$ and each leaf bag X_v has size 3. Hence, $\mathcal{T}$ is a tree decomposition of $S(G)$ of width $n - 1$. This implies $\mathrm{tw}(S(G)) \leq n - 1$, which completes the proof of (i).

For the first inequality in (ii), observe that a largest independent set in G induces a $K_{\alpha(G)}^{(1)}$ in $S(G)$. Thus, $K_{\alpha(G)}^{(1)}$ is an induced minor of $S(G)$, and by Lemma 3.2 and Corollary 3.6, we have $\text{tree-}\alpha(S(G)) \geq \text{tree-}\alpha(K_{\alpha(G)}^{(1)}) = \alpha(G) - 1$.

To prove the second inequality in (ii), we take the same tree decomposition $\mathcal{T}$ of $S(G)$ as in the proof of (i) and see that the bag X induces a subgraph with independence number $\alpha(G)$ and each leaf bag X_v induces a subgraph of independence number 2. As long as $\alpha(G) \geq 2$, the second inequality of (ii) follows immediately. If $\alpha(G) = 1$, then $G = K_n$, thus $S(G) = K_n$, and the claim follows by Observation 3.1.

Now we prove (iii). We have tree-$\chi(G) \le$ tree-$\chi(S(G))$ since G is a subgraph of $S(G)$ and tree-χ is monotone under taking subgraphs. To show tree-$\chi(S(G)) \le$ tree-$\chi(G)$, let $\mathcal{T}_G$ be a tree-χ-optimal tree decomposition of G. Since G is not edgeless, we have $\chi(\mathcal{T}_G) \ge 2$ by Lemma 6.2. If G is the complete graph, then $S(G) = G$ and hence tree-$\chi(S(G)) =$ tree-$\chi(G)$. So we may assume that there exist two non-adjacent vertices u and v in G. Let w be the corresponding subdivision vertex in $S(G)$. We construct a tree decomposition $\mathcal{T}_{S(G)}$ of $S(G)$ from $\mathcal{T}_G$ with $\chi(\mathcal{T}_{S(G)}) \le \chi(\mathcal{T}_G)$. If there is a bag X_t in $\mathcal{T}_G$ with $u, v \in X_t$, we add a leaf bag to X_t containing u, v, w. Such a leaf bag induces a graph that is 2-colorable. If there is no bag in $\mathcal{T}_G$ containing both u and v, then we add w to every bag. This does not increase the chromatic number of the tree decomposition, since $\chi(\mathcal{T}_G) \ge 2$ and w has at most one neighbor in each bag. We repeat this process for every pair of non-adjacent vertices in G. We see that $\mathcal{T}_{S(G)}$ is a valid tree decomposition of H satisfying $\chi(\mathcal{T}_{S(G)}) \le \chi(\mathcal{T}_G) =$ tree-$\chi(G)$, which implies tree-$\chi(S(G)) \le$ tree-$\chi(G)$. This completes the proof of (iii). ∎

Our goal is to find a graph G such that $S(G)$ satisfies

$$\operatorname{tw}(S(G)) + 1 \ge c\, \text{tree-}\alpha(S(G))\, \text{tree-}\chi(S(G)) \tag{6.1}$$

for some constant $c \in \mathbb{R}$ as large as possible. By Lemma 6.9, we have tree-$\alpha(S(G)) \le \alpha(G)$ and tree-$\chi(S(G)) =$ tree-$\chi(G)$, so basically we want G to have small $\alpha(G) \cdot$ tree-$\chi(G)$ compared to n. If tree-$\chi(G) = 1$, then $G = E_n$ by Lemma 6.2 and hence $\alpha(G) = n$, so this will not work. Let's focus on the case where tree-$\chi(G) = 2$. We want to have two disjoint subsets $A, B \subseteq V(G)$ such that $G - A$ and $G - B$ are bipartite graphs and A and B are anticomplete to each other. In such a case, tree-$\chi(G) = 2$ is always satisfies, as shown in the following lemma.

Lemma 6.10: *Let G be a graph and let $A, B \subseteq V(G)$ be two disjoint, anticomplete subsets such that $G - A$ and $G - B$ are bipartite graphs. Then* tree-$\chi(G) \le 2$.

Proof. If G is edgeless, then tree-$\chi(G) = 1$ by Lemma 6.2. So we may assume that G contains at least one edge. We construct a tree decomposition of G with three bags: $X_1 := V(G) \setminus (A \cup B)$, that is connected to two leaf bags $X_2 := V(G) \setminus A$ and $X_3 := V(G) \setminus B$. Since A and B are anticomplete, all conditions of the definition of a tree decomposition are satisfied. Since $G - A$ and $G - B$ are bipartite, it follows that $G - (A \cup B)$ is bipartite, too. Thus, all the bags X_1, X_2 and X_3 induce subgraphs in G with chromatic number at most 2. This proves the claim. ∎

Now if we find such a graph G with $\alpha(G) = c'n$ for some $c' \in \mathbb{R}$, we get tree-$\alpha(S(G))$ tree-$\chi(S(G)) \le 2\alpha(G) = 2c'n$, and since $\operatorname{tw}(S(G)) = n - 1$ by Lemma 6.9, we want $2c'n < n$ and hence $c' < 1/2$.

But how small can c' be under the given circumstances? Since $G - A$ is bipartite, we have

$$\alpha(G) \ge \frac{1}{2}(n - |A|).$$

Analogously, we obtain

$$\alpha(G) \ge \frac{1}{2}(n - |B|).$$

Also, since A and B are anticomplete to each other and A and B are bipartite themselves (otherwise, $A \subseteq V(G - B)$ would not induce a bipartite graph), we get

$$\alpha(G) \ge \frac{1}{2}|A| + \frac{1}{2}|B|.$$

Adding these three inequalities yields $3\alpha(G) \ge n$ which implies $\alpha(G) \ge \frac{1}{3}n$. So, the best constant we can hope for is $c' = \frac{1}{3}$. In fact, we will see later that $c' = \frac{1}{3}$ is not reachable. But first, we give a graph G with $\alpha(G) = \frac{3}{8}n$, hence $c' = \frac{3}{8}$, which is best possible, which we also prove later.

Consider the graph in Figure 6.2, which is known as the *Wagner graph* W_8. It is named after the German mathematician Klaus Wagner[1]. Wagner became famous for his results in graph theory, including Wagner's theorem, which characterizes planar graphs by the two forbidden minors K_5 and $K_{3,3}$. Also, the Robertson-Seymour theorem was known as Wagner's conjecture before it was proven.

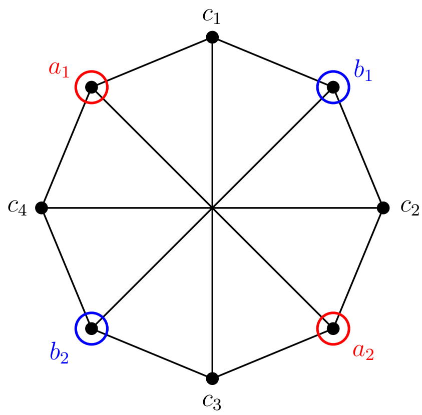


Figure 6.2: The Wagner graph W_8.

We have to show that W_8 fulfills all the properties discussed above. First of all, we set $A := \{a_1, a_2\}$ and $B := \{b_1, b_2\}$ and see that $A \cap B = \emptyset$, A and B are anticomplete to each other and $W_8 - A$ and $W_8 - B$ are bipartite. Lemma 6.10 implies that tree-$\chi(W_8) \le 2$. One can easily verify that $\alpha(W_8) = 3 = \frac{3}{8}|V(W_8)|$, hence $c' = \frac{3}{8}$. By Lemma 6.9 and the discussion above, we get $\text{tw}(S(W_8)) = 7$ and tree-$\alpha(S(W_8))$ tree-$\chi(S(W_8)) \le 2\alpha(W_8) = 6$. So until now, there is no improvement compared to the graph in Figure 6.1.

We generalize this idea as follows: Let now G_k be the graph obtained by taking k disjoint copies of W_8. Clearly, $|V(G_k)| = 8k$ and $\alpha(G_k) = 3k$. In each copy of W_8, we pick vertices a_1, a_2 and put them in a set A, and pick vertices b_1, b_2 and put them in a set B, just as we did before. In this way, we obtain two disjoint, anticomplete sets A, B with $G_k - A$ and $G_k - B$ bipartite. Then, we have $\text{tw}(S(G_k)) = 8k - 1$ and tree-$\alpha(S(G_k))$ tree-$\chi(S(G_k)) \le 2\alpha(G_k)$. Take for example $k = 10$, then $\text{tw}(S(G_k)) = 79$ and tree-$\alpha(S(G_k))$ tree-$\chi(S(G_k)) \le 60$. As k grows, $\text{tw}(S(G_k))$ gets arbitrary far away from tree-$\alpha(S(G_k))$ tree-$\chi(S(G_k))$. The graph $S(G_k)$ satisfies

$$\text{tw}(S(G_k)) + 1 \ge \frac{4}{3}\,\text{tree-}\alpha(S(G_k))\,\text{tree-}\chi(S(G_k)),$$

hence, $c = 4/3$ for the constant c in the Inequality 6.1.

Our construction started with the Wagner graph W_8 that has $\alpha(W_8) = \frac{3}{8}|V(W_8)|$. Can we find another graph with a smaller independence number and thereby improve the constant $c = 4/3$? This is not possible, as shown in the following theorem.

Theorem 6.11: *Let G be a non-edgeless graph on $n \ge 3$ vertices, and let $A, B \subseteq V(G)$ be two vertex-disjoint subsets, such that A and B are anticomplete to each other and $G - A$ and $G - B$ are bipartite graphs. Then $\alpha(G) \ge \frac{3}{8}n$.*

[1] *https://en.wikipedia.org/wiki/Klaus_Wagner*

Proof. Let $C := V(G) \setminus (A \cup B)$. Clearly, $G[A \cup C] = G - B$ and $G[B \cup C] = G - A$. Since $G[A \cup C]$ is bipartite, there is a coloring $\phi_A : A \cup C \to \{1, 2\}$. Analogously, there is a coloring $\phi_B : B \cup C \to \{1, 2\}$. Now, we define the coloring ϕ with

$$\phi(v) := \begin{cases} \phi_A(v), & \text{if } v \in A. \\ \phi_B(v), & \text{if } v \in B. \\ (\phi_A(v), \phi_B(v)), & \text{if } v \in C. \end{cases}$$

Note that two vertices $c_1, c_2 \in C$ can only be adjacent if $\phi(c_1) = (1, 1)$ and $\phi(c_2) = (2, 2)$ or $\phi(c_1) = (1, 2)$ and $\phi(c_2) = (2, 1)$. Otherwise, if e.g. $\phi(c_1) = (1, 2)$ and $\phi(c_2) = (1, 1)$, then $\phi_A(c_1) = \phi_A(c_2) = 1$, a contradiction. We denote by a_1 and a_2 the number of vertices in A that have colors 1 and 2 respectively. We do the same for vertices in B and C. Observe that $a_1+a_2 = |A|$, $b_1+b_2 = |B|$ and $c_{1,1}+c_{1,2}+c_{2,1}+c_{2,2} = |C|$. Then we obtain the following inequalities:

$$\begin{aligned} \alpha(G) &\geq a_1 + b_1 + c_{2,2}, \\ \alpha(G) &\geq a_1 + b_2 + c_{2,1}, \\ \alpha(G) &\geq a_2 + b_1 + c_{1,2}, \\ \alpha(G) &\geq a_2 + b_2 + c_{1,1}. \end{aligned}$$

Similarly, we get

$$\begin{aligned} \alpha(G) &\geq a_1 + c_{2,1} + c_{2,2}, \\ \alpha(G) &\geq a_2 + c_{1,1} + c_{1,2}, \\ \alpha(G) &\geq b_1 + c_{1,2} + c_{2,2}, \\ \alpha(G) &\geq b_2 + c_{1,1} + c_{2,1}. \end{aligned}$$

Adding these 8 inequalities, we have

$$\begin{aligned} 8\alpha(G) &\geq 3a_1 + 3a_2 + 3b_1 + 3b_2 + 3c_{1,1} + 3c_{1,2} + 3c_{2,1} + 3c_{2,2} \\ &= 3(a_1 + a_2 + b_1 + b_2 + c_{1,1} + c_{1,2} + c_{2,1} + c_{2,2}) \\ &= 3(|A| + |B| + |C|) = 3n. \end{aligned}$$

Thus, $\alpha(G) \geq \frac{3}{8}n$. This completes the proof. ■

In view of Lemma 6.10 and Theorem 6.11, the Wagner graph W_8 is best possible. So we cannot get any better bounds with graphs that have two subsets A and B satisfying the discussed properties. In the next section, we use another approach to improve our constant c from Inequality 6.1.

6.4 Another lower bound using shift graphs

Remember that our goal is to find a graph G such that $\operatorname{tw}(S(G)) + 1 \geq c \text{ tree-}\alpha(S(G)) \text{ tree-}\chi(S(G))$ for some constant $c \in \mathbb{R}$ as large as possible. In the last section we showed that $c \geq \frac{4}{3}$ is possible; we now show how to improve this value.

If G is non-edgeless and has $n \geq 3$ vertices, then Lemma 6.9 implies $\operatorname{tw}(S(G)) = n - 1$, tree-$\chi(S(G)) =$ tree-$\chi(G)$ and tree-$\alpha(S(G)) \leq \alpha(G)$. Thus, we want tree-$\chi(G) \cdot \alpha(G)$ to be as small as possible compared to n.

For an integer $N \geq 2$, the *shift graph* S_N is a graph whose vertex set consists of real intervals $[a, b]$ with $1 \leq a < b \leq N$ and two vertices $[a, b]$ and $[c, d]$ are adjacent if and only if $b = c$ or $d = a$. Shift graphs were introduced by Erdős and Hajnal in [EH68]. They form a class of graphs that is triangle-free but has arbitrary large chromatic number.

We prove that every shift graph has tree-chromatic number equal to 2. It follows that there exist graph classes with bounded tree-chromatic number but unbounded chromatic number.

Theorem 6.12: *[Sey16] Every shift graph S_N has tree-chromatic number at most 2.*

Proof. We give a tree decomposition $\mathcal{T}$ of S_N with $\chi(\mathcal{T}) = 2$. Let T be a path with vertices $t_1, \dots, t_N$ that are adjacent in the obvious way. For $1 \leq i \leq N$, let X_i contain all vertices $[a, b] \in V(S_N)$ with $a \leq i \leq b$. We claim that $\mathcal{T} = (T, X_t)$ is a tree decomposition of S_N. First, observe that $[a, b] \in X_i$ if and only if $a \leq i \leq b$. If $[a, b][b, c] \in E(S_N)$ is an edge, then $[a, b]$ and $[b, c]$ are both contained in X_b. Also, if $i < j < k$ and $[a, b] \in X_i \cap X_k$, then $a \leq i \leq b$ and $a \leq k \leq b$, hence $a \leq i < j < k \leq b$, so $[a, b] \in X_j$ (that is condition (iii)* of the definition of a tree decomposition). Thus, $\mathcal{T}$ is a tree decomposition of S_N.

Now for $1 \leq i \leq N$, X_i is the union of the two sets $\{[a, b] \mid a < i \leq b\}$ and $\{[a, b] \mid a \leq i < b\}$. These sets are independent in S_N. Therefore, every bag induces a bipartite graph in S_N, which implies $\chi(\mathcal{T}) = 2$, and we are done. ∎

In [ARS22] it is shown that there exists an induced subgraph S'_N of S_N with

$$\lim_{N \to \infty} \frac{\alpha(S'_N)}{|V(S'_N)|} = \frac{1}{4}.$$

The proof of this result uses random graphs and is not constructive. Nevertheless, we use this result to get a graph S'_N with tree-$\chi(S'_N) = 2$ and $\alpha(S'_N) = \frac{1}{4}|V(S'_N)|$ as $N \to \infty$. Collecting all these results, the graph $S(S'_N)$ satisfies

$$\begin{aligned} \operatorname{tw}(S(S'_N)) + 1 &= |V(S'_N)| = 4\alpha(S'_N) \\ &\geq 4 \,\text{tree-}\alpha(S(S'_N)) \\ &= 2 \,\text{tree-}\chi(S'_N) \,\text{tree-}\alpha(S(S'_N)) \\ &= 2 \,\text{tree-}\chi(S(S'_N)) \,\text{tree-}\alpha(S(S'_N)), \end{aligned}$$

as N tends to infinity, improving the constant c in Inequality 6.1 to be equal to 2.

6.5 Finding upper bounds

In this section we want to find upper bounds for the treewidth of a graph in terms of its tree-independence number and its tree-chromatic number. More precisely, we want to find a function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that $\operatorname{tw}(G) \leq f(\text{tree-}\alpha(G), \text{tree-}\chi(G))$. All the lower bounds that we constructed use G-subdivisions of K_n, so we first consider the limitations of this construction by providing an upper bound.

Theorem 6.13: *Let G be a non-edgeless graph on $n \geq 3$ vertices and let $S(G)$ be the G-subdivision of K_n. Then,* $\operatorname{tw}(S(G)) \in O(\text{tree-}\alpha(S(G))^2 \,\text{tree-}\chi(S(G)))$.

Proof. By Lemma 6.9, we have $\operatorname{tw}(S(G)) + 1 = n$, tree-$\chi(S(G))$ = tree-$\chi(G)$ and tree-$\alpha(S(G)) \geq \alpha(G) - 1$.

Now let $\mathcal{T} = (T, X_t)$ be a tree-χ-optimal tree decomposition of G. For each bag X_t and for any pair of vertices $u, v \in X_t$, we add an edge uv in G. By this procedure, we obtain a graph G' with $|V(G')| = |V(G)|$, $\alpha(G) \geq \alpha(G')$ and $\mathcal{T}$ is a tree decomposition of G' where every bag induces a clique in G'. Lemma 2.9 implies that G' is chordal.

Now, let X_t be a largest bag of $\mathcal{T}$. Clearly, $|X_t| = \omega(G')$ and we get

$$\omega(G') = |X_t| \leq \alpha(G[X_t]) \cdot \chi(G[X_t]) \leq \alpha(G) \cdot \text{tree-}\chi(G),$$

where the first inequality holds in any graph. Since G' is chordal, it is perfect by Corollary 2.2 and therefore satisfies $|V(G')| \leq \alpha(G')\omega(G')$ (see [Lov72]). Putting all this together, we obtain

$$\begin{aligned} \operatorname{tw}(S(G)) + 1 &= n = |V(G)| = |V(G')| \\ &\leq \alpha(G')\omega(G') \\ &\leq \alpha(G)\alpha(G) \,\text{tree-}\chi(G) \\ &\leq (\text{tree-}\alpha(S(G)) + 1)^2 \,\text{tree-}\chi(S(G)), \end{aligned}$$

so $\mathrm{tw}(S(G)) \in O(\text{tree-}\alpha(S(G))^2\, \text{tree-}\chi(S(G)))$, which is what we wanted to show. ∎

If we consider the general case, we are not able to give a polynomial function f. Nevertheless, we can give an exponential bound.

Theorem 6.14: *Any graph G satisfies*

$$\mathrm{tw}(G) \in O(4^{\text{tree-}\chi(G)+\text{tree-}\alpha(G)}).$$

Proof. Let G be a graph and let $\mathcal{T}$ be a tree-α-optimal tree decomposition of G, i.e. $\alpha(\mathcal{T}) = \text{tree-}\alpha(G)$. Every bag X_t of $\mathcal{T}$ induces a subgraph $G[X_t]$ with $\alpha(G[X_t]) \leq \text{tree-}\alpha(G)$ and $\omega(G[X_t]) \leq \omega(G) \leq \text{tree-}\chi(G)$. By Ramsey's Theorem (see also proof of Lemma 5.3), we have $|X_t| \leq R(\text{tree-}\chi(G) + 1, \text{tree-}\alpha(G) + 1) - 1$. Since $R(m, m) \leq 4^m$ (see e.g. [AZ18]) and $R(m, n) \leq \max\{R(m, m), R(n, n)\}$, we obtain

$$R(m, n) \leq \max\{R(m, m), R(n, n)\} \leq \max\{4^m, 4^n\} \leq 4^{m+n}.$$

If we set $m = \text{tree-}\chi(G) + 1$ and $n = \text{tree-}\alpha(G) + 1$, we get $|X_t| \leq 4^{\text{tree-}\chi(G)+\text{tree-}\alpha(G)+2}$. This holds for every bag X_t, in particular it holds for a largest bag and hence $\mathrm{tw}(G) + 1 \leq 4^{\text{tree-}\chi(G)+\text{tree-}\alpha(G)+2}$, i.e. $\mathrm{tw}(G) \in O(4^{\text{tree-}\chi(G)+\text{tree-}\alpha(G)})$. ∎

So, in the general case, there is a huge gap between the lower bound (a linear function in tree-α and tree-χ) and the upper bound (an exponential function in tree-α and tree-χ). We ask a natural follow-up question:

Question 6.15: *Given a graph G, is there any polynomial function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that* $\mathrm{tw}(G) \leq f(\text{tree-}\alpha(G), \text{tree-}\chi(G))$*?*

7 The Central Bag Method

In this chapter, we will introduce a concept called the central bag method. It has application in different papers, including [ACV22 | Abr+24b | CAHS22 | ACDR22]. The authors of these papers developed the central bag method; in particular, Abrishami wrote a survey about this method in [Abr23]. Most definitions and proofs are taken from this survey.

The method was invented to bound the treewidth of graph classes. The main idea is as follows: Assume we are given a graph class $\mathcal{G}$ and we need to find out whether the treewidth of graphs in $\mathcal{G}$ is bounded or not. In general, this is not an easy task. Instead, we try to find a "core" of a graph $G \in \mathcal{G}$ for which the task of determining its treewidth is easier than determining the treewidth of G itself. Under certain conditions, we can then extend the solution of this core to a solution of G.

In fact, we generalize this method in a way, such that we can not only bound the treewidth of graph classes, but also the tree-independence number and the tree-chromatic number.

7.1 The tree decomposition point of view

In the central bag method, everything is defined in terms of separations. Before we get there in the next section, we want to give an intuition of the new concept using tree decompositions. The reason why we do not work with tree decompositions directly, is that we often know more about the separations of a graph class than we know about its tree decompositions. We will see in this section that these concept are in fact equivalent. We follow the notation of [Abr23].

Let G be a graph. A function $w : V(G) \to \mathbb{R}$ is called a *weight function on G*. Given a weight function w and a subset $X \subseteq V(G)$, we define $w(X) := \sum_{x \in X} w(x)$. A weight function w on G is a *normal weight function* if $w : V(G) \to [0, 1]$ and $w(V(G)) = 1$.

From now on, unless stated otherwise, let G be a graph and let w be a normal weight function on G. A set $X \subseteq V(G)$ is called *w-balanced separator of G* if for every component D of $G - X$, it holds that $w(D) \leq 1/2$. The existence of w-balanced separators of small size for every normal weight function w is crucial in the central bag method, since it is equivalent of having bounded treewidth. We will show this equivalence later. The following lemma is a well-known result and it is used very frequently when working with tree decompositions.

Lemma 7.1: *[Die17][Abr23] Let (T, X_t) be a tree decomposition of a graph G, let $t_1t_2 \in E(T)$ be an edge and let T_1 and T_2 be the two components of $T - t_1t_2$ with $t_1 \in V(T_1)$ and $t_2 \in V(T_2)$. Let $C := X_{t_1} \cap X_{t_2}$, $A := \bigcup_{t \in V(T_1)} X_t \setminus C$ and $B := \bigcup_{t \in V(T_2)} X_t \setminus C$. Then (A, C, B) is a separation of G.*

Proof. First we show that A and B are disjoint. Assume that $v \in A \cap B$ for the sake of contradiction. Then $v \in \bigcup_{t \in V(T_1)} X_t$ and $v \in \bigcup_{t \in V(T_2)} X_t$. Condition (iii) of the definition of tree decompositions implies that $v \in X_{t_1}$ and $v \in X_{t_2}$, a contradiction. Hence, A and B are disjoint.

Now we need to show that A and B are anticomplete to each other. Assume that $ab \in E(G)$ is an edge with $a \in A$ and $b \in B$. By condition (ii) of the definition of a tree decomposition, there is a node $t \in V(T)$ with $a, b \in X_t$. Assume w.l.o.g. that $t \in V(T_1)$. Then $b \in \bigcup_{t \in V(T_1)} X_t$ and by condition (iii) of the definition of a tree decomposition it follows that $b \in X_{t_1}$ and $b \in X_{t_2}$, a contradiction. Therefore, A and B are anticomplete. This shows that (A, C, B) is a separation of G. ∎

The previous lemma tells us that every edge of a tree decomposition, or rather the intersection of the bags of its two endpoints, induces a separation of G. Therefore, if $\mathcal{T} = (T, X_t)$ is a tree decomposition of G, we define $\tau(\mathcal{T})$ as the set of all separations of G induced by the edges of T. More precisely, for $e = t_1t_2 \in E(T)$, we set $C_e := X_{t_1} \cap X_{t_2}$, $A_e := \bigcup_{t \in V(T_1)} X_t \setminus C_e$ and $B_e := \bigcup_{t \in V(T_2)} X_t \setminus C_e$, just as we did in Lemma 7.1. The *corresponding separation to* e is the triple (A_e, C_e, B_e) and we denote it by S_e. Then $\tau(\mathcal{T}) := \{S_e \mid e \in E(T)\}$ is the *collection of separations corresponding to* $\mathcal{T}$. We say that $\mathcal{T}$ is w*-unbalanced*, if for every separation $(A, C, B) \in \tau(\mathcal{T})$ it holds that $w(A) > \frac{1}{2}$ or $w(B) > \frac{1}{2}$. On the other hand we say that $\mathcal{T}$ is w*-balanced*, if there exists a separation $(A, C, B) \in \tau(\mathcal{T})$ with $w(A) \leq \frac{1}{2}$ and $w(B) \leq \frac{1}{2}$.

If $\mathcal{T}$ is w-unbalanced, every edge of T induces a separation that has a large A-part or a large B-part with respect to w. Therefore, we can define the directed tree $\overrightarrow{T}$ obtained from T by directing each edge towards the larger side. We call $\overrightarrow{T}$ the w*-direction of* T.

We now prove some properties of $\overrightarrow{T}$. The first one holds for every directed tree.

Lemma 7.2: *[Abr23] Every directed tree T has a sink, that is a vertex v with only incoming edges.*

Proof. We assume that T does not have a sink. Let $P = (v_1, \dots, v_k)$ be a longest directed path in T. Since T does not have a sink, the vertex v_k has an outgoing edge to a neighbour x. If $x \notin V(P)$, then $P' = (v_1, \dots, v_k, x)$ is longer than P, a contradiction. If $x \in V(P)$, then T contains a cycle, contradicting the fact that T is a tree. Therefore, T must have a sink. ∎

Lemma 7.3: *[Abr23] Let G be a graph, let $\mathcal{T} = (T, X_t)$ be a tree decomposition of G and let w be a normal weight function on G such that $\mathcal{T}$ is w-unbalanced. Let $\overrightarrow{T}$ be the w-direction of T. Then there exists a vertex $r \in V(\overrightarrow{T})$ such that every path P in T that ends in r is a directed path to r in $\overrightarrow{T}$.*

Proof. We start by proving the following property of $\overrightarrow{T}$.

(1) Every vertex $v \in V(\overrightarrow{T})$ has at most one outgoing edge.

Assume that v has two outgoing edges with corresponding neighbours v_1 and v_2. Let S_1 be the separation corresponding to the edge vv_1 and let S_2 be the separation corresponding to the edge vv_2. Let T_1 and T_2 be the components of $\overrightarrow{T} - \{v\}$ containing v_1 and v_2, respectively. Since vv_1 is directed towards v_1, it follows that $w(\bigcup_{t \in V(T_1)} X_t \setminus (X_v \cap X_{v_1})) > \frac{1}{2}$. Analogously, we have $w(\bigcup_{t \in V(T_2)} X_t \setminus (X_v \cap X_{v_2})) > \frac{1}{2}$. Condition (iii) of the definition of a tree decomposition implies that $(\bigcup_{t \in V(T_1)} X_t) \cap (\bigcup_{t \in V(T_2)} X_t) \subseteq X_{v_1} \cap X_v \cap X_{v_2}$ and therefore $\bigcup_{t \in V(T_1)} X_t \setminus (X_{v_1} \cap X_v)$ is disjoint from $\bigcup_{t \in V(T_2)} X_t \setminus (X_{v_2} \cap X_v)$. But then $w(V(G)) > 1$, a contradiction to the fact that w is a normal weight function on G. This finishes the proof of (1).

Now, by Lemma 7.2, $\overrightarrow{T}$ has a sink s. We prove the statement of the lemma by induction on the length of the path. In the base case, consider all paths of length 1 in T to s. Since s is a sink, every edge is directed towards s, which proves the base case. Now let $P = (v_1, \dots, v_{k-1}, s)$ be a path of length k from v_1 to s. By induction, the path $(v_2, \dots, v_{k-1}, s)$ is directed towards s. By (1), since v_2v_3 is directed towards v_3, we conclude that the edge v_1v_2 is directed towards v_2. Hence P is a directed path towards s. This completes the proof. ∎

We call the vertex r in the previous lemma the w*-heavy vertex of* T and the bag X_r the w*-heavy bag of* $\mathcal{T}$. This bag X_r is essentially our "central bag", the induced subgraph to which we want to reduce our problem to. Although we will define the central bag in another way later, the way of thinking about it as a heavy vertex in a tree decomposition gives us good intuition.

Now we prove the equivalence between the existence of balanced separators and bounded graph invariants, namely the treewidth, tree-independence number and tree-chromatic number.

Lemma 7.4: *[Abr23][RS86] Let G be a graph, let k be a positive integer and let $\rho \in \{\text{tw}, \text{tree-}\alpha, \text{tree-}\chi\}$. Suppose that $\rho(G) \leq k$. Then for every normal weight function w on G, there exists a set $X_w \subseteq V(G)$ such that X_w is a w-balanced separator of G with*

(1) $|X_w| \leq k + 1$, *in case* $\rho = \text{tw}$,

(2) $\alpha(X_w) \leq k$, *in case* $\rho = \text{tree-}\alpha$,

(3) $\chi(X_w) \leq k$, *in case* $\rho = \text{tree-}\chi$.

Proof. Throughout the proof, we distinguish three cases: (1): $\rho = \text{tw}$, (2): $\rho = \text{tree-}\alpha$, (3): $\rho = \text{tree-}\chi$. Let $\mathcal{T} = (T, X_t)$ be a tree decomposition of G such that

(1) the width of $\mathcal{T}$ is at most k.

(2) $\alpha(\mathcal{T}) \leq k$.

(3) $\chi(\mathcal{T}) \leq k$.

First, assume that $\mathcal{T}$ is w-balanced. Then, there exists an edge $t_1t_2 \in E(T)$ and a corresponding separation $S_e = (A_e, C_e, B_e) \in \tau(\mathcal{T})$ with $w(A_e) \leq \frac{1}{2}$ and $w(B_e) \leq \frac{1}{2}$. Since $C_e = X_{t_1} \cap X_{t_2}$ and $\rho(G) \leq k$, we conclude that

(1) $|C_e| \leq k + 1$. Now $X_w = C_e$ is a w-balanced separator of G of size at most $k + 1$.

(2) $\alpha(C_e) \leq \max\{\alpha(X_{t_1}), \alpha(X_{t_2})\} \leq k$. Now $X_w = C_e$ is a w-balanced separator of G of independence number at most k.

(3) $\chi(C_e) \leq \max\{\chi(X_{t_1}), \chi(X_{t_2})\} \leq k$. Now $X_w = C_e$ is a w-balanced separator of G of chromatic number at most k.

So we can assume that $\mathcal{T}$ is w-unbalanced. Let r be the w-heavy vertex of T and let $t_1, \ldots, t_m$ be the neighbours of r in T. Each edge $rt_i \in E(T)$ induces a corresponding separation $S_i = (A_i, C_i, B_i)$ of G. We may assume w.l.o.g. that $w(A_i) < \frac{1}{2}$ for each $1 \leq i \leq m$. The connected components of $G - X_r$ are exactly the sets $A_1, \ldots, A_m$. Hence, $X_w = X_r$ is a w-balanced separator of G of

(1) size at most $k + 1$.

(2) independence number at most k.

(3) chromatic number at most k.

This finishes the proof. ∎

We state the treewidth-version of the converse direction without a proof here.

Lemma 7.5: *[Abr+24b] Let G be a graph, let k be a positive integer and assume that G has a w-balanced separator of size at most k for every normal weight function w. Then* $\text{tw}(G) \leq 2k$.

Now we prove the tree-α- and tree-χ-version. The proof is an adjustment from [Abr+24a].

Lemma 7.6: *[Abr+24a] Let G be a graph, let k be a positive integer and let $\rho \in \{\alpha, \chi\}$. Suppose that G has a w-balanced separator X_w with $\rho(G[X_w]) \leq k$ for every normal weight function w. Then,*

(1) $\text{tree-}\alpha(G) \leq 5k$, *in case* $\rho = \alpha$,

(2) tree-$\chi(G) \leq 5k$*, in case* $\rho = \chi$*.*

Proof. During this proof, we distinguish the cases when (1): $\rho = \alpha$ and (2): $\rho = \chi$. In case (1), we show the tree-α-version of the proof; in case (2), we show the tree-χ-version.

We recursively show the following stronger statement:

(*) Let (G, A) be a pair where G is a graph as in the lemma and $A \subseteq V(G)$ has $\rho(G[A]) \leq 4k$. Then there exists a tree decomposition $\mathcal{T} = (T, X_t)$ of G with $\rho(\mathcal{T}) \leq 5k$ such that $A \subseteq X_t$ for some $t \in V(T)$. This implies (1) tree-$\alpha(G) \leq 5k$ or (2) tree-$\chi(G) \leq 5k$.

The statement (*) clearly holds if $\rho(G) \leq 4k$. If $\rho(G) > 4k$, we pick a set $A \subseteq V(G)$ with $|A| = 4k$ and $\rho(G[A]) = 4k$. We need to show that there exists a tree decomposition $\mathcal{T} = (T, X_t)$ of G with $\rho(\mathcal{T}) \leq 5k$ such that $A \subseteq X_t$ for some $t \in V(T)$. We define a weight function $w : V(G) \to [0, 1]$ with

$$w(v) := \begin{cases} \frac{1}{4k}, & \text{if } v \in A \\ 0, & \text{if } v \notin A. \end{cases}$$

Observe that w is a normal weight function on G. Recall that, by assumption, G has a w-balanced separator X_w with $\rho(G[X_w]) \leq k$. Thus, for every component D of $G - X_w$ we have $w(D) \leq 1/2$ and so there are at least $2k$ vertices of A outside of D. Since $\rho(G[X_w]) \leq k$, there are at least k vertices $v_1, \ldots, v_k \in A$ which do not belong to $D \cup X_w$.

Let us now consider the set $Y := A \cap D$. The vertices $v_1, \ldots, v_k$ do not belong to Y, so $\rho(G[Y]) \leq 4k - k = 3k$. This implies $\rho(G[Y \cup X_w]) \leq \rho(G[Y]) + \rho(G[X_w]) \leq 3k + k = 4k$ and we recursively find a tree decomposition of $D \cup X_w$ by applying (*) to the pair $(D \cup X_w, Y \cup X_w)$.

We do the previous procedure for each component D_i of $G - X_w$, obtaining for each i a set Y_i analogously to Y and a tree decomposition $\mathcal{T}_i$ of $D_i \cup X_w$ with a node $t_i \in V(T_i)$ such that $Y_i \cup X_w \subseteq X_{t_i}$. We then create a tree decomposition of G, obtained from the union of the T_i's and adding a node t that is adjacent to all t_i's with $X_t := A \cup X_w$. Observe that $\rho(G[X_t]) = \rho(G[A \cup X_w]) \leq \rho(G[A]) + \rho(G[X_w]) \leq 4k + k = 5k$. This yields a tree decomposition of G with the desired properties, which completes the proof. ∎

Together, the Lemmas 7.4, 7.5 and 7.6 show the equivalence of graphs having bounded treewidth, tree-independence number or tree-chromatic number and the existence of w-balanced separators of bounded size, independence number or chromatic number, respectively. This characterization of the different graph invariants is powerful in a sense that we often know more about separators of graph classes than we know about its tree decompositions.

Given a separation (A, C, B), it does not really matter which part of the graph is A and which other part of the graph is B; it is just a naming of the parts. Therefore, we fix the following convention for the rest of this chapter: The A-part in a separation is always the part with lower weight, i.e. $w(A) \leq w(B)$. Now, two separations (A_1, C_1, B_1) and (A_2, C_2, B_2) are *non-crossing* if $A_1 \cup C_1 \subseteq B_2 \cup C_2$ and $A_2 \cup C_2 \subseteq B_1 \cup C_1$. A collection $\mathcal{S}$ of separations of G is called *laminar* if the separations of $\mathcal{S}$ are pairwise non-crossing.

Intuitively, two separations are non-crossing if the small part of one separations is contained in the large part of the other one. We want to illustrate this concept on an example.

Consider the graph G (that was introduced in Figure 2.1) and another possible tree decomposition of G as shown in Figure 7.1.

If we assume for the moment, that w is a normal weight function that assigns each vertex the same weight, then this tree decomposition admits the following corresponding collection of separations $\mathcal{S} = \{S_1, S_2, S_3\}$ with

$$\begin{aligned} S_1 &= (\{1, 6\}, \{2, 7\}, \{3, 4, 5, 8, 9\}), \\ S_2 &= (\{9\}, \{7, 8\}, \{1, 2, 3, 4, 5, 6\}), \\ S_3 &= (\{4, 5\}, \{3, 8\}, \{1, 2, 6, 7, 9\}). \end{aligned}$$

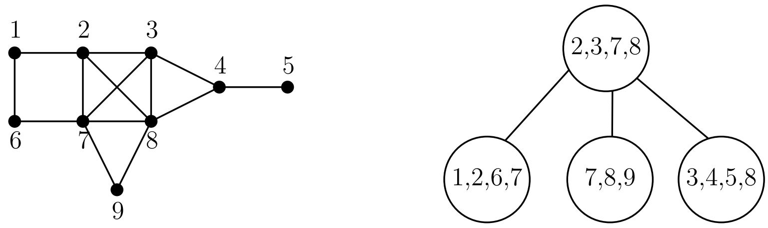


Figure 7.1: Example graph G and another possible tree decomposition of G.

As one easily checks, S_1, S_2 and S_3 are pairwise non-crossing and thus $\mathcal{S}$ is a laminar collection of separations. On the other hand, take an arbitrary laminar collection of separations, say $\mathcal{S}' = \{S_1', S_2'\}$ with

$$\begin{aligned} S_1' &= (\{1,6\}, \quad \{2,3,7,8\}, \quad \{4,5,9\}), \\ S_2' &= (\{5\}, \quad \{3,4,8\}, \quad \{1,2,6,7,9\}). \end{aligned}$$

We see that there exists a tree decomposition $\mathcal{T}$ of G with $\tau(\mathcal{T}) = \mathcal{S}'$; it is shown in Figure 7.2.

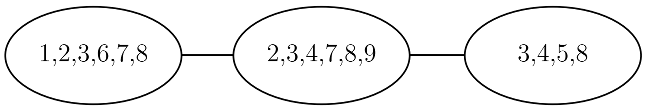


Figure 7.2: Tree decomposition $\mathcal{T}$ of G with $\tau(\mathcal{T}) = \mathcal{S}'$.

This connection between tree decomposition and laminar collection of separations holds in general. Given a graph G and a tree decomposition $\mathcal{T}$ of G, it holds that $\tau(\mathcal{T})$ is laminar. Also, no matter which laminar collection of separations $\mathcal{S}$ we choose, there exists a tree decomposition whose corresponding collection of separations is equal to $\mathcal{S}$. We record this correspondence in the following theorem.

Theorem 7.7: *[Abr23][RS91] Let G be a graph and let $\mathcal{T}$ be a tree decomposition of G. Then $\tau(\mathcal{T})$ is laminar.*

On the other hand, for every laminar collection of separations $\mathcal{S}$ of G, there exists a tree decomposition $\mathcal{T}$ of G with $\tau(\mathcal{T}) = \mathcal{S}$.

7.2 Everything in the language of separations

In this section we do not talk about tree decompositions anymore, but rather about collections of separations. Even the central bag is not defined as a bag of a tree decomposition; it is defined in terms of separations.

Again, let G be a graph and let w be a normal weight function on G. Also, let $\mathcal{S}$ be a collection of separations of G. The *central bag for $\mathcal{S}$*, denoted as $\beta_{\mathcal{S}}$, is defined as

$$\beta_{\mathcal{S}} := \bigcap_{S \in \mathcal{S}} (B(S) \cup C(S)).$$

To get an intuition of this concept we refer back to the tree decomposition in Figure 7.1 and its corresponding collection of separations $\mathcal{S}$. We see that

$$\begin{aligned} \beta_{\mathcal{S}} &= (B(S_1) \cup C(S_1)) \cap (B(S_2) \cup C(S_2)) \cap (B(S_3) \cup C(S_3)) \\ &= \{2,3,4,5,7,8,9\} \cap \{1,2,3,4,5,6,7,8\} \cap \{1,2,3,6,7,8,9\} \\ &= \{2,3,7,8\}. \end{aligned}$$

Recall that (in this example) we are assuming that the weight function w assigns each vertex the same weight, i.e. $w(v) = \frac{1}{9}$ for all $v \in V(G)$. Then this central bag is exactly what we would expect it to be, since the w-direction of T would direct each edge towards the bag with vertices $\{2, 3, 7, 8\}$. So the central bag corresponds to the w-heavy bag of the tree decomposition.

We make the same observation if we look at the tree decomposition in Figure 7.2: Its corresponding collection of separations $\mathcal{S}$ admits a central bag $\beta_{\mathcal{S}} = \{2, 3, 4, 7, 8, 9\}$, which again corresponds to the w-heavy bag of the w-direction of T.

We make the following simple observation.

Observation 7.8: *Let $\beta_{\mathcal{S}}$ be the central bag for a graph G and a collection of separations $\mathcal{S}$. Then $V(G) \setminus \beta_{\mathcal{S}} = \bigcup_{S\in\mathcal{S}} A(S)$.*

We want our collections of separations to fulfill some additional properties. Let G be a graph, let w be a normal weight function on G and let k be a positive integer. Let $\rho \in \{|\cdot|, \alpha, \chi\}$. A collection of separations $\mathcal{S}$ of G is *k-aligned* if the following conditions are satisfied:

(1) For every $S \in \mathcal{S}$ it holds that

  (i) $C(S) \cap \beta_{\mathcal{S}}$ is connected,

  (ii) there exists a set $\delta(S) \subseteq C(S)\cap\beta_{\mathcal{S}}$ such that $\rho(\delta(S)) \leq k$ and $(A(S)\cup(C(S)\setminus\delta(S)), \delta(S), B(S))$ is a separation of G,

  (iii) $w(A(S) \cup (C(S) \setminus \delta(S))) < \frac{1}{2}$, and

(2) for every component D of $\bigcup_{S\in\mathcal{S}} A(S)$, there exists a separation $S \in \mathcal{S}$ such that $D \subseteq A(S)$.

Given the graph G of Figure 7.1, together with its laminar collection of separations $\mathcal{S} = \{S_1, S_2, S_3\}$ and the corresponding central bag $\beta_{\mathcal{S}} = \{2, 3, 7, 8\}$, we see that $\mathcal{S}$ is 2-aligned, with $\delta(S) = C(S)$ for all $S \in \mathcal{S}$. But $\mathcal{S}$ is not 1-aligned, because any proper subset of $C(S)$ (for all $S \in \mathcal{S}$) does not separate the graph and therefore violates condition (1ii).

We note that condition (1i) is not necessary in our proofs, but it is required in most applications. Therefore, it was chosen to be part of the definition.

Condition (1iii) of the definition of k-aligned implies that $w(A) < \frac{1}{2}$. In most applications, that is achieved by ensuring that $w(B) > \frac{1}{2}$. This matches our convention that $w(A) \leq w(B)$ holds for every separation (A, C, B).

Condition (2) is satisfied whenever we deal with a laminar collection of separations, as shown in the following lemma.

Lemma 7.9: *[Abr23] Let G be a graph and let $\mathcal{S}$ be a laminar collection of separations. Then $\mathcal{S}$ satisfies condition (2) of the definition of k-aligned.*

Proof. Let D be a component of $\bigcup_{S\in\mathcal{S}} A(S)$. Clearly, there exists $S_1 \in \mathcal{S}$ such that $D\cap A(S_1) \neq \emptyset$. Since S_1 is a separation, we have $N(A(S_1)) \subseteq C(S_1)$. Since $\mathcal{S}$ is laminar, we have $A(S_1) \cup C(S_1) \subseteq B(S_2) \cup C(S_2)$ for all $S_1 \neq S_2 \in \mathcal{S}$, so $C(S_1) \subseteq B(S_2) \cup C(S_2)$. Suppose that there is a vertex $v \in C(S_1) \cap A(S_2)$ for any $S_1 \neq S_2$. Then, $v \in A(S_2)$ and $v \in B(S_2) \cup C(S_2)$, a contradiction. Therefore, $C(S_1) \cap A(S_2) = \emptyset$ for all $S_1 \neq S_2 \in \mathcal{S}$.

Now assume that there is a vertex $v \in D$ and $v \notin A(S_1)$. Note that $v \notin B(S_1)$, since D is a connected component. Thus, $v \in C(S_1)$, which implies $v \notin A(S_2)$ for all $S_2 \in \mathcal{S}$, i.e. $v \notin \bigcup_{S\in\mathcal{S}} A(S)$, a contradiction. We conclude that $D \subseteq A(S_1)$, which finishes the proof. ∎

Assume we are given a collection of separations $\mathcal{S}$ that is k-aligned. For every separation $S \in \mathcal{S}$, we set $A^*(S) := A(S) \cup (C(S) \setminus \delta(S))$. So, $A^*(S)$ is the new A-part of the new separation. Furthermore, we define the *anchor map* $\delta^*(S)$ that intuitively chooses a representative of a separation S, that we call the *anchor for* S. Formally, $\delta^*(S)$ maps a separation S to a vertex $v \in \delta(S)$.

The following observation follows by condition (1ii) of the definition of k-aligned and by construction $\delta^*(S) \in \delta(S)$.

Observation 7.10: *Let G be a graph, let w be a normal weight function on G and let $\mathcal{S}$ be a k-aligned collection of separations of G. Then, for all $S \in \mathcal{S}$, the anchor of S is in $\beta_{\mathcal{S}}$.*

Remember that we want to reduce the problem for the graph G to a core part of G, the central bag $\beta_{\mathcal{S}}$. Therefore, we want to infer a new weight function $w_{\mathcal{S}}$ on $\beta_{\mathcal{S}}$, which we call the *inherited weight function for* $\mathcal{S}$. We fix an ordering $\mathcal{O}$ of the vertices of G. For each component D of $\bigcup_{S \in \mathcal{S}} A(S)$, let $f(D)$ denote the minimum vertex with respect to the ordering, such that $D \subseteq \bigcup_{S \in \delta^{*-1}(v)} A(S)$. By $A_{\mathcal{O}}(v)$, we denote the union of the components D of $\bigcup_{S \in \mathcal{S}} A(S)$ such that $f(D) = v$. By Observation 7.10, the anchor for each separation is in $\beta_{\mathcal{S}}$. Thus, $\{A_{\mathcal{O}}(v) \mid v \in \beta_{\mathcal{S}}\}$ is a partition of $\bigcup_{S \in \mathcal{S}} A(S)$. Now, for all $v \in \beta_{\mathcal{S}}$, we define

$$w_{\mathcal{S}}(v) := w(v) + w(A_{\mathcal{O}}(v)).$$

Let us take a look at an example. We consider the graph G of Figure 7.1 and its 2-aligned collection of separations $\mathcal{S} = \{S_1, S_2, S_3\}$. As anchors, we set $\delta^*(S_1) := 2, \delta^*(S_2) := 7$ and $\delta^*(S_3) := 3$. Assume that the ordering $\mathcal{O}$ is the identity. There are three components $D_1, D_2, D_3 \in \bigcup_{S \in \mathcal{S}} A(S)$, namely $D_1 = \{1, 6\}$, $D_2 = \{9\}$ and $D_3 = \{4, 5\}$. Therefore, we have $f(D_1) = 2$, $f(D_2) = 7$ and $f(D_3) = 3$. Also, we have $A_{\mathcal{O}}(2) = D_1$, $A_{\mathcal{O}}(3) = D_3$ and $A_{\mathcal{O}}(7) = D_2$ and $A_{\mathcal{O}}(v) = \emptyset$ for all $v \notin \{2, 3, 7\}$. Assuming that the weight function w on G is assigning each vertex the same weight $1/9$, we obtain the inherited weight function $w_{\mathcal{S}}$ on $\beta_{\mathcal{S}}$ with

$$w_{\mathcal{S}}(2) = w(2) + w(A_{\mathcal{O}}(2)) = w(2) + w(D_1) = \frac{1}{9} + \frac{2}{9} = \frac{3}{9}$$

and in the same way

$$w_{\mathcal{S}}(3) = \frac{3}{9}, w_{\mathcal{S}}(7) = \frac{2}{9}, w_{\mathcal{S}}(8) = \frac{1}{9}.$$

Note that $w_{\mathcal{S}}$ is a normal weight function on $\beta_{\mathcal{S}}$, which holds in general and we prove that in the following lemma.

Lemma 7.11: *[Abr23] Let G be a graph, let w be a normal weight function on G and let $\mathcal{S}$ be a k-aligned collection of separations of G. Then, $w_{\mathcal{S}}$ is a normal weight function on $\beta_{\mathcal{S}}$.*

Proof. Clearly, we have that $w_{\mathcal{S}} : \beta_{\mathcal{S}} \to [0, 1]$, since w is a normal weight function and by definition of $w_{\mathcal{S}}$. We show that $w_{\mathcal{S}}(\beta_{\mathcal{S}}) = 1$. By definition of $w_{\mathcal{S}}$, we have

$$\begin{aligned} w_{\mathcal{S}}(\beta_{\mathcal{S}}) &= \sum_{v \in \beta_{\mathcal{S}}} w_{\mathcal{S}}(v) \\ &= \sum_{v \in \beta_{\mathcal{S}}} w(v) + \sum_{v \in \beta_{\mathcal{S}}} w(A_{\mathcal{O}}(v)). \end{aligned}$$

Recall that $\{A_{\mathcal{O}}(v) \mid v \in \beta_{\mathcal{S}}\}$ is a partition of $\bigcup_{S\in\mathcal{S}} A(S)$ and thus $\bigcup_{v\in\beta_{\mathcal{S}}} A_{\mathcal{O}}(v) = \bigcup_{S\in\mathcal{S}} A(S)$. Therefore, and by Observation 7.8, we have

$$\begin{aligned} w_{\mathcal{S}}(\beta_{\mathcal{S}}) &= \sum_{v\in\beta_{\mathcal{S}}} w(v) + \sum_{v\in\beta_{\mathcal{S}}} w(A_{\mathcal{O}}(v)) \\ &= \sum_{v\in\beta_{\mathcal{S}}} w(v) + \sum_{v\in V(G)\setminus\beta_{\mathcal{S}}} w((v)) \\ &= \sum_{v\in V(G)} w(v) \\ &= 1, \end{aligned}$$

since w is a normal weight function on G. This concludes the proof. ∎

We are almost ready to prove that finding a solution for the central bag helps us to find a solution for the original graph. We need just one more definition. Let $\mathcal{S}$ be a k-aligned collection of separations with an anchor map δ^* and let $X \subseteq \beta_{\mathcal{S}}$. We say that a separation $S \in \mathcal{S}$ *crosses* X if either $\delta^*(S) \in X$ or if there exist two distinct components D_1, D_2 of $\beta_{\mathcal{S}} \setminus X$ such that $C(S) \cap D_1 \neq \emptyset$ and $C(S) \cap D_2 \neq \emptyset$.

Figure 7.3 shows an example. The central bag $\beta_{\mathcal{S}}$ is separated by a set X into two parts Q_1 and Q_2. The separation S_1 does not cross X, the separation S_3 crosses X and separation S_2 crosses X if $\delta^*(S_2) = u$.

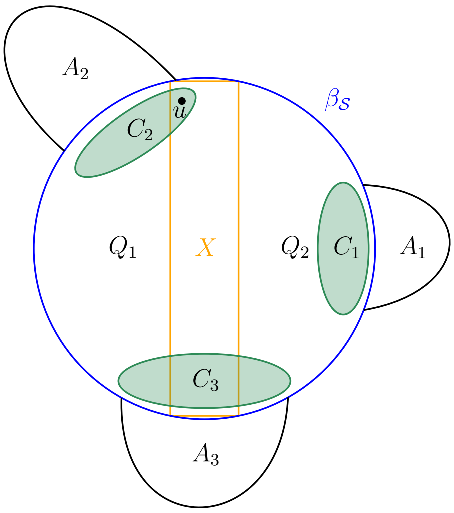


Figure 7.3: An illustration for the central bag $\beta_{\mathcal{S}}$, a set $X \subseteq \beta_{\mathcal{S}}$ that separates $\beta_{\mathcal{S}}$ into two parts Q_1 and Q_2, and three separations of G that cross or do not cross X.

Finally, we are ready to prove the following main theorem of this chapter.

Theorem 7.12: *[Abr23] Let G be a graph, let w be a normal weight function on G, let $\mathcal{S}$ be a k-aligned collection of separations of G, let δ^* be the anchor map for $\mathcal{S}$ and let $w_{\mathcal{S}}$ be the inherited weight function for $\mathcal{S}$. Let $\rho \in \{|\cdot|, \alpha, \chi\}$. If $G[\beta_{\mathcal{S}}]$ has a $w_{\mathcal{S}}$-balanced separator $X_{w_{\mathcal{S}}}$ with $\rho(X_{w_{\mathcal{S}}}) \leq \gamma$, then G has a w-balanced separator X_w with $\rho(X_w) \leq \gamma + ck$, where c denotes the number of separations of $\mathcal{S}$ that cross $X_{w_{\mathcal{S}}}$ in $\beta_{\mathcal{S}}$.*

Proof. Let $\mathcal{S}' \subseteq \mathcal{S}$ denote the set of all separations S of $\mathcal{S}$ such that $C(S)$ crosses $X_{w_{\mathcal{S}}}$. We define

$$Y := X_{w_{\mathcal{S}}} \cup \left(\bigcup_{S\in\mathcal{S}'} \delta(S) \right).$$

By condition (1ii) of the definition of k-aligned, it follows that $\delta(S) \subseteq \beta_{\mathcal{S}}$ for all $S \in \mathcal{S}$ and hence $Y \subseteq \beta_{\mathcal{S}}$.

We prove that Y is a w-balanced separator of G with $\rho(Y) \leq \gamma + ck$. We first show that $\rho(Y) \leq \gamma + ck$. By definition, $\rho(Y) \leq \rho(X_{w_{\mathcal{S}}}) + |\mathcal{S}'| \cdot \max_{S \in \mathcal{S}'} \rho(\delta(S))$. Since $\mathcal{S}$ is k-aligned, it follows that $\rho(\delta(S)) \leq k$ for all $S \in \mathcal{S}$. By assumption, we have $\rho(X_{w_{\mathcal{S}}}) \leq \gamma$ and $\mathcal{S}' \leq c$. Thus, $\rho(Y) \leq \gamma + ck$.

Now we show that Y is a w-balanced separator of G. Let M be a component of $G - Y$. We show that $w(M) \leq 1/2$.

Suppose that $w(M) > 1/2$. Let $Q_1, \ldots, Q_m$ be the components of $\beta_{\mathcal{S}} \setminus X_{w_{\mathcal{S}}}$ and let $D_1, \ldots, D_\ell$ be the components of $G - \beta_{\mathcal{S}}$. Observe that $M \subseteq (\bigcup_{i=1}^{m} Q_i) \cup (\bigcup_{i=1}^{\ell} D_i)$. Also, since $Y \subseteq \beta_{\mathcal{S}}$, if $D_i \cap M \neq \emptyset$, then $D_i \subseteq M$. By condition (2) of the definition of k-aligned, it follows that for every $1 \leq i \leq \ell$, there exists a separation $S_i \in \mathcal{S}$ such that $D_i \subseteq A(S_i)$.

Now we distinguish several cases. First, we assume that $Q_1 \cap M \neq \emptyset$ and $Q_2 \cap M \neq \emptyset$. Since Q_1 and Q_2 are components of $\beta_{\mathcal{S}} \setminus X_{w_{\mathcal{S}}}$, it follows that there exists D_i such that $D_i \cap M \neq \emptyset$, $C(S_i) \cap Q_1 \neq \emptyset$ and $C(S_i) \cap Q_2 \neq \emptyset$. Thus, S_i crosses $X_{w_{\mathcal{S}}}$ and therefore $\delta(S_i) \subseteq Y$. Condition (1ii) of the definition of k-aligned implies that $(A^*(S_i), \delta(S_i), B(S_i))$ is a separation of G. Since $M \cap A^*(S_i) \neq \emptyset$ and $\delta(S_i) \subseteq Y$, it follows that $M \subseteq A^*(S_i)$. Now condition (1iii) of the definition of k-aligned implies that $w(M) \leq w(A^*(S_i)) \leq 1/2$, a contradiction.

So we may assume w.l.o.g. that $Q_i \cap M = \emptyset$ for $2 \leq i \leq m$. Then we have to distinguish two cases: First, suppose that $Q_1 \cap M = \emptyset$. Then there exists $1 \leq i \leq \ell$ such that $M = D_i$. Since $D_i \subseteq A(S_i)$, we obtain $w(M) = w(D_i) \leq w(A(S_i)) \leq 1/2$, a contradiction.

We may therefore assume that $Q_1 \cap M \neq \emptyset$. Let

$$M' := Q_1 \cup \bigcup_{S \in \mathcal{S}, C(S) \subseteq Q_1 \cup X_{w_{\mathcal{S}}}} A^*(S).$$

By definition, $M \subseteq M'$, and we get

$$\begin{aligned} w(M) &\leq w(M') && \text{because } M \subseteq M' \\ &= w(Q_1) + \sum_{S \in \mathcal{S}, C(S) \subseteq Q_1 \cup X_{w_{\mathcal{S}}}} w(A^*(S)) && \text{by definition of } M' \\ &\leq w_{\mathcal{S}}(Q_1) && \text{by definition of } w_{\mathcal{S}} \\ &\leq \tfrac{1}{2} && \text{since } X_{w_{\mathcal{S}}} \text{ is a } w_{\mathcal{S}}\text{-balanced separator of } \beta_{\mathcal{S}}, \end{aligned}$$

again, a contradiction. This completes the proof. ∎

7.3 An application of the central bag method

Unfortunately, the real world applications of the central bag method are very advanced and beyond the scope of this thesis. Therefore, we consider a very simple toy example and apply the central bag method on this graph class. We need the following two definitions.

A graph G is called *outerplanar*, if G is planar and if it can be drawn in the plane such that all vertices of G lie on the boundary of the outer face. For a positive integer k, a graph G is called *k-connected*, if $|V(G)| \geq k+1$ and $G - S$ is connected for all $S \subseteq V(G)$ with $|S| \leq k-1$. Clearly, a graph is 1-connected if and only if it is connected (except for the case when G consists of a single vertex: then G is connected but not 1-connected, since it contains only one vertex).

Now, let $\mathcal{G}$ be the class of outerplanar, 2-connected graphs. Assume that our task is to determine whether $\mathcal{G}$ has bounded treewidth or not. In fact, we know that every outerplanar graph has treewidth at most 2, but let's forget about this for the moment in order to show an application of the central bag method.

One can show that every graph $G \in \mathcal{G}$ has a Hamiltonian cycle that forms the boundary of the outer face (see [Sys79]). Figure 7.4 shows two graphs: the left one is outerplanar, but not 2-connected and contains no Hamiltonian cycle. The right graph is outerplanar and 2-connected and therefore contains a Hamiltonian cycle, which is visualized by red edges.

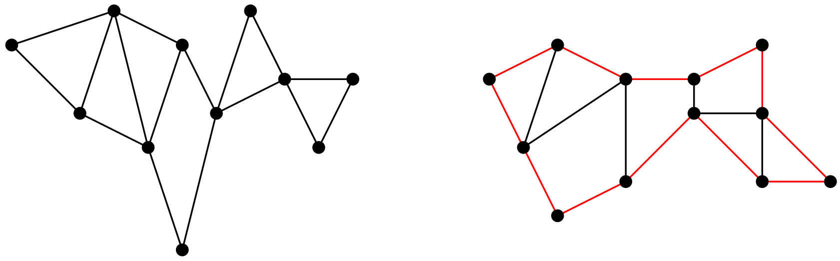


Figure 7.4: Two outerplanar graphs: the left one contains no Hamiltonian cycles; the right one is 2-connected and thus contains a Hamiltonian cycle.

Let C denote the Hamiltonian cycle of $G \in \mathcal{G}$ and let w be a normal weight function on G (w is an arbitrary weight function, but in examples we may assume that w assigns each vertex the same weight: $w(v) := \frac{1}{n}$ for each $v \in V(G)$ and $n = |V(G)|$). Observe that every edge $e = uv \in E(G) \setminus E(C)$ separates G. Thus, we set $S_e := (A_e, C_e, B_e)$ with $C_e := \{u, v\}$ and we choose A_e such that $w(A_e) \leq 1/2$. We call S_e the *canonical separation for* e. Now let $\mathcal{S}$ be a collection of separations with $S_e \in \mathcal{S}$ if and only if $e \in E(G) \setminus E(C)$ and S_e is the canonical separation for e. We may assume that $\mathcal{S}$ contains at least one separation, since otherwise, if $\mathcal{S} = \emptyset$, G is a cycle and therefore $\operatorname{tw}(G) = 2$, so we are done.

Consider the right graph from Figure 7.4 The collection of separations $\mathcal{S}$ for this graph consists of all black edges; these are exactly the edges that disconnect the graph. Figure 7.5 shows the central bag $\beta_{\mathcal{S}}$ for this graph.

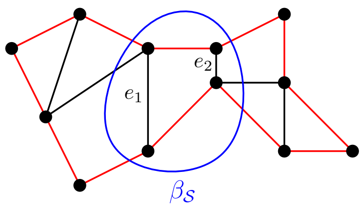


Figure 7.5: The central bag for the example graph from Figure 7.4 induces a cycle.

Observe that the collection of separations $\mathcal{S}' := \{S_{e_1}, S_{e_2}\}$ yields the same central bag as $\mathcal{S}$, because for every separation $S \notin \mathcal{S}'$ there is a separation $S' \in \mathcal{S}'$ such that $A(S) \subseteq A(S')$ and hence $B(S') \cup C(S') \subseteq B(S) \cup C(S)$. Thus, the separation S plays no important role for the central bag: it does not "contribute" anything new to $\beta_{\mathcal{S}}$. Therefore we define the collection of separations $\mathcal{S}' := \{S_e \in \mathcal{S} \mid C(S_e) \subseteq \beta_{\mathcal{S}}\}$. Indeed, the central bag for $\mathcal{S}'$ is the same as the central bag for $\mathcal{S}$. We prove that this holds in general.

Lemma 7.13: *Let G be a graph, let $\mathcal{S}$ be a collection of separations of G and let $\mathcal{S}' := \{S \in \mathcal{S} \mid C(S) \subseteq \beta_{\mathcal{S}}\}$. Then $\beta_{\mathcal{S}} = \beta_{\mathcal{S}'}$.*

Proof. By Observation 7.8 we have $\beta_{\mathcal{S}} = V(G) \setminus \bigcup_{S \in \mathcal{S}} A(S)$ and $\beta_{\mathcal{S}'} = V(G) \setminus \bigcup_{S' \in \mathcal{S}'} A(S')$. We prove that $\bigcup_{S \in \mathcal{S}} A(S) = \bigcup_{S' \in \mathcal{S}'} A(S')$, which implies the claim. Clearly, since $\mathcal{S}' \subseteq \mathcal{S}$, we have $\bigcup_{S' \in \mathcal{S}'} A(S') \subseteq \bigcup_{S \in \mathcal{S}} A(S)$.

Now we show that $\bigcup_{S \in \mathcal{S}} A(S) \subseteq \bigcup_{S' \in \mathcal{S}'} A(S')$. Let $v \in \bigcup_{S \in \mathcal{S}} A(S)$. Then there exists a separation $S_1 \in \mathcal{S}$ with $v \in A(S_1)$. Suppose that $v \notin \bigcup_{S' \in \mathcal{S}'} A(S')$, so for all $S' \in \mathcal{S}'$ it holds that $v \notin A(S')$. It follows that $S_1 \notin \mathcal{S}'$, so $C(S_1) \nsubseteq \beta_{\mathcal{S}} = \bigcap_{S \in \mathcal{S}}(B(S) \cup C(S))$. Hence, there is a vertex $x_1 \in C(S_1)$ and a separation $S_2 \in \mathcal{S}$ with $x_1 \in A(S_2)$. If $S_2 \in \mathcal{S}'$, then $v \in \bigcup_{S' \in \mathcal{S}'} A(S')$, a contradiction. Thus, $S_2 \notin \mathcal{S}'$. Then $C(S_2) \nsubseteq \beta_{\mathcal{S}}$ and again, there is a vertex $x_2 \in C(S_2)$ and a separation $S_3 \in \mathcal{S}$ with $x_2 \in A(S_3)$. We repeat this process until we eventually find a separation $S' \in \mathcal{S}'$ with $v \in A(S')$, a contradiction. Thus, $v \in \bigcup_{S' \in \mathcal{S}'} A(S')$, which completes the proof. ■

By Lemma 7.13, we can work with $\mathcal{S}'$ instead of $\mathcal{S}$ from now on. We see in Figure 7.5 that the central bag $\beta_{\mathcal{S}}$ induces a cycle in G and the following lemma states that this holds in general in the graph class $\mathcal{G}$. Since $\beta_{\mathcal{S}} = \beta_{\mathcal{S}'}$, we do not need the collection $\mathcal{S}'$ in the proof of this lemma.

Lemma 7.14: *Let $G \in \mathcal{G}$ be a graph with a corresponding collection of separations $\mathcal{S}$. Then, $\beta_{\mathcal{S}}$ induces a cycle in G.*

Proof. Let $S \in \mathcal{S}$ be a separation. Note that for every vertex $v \in A(S)$, we have $v \notin \beta_{\mathcal{S}}$ by the definition of the central bag. Now let $G' := G - A(S)$. Since G' is an induced subgraph of G and since G is outerplanar, it follows that G' is outerplanar. Also, G' contains a Hamiltonian cycle. To see that, let $C(S) = \{u, v\}$ and let H denote the Hamiltonian cycle in G. If we remove the vertices in $A(S)$ from H, we obtain a Hamiltonian path P from u to v in G'. But then $H' := P + uv$ forms a Hamiltonian cycle in G'. Clearly, removing any vertex from C' does not disconnect G', hence G' is 2-connected.

So, if we remove the set $A(S)$ from G for any $S \in \mathcal{S}$, we obtain a smaller graph G' that is 2-connected and outerplanar. Therefore, we can recursively remove the sets $A(S)$ for every $S \in \mathcal{S}$ and we end up in a cycle, that contains exactly the vertices of the central bag $\beta_{\mathcal{S}}$.

If there is a chord $e = uv$ in $G[\beta_{\mathcal{S}}]$, then there is a corresponding separations $S_e \in \mathcal{S}$, that separates $G[\beta_{\mathcal{S}}]$ into two components $A(S_e)$ and $B(S_e)$. But this is a contradiction, since we just removed all sets $A(S)$, in particular the set $A(S_e)$. Hence, there is no chord in $G[\beta_{\mathcal{S}}]$. This completes the proof. ■

Now we refer back to the collection $\mathcal{S}'$ and prove that it satisfies all properties we need for the central bag method.

Lemma 7.15: *The collection of separations $\mathcal{S}'$ is laminar.*

Proof. Let $S_1', S_2' \in \mathcal{S}'$ be two distinct separations. We need to show that $A(S_1') \cup C(S_1') \subseteq B(S_2') \cup C(S_2')$ and $A(S_2') \cup C(S_2') \subseteq B(S_1') \cup C(S_1')$.

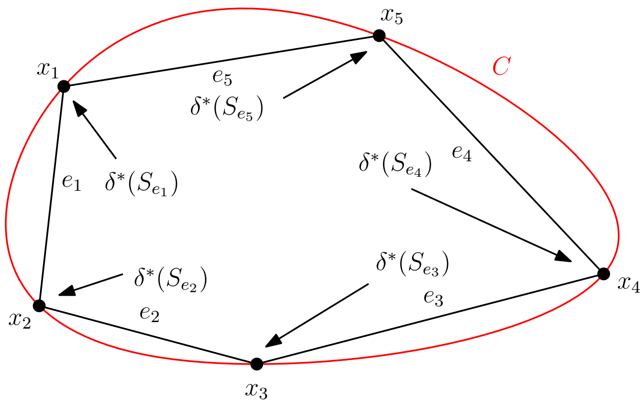
Assume w.l.o.g. that $A(S_1') \cup C(S_1') \nsubseteq B(S_2') \cup C(S_2')$. Then there exists a vertex $v \in A(S_1') \cup C(S_1')$ and $v \notin B(S_2') \cup C(S_2')$, which implies $v \in A(S_2')$ and hence $v \notin \beta_{\mathcal{S}}$. Now if $v \in C(S_1')$, then $v \in \beta_{\mathcal{S}}$ by the definition of $\mathcal{S}'$, a contradiction. So $v \in A(S_1')$. But then we have $A(S_1') \subseteq A(S_2')$ which implies $S_1' \notin \mathcal{S}'$, a contradiction. Therefore, $A(S_1') \cup C(S_1') \subseteq B(S_2') \cup C(S_2')$ and the other case works analogously. ■

Lemma 7.16: *The collection of separations $\mathcal{S}'$ is 2-aligned.*

Proof. Let $S_e \in \mathcal{S}'$ be a canonical separation for $e = uv \in E(G)$. Clearly, $C_e \cap \beta_{\mathcal{S}}$ is connected, so condition (1i) of the definition of k-aligned is fulfilled. We set $\delta(S_e) := C(S_e) \subseteq \beta_{\mathcal{S}}$ and so condition (1ii) of the definition of k-aligned is satisfied with $k = 2$, since $|C(S_e)| = 2$ and $(A(S_e) \cup (C(S_e) \setminus \delta(S_e)), \delta(S_e), B(S_e)) = (A(S_e), C(S_e), B(S_e))$ is a separation of G. We have $w(A(S_e) \cup (C(S_e) \setminus \delta(S_e))) = w(A(S_e)) \leq 1/2$, so condition (1iii) is satisfied, too.

Since $\mathcal{S}'$ is laminar by Lemma 7.15, it follows by Lemma 7.9 that $\mathcal{S}'$ satisfies condition (2) of the definition of k-aligned. This completes the proof. ■

For each pair of distinct separations $S_1', S_2' \in \mathcal{S}'$, we set the anchors $\delta^*(S_1')$ and $\delta^*(S_2')$ such that $\delta^*(S_1') \neq \delta^*(S_2')$. To see that such a choice is possible, note that $\delta^*(S_1') \in \delta(S_1') := C(S_1')$ and $\delta^*(S_2') \in \delta(S_2') := C(S_2')$. There are two possible cases: $C(S_1') \cap C(S_2') = \emptyset$ and $|C(S_1') \cap C(S_2')| = 1$ (note that $|C(S_1') \cap C(S_2')| = 2$ is not possible, since otherwise $C(S_1') = C(S_2')$ and therefore $S_1' = S_2'$, a contradiction because S_1' and S_2' are distinct separations). In the case when $C(S_1') \cap C(S_2') = \emptyset$, we are free in the choice of $\delta^*(S_1')$ and $\delta^*(S_2')$. In the worst case, for each separation $S_1' \in \mathcal{S}'$ with $C(S_1') = \{u, v\}$ there are two separations $S_2', S_3' \in \mathcal{S}'$ with $C(S_1') \cap C(S_2') = \{u\}$ and $C(S_1') \cap C(S_3') = \{v\}$. In this case, the separations in $\mathcal{S}'$ form a cycle of length $|\mathcal{S}'|$. But then, for each separation $S' \in \mathcal{S}'$ we can set $\delta^*(S')$ as a vertex of that cycle, where distinct separations get distinct anchors. Figure 7.6 shows an example of such a situation. The Hamiltonian cycle of the graph is visualized in red.



$$\mathcal{S}' = \{S_{e_1}, S_{e_2}, S_{e_3}, S_{e_4}, S_{e_5}\} \text{ and } \delta^*(S_{e_i}) = x_i$$

Figure 7.6: An illustration for the choice of the anchors in a collection of separation $\mathcal{S}'$.

Since $\beta_{\mathcal{S}'} = \beta_{\mathcal{S}}$ induces a cycle by Lemma 7.14, it follows that $\text{tw}(G[\beta_{\mathcal{S}}]) = 2$. Now Lemma 7.4 implies that there exists a balanced separator X of $G[\beta_{\mathcal{S}}]$ of size at most 3.

The last ingredient to show that G has bounded treewidth is the number of separations that cross X.

Lemma 7.17: *At most 3 separations in $\mathcal{S}'$ cross X.*

Proof. Let $S' \in \mathcal{S}'$. Assume there are two distinct components D_1 and D_2 of $\beta_{\mathcal{S}} \setminus X$. Recall that $C(S')$ induces an edge $xy \in E(G)$. If $C(S') \cap D_1 \neq \emptyset$ and $C(S') \cap D_2 \neq \emptyset$, then we may assume w.l.o.g. that $x \in D_1 \subseteq \beta_{\mathcal{S}}$ and $y \in D_2 \subseteq \beta_{\mathcal{S}}$. This contradicts the fact that $\beta_{\mathcal{S}}$ induces a cycle by Lemma 7.14 (see Figure 7.7 for an illustration).

Since $|X| \leq 3$, and since each separation in $\mathcal{S}'$ gets a different anchor, there are at most 3 separations $S_1', S_2', S_3' \in \mathcal{S}'$ such that $\delta^*(S_1'), \delta^*(S_2'), \delta^*(S_3') \in X$. This means that at most 3 separations in $\mathcal{S}'$ cross X, which completes the proof. ∎

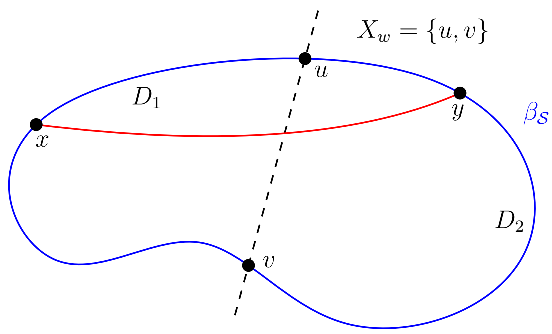


Figure 7.7: An illustration for the proof of Lemma 7.17.

We can now apply Theorem 7.12 with $k = 2$, $\gamma \leq 3$ and $c \leq 3$ to obtain a w-balanced separator of G of size $\gamma + ck \leq 9$. Since w is an arbitrary weight function, Lemma 7.5 implies that $\mathrm{tw}(G) \leq 18$. Clearly, this result is not optimal, since the treewidth of any outerplanar graph is at most 2. But we are mainly interested in whether the treewidth of a graph class is bounded or not, and not so much in the exact value. For that reason, our result is what we wanted to show, and it is an application of the central bag method, which is what we wanted to present.

In this example, we showed how to bound the treewidth of the graph class $\mathcal{G}$ using the central bag method. Our framework allows us to bound the tree-independence number or the tree-chromatic number of $\mathcal{G}$. In this particular example, there would have been no interesting difference in either approach. Additionally, by bounding the treewidth of $\mathcal{G}$ we already bounded the tree-independence number and the tree-chromatic number of G using the Observations 3.1 and 6.1. But in general, there could be structure theorems of graph classes that ensure us the existence of separations C that are not bounded in size, but maybe $\alpha(G[C]) \leq k$ or $\chi(G[C]) \leq k$ for some constant k. In such a case, we could try to apply the central bag method with respect to α or χ, in order to bound the tree-independence number or the tree-chromatic number of the given graph class.

8 Conclusion and Future Work

In this thesis, we gave an introduction to the tree-independence number and the tree-chromatic number of a graph. We proved some basic properties of these invariants and showed the connection to other concepts in structural graph theory, e.g. (tw, ω)-bounded graph classes. Although we did not talk about computational complexity a lot, the problems of computing tree-$\alpha(G)$ and tree-$\chi(G)$, given a graph G, are **NP**-hard in general. At least, for the tree-independence number, deciding whether a graph G satisfies tree-$\alpha(G) \leq 1$ is computable in linear time, since those are exactly the chordal graphs. Deciding whether tree-$\alpha(G) \leq k$ for $k \geq 4$ is **NP**-hard. The computational complexity for the cases $k \in \{2, 3\}$ are still unknown and open problems in current research. Clearly, deciding whether tree-$\chi(G) \leq 1$ for a given graph G is computable in linear time as well, since those are exactly the edgeless graphs. But already the complexity of deciding tree-$\chi(G) \leq 2$ is unknown. It is conjectured to be **NP**-complete in [HRWY21].

By Ramsey's theorem, it follows that every tree-α-bounded graph class is (tw, ω)-bounded. It was conjectured that the converse also holds, i.e. that every (tw, ω)-bounded graph class is tree-α-bounded. It was shown by Chudnovsky and Trotignon in [CT24] that the conjecture is false. We showed their construction using the concept of a layered wheel, which is a nice tool to study on its own.

For the tree-chromatic number, we were able to give a characterization of the class of perfect graphs in terms of tree-χ, that reminds us of the Strong Perfect Graph Theorem. In fact, we used the SPGT in our characterization. Furthermore, we were able to answer a question (in the negative) from [DMŠ24a]: Does $\mathrm{tw}(G) + 1 \leq$ tree-$\alpha(G)$ tree-$\chi(G)$ hold for any graph G? By constructing certain graphs, namely the G-subdivisions of K_n for some graphs G, we showed that $\mathrm{tw}(S(G)) \geq \frac{4}{3}$ tree-$\alpha(S(G))$ tree-$\chi(S(G))$. We could improve this bound further by using shift graphs, and we obtained $\mathrm{tw}(S(S'_n)) \geq 2$ tree-$\alpha(S(S'_n))$ tree-$\chi(S(S'_n))$, where S'_n is an induced subgraph of a shift graph.

On the other hand, we gave upper bounds for the treewidth of a graph in terms of tree-α and tree-χ. In particular, we showed that $\mathrm{tw}(S(G)) \in O(\text{tree-}\alpha(S(G))^2\ \text{tree-}\chi(S(G)))$, where $S(G)$ is the G-subdivision of K_n. In general, we showed that $\mathrm{tw}(G) \in O(4^{\text{tree-}\chi(G)+\text{tree-}\alpha(G)})$ holds for any graph G. Obviously, the gap between the lower bound and the upper bound is huge in general, as the upper bound is an exponential function and the lower bound is a linear function in tree-α and tree-χ. Therefore, closing this gap is a natural next thing to do. Especially, we ask (see Question 6.15): Is there any polynomial function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that $\mathrm{tw}(G) \leq f(\text{tree-}\alpha(G), \text{tree-}\chi(G))$ for every graph G?

Finally, we studied the central bag method that was developed to bound the treewidth of graph classes. We were able to generalize this concept in order to bound tree-α and tree-χ of graph classes, too. With this tool in our pocket, we might be able to prove that certain graph classes are bounded in one of the parameters tw, tree-α or tree-χ.

Bibliography

[Abr+24a] Tara Abrishami, Bogdan Alecu, Maria Chudnovsky, Sepehr Hajebi, Sophie Spirkl, and Kristina Vušković. "Tree independence number I. (Even hole, diamond, pyramid)-free graphs". In: *Journal of Graph Theory* Volume 106 (2024), pp. 923–943. eprint: *https://onlinelibrary.wiley.com/doi/pdf/10.1002/jgt.23104*.

[Abr+24b] Tara Abrishami, Maria Chudnovsky, Cemil Dibek, Sepehr Hajebi, Paweł Rzążewski, Sophie Spirkl, and Kristina Vušković. "Induced subgraphs and tree decompositions II. Toward walls and their line graphs in graphs of bounded degree". In: *Journal of Combinatorial Theory, Series B* Volume 164 (2024), pp. 371–403. ISSN: 0095-8956. DOI: *https://doi.org/10.1016/j.jctb.2023.10.005*.

[Abr23] Tara Abrishami. *Central bags and related topics.* 2023.

[ACDR22] Tara Abrishami, Maria Chudnovsky, Cemil Dibek, and Pawel Rzazewski. "Polynomial-time algorithm for Maximum Independent Set in bounded-degree graphs with no long induced claws". In: *Proceedings of the 2022 ACM-SIAM Symposium on Discrete Algorithms, SODA 2022, Virtual Conference / Alexandria, VA, USA, January 9 - 12, 2022.* Edited by Joseph (Seffi) Naor and Niv Buchbinder. SIAM, 2022, pp. 1448–1470. DOI: *10.1137/1.9781611977073.61*.

[ACV22] Tara Abrishami, Maria Chudnovsky, and Kristina Vušković. "Induced subgraphs and tree decompositions I. Even-hole-free graphs of bounded degree". In: *Journal of Combinatorial Theory, Series B* Volume 157 (2022), pp. 144–175. ISSN: 0095-8956. DOI: *https://doi.org/10.1016/j.jctb.2022.05.009*.

[ARS22] Andrii Arman, Vojtěch Rödl, and Marcelo Tadeu Sales. "Independent Sets in Subgraphs of a Shift Graph". In: *The Electronic Journal of Combinatorics* Volume 29 (2022). DOI: *https://doi.org/10.37236/10453*.

[AZ18] Martin Aigner and Günter M Ziegler. *Proofs from the Book.* Springer, 2018.

[BB73] Umberto Bertelè and Francesco Brioschi. "On non-serial dynamic programming". In: *Journal of Combinatorial Theory, Series A* Volume 14 (1973), pp. 137–148. ISSN: 0097-3165. DOI: *https://doi.org/10.1016/0097-3165(73)90016-2*.

[CAHS22] Maria Chudnovsky, Tara Abrishami, Sepehr Hajebi, and Sophie Spirkl. "Induced subgraphs and tree decompositions III. Three-path-configurations and logarithmic treewidth". In: *Advances in Combinatorics* (Sept. 2022), pp. 1–29. ISSN: 2517-5599. DOI: *10.19086/aic.2022.6*.

[CRST06] Maria Chudnovsky, Neil Robertson, Paul Seymour, and Robin Thomas. "The strong perfect graph theorem". In: *Annals of mathematics* (2006), pp. 51–229.

[CT24] Maria Chudnovsky and Nicolas Trotignon. *On treewidth and maximum cliques.* 2024. arXiv: *2405.07471*.

[Dal+24] Clément Dallard, Fedor V. Fomin, Petr A. Golovach, Tuukka Korhonen, and Martin Milanic. "Computing Tree Decompositions with Small Independence Number". In: *51st International Colloquium on Automata, Languages, and Programming, ICALP 2024, July 8-12, 2024, Tallinn, Estonia.* Edited by Karl Bringmann, Martin Grohe, Gabriele Puppis, and Ola Svensson. Vol. 297. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2024, 51:1–51:18. DOI: *10.4230/LIPICS.ICALP.2024.51.*

[Die17] R. Diestel. *Graph Theory.* Springer, 2017. ISBN: 9783662536216.

[DMŠ24a] Clément Dallard, Martin Milanič, and Kenny Štorgel. "Treewidth versus clique number. II. Tree-independence number". In: *Journal of Combinatorial Theory, Series B* Volume 164 (2024), pp. 404–442. ISSN: 0095-8956. DOI: *https://doi.org/10.1016/j.jctb.2023.10.006.*

[DMŠ24b] Clément Dallard, Martin Milanič, and Kenny Štorgel. "Treewidth versus clique number. III. Tree-independence number of graphs with a forbidden structure". In: *Journal of Combinatorial Theory, Series B* Volume 167 (2024), pp. 338–391. ISSN: 0095-8956. DOI: *https://doi.org/10.1016/j.jctb.2024.03.005.*

[DN19] Zdeněk Dvořák and Sergey Norin. "Treewidth of graphs with balanced separations". In: *Journal of Combinatorial Theory, Series B* Volume 137 (2019), pp. 137–144. ISSN: 0095-8956. DOI: *https://doi.org/10.1016/j.jctb.2018.12.007.*

[EH68] Paul Erdos and András Hajnal. "On chromatic number of infinite graphs". In: *Theory of Graphs (Proc. Colloq., Tihany, 1966).* 1968, pp. 83–98.

[Gav74] Fănică Gavril. "The intersection graphs of subtrees in trees are exactly the chordal graphs". In: *Journal of Combinatorial Theory, Series B* Volume 16 (1974), pp. 47–56. ISSN: 0095-8956. DOI: *https://doi.org/10.1016/0095-8956(74)90094-X.*

[HRWY21] Tony Huynh, Bruce Reed, David R. Wood, and Liana Yepremyan. "Notes on Tree- and Path-Chromatic Number". In: *2019-20 MATRIX Annals.* Edited by Jan de Gier, Cheryl E. Praeger, and Terence Tao. Cham: Springer International Publishing, 2021, pp. 489–498. ISBN: 978-3-030-62497-2. DOI: *10.1007/978-3-030-62497-2_30.*

[KN12] S.O. Krumke and H. Noltemeier. *Graphentheoretische Konzepte und Algorithmen.* Vieweg+Teubner Verlag, 2012. ISBN: 9783834818492.

[Kur30] Casimir Kuratowski. "Sur le problème des courbes gauches en Topologie". fre. In: *Fundamenta Mathematicae* Volume 15 (1930), pp. 271–283.

[Lim+24] Paloma T. Lima, Martin Milanic, Peter Mursic, Karolina Okrasa, Pawel Rzazewski, and Kenny Storgel. "Tree Decompositions Meet Induced Matchings: Beyond Max Weight Independent Set". In: *32nd Annual European Symposium on Algorithms, ESA 2024, September 2-4, 2024, Royal Holloway, London, United Kingdom.* Edited by Timothy Chan, Johannes Fischer, John Iacono, and Grzegorz Herman. Vol. 308. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2024, 85:1–85:17. DOI: *10.4230/LIPICS.ESA.2024.85.*

[Lov72] L Lovász. "A characterization of perfect graphs". In: *Journal of Combinatorial Theory, Series B* Volume 13 (1972), pp. 95–98. ISSN: 0095-8956. DOI: *https://doi.org/10.1016/0095-8956(72)90045-7.*

[Ram87] F. P. Ramsey. "On a Problem of Formal Logic". In: *Classic Papers in Combinatorics.* Edited by Ira Gessel and Gian-Carlo Rota. Boston, MA: Birkhäuser Boston, 1987, pp. 1–24. ISBN: 978-0-8176-4842-8. DOI: *10.1007/978-0-8176-4842-8_1.*

[RS04] Neil Robertson and P.D. Seymour. "Graph Minors. XX. Wagner's conjecture". In: *Journal of Combinatorial Theory, Series B* Volume 92 (2004), pp. 325–357. ISSN: 0095-8956. DOI: *https://doi.org/10.1016/j.jctb.2004.08.001*.

[RS86] Neil Robertson and P.D Seymour. "Graph minors. II. Algorithmic aspects of tree-width". In: *Journal of Algorithms* Volume 7 (1986), pp. 309–322. ISSN: 0196-6774. DOI: *https://doi.org/10.1016/0196-6774(86)90023-4*.

[RS91] Neil Robertson and P.D Seymour. "Graph minors. X. Obstructions to tree-decomposition". In: *Journal of Combinatorial Theory, Series B* Volume 52 (1991), pp. 153–190. ISSN: 0095-8956. DOI: *https://doi.org/10.1016/0095-8956(91)90061-N*.

[RTL76] Donald J. Rose, R. Endre Tarjan, and George S. Lueker. "Algorithmic Aspects of Vertex Elimination on Graphs". In: *SIAM Journal on Computing* Volume 5 (1976), pp. 266–283. eprint: *https://doi.org/10.1137/0205021*.

[Sey16] Paul Seymour. "Tree-chromatic number". In: *Journal of Combinatorial Theory, Series B* Volume 116 (2016), pp. 229–237. ISSN: 0095-8956. DOI: *https://doi.org/10.1016/j.jctb.2015.08.002*.

[SS20] Alex Scott and Paul Seymour. "A survey of -boundedness". In: *Journal of Graph Theory* Volume 95 (2020), pp. 473–504. eprint: *https://onlinelibrary.wiley.com/doi/pdf/10.1002/jgt.22601*.

[Sys79] Maciej M. Sysło. "Characterizations of outerplanar graphs". In: *Discrete Mathematics* Volume 26 (1979), pp. 47–53. ISSN: 0012-365X. DOI: *https://doi.org/10.1016/0012-365X(79)90060-8*.

[TW15] Volker Turau and Christoph Weyer. *Algorithmische Graphentheorie*. Berlin, München, Boston: De Gruyter, 2015. ISBN: 9783110417326. DOI: *doi:10.1515/9783110417326*.

[Yol18] Nikola Yolov. "Minor-matching hypertree width". In: *Proceedings of the 2018 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*. SIAM, 2018, pp. 219–233. eprint: *https://epubs.siam.org/doi/pdf/10.1137/1.9781611975031.16*.