Karlsruhe Institute of Technology

Induced Universal Graphs for edge-colored Complete Graphs

Master Thesis of

Christian Kouekam

At the Department of Informatics
Institute of Theoretical Informatics

Reviewers: PD Dr. Torsten Ueckerdt
Prof. Dr. Peter Sanders
Advisors: Prof. Dr. Maria Axenovich

Time Period: 1st November 2020 – 30th July 2021

Statement of Authorship

Ich versichere wahrheitsgemäß, die Arbeit selbstständig verfasst, alle benutzten Hilfsmittel vollständig und genau angegeben und alles kenntlich gemacht zu haben, was aus Arbeiten anderer unverändert oder mit Abänderungen entnommen wurde sowie die Satzung des KIT zur Sicherung guter wissenschaftlicher Praxis in der jeweils gültigen Fassung beachtet zu haben.

Karlsruhe, July 30, 2021

Abstract

A graph U is an induced-universal graph for a family of graphs $\mathcal{F}$ if every graph in $\mathcal{F}$ is an induced subgraph of U. We define by $g_v(\mathcal{F})$ the minimum number of vertices of an induced-universal graph for $\mathcal{F}$.
Alon investigated induced-universal graphs for the family $\mathcal{F}(n)$ of undirected graphs n vertices, and showed the lower and upper bounds of $g_v(\mathcal{F}(n))$ have the same order of magnitude. Furthermore, he suggested that his proof can be expanded to other types of graphs.
In this thesis, we take a look at the family $\mathcal{K}(k, r)$ of complete graphs on k vertices and edges colored by r colors and, by adapting Alon's proof, show that $g_v(\mathcal{K}(k, r)) = (1 + o(1))r^{(k-1)/2}$.

Deutsche Zusammenfassung

Ein graph U heißt induzierter universaler graph für eine Familie von Graphen $\mathcal{F}$, falls jeder Graph in $\mathcal{F}$ ein induzierter Teilgraph von U ist. Wir definieren $g_v(\mathcal{F})$ als die minimale Anzahl an Knoten eines induzierten universalen Graphen für $\mathcal{F}$.
Alon untersuchte indizierte universale Graphen für die Familie $\mathcal{F}(n)$ von ungerichteten Graphen auf n Knoten und zeigte, dass die untere Schranke und die Obere Schranke von $g_v(\mathcal{F}(n))$ dieselbe Größenordnung haben. Darüber hinaus, deutet er hin, dass seine Methoden für andere Typen von Graphen angewandt werden können.
In dieser Masterarbeit analysieren wir komplette Graphen auf k Knoten und Kanten gefärbt mit r Farben und zeigen, dass g_v für diese Familie Größenordnung $(1 + o(1))r^{(k-1)/2}$ hat.

Contents

1. Introduction

1.1. Motivation

How to represent data is a fundamental data structuring problem in computer science. A conventional approach to this problem is to find and exploit structural uniformities in families of data structures. These uniformities allow us for example to statically allocate storage for the entire family instead of dynamically allocating for each member of the family [CRS83]. However, we still require an exponential number of memory locations to represent the family. This motivates the problem of minimizing the number of memory locations needed. We look for a "universal" structure that can simulate every structure in a family.
A data graph is obtained from a data structure by masking out the data items which appear at the nodes of the structure and concentrating only on the linkages. Data graphs are particularly interesting in a computational environment as many important properties of data structures are independent of the explicit data items at the nodes of the structure [Ros71]. We may formalize a definition of universality for data graphs as follows. Given a collection of graphs Γ, each member of Γ containing labelled vertices possibly shared with other members, a graph U is universal for Γ if it contains every graph of Γ as a subgraph. Our goal is thus is minimize properties of a universal graph depending on the specification at hand. Very often, we seek to optimize the number of vertices or the number of edges.
Moving on from data representation, universality is also an important component in the study of VLSI circuit design. The circuitry of a computer chip may be viewed as a graph. Designing a circuit that simulate a family of other circuits is then equivalent to designing a graph that is universal for the family of the corresponding graphs of the other circuits. It is expensive to design new computer chips. The cost of replication of the chip is relatively low. Universal graphs come here into play as they allow manufacturers to design chips with many functionalities that can be later on configured to the need of the customers.

Universal graphs enjoy a broad spectrum of research in the field of graph theory. They belong to the area of extremal graph theory as we not only try to find a graph that "contains" all graphs in a specific family of graphs, but also optimize (minimize) certain of its properties. It can be interesting to broaden the concept of "containment" beyond the relation of simple subgraph. Another type of containment are homomorphic subgraphs where a graph G contains another graph H if G contains a homomorphic copy of H. In this thesis, we are particularly interested in induced-universal graphs.

Let $\mathcal{F}$ be a family of graphs. A graph G is called *induced universal* for the family $\mathcal{F}$ if every member of $\mathcal{F}$ is isomorphic to an induced subgraph of G. We denote by $g_v(\mathcal{F})$ the minimum number of vertices in an induced-universal graph for $\mathcal{F}$ and by $g_e(\mathcal{F})$ the minimum number of edges in an induced-universal graph $\mathcal{F}$. Often in this thesis, we will only investigate the minimum number of vertices.

1.2. Overview of related work

Universal graphs were introduced by Rado [Rad64] in 1964. He provided a universal graph for the family of all finite or countably infinite graphs. Since then, the notion of universal graph has been studied for diverse families of graphs: planar graphs, bipartite graphs, hypergraphs, graphs with bounded maximum-degree, trees or more generally graphs with bounded arboricity, etc...

Let $\mathcal{F}(n)$ be the family of all undirected graphs on n vertices. Moon [Moo65] observed in 1965 that $2^{(n-1)/2} \leq g_v(\mathcal{F}(n)) \leq O(n2^{n/2})$. Using a probabilistic approach, Bollobás and Thomason [BT81] showed that the binomial random graph on $n^2 2^{n/2}$ vertices with probability $p = 0.5$ is with high probability induced-universal for $\mathcal{F}(n)$ (as n tends to infinity). This result was later improved to $O(n2^{n/2})$ by Brightwell and Kohayakawa [BK93]. Using adjacency labeling schemes, Alstrup, Kaplan, Thorup and Zwick [AKTZ15] further improved the upper bound to $16 \cdot 2^{n/2}$. The latest result on this family of graphs is due to Alon [Alo17] who showed that upper and lower bounds have the same order, $g_v(\mathcal{F}(n)) = (1 + o(1))2^{(n-1)/2}$.

Moving on to the family bipartite graphs $\mathcal{B}(n)$, using simple counting arguments, we can see that $g_v(\mathcal{B}(n)) \geq 2^{n/4}(1 - O(1/n))$. Analyzing characteristics of infinite hereditary classes, Lozin and Rudolf showed that $g_v(\mathcal{B}(n)) \leq O(n^2 2^{n/4})$. Alstrup, Kaplan, Thorup and Zwick [AKTZ15] improved this bound to $g_v(\mathcal{B}(n)) \leq O(c2^{n/4})$ for some absolute constant less than 100. Later on, Alon [Alo17] proved that $g_v(\mathcal{B}(n)) = (1 + o(1))2^{n/4}$.

The next family we consider is the family of graphs with bounded maximum degree. Let $\mathcal{F}(n, d)$ be the family of all graphs on n vertices with maximum degree at most d, d even. Petersen's 2-factor theorem allows us to decompose graphs in $\mathcal{F}(n, d)$ into several spanning subgraphs with maximum degree at most 2. Butler [But09] constructed an induced-universal graphs for the family $\mathcal{F}(n, 2)$. Then using a reduction technique by Chung [Chu90], he showed that $\Omega(n^{d/2}) \leq g_v(\mathcal{F}(n, d)) \leq O(n^{d/2})$. Esperet, Labourel and Ochem later improved on the upper bound by a multiplicative factor by construction an induced-universal for $\mathcal{F}(n, 2)$ that requires less vertices. Using a different reduction and construction, Alon and Nenadov [AN17] showed that those bounds even when d is odd.

The family of planar graphs has also seen major improvements over the years. Let $\mathcal{P}(n)$ denote the family of planar of n. Gonçalves [Gon06, Gon09] showed that planar can be edge-decomposed into three spanning forests. Using this fact with a later result on forests by Bonichon, Gavoille and Labourel [BGL07], we can show that $g_v(\mathcal{P}(n)) \leq O(n^3)$. Gavoille and Labourel [GL07] improved this result shortly after, $g_v(\mathcal{P}(n)) \leq O(n^{2+o(1)})$. The best result to date is very recent, February 2021. Dujmović, Esperet, Gavoille, Joret, Micek and Morin proved that $g_v(\mathcal{P}(n)) \leq O(n^{1+o(1)})$.

We conclude this section with forests. Using adjacency labeling schemes, it can easily be shown that the family of forests on n vertices admits an induced-universal graph on at most n^2 vertices. In 2017, Alstrup, Dahlgaard and Knudsen showed an adjacency labeling schemes for the family of forests on n vertices that only requires $\log n + O(1)$ bits, proving that $g_v \leq O(n)$. This result can be extrapolated to graphs on n vertices with arboricity k, ending with the bounds $\frac{n^k}{2^{O(k^2)}} \leq g_v \leq O(n^k)$. As for graphs on n vertices of treewidth k, Gavoille and Labourel [GL07] proved that $n2^{\Omega(k)} \leq g_v \leq n(\log \frac{n}{k})^{O(k)}$.

1.3. Contribution and Outline

We mentioned earlier that Alon showed for the family $\mathcal{F}(n)$ of undirected graphs on n, we have $g_v(\mathcal{F}(n)) = (1 + o(1))2^{(n-1)/2}$. Similar arguments can be made for differents classes to provide asymptotically tight bounds. In this thesis, we focus on the case of edge-colored complete graphs and provide a detailed proof to

$$r^{(k-1)/2} \leq g_v(\mathcal{K}(k,r)) \leq r^{(k-1)/2}\left(1 + O\left(\frac{\log_r^{3/2} k}{\sqrt{k}}\right)\right).$$

$\mathcal{K}(k,r)$ is the family of complete graphs on k vertices and edges colored with r colors.

In Chapter 2, we introduce the notions and concepts necessary for the comprehension of the thesis. This includes basic definitions in graph theory, homomorphism of edge-colored graphs and induced-universal graphs.

In Chapter 3, we introduce implicit graph representations with a heavy focus on adjacency labeling schemes. We provide a few examples and also show a generalization of this concept to informative labeling schemes.

In Chapter 4, we provide more details to previous work on induced-universal graphs. We provide a complete construction by Butler of induced-universal graphs for the family of graphs with bounded maximum degree. We also provide an analysis of this construction and discuss ways to improve it. We show a reduction by Chung that allows us a construct an induced-universal graph from a universal graph. We show how this technique can be used for the family of planar graphs. Finally, we take a look at hereditary families of graphs.

In Chapter 5, we adapt Alon's proof for the family of undirected graphs to the setting of edge-colored complete graphs. Wherever necessary, we expanded on his proof to provide more details for the reader's comprehension.

2. Preliminaries

In this chapter, we introduce theoretical notions used throughout this thesis. In this thesis, unless stated otherwise, all log are of base 2.

2.1. Basic notions

A *graph* G is a pair of sets $V(G), E(G)$. We called the elements of $V(G)$ *vertices* and the elements of $E(G)$ *edges*. An edge is a set of two vertices. For an edge u, v, we often simply write uv. If a graph G has an edge uv, we say that the vertices u and v are *connected* or adjacent. A *loop* is an edge connecting a vertex to itself. We say that a graph has *multiple edges* if two or more edges connects the same two vertex with the graph. We call a graph without loops and multiple edges a *simple graph*. Throughout this thesis, unless stated otherwise, all graphs we handle are simple graphs.
We call a graph *finite* if the set of its vertices is finite. A graph that is not finite is *infinite*. The *neighborhood* $N(u)$ of a vertex u is the set of vertices adjacent to u in G. If a vertex v is in the neighborhood of a vertex u, we say that v is a neighbor of u. The *degree* of a vertex u in G is the number of vertices in its neighborhood.The maximum degree of a graph G is the maximum of the degrees of its vertices. The minimum degree of a graph is minimum of the degrees of its vertices.
A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If H is a subgraph of G, we write $H \subseteq G$. We say that a subgraph H of G is an *induced subgraph* if $E(H) = \{uv \in E(G) \mid u, v \in E(H)\}$. As such, if H is an induced subgraph of G, then $uv \in E(H)$ if and only if $uv \in E(G)$. For a subset $K \subseteq V(G)$, we call the induced subgraph $G[K]$ of G with vertex set K *induced subgraph of G on K*. We also say that K *induces* $G[K]$. For the induced subgraph of G obtained by removing a subset X of the vertices, we write $G - X$.

An *homomorphism* from a graph G to another graph H is a map $f : V(G) \to V(H)$ such that if $uv \in E(G)$ then $f(u)f(v) \in E(H)$. An *isomorphism* from a graph G to another H is a bijective homomorphism from G to H. An isomorphism is an *automorphism* if $G = H$.

A *path* in a graph is a sequence $(v_1, \dots, v_q)$ of vertices of G such that vertex v_i is connected to vertex $v_{i+1}, i = 1, \dots, q-1$. If we add the edge v_1v_q, we obtain a cycle. A graph without cycle is called *acyclic*. The length of a path is its number of edges. The distance between two vertices in a graph is the minimum length of a path connecting them. A *Hamiltonian path* is path that visits each vertex exactly once. A *Hamiltonian cycle* is a Hamiltonian path that is a cycle. A *connected component* of a graph is an induced subgraph in which any two vertices are connected to each other by a path. A graph is *connected* if it only had one connected component. Otherwise, we call it *disconnected*. A graph is *k-connected* if we need to remove at least k vertices to make it disconnected.

A *matching* $M \in E(G)$ is a subset of the edges without common vertices. A matching M is *(inclusion) maximal* if there is no other matching $\overline{M}$ with $M \subset \overline{M}$.

2.2. Basic families

A graph is *complete* if it contains every possible edges. We denote K_n the complete graph on n vertices. A *clique* is a subset of vertices that induces a complete graph. An *independent set* is a set of vertices in a graph, no two of which are adjacent. The *complement* of a graph G is a graph H on the same vertices such that $uv \in E(G)$ if and only in $uv \notin E(H)$. Cliques and independent sets are complement of one another. We call a graph *bipartite* if its vertices can be divided into two disjoint and independent sets A and B such that every edge connects a vertex in A to a vertex in B. The *complete bipartite graph* is the graph with all possible edges between A and B. We write $K_{n,m}$ if $|A| = n$ and $|B| = m$. A *star* S_k is the complete bipartite graph $K_{1,k}$. A graph is said to be *regular* if each of its vertices has the same number of neighbors. A regular graph with vertices of degree k is called *k-regular*. A *k-degenerate* graph is a graph in which every subgraph has a vertex of degree at most k.

A *forest* is an acyclic graph. A *tree* is a connected graph without cycles (i.e. a connected forest). In a forest, a *leaf* is a vertex degree 1. A *caterpillar* is a tree for which removing the leaves and incident edges produces a path graph. For $k \in \mathbb{N}$ a graph G is a *k-tree* if and only if it is the complete graph K_{k+1} or a vertex v exists such that $N(v)$ induces a copy of K_k and $G - v$ is a k-tree. The definition of trees and 1-trees are equivalent. The *arboricity* of a graph is the minimum number of forests into which its edges can be partitioned.
The *k-power* G^k of a graph G is another graph with the same set of vertices, but in which two vertices are adjacent when their distance in G is at most k.

A graph G can be drawn on the plane with its vertices mapped to points and its edges mapped to curves connecting the corresponding endpoints. G is if it can be drawn on the plane in such a way that its edges intersect only at their endpoints.

A graph is *covered* by subgraphs $G_1, \dots, G_k$ of G if every edge of G belongs to one of these subgraphs. A graph G is (t, D)*-coverable* if it can be covered by t forests and a graph H of maximum degree D. A graph is $F(d_1, \dots, d_k)$*-coverable* if it can be covered by k forests $F_1, \dots, F_k$ such that F_i has maximum degree d_i for all $i = 1, \dots, k$. $d_i = \infty$ is possible.
A class of graph is X is called hereditary if $G \in X$ implies $H \in X$ for every graph H isomorphic to an induced subgraph of G.

The *binomial random graph* model $G(n, p)$ is a model for generating random graphs on n vertices with each edge included in the graph with probability p, independently from every other edge.

2.3. Edge colored graphs

An *edge colored graph* is a pair (G, c) consisting of a simple graph G and a coloring c of its edges. We say that (H, c_H) is an induced subgraph of (G, c) if H is an induced subgraph of G and $c_H(uv) = c(uv)$ for every edge $uv \in H$. A *graph isomorphism* from an edge colored graph (G, c_1) to an edge colored graph (H, c_2) is a bijection $g : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $g(u)g(v) \in E(H)$, and $c_1(uv) = c_2(g(u)g(v))$ for every edge uv of

G. An *automorphism* of an edge colored graph is a graph isomorphism with itself. Let X be a family of edge colored graphs. Analogously to simple graphs, we call an edge colored graph (G, c) induced universal for the family X if every member of X is isomorphic to an induced subgraph of (G, c). The edge colored graph pictured in Figure 2.1 has exactly two automorphisms, namely *id* and (14).

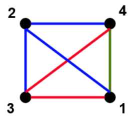


Figure 2.1.: An edge colored graph with exactly two automorphisms: *id* and (14).

For an edge coloring c with color set $\{1, \ldots, r\}$ and for a set $\mathcal{P} = \{p_1, \ldots, p_r\}$ with $0 \leq p_i \leq 1, 1 \leq i \leq r$, and $p_1 + \cdots + p_r = 1$, a *multinomial random graph* $\overline{G}(n, \mathcal{P})$ is an edge colored graph (G, c) where G is a complete graph on n vertices and c independently assigns label i (color) to each edge with probability p_i. If $p = p_1 = p_2 = \cdots = p_r = \frac{1}{r}$, we simply write $\overline{G}(n, p)$.

2.4. Universal Graphs

Let $\mathcal{F}$ be a family of graphs.A graph G is called *universal* for the family $\mathcal{F}$ if G contains every member of $\mathcal{F}$ as a subgraph. G is called *induced universal* for the family $\mathcal{F}$ if every member of $\mathcal{F}$ is isomorphic to an induced subgraph of G. We denote by $g_v(\mathcal{F})$ the minimum number of vertices in an induced-universal graph for $\mathcal{F}$ and by $g_e(\mathcal{F})$ the minimum number of edges in an induced-universal graph $\mathcal{F}$. We denote by $f_v(\mathcal{F})$ the minimum number of vertices in a universal graph for $\mathcal{F}$ and by $f_e(\mathcal{F})$ the minimum number of edges in a universal graph $\mathcal{F}$.

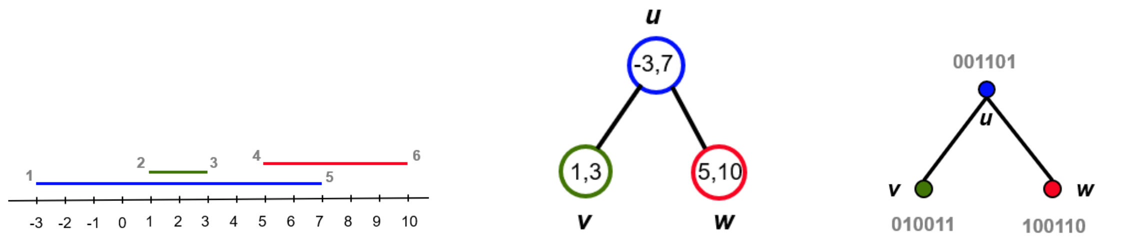
3. Implicit Graph Representation

Traditionally, a simple graph is represented by its vertices and its edges. In a computer memory, the names of the vertices, which we refer to as labels, are holders that allow data on the edges to encode the structure of the graph. They betray nothing about the graph itself. For a graph on n vertices and $\log n$ bit labels, we may need $O(n^2 \log n)$ space to store the graph if we decide to store each edge individually. This space requirement can be slightly decreased if we use a different data structure such as an adjacency list. It can be further decreased if we are handling specific classes of graphs. Finding efficient represention of various classes of graphs is a fundamental data structuring question in computer science.

If we have an understanding, even partial, of the structure of the graphs in the family we are investigating, we may approach this question differently. Instead of describing all the edges explicitly, we may only need to store a subset of the edges and determine the rest of the edges upon reconstruction of the graph. Suppose a graph G is transitive and has edges ab, bc. By transitivity, G also has the edge ac. Instead of storing ab, bc and ac, we only need to store ab and bc. Let us consider a more sophisticated example, the case of interval graphs.

3.1. Interval Graphs

We recall that an interval graph is a graph where each vertex can be associated with an interval on the real line in such way that two vertices are adjacent if and only if the associated intervals have a nonempty intersection. For an interval graph on n vertices, consider the set of n intervals that are associated with the vertices (interval model) and enumerate the endpoints of the intervals from left to right and label each vertex with the two endpoints of its interval (concatenated in binary). This associates with each vertex a some interval $I(a) \subseteq [1, 2n]$ with integer endpoints (see Figure 3.1). Clearly, a and b are adjacent if and only if $I(a) \cap I(b) \neq \emptyset$. The label of a vertex u is the concatenation of its associated endpoints in binary. The interval model can also be reconstructed from the set of labels of the vertices. In our example, the vertex u is associated to the interval $[-3, 7]$, the vertex v to $[1, 3]$ and the vertex w to $[5, 10]$. Then we have $L(u) = 001 \,|\, 101$, $L(v) = 010011$ and $L(w) = 100110$. With this representation, we may check whether two vertices are adjacent by comparing the labels they are associated with and verifying that they define overlapping intervals. Although our graph may have a quadratic number of edges, we only need $2 \log n + O(1)$ bits per vertex to represent our graph. This number is asymptotically tight [GP08]. In this graph representation, the labels of the vertices



(a) interval model

(b) graph with vertices associated to intervals

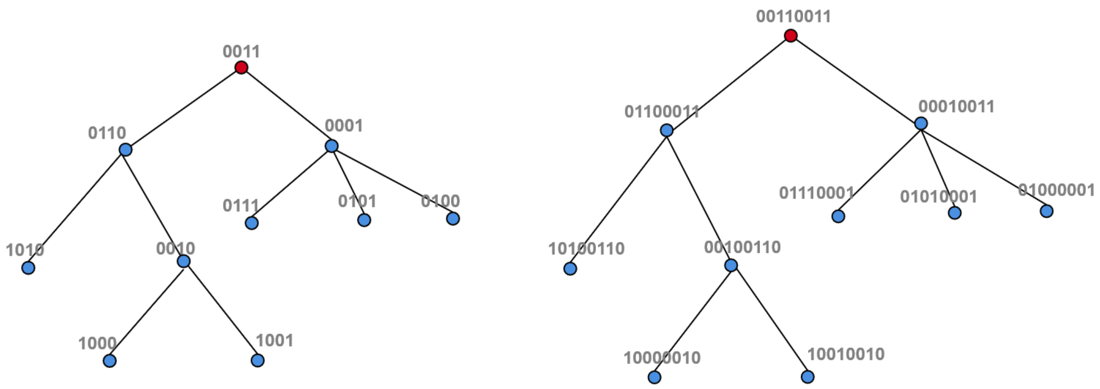
(c) labelled vertices according to our enumeration

Figure 3.1.: An interval graph and an adjacency labeling scheme of its vertices.

give enough information so that some properties (adjacency in our case) can be efficiently determined. We call this type of representation *implicit graph representation*.

3.2. Adjacency Labeling Scheme

A family of simple graphs $\mathcal{F}$ admits an *adjacency labeling scheme* if, for every graph $G \in \mathcal{F}$, we can assign labels to vertices of G so that for any two vertices u, v, whether u and v are adjacent can be determined by an efficient algorithm that examines only their labels. Adjacency labeling schemes are implicit graph representations. Each vertex is associated to some information so that adjacency can be algorithmically efficiently determined without the need of the global data-structure. Let us consider a tree on n vertices. We want to label its vertices such that we can efficiently determine adjacency. To this effect, we root the tree and prelabel each vertex with arbitrary, distinct, positive integers. We set the label of each vertex as its prelabel appended with the prelabel of its parent (the root is its own parent). Then two vertices are adjacent if and only if the first part of one label is identical to the second part of the other label (see Figure 3.2).



(a) tree rooted on red vertex with prelabelled vertices

(b) vertices are labelled

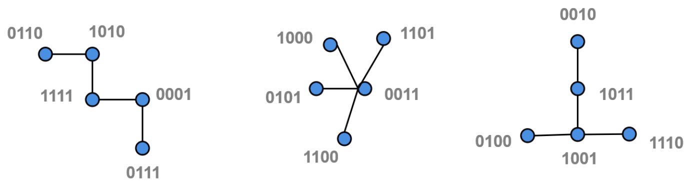
Figure 3.2.: A rooted tree on 10 vertices with labelled vertices.

The importance of shorter label sequences cannot be understated. Besides memory storage, BFS traversal can be done in $O(n)$ time for certain classes of graphs given a succinct representation [ACJS19, RLDL94]. Adjacency labeling schemes are also tightly connected to universal graphs. Indeed, a family of graphs $\mathcal{F}$ admits a labeling scheme with L-bit labels if and only if $\mathcal{F}$ has an induced universal graph with at most 2^L vertices. Given an adjacency labeling scheme with L-bit labels for a family of graphs, the graph whose vertices are all possible L-bit labels in which two vertices are adjacent if and only if their labels correspond to adjacent vertices is an induced universal graph for the family. Thus, the shorter the length of the label, the smaller the resulting induced universal graph. We formalize this notion in the following theorem.

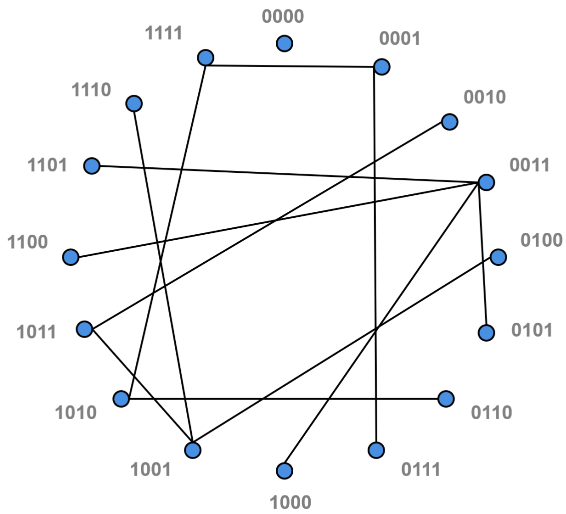
Theorem 3.1. *A graph family $\mathcal{F}$ admits a labeling scheme with L-bit labels if and only if $\mathcal{F}$ admits an induced universal graph with at most 2^L vertices.*

Proof. Let $\mathcal{F}$ be a graph family which admits a labeling scheme with L-bit labels. We label the vertices of $\mathcal{F}$ randomly. The graph whose vertices are all possible labels and for which two vertices are adjacent if and only if their labels correspond to adjacent vertices in $\mathcal{F}$ is an induced universal graph for $\mathcal{F}$. This graph has at most 2^L vertices. Conversely, if $\mathcal{F}$ admits an induced universal graph U with at most 2^L vertices. We can label its vertices with L-bit labels. The labeling scheme which determines adjacency if and only if the corresponding vertices in U are adjacent is an adjacency labeling scheme for $\mathcal{F}$. □

The family of non-isomorphic trees with 5 vertices consists of three graphs: a path, a star and a caterpillar. Thus, we need 15 labels (4 bits) to represent all its vertices. By Theorem 3.1, this family admits an induced-universal graph on 16 vertices.



(a) family of non-isomorphic trees with 5 vertices



(b) induced-universal graph for the family

Figure 3.3.: The family of non-isomorphic trees with 5 vertices (randomly labelled) with an induced-universal graph for the family according to our random labels.

Before we formally define adjacency labeling schemes, we consider a general and broader concept.

3.3. Informative Labeling Schemes

Until now, we have described adjacency labeling schemes as they pertain to simple graphs. However, they are not suitable to describe edge colored graphs. For an edge colored graph

(G, c) and a pair of vertices $u, v \in G$, we are not only interested in whether u and v are adjacent, but also in the color $c(uv)$ of the edge connecting them. Therefore, we want our labeling scheme to encode the color of the edge in the label.

Peleg [Pel00] formally introduced the concept of *informative labeling schemes*.
A vertex-labeling of the graph G is a function L assigning a label $L(u)$ to each vertex u of G. A *labeling scheme* is composed of two major components:

- A marker algorithm $\mathcal{M}$, which given a graph G, selects a label assignment $L = \mathcal{M}(G)$ for G.
- A decoder algorithm $\mathcal{D}$, which given a set of labels $\hat{L} = \{L_1, \ldots, L_k\}$, returns a value $\mathcal{D}(\hat{L})$. The time complexity of the decoder is required to be polynomial in its input.

Let f be a function defined on sets of vertices in a graph. Given a family $\mathcal{G}$ of weighted graphs, an f *labeling scheme* for $\mathcal{G}$ is a marker-decoder pair $\langle \mathcal{M}_f, \mathcal{D}_f \rangle$ with the following property. Consider any graph $G \in \mathcal{G}$, and let $L = \mathcal{M}_f(G)$ be the vertex labeling assigned by the marker $\mathcal{M}_f$ to G. Then for any set of vertices $W = \{v_1, \ldots, v_k\}$ in G, the value returned by the decoder $\mathcal{D}_f$ on the set of labels $\hat{L}(W) = \{L(v) \mid v \in W\}$ satisfies $\mathcal{D}_f(\hat{L}(W)) = f(W)$. The decoder $\mathcal{D}_f$ is independent of G or its number of vertices.

Let us consider a boolean function f that determines adjacency for a graph G, i.e. $f(\{u, v\}) = \mathsf{TRUE}$ if and only if $uv \in E(G)$. If we feed our decoder algorithm pairs of labels, our labeling scheme determines adjacency. We can see thus that adjacency labeling schemes are just a special case of informative labeling schemes. Indeed, two vertices u and v are adjacent if and only if $\mathcal{D}_f(\hat{L}(\{u, v\})) = f(\{u, v\}) = \mathsf{TRUE}$.

A weighted graph is a graph in which a number is assigned to each edge. Formally, it is a triple $G(V, E, w)$ where V is the set of vertices, E the set of edges and w is a weight function $w : E(G) \to \mathbb{R}$. The length of a path is the combined weight of the edges composing it. The distance between two vertices u, v, denoted $dist(u, v, G)$, is defined as the length of the shortest path connecting them. Another example of informative labeling schemes are so-called distance labeling schemes. In this case, the function f computes the distance between vertices of a weighted graph G. Given only the labels of two vertices u and v, our decoder $\mathcal{D}$ returns $f(\{u, v\}) = dist(u, v, G)$. Using distance labeling schemes, Peleg [Pel99] showed that the family of n-vertex weighted trees admits a distance labeling scheme with $O(M \log n + \log^2 n)$ bit labels.

Lemma 3.2 (Peleg [Pel99])**.** *The family of n-vertex weighted trees admits a distance labeling scheme with $O(M \log n + \log^2 n)$ bit labels, where M denotes the maximum number of bits required for representing a weight in the graph.*

In the case of an edge colored graph (G, c), we want the function f to determine the color of edges. With only the labels of vertices u, v as input, our decoder returns $\mathcal{D}(\{L(u), L(v)\}) = f(\{u, v\}) = c(uv)$ if u and v are have a connecting edge, $\perp$ otherwise. We shall revisit informative labeling scheme in Section **??** to find an induced-universal graph for the family of edge-colored complete graphs.

4. Related Work

Induced-universal graphs have been investigated for a wide variety of families of graphs. In this chapter, we explore interesting approaches developed over the years (both implicit and explicit constructions). We take a look at hereditary classes of graphs, families of graphs with bounded degree and a technique to construct induced-universal graphs from universal graphs.

4.1. Graph of bounded-degree

We consider the family of bounded-degree graphs with n vertices. For positive integers d and n, let $\mathcal{H}(n,d)$ denote the family of all graphs on n vertices with maximum degree at most k. Here, we consider d a constant and n an arbitrarily large number. Universal graphs for $\mathcal{H}(n,d)$ have been considered in various papers. They relied mainly of probabilistic methods [ACK$^+$00, ACK$^+$01, AA02, AC07, CK99, Cap02]. The study of induced-universal graphs for $\mathcal{H}(n,d)$ however has been a bit sparse. The first results on induced-universal graphs for bounded-degree graphs are comparatively very recent. The first studies were conducted by Butler [But09] in 2009. He provided narrow bounds on $g_v(\mathcal{H}(n,d))$ and $g_e(\mathcal{H}(n,d))$.
We know from Janson et al. [JLR11] that there are

$$(1+o(1))\sqrt{2}e^{-(d^2-1)/4}\left(\frac{d^{d/2}}{e^{d/2}d!}\right)^n n^{dn/2}$$

labeled d-regular graphs on n vertices when dn is even. Any induced-universal graph U for $\mathcal{H}(n,d)$ satisfies $\frac{|V(U)|^n}{n!} \geq \binom{|V(U)|}{n} \geq |\mathcal{H}(n,d)|$. The following counting argument naturally follows and yields a lower bound on $g_v(\mathcal{H}(n,d))$,

$$\frac{|V(U)|^n}{n!} \geq \binom{|V(U)|}{n} \geq |\mathcal{H}(n,d)| \geq \sqrt{2}e^{-(d^2-1)/4}\left(\frac{d^{d/2}}{e^{d/2}d!}\right)^n n^{dn/2}/n!.$$

Thus, we have $g_v(\mathcal{H}(n,d)) \geq cn^{d/2}$ where c is a constant depending only on d.

4.1.1. A first upper bound

Butler showed that this is the correct order of magnitude for $g_v(\mathcal{H}(n,d))$ when d is even, namely $g_v(\mathcal{H}(n,d)) < O(n^{d/2})$.

Theorem 4.1 (Butler [But09]). *We have*

$$g_v(\mathcal{H}(n,d)) \leq Cn^{\lfloor (d+1)/2 \rfloor}, \quad \textit{and} \quad g_e(\mathcal{H}(n,d)) \leq Dn^{2\lfloor (d+1)/2 \rfloor - 1},$$

where C and D are constants which depend only on d.

To prove this, he gives an explicit construction of an induced-universal graph for $\mathcal{H}(n,d)$ and show that it indeed contains every graph of $\mathcal{H}(n,d)$ as an induced subgraph. There are three major steps for the proof. Using Petersen's 2-factor theorem, we can decompose d-regular graphs into several 2-regular graphs. Then we construct an induced-universal graph for 2-regular graphs. Finally, using the reduction of Chung [Chu90], we can recompose our desired induced-universal graphs from those induced-universal graphs for 2-regular graphs. We start with the reduction of Chung.

4.1.1.1. Reduction of Chung

Investigating relations between universal graphs and induced-universal graphs, together with the techniques in [KNR92], Chung gave the following approach.

Theorem 4.2. *Let U_i be an induced-universal graph for the family of graphs $\mathcal{F}_i$ for $i = 1, 2, \ldots, k$. $\mathcal{H}$ is a family such that $H \in \mathcal{H}$ can be decomposed into edge-disjoint spanning subgraphs $H_1, H_2, \ldots, H_k$ where $H_i \in \mathcal{F}_i$.*
Let W be the graph with vertices $(u_1, u_2, \ldots, u_k)$ where $u_i \in U_i$ for all i, and an edge connecting $(u_1, u_2, \ldots, u_k)$ and $(u'_1, u'_2, \ldots, u'_k)$ if and only if there is an edge connecting u_i to u'_i for some $i = 1, 2, \ldots, k$. Then W is an induced-universal graph for $\mathcal{H}$ and we have

$$|V(W)| = \prod_{i=1}^{k} |V(U_i)|, \quad \textit{and} \quad |E(W)| \leq \sum_{i=1}^{k} |E(U_i)| \prod_{j \neq i} |V(U_j)|^2$$

Proof. Clearly, $|V(W)| = \prod_{i=1}^{k} |V(U_i)|$. For a fixed i, we observe that an edge $u_i u'_i$ in U_i creates at most $\prod_{j \neq i} |V(U_j)|^2$ new edges in W. Thus, $|E(W)| \leq \sum_{i=1}^{k} |E(U_i)| \prod_{j \neq i} |V(U_j)|^2$.

We now show that W is an induced-universal graph for $\mathcal{H}$. Let $H \in \mathcal{H}$. By assumption, H can be edge-partitioned into $H_1, H_2, \ldots, H_k$ and the vertex set of H_i is the same as $V(H)$ for all $i = 1, \ldots, k$. Let H_i denote the mapping from $V(H_i)$ to $V(U_i)$ such that $h_i(v)$ and $h_i(w)$ are adjacent in U_i if and only if v and w are adjacent in H_i. In other words, $h_i(v)$ denotes the vertex that $v \in H$ maps to when H_i is embedded into U_i. The following statements are equivalent:

1. $(h_1(v), h_2(v), \ldots, h_k(v))$ and $(h_1(w), h_2(w), \ldots, h_k(w))$ are adjacent in W.
2. $h_i(v)$ and $h_i(w)$ are adjacent for some $i = 1, 2, \ldots, k$.
3. v and w are adjacent in H_i for some $i = 1, 2, \ldots, k$.
4. v and w are adjacent in H.

$1 \Leftrightarrow 2$ follows from the definition of W, $2 \Leftrightarrow 3$ holds because $h_i, i = 1, \ldots, k$, are mappings. $3 \Leftrightarrow 4$ because $H_i, i = 1, \ldots, k$, are edge-decomposition of H. Thus, $(h_1(v), h_2(v), \ldots, h_k(v))$ and $(h_1(w), h_2(w), \ldots, h_k(w))$ are adjacent in W if and only if v and w are adjacent in H, and W is an induced-universal graph for $\mathcal{H}$. □

If we can decompose graphs of $\mathcal{H}(n,d)$ into edge disjoint smaller graphs and find induced-universal for each of these small graphs, then we can construct an induced-universal graph for $\mathcal{H}(n,d)$.

4.1.1.2. Decomposition of graphs

Here, we show how to decompose graphs of maximum degree at most d into $\lfloor (d+1)/2 \rfloor$ 2-regular graphs.

Lemma 4.3. *Let G be a graph n vertices with maximum degree at most d. Then G can be decomposed into $\lfloor (d+1)/2 \rfloor$ edge disjoint subgraphs where the maximum degree of each such subgraph is at most 2.*

Proof. This lemma is a simple corollary of Peterseon's 2-factor theorem for hypergraphs which states that every $(2k)$-regular hypergraph can be decomposed into k edge-disjoint 2-regular graphs. Let G be a graph n vertices with maximum degree at most d. We add edges randomly (possibly multi-edges and loops) until every vertex has degree d if d is even or degree $d+1$ if d is odd. Our new hypergraph is now d-regular (or $(d+1)$-regular if d is odd). We can now apply Peterson's 2-factor theorem and obtain an decomposition into $\lfloor (d+1)/2 \rfloor$ 2-regular edge-disjoint graphs. From each 2-regular graphs, we remove the edges that we initially added. This results in a decomposition of G into $\lfloor (d+1)/2 \rfloor$ edge-disjoint graphs with maximum degree at most 2. □

4.1.1.3. Induced-universal graph for graphs with maximum degree at most 2

We construct an induced-universal graph for the family of graphs on n vertices with maximum degree at most 2, $\mathcal{H}(n, 2)$.

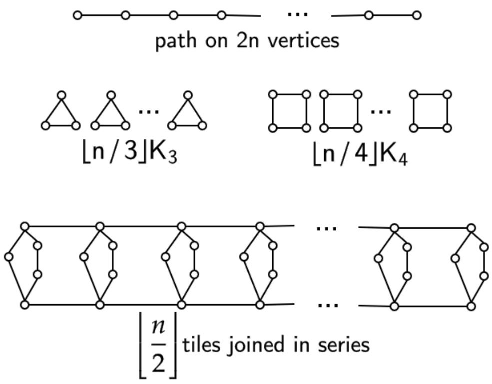


Figure 4.1.: An induced-universal graph for the family of graphs $\mathcal{H}(n, 2)$.

Let U denote the graph in Figure 4.1. We claim that U is induced-universal for $\mathcal{H}(n, 2)$. To see this, we use the following characteristic of graphs in $\mathcal{H}(n, 2)$: a graph G with maximum degree at most 2 is a collection of cycles and paths. Obviously, we can embed the paths of G into the path of $2n$ vertices. 3-cycles and 4-cycles can be respectively embedded into the $\lfloor n/3 \rfloor K_3$ and $\lfloor n/4 \rfloor K_4$. We now consider cycles of length $b_1, \dots, b_q$ with $b_i \geq 5, i = 1, \dots, q$. If the cycle is odd, then it has length $3 + 2k$ for some $k \geq 2$. We need k tiles. We take 2 edges of the first tile, then use $2(k-1)$ of the edges connecting the tles and finally 3 edges on the last tiles. If the cycle is even, then it has length $2 + 2k$ for some $k \geq 2$. We need k tiles. We take 2 edges of the first tile, then use $2(k-1)$ of the edges connecting the tles and finally 2 edges on the last tiles (see Figure 4.1). A cycle b_j needs $\lfloor b_j/2 \rfloor - 1$ tiles. Between two induced cycles, we need to skip a tile. Thus, we need at most $\lfloor \frac{b_1}{2} \rfloor + \cdots + \lfloor \frac{b_q}{2} \rfloor - 1 \leq \frac{1}{2}(b_1 + \cdots + b_q) - 1 \leq \lfloor \frac{n}{2} \rfloor$ tiles. We also have:

$$|V(U)| = 2n + 3\lfloor \frac{n}{3} \rfloor + 4\lfloor \frac{n}{4} \rfloor + 5\lfloor \frac{n}{2} \rfloor \leq 6.5n,$$

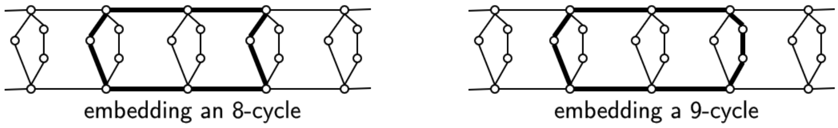


Figure 4.2.: Embedding of long cycles into U.

$$|E(U)| = (2n-1) + 3\lfloor\frac{n}{3}\rfloor + 4\lfloor\frac{n}{4}\rfloor + (7\lfloor\frac{n}{2}\rfloor - 2) \leq 7.5n.$$

We have now gathered all the necessary ingredients to prove Theorem 4.1. Let G be a graph in $\mathcal{H}(n,d)$.

1. Apply Lemma 4.3 to decompose G into $\lfloor(d+1)/2\rfloor$ edge disjoint subgraphs with maximum degree at most 2.
2. The graph U in Figure4.1 is an induced-universal graph for $\mathcal{H}(n,2)$.
3. Apply Theorem 4.1 with $\lfloor(d+1)/2\rfloor$ copies of U to construct an induced-universal graph for $\mathcal{H}(n,d)$.

4.1.2. Improving the bound

The bounds offered by Butler are not tight. In this subsection, we explore the improvements that can be made. There are two major factors to consider.
First, Butler's strategy yields an upper bound of $O(n^{\lfloor(d+1)/2\rfloor})$ for $g_v(\mathcal{H}(n,d))$. This order matches that of the lower bound when n is even. When n is odd, the bound is of order $O(n^{d/2+1/2})$. Unfortunately, we cannot use his approach to bridge this gap. It would necessitate the graph W in Theorem 4.2 to need vertices with $d/2$ coordinates (we cannot have half coordinates). Alon and Capalbo [AC07] offered a new type of decomposition which decomposes a graph with maximum degree at most d into d subgraphs. We shall expand a little on this below.
Second, to minimize $|V(W)|$ in Theorem 4.2, our only avenue is to minimize $|V(U_i)|$ for some $i = 1, \ldots, k$. Thankfully, the induced-universal graph for $\mathcal{H}(n,2)$ proposed by Butler is in no way optimal. Esperet, Labourel and Ochem [ELO08] constructed an induced-universal graph requiring less vertices. This results in an improvement of our upper bound by a multiplicative factor.

4.1.2.1. Case when n is odd

Alon and Nenadov [AN17] gave in 2017 an upper bound of $g_v(\mathcal{H}(n,d))$ with order matching that of the lower bound when n is odd. Formally, we have the following theorem.

Theorem 4.4 (Alon, Nenadov [AN17])**.** *There is a constant $c > 1$ such that for every integer $d \geq 2$ and $n \in \mathbb{N}$ there exists an induced-universal graph W for $\mathcal{H}(n,d)$ with at most $(cd)^d n^{d/2}$ vertices.*

We will only offer an outline of the proof of this theorem as it is quite expansive. For more details, we refer the reader to Alon's and Nenadov's paper [AN17]. We start with an idea similar to Butler's: decomposition of graph and patching together small induced-universal graphs. Instead of Peterson's 2-factor theorem, we decompose our graphs into so-called "thin" graphs.
An *augmentation* of a graph H is any graph obtained from H by choosing an arbitrary

Figure 4.3.: A graph on the left with an augmentation on the right. New added vertices are in red and the edges of the matching in bold.

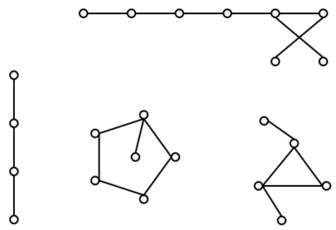


Figure 4.4.: A thin graph with 4 components.

(possibly empty) subset $U \subset V(H)$, adding a new set U' of $|U|$ vertices, and adding a matching between U and U' (see Figure 4.3). Obviously, any graph is an augmentation of itself.

We call a graph *thin* if its maximum degree is at most 3 and each connected component of it is either an augmenation of a path or a cycle, or a graph with at most two vertices of degree 3 (see Figure 4.4).

Any thin graph H on n vertices has an (H, P)-homomorphism to a path P on n vertices with a loop at each vertex, such that the inverse image of any vertex of P consists of at most 4 vertices. We can see that every thin graph on at most n vertices is a subgraph of P_n^4, the 4th power of a path with n vertices. We recall that the kth power of a graph is a graph with the same set of vertices and an edge between two vertices if and only if there is path of length at most k between them. The following theorem by Alon and Capalbo [AC07] states that we can decompose any graph with degree at most d into d thin subgraphs.

Theorem 4.5. *Let $d \geq 2$ be an integer, and let H be an arbitrary graph of maximum degree at most d. Then there are d spanning subgraphs $H_1, H_2, \ldots, H_d$ of H such that each H_i is thin, and every edge of H lies in precisely two graphs H_i.*

With this decomposition, in order to construct an induced-universal graph W following Theorem 4.2's approach and achieve the bound in Theorem 4.4, we need each coordinate to represent a graph of size $O(\sqrt{n})$. We also require W to have an edge between two vertices if they are adjacent in at least two coordinates. Obviously, Petersen's 2-factor theorem does not hold for thin graphs in general. Thus, we also need to find an alternative to embed our thin graphs into graphs U_i of size $O(\sqrt{n}$. Alon and Capalbo achieve this using homomorphisms of graphs and results on non-bipartite r-regular *Ramanujan* graphs. A r-regular graph is a *Ramanujan* graph if all non-trivial eigenvalues have absolute value at most $2\sqrt{r-1}$.

4.1.2.2. Case when n is even

As already mentioned above, when n is even, the simplest way to improve our upper bound is to reduce the number of vertices of the induced-universal graph for $\mathcal{H}(n, 2)$. Slightly modifying Butler's construction, Esperet, Labourel and Ochem [ELO08] gave a construction requiring at most $5(\lfloor n/2 \rfloor + 5)$ vertices and $9(\lfloor n/2 \rfloor + 5) - 4$ edges.

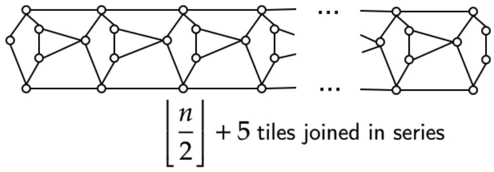
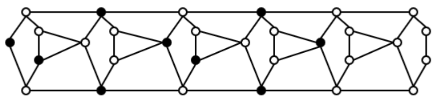


Figure 4.5.: A smaller induced-universal graph for $\mathcal{H}(n, 2)$.

Theorem 4.6 (Esperet, Labourel and Ochem [ELO08]). *The graph depicted in Figure 4.5 is induced-universal for $\mathcal{H}(n, 2)$.*

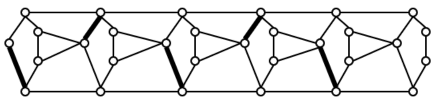
Proof. Let U be depicted in Figure 4.5 and let $G \in \mathcal{H}(n, 2)$. Our goal is to show that we can embed any graph of $\mathcal{H}(n, 2)$ into U. Let n_i denote the number of components of G on i vertices. We recall that graphs in $\mathcal{H}(n, 2)$ are a collection of cycles and paths.

- For components of size 1, we need $\lceil n/2 \rceil + 1$ tiles for our embedding. One tile covers at most 2 vertices and we need an additional tile to ensure that our embedding is correct.



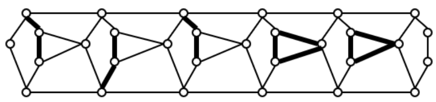
$\lceil n/2 \rceil + 1$ tiles.

- For components of size 2, we require $n_2 + 1$ tiles for our embedding.



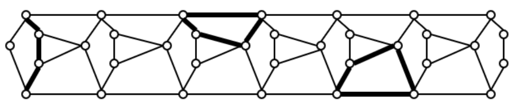
$n_2 + 1$ tiles.

- For components of size 3, we require $n_3 + 1$ tiles for our embedding.



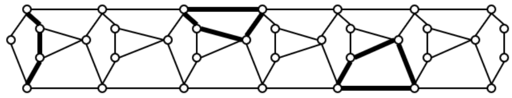
$n_3 + 1$ tiles.

- For components of size 4, we require $2n_4 + 1$ tiles for our embedding.



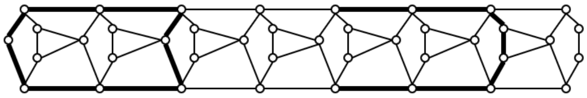
$2n_4 + 1$ tiles.

- For components of size 5, we require $2n_5$ tiles for our embedding.



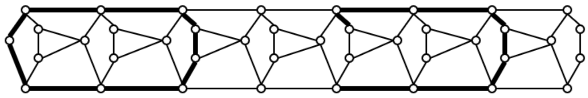
$2n_5$ tiles.

- Each component of size $2k, k \geq 3$, requires k tiles. Thus, we need kn_{2k} tiles.



Embedding of paths and cycles of length 8. It requires 8 tiles.

- Each component of size $2k+1, k \geq 3$, requires k tiles. Thus, we need kn_{2k+1} tiles.



Embedding of paths and cycles of length 8. It requires 8 tiles.

We now count the number of tiles required. In total, we need

$$\begin{aligned}
&\frac{n}{2} + 2 + n_2 + 1 + n_3 + 1 + 2n_4 + 1 + 2n_5 + \sum_{k=3}^{\lfloor n/2 \rfloor} kn_{2k} + \sum_{k=3}^{\lfloor n/2 \rfloor} kn_{2k+1} \\
&\leq 5 + \sum_{i=1}^{n} i\frac{n_i}{2} \\
&\leq 5 + \left\lfloor \frac{n}{2} \right\rfloor
\end{aligned}$$

tiles. Here, we used the known fact $\sum_{i=1}^{n} in_i = n$. Thus, W has at most $5(\lfloor n/2 \rfloor + 5)$ vertices and $9(\lfloor n/2 \rfloor + 5) - 4$ edges. □

Inserting these numbers into Theorem 4.2, we see that $g_v(\mathcal{H}(n,d)) \leq (1 + o(1)) \left(\frac{5n}{2}\right)^{d/2}$ and $g_e(\mathcal{H}(n,d)) \leq \left(\frac{9d}{10} + o(1)\right) \left(\frac{5n}{2}\right)^{d-1}$. These results are not sharp. Esperet, Labourel and Ochem conjecture that there exists an induced-universal graph for $\mathcal{H}(n,2)$ with only $2n + o(n)$ vertices [ELO08].

4.2. From Universal Graphs to Induced-Universal Graphs

In this section, we analyse relations between universal graphs and induced-universal graphs. In particular, we will see a construction by Chung [Chu90] that aims to bridge the gap between universal graphs and induced-universal graphs for particular families of graphs.

We recall that a graph is universal U for a family of graphs $\mathcal{F}$ if U contains every graph of $\mathcal{F}$ as a subgraph. In other words, for any graph $G \in \mathcal{F}$ there is a one-to-one mapping σ from $V(G)$ to $V(U)$ such that $\sigma(u)$ is adjacent to $\sigma(v)$ in U if u is adjacent to v in G. This contrasts from induced-universal graphs where we would require our mapping to satisfy adjacency *if and only if* u is adjacent to v in G. For a family of graphs $\mathcal{F}$, we denote by $f_v(\mathcal{F})$ and $f_e(\mathcal{F})$ respectively the minimum number of vertices and the minimum number of edges of a universal graph for $\mathcal{F}$. We denote by $f_{e,v}(\mathcal{F})$ the minimum number of edges in a universal graph for $\mathcal{F}$ on $f_v(\mathcal{F})$ vertices. Since an induced subgraph is still a subgraph, we obviously have $f_e \leq g_e$ and $f_v \leq g_v$ for any family of graphs.

The gap between f_v and g_v varies drastically depending on the family we are investigating. For example, we saw in Section 4.1 that $g_v(\mathcal{H}(n,d)) \in O(n^{d/2})$. Kohayakawa, Rödl and Schacht [KRSS11] showed in 2011 that $f_v(\mathcal{H}(n,d)) \in O(n)$. In fact, they proved a much stronger statement. For every $d \geq 2$ and fixed integer r, there exists constants B and C such that for every n and N satisfying $N \geq Bn$ there exists a graph G on N vertices and at most $CN^{2-1/d} \log^{1/d} N$ edges such that any r-coloring of the edges of G contains a monochromatic universal graph for $\mathcal{H}(n,d)$. B and C are constants depending only on d and r. If we consider the class of planar graphs on n vertices $\mathcal{P}(n)$, the gap between f_v and g_v becomes much smaller. In fact, they both have the same order. Dujmović, Joret, Micek, Morin, Ueckerdt and Wood [DJM+20] proved in 2020 that every planar graph is a subgraph of the strong product between a graph of treewidth at most 8 and a path. Very recently, this fact was used by Esperet et al. [EJM20] to show that $f_v(\mathcal{P}(n)) \in O(n^{1+o(1)})$, and by Dujmović et al. [DJM+20] to show that $g_v(\mathcal{P}(n)) \in O(n^{1+o(1)})$.

4.2.1. Constructing Induced-Universal Graphs from Universal Graphs

We now take a look at another of Chung's reduction showing that an induced-universal graph can be constructed from a universal graph for graphs with bounded arboricity. This translates to the relation $g_v \leq 2f_{e,v} + f_v$ for the family of acyclic graphs.

Theorem 4.7 (Chung [Chu90])**.** *Let $\mathcal{A}_k$ denote the family of graphs with arboricity at most k. Let U be a universal graph for $\mathcal{A}_k$. Then we have*

$$g_v(\mathcal{A}_k) \leq \sum_i (d_i + 1)^k$$

and

$$g_e(\mathcal{A}_k) \leq \sum_{v_i \sim v_j} (d_i + 1)^k d_k^{k-1}$$

where d_i denotes the degree of the ith vertex in U and $v_i \sim v_j$ denotes that v_i and v_j are adjacent.

Proof. We construct an induced-universal graph W for $\mathcal{A}_k$ as follows.

- $V(W) = \{(u_0, u_1, \ldots, u_k) \mid u_0$ is a vertex of $U, u_i, i \neq 0$, is either $*$ (a special symbol) or is a neighbor of $u_0\}$.
- Two vertices $(u_0, u_1, \ldots, u_k)$ and $(u'_0, u'_1, \ldots, u'_k)$ are adjacent if $u_0 = u'_i$ or $u'_0 = u_i$ for some $i \neq 0$.

We need to show that contains every graph from $\mathcal{A}_k$ as an induced subgraph. Let $G \in \mathcal{A}_k$. Then G can be partitioned into k edge-disjoint spanning forests $F_1, \ldots, F_k$. In constructing W and its vertices $(u_0, u_1, \ldots, u_k)$, our aim is to have each dimension (except the first) represent the forest F_i and model its adjacency.
In each F_i we orient the edges so that each connected component is an out-tree with a root, i.e. all edges are oriented from the root to the leaves. U is universal for $\mathcal{A}_k$, so there is a mapping α from $V(G)$ to $V(U)$ so that $\alpha(u)$ and $\alpha(v)$ are adjacent if u and v are adjacent. Let $\sigma : V(G) \to V(W)$ map each vertex u to $(\alpha(u), \beta(v_1), \ldots, \beta(v_k))$ where v_i is the parent of u in F_i and $\beta(v_i) = \alpha(v_i)$ if u is not a root, and $\beta(v_i) = *$ otherwise.
If u and w are adjacent in G, then $\alpha(u)$ and $\alpha(w)$ are adjacent in U. There must be some forest F_i where u is the parent of w or w is the parent of u. Say w is the parent of u in F_i.

Then $\sigma(u)$ has the form $(\alpha(u), \dots, \alpha(w), \dots)$ where $\alpha(w)$ sits at the ith coordinate. Hence, $\sigma(u)$ and $\sigma(w)$ are adjacent in W.
If $\sigma(u) = (\alpha(u), \alpha(v_1), \dots, \alpha(v_k))$ and $\sigma(w) = (\alpha(w), \alpha(x_1), \dots, \alpha(x_k))$ are adjacent in W, then, by construction of W, $\alpha(u) = \alpha(x_i)$ or $\alpha(w) = \alpha(v_i)$ for some $i = 1, \dots, k$. Say $\alpha(w) = \alpha(v_i)$. Then by definition of σ, w is the parent of u in F_i. Thus, u and w are adjacent in G and W contains G as an induced subgraph.
We now count the vertices and edges of W to conclude the proof of our theorem. The i vertex v_i in G has degree at most d_i in G and, thus, degree at most d_i in each forest decomposition of G. Hence, the ith vertex generates at most $(d_i+1)^k$ vertices in W (we need to account for the special character $*$). We have $|V(W)| \leq \sum_i (d_i + 1)^k$. The fixed vertex $(u, v_1, \dots, v_{i-1}, w, v_{i+1}, \dots, v_k)$ has neighbors of the form $(w, x_1, \dots, x_{j-1}, u, v_{j+1}, \dots, x_k)$. Thus, we have $|E(W)| \leq \sum_{v_i \sim v_j} (d_i + 1)^k \cdot d_j^{k-1}$. □

An obvious corollary of Theorem 4.7 is the following.

Corollary 4.8. *Let A denote a family of acyclic graphs and let U be a universal graph for A. Then we have*

$$g_v(A) \leq 2\,|E(U)| + |V(U)|\,,$$
$$g_v(A) \leq 2 f_{e,v}(A) + f_v(A).$$

4.2.2. Applications

The construction in Theorem 4.7 has a lot of applications in the literature. Indeed, If a family of graphs with arboricity k admits a universal graph with bounded-degree d, then the induced-universal graph for that family has $O(n(d+1)^k)$ vertices. An interesting family we can, although indirectly, apply this logic to is the family of planar graphs.

We recall that a graph G is covered by subgraphs $G_1, \dots, G_k$ of G if every edge of G belongs to one of these subgraphs. A graph $F(d_1, \dots, d_k)$-coverable if it can be covered by k forests $F_1, \dots, F_k$ such that the maximum degree of each F_i is at most $d_i, i = 1, \dots, k$.

Theorem 4.9 (Gonçalves [Gon06, Gon09])**.** *Planar graphs are $F(\infty, \infty, 4)$-coverable.*

To prove this theorem, Gonçalves shows that every triangulation can be decomposed into three graphs satisfying specific conditions. We can then proceed by induction to conclude the proof.

Theorem 4.10. *The family of planar graphs on n vertices $\mathcal{P}(n)$ admits an induced-universal graph on $O(n^3)$ vertices.*

Proof. We construct a labeling scheme for $\mathcal{P}(n)$ as follows. Let G be a graph from $\mathcal{P}(n)$. We know by Theorem 4.9 that G is $F(\infty, \infty, 4)$-coverable. In other words, G can be decomposed into three forests F_1, F_2, F_3, one of which is a spanning forest of degree at most 4. Let F_3 be the forest of degree at most 4. Using the Traversal and Jumping technique, Bonichon, Gavoille and Labourel [BGL07] showed bounded degree forests enjoy an adjacency labeling scheme $\mathcal{L}$ with labels of length $\log(n) + O(1)$ bits. For a vertex u, let the label u in G be its label in F_3 plus the label of its parent in F_1 and F_2. It is easy to verify that this is indeed an adjacency labeling scheme for $\mathcal{P}(n)$ and it requires $3 \log n + O(1)$ bits.
By Theorem 3.1, $\mathcal{P}(n)$ admits an induced-universal graph with at most $2^{3 \log n + O(1)} = c \cdot n^3$ vertices for a constant c. □

This bound has long since been improved by Dujmović et al. [DEG+20] to $g_v(\mathcal{P}(n)) \in O(n^{1+o(1)})$.

4.3. Hereditary families of graphs

To find an induced-universal graph for a given family of graphs, the main strategy often used is to try to infer as much information as possible on the underlying structure of the graphs composing said family, hoping to find helpful characteristics. For a family of graphs $\mathcal{F}$ and an induced-universal graph U for $\mathcal{F}$, we know that $|\mathcal{F}| \leq |U|^n$, offering us a natural lower bound on $|U|$. In this section, we take interest in hereditary classes of graphs. We will analyze their speed of growth, classify them by their size, fully characterize so-called “exponential” classes and from this characterization build an induced-universal for exponential classes of graphs. We recall that a class of graphs is hereditary if it contains with each graph G all induced subgraphs of G.

We start with some definitions. For a class of graphs $\mathcal{X}$, let $\mathcal{X}_n$ be the class of graphs in $\mathcal{X}$ on n vertices. The *entropy* of a class of graphs $\mathcal{X}$ is defined as

$$h(\mathcal{X}) = \lim_{n\to\infty} \frac{(\log|\mathcal{X}_n|)}{\binom{n}{2}}.$$

We denote by $\mathcal{E}_{i,j}$ the class graphs which permit a partition of the vertex set into $i+j$ subsets inducing i cliques and j independent sets. For example, $\mathcal{E}_{0,2}$ is the class of bipartite graphs, $\mathcal{E}_{1,1}$ is the class of split graphs [FH77] and $\mathcal{E}_{2,0}$ is the class of co-bipartite graphs. For an *infinite* hereditary a class of graphs $\mathcal{X}$ different from the class of graphs, we define the index $k(\mathcal{X})$ as the largest integer k such that a class $\mathcal{E}_{i,j} \subseteq \mathcal{X}$ with $i+j=k$. We set $k(\mathcal{X}) = 0$ for any finite hereditary class of graphs and $k(\mathcal{X}) = \infty$ for the class of all graphs. Alekseev [Ale92] provided a way to evaluate the entropy of hereditary classes of graphs.

Theorem 4.11 (Alekseev [Ale92])**.** *For every infinite hereditary classes of graphs (except the class of all graphs), we have*

$$h(\mathcal{X}) = 1 - \frac{1}{k(\mathcal{X})}.$$

We can thus partition the classes of hereditary graphs into a countable set layers. The kth layer consists of the classes with entropy $h(\mathcal{X}) = 1 - 1/k(\mathcal{X})$. Only the classes $\mathcal{E}_{i,j}, i+j=k$ are minimal within the kth layer. The minimal classes of second layer are bipartite graphs, split graphs and co-bipartite graphs.
Of particular interest are the classes of the first layer. In this layer resides classes of graphs that do not contain $\mathcal{E}_{2,0}, \mathcal{E}_{1,1}$ and $\mathcal{E}_{0,2}$: forests, planar graphs, interval graphs, line graphs, cographs, etc. These graphs do not all have the same rate of growth as n tends to infinity. Scheinerman [SZ94] identified four main growth rates:

- the *constant* tier contains classes $\mathcal{X}$ with $\log|\mathcal{X}| = O(1)$,
- the *polynomial* tier contains classes $\mathcal{X}$ with $\log|\mathcal{X}| = O(\log n)$,
- the *exponential* tier contains classes $\mathcal{X}$ with $\log|\mathcal{X}| = O(n)$,
- the *factorial* tier contains classes $\mathcal{X}$ with $\log|\mathcal{X}| = O(n\log n)$.

Alexseev [Ale97] provided a description of the minimal classes of these tiers, leading the following theorem.

Theorem 4.12. *For each exponential class $\mathcal{X}$, there is a constant p such that every graph $G \in \mathcal{X}$ can be partitioned into at most p subsets each of which is either an independent set or a clique and between any two subsets there are either all possible edges or none of them.*

Given this full characterization, Lozin and Rudolf [LR07] constructed an induced-universal graph for the exponential class of graphs on n vertices.

Theorem 4.13 (Lozin and Rudolf [LR07]). *Let $\mathcal{X}$ be an exponential class of graphs and p a constant associated to it. Let Γ_k be the class of all graphs on p vertices and let $\overline{\Gamma}_p$ contain exactly one graph from each isomorphism class of Γ_k. The graph W defined as follows*

- *The vertex set of W is $V(W) = \{(\overline{G}, i, j, \delta) \mid \overline{G} \in \overline{\Gamma}_k,\ i \in [p],\ j \in [n],\ \delta \in \{0, 1\}\}$.*
- *Two distinct vertices $(\overline{G}_1, i_1, j_1, \delta_1)$ and $(\overline{G}_2, i_2, j_2, \delta_2)$ are adjacent in W if and only if $\overline{G}_1 = \overline{G}_2$ and either $i_1 i_2 \in E(\overline{G}_1)$ or $i_1 = i_2, \delta_1 = \delta_2 = 1$.*

is induced-universal for the class $\mathcal{X}_n$.

We give here Lozin's and Rudolf's proof to the theorem.

Proof. Let G be a graph in $\mathcal{X}$. Then G can be partitioned into independent sets $V_1, \ldots, V_r$ and cliques $V_{r+1}, \ldots, V_t$ with $t \leq p$ such that if two vertices u and v belong to the same subset V_i, then they share the same neighborhood in $G \setminus V_i$. For each subset V_i, we define a bijection $\phi_i : V_i \to [V_i] \subset [n]$. Let H be the graph obtained from G by contracting each subset V_j into a single vertex v_j. H has at most p vertices. There is an isomorphism $\psi : V(H) \to V(\overline{H}) = [p]$ for some $\overline{H} \in \overline{\Gamma}_p$.
Mapping a vertex $v \in V_i$ to $(\overline{H}, \psi(v_i), \phi(v_i), \delta_i)$, where $\delta_i = 0$ if $i \leq r$ and $\delta_i = 1$ otherwise, provides us with an embedding of G into W. $\square$

5. Case of edge-colored Complete Graphs

Let $\mathcal{F}(k)$ the family of undirected graphs on k vertices. In 2017, Alon [Alo17] extensively studied this class of graphs and showed that $g_v(\mathcal{F}(k)) = (1 + o(1))2^{(k-1)/2}$, improving on previous results. His proof can be expanded to different settings. In this chapter, we focus on the case of edge colored complete graphs and will show in detail how to make to adapt his proof.

We recall that for a fixed positive integer r, $\mathcal{K}(k, r)$ denotes the set of all complete graphs on k vertices with edges colored with the r colors $\{1, \ldots, r\}$. It is a known fact that a complete graph on k vertices has $\binom{k}{2}$ edges. Since each edge can take r colors and there are $k!$ permutations of the vertices, $\mathcal{K}(k, r)$ contains at least $r^{\binom{k}{2}}/k!$ graphs. Furthermore, an induced universal graph for $\mathcal{K}(k, r)$ must have at least $|\mathcal{K}(k, r)|$ induced subgraphs with k vertices. With these simple counting arguments, it is easy to see that

$$\frac{g_v(\mathcal{K}(k, r))^k}{k!} \geq \binom{g_v(\mathcal{K}(k, r))}{k} \geq |\mathcal{K}(k, r)| \geq \frac{r^{\binom{k}{2}}}{k!}.$$

Thus, we have $g_v(\mathcal{K}(k, r)) \geq r^{(k-1)/2}$.

Theorem 5.1 (Alon [Alo17])**.** *For every fixed r and large k, we have*

$$r^{(k-1)/2} \leq g_v(\mathcal{K}(k, r)) \leq r^{(k-1)/2}\left(1 + O\left(\frac{\log_r^{3/2} k}{\sqrt{k}}\right)\right).$$

We define a few notions before starting with our proof. Let r be a fixed positive integer. We call an edge-colored graph $(G, c) \in \mathcal{K}(k, r)$ *asymmetric* if every induced subgraph of (G, c) has at most k^{4m} automorphisms, where $m = 2\sqrt{k \log_r k}$. In particular, an asymmetric graph has at most k^{4m} automorphisms. Since any graph k vertices has at most $k!$ automorphisms, we have $k^{4m} \leq k!$. This holds for $k = 1$ or extremely large k, justifying the condition of the theorem. An edge colored graph in $\mathcal{K}(k, r)$ is *symmetric* if it is not asymmetric. We denote by $\mathcal{H}(k, r)$ the family of all asymmetric complete graphs on k vertices with edges colored with the r colors $\{1, \ldots, r\}$, and by $\mathcal{T}(k, r) = \mathcal{K}(k, r) - \mathcal{H}(k, r)$ the family of all symmetric complete graphs on k vertices with edges colored with the r colors $\{1, \ldots, r\}$.

We recall that for an edge coloring c with color set $\{1, \ldots, r\}$ and for a set $\mathcal{P} = \{p_1, \ldots, p_r\}$ with $0 \leq p_i \leq 1, 1 \leq i \leq r$, and $p_1 + \cdots + p_r = 1$, a *multinomial random graph* $\overline{G}(n, \mathcal{P})$ is an edge colored graph (G, c) where G is a complete graph on n vertices and c independently assigns label i (color) to each edge with probability p_i. If $p = p_1 = p_2 = \cdots = p_r = \frac{1}{r}$, we simply write $\overline{G}(n, p)$. Obviously, each graph $\overline{G}(n, \mathcal{P})$ is an element of $\mathcal{K}(n, r)$. For a graph $(G, c) \in \mathcal{K}(n, r)$, let G_i be the subgraph that only contains edges of color i, $1 \le i \le r$. The $\overline{G}(n, \mathcal{P})$ model assigns to a graph G the probability

$$\mathsf{Prob}[G] = \prod_i p_i^{|E(G_i)|}.$$

If $r = 2$, we have a *binomial random graph.*

Our induced universal graph for $\mathcal{K}(k, r)$ consists of two vertex disjoint parts. One part is induced universal for asymmetric graphs and the other part induced universal for symmetric graphs. We will show that, for an appropriate n, the multinomial random graph $\overline{G}(n, 1/r)$ is induced universal for $\mathcal{H}(k, r)$ with high probability. Using information on the structure of symmetric graphs, we will then give an explicit construction of an induced universal graph for $\mathcal{T}(k, r)$.

5.1. Asymmetric graphs

In this section, we are interested in finding an induced universal graph for the family of asymmetric graphs. Let n be the smallest integer that satisfies the following inequality

$$\binom{n}{k} \frac{k!}{k^{8m}} r^{-\binom{k}{2}} \ge 1. \tag{5.1}$$

For the smallest n satisfying this condition, we have $\binom{n+1}{k} \frac{k!}{k^{8m}} r^{-\binom{k}{2}} \Big/ \binom{n}{k} \frac{k!}{k^{8m}} r^{-\binom{k}{2}} = \frac{n+1}{n-k+1}$. This ratio is close to 1 as n is much larger than k. Thus, the left-hand side of (5.1) for this smallest n is $1 + o(1)$. Solving for n, we have

$$\begin{aligned}
\binom{n}{k} \frac{k!}{k^{8m}} r^{-\binom{k}{2}} &= 1 + o(1) \\
n^k \frac{1}{k^{8m}} \cdot r^{k(k-1)/2} &= 1 + o(1) \\
n^k &= r^{k(k-1)/2} \cdot k^{8m} \cdot (1 + o(1)) \\
n &= r^{(k-1)/2} \cdot k^{8m/k} \cdot (1 + o(1))^{1/k} \\
n &= r^{(k-1)/2} \cdot e^{16 \frac{\log_r^{3/2} k}{\sqrt{k}}} \cdot (1 + o(1))^{1/k} \\
n &= r^{(k-1)/2} \cdot \left(1 + O\left(\frac{\log_r^{3/2} k}{\sqrt{k}}\right)\right)
\end{aligned}$$

In particular, $n = (1 + o(1)) \cdot r^{(k-1)/2}$. We prove that the multinomial random graph $\overline{G}(n, 1/r)$, for $n = (1+o(1))r^{(k-1)/2}$, contains with high probability an induced copy of every asymmetric graph with k vertices. Using Talagrand's inequality and other probabilistic arguments, we show that the probability that any specific graph of $\mathcal{H}(k, r)$ does not appear in $\overline{G}(n, 1/r)$ is very small.

Theorem 5.2. *Let $n = (1 + o(1)) \cdot r^{(k-1)/2}$. The multinomial random graph $(G, c) = \overline{G}(n, 1/r)$ is, with high probability, induced universal for $\mathcal{H}(k, r)$.*

We need the following lemma to prove Theorem 5.2.

Lemma 5.3. *Let $(H, c_H) \in \mathcal{H}(k, r)$. Let K and K' be two graphs with labelled vertices, each of size k, where $|V(K) \cap V(K')| = k - i$. Then the number of pairs of colorings (c_1, c_2), from the color set $\{1, \ldots, r\}$, so that the edge colored graph (K, c_1) is isomorphic to (H, c_H) and the edge colored graph (K', c_2) is also isomorphic to (H, c_H) is at most*

$$\frac{k!}{|Aut(H, c_H)|} k^i k^{4m}.$$

Proof. There are $k!$ ways to assign labels to the vertex set of K and each automorphism class of (H, c_H) is represented only once on K. Thus, there are exactly $\frac{k!}{|Aut(H,c_H)|}$ copies of (H, c_H) on K. Each such copy represents an edge coloring c_1 of K. For each fixed coloring c_1 of K, we bound the number of possible embeddings of (H, c_H) into K'. Each such embedding will represent a coloring c_2 of K' satisfying the condition of the lemma. Since $|V(K) \cap V(K')| = k - i$, there are at most $k(k-1) \cdots (k-i+1) < k^i$ ways to choose the vertices of H mapped to the vertices $V(K') - V(K)$. Fix a set T of these i vertices and their embedding. In order to complete the embedding, the induced colored subgraph of (K, c_1) on the set of vertices $V(K) \cap V(K')$ has to be isomorphic to the induced colored subgraph of (H, c_H) on $V(H) - T$. The number of embeddings of these $k - i$ vertices corresponds to the number of automorphisms of this induced colored subgraph of (H, c_H). It is at most k^{4m} since (H, c_H) is asymmetric. $\square$

We can now proceed with the proof of Theorem 5.2. Let (H, c_H) be a fixed member of $\mathcal{H}(k, r)$ and let $s = |Aut((H, c_H))|$ be the size of its automorphism group. Let $(G, c) = \overline{G}(n, 1/r)$ be a multinomial random graph, where $n = (1 + o(1)) \cdot r^{(k-1)/2}$.
For every subset $K \subset V(G)$ of size k, let X_K be the indicator random variable that takes value 1 if the induced colored subgraph of (G, c) on K is isomorphic to (H, c_H). Let the random variable X be equal to the number of copies of (H, c_H) in (G, c). Thus $X = \sum_K X_K$, where the summation is over all subsets $K \subset V(G)$ of size k. There are $k!/s$ copies of (H, c_H) on K. The color of each edge of the induced colored subgraph of (G, c) on K is chosen independently with probability $1/r$. Therefore, the expected value of each X_K is

$$E(X_K) = \frac{k!}{s} r^{-\binom{k}{2}}.$$

By linearity of expectation, we have

$$E(X) = \binom{n}{k} \frac{k!}{s} r^{-\binom{k}{2}}.$$

It holds

$$E(X) = \binom{n}{k} \frac{k!}{s} r^{-\binom{k}{2}} = \binom{n}{k} \frac{k!}{k^{8m}} r^{-\binom{k}{2}} \frac{k^{8m}}{s} \geq \frac{k^{8m}}{s} \geq \frac{k^{8m}}{k^{4m}} \geq k^{4m}.$$

Here, we used (5.1) and the inequality $s \leq k^{4m}$. Since $n = (1 + o(1)) \cdot r^{(k-1)/2}$, we have that $k = (2 + o(1)) \log_r n$ and thus,

$$E(X) = \binom{n}{k} \frac{k!}{s} r^{-\binom{k}{2}} = \binom{n}{k} \frac{k!}{k^{8m}} r^{-\binom{k}{2}} \frac{k^{8m}}{s} = (1 + o(1)) \frac{k^{8m}}{s} \leq (1 + o(1)) k^{8m} < n^{0.01}.$$

The last inequality holds since k^{8m} has order $(\log_r n)^{\sqrt{(\log_r n) \cdot (\log_r \log_r n)}}$.

We say that two copies of (H, c_H) in (G, c) have a *nontrivial intersection* if they share at least two vertices. Let the random variable Z be equal to the number of pairs of copies of (H, c_H) in (G, c) that have a nontrivial intersection. We have $Z = \sum_{K,K'} X_K X_{K'}$, where the summation is over all pairs of k-subsets of $V(G)$ that satisty $2 \leq |K \cap K'| \leq k-1$. Let $\mu = E(X)$ and $\Delta = E(Z)$. Then $\Delta = \sum_{j=2}^{k-1} \Delta_j$ where Δ_j is the expected number of pairs K, K' with $X_K = X'_K = 1$ and $|K \cap K'| = j$.

We claim that for each $2 \leq j \leq k-1$,

$$\Delta_j \leq \mu \frac{1}{n^{0.48}}. \tag{5.2}$$

We consider two cases:

- **Case 1:** $2 \leq j \leq 3k/4$.
  There are $\binom{n}{k}$ ways to choose the set K, and $\binom{k}{j}\binom{n-k}{k-j}$ ways to choose K' with $|K \cap K'| = j$ (choose j vertices from K for the intersection, the remaining vertices are chosen from $G - K$). There are $\frac{k!}{s}$ ways to place a copy of (H, c_H) in K and $\frac{k!}{s}$ ways to place a copy of (H, c_H) in K' (we are overcounting here, as these two copies have to agree on the edges in their common part). This determines the color of each edge in the induced colored subgraph of (G, c) on K and on K'. The probability that (G, c) indeed has exactly these edges is $r^{-\binom{k}{2}} \cdot r^{-\binom{k}{2}+\binom{j}{2}}$. Thus, we have

  $$\Delta_j \leq \binom{n}{k} \frac{k!}{s} r^{-\binom{k}{2}} \binom{k}{j} \binom{n-k}{k-j} \frac{k!}{s} r^{-\binom{k}{2}+\binom{j}{2}}.$$

  We have $k = (2 + o(1)) \log_r n$ and since $j \leq 3k/4$, it follows that

  $$r^{(j-1)/2} \leq r^{(\frac{3k}{4}-1)/2} \leq r^{\left(\frac{3}{4}+\frac{3}{8}o(1)\right)\log_r n - \frac{1}{2}} = n^{\frac{3}{4}+\frac{3}{8}o(1)} r^{-\frac{1}{2}} \leq n^{\frac{3}{4}+o(1)}.$$

  Therefore,

  $$\begin{aligned} \frac{\Delta_j}{\mu^2} &\leq \frac{\binom{n}{k}\frac{k!}{s} r^{-\binom{k}{2}} \binom{k}{j}\binom{n-k}{k-j}\frac{k!}{s} r^{-\binom{k}{2}+\binom{j}{2}}}{\left(\binom{n}{k}\frac{k!}{s} r^{-\binom{k}{2}}\right)^2} \\ &\leq \frac{\binom{k}{j}\binom{n-k}{k-j} r^{-\binom{j}{2}}}{\binom{n}{k}} \leq \left(\frac{k^2 r^{(j-1)/2}}{n}\right)^j \leq \left(\frac{1}{n^{1/4-0.005}}\right)^j \leq \frac{1}{n^{0.49}}. \end{aligned}$$

  Recall that $\mu \leq n^{0.01}$ and thus

  $$\frac{\Delta_j}{\mu} = \mu \frac{\Delta_j}{\mu^2} \leq \frac{1}{n^{0.48}}.$$

- **Case 2:** $j = k - i, i \leq k/4$.
  There are $\binom{n}{k}$ ways to choose the set K, and $\binom{k}{j}\binom{n-k}{k-j}$ ways to choose K' with $|K \cap K'| = j$. By Lemma 5.3, for each such choice there are at most $\frac{k!}{s} k^i k^{4m}$ ways to place copies of H in K and K'. The probability that this coincides with edges of the induced colored subgraph of (G, c) on K and on K' is $r^{-2\binom{k}{2}+\binom{j}{2}}$. Thus, we have

  $$\Delta_j \leq \binom{n}{k}\binom{k}{j}\binom{n-k}{k-j} \frac{k!}{s} k^i k^{4m} r^{-2\binom{k}{2}+\binom{j}{2}}.$$

  We have

  $$-\binom{k}{2} + \binom{j}{2} = \frac{-k^2 + k + j^2 - j}{2} = \frac{-k^2 + k + (k-i)^2 - (k-i)}{2} \leq -ki + i^2$$

and

$$nr^{-(k-i)} \le nr^{-(k-\frac{k}{4})} = nr^{-\frac{3}{4}(2+o(1))\log_r n} = n \cdot n^{-\frac{3}{4}(2+o(1))} = \frac{1}{n^{0.5+o(1)}}.$$

Recall that $\mu \ge k^{4m} > 1$ and thus

$$\begin{aligned}
\frac{\Delta_j}{\mu^2} < \frac{\Delta_j}{\mu} &\le \binom{k}{j}\binom{n-k}{k-j} k^i k^{4m} r^{-\binom{k}{2}+\binom{j}{2}} \\
&\le \binom{k}{i}\binom{n-k}{i} k^i k^{4m} r^{-i(k-i)} && (j = k-i) \\
&\le k^i (n-k)^i k^i k^{4m} r^{-i(k-i)} \\
&\le (k^2 n r^{-(k-i)})^i k^{4m} \le \frac{1}{n^{0.5-o(1)}} \le \frac{1}{n^{0.48}}.
\end{aligned}$$

We recall Chebyshev's inequality, Markov's inequality and Talagrand's inequality.

Lemma 5.4 (Chebyshev's Inequality). *Let X be random variable with expected value μ and variance σ^2. Then for any real number $t > 0$,*

$$\mathsf{Prob}[|X-\mu| \ge t\sigma] \le \frac{1}{t^2}.$$

Lemma 5.5 (Markov's Inequality). *For a positive random variable $X \ge 0$, with finite mean, we have*

$$\mathsf{Prob}[X \ge t] \le \frac{E(X)}{t}.$$

Theorem 5.6 (Talagrand's Inequality). *Let $\Omega = \prod_{i=1}^{p} \Omega_i$, where each Ω_i is a probability space and Ω has the product measure, and let $h : \Omega \to R$ be a function. Assume that h is Lipschitz, that is, $|h(x) - h(y)| \le 1$ whenever x, y differ in at most one coordinate. For a function $f : N \to N, h$ is f-certifiable if whenever $h(x) \ge s$ there exists $I \subseteq \{1, \dots, p\}$ with $|I| \le f(s)$ so that for every $y \in \Omega$ that agrees with x on the coordinates I we have $h(y) \ge s$. Suppose that h is f-certifiable and let Y be the random variable given by $Y(x) = h(x)$ for $x \in \Omega$. Then for every b and t*

$$\mathsf{Prob}[Y \le b - t\sqrt{f(b)}] \cdot \mathsf{Prob}[Y \ge b] \le e^{-t^2/4}.$$

Returning to our proof, the variance of the random variable X satisfies

$$\mathsf{Var}(X) = \mathsf{Var}(\sum_K X_K) = \sum_K \mathsf{Var}(X_K) + \sum_{K,K'} \mathsf{Cov}(X_K, X_{K'})$$

where the summation is over all ordered pairs K, K' where $2 \le |K \cap K'| \le k-1$. For $|K \cap K'| \in \{0,1\}$, X_K and $X_{K'}$ are independent and thus have covariance zero. We have

$$\mathsf{Cov}(X_K, X_{K'}) = E(X_K X_{K'}) - E(X_K)E(X_{K'}) \le E(X_K X_{K'})$$

and since X_K is an indicator random variable, it holds

$$\mathsf{Var}(X_K) \le E(X_K).$$

It follows from the claim (5.2) that

$$\begin{aligned}
\mathsf{Var}(X) &= \sum_K \mathsf{Var}(X_K) + \sum_{K,K'} \mathsf{Cov}(X_K, X_{K'}) \\
&\leq \sum_K E(X_K) + \sum_{K,K'} E(X_K X_{K'}) \\
&= \mu + \Delta \qquad\qquad (\Delta = E(Z) = \sum_{K,K'} E(X_K X_{K'})) \\
&\leq \mu + \frac{k-2}{n^{0.48}}\mu \\
&\leq (1+o(1))\mu.
\end{aligned}$$

Therefore, by Chebyshev's inequality and the fact that μ is large, we have

$$\mathsf{Prob}[|X-\mu| \geq \frac{1}{4}\mu] \leq 16\frac{\mathsf{Var}(X)}{\mu^2} \leq 16\frac{1+o(1)}{\mu} \ll \frac{1}{4}.$$

Thus, $\mathsf{Prob}\left[X \in [\frac{3}{4}\mu, \frac{5}{4}\mu]\right] \gg \frac{3}{4}$ and the probability that $X \geq 3\mu/4$ is much bigger than 3/4. By Markov's inequality, we have

$$\mathsf{Prob}[\Delta \geq \frac{\mu}{4}] \leq \frac{4\Delta}{\mu} \ll \frac{1}{4} \implies \mathsf{Prob}[\Delta \leq \frac{\mu}{4}] \gg \frac{3}{4}.$$

Hence, both events happen simutaneously with probability greater than 1/2, that is, the number of copies of (H, c_H) in (G, c) is at least $3\mu/4$ and the number of pairs of copies of (H, c_H) in G with nontrivial intersection is smaller than $\mu/4$. If we remove one copy of (H, c_H) from each pair with a nontrivial intersection, then G contains a family of at least $\mu/2$ copies of (H, c_H) with no two having a nontrivial intersection.
Let $h((G, c))$ be the maximum cardinality of a family of copies of (H, c_H) in (G, c) in which no two members have a nontrivial intersection, and let Y be the random variablle $Y = h((G, c))$. We want to apply Talagrand's inequality to conclude our proof.
If the value of c changes for one edge of G, this change affects at most one copy of (H, c_H) in (G, c). Thus, the value of Y can change by at most 1 if we change the color of an edge in G and h is Lipschitz. Let L be a set of n labelled vertices. Consider the function $f(x) = x\binom{k}{2}$ and a multinomial random graph (U, c_U) on L. If $h((U, c_U)) \geq s$, let I be the set of edges from s copies of (H, c_H) in (U, c_U) in which no two members have a nontrivial intersection. Obviously, $|I| = s\binom{k}{2} \leq f(s)$. For every multinomial random graph (W, c_W) on L that agrees with (U, c_U) on I, i.e. $c_U(uv) = c_W(uv)$ for every $uv \in I$, we have $h((W, c_W)) \geq s$. Thus, h is f-certifiable for $f(s) = s\binom{k}{2}$. All the conditions for Talagrand's theorem are met, and we have with $b = \mu/2$ and $t = \sqrt{\mu}/k$

$$\begin{aligned}
&\mathsf{Prob}\left[Y \leq \frac{\mu}{2} - \frac{\sqrt{\mu}}{k}\sqrt{f(\frac{\mu}{2})}\right] \cdot \mathsf{Prob}\left[Y \geq \frac{\mu}{2}\right] \leq e^{-\mu/4k^2} \\
\implies &\mathsf{Prob}\left[Y \leq \frac{\mu}{2}\left(1 - \frac{\sqrt{k(k-1)}}{k}\right)\right] \cdot \mathsf{Prob}\left[Y \geq \frac{\mu}{2}\right] \leq e^{-\mu/4k^2}.
\end{aligned}$$

As we have seen above, the probability that $Y \geq \mu/2$ is greater than 1/2. We can conclude that the probability that $Y = 0$ is smaller than $e^{-\mu/4k^2}$ which is much smaller than r^{-k^2}. As $Y = 0$ if and only if there is no copy of (H, c_H) in (G, c), and as the total number of graphs in $\mathcal{H}(k, r)$ is smaller than $r^{\binom{k}{2}}$, we conclude that (G, c) is induced universal for $\mathcal{H}(k, r)$ with high probability.

5.2. Symmetric graphs

In this section, we use information on the structure of graphs with a lot of automorphisms to construct an adjacency labeling scheme for symmetric graphs. We recall that $\mathcal{T}(k,r)$ is the family of all symmetric complete graphs on k vertices with edges colored with the r colors $\{1,\ldots,r\}$.
Let S_p denote the symmetric group on $\{1,\ldots p\}$. For a permutation $h \in S_p$ define its support $supp(h)$ by

$$supp(h) = \{i \in \{1,\ldots,n\} : h(i) \neq i\}.$$

The minimal degree of a permutation group is the size of the minimum support of a nontrivial (non-identity) element of H. The following lemma states that large permutation groups may have nontrivial elements with small supports.

Lemma 5.7. *For any $p > 1$ and t, any subgroup S of size at least p^{4t} of the symmetric group S_p contains a permutation with at least t and at most $p - 3t$ fixed points.*

Proof. The subgroup S is a group of permutations of $[p]$. Consider all t-permutations of $[p]$ and for each ordering $T = (a_1,\ldots,a_t), a_i \in [p], i \in [t]$, the permutations in S satisfying $\sigma(i) = a_i$. There are $p(p-1)\cdots(p-t+1)$ t-permutations of $[s]$. By the pigeonhole principle, there is a t-element subset $A = \{a_1,\ldots,a_t\}$ of $[p]$ so that there are at least $\frac{|S|}{p(p-1)\cdots(p-t+1)} > p^{3t}$ permutations σ in S satisfying $\sigma(i) = a_i$ for all $i \in [t]$. For any two such permutations σ_1, σ_2, the product $\sigma_1\sigma_2^{-1}$ fixes all points of A. Let S' be the subgroup of S that fixes all points of A. Then $|S'| > p^{3t}$. The number of permutations in S' that fixes all points but at most i is at most $\binom{p-t}{i}i! < p^i$ (choose i non-fixed points in $[p] - A$). The number of permutations in S' that fixes all points of $[p]$ but at most $3t - 1$ is at most p^{3t-1}. However, we know that $|S'| > p^{3t} > p^{3t-1}$. Thus, there must be a permutation in S' that fixes at most $p - 3t$ points, i.e. a permutation with at least $3t$ non-fixed elements. $\square$

We recall that an edge colored graph on k vertices is asymmetric if no induced subgraph of it has more than k^{4m} automorphisms, where $m = 2\sqrt{k \log k}$. A graph is symmetric if it is not asymmetric. We denote by $\mathcal{T}(k,r)$ the family of all symmetric complete graphs on k vertices with edges colored with the r colors $\{1,\ldots,r\}$.

We provide the next Corollary and Lemma complete with Alon's proof of them.

Corollary 5.8. *Let (T,c) be a graph in $\mathcal{T}(k,r)$. Then there are three pairwise disjoint sets of vertices A, B, C of T, each of cardinality m, so that the following holds. There is a numbering of the elements of A, B, C : $A = \{a_1,\ldots,a_m\}, B = \{b_1,\ldots,b_m\}$ and $C = \{c_1,\ldots,c_m\}$ such that for any $1 \le i,j \le m$, we have $c(a_ib_j) = c(a_ic_j)$, i.e. the edge a_ib_j shares the same color with the edge a_ic_j.*

Proof. By the definition of $\mathcal{T}(k,r)$, there is an induced colored subgraph (T',c) of (T,c) on $p \le k$ vertices whose group of automorphisms S is of size at least $k^{4m} \ge p^{4m}$. By Lemma 5.7 this group contains a permutation σ with at least m and at most $p - 3m$ fixed points. Let $A = \{a_1,\ldots,a_m\}$ be m of these fixed points and consider the expression of σ as a product of nontrivial cycles. A cycle of length q contains q points. There are at least $3m$ non-fixed points. Thus, the total length of these cycles is at least $3m$. From each cycle $(w_1,w_2,\ldots,w_l)$ of length l, define $\lfloor l/2 \rfloor$ disjoint pairs

$$(w_1,w_2),(w_3,w_4),\ldots,(w_{2\lfloor l/2\rfloor-1}, w_{2\lfloor l/2\rfloor}).$$

Altogether we get at least m such pairs since we lose at most one point per cycle. Let $(b_ic_i), (1 \le i \le m)$ be m of them. Observe now that for every $1 \le j \le m$, σ maps b_j to c_j

and fixes all elements a_i. As σ is an automorphism of (T, c) this means that for every i, the edge a_ib_j has the same color as the edge a_ic_j. $\square$

Lemma 5.9. *Let T' be the graph obtained from a complete graph on k vertices by removing all edges of a complete bipartite graph $K_{m,m}$ in it, where, as before, $m = 2\sqrt{k \log_r k}$. Let $[k] = \{1, \dots, k\}$ be the set of vertices of T' and suppose $A' = \{1, \dots, m\}, B' = \{m+1, \dots, 2m\}, C' = \{2m+1, \dots, 3m\}$ and $D' = \{3m+1, \dots, k\}$, where there are no edges between A' and C' and all other pairs of vertices of T' are adjacent. Then there is an orientation T'' of the edges of T' in which all edges between A' and B' are oriented from A' to B' and the maximum outdegree of a vertex in T'' is at most $\frac{k}{2} - \frac{m^2}{k} + O(1) < \frac{k}{2} - 2\log k$.*

Proof. We need two facts.

- **Fact 1:** For any integer $g > 1$, the edges of the complete graph K_g on g vertices can be oriented so that every outdegree is at most $g/2$.
  We recall that an undireced graph has an Eulerian cycle if and only if every vertex has even degree, and all its vertices with nonzero degree belong to a single connected component. If g is odd, then K_g has an Eulerian orientation in which every outdegree is exactly $(g-1)/2$. If g is even, omit a vertex from an Eulerian orientation of K_{g+1} to get an orientation as needed.

- **Fact 2:** For any positive integers p, q and $s \le q$ there is a bipartite graph with classes of vertices P and Q of sizes p and q, respectively, so that every vertex of P has degree exactly s and every vertex of Q has degree either $\lfloor ps/q \rfloor$ or $\lceil ps/q \rceil$.
  To prove this fact, we number the vertices of $Q : u_1, u_2, \dots, u_q$ and connect, for each i, vertex number i of P to the vertices
  $$u_{(i-1)s+1}, u_{(i-1)s+2}, \dots, u_{is}$$
  where the indices are reduced modulo q.

We now construct an orientation T'' of T'. Using Fact 1, orient the edges of the complete graph on A' so that each outdegree is at most $m/2$, orient the edges of the complete graph on $B' \cup C'$ so that every outdegree is at most m, and orient the edges of the complete graph on D' so that every outdegree is at most $(k-3m)/2$. Orient all edges between A' and B' from A' to B'. By Fact 2, for any real $x \in (0, 1)$, the edges of the complete bipartite graph with vertex classes A' and D' can be oriented so that the outdegree of each vertex of A' is at most $s = x\,|D'| + 1 = x(k-3m) + 1$ and the outdegree of each vertex of D' is at most $m - \lfloor \frac{mr}{k-3m} \rfloor = (1-x)m + O(1)$. Similarly, for any real $y \in (0, 1)$, the edges of the complete bipartite graph with vertex classes $B' \cup C'$ and D' can be oriented so that the outdegree of each vertex of $B' \cup C'$ is at most $y(k-3m) + 1$ and the outdegree of each vertex of D' is at most $(1-y)2m + O(1)$. In the resulting orientation, the outdegrees of the vertices of A', $B' \cup C'$ and D' are bounded, up to absolutely bounded additive terms, by

$$\frac{3}{2}m + x(k-3m), \;\; m + y(k-3m) \;\; \text{and} \;\; \frac{k-3m}{2} + (1-x)m + (1-y)2m,$$

respectively. Since $m = o(k)$, there are $x, y \in (0, 1)$ so that these 3 quantities are equal. We have:

$$x = \frac{k^2 - 3km - 2m^2}{2k(k-3m)} = \frac{1}{2} - \frac{m^2}{k(k-3m)} - \frac{3m}{2(k-6m)},$$
$$y = \frac{k^2 - 2km - 2m^2}{2k(k-3m)} = \frac{1}{2} - \frac{m^2}{k(k-3m)} - \frac{m}{k-3m}.$$

With these x and y, all three quantities above are exactly $\frac{k}{2} - \frac{m^2}{k}$. Therefore, there is an orientation T'' in which all outdegrees are equal, up to an $O(1)$ additive error. The total number of edges is $\binom{k}{2} - m^2$. This implies that every outdegree in T'' is at most $\frac{k}{2} - \frac{m^2}{k} + O(1) = \frac{k}{2} - \frac{4k \log_r k}{k} + O(1) < \frac{k}{2} - 2\log_r k$. $\square$

We can now fully properly describe a labeling scheme for the family $\mathcal{T}(k,r)$. We start with the marker algorithm $\mathcal{M}$. Let $(T,c) \in \mathcal{T}(k,r)$. By Corollary 5.8, T contains three disjoint subsets of vertices $A = \{a_1, \ldots, a_m\}, B = \{b_1, \ldots, b_m\}$ and $C = \{c_1, \ldots, c_m\}$ such that for any $1 \leq i, j \leq m$, $c(a_i b_j) = c(a_i c_j)$. Number the vertices of T by the integers $1, 2, \ldots, k$ so that a_i gets the number i, b_i gets the number $m+i$, c_i gets the number $2m+i$ and the rest of the numbering is arbitrary. We want to apply Lemma 5.9 and construct an orientation T'' satisfying its condition. Let T' be the graph obtained from T by removing all edges between A and C. Then by Lemma 5.9, there is orientation T'' of the edges of T' in which all edges between A and B are oriented from A to B and the maximum outdegree of a vertex in T'' is at most $\frac{k}{2} - \frac{m^2}{k} + O(1)$. The label of vertex number i of (T,c) assigned by the marker algorithm $\mathcal{M}$ is the number i in binary followed by the binary representation of the color $c(j)$ of each outneighbor j of i in (T'',c), in order.
The decoder algorithm $\mathcal{D}$, which knows (T'',c), works as follows. Given the labels of two vertices u, v, if it is not the case that one of them lies in A and the other in C, then one of the labels contains the information about the color of the edge between the two vertices (since one of the vertices is an outneighbor of the other in (T'',c)). The decoder returns this color as output. If one vertex lies in A and the other in C, then one vertex is a_i and the other c_j. The label of a_i determines the color $c(a_i, b_j)$. By Corollary 5.8, we have $c(a_i, b_j) = c(a_i, c_j)$. The decoder returns $c(a_i, b_j)$ as output. Thus, $(\mathcal{M}, \mathcal{D})$ is a valid labeling scheme for $\mathcal{T}(k,r)$.
For the label of vertex i, we need $\log_2 k$ bits for the number i and $\log_2 r$ bits for the color of each outneighbor in (T'',c). The maximum outdegree of (T'',c) is $\frac{k}{2} - \frac{m^2}{k} + O(1) < \frac{k}{2} - 2\log_r k$. Thus, the length of each label is at most $\log_2 k + (k/2 - 2\log_r k)\log_2 r$ bit. It follows from Theorem 3.1 that T admits an induced universal graph with at most

$$2^{\log_2 k + (\frac{k}{2} - 2\log_r k)\log_2 r} = k \cdot r^{\frac{k}{2} - 2\log_r k} = \frac{1}{k} r^{k/2}$$

vertices.
This completes the proof of Theorem 5.2.

6. Conclusion

In this thesis, we surveyed a multitude of families of graphs, bounded degree-graphs, planar graphs, forests, infinite hereditary graphs just to name a few. For each of those families, we analyzed previous work on their induced-universal graphs. Table 6.1 provides an overview on lower and upper bounds on the minimum number of vertices of an induced-universal graph for various classes of graphs.

Table 6.1.: Bounds on the number of vertices of induced-universal graphs for various families of graphs. In the case of graphs of maximum degree d, d is a constant. In the result for families of graphs excluding a fixed minor, the $O(1)$ term in the exponent depends on the fixed minor excluded.

Graph family	**Lower bound**	**Upper bound**	**Reference**
General graphs	$2^{\frac{n-1}{2}}$	$O(2^{\frac{n-1}{2}})$	Moon [Moo65], Alon [Alo17]
Tournaments	$2^{\frac{n-1}{2}}$	$O(2^{\frac{n-1}{2}})$	Moon [Moo15], Alon [Alo17]
Bipartite graphs	$\Omega(2^{\frac{n}{4}})$	$O(2^{\frac{n}{4}})$	Alon [Alo17]
Graphs of max degree d, d even	$\Omega(n^{\frac{d}{2}})$	$O(n^{\frac{d}{2}})$	Butler [But09]
Graphs of max degree d, d odd	$\Omega(n^{\frac{d}{2}})$	$O(n^{\frac{d}{2}})$	Butler [But09], Alon and Nenadov [AN17]
Graphs of max degree 2	$\left\lceil \frac{11n}{6} \right\rceil$	$\left\lceil \frac{5n}{2} \right\rceil + O(1)$	Esperet, Labourel and Ochem [ELO08]
Graphs excluding a fixed minor	$\Omega(n)$	$n^2(logn)^{O(1)}$	Gavoille and Labourel [GL07]
Planar graphs	$\Omega(n)$	$O(n^{1+o(1)})$	Dujmović, Esperet, Gavoille, Joret, Micek and Morin [DEG$^+$20]
Outerplanar graphs	$\Omega(n)$	$O(n^{1+o(1)})$	Dujmović, Esperet, Gavoille, Joret, Micek and Morin [DEG$^+$20]
Outerplanar graphs of bounded degree	$\Omega(n)$	$O(n)$	Chung [Chu90]
Graphs of treewidth k	$n2^{\Omega(k)}$	$n(log\frac{n}{k})^{O(k)}$	Gavoille and Labourel [GL07]
Graphs of arboricity k	$\frac{n^k}{2^{O(k^2)}}$	$O(n^k)$	Alstrup, Dahlgaard and Knudsen [ADK17]
Forests	$\Omega(n)$	$O(n)$	Alstrup, Dahlgaard and Knudsen [ADK17]

In Chapter 5, we provided a detailed proof of Alon's suggestion that his work on undirected graphs can be expanded to edge-colored complete graphs, yielding an upper bound of $(1 + o(1)) \cdot r^{(k-1)/2}$. To this purpose, we introduced the concept of multinomial random graphs and showed how to adapt each theorem and lemma from Alon's paper.

Throughout this thesis, we saw that there is three main groups of thinking in approaching induced-universal graphs (or universal graphs for that matter). We can try to provide a very explicit construction of the induced-universal graph and then prove that every graph in the family we are investigating appear as an induced subgraph in our induced-universal graph. We also can try a probabilistic approach by showing that a particular graph is induced-universal with high probability. Lastly, we can try to implicit generate the graph by providing a labeling scheme for the family at hand with as few bits as possible.

A few questions arose throughout the thesis.

- It is easy to that Butler's construction in Section 4.1 can be extrapolated to edge-colored complete graphs where each color defines a graph with bounded maximum degree.
  Let $\mathcal{H}$ be a family of complete graphs on n vertices whose edges are colored by k colors $\{1, \ldots, k\}$. If each color $i \in \{1, \ldots, k-1\}$ defines a family with bounded maximum degree. Obviously, Petersen's 2-factor theorem is no longer applicable for the graph defined by color k. Then what bounds does $g_v(\mathcal{H})$ satisfy?

- Alon's proof in Chapter 5 sets requirements for a few variables. For example, r is fixed, k is large, m depends on k. We saw that k indeed has to be large for the proof to work. A graph can only be asymmetric if $k! > k^{4m}$. This inequality hold for $k = 1$ and very large k. Also if we change m, we may break a few components of the proof. For example, Lemma 5.9 can only hold if m is very small compared to k. What can we say if we let r be a function of k?

Bibliography

[AA02] Noga Alon and Vera Asodi. Sparse universal graphs. *Journal of Computational and Applied Mathematics*, 142(1):1–11, 2002.

[AC07] Noga Alon and Michael Capalbo. Sparse universal graphs for bounded-degree graphs. *Random Structures & Algorithms*, 31(2):123–133, 2007.

[ACJS19] Hüseyin Acan, Sankardeep Chakraborty, Seungbum Jo, and Srinivasa Rao Satti. Succinct data structures for families of interval graphs. In *Workshop on Algorithms and Data Structures*, pages 1–13. Springer, 2019.

[ACK+00] Noga Alon, Michael Capalbo, Yoshiharu Kohayakawa, Vojtech Rodl, Andrzej Rucinski, and Endre Szemerédi. Universality and tolerance. In *Proceedings 41st Annual Symposium on Foundations of Computer Science*, pages 14–21. IEEE, 2000.

[ACK+01] Noga Alon, Michael Capalbo, Yoshiharu Kohayakawa, Vojtěch Rödl, Andrzej Ruciński, and Endre Szemerédi. Near-optimum universal graphs for graphs with bounded degrees. In *Approximation, Randomization, and Combinatorial Optimization: Algorithms and Techniques*, pages 170–180. Springer, 2001.

[ADK17] Stephen Alstrup, Søren Dahlgaard, and Mathias Bæk Tejs Knudsen. Optimal induced universal graphs and adjacency labeling for trees. *Journal of the ACM (JACM)*, 64(4):1–22, 2017.

[AKTZ15] Stephen Alstrup, Haim Kaplan, Mikkel Thorup, and Uri Zwick. Adjacency labeling schemes and induced-universal graphs. In *Proceedings of the forty-seventh annual ACM symposium on Theory of Computing*, pages 625–634, 2015.

[Ale92] Vladimir Evgen'evich Alekseev. Range of values of entropy of hereditary classes of graphs. *Diskretnaya Matematika*, 4(2):148–157, 1992.

[Ale97] Vladimir Evgen'evich Alekseev. On lower layers of a lattice of hereditary classes of graphs. *Diskretnyi Analiz i Issledovanie Operatsii*, 4(1):3–12, 1997.

[Alo17] Noga Alon. Asymptotically optimal induced universal graphs. *Geometric and Functional Analysis*, 27(1):1–32, 2017.

[AN17] Noga Alon and Rajko Nenadov. Optimal induced universal graphs for bounded-degree graphs. In *Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1149–1157. SIAM, 2017.

[BGL07] Nicolas Bonichon, Cyril Gavoille, and Arnaud Labourel. Short labels by traversal and jumping. *Electronic Notes in Discrete Mathematics*, 28:153–160, 2007.

[BK93] Graham R Brightwell and Yoshiharu Kohayakawa. Ramsey properties of orientations of graphs. *Random Structures & Algorithms*, 4(4):413–428, 1993.

[BT81] Béla Bollobás and Andrew Thomason. Graphs which contain all small graphs. *European Journal of Combinatorics*, 2(1):13–15, 1981.

[But09] Steve Butler. Induced-universal graphs for graphs with bounded maximum degree. *Graphs and Combinatorics*, 25(4):461–468, 2009.

[Cap02] Michael Capalbo. Small universal graphs for bounded-degree planar graphs. *Combinatorica*, 22(3):345–359, 2002.

[Chu90] Fan RK Chung. Universal graphs and induced-universal graphs. *Journal of Graph Theory*, 14(4):443–454, 1990.

[CK99] Michael R Capalbo and S Rao Kosaraju. Small universal graphs. In *Proceedings of the thirty-first annual ACM symposium on Theory of computing*, pages 741–749, 1999.

[CRS83] FRK Chung, AL Rosenberg, and Lawrence Snyder. Perfect storage representations for families of data structures. *SIAM Journal on Algebraic Discrete Methods*, 4(4):548–565, 1983.

[DEG+20] Vida Dujmović, Louis Esperet, Cyril Gavoille, Gwenaël Joret, Piotr Micek, and Pat Morin. Adjacency labelling for planar graphs (and beyond). In *2020 IEEE 61st Annual Symposium on Foundations of Computer Science (FOCS)*, pages 577–588. IEEE, 2020.

[DJM+20] Vida Dujmović, Gwenaël Joret, Piotr Micek, Pat Morin, Torsten Ueckerdt, and David R Wood. Planar graphs have bounded queue-number. *Journal of the ACM (JACM)*, 67(4):1–38, 2020.

[EJM20] Louis Esperet, Gwenaël Joret, and Pat Morin. Sparse universal graphs for planarity. *arXiv preprint arXiv:2010.05779*, 2020.

[ELO08] Louis Esperet, Arnaud Labourel, and Pascal Ochem. On induced-universal graphs for the class of bounded-degree graphs. *Information Processing Letters*, 108(5):255–260, 2008.

[FH77] Stéphane Foldes and Peter L Hammer. Split graphs having dilworth number two. *Canadian Journal of Mathematics*, 29(3):666–672, 1977.

[GL07] Cyril Gavoille and Arnaud Labourel. Shorter implicit representation for planar graphs and bounded treewidth graphs. In *European Symposium on Algorithms*, pages 582–593. Springer, 2007.

[Gon06] Daniel Gonçalves. *Etudes de différents problèmes de partition de graphes*. PhD thesis, Bordeaux 1, 2006.

[Gon09] Daniel Gonçalves. Covering planar graphs with forests, one having bounded maximum degree. *Journal of Combinatorial Theory, Series B*, 99(2):314–322, 2009.

[GP08] Cyril Gavoille and Christophe Paul. Optimal distance labeling for interval graphs and related graph families. *SIAM Journal on Discrete Mathematics*, 22(3):1239–1258, 2008.

[JLR11] Svante Janson, Tomasz Luczak, and Andrzej Rucinski. *Random graphs*, volume 45. John Wiley & Sons, 2011.

[KNR92] Sampath Kannan, Moni Naor, and Steven Rudich. Implicat representation of graphs. *SIAM Journal on Discrete Mathematics*, 5(4):596–603, 1992.

[KRSS11] Yoshiharu Kohayakawa, Vojtěch Rödl, Mathias Schacht, and Endre Szemerédi. Sparse partition universal graphs for graphs of bounded degree. *Advances in Mathematics*, 226(6):5041–5065, 2011.

[LR07] Vadim V Lozin and Gábor Rudolf. Minimal universal bipartite graphs. *Ars Combinatoria*, 84:345–356, 2007.

[Moo65] JW Moon. On minimal n-universal graphs. *Glasgow Mathematical Journal*, 7(1):32–33, 1965.

[Moo15] John W Moon. *Topics on tournaments in graph theory*. Courier Dover Publications, 2015.

[Pel99] David Peleg. Proximity-preserving labeling schemes and their applications. In *International Workshop on Graph-Theoretic Concepts in Computer Science*, pages 30–41. Springer, 1999.

[Pel00] David Peleg. Informative labeling schemes for graphs. In *International Symposium on Mathematical Foundations of Computer Science*, pages 579–588. Springer, 2000.

[Rad64] Richard Rado. Universal graphs and universal functions. *Acta Arithmetica*, 4(9):331–340, 1964.

[RLDL94] C Rhee, Y Daniel Liang, Sudarshan K. Dhall, and S Lakshmivarahan. Efficient algorithms for finding depth-first and breadth-first search trees in permutation graphs. *Information Processing Letters*, 49(1):45–50, 1994.

[Ros71] Arnold L Rosenberg. Data graphs and addressing schemes. *Journal of Computer and System Sciences*, 5(3):193–238, 1971.

[SZ94] Edward R Scheinerman and Jennifer Zito. On the size of hereditary classes of graphs. *Journal of Combinatorial Theory, Series B*, 61(1):16–39, 1994.

Appendix

A. Appendix Section 1