Edge Densities and The Density Formula

Master's Thesis of

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I declare that I have developed and written the enclosed thesis completely by myself. I have not used any other than the aids that I have mentioned. I have marked all parts of the thesis that I have included from referenced literature, either in their original wording or paraphrasing their contents. I have followed the by-laws to implement scientific integrity at KIT.

Karlsruhe, September 30, 2024

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(Benedikt Hahn)

Abstract

Recently, Kaufmann et al. introduced the "Density Formula". It relates edges, vertices, crossings, and cells in drawings of graphs. They use it to unify edge-density proofs of a wide range of beyond-planar graph classes — whose members can be drawn while satisfying some property — into a single framework. Their proofs of earlier known results are comparatively streamlined and for some classes they are able to improve on the previous best bounds. However, in some cases, their results rely on a slight assumption on the drawings. We prove that this assumption can be lifted for certain classes, strengthening the Density Formula results for 1-planar, 2-planar and quasiplanar graphs to match the most general bounds from the literature.

Further, we consider *outer* variants of beyond-planar graph classes, in which drawings are restricted so that every vertex is incident to the outer cell. We derive edge density bounds of ten outer classes, of which seven are new and all but two are tight. All but one of our bounds are derived using a simple framework for translating Density Formula results into the outer setting.

We also introduce new topological restrictions — forbidden cell types — for drawings. We study the edge densities of the resulting graph classes and establish upper bounds that are tight up to lower-order terms for all but three cell types.

Finally, to show the optimality of our bounds, we provide many lower bound constructions. In addition, we give a lower bound for the edge density of simple quasiplanar graphs, improving the previous best bound in the leading coefficient.

Zusammenfassung

Vor kurzem haben Kaufmann et al. die "Density Formula" vorgestellt, die Knoten, Kanten, Kreuzungen und Zellen in Zeichnungen von Graphen in Zusammenhang bringt. Diese verwenden sie um eine Vielzahl Ergbenisse zu Kantendichten von "Beyond-Planar" Graphklassen — deren Graphen Zeichnungen mit gewissen Eigenschaften zulassen — mit einer einheitlichen Methode zu beweisen. Die resultierenden Beweise sind vergleichsweise kurz und für mehrere Klassen verbessern die Autoren die bisher besten Schranken. In einigen Fällen benötigen die Autoren jedoch eine Annahme an die Zeichnungen der Graphen. Wir zeigen, dass diese Annahme für viele Klassen aufgehoben werden kann. Damit verallgemeinern wir die mit der Density Formula erzielten Ergebnisse für 1-planare, 2-planare und quasiplanare Graphen und gleichen sie somit an die allgemeinsten in der Literatur bekannten Schranken an.

Außerdem betrachten wir *äußere* Varianten von Beyond-Planar Graphklassen, in denen in Zeichnungen gefordert wird, dass jeder Knoten zur äußeren Facette inzident ist. Wir bestimmen obere Schranken an die Kantendichten von zehn solcher Klassen. Von diesen sind sieben neu und alle bis auf zwei scharf. Fast alle unserer Schranken wurden mit einer einfachen Methode bestimmt, die es erlaubt Ergebnisse die mit der Density Formula bewiesen wurden auf äußere Varianten zu übertragen.

Weiterhin führen wir neue topologische Einschränkungen — verbotene Zelltypen — für Zeichnungen ein. Wir untersuchen die Kantendichten der abgeleiteten Graphklassen und bestimmen für alle außer drei dieser Klassen obere Schranken die scharf bis auf Terme niedrigerer Ordnung sind.

Um die Optimalität unserer oberer Schranken zu belegen, präsentieren wir außerdem eine Vielzahl Konstruktionen für untere Schranken. Zusätzlich zeigen wir eine untere Schranke für die Kantendichten einfacher quasiplanarer Graphen, mit der wir die bisher beste Schranke im Leitkoeffizient verbessern.

Contents

1 Introduction

When dealing with relational data, connections between concepts, or complex systems in general, getting a good grasp on the underlying structure of the system is often paramount. To this end, visualizations in the form of drawings of graphs are often employed. While this can be done successfully, using a drawing that is hard to take in visually or even conceals some important structure in the underlying graph, can lessen its positive effect. For example, cognitive scientists have established evidence that the number of crossings and their features, e.g. the size of angles at crossings, impact the effectiveness of a visualization [Pur00 , WPCM02 , HEH14 , Hua07]. Usually, finding a drawing of a graph that optimizes just one of the relevant criteria is computationally difficult. For example, determining the minimum number of crossings needed to draw a graph is NP-hard in general [GJ83] and even for complete graphs, these numbers are unknown (see e.g. [Buc+13]).

Instead of minimizing some criterion, an undesired configuration in a drawing can also be forbidden entirely. For example, graphs which admit drawings without crossings are called *planar*. This family of graphs has been studied extensively and shows a wide range of structural properties. However, there are also non-planar graphs that admit easily understandable drawings. Thus, many other, less restrictive forbidden configurations were considered, especially in recent years. As these forbidden patterns give rise to graph classes that usually contain planar graphs, they are often called *beyond-planar graph classes* (for an overview see [DLM20]). One example are *quasiplanar graphs*, in which no three edges are permitted to pairwise intersect.

One of the key features of a beyond-planar graph class is the maximum number of edges its member graphs can contain when the number of vertices is fixed. This is the class' *edge density*. Intuitively, it measures the complexity of graphs that can be drawn without violating the forbidden pattern. Thus, knowing the edge density of a beyond-planar graph class, one has a simple condition for when a graph can not be visualized while avoiding the forbidden pattern. However, for many beyond-planar graph classes, the edge densities are still not fully understood.

Recently, the "Density Formula" was introduced in [Kau+24]. With it, the authors reprove and improve on many bounds for edge densities of beyond-planar graph classes using a single framework. This is in stark contrast to the previous state of edge density proofs, which were distinctly non-unified (see for example [Ari+12 , Bin+24 , PT97 , Tót23]). Part of this thesis aims to integrate further edge density results into the Density Formula approach.

1.1 Contribution

Upper bounds proven via the Density Formula often rely on the assumption that drawings are "non-homotopic". We show that this assumption can be lifted when the graph class is defined via a "forbidden combinatorial crossing configuration" and apply this to generalize multiple edge density results to arbitrary drawings.

We introduce a new method based on the Density Formula for translating established edge density results of beyond-planar graph classes to their *outer* variants in which every vertex is incident to the outer cell. We demonstrate its effectiveness by proving nine upper bounds for outer variants, of which seven are tight and only two were previously known. Via a different method, we also determine the exact edge density of outer quasiplanar graphs. Further, we suggest a new collection of graph classes defined by forbidden cell types (topological patterns) in drawings. We study their edge densities and for all but three of them either establish that the edge density is quadratic or provide upper bound proofs that are tight up to an additive constant. Finally, we provide many tight lower bound constructions for the above families and additionally for simple quasiplanar graphs, improving the density in the leading coefficient.

1.2 Related Work

Apart from outerplanar graphs, the first outer drawing class was considered in the 80s by Eggleton [Egg86] (although we were unable to access the work and confirm this), who gives structural properties of graphs that admit an outer drawing with at most one crossing per edge (called outer *1-planar*). Their density was determined in [Aue+16] where many further algorithmic and structural properties of outer 1-planar graphs are explored. Further, in contrast to unrestricted 1-planar and k-planar graphs, whose recognition is NP-hard [GB07 , UW21], both outer 1-planarity and outer 2-planarity can be tested for in linear time [Hon+15 , HN19].

In general, it is common that a hard problem in a beyond-planar graph class is easier in its outer variant. Another example of this is for k-quasiplanar graphs which can be drawn such that no set of k edges cross pairwise. It is conjectured that for any k, there is some c_k such that the edge density of k-quasiplanar graphs is at most $c_k n$. This was first posed as a question by Gärtner in a paper of Pach [Pac91] but is only proven for $k \leq 4$ [AT07 , Ack09]. For outer k-quasiplanar graphs, however, Capolyeas and Pach [CP92] were able to confirm this by deriving the exact densities. Other outer variants whose edge densities were considered include k^+-real face graphs [Bin+24] and fan-planar graphs [Bin+15].

Other edge density results for beyond-planar graph classes that are not superseded by [Kau+24] are found for example in [ZL13] for IC graphs, in [Zha14] for NIC graphs and in [CHKK15] for fan-crossing free graphs. For bipartite variants, the edge density of 1-planar graphs can be found in [Kar14] and a treatment of multiple other classes is given in [Ang+19]. Another variant commonly considered is straight-line drawings. The edge density of straight-line RAC graphs (edges may only cross at right angles) is discussed in [DEL11]. Drawings of 1-planar graphs with only straight lines and their density are considered in [Did13].

1.3 Outline

After introducing some basic notions necessary for this thesis in Chapter 2, we generalize some edge density results for non-homotopic drawings to an arbitrary setting in Chapter 3. Afterwards, we present multiple edge density results for various outer variants of beyond-planar graph classes in Chapter 4. We give an improved lower bound for simple quasiplanar graphs in Chapter 5. In Chapter 6 we consider edge densities of drawings in which a single cell type is forbidden. Finally, in Chapter 7, we discuss questions left unanswered by this thesis and give ideas for further work.

2 Preliminaries

This chapter gives an introduction to notions from graph theory, combinatorial geometry, and graph drawing that are necessary for this thesis.

2.1 Graph Theory

A *(simple) graph* $G = (V, E)$ consists of a ground set $V =: V(G)$ of *vertices*, and a set of *edges* $E =: E(G) \subseteq \binom{V}{2}$. For us, the vertex set V will be finite and we consider graphs up to isomorphism, i.e., relabelling of vertices. Thus, we will assume $V = [n]$ and call G an n-vertex graph. An edge *connects* two vertices and is *incident* to them. Two vertices connected by an edge are called *adjacent* and edges incident to a common vertex are also called *incident* and otherwise *independent*. For brevity, we will write uv for an edge $\{u, v\}$. However, note that edges are undirected so $uv = vu$.

At multiple points in the thesis, we will talk about graphs that differ from the above definition. That is, we will allow multiple edges connecting the same two vertices — called *parallel edges*. We call a graph in which parallel edges are allowed a *multigraph* or *non-simple*. Formally, the edge set of a multigraph can be defined as a multiset of elements in $\binom{V}{2}$. Whenever considering multigraphs, we make the distinction to simple graphs explicit. However, if not specified, such a distinction is not made and our graphs are assumed to be simple. Since many of our results carry over from [Kau+24], let us make explicit that "graphs" in their terminology are "multigraphs" in ours.

For some vertex v in a graph G, $\deg(v)$ is the *degree* of v, i.e. the number of edges it is incident to. Further, $\delta(G)$ is the minimum degree of G. A basic theorem we employ a few times throughout the thesis is the Handshake Lemma. It can be easily proven by double-counting the edge-vertex incidences in a graph.

Theorem (Handshake Lemma)**:** *In any multigraph $G = (V, E)$,*

$$|E| = \frac{1}{2} \sum_{v \in V} deg(v).$$

A *graph family* or *family (of graphs)* $\mathcal{F}$ is a set of graphs. The *(edge) density* of such a family is the function

$$\begin{aligned} f : \mathbb{N} &\to \mathbb{N} \\ n &\mapsto \max_{\substack{G \in \mathcal{F} \\ |V(G)| = n}} |E(G)|. \end{aligned}$$

In our theorems, we will be precise for which n an upper or lower bound on an edge density holds. However, when we say "The edge density of [...] graphs is f", this is usually meant less formally. We allow ourselves to omit that f is not a valid upper bound for small n and only for infinitely many n (and not all) are we aware of a lower bound construction on n vertices and $f(n)$ edges. Thus, while not fully precise, such assertions still explain the general behaviour of the edge density.

2.2 Graph Drawing

An *(arbitrary) drawing* Γ of a graph $G = (V, E)$ is formally a map

$$\Gamma : V \to \mathbb{R}^2,$$
$$\Gamma : E \to \mathcal{C}^0([0,1], \mathbb{R}^2),$$

from vertices to points and edges to continuous paths in the plane. Let $\Gamma(G) \coloneqq \cup_{v \in V} \Gamma(v) \cup_{e \in E} \Gamma(e)([0,1]) \subseteq \mathbb{R}^2$ be the *image* of Γ. A point $p \in \mathbb{R}^2$ for which two edges $e_1, e_2 \in E$ and real numbers $t_1, t_2 \in (0,1)$ exist such that $(e_1, t_1) \neq (e_2, t_2)$ and $\Gamma(e_1)(t_1) = \Gamma(e_2)(t_2)$ is called a *crossing*. We say that e_1 and e_2 *cross*. The set of crossings is denoted as $\mathcal{X}(\Gamma)$ or just $\mathcal{X}$ if the drawing is clear from context. For any drawing, we require that

- Γ is injective on V.
- for any edge $e \in E$, $\Gamma(e) : [0,1] \to \mathbb{R}^2$ is injective.
- for any $uv = e \in E$, $\Gamma(e)(0) = \Gamma(u)$ and $\Gamma(e)(1) = \Gamma(v)$ or vice versa.
- for any edge $e \in E$ and any $t \in (0,1)$, $\Gamma(e)(t) \neq \Gamma(v)$ for any vertex $v \in V$.
- at any crossing p there are exactly two distinct edge-time pairs $(e_1, t_1), (e_2, t_2)$ with $\Gamma(e_1)(t_1) = \Gamma(e_2)(t_2) = p$.
- there are only finitely many crossings.
- for any $e_1, e_2 \in E$, $t_1, t_2 \in (0,1)$ with $(e_1, t_1) \neq (e_2, t_2)$ and $p \coloneqq \Gamma(e_1)(t_1) = \Gamma(e_2)(t_2)$, there is some neighbourhood $U_p \subseteq \mathbb{R}^2$ of p such that $X \coloneqq U_p \cap \Gamma(G) = \Gamma(e_1)(I_1) \cup \Gamma(e_2)(I_2)$ for some intervals $I_1, I_2 \subseteq (0,1)$, X is homeomorphic to a "cross" $[-1,1] \times \{0\} \cup \{0\} \times [-1,1]$ and the ends of the cross alternately belong to e_1 and e_2. That is, when two edges touch, they also have to cross.
- every image of an edge under Γ is "well-behaved" as a curve. That is, it has finite length, positive but bounded speed at every point, its image has no accumulation point outside the image, etc.

We will often make no clear distinction between a drawing and the underlying graph. For example, we will talk about the edge of a drawing to mean either the edge of the underlying graph, the respective curve in $\mathbb{R}^2$, or its image.

In most settings, we are only interested in the combinatorial properties of a drawing, e.g. which edges cross another and in which order the crossings appear along an edge. Thus, using a stereographic projection, we can alternatively assume that a drawing maps onto a sphere $\mathcal{S}^2$ instead of $\mathbb{R}^2$.

The crossings along an edge in a drawing split it into *edge segments*. The first and last edge segments are called *outer* and any other edge segment is *inner*. Denote by $\mathcal{S}$ the set of all edge segments. An edge is *planar* if it has no crossing in Γ. Let $E_p \subseteq E$ be the set of planar edges and $E_x = E \setminus E_p$ the crossed edges. If $E_x = \emptyset$, the drawing is called *plane*. A graph that admits a plane drawing is called *planar*.

A *cell* in a drawing Γ of a graph G is a connected component of $\mathbb{R}^2 \setminus \Gamma(G)$. We call the unique unbounded component the *outer cell* of Γ. A drawing is *outer* if and only if every vertex lies on the boundary of the outer cell. If Γ is plane, cells are also called *faces*. For a cell c, its *boundary* ∂c is made up of closed curves, each of which is a sequence of edge segments

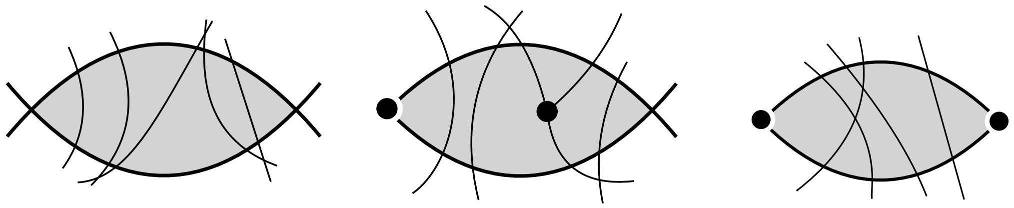


Figure 2.1: The three types of lenses in drawings of multigraphs. The first lens is empty and the others are not, since they contain crossings and vertices.

separated by crossings and vertices, which are *incident* to c. We call these sequences the *(boundary) chains* of the cell. Note that if Γ is connected, each cell is only bounded by a single chain. Note further that the same edge segment or vertex can occur multiple times in the same chain in which case we call the cell *degenerate*. For a cell c we denote by $e(c)$ and $v(c)$ the number of edge segments and vertices on ∂c counted with multiplicity. The *size* of a cell c is $\|c\| \coloneqq v(c) + e(c)$. By $\mathcal{C}_k, \mathcal{C}_{\geq k}$ we denote the set of all cells of size k and at least k in a drawing. We emphasize that with our definition, a crossing does not count towards the size of a cell.

We say that two cells are of the same *type* if their number of boundary chains and the order of vertex- and crossing-incidences thereon coincide. Note that when talking about the type of a cell, we "forget" when multiple vertex-, crossing- or edge segment-incidences are with the same vertex, crossing or edge segment. We denote a cell type by a pictogram in which we inscribe its size, e.g. , , and . While these only depict inner cells, they can also describe the type of the outer cell. It turns out that in connected drawings on at least three vertices, there are only a few small cell types.

Observation 2.1: *In any drawing of a simple connected graph on at least 3 vertices*

- $\mathcal{C}_1 = \emptyset$,
- $\mathcal{C}_2$ *is the set of -cells,*
- $\mathcal{C}_3$ *is the set of - and -cells,*
- $\mathcal{C}_4$ *is the set of -, and -cells,*
- $\mathcal{C}_5$ *is the set of -, -, and -cells.*

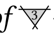
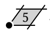
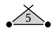
If no cells of type $\mathfrak{c}$ occur in a drawing, we call it $\mathfrak{c}$*-free*. Further, a graph that admits a $\mathfrak{c}$-free drawing is also called $\mathfrak{c}$*-free*.

2.2.1 Types of drawings

Often, we further limit the previously defined arbitrary drawings to two increasingly restrictive drawing styles. These restrictions concern *lenses*. A lens is a region bounded by a closed curve that can be split into two parts, each being part of a single edge. In other words, a lens is a region formed by two edges with two intersections, illustrated in Figure 2.1. We call a lens empty if its interior contains neither a crossing nor a vertex.

A drawing is *non-homotopic* if it contains no empty lenses. Thus, - and -cells do not occur in non-homotopic drawings.

Observation 2.2: *In any non-homotopic drawing of a connected multigraph on at least 3 vertices*

- $\mathcal{C}_1 = \mathcal{C}_2 = \emptyset$,
- $\mathcal{C}_3$ *is the set of*  *-cells,*
- $\mathcal{C}_4$ *is the set of*  *-, and*  *-cells,*
- $\mathcal{C}_5$ *is the set of*  *-,*  *-, and*  *-cells.*

By contrast, a drawing is *simple* if it does not contain any lens. Thus, the underlying graph of any simple drawing has no parallel edges and is itself simple. We will also use the adjective non-homotopic for a graph to denote that a non-homotopic drawing of the graph with certain properties exists. For example, a non-homotopic quasiplanar graph admits a non-homotopic quasiplanar drawing. For the adjective simple, we refrain from similar usage to avoid confusion with simple graphs. Let us remark that had we not forbidden self-intersection of edges already for arbitrary drawings, we would exclude them explicitly for non-homotopic and simple drawings.

In graph drawing, one often also considers some geometric restrictions to the way edges and vertices are drawn. We will not encounter this frequently in the present thesis, so there are only a few definitions we need. A drawing is called *straight-line* if every edge is drawn as a straight line segment. If all vertices in a drawing lie on a common circle, we call it a *circular layout*. In a k*-bend drawing* each edge is a polyline with at most k bends. That is, each edge can be divided into at most $k + 1$ parts such that each part is a straight-line segment.

2.2.2 Beyond-planar graph classes

Frequently in Graph Drawing, one is interested in drawings that satisfy a specific property P. In this context, it is useful to ask which graphs even admit a drawing that adheres to P. We call the family of graphs that can be drawn while satisfying P a *beyond-planar graph class*. This is justified because P will usually be a forbidden crossing pattern or restrict some property of crossings, in which case the resulting family is a superset of planar graphs. We now define all beyond-planar graph classes that appear in this thesis. We do this implicitly by only presenting the relevant properties for drawings. The names will always carry over to graphs and the relevant graph classes.

A drawing is k*-quasiplanar* if there are no k edges which cross pairwise. So, 2-quasiplanarity is the usual planarity-restriction. We will only consider 3-quasiplanar graphs in this thesis, which we henceforth call *quasiplanar*. A drawing is k*-planar* if there are at most k crossings on each edge. Thus, 0-planar drawings are just planar drawings. We call a drawing k^+*-face real* if every cell is incident to at least k vertices. A drawing is k*-bend RAC (Right Angle Crossing)* if it is k-bend, crossings do not occur on bends and all crossings are at a right angle. We call a drawing *fan-crossing* if for every edge e, the edges crossing e form a star (i.e. are incident to a common vertex) in the underlying graph. In a *fan-crossing free* drawing on the other hand, if an edge e crosses edges f_1 and f_2 then f_1 and f_2 are independent in the underlying graph.

2.3 The Density Formula

The *Density Formula* is a tool introduced by Kaufmann et al. [Kau+24] which will be useful to us for proving edge density bounds of beyond-planar graph classes.

Theorem (Density Formula)**:** *Let t be a real number and Γ some connected drawing of a multigraph $G = (V, E)$ with at least one edge. Then,*

$$|E| = t(|V| - 2) - \sum_{c \in \mathcal{C}} \left(\frac{t-1}{4} \, \|c\| - t \right) - |\mathcal{X}|.$$

The authors most often use the Density Formula by choosing some t, then upper bounding the positive terms in the sum by the number of crossings and ignoring the negative terms. For example, for $t = 3, 4, 5$, one obtains the following inequalities.

Observation 2.3: *For any connected drawing of a multigraph $G = (V, E)$ with at least one edge*

$$|E| \leq 3(|V| - 2) + \frac{3}{2}|\mathcal{C}_3| + |\mathcal{C}_4| + \frac{1}{2}|\mathcal{C}_5| - |\mathcal{X}| \tag{2.1}$$

$$|E| \leq 4(|V| - 2) + \frac{7}{4}|\mathcal{C}_3| + |\mathcal{C}_4| + \frac{1}{4}|\mathcal{C}_5| - |\mathcal{X}| \tag{2.2}$$

$$|E| \leq 5(|V| - 2) + 2|\mathcal{C}_3| + |\mathcal{C}_4| - |\mathcal{X}|. \tag{2.3}$$

For many beyond-planar graph classes, these inequalities can be used to find upper bounds on their edge densities by relating the number of cells of different sizes and the number of crossings. We exploit this proof structure in Chapter 4 to translate results from Kaufmann et al. to the outer setting.

3 Generalizing results

In this chapter, we inspect the relationship between arbitrary and non-homotopic drawing variants of beyond-planar graph classes. We find that under some assumptions on the property P, the family of graphs that admit an arbitrary drawing satisfying P coincides with those graphs with a non-homotopic drawing that satisfies P. We obtain the same result when further restricting to outer drawings. Recall that a drawing is outer if and only if every vertex is incident to the outer cell. At the end of the chapter, we apply these results to translate edge density proofs for non-homotopic drawings that are obtained via the Density Formula to arbitrary drawings.

Definition 3.1: *Let P be a property of drawings. For some drawing Γ of a graph $G = (V, E)$, define the multiset*

$$C_\Gamma \coloneqq \{k \cdot \{e, f\} \mid e, f \in E \text{ cross } k \text{ times in } \Gamma\}$$

of size $|\mathcal{X}|$. Here, $k \cdot \{e, f\}$ denotes that C_Γ contains the set $\{e, f\}$ k times. We call P a forbidden combinatorial crossing configuration *(*fccc *for short) if it has the following properties.*

1. *Whether P holds for Γ only depends on G and C_Γ.*
2. *Let two graphs $G = (V, E)$ and $G = (V, E')$ with $E \subset E'$ be drawn in Γ, Γ' such that $C_{\Gamma'} \subset C_\Gamma$. Then, if P holds for Γ, it must also hold for Γ'. In other words, P is closed under adding planar edges and removing crossings.*

The majority of beyond-planar graph classes we introduced in Chapter 2 are defined via a fccc. Other classes like k-bend RAC and k^+-real face are inherently topological and therefore do not fulfil 1. An example of a property that fulfils 1 but not 2 is "any two edges cross an even number of times".

Observation 3.2: *Let $k \in \mathbb{N}_0$ be any natural number. The drawing properties k-planar, k-quasiplanar, fan-crossing, and fan-crossing free are forbidden combinatorial crossing configurations. On the other hand, k^+-real face for $k \geq 1$ and k-bend RAC are not fcccs.*

Proof. Let Γ be some drawing of a graph G and define for an edge $e \in E(G)$ the set of all its crossing edges $E_e \coloneqq \{e' \in E(G) \mid \{e, e'\} \in C_\Gamma\}$. Note that E_e only depends on G and C_Γ.

We first reformulate the properties claimed to be fccc so that they only depend on G and C_Γ. This proves that they all fulfil 1. The drawing Γ is k-**planar** if and only if for every edge $e \in E(G)$, C_Γ contains at most k sets with e as a member. The drawing is k-quasiplanar if and only if there is no k-element set $A \subseteq E(G)$ with $2^{\binom{A}{2}} \subseteq C_\Gamma$, where 2^X denotes the powerset of X. The drawing is fan-crossing if and only if for every edge $e \in E(G)$ there is some $v \in V(G)$ such that e' is incident to v for all $e' \in E_e$. Finally, the drawing is fan-crossing free if and only if all edges in E_e are independent in G.

For 2, note that none of the four properties depend on the planar edges in a drawing and, by the above formulations, the properties are closed under taking subsets of C_Γ. This shows that these properties are fccc as claimed.

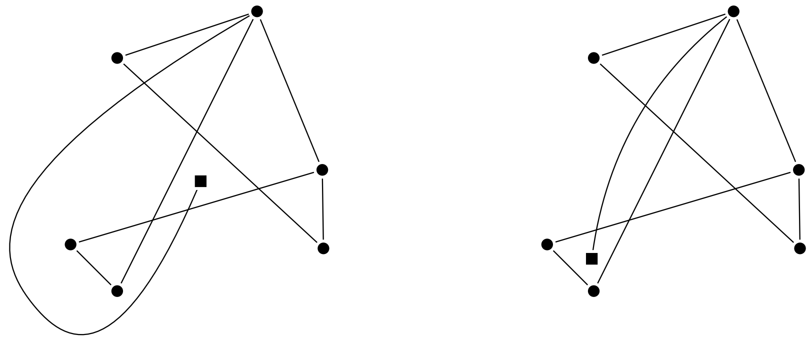


Figure 3.1: Two drawings of the same graph. Although they coincide in C_Γ, the left one is 1^+-real face while the right one is not.

On the other hand, k-bend RAC fails 1 as both polylines and angles at crossings are inherently topological properties and thus cannot be encoded only in G and C_Γ. The property k^+-real face also fails 1 if $k \geq 1$, as properties of cells cannot be determined from G and C_Γ alone. For example, Figure 3.1 depicts two drawings of the same graph, only one of which is 1^+-real face, but which coincide in C_Γ. Similar examples can be constructed for any $k \geq 1$. ∎

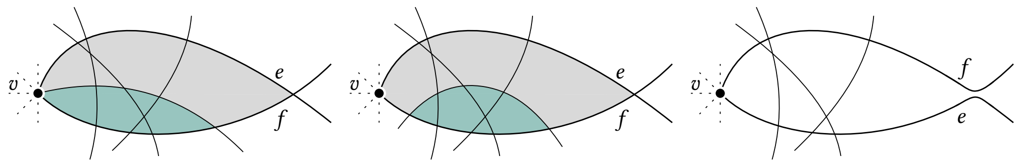
We now begin with proving the key theorem of this chapter. That is, if a drawing of a graph G satisfies some fccc P, then there is also a non-homotopic drawing of G that fulfils P. First, we show that lenses without interior vertices can be eliminated without changing the structure of the drawing. Here we use that multigraphs are disallowed.

Lemma 3.3: *Let G be a graph which admits a drawing Γ that contains a lens without a vertex in its interior. Then, there is some drawing Γ' of G with $C'_\Gamma \subsetneq C_\Gamma$ that contains only lenses with interior vertices. Additionally, if Γ is outer, then Γ' preserves the outer property.*

Proof. We start with Γ and eliminate lenses without interior vertices one after another. Throughout, we show that the subset of C_Γ that is our current multiset of crossing edge-pairs shrinks with each eliminated lens. Thus, this process has to end after finitely many steps. Further, we show that the outer property is preserved in each step, thereby finishing the proof.

Let l be an inclusion-minimal lens without a vertex in its interior. Let e and f be the two edges that bound l. Further, let $p, q \in \mathbb{R}^2$ be the intersections of e and f at the tips of the lens. Depending on the type of lens, there might be a vertex v at either p or q. We are agnostic about whether such a vertex exists and thus all references to v can be ignored when l is bounded by two crossings.

Since l contains no vertices, no edge can have an endpoint in the interior of l. So, an edge incident to v cannot leave v through l, since it would cross either e or f, creating a smaller lens contained in l (see Figure 3.2a) — a contradiction to inclusion-minimality of l. Further, an edge cannot enter and subsequently exit l through e (and similarly with f). Otherwise, as depicted in Figure 3.2b, it would form a smaller lens with e that is entirely contained within l. This, again, contradicts inclusion-minimality. Thus, e and f are crossed by the same (multi-)set of edges on the boundary of l. By switching the paths that e and f take along the boundary of l (see Figure 3.2c), at least one of their crossings is eliminated and no new crossing is introduced. So, in the resulting drawing the multiset of edge crossings is a proper subset of C_Γ. Note that switching e and f does not destroy any cell-vertex incidence. So, if Γ was outer, this property is retained. As argued in the beginning, this finishes the proof. ∎



(a) If the lens is bounded by a vertex v, an incident edge that leaves l through the lens creates a smaller lens.

(b) If an edge enters and exits the lens through the same boundary edge, it creates a smaller lens.

(c) By switching the paths of the boundary edges, the lens is destroyed while getting rid of one crossing.

Figure 3.2: The steps in the proof of Lemma 3.3. An inclusion-minimal lens without a vertex in its interior cannot be crossed by edges in certain ways (Subfigure a and b). Thus, the lens can be eliminated without introducing new crossings (Subfigure c).

With this we can show our desired result.

Theorem 3.4: *Let P be a forbidden combinatorial crossing configuration and G a graph that admits a drawing satisfying P. Then, G also admits a non-homotopic drawing that satisfies P.*

Proof. Take a drawing Γ of G that satisfies P. By Lemma 3.3, there is a drawing Γ' of G that contains no empty lenses — thus it is non-homotopic — and fulfils $C_{\Gamma'} \subsetneq C_\Gamma$. By the latter property and since P is a fccc, Γ' also satisfies P. ∎

What Theorem 3.4 shows, is that for any fccc P, the family of graphs that admit an arbitrary drawing satisfying P coincides with the respective non-homotopic family. Thus, in particular, the edge densities of two such families coincide. Before applying this knowledge, we prove an analogous result for outer drawings which we will apply later in Chapter 4.

Theorem 3.5: *Let P be a forbidden combinatorial crossing configuration and G a graph that admits an outer drawing Γ that satisfies P. Then G also admits an outer non-homotopic drawing that satisfies P.*

Proof. Take an outer drawing Γ of G that satisfies P. By Lemma 3.3, there is an outer drawing Γ' of G that is non-homotopic and fulfils $C'_\Gamma \subsetneq C_\Gamma$. By the latter property and since P is a fccc, Γ' also satisfies P. ∎

We now take the edge density results in [Kau+24] for non-homotopic 1-planar, 2-planar, and quasiplanar graphs and generalize them to the arbitrary setting. While none of the resulting theorems were previously unknown, this represents their first proofs via the Density Formula. Let us note that these results are not strict generalizations, since the results in Kaufmann et al. also apply to non-homotopic multigraphs.

3.1 1- and 2-planar graphs

The tight edge density upper bounds for 1- and 2-planar graphs were first proven in [PT97].

Theorem 3.6: *Any 1-planar graph on $n \geq 3$ vertices contains at most $4n - 8$ edges.*

Proof. Kaufmann et al. [Kau+24] proved the upper bound in the non-homotopic setting. By Observation 3.2, 1-planarity is a fccc and so with Theorem 3.4 the result translates to the arbitrary setting. ∎

Theorem 3.7: *Any 2-planar graph on $n \geq 3$ vertices contains at most $5n - 10$ edges.*

Proof. Kaufmann et al. [Kau+24] proved the bound in the non-homotopic setting. In Observation 3.2 we noted that 2-planarity is a fccc and so Theorem 3.4 gives the result for the arbitrary setting. ∎

3.2 Quasiplanar graphs

The following tight upper bound on the edge density of quasiplanar graphs were previously proven by Ackerman and Tardos [AT07] .

Theorem 3.8: *Any quasiplanar graph on $n \geq 3$ vertices contains at most $8n - 20$ edges.*

Proof. For the non-homotopic setting, this bound was proven by Kaufmann et al. [Kau+24]. This can be translated to the arbitrary setting by involving Theorem 3.4 and noting that quasiplanarity is a fccc (Observation 3.2). ∎

4 Outer variants

In this chapter, we consider many classic beyond-planar graph classes and their edge density when restricted to outer variants (the resulting classes might be called beyond-outerplanar graph classes). An outer variant of a beyond-planar graph class contains those graphs which admit an outer drawing under the restrictions given by the graph class. A well-known such variant is the class of outerplanar graphs, whose edge density is $2n - 3$ for $n \geq 2$ in contrast to planar graphs with at most $3n - 6$ edges if $n \geq 3$.

4.1 Applying the Density Formula

The bulk of our results are easily derived from the proofs by Kaufmann et al. [Kau+24] by applying knowledge about the outer cell in the Density Formula. All we need is that the outer cell has large size.

Lemma 4.1: *In an outer drawing of a connected graph on $n \geq 2$ vertices, the outer cell has size at least $2n$.*

Proof. Since the graph is connected, the outer cell c_o is bounded by a single boundary chain. Because of the outer property, this chain contains at least n vertices, so $v(c_o) \geq n$. Between two vertices on the chain, there is at least one edge segment. So, $e(c_o) \geq n$ and in total $\|c_o\| \geq 2n$. ∎

Let $\mathcal{C}_{\text{in}}$ be the set of all inner cells. Then, in an outer drawing of a connected graph on $n \geq 2$ vertices, the Density Formula can be rewritten to

$$\begin{aligned}
|E| &\leq t(|V| - 2) - \sum_{c \in \mathcal{C}_{\text{in}}} \left(\frac{t-1}{4} \|c\| - t \right) - \frac{t-1}{2}|V| + t - |\mathcal{X}| \\
&= \frac{t+1}{2}|V| - t - \sum_{c \in \mathcal{C}_{\text{in}}} \left(\frac{t-1}{4} \|c\| - t \right) - |\mathcal{X}|.
\end{aligned}$$

Under the assumption $|V| \geq 3$ and choosing $t = 3, 4, 5$, we obtain formulas similar to those in Observation 2.3 by noticing that the outer cell has size at least 6 and so is not already counted by the relevant $\mathcal{C}_i$.

Observation 4.2: *For any outer drawing of a connected graph $G = (V, E)$ with $|V| \geq 3$ and at least one edge*

$$\begin{aligned}
|E| &\leq 2|V| - 3 + \frac{3}{2}|\mathcal{C}_3| + |\mathcal{C}_4| + \frac{1}{2}|\mathcal{C}_5| - |\mathcal{X}| \\
|E| &\leq \frac{5}{2}|V| - 4 + \frac{7}{4}|\mathcal{C}_3| + |\mathcal{C}_4| + \frac{1}{4}|\mathcal{C}_5| - |\mathcal{X}| \\
|E| &\leq 3|V| - 5 + 2|\mathcal{C}_3| + |\mathcal{C}_4| - |\mathcal{X}|.
\end{aligned}$$

Now let us see via an example how an edge density proof that relies on the Density Formula can be translated into an edge density proof for the outer variant. For this, we consider outerplanar graphs.

Theorem 4.3: *A connected outerplanar graph on $n \geq 3$ vertices has at most $2n - 3$ edges.*

Proof. To demonstrate our general framework, let us first consider a proof for the edge density of planar graphs. We will use the first formula in Observation 2.3 for $t = 3$.

To prove $|E| \leq 3|V| - 6$ for connected planar graphs, we are left with showing

$$\frac{3}{2}|\mathcal{C}_3| + |\mathcal{C}_4| + \frac{1}{2}|\mathcal{C}_5| - |\mathcal{X}| \leq 0. \tag{4.1}$$

This inequality immediately follows from the fact that a planar drawing on at least three vertices contains no cells of size less than 6.

Now, getting back to proving the edge density of outerplanar graphs, we will use the inequality for $t = 3$ from Observation 4.2. Note that to obtain a $2n - 3$ upper bound we again need to prove Equation (4.1). Thus, by viewing the proof of Equation (4.1) as a black box, the result is obtained for connected outerplanar graphs. ∎

We will now apply this framework to derive upper bounds for edge densities of outer variants. All arguments in the remainder of Section 4.1 are identical to Theorem 4.3 and we thus omit the proofs. Proofs of the counterparts for non-outer multigraphs we rely on are found in [Kau+24]. Additionally, in the case of outer 1-, and 2-planar graphs we apply Theorem 3.5 to generalize from non-homotopic to arbitrary drawings of graphs. Slightly more involved arguments using our method will be given in Section 4.2.

4.1.1 k-planar graphs

We begin with outer k-planar graphs which have been introduced by Eggleton [Egg86]. A tight upper density result for outer 1-planar graphs has been proven before for arbitrary drawings of simple graphs by Auer et al. [Aue+16].

Theorem 4.4: *Let G be a simple connected outer 1-planar graph or a connected non-homotopic outer 1-planar multigraph on $n \geq 3$ vertices. Then, G has at most $\frac{5}{2}n - 4$ edges.*

There is essentially no literature about outer 2-planar graphs, so, to the best of our knowledge, the following result is new.

Theorem 4.5: *Let G be a simple connected outer 2-planar graph or a connected non-homotopic outer 2-planar multigraph on $n \geq 3$ vertices. Then, G has at most $3n - 5$ edges.*

4.1.2 Fan-crossing graphs

For simple outer fan-crossing drawings of graphs, Binucci et al. [Bin+15] proved a density of $3n - 5$. We give the same result.

Theorem 4.6: *A connected multigraph on $n \geq 3$ vertices with a simple outer fan-crossing drawing has at most $3n - 5$ edges.*

4.1.3 1^+- and 2^+-real face graphs

Recently, the class of k^+-real face graphs in which every cell is incident to at least k vertices was introduced by Binucci et al. [Bin+24]. They prove — among many other results — density results for simple drawings of simple outer k^+-real face graphs for every k. We start with a bound for $k = 1$ which is worse than the $3n - 6$ bound given by Binucci et al. but also applies to a more general setting. We do not know whether our bound is optimal.

Theorem 4.7: *A connected non-homotopic outer 1^+-real face multigraph on $n \geq 3$ vertices has at most $3n - 5$ edges.*

Next, we consider $k = 2$ where our bound is tight. As it applies to non-homotopic drawings of multigraphs, this is a slight generalization of the result for simple drawings of simple graphs in [Bin+24]. We will consider k^+-real face graphs for $k \geq 3$ in a later section, as the argument will differ slightly from the established framework.

Theorem 4.8: *A connected non-homotopic outer 2^+-real face multigraph on $n \geq 3$ vertices has at most $\frac{5}{2}n - 4$ edges.*

4.1.4 0- and 1-bend RAC graphs

Finally, we consider RAC graphs. While they have been extensively studied in graph drawing literature, as far as we are aware there are no results concerning their outer variant. Thus, the following results are novel.

Theorem 4.9: *A connected outer 0-bend RAC multigraph on $n \geq 3$ vertices has at most $\frac{5}{2}n - 4$ edges.*

Theorem 4.10: *A connected non-homotopic outer 1-bend RAC multigraph on $n \geq 3$ vertices has at most $3n - 5$ edges.*

We tend to outer 2-bend RAC graphs in the next section because the proof involves a few more steps.

4.1.5 Lower bounds

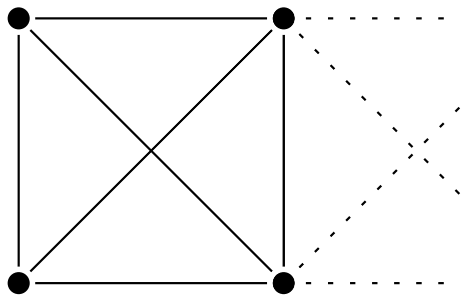
Before moving on to proving further upper bounds by other methods, we discuss optimality of the upper bounds presented above. For many of the upper bounds, we can indeed construct a matching family of lower bound examples. We start with a family of outer graphs with $\frac{5}{2}n - 4$ edges.

Construction 4.11: *For infinitely many n, there is a simple outer 1-planar, 2^+-real face, 0-bend RAC graph on n vertices with $\frac{5}{2}n - 4$ edges.*

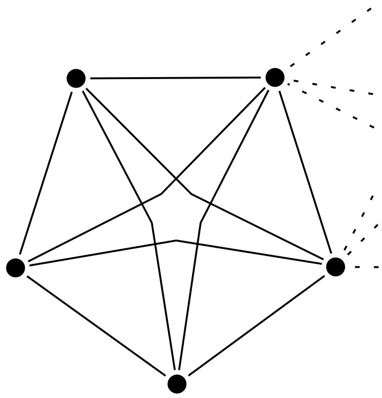
Proof. As a base graph for our construction, we use K_4. Draw the graph with the vertices arranged in a square and all edges straight. This drawing clearly fulfils all desired conditions. We construct further drawings, as depicted in Figure 4.1a, by glueing copies of the drawing along uncrossed edges. Each further copy adds 2 vertices and 5 edges, thus fitting the 5 : 2-ratio. ∎

Similarly, a family of outer graphs with $3n - 5$ edges can be constructed.

Construction 4.12: *For infinitely many n, there is a simple outer fan-crossing 2-planar, 1-bend RAC graph on n vertices with $3n - 5$ edges.*



(a) Outer family of graphs with density $\frac{5}{2}n - 4$.



(b) Outer family of graphs with density $3n - 5$.

Figure 4.1: Constructing chains of these graphs as indicated gives rise to families of outer graphs. These families give tight lower bound examples for multiple of our upper bounds on outer variants.

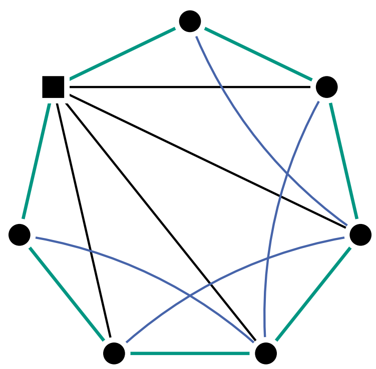


Figure 4.2: Example for $n = 7$ of an outer 1^+-real face graph with n vertices and $3n - 6$ edges. This can be attained for any n by taking an n-cycle, then connecting one vertex with every other and finally connecting vertices at distance 2 except those at the high-degree vertex.

Proof. Note that K_5 achieves the required density. We use it as a base graph. To obtain a drawing of K_5, take a drawing of C_5 as a regular pentagon and add the remaining edges with one bend each inside such that every crossing is at a right angle. The resulting drawing is outer, 1-bend RAC, and each edge is crossed at most twice. Now, as depicted in Figure 4.1b, construct further graphs in the family by glueing copies of the drawing along uncrossed edges. Each further copy adds 3 vertices and 9 edges, thus attaining the 3 : 1-ratio. ∎

The only previous bound that is not matched is in the outer 1^+-real face setting. Here, Binucci et al. [Bin+24] prove that the edge density of simple outer 1^+-real face drawings of simple graphs is $3n - 6$. Figure 4.2 shows how such graphs can be constructed. Since our result is in a more general setting, it might be possible to fit in one extra edge. We believe this is unlikely, however.

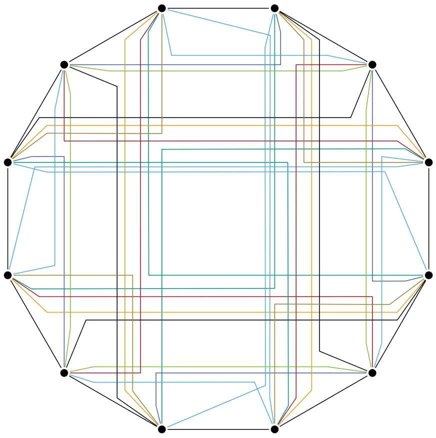


Figure 4.3: An outer 2-bend RAC graph with 12 vertices and 40 edges. By glueing copies of this graph along uncrossed edges, a family of graphs with $4|V| - 8$ edges is constructed.

4.2 More involved arguments

In this section, we prove further density results for outer variants with methods more involved than the simple approach used in the last section. These will come in the form of less direct usages of the Density Formula and later a discharging argument.

4.2.1 2-bend RAC graphs

We begin with 2-bend RAC graphs. Here, the argument mostly remains the same as in the last section. However, we give the proof explicitly, as it provides an unusual bound. The main difference in the argument is that the proof we rely on in [Kau+24] upper bounds the right-hand terms of the equation for $t = 5$ in Observation 2.3 by a value different from zero.

Theorem 4.13: *A connected non-homotopic outer 2-bend RAC multigraph on $n \geq 3$ vertices has at most $6n - 8$ edges.*

Proof. Recall that in a drawing, E_x and E_p are the sets of edges that are crossed and planar, respectively. We use the Density Formula for $t = 5$ in Observation 4.2. In [Kau+24, Lemma 4.1] the relation $2|\mathcal{C}_3| + |\mathcal{C}_4| - |\mathcal{X}| \leq \frac{1}{2}(|E_x| + 1)$ is proven for connected non-homotopic 2-bend RAC multigraphs. We thus obtain

$$|E| \leq 3|V| - 5 + \frac{1}{2}|E_x| + 1.$$

Now the argument follows analogously

$$|E| \leq |E| + |E_p| = 2|E| - |E_x| \leq 2(3|V| - 4) = 6|V| - 8. \qquad \blacksquare$$

While we do not believe that our upper bound is tight (see Chapter 7), we think it might be tight up to an additive constant. A fitting candidate graph for a glueing construction which would achieve this is K_{11}. If K_{11} admits an outer 2-bend RAC drawing, glueing copies of this drawing along uncrossed edges as in Figure 4.1 would give a family of graphs with $6n - 11$ edges.

The best lower bound construction we are aware of also uses the glueing construction. The base graph, depicted in Figure 4.3 has 12 vertices and is adapted from a construction for 2-bend RAC graphs in [Ari+12]. Graphs in the resulting family have $4n - 8$ edges.

4.2.2 k^+-real face graphs for $k \geq 3$

Next, we consider k^+-real face graphs for $k \geq 3$. The upper bound we derive here using the Density Formula generalizes the result in [Bin+24] from simple to non-homotopic drawings.

Theorem 4.14: *For $k \geq 3$, $n \geq 3$, any connected non-homotopic k^+-real face multigraph on n vertices has at most $\frac{k-1}{k-2}n - \frac{k}{k-2}$ edges.*

Proof. Note that in a k^+-real face drawing any cell has size at least $2k$ and the outer cell c_0 has size at least $2|V|$. Choosing $t = \frac{2k}{2k-4}$, we obtain $\frac{t-1}{4} = \frac{1}{2k-4}$ and an application of the Density Formula yields the Theorem

$$
\begin{aligned}
|E| =& \frac{k}{k-2}(|V|-2) - \sum_{c \in \mathcal{C}_{\text{in}}} \left(\frac{\|c\|}{2k-4} - \frac{2k}{2k-4} \right) - \frac{\|c_0\|}{2k-4} + \frac{k}{k-2} \\
\leq& \frac{k}{k-2}(|V|-2) - \frac{|V|}{k-2} + \frac{k}{k-2} \\
=& \frac{k-1}{k-2}|V| - \frac{k}{k-2}.
\end{aligned}
$$

∎

We do not want to retread the arguments by Binucci et al. and therefore refer to their paper for matching lower bound examples.

4.2.3 Quasiplanar graphs

Now, we tend to outer quasiplanar (multi-)graphs. First, we show that we may assume the outer cell of an outer quasiplanar drawing to be bounded by a planar n-cycle. The same holds for all families of outer (multi-)graphs defined by some fccc.

Lemma 4.15: *Let P be a forbidden combinatorial crossing configuration and G be a possibly non-simple graph on at least 3 vertices with an outer drawing Γ that satisfies P. Then, after possibly adding edges to G, the resulting graph admits an outer drawing Γ' that satisfies P in which the outer cell is bounded by a cycle on n vertices. That is, the drawing is connected, every vertex is incident exactly once to the outer cell, and there is no crossing incident to the outer cell.*

The final drawing retains simplicity and non-homotopicity if either were properties of Γ. Further, the underlying graph retains simplicity, if G was simple.

Proof. Consider some outer drawing Γ of G that satisfies P. If Γ is not connected, choose two vertices from different connected components and connect them via a planar edge in the outer cell. This retains that the drawing is outer but reduces the number of connected components. Thus, repeating this until there is only a single component, we obtain a connected drawing Γ'.

Now there is a single cyclic sequence S of vertices on the unique boundary chain of the outer cell of Γ'. Suppose some vertex v exists that appears multiple times on S and so $S = vUvW$. Both U and W contain at least one vertex as otherwise the two occurrences of v would coincide. Because both appearances of v on S are incident to the outer face, U and W share no vertices and there is no edge between the vertices of the two subsequences. Now, by drawing a planar edge from the last vertex in U to the first vertex in W, the second occurrence of v on the cyclic sequence is removed while keeping the drawing outer. Further, as no crossing is added, the drawing still satisfies P. Since this strictly reduces the size of the cyclic sequence, this process can be repeated until obtaining Γ'' — a drawing in which no vertex has multiple incidences with the outer cell.

If Γ'' has no crossing incident to the outer cell, we are done. Otherwise, let $v_0, v_1, \ldots, v_{n-1}$ be the cyclic sequence of vertices along the outer cell. In what follows, all index calculations are modulo n. For every $i \in [n]$, draw an edge $e_i \coloneqq v_i v_{i+1}$ through the outer cell of Γ''. By tracing the new edges sufficiently close to the boundary of the outer cell, all e_i are planar and the drawing remains outer quasiplanar. By next removing for every $i \in [n]$ an edge $v_i v_{i+1}$ that was already present in Γ'' (if any such edge exists), we obtain Γ'''. Note that because of the edge removal, no new parallel edges are created in the underlying graph. Further, note that the underlying graph does not lose any edges in this step. Some edges are redrawn, however, with less crossings than before.

The e_i form an n-cycle on the boundary of Γ'''. As argued throughout, Γ''' is an outer drawing whose underlying graph has G as a subgraph. Also, as all added edges are planar and $C'''_\Gamma \subseteq C_\Gamma$, the resulting drawing satisfies P and is non-homotopic and simple if Γ was. ∎

Our best upper bound for the density of outer quasiplanar graphs is not proven via the Density Formula but instead with a discharging argument. Before we present this, let us first give an easier argument for a slightly worse bound that can be obtained with a Density Formula argument. We use the fact that any — not necessarily simple — n-vertex non-homotopic quasiplanar multigraph has at most $8n - 20$ edges as proven in [Kau+24].

Lemma 4.16: *Any non-homotopic outer quasiplanar multigraph on $n \geq 3$ vertices has at most $4.5n - 10$ edges.*

Proof. Let G be a non-homotopic outer quasiplanar multigraph on n vertices. By Lemma 4.15 there is a multigraph $G' \supseteq G$ that admits a non-homotopic outer quasiplanar drawing Γ' whose outer cell is bounded by an n-cycle C.

Let $E_{\text{in}} \subseteq E(G')$ be the set of edges not on the n-cycle in Γ'. Consider Γ' as drawn on the lower hemisphere of $\mathbb{S}^2$ such that the image of C in G' under Γ' is the equator. Let Γ'' be created from Γ' by creating for every edge $e \in E_{\text{in}}$ a copy $\bar{e} \in \bar{E}_{\text{in}}$ and mirroring it along the centre of the sphere.

The new drawing Γ'' is quasiplanar as no edge crosses the equator and the lower and upper hemispheres are copies of Γ' which was quasiplanar by assumption. For the same reason, non-homotopicity is not broken by any set of edges living entirely in one hemisphere. So, if an empty lens exists in Γ'' it must span both hemispheres. Such a lens would thus be bounded by some $e \in E_{\text{in}}$ and $\bar{f} \in \bar{E}_{\text{in}}$ neither of which are edges in C.

If the vertices u and v that e and f share have distance at least 2 on C, a lens spanned by e and f contains some other vertex in its interior and is thus non-empty. If instead u, v are neighbours on the n-cycle, let $g \notin \{e, f\}$ be the edge that connects them in C. Then, the lens spanned by e and f contains the lens spanned by e and g which was already present in Γ'. So, as Γ' is non-homotopic, the lens is non-empty.

Now we can apply the result about non-homotopic quasiplanar multigraphs from [Kau+24]. The underlying graph G'' of the drawing Γ'' has $|E(G'')| = 2|E(G')| - n \geq 2|E(G)| - n$ edges. So, we obtain

$$\begin{aligned} |E(G)| &\leq \frac{1}{2}|E(G'')| + \frac{n}{2} \\ &\leq \frac{1}{2}(8n - 20) + \frac{n}{2} = 4.5n - 10. \end{aligned}$$

∎

Next, we turn to prove a better, tight bound for the edge density of outer quasiplanar graphs. The proof is via discharging and only a slight variation of the edge density proof for non-outer quasiplanar graphs in [AT07]. To prepare, we prove a general identity that is also one of the core ingredients in deriving the Density Formula.

Lemma 4.17: *For any drawing Γ of a connected mutligraph G with at least one edge,*

$$|V(G)| - 2 = \sum_{c \in \mathcal{C}} \left(\frac{1}{4} \, \|c\| - 1 \right).$$

Proof. Recall that we denote by $\mathcal{S}$, $\mathcal{X}$ and $\mathcal{C}$ the sets of edge segments, crossings, and cells in Γ. Consider the *planarization* of Γ. That is, the planar graph G' whose vertices are the vertices of G and the crossings in Γ and whose edges are the edge segments of Γ.

Note that by applying the Handshake Lemma on G' we get

$$\begin{aligned} |\mathcal{S}| = |E(G')| &= \frac{1}{2} \left(\sum_{v \in V(G)} \deg_G(v) + 4|\mathcal{X}| \right) \\ &= |E(G)| + 2|\mathcal{X}|. \end{aligned}$$

With this equality and further double counting of edge segment-cell and vertex-cell incidences, we can rewrite the sum in the statement as

$$\begin{aligned} \sum_{c \in \mathcal{C}} \left(\frac{1}{4} \, \|c\| - 1 \right) &= \sum_{c \in \mathcal{C}} \left(\frac{1}{4} (e(c) + v(c)) - 1 \right) \\ &= \frac{1}{4} (2|\mathcal{S}| + 2|E(G)|) - |\mathcal{C}| \\ &= \frac{1}{4} (4|E(G)| + 4|\mathcal{X}|) - |\mathcal{C}| \\ &= |E(G)| + |\mathcal{X}| - |\mathcal{C}|. \end{aligned}$$

Let Γ' be the planar drawing of G' induced by Γ. Then, the faces of Γ' are precisely the cells of Γ when viewed as subsets of $\mathbb{R}^2$. Denote the set of faces in Γ' as $F(G')$. Using the previous equalities and Euler's formula for planar graphs, one thus obtains

$$\begin{aligned} \sum_{c \in \mathcal{C}} \left(\frac{1}{4} \, \|c\| - 1 \right) &= |E(G)| + |\mathcal{X}| - |\mathcal{C}| \\ &= |\mathcal{S}| - |\mathcal{X}| - |F(G')| \\ &= |E(G')| - (|V(G')| - |V(G)|) - |F(G')| \\ &= |V(G)| - 2, \end{aligned}$$

as desired. ∎

Now, to the proof of our tight upper bound. Note that this result is a special case of a theorem by Capoyleas and Pach [CP92] who for any k give tight density bounds for outer k-quasiplanar graphs. Our approach is closer to a Density Formula approach, however. We use discharging with an initial assignment of charges and a first discharging step exactly as in [AT07]. The second step then incorporates the special structure of the outer drawing.

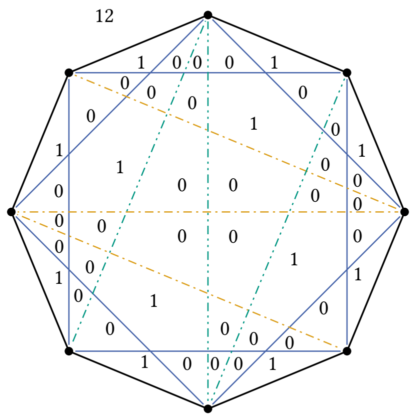


Figure 4.4: A simple graph on 8 vertices with the initial charge *ch* in the proof of Theorem 4.18.

Theorem 4.18: *Let $n \geq 4$ be a natural number. Let G be either a simple outer quasiplanar graph or an outer non-homotopic quasiplanar multigraph on n vertices. Then G has at most $4n - 10$ edges.*

Proof. Since quasiplanarity is defined via a fccc, by Theorem 3.5 G admits an outer non-homotopic quasiplanar drawing Γ. If the outer cell of Γ is not bounded by an n-cycle, we instead work on the supergraph G' of G and drawing Γ' obtained from Lemma 4.15. Note that this retains non-homotopicity. Let C denote the n-cycle bounding the outer cell of Γ'.

Before we begin the discharging argument, let us argue why we may assume $\delta(G') \geq 3$. To this end, let v be a vertex only adjacent to its two neighbours on C. We apply induction on $G' - v$ and obtain that G' has at most $4(n-1) - 10 + 2 \leq 4n - 10$ edges. Note that the base case $n = 4$ is trivial.

Now, we assign to each cell an initial charge $ch(c) \coloneqq \|c\| - 4$. Throughout the proof, we give the changing charge distribution for the example graph depicted in Figure 4.4.

Claim 1: The total charge distributed initially is

$$\sum_{c \in \mathcal{C}} (\|c\| - 4) = 4n - 8.$$

Proof of Claim. The claim immediately follows from multiplying both sides of Lemma 4.17 by 4. ▢

In the first discharging step, we aim to redistribute the charges such that every cell has charge at least $\frac{\nu(c)}{5}$. As Γ is non-homotopic and quasiplanar, it contains no cells of size 3 or less by Observation 2.2. Thus, the only cell type which does not have charge $\frac{\nu(c)}{5}$ initially is the -cell. Cells of type have initial charge 0 as desired and for any cell c of size at least 5, the inequality can be confirmed by noticing $\nu(c) \leq \frac{\|c\|}{2}$ and a short calculation:

$$\begin{aligned} \frac{\nu(c)}{5} \leq \frac{\|c\|}{10} &\leq \|c\| - 4 \\ \Leftrightarrow \quad \|c\| &\geq \frac{40}{9} \approx 4.4 \end{aligned}$$

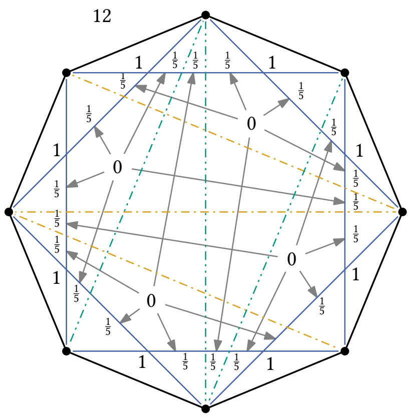


Figure 4.5: Resulting charge ch' after the first discharging step from the proof of Theorem 4.18. Each arrow indicates a charge of $\frac{1}{5}$ being sent from a donor cell to another cell.

We assign to each $\triangle_4$-cell a *donor*-cell from which it receives the required $\frac{1}{5}$ charge. Consider any $\triangle_4$-cell c. Let e, f be the two edges incident to the vertex on c and consider the cell c' on the other side of the inner edge segment on c. If c' is not a $\boxed{4}$-cell, set c' as the donor of c. If instead c is a $\boxed{4}$-cell, consider the next cell after c' to which e and f are incident. Repeat this process until a cell is reached that is not a $\boxed{4}$-cell and assign it as a donor to c. Note, that a $\triangle_4$-cell cannot be the donor of another $\triangle_4$-cell as e and f would then span an empty lens.

Now, gather for each $\triangle_4$-cell $\frac{1}{5}$ charge from its donor and denote the resulting charge distribution by ch'. We think of a donor as sending charge *through* its incident inner edge segments. Thus, any cell loses at most $\frac{e_{in}(c)}{5}$ charge in this first discharging step. For our example graph, the charges after the first step are depicted in Figure 4.5.

Claim 2: For any cell c, its charge after the first step is $ch'(c) \geq \frac{v(c)}{5}$.

Proof of Claim. For $\boxed{4}$- and $\triangle_4$-cells this is immediate from the construction. For any other cell c — again by Observation 2.2 and quasiplanarity — $\|c\| \geq 5$ and by using $e_{in}(c)+v(c) \leq \|c\|$ we obtain

$$\begin{aligned} ch'(c) - \frac{v(c)}{5} &= \|c\| - 4 - \frac{e_{in} + v(c)}{5} \\ &\geq \frac{4}{5}\|c\| - 4 \geq 0 \end{aligned}$$

and thus $ch'(c) \geq \frac{v(c)}{5}$. ▢

In the second and final discharging step, we collect for each vertex the excess charge of its incident inner cells. That is, each inner cell c distributes $ch'(c) - \frac{v(c)}{5}$ charge equally between its incident vertices. We denote the resulting charge distribution (now a function on cells *and* vertices) by ch''. Note that Claim 2 also holds for ch''. Charges after the second step are shown for our example graph in Figure 4.6.

Claim 3: For any vertex v, $ch''(v) \geq \frac{3}{5}$.

Proof of Claim. Let v be any vertex. Since we assume that every vertex has degree at least 3, v is incident to at least two inner cells. Consider the two inner cells incident to the previous and next vertex on C. We will argue that both cells send at least $\frac{3}{10}$ charge to v, thus proving the inequality.

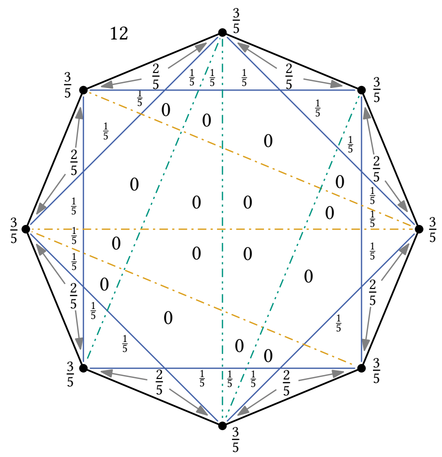


Figure 4.6: Resulting charge ch'' after the second and final discharging step ch'' from the proof of Theorem 4.18. Each arrow indicates a charge of $\frac{3}{10}$ being sent from a cell to a vertex.

Let c be one such cell. Note that $e(c) \geq 3$ as c is incident to a planar edge uv on C and — since $\delta(G') \geq 3$ — two edge segments incident to u and v respectively that are distinct by non-homotopicity. Note further that $e_{in}(c) \leq e(c) - 3$ as each of these three edge segments is incident to a vertex. With this, we can bound the charge at c after the first step

$$\begin{aligned}
ch'(c) &\geq \|c\| - 4 - \frac{1}{5}e_{in}(c) \\
&\geq e(c) + v(c) - 4 - \frac{1}{5}(e(c) - 3) \\
&= \frac{4}{5}e(c) + v(c) + \frac{3}{5} - 4 \\
&\geq \frac{12}{5} + v(c) + \frac{3}{5} - 4 \\
&= v(c) - 1.
\end{aligned}$$

Finally, using $v(c) \geq 2$, the charge sent to each incident vertex of c is

$$\begin{aligned}
\frac{1}{v(c)}\left(ch'(c) - \frac{v(c)}{5}\right) &\geq \frac{1}{v(c)}\left(v(c) - 1 - \frac{v(c)}{5}\right) \\
&= \frac{4}{5} - \frac{1}{v(c)} \\
&\geq \frac{4}{5} - \frac{1}{2} = \frac{3}{10}.
\end{aligned}$$

So, in total $ch''(v) \geq \frac{3}{5}$, proving the Claim. ▢

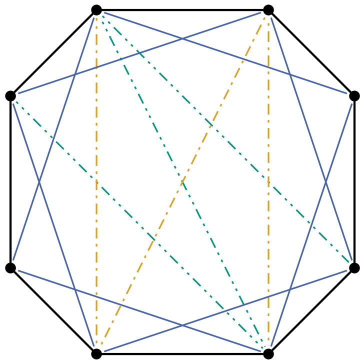


Figure 4.7: A simple graph on 8 vertices illustrated in an outer quasiplanar drawing. The depicted graph is densest possible with this property as it matches the bound in Theorem 4.18. It is also the example graph used throughout the proof of the Theorem.

To finish the proof, notice that the outer cell c_0 was initially assigned $2n-4$ charge and did not send any charge to other cells or vertices. Thus, $ch''(c_0) = \frac{v(c_0)}{5} + \frac{9}{5}n - 4$. Using all previous claims, we thus obtain

$$\begin{aligned} 4n - 8 = \sum_{c \in C} ch(c) &= \sum_{v \in V} ch''(v) + \sum_{c \in C} ch''(c) \\ &\geq \sum_{c \in C} \frac{v(c)}{5} + \sum_{v \in V} \frac{3}{5} + \frac{9}{5}n - 4 \\ &= \frac{2}{5}|E| + \frac{12}{5}n - 4 \end{aligned}$$

where we double count the vertex-cell incidences for the last equality. Solving for $|E|$ we get

$$|E| \leq \frac{5}{2}\left(\frac{8}{5}n - 4\right) = 4n - 10$$

as desired. ∎

The next construction shows that the bound in Theorem 4.18 is optimal. For $n = 8$ an example achieving the bound is depicted in Figure 4.7.

Construction 4.19: *For infinitely many n there is a simple n-vertex outer quasiplanar graph with $4n - 10$ edges.*

Proof. We construct the family by taking subdrawings of a family of quasiplanar graphs given in [Kau+24]. Thus, quasiplanarity of the drawing is immediate.

Start with an n-cycle C drawn as a regular simple polygon. Connect any pair of vertices at distance 2 on C by a straight line. Now, add two zig-zag paths on the inside of C such that all the edges connect vertices with distance at least 3 on C.

The n-cycle and the edges of "length" 2 give $2n$ edges. Further, each zig-zag path introduces $n - 5$ further edges. We thus obtain the claimed density. ∎

Note that our lower bound construction produces simple drawings. Hence — in contrast to the non-outer case — the edge density of outer quasiplanar graphs does not change when restricting to simple drawings.

5 Lower bounds for simple quasiplanar graphs

This chapter presents a short detour from the main topics of this thesis. Kaufmann et al. [Kau+24] give the only known examples of non-homotopic quasiplanar drawings which achieve the optimal upper bound of $8n - 20$ edges. However, their underlying graphs have parallel edges and are thus not simple..

Indeed, the exact edge density of *simple* graphs that admit quasiplanar drawings is currently unknown. A first lower bound construction with $7n - 29$ edges is given by Ackerman and Tardos in [Ack09]. By removing parallel edges in the $8n - 20$ construction of Kaufmann et al., one obtains a slightly better lower bound of $7n - 28$.

In this chapter, we improve on this in the linear term by constructing a family of simple graphs with $7.5n - 28$ edges that admit quasiplanar drawings.

Construction 5.1: *For every even $n \geq 16$, there is a simple graph on $7.5n - 28$ edges that admits a non-homotopic quasiplanar drawing.*

Proof. Let n be even. We describe a drawing on n vertices with the desired properties but only outline the edges' paths. Figure 5.1 depicts the paths more precisely. We draw

1. an $\frac{n}{2}$-cycle C_0 drawn as a circle with equidistant points.
2. another $\frac{n}{2}$-cycle C_1, a concentric copy of C_0 with larger diameter such that vertices on C_0 and C_1 are radially aligned.

Call two aligned vertices on C_0 and C_1 *partners*. Further, for any vertex x let us denote the vertex reached by taking k clockwise steps on the cycle of x as x_{+k}. Also denote by x'_{+k} the partner of x_{+k}. We continue drawing

3. an edge between any pair of partners.
4. for any vertex v, a straight edge to v'_{+1}.
5. for any vertex v on C_0 an edge from v to v_{+2}, drawn through the interior of C_0.
6. for any vertex v on C_1 an edge from v to v_{+2}, drawn through the exterior of C_1.
7. for any vertex v on C_0, an edge to v'_{+2}. The edge first moves through the space between C_0 and C_1, crosses C_1 between the partner of v and its neighbour and then continues through the exterior of C_1.
8. for any vertex v on C_1, an edge to v'_{+2}. Edges are drawn symmetrically to the last step with the roles of C_0 and C_1 reversed.
9. an edge between v and v_{+3} for any vertex v on C_0. Going clockwise, the edge first passes through the space between C_0 and C_1. It crosses C_0 between the two vertices on C_0 after the starting vertex and then goes through the interior of C_0.

10 an edge between v and v_{+3} for any vertex v on C_1. The edge travels as in the previous step with the roles of C_0 and C_1 reversed.

11 a zig-zag path in the interior of C_0 whose edges connect vertices of distance at least 4 on C_0. By *zig-zag path* we mean a path that starts at some vertex v_0 on C_0, connects it to the vertex v_1 reached by 4 clockwise steps, then connects v_1 to v_2 reached from v_1 by 5 counterclockwise steps. This goes on — increasing the number of steps by 1 for each new vertex and alternating between clockwise and counterclockwise — until two vertices are connected that are at distance 4 on C_0.

12 a second zig-zag path in the interior of C_0 that starts at some other vertex such that no parallel edges are created.

13 a zig-zag path in the exterior of C_1, again connecting vertices of distance at least 4 on C_1.

14 another such zig-zag path in the exterior of C_1.

The first three steps add $\frac{n}{2}$ edges each, the next step n edges, the six steps after that again $\frac{n}{2}$ edges each, and the last four steps $\frac{n}{2} - 7$ edges each. Thus, in total, the drawing contains

$$13\frac{n}{2} + n - 4 \cdot 7 = 7.5n - 28$$

edges. As each step connects vertices at different distances, the underlying graph is simple.

We now argue quasiplanarity of the drawing. First, consider all edges except those in zig-zag paths. These edges can be coloured in three colours as depicted in Figure 5.1 such that edges of the same colour do not cross. Now, the edges coloured black, that is those from 3, only cross two edges each, which in turn do not cross. So, as any three pairwise crossing edges would involve one black edge, we have confirmed quasiplanarity except for the zig-zag paths.

Now consider without loss of generality the zig-zag paths P and P' in the interior of C_0 and some edge $e = uv$ in P. We argue that e is not part of three pairwise crossing edges. Observe that e is crossed by three groups of edges: 1. Edges in P', 2. edges that are not on zig-zag paths and cross e at u, and 3. edges that are not on zig-zag paths and cross e at v.

We argue that no edges from 1., 2., or 3. cross. To start, no pair of edges in the same group cross since P' is drawn without self-crossing and edges in 2. and 3. have the same colour in Figure 5.1. Next, an edge from 2. cannot cross an edge from 3. since u and v have distance at least 4 on C_0, which two edges cannot span while crossing. Further, note that no edge in 1. is incident to u or v. However, edges in 2. and 3. cross edges from zig-zag paths only at u and v, so they cannot cross edges from 1. Thus, e is not part of three pairwise crossing edges and in total, the drawing is quasiplanar.

Note that the existence of a non-homotopic drawing is guaranteed by Theorem 3.4. Still, the given drawing is already non-homotopic. To argue this, it can be observed that when constructing the drawing iteratively as above, its simplicity is only broken by the edges added in 8 and 9. Each such edge is part of four lenses. One with an edge that has a common vertex and three with edges it crosses twice. However, as highlighted in Figure 5.1, the four lenses are non-empty and thus the drawing is non-homotopic. ∎

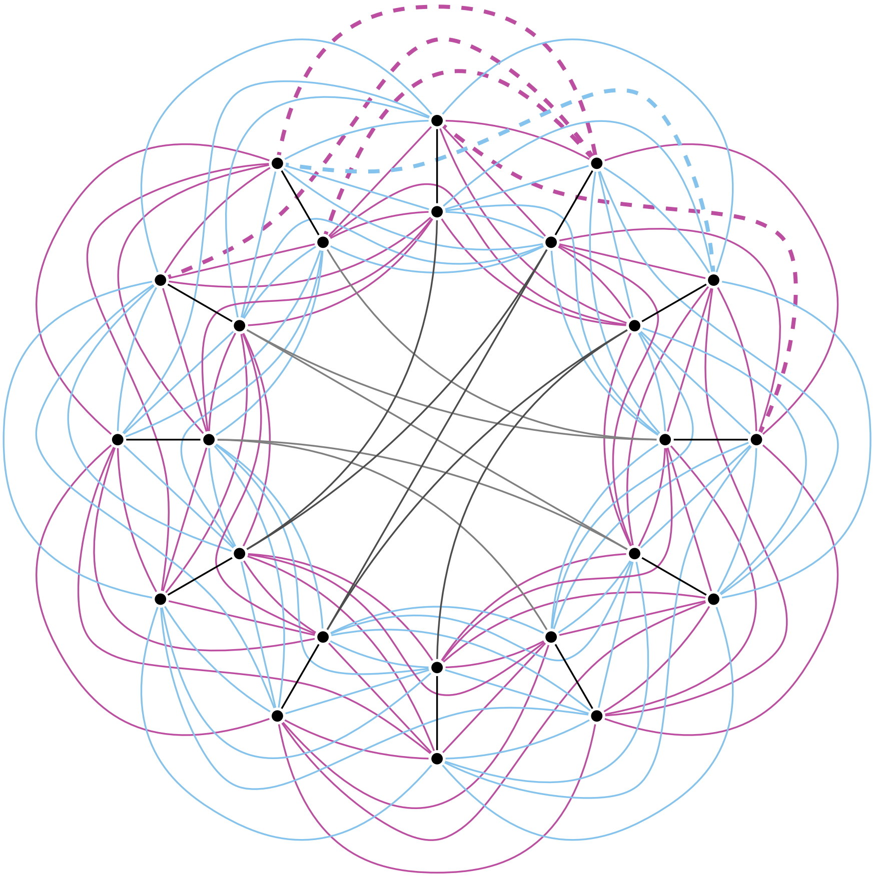


Figure 5.1: The drawing constructed in Construction 5.1 for $n = 24$ with the outer zig-zag paths omitted for visual clarity. The edges are coloured such that identically coloured edges do not cross. Although a global colouring of this kind can only be found if 4 divides n, locally such a colouring is always possible. The thick dashed cyan line forms lenses with the four thick dashed magenta lines, none of which are empty, however.

6 Forbidding cells

In this chapter, we investigate edge densities of graphs that admit drawings not including some fixed cell type $\mathfrak{c}$. Recall that a graph is called $\mathfrak{c}$*-free* if it admits a drawing without cells of type $\mathfrak{c}$. To keep the scope of this chapter manageable, we restrict our discussion to simple graphs. In this setting, we will find that a forbidden cell either admits drawings of graphs with quadratic or linear edge density. Our results are collected in Table 6.1.

6.1 Arbitrary drawings

Before considering non-homotopic or simple drawings in the following sections, we prove that edge densities of arbitrary drawings without a single cell type are always quadratic.

Theorem 6.1: *For any cell type $\mathfrak{c}$, the edge density of $\mathfrak{c}$-free graphs is in $\Theta(n^2)$.*

Proof. Let $\mathfrak{c}$ be any cell type, and $k = \|\mathfrak{c}\|$ the size of $\mathfrak{c}$. The implied upper bound of $\mathcal{O}(n^2)$ is trivial, as the number of edges in a graph is at most quadratic. Thus, in future proofs, we omit this step.

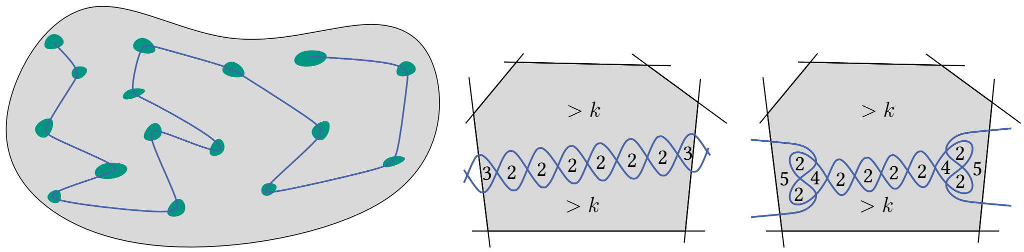
For the lower bound, we distinguish two cases. If $k \leq 2$, consider straight line drawings of K_n with vertices on a circle as in Figure 6.2. For $n \geq 3$ the boundary of any cell in such drawings is a convex polygon and so every cell has size at least 3.

If otherwise $k \geq 3$ we prove that for any $n \geq 3$, there is some graph on n vertices with $\binom{n-4}{2} + 2$ edges and an arbitrary $\mathfrak{c}$-free drawing. To this end, consider any drawing Γ of K_{n-4}. Then, take a curve s that adheres to all the properties we assume of edges in drawings and that passes through every cell of type $\mathfrak{c}$ in Γ. This is depicted in Figure 6.1a. Take a "thickening" s' of s, i.e. choose a sufficiently small width $w \in \mathbb{R}$ such that $\bigcup_{t\in[0,1]} B_w(s(t))$ only intersects edges of Γ where s does.

We alter Γ by adding two vertices v_1, v_2 on one end of s' and two further vertices u_1, u_2 on the other end. Further, draw two edges $e_1 = v_1u_1, e_2 = v_2u_2$ through s' which intersect $k+1$ times whenever s passes through a cell. If $\mathfrak{c} =$ , we additionally add some crossings whenever s' crosses an edge in Γ as depicted in Figure 6.1. Thus, if $\mathfrak{c} \neq$  any cell in the

Drawing \ Forbidden cell					Other
arbitrary	$\Theta(n^2)$	$\Theta(n^2)$	$\Theta(n^2)$	$\Theta(n^2)$	$\Theta(n^2)$
non-homotopic	$8n-20$	$\Theta(n^2)$	$\geq 6n-12$	$6n-12$	$\Theta(n^2)$
simple	$6.5n-20$	$\geq 4.2(n-2)$	$\geq 6n-12$	$6n-12$	$\Theta(n^2)$

Table 6.1: Overview of edge density bounds for simple n-vertex graphs that admit an arbitrary/non-homotopic/simple drawing (rows) without some fixed cell type (columns). An entry $\Theta(n^2)$ indicates quadratic asymptotic growth of the edge density. Entries preceded by "$\geq$" claim a lower bound for infinitely many n. An entry without this prefix is an upper bound for $n \geq 4$ as well as a lower bound for infinitely many n.



(a) **(b)** **(c)**

Figure 6.1: In the proof of Theorem 6.1, undesired cells of size k are eliminated by drawing a path through all of them (a) and replacing it with two edges which cross $k+1$ times in each cell the path traverses (b). If -cells are forbidden, the pattern the two edges form when crossing an edge in the original drawing is slightly altered (c).

resulting drawing that is incident to e_1 or e_2 is either one of the many - and -cells created by the intersection of e_1 and e_2 or has size at least $k+1$. If otherwise $\mathfrak{c}$ = , instead of -cells, the construction creates - and -cells. As s visits all cells of type $\mathfrak{c}$ in Γ, this cell type is not present in the altered drawing. Finally, because the number of vertices and edges is as above and the construction works for any n we have found a family with the desired properties and edge density $\Theta(n^2)$. ∎

For some cell types $\mathfrak{c}$, we will later give simpler constructions of arbitrary $\mathfrak{c}$-free drawings with quadratic edge density. In light of the previous theorem, these will not give us new results. Instead, the goal of presenting them will be either for contrast to the non-homotopic or simple case, or to introduce a new method of eliminating cells.

6.2 Uncommon cell types

The results in this chapter will be mostly proven by giving explicit drawings of families of dense graphs without the forbidden cell type. All of these give quadratic lower bounds in the setting of simple drawings. Naturally, these translate to the less restrictive settings of non-homotopic and arbitrary drawings.

There are alternative proofs for Lemma 6.2 and Lemma 6.3 which show that any drawing containing the forbidden cell type can be augmented by adding a linear number of vertices and edges so that all occurrences of the forbidden cells vanish. While these proofs show more general results, we do not give them here, as their ideas are simple and do not assist our aim of computing edge densities further.

Lemma 6.2: *For any cell type $\mathfrak{c}$ whose boundary contains at least two vertices, the edge density of graphs that admit a simple $\mathfrak{c}$-free drawing is in $\Theta(n^2)$.*

Proof. Let $\mathfrak{c}$ be such a cell type and $u, v \in \partial\mathfrak{c}$ two vertices on its boundary.
If u and v are connected by an edge on $\mathfrak{c}$, this edge is planar. Thus, taking a drawing Γ of a complete graph on n vertices with p planar edges, we achieve a drawing without cells of type $\mathfrak{c}$ by adding p edges on $2p$ new vertices to cross all planar edges. We are left with an altered drawing on at most $n+2p \leq n+2\cdot 3(n-6) \in \mathcal{O}(n)$ vertices and at least $\frac{n(n-1)}{2} \in \Omega(n^2)$ edges. This proves the claim in this case.

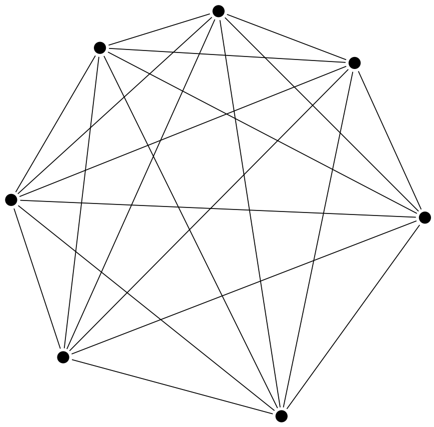


Figure 6.2: A circular layout straight line drawing of a complete graph contains only few different cell types incident to at least one vertex. This is used in Lemma 6.2 and Lemma 6.3.

If u and v are not connected, we consider a straight-line drawing Γ of a complete graph on n vertices in a circular layout as shown in Figure 6.2. If Γ does not contain a copy of $\mathfrak{c}$, we are done. Otherwise, $\mathfrak{c}$ is the outer cell. Thus, any other choice of n does not contain the forbidden cell type, so for n large enough we have examples with a quadratic number of edges. ∎

Lemma 6.3: *For any cell type $\mathfrak{c}$ whose boundary contains one vertex except the -cell, the edge density of simple $\mathfrak{c}$-free graphs is in $\Theta(n^2)$.*

Proof. Consider the circular layout drawing of K_n in Figure 6.2. The only cells with one incident vertex it contains are -cells, finishing the proof. ∎

To show quadratic edge density for cells without incident vertices, we consider a different drawing of the complete graph.

Lemma 6.4: *For any cell type $\mathfrak{c}$ whose boundary contains no vertices except the -, and -cell, the edge density of simple $\mathfrak{c}$-free graphs is in $\Theta(n^2)$.*

Proof. We consider 3-bend drawings of K_n as depicted for $n = 8$ in Figure 6.3 where all vertices lie on a straight line l.

Note that no two incident edges cross. Since any other pair of edges cross if and only if their endpoints alternate on l and then only once, the drawing is simple. Finally, because all cells with no incident vertices are -, -, and -cells, the claim follows. ∎

With this, -, -, - and -cells are the only cells for which we have not yet determined the edge densities. For - and -cells we will fully characterize the respective edge densities for non-homotopic and simple drawings. In the case of - and -cells, we can only present partial results, however.

6.3 The -cell

For the case of -cells, one has to look no further than the proof of the first tight upper bound for the edge density of quasiplanar graphs by Ackerman and Tardos [AT07].

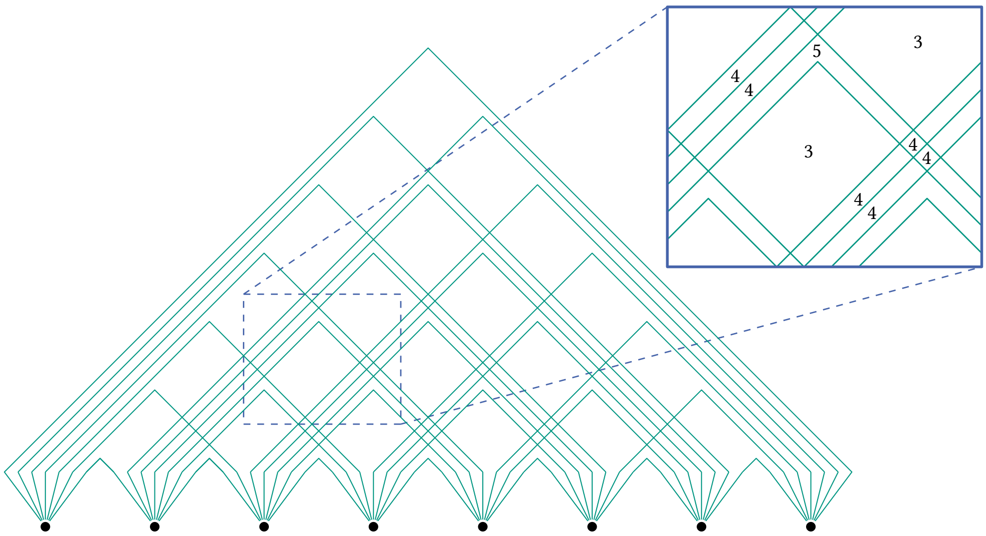


Figure 6.3: A 3-bend drawing of a complete graph. All cells without incident vertices have size 3, 4 or 5.

In their paper, they recognize *"We have not used the quasi-planarity assumption in full generality [...]. All we used is the assumption that the drawing does not yield 0-triangles, i.e., three pairwise crossing edges determining an empty triangle with no vertex inside."*

By "0-triangle" they denote a cell with three incident edge segments and no incident vertex, i.e., a $\triangledown$-cell.

6.3.1 Non-homotopic drawings

Ackerman and Tardos [AT07] prove that any graph on $n \geq 3$ vertices and drawn without $\triangledown$-cells has at most $8n - 20$ edges. They implicitly use that their drawings are non-homotopic. Formally, they assume that the number of crossings in their quasiplanar drawing is minimal. However, they only apply this to rule out empty lenses and cells of size 1 similar to what we did in Theorem 4.18. We thus obtain the following without proof.

Theorem 6.5: *A non-homotopic $\triangledown$-free graph on $n \geq 3$ vertices contains at most $8n - 20$ edges.*

For non-homotopic multigraphs, this bound is tight as already apparent from the respective examples for quasiplanar graphs from [Kau+24]. However, in contrast to quasiplanar graphs (see Chapter 5), we can even give examples of simple graphs which achieve this bound up to an additive constant. The previously best known lower bound was $7.5n - \mathcal{O}(1)$, found also in [AT07].

Construction 6.6: *For infinitely many n, there is a simple graph on n vertices and $8n - 28$ edges that admits a non-homotopic drawing without $\triangledown$-cells.*

Proof. Take Construction 5.1 as a base drawing. Recall that C_0 is the inner cycle, C_1 the outer cycle and two aligned vertices on the cycles are *partners*. Also, recall that we denote a vertex as x_{+k} if it is reached by taking k clockwise steps from x on its cycle. Further, denote by x'_{+k} the partner of x_{+k}.

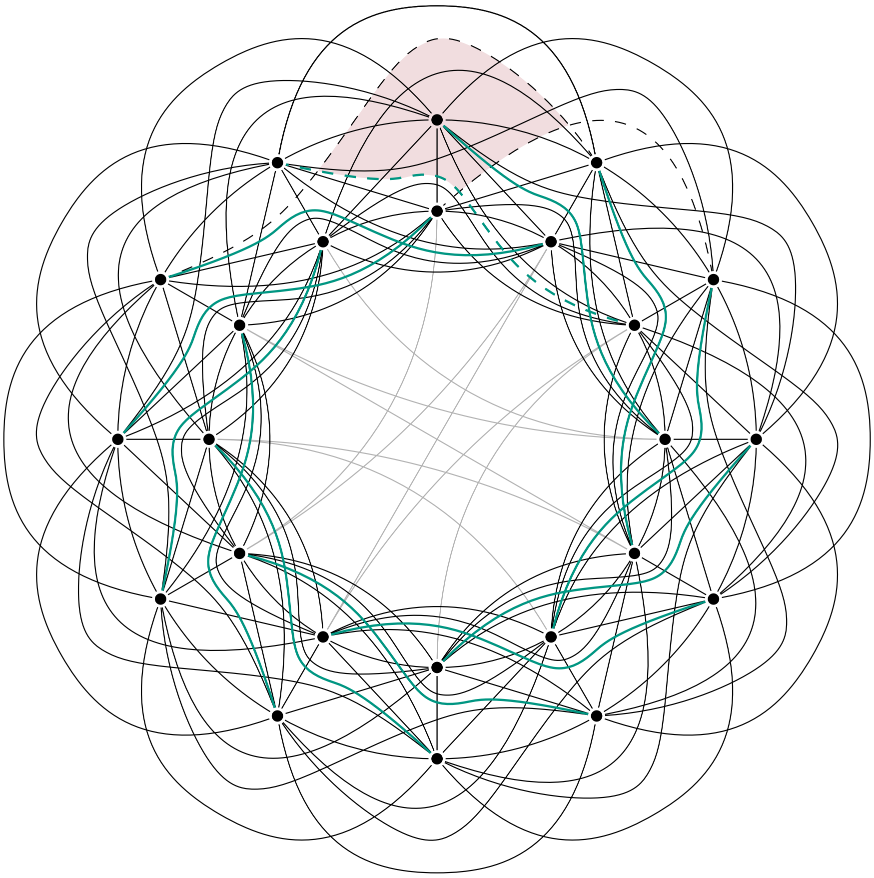


Figure 6.4: A graph with $n = 24$ vertices drawn as in Construction 6.6 with no $\triangledown$-cells. The drawing is as in Figure 5.1 with some additional edges, drawn thick and green. The drawing is not quasiplanar as shown by the three dashed edges. However, each region bounded by a triple of edges which pairwise cross contains a vertex. This region is marked for the highlighted triple with a pink background.

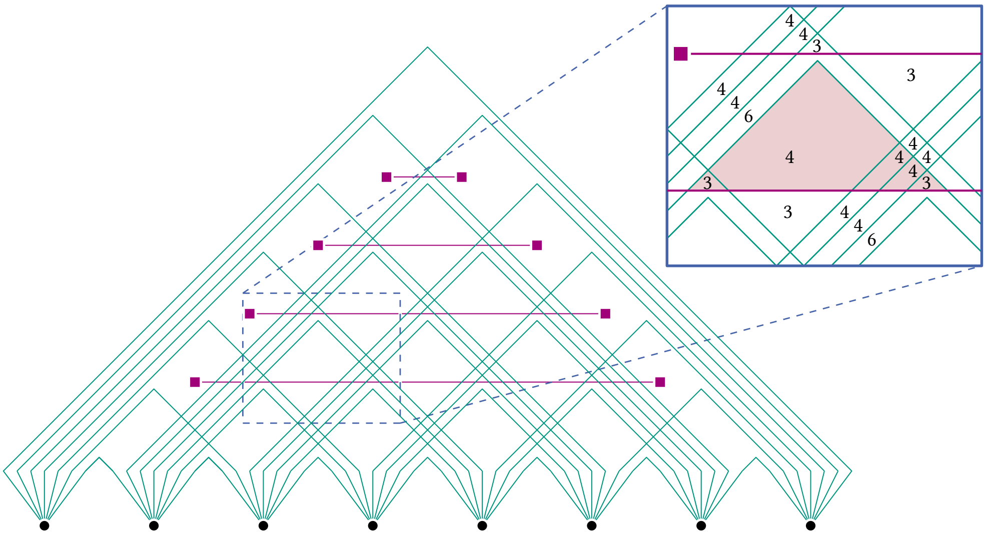


Figure 6.5: By adding $n-4$ edges, all ⬠(5)-cells can be eliminated in drawings of K_n as depicted in Figure 6.3. This introduces empty lenses (the marked triangle in the zoomed-in region), however.

Now, for any vertex v on C_1 draw an edge from v to v'_{+3} that first moves through the space between C_0 and C_1, crosses $v_{+1}v'_{+1}$, enters the interior of C_0 through $v'_{+1}v'_{+2}$ and then ends in v'_{+3}. The resulting drawing is shown in Figure 6.4. Checking along the paths of the newly added edges, it can be verified that the drawing remains ▽(3)-free. Note that this process adds $\frac{n}{2}$ edges, none of which are parallel to each other or previous edges. Therefore we obtain the claimed density. ∎

6.3.2 Simple drawings

Here, Ackerman and Tardos [AT07] provide a better upper bound which again holds in the ▽(3)-free setting.

Theorem 6.7: *A graph on $n \geq 4$ vertices that admits a simple ▽(3)-free drawing contains at most $6.5n - 20$ edges.*

The tightness of this bound — even for simple graphs — has already been implicitly established by Kaufmann et al. in [Kau+24] who give an infinite family with the desired properties.

6.4 The ⬠(5)-cell

We will prove that the density of ⬠(5)-free graphs depends on whether we restrict to non-homotopic drawings or not. For the arbitrary case, as an alternative to Theorem 6.1, we give a simpler construction without ⬠(5)-cells.

Theorem 6.8: *The edge density of ⬠(5)-free graphs is in $\Theta(n^2)$.*

Proof. Consider the drawings of K_n used in Lemma 6.4. There are $n-4$ layers of ⬠(5)-cells, each of which can be eliminated by drawing a horizontal edge. The resulting drawing in which each new edge is adjacent to two new vertices is depicted in Figure 6.5.

Evidently, no new 5-cells are introduced. As the number of new vertices is linear in n, the density of this family remains in $\Theta(n^2)$.

Note that the drawing is not non-homotopic as the new edges form empty lenses. ∎

The following proof for the edge density of non-homotopic 5-free cells uses a similar core idea as the proof of Theorem 4.18. We will not present this in the form of discharging, however (although it is possible) but by relating the number of cells of different types.

Theorem 6.9: *For $n \geq 3$, any n-vertex graph that admits a non-homotopic drawing without 5-cells has at most $6n-12$ edges.*

Proof. Consider any n-vertex graph G and a non-homotopic drawing Γ of G without 5-cells. We may assume that G is connected as otherwise an induction on the connected components completes the argument.

The proof can essentially be reduced to proving the following claim.

Claim 1: We denote by $\mathcal{C}_{\geq 5}$ the set of cells with size at least 5. Then,

$$3\#\text{3-cells} + \#\text{4-cells} \leq \sum_{c \in \mathcal{C}_{\geq 5}} (3e(c) + 2v(c) - 12)\,.$$

Suppose, we had a proof of this claim. Then, since there are no cells of size less than 3 by Observation 2.2 and for 4-cells, $3e(c) + 2v(c) - 12 = 0$, we equivalently get

$$0 \leq -\frac{3}{2}\#\text{3-cells} - \frac{1}{2}\#\text{4-cells} + \sum_{c \in \mathcal{C}_{\geq 5}} \left(\frac{3}{2}e(c) + v(c) - 6\right) = \sum_{c \in \mathcal{C}} \left(\frac{3}{2}e(c) + v(c) - 6\right).$$

Used in conjunction with the identity in Lemma 4.17 multiplied on both sides by 6, the Theorem follows

$$\begin{aligned} |E| = \frac{1}{2}\sum_{c \in \mathcal{C}} v(c) &\leq \sum_{c \in \mathcal{C}} \left(\frac{3}{2}e(c) + \frac{3}{2}v(c) - 6\right) \\ &= \sum_{c \in \mathcal{C}} \left(\frac{3}{2}\,\|c\| - 6\right) = 6(n-2). \end{aligned}$$

We are thus only left with proving the claim.

Proof of Claim. Note that $3\#$3-cells $+ \#$4-cells counts the number of inner edge segments that are incident to 3- or 4-cells.

To each such segment, we assign a cell of size at least 5 in the same manner as in the first discharging step of Theorem 4.18. Consider an inner edge segment s at a 3- or 4-cell c_0 and let e be the two edges on c_0 that cross s. The cell c_1 on the other side of s is also incident to e and f. If c_1 is a 4-cell, consider the next cell c_2 when following e and f through c_1. Otherwise assign c_1 to s. Repeat this process for 4-cells, until a cell c_k that is not a 4-cell is encountered and assign it to s. Note that $c_k \in \mathcal{C}_{\geq 5}$ since if it was a 3- or 4-cell, e and f would form an empty lens — contradicting non-homotopicity.

Since the above assignment is unambiguous and the "paths" of 4-cells can also be uniquely followed backwards, each cell $c \in \mathcal{C}_{\geq 5}$ is assigned at most $e_{in}(c)$ inner edge segments and therefore

$$3\#\text{3-cells} + \#\text{4-cells} \leq \sum_{c \in \mathcal{C}_{\geq 5}} e_{in}(c).$$

We prove $e_{in}(c) \leq 3e(c) + 2v(c) - 12$ for each $c \in \mathcal{C}_{\geq 5}$ via a case analysis. This then implies the claim. Note that the inequality does not hold for -cells, so here is where we use -freeness.

If $\|c\| \geq 6$, then

$$3e(c) + 2v(c) - 12 = e(c) + 2\|c\| - 12 \geq e(c) \geq e_{in}(c).$$

If c is a -cell, then

$$3e(c) + 2v(c) - 12 = 2 = e_{in}(c).$$

If otherwise c is a -cell, then

$$3e(c) + 2v(c) - 12 = 1 \geq 0 = e_{in}(c).$$ ▢

By the previous argument, this completes the proof. ∎

The next construction implies that this bound is optimal not only for non-homotopic but also for simple graphs.

Construction 6.10: *There are infinitely many n for which there is a simple graph on n vertices and* $6n - 12$ *edges with a simple -free drawing.*

Proof. To see this, take any graph G with a plane triangulation. Then draw for each vertex v and each incident face f an edge starting in v. Let e be the edge opposite v on f, f' the face on the other side of e, and v' the vertex opposite to e on f'. Draw the new edge by starting in v, traversing f and f' by crossing e and ending in v'.

Note that we need some structural assumptions on our starting triangulation of G to not obtain parallel edges or loops during the construction. To avoid loops, no edge e should be opposite to the same vertex on both its incident faces. For no parallel edges to be created, there may not be two edges opposite to the same two vertices on their incident faces. Constructing triangulations that fulfil these assumptions is not difficult. An infinite family emerges, for example, by taking a plane triangulation of the icosahedral graph and iteratively replacing the innermost face by plane triangulations of the icosadrehal graph.

Figure 6.6 depicts how the construction affects the surroundings of a face in the plane triangulation. It is apparent that indeed no -cells are created and no - and -cells form any lenses.

For the edge density, note that the procedure doubles the degree of each vertex. Since the number of edges in plane triangulations on $n \geq 3$ vertices is $3n - 6$, the resulting drawing has $6n - 12$ edges, as desired. ∎

6.5 The -cell

We will only mention preliminary results for this cell type as we were unable to determine any non-trivial upper bounds or superlinear lower bounds. The best lower bound we are aware of stems from the family of graphs obtained in Construction 6.10 as the simple drawings constructed of these graphs are -free. We thus obtain the following theorem which we state without proof.

Theorem 6.11: *There are infinitely many n such that there is a graph on n vertices and* $6n - 12$ *edges that admits a simple -free drawing.*

We postpone a discussion of why better lower and upper bounds might be hard to find to Chapter 7.

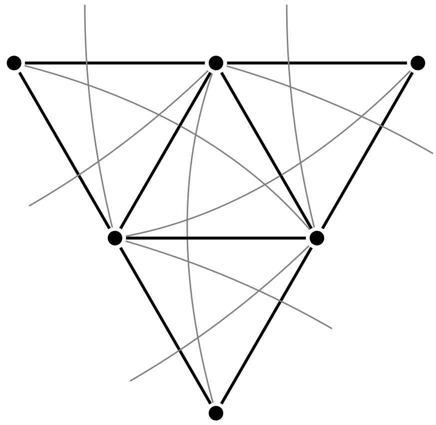


Figure 6.6: Illustration of the local construction process used for non-homotopic drawings of graphs with $6n - 12$ edges and no -cell.

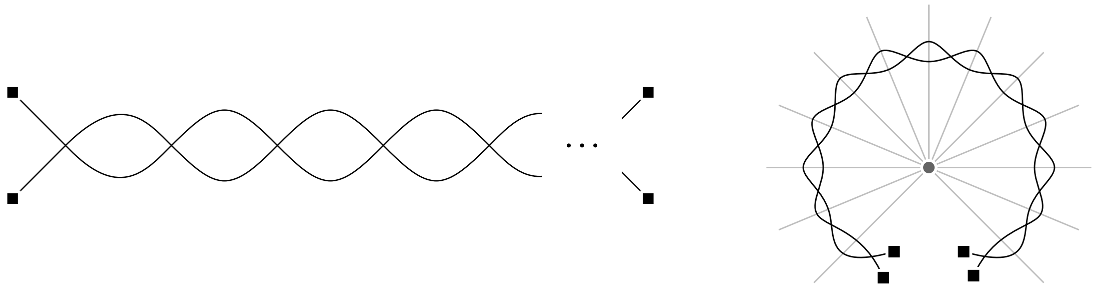


Figure 6.7: Illustration of the wave pattern used in the proof of Lemma 6.12 and how the pattern can be bent around a vertex to eliminate any occurrence of -cells.

6.6 The -cell

We will consider arbitrary, non-homotopic, and simple drawings independently in this section, as the relatively straightforward approach we use to deal with arbitrary and non-homotopic drawings is insufficient to deal with simple drawings.

6.6.1 Arbitrary and non-homotopic drawings

Lemma 6.12: *Any drawing of a graph on n vertices can be made -free by adding* $4n$ *vertices and* $2n$ *edges. In particular, the edge density of graphs that can be drawn without -cells is in* $\Theta(n^2)$.

Proof. Take for each vertex v a neighbourhood U_v around its position in the plane such that its intersection with the drawing only contains v and a single connected segment of each incident edge of v. W.l.o.g. these neighbourhoods are disjoint as otherwise one can shrink them while preserving the conditions.

In each U_v, draw two new edges e_1, e_2 on four new vertices. The two edges form a wave pattern, crossing each other multiple times as depicted in Figure 6.7. This pattern is then bent into a circle around v such that e_1 and e_2 alternate being closer or farther from v. Then, let the edges incident to v cross over this circle before leaving U_v so that no two edges cross the same edge segment of e_1 and e_2.

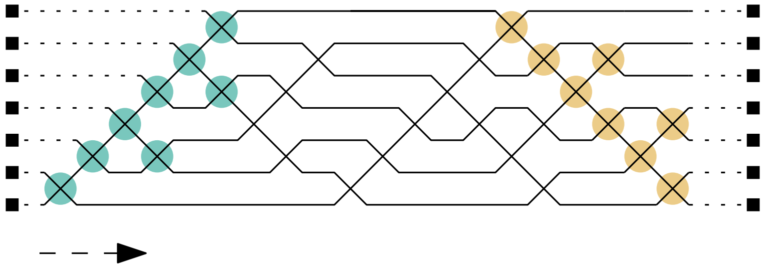


Figure 6.8: Illustration of the braid pattern used in proving Theorem 6.13. When it is drawn from left to right, no crossing except those marked green on the left can introduce an empty lens.

In the resulting drawing, there is no -cell incident to v as all edges incident to v cross different edge segments first. Further, the four new vertices in U_v are only incident to one cell, which they all share. Thus, they are also not incident to any -cell. There is no -cell in the drawing and so the Lemma holds. ∎

The construction in the previous proof does not result in a non-homotopic drawing, as the wave pattern creates empty lenses. However, a more complex edge pattern will allow us to apply the same idea while preserving non-homotopicity.

Theorem 6.13: *Any non-homotopic drawing of a graph on n vertices can be made -free while preserving non-homotopicity by adding $14n$ vertices and $7n$ edges. In particular, the edge density of graphs that admit non-homotopic drawings without -cells is in $\Theta(n^2)$.*

Proof. We proceed as in the proof of Lemma 6.12, only changing the pattern we put around each vertex.

Consider the partial braid pattern consisting of 7 curves depicted in Figure 6.8. Drawing this pattern from left to right, notice that under the assumption that the crossings on the left marked in green can be drawn without introducing empty lenses, the remaining crossings can all be drawn without breaking non-homotopicity — no matter how the edges behave left of the green crossings. This holds in particular for the crossings on the right marked in orange, which form the same pattern as the green crossings, only mirrored along a horizontal axis. By induction, the pattern can thus be repeated indefinitely if the green crossings can be drawn a first time. This can be accomplished by having no crossings preceding to the left.

Note that in this braid pattern, the number of distinct edge segments on the bottom can be increased indefinitely. So, bending one such pattern around each vertex as in Lemma 6.12 and drawing the edges straight through the braid pattern, we are left with no -cells. The resulting drawing is non-homotopic as both the braid pattern and original drawing are non-homotopic and edges crossing a braid pattern cross each braid edge at most once. ∎

6.6.2 Simple drawings

In the remainder of this chapter, we consider the edge density of -free simple drawings. For this, we assume all drawings referred to from now on to be simple. Let us already remark that we found neither a subquadratic upper bound nor a superlinear lower bound in this setting. Nonetheless, we can make some observations that we believe give further insight into the problem.

We begin by trying to approach the problem as in the arbitrary and non-homotopic setting. Thus, our goal is to find a pattern similar to the braid or wave pattern. We will argue why a pattern with the properties we desire cannot exist.

The general framework is as before: For some vertex v in our drawing, we want a configuration of edges on a small number of vertices that has a comparatively large number of edge segments that are incident to the outer cell. If we have this, bending the pattern around v, we would redirect the edges incident to v such that each edge crosses a unique outer edge segment of the configuration first. Thus, the number of outer edge segments in our pattern cannot be smaller than the degree of v.

We only consider patterns with a linear edge density as we do not want to rely on the existence of -free graphs with superlinear edge density. Thus, we may assume — by splitting each vertex v into $\deg(v)$ vertices of degree 1 — that our pattern is a simple drawing of k independent edges on $2k$ vertices. Such a configuration can be viewed as a collection of curves in the plane and is called a *pseudoline arrangement* in the literature.

If one could actually achieve a superlinear number of outer edge segments in a pseudoline arrangement on k pseudolines, we could obtain a superlinear edge density for -free simple drawings using the strategy used for the arbitrary and non-homotopic case. However, we can use the "Zone Theorem" for pseudoline arrangements to show that this is not possible.

Theorem 6.14 (Zone Theorem)**:** *Consider a pseudoline arrangement on $k+1$ pseudolines. Let p be one of the pseudolines and $C(p)$ the set of cells that p is incident to. Then,*

$$\sum_{c \in C(p)} e(c) \leq 9.5k - \mathcal{O}(1)$$

where we denote by $e(c)$ the number of pseudoline segments incident to c counted with multiplicity.

A proof of this is given by Bern et al. [BEPY90]. From this, we immediately get a bound on the number of outer edge segments in a graph with k edges.

Corollary 6.15: *In any simple drawing of a graph with k edges, the outer cell has at most $9.5k - \mathcal{O}(1)$ incident edge segments (counted with multiplicity).*

Proof. Turn such a drawing Γ into a pseudoline arrangement Γ' on k pseudolines by removing small neighbourhoods around each vertex. Choose the neighbourhoods sufficiently small such that no crossing is removed. Then, the number of pseudoline segments on the outer cell of Γ' is larger than the number of edge segments on the outer cell of Γ.

Add on the outer cell of Γ' a $k+1$th pseudoline p that does not intersect any other pseudoline. We apply Theorem 6.14 with p. Since $C(p)$ is just the outer cell the statement follows for Γ' and by the monotonicity argued above also for Γ. ∎

Now, we can formally prove the impossibility of the pattern approach.

Lemma 6.16: *Let G be a graph with a simple drawing Γ, $\mathcal{F}$ a family of graphs, and $c \in \mathbb{N}$ such that for any graph $H \in \mathcal{F}$, $|E(H)| \leq c|V(H)|$. Create a new drawing Γ' by adding in a small enough neighbourhood around each vertex v in Γ a graph $F_v \in \mathcal{F}$ such that*

- *the F_v are drawn simply,*
- *every edge incident to v crosses some edge segment in the drawing of F_v first,*
- *distinct edges incident to v cross different edge segments first.*

Then, Γ' *is a drawing of a graph* G' *with* $|E(G')| \leq \frac{23}{4}c|V(G')|$ *edges.*

Proof. Let Γ' be the drawing of a graph G' as constructed in the claim from a graph G and a family with edge density $\leq cn$. Denote the total number of added vertices and edges by $n' \coloneqq \sum_{v \in V(G)} |V(F_v)|$ and $m' \coloneqq \sum_{v \in V(G)} |E(F_v)|$. Corollary 6.15 tells us that $|E(F_v)| \geq \frac{2}{19}\deg(v)$ and so (applying the Handshake Lemma) $m' \geq \frac{4}{19}|E(G)|$. From the edge density of the family, we also get $n' \geq \frac{m'}{c}$. Therefore, in total

$$\frac{|E(G')|}{|V(G')|} = \frac{|E(G)| + m'}{|V(G)| + n'} \leq \frac{\frac{23}{4}m'}{|V(G')| + n'} \leq \frac{\frac{23}{4}m'}{\frac{m'}{c}} = \frac{23}{4}c$$

as claimed. ∎

We remark that while we have shown the impossibility of constructing dense non-homotopic graphs using patterns of global linear density, we did not rule out the feasibility of an iterative approach. Starting with patterns of edge density $c_0 n$ it might be possible to construct graphs of edge density $c_1 n$, $c_1 > c_0$ which could then be used as patterns for constructing even denser graphs and so on. We will not investigate this further, however, as we see little chance of this succeeding. Instead, we take the remainder of this section to investigate a different approach.

Recall that for -cells, there is a generalized pattern which is the characteristic forbidden configuration of quasiplanar graphs. In Section 6.3 were able to capitalize on the fact that edge density proofs for quasiplanar drawings only used that the -cell was forbidden. For -cells, there is a similar generalized combinatorial configuration. It consists of an edge that crosses two edges incident to the same vertex. We call this pattern a *fan* and graphs that can be drawn without it are *fan-crossing free*. Cheong et al. proved in [CHKK15] that the edge density of $n \geq 3$-vertex fan-crossing free graphs is $4n - 8$. In contrast to the -case, their proof uses that fans (and not only -cells) are forbidden multiple times. This cannot be avoided, as there are -free graphs with more than $4n - 8$ edges. The following construction is the best lower bound we are aware of.

Construction 6.17: *For infinitely many n, there is an n-vertex graph with* $4.2(n-2)$ *edges that admits a simple -free drawing.*

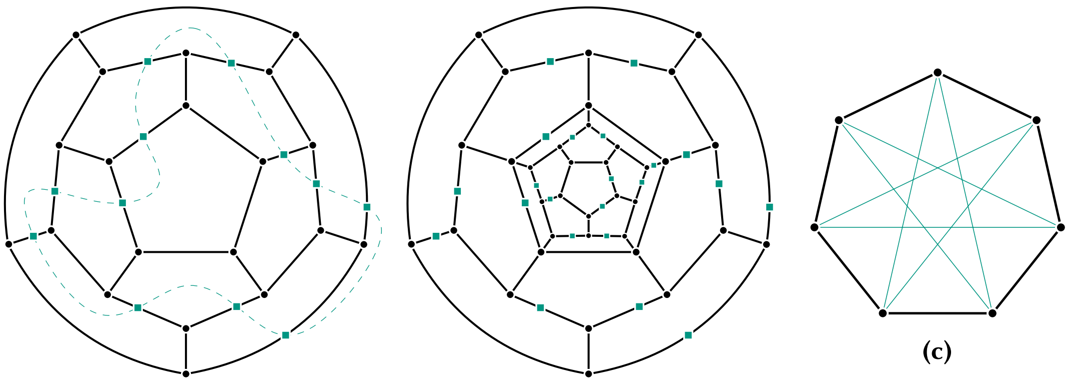
Proof. Take a graph G with a plane drawing Γ in which every face is a 7-cycle. We require that any two vertices of G have distance 3 along at most one face of Γ. This condition is to avoid adding parallel edges in the next step. Constructing arbitrarily large graphs and drawings which satisfy this is not hard, see Figure 6.9. Let n, m, f be the number of vertices, edges and faces of G and Γ. By double counting the edge-face incidences in Γ we have $2m = 7f$. With Euler's formula, we get $m = n + f - 2 = n + \frac{2}{7}m - 2$ and so

$$m = \frac{7}{5}(n-2).$$

On every face of Γ, connect every vertex-pair at distance 3 as depicted in Figure 6.9. The resulting drawing is simple, -free, and its number of edges is

$$m + 7f = 3m = \frac{21}{5}(n-2) = 4.2(n-2)$$

as desired. ∎



(a) (b) (c)

Figure 6.9: Construction of simple $\triangle_4$-free drawings of graphs with edge density $4.2(n-2)$. Subfigure (a) is a plane drawing of a dodecahedral graph with a Hamiltonian path through its faces that subdivides each edge it crosses. Thus, each face is bounded by a 7-cycle. This property is retained by iteratively copying this drawing into the innermost face as in (b). Adding 7 edges to each face in these drawings as in (c), a family of $\triangle_4$-free drawings with no parallel edges and the desired edge density is obtained.

While the proof of Cheong et al. cannot be translated to our setting, we will investigate the *n-stars* they introduce. Our hope is that upper bounds for the edge density in this somewhat simpler setting translate can be related to the edge density of $\triangle_4$-free drawings in general. We do not have proof of this fact, however.

Definition 6.18 (n-stars)**:** *An n-*star *(P, E) is a simple n-gon P (usually regular) and a set of* arrows *E. An arrow is a ray starting at a vertex of P, pointing into the interior of P and leaving it through one of its sides. We require that the drawing obtained when interpreting the arrows and sides of the polygon as edges is simple. We further require that no $\triangle_4$-cells occur in any n-star. Thus, two arrows starting at the same vertex cannot both have their first crossing with the same side of the polygon or arrow in the n-star.*

An arrow $e \in E$ is the *witness* of an arrow $f \in E$ if e is the first arrow that f crosses. We call an arrow *planar* if it first crosses a side of the polygon, i.e. it does not cross any other arrow. The number of planar arrows in an n-star is linear in n.

Lemma 6.19: *In an n-star with $n \geq 2$, the number of planar arrows is at most $n-2$.*

Proof. We use induction over n. Because of the simplicity of our drawings, an arrow cannot exit through a side of the polygon that is next to the arrow's starting vertex. Thus, for $n = 2$, there can be no planar arrow.

Now let $n \geq 3$ be arbitrary and (P, E) an n-star with some planar arrow $e \in E$. Cutting P into two halves along e, we essentially obtain a k-star and an $n+1-k$-star for some $k, n+1-k \geq 2$. By induction, the n-star thus contains at most $1 + (k-2) + (n+1-k-2) = n-2$ arrows. ∎

Achieving a linear amount of arrows in n-stars without planar arrows is not difficult. Consider for example an n-star with one arrow at each vertex that runs straight through the side of the polygon after the next vertex in clockwise order. Thus, the previous Lemma tells

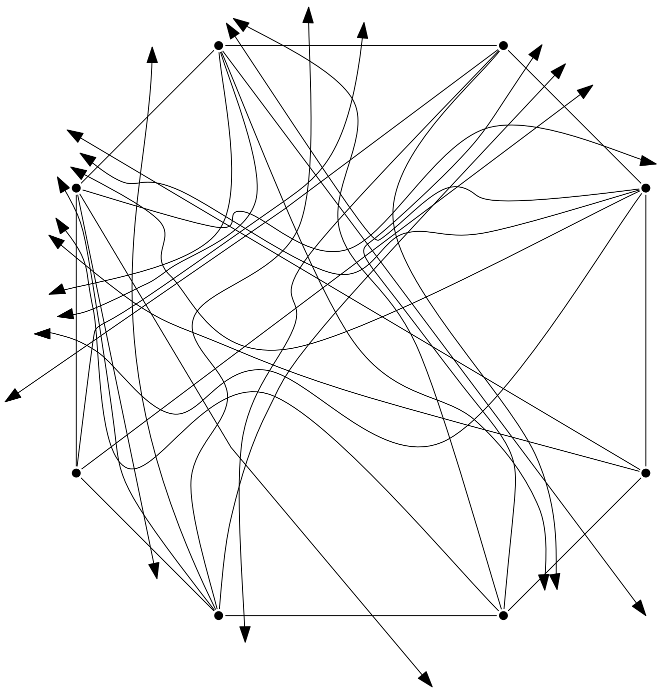


Figure 6.10: An 8-star with 25 arrows. Note that more arrows can be added to the drawing but we only need that $25 > 3 \cdot 8$ so we prefer to retain some visual clarity.

us that ignoring planar arrows does not change the asymptotics of the maximum number of arrows in an n-star. So, we will assume from now on that there are no planar arrows in n-stars and accordingly define

$$s(n) \coloneqq \max\{|E| \mid (P, E) \text{ is an } n\text{-star without planar arrows}\}.$$

With this convention, we can now assign to every arrow e a unique witness we call $w(e)$.

Let us note note that not only do our n-stars differ in their definition from those by Cheong et al. but ours can also contain strictly more arrows. In their paper, Cheong et al. prove that their n-stars contain at most $3n - 8$. In contrast, $s(n)$ grows at least linearly in n with a factor strictly above 3. We prove this by first showing superadditivity.

Lemma 6.20: *The function s is superadditive. That is, for all $n_1, n_2 \in \mathbb{N}$,*

$$s(n_1) + s(n_2) \leq s(n_1 + n_2).$$

Proof. Take two polygons P_1, P_2 of size n_1 and n_2 joined at a vertex v. Now, draw $s(n_1)$ arrows in P_1 and $s(n_2)$ arrows in P_2. Split v into v_1 and v_2 and let each arrow incident to v now begin at v_1. The split does not introduce any -cells as all arrows and sides existed before the split. Thus, the construction yields an $n_1 + n_2$-star with $s(n_1) + s(n_2)$ arrows which proves the claim. ∎

With this and and using the 8-star in Figure 6.10 we thus obtain a lower bound for our n-stars.

Construction 6.21: *For infinitely many n, there is an n-star with $3.125n$ arrows.*

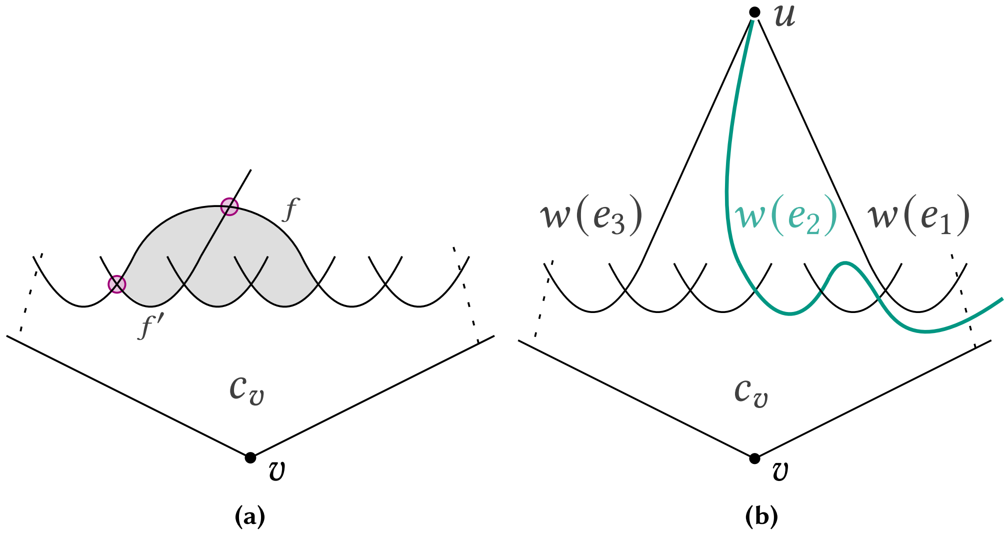


Figure 6.11: Illustration of the contradictions arrived at in the proof of Lemma 6.22. In (a), f' needs to leave the grey area by crossing f twice. Subfigure (b) depicts why three arrows starting at the same vertex cannot all be witnesses to arrows at some other vertex.

Proof. It can be checked that the creature depicted in Figure 6.10 is indeed an 8-star. Thus, $s(8) \geq 25 = 3.125 \cdot 8$. By Lemma 6.20, s is superadditive and so for any n that is a multiple of 8, $s(n) \geq 3.125n$. ∎

Finally, let us give an upper bound. *A priori*, it is unclear whether the number of arrows in n-stars is bounded at all. The following proves that — in accordance with graphs —, their density grows at most quadratic in n.

Lemma 6.22: *The number of arrows in an n-star is at most quadratic in n. More precisely,*

$$s(n) \leq 2n(n-1).$$

Proof. Let S be some n-star. We show that the number of arrows at a single vertex of S is at most $2(n-1)$. As there are n vertices, this then finishes the proof.
Suppose there was a vertex v of S with at least $2(n-1)+1$ incident arrows. As the drawing is simple, no arrow at v is witnessed by another arrow starting in v. Thus, by pigeonhole principle, there are three edges e_1, e_2, e_3 starting in v whose witnesses $w(e_i)$ all start in the same vertex $u \neq v$.
First, we argue that $w(e_i) \neq w(e_j)$ for $i \neq j$. Suppose this was not the case and w.l.o.g. $f = w(e_1) = w(e_2)$. Denote by c_v the cell in the interior of S incident to v that is created when removing the first segment (including the crossing) of all arrows starting in v. Then f occurs as at least two distinct segments on ∂c_v. These are separated by the occurrence of at least one segment of another edge f'. However, for f' to escape c_v it has to cross f twice as depicted in Figure 6.11a. This contradicts simplicity and thus the $w(e_i)$ are distinct.
Now, let the e_i be indexed such that the $w(e_i)$ leave u clockwise in the order they are indexed. As $w(e_2)$ cannot cross $w(e_1)$ or $w(e_3)$ by simplicity, it can only reach the boundary of S through c_v. This would then make it impossible either for $w(e_1)$ or $w(e_3)$ to appear on the boundary of c_v — depicted in Figure 6.11b — a contradiction to our assumption. ∎

7 Conclusion and outlook

In this thesis, we have proven edge density results for various beyond-planar graph classes. Using the Density Formula, we were able to generalize results, improve on others and also gave some first upper bounds, especially for outer variants. In doing so, we further established the usefulness of the Density Formula. Additionally, as our second main contribution, we considered new families of drawings and graphs, respectively, in which we forbade certain cell types.

However, some parts of our discussion of these topics leave unanswered questions. In this chapter, we give an overview of the most glaring and important of these gaps and present the problems we had in trying to fill them. Furthermore, we discuss other possible applications of the Density Formula for edge densities.

7.1 Lower bounds for outer 2-bend RAC graphs

In Section 4.2.1, we presented a $6n - 8$ upper bound for the edge density of non-homotopic 2-bend RAC graphs but only a lower bound of $4n - 8$. We believe that our upper bound is not tight. An indication of this is that the corresponding upper bound for non-outer 2-bend RACs in [Kau+24] does not match the lower bound either. However, the linear term of the lower bound example constructed by Kaufmann et al. does match the upper bound. We do, however, not know of any family achieving this in the outer case either.

7.2 Simple quasiplanar graphs

In Chapter 5 we were able to improve the lower bound for the edge density of simple quasiplanar graphs from $7n - 28$ to $7.5n - 28$. The upper bound remains at $8n - 20$, however, currently only matched by examples of non-homotopic drawings of multigraphs. We suspect that even for simple graphs, the upper bound is tight up to an additive constant.

7.3 Non-homotopic and simple $\boxed{4}$-free drawings

A proof of a non-trivial upper bound seems elusive. As a possible reason for this, recall that the discharging scheme used in Theorem 4.18 seemed to ignore $\boxed{4}$-cells entirely. Charge was never taken from or distributed to them. They were only the carrier through which charge flowed. Similarly, $\boxed{4}$-cells can be ignored in the Density Formula: For any t, a $\boxed{4}$-cell contributes +1 to the sum on the right side of the Density Formula. At the same time, the $-|\mathcal{X}|$ term in the Density Formula can be evenly distributed such that every cell-crossing incidence contributes $-\frac{1}{4}$. Since a $\boxed{4}$-cell has 4 crossing incidences, these contribute -1 in total, cancelling the contribution of the $\boxed{4}$-cell. Thus, any proof of a non-trivial upper bound would likely need to use some method not considered in this thesis.

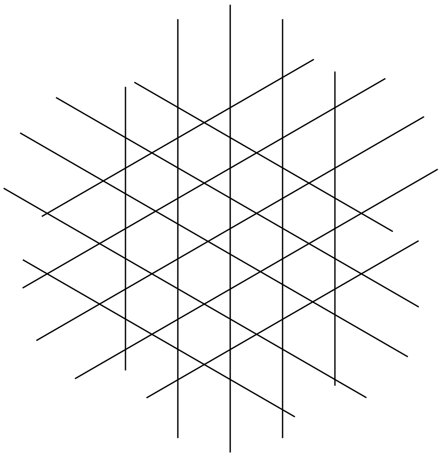


Figure 7.1: Three bundles of edges can cross without introducing $\boxed{4}$-cells, while two such bundles cannot.

We therefore believe it plausible that even for simple $\boxed{4}$-free drawings the edge density is quadratic. All simple drawings of dense graphs we are aware of contain many $\boxed{4}$-cells, however. In chaotic drawings, e.g. the circular layout drawings, $\boxed{4}$-cells are abundant because they are small enough to occur randomly. In more uniform drawings, e.g. the three-bend drawings of complete graphs, $\boxed{4}$-cells occur because one often chooses multiple edges to run somewhat parallel and so if multiple such bundles cross, many $\boxed{4}$-cells appear. It might be possible to avoid this by always having three such bundles cross at once as in Figure 7.1 to only create cells of size 3 and 6. Our attempts at incorporating this idea in a drawing of a dense graph have not yet been successful, however.

7.4 Simple $\triangle_4$-free drawings

Similarly to the case of $\boxed{4}$-free drawings, this setting seems impenetrable both by a discharging approach and an application of the Density Formula. In the discharging approach, the first discharging step fails because a $\pentagon_5$-cell that spans corridors on all sides with $\triangledown_3$-cells would obtain a negative charge. Similarly, the Density Formula approach for the usual choices of t seems to fail because either the number of $\triangledown_3$-cells or corridors of $\triangledown_3$- and $\pentagon_5$-cells cannot be accounted for. Thus, we know of no “intuitive” reason why the density of $\triangle_4$-free drawings would be linear.

On the other hand, constructing simple $\triangle_4$-free drawings with superlinear edge density seems challenging. This is mostly because we were able to rule out the approach of adding a pattern around each vertex. Thus, in any such construction, there cannot be a clear separation between the edges which contribute to the superlinear edge density and those which are witness (using this term as in the context of n-stars) to the edges incident to some fixed vertex. This implies that many edges must pass close to vertices they are not incident to.

While the setting of n-stars seems somewhat simpler, we were equally unable to provide good upper and lower bounds for their edge density. This is also true in some further restrictions to the n-star setting we considered, e.g., when vertices are added iteratively. Even with better bounds, the usefulness of n-stars in the case of $\triangle_4$-free drawings hinges on some connection between their edge densities. During the majority of writing this thesis, we believed we had such a relation. We had a faulty proof — based on a section by Cheong et al.

[CHKK15] — that upper bounds for the edge density of n-stars carry over in order of growth to upper bounds for the general -free setting. Only shortly before this thesis' completion did we notice the error which did not appear to be easily fixable.

7.5 Further Density Formula application

A main motivation for this thesis was the prospect of using the Density Formula to prove other edge density results than those found in [Kau+24]. We were able to achieve this to some extent by lifting a restriction to non-homotopic drawings in some cases and finding edge densities of outer variants in many others. However, throughout our efforts, we have considered other beyond-planar graph classes, for which we failed to give Density Formula proofs.

- Fan-crossing free graphs have edge density $4n - 8$ as determined in [CHKK15]. Finding a Density Formula proof for this might also lead to an approach for simple drawings of -free graphs as these can be considered a special case of fan-crossing free graphs.
- For graphs that admit arbitrary 1- and 2-bend RAC drawings, the best known density upper bounds are $5.5n - 11$ and $20n - 24$ respectively, proven in [ABFK20] and [Tót23]. For 1-bend RAC drawings, the linear term is optimal as proven also in [ABFK20]. Applying the Density Formula as for the non-homotopic variants fails since one of the main ideas of the proof in [Kau+24] uses the absence of empty lenses.
- Bipartite variants (considered for example in [Ang+19, Kar14]) might provide another general setting in which the Density Formula could be applied effectively.
- The proofs of Theorems 6.9 and 4.18 rely on the discharging method. However, a full reformulation using only the Density Formula (or a variation) might be possible. We believe this is indicated by the fact that in Theorem 6.9 we could phrase the discharging step as an inequality (Claim 1) very similar to those found when working with the Density Formula. Further, Lemma 4.17 which is used in both discharging proofs is also essential in deriving the Density Formula. The main reason why we were unable to provide a full mapping is that the Density Formula does not only use Lemma 4.17 but is made up of another equation which gives the relation to $|E|$. In contrast, both our discharging proofs introduce a relation to $|E|$ instead via double counting vertex-cell incidences.

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