Karlsruher Institut für Technologie

Master Thesis

The Ramsey Turnaround Numbers

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Abstract

The Ramsey turnaround game, first introduced by Mirbach, is a two-player game denoted by $\mathcal{G}(G, n, f, q)$. The Builder and the Painter play on n vertices and q colors. In each step, Builder exposes an edge and forbids f of the q colors, and Painter colors the edge in one of the remaining colors. For a given graph G, the goal of Painter is to force a monochromatic copy of G, while Builder wants to avoid it as long as possible. As a variant of online Ramsey numbers, the Ramsey turnaround number $\mathfrak{R}_f(G, n, q)$ equals the minimum number of exposed edges in the game $\mathcal{G}(G, n, f, q)$, that Painter needs to create a monochromatic copy of G no matter what Builder's strategy is. In the thesis, we determined lower and upper bounds on Ramsey turnaround numbers with different parameters on general and specific graph classes, including complete graphs, paths, cycles, stars and matchings. We also considered and compared offline and online strategies for both Builder and Painter, and showed that online strategies can prove strictly stronger bounds than offline ones.

Contents

1 Introduction

Beck [6], and Kurek and Rucinski [33] introduced the original *Ramsey game* independently. The game has two players, Builder and Painter, and a graph G is fixed. In each round, Builder exposes and edge and Painter colors it in either red or blue. The aim of Builder is to force a monochromatic copy of G, while Painter tries to avoid this as long as possible. The *online Ramsey number* of a graph G is the smallest number of rounds, in which Builder can force a monochromatic copy of G regardless of Painter's moves.

In this thesis, we turn the game around and investigate the *Ramsey turnaround game* $\mathcal{G}(G, n, f, q)$, first introduced by Mirbach [34]. The game is played on a board of n vertices, and with q colors. In each round, Builder exposes and edge, and Painter colors it using one of the q colors. However, unlike in the Ramsey game, in the Ramsey turnaround game Painter is the one trying to create a monochromatic copy of G, and Builder tries to avoid it as long as possible. To prevent the Painter from choosing the same color every time, the forbiddance number f is also given, and thus Builder forbids the usage of f of the q colors in each round for the exposed edge. We also set $n \geq r(G, q)$, where $r(G, q)$ denotes the monochromatic Ramsey number on q colors, to guarantee that Painter always reaches his goal and thus ends the game. The *Ramsey turnaround number* $\mathfrak{R}_f(G, n, q)$ is the smallest number of rounds that Painter needs in the game $\mathcal{G}(G, n, f, q)$ to reach his goal. In the whole thesis we refer to Builder as a *she* and to Painter as a *he*.

In this thesis, we investigate the Ramsey turnaround number $\mathfrak{R}_f(G, n, q)$ with various parameters and graph families, and prove lower and upper bounds. The extremal number $\mathrm{ex}(n, G)$ gives a trivial lower bound, as Builder can expose all edges of an extremal graph so that no copy of G is exposed. As a trivial upper bound we have $\binom{n}{2}$, as from the condition $n \geq r(G, q)$ follows, that any coloring of all edges of the complete graph K_n in q colors contains a monochromatic copy of G. Thus the Ramsey turnaround number is well-defined.

We investigate G as the concrete instance of $2K_2$, i.e. the graph of two independent edges, with different values for the forbiddance number f and the number of colors q. In Theorems 5.1 and 5.6 we prove, respectively,

$$\mathfrak{R}_1(2K_2, n, 2) = 2n - 2$$

and

$$n + 2 \leq \mathfrak{R}_1(2K_2, n, 3) \leq n + 3.$$

We study the two-color case as well, i.e. the game $\mathcal{G}(G, n, 1, 2)$. Here Builder determines the outcome of the game completely, as by forbidding one color she forces Painter to choose the other one. In this setup, we have results showing a strong relation between the game and the chromatic Ramsey number $R_\chi(G)$. Theorem 6.8 proves, using results of Erdős et al. [21], Hancock et al. [30] and Gaa [27], that for any non-bipartite graph G we have

$$\mathfrak{R}_1(G, n, 2) = \left(1 - \frac{1}{R_\chi(G) - 1}\right)\binom{n}{2}(1 + o(1)) + 1.$$

We also consider the monotonicity behavior of Ramsey turnaround numbers, as well as lower and upper bounds for general G. In Theorems 7.9 and 7.7 we prove the following general bounds:

$$\frac{n((f+1)(\Delta(G)-1)+1)}{4}+1 \leq \mathfrak{R}_f(G,n,q) \leq \min\Big\{(f+1)\mathrm{ex}(n,G); \mathrm{ex}(n,K_{r(G,f+1)})\Big\}+1.$$

Our results with G as a complete graph, i.e. with the Ramsey turnaround number $\mathfrak{R}_f(K_t,n,q)$, belong to our main contributions. We proved lower bounds for different parameter values of f,q and t using matchings, balanced colorings and the probabilistic method. Table 1 presents our lower and upper bound results for complete graphs.

Theorem	Bound	Method	Type	Comment
8.3	$\left(1-\frac{1}{2t-3}\right)\binom{n}{2}$	matchings	LB	$q \leq 2t-3$
8.10	$\left(1-\frac{1}{(\frac{t}{2})^2}\right)\binom{n}{2}$	balanced colorings	LB	$q \leq \frac{t}{2}$
8.12	$\left(1-\frac{1}{(t-t^{0.525})^2}\right)\binom{n}{2}$	balanced colorings	LB	$q \leq t-t^{0.525}-1$, t large enough
8.13	$\left(1-\frac{1}{t^{\frac{t^{1-\epsilon}}{\ln t}}}\right)\binom{n}{2}$	probabilistic	LB	$t>t_0$ for some t_0 with $q \in o(t_0^{\epsilon})$
8.15	$\left(1-\frac{1}{q^{qt}}\right)\binom{n}{2}$	q-color Ramsey	UB	

Table 1: Lower and upper bound results for $\mathfrak{R}_1(K_t,n,q)$.

We also study and prove bounds for other graph families. Theorems 9.2, 9.4, 9.6 and 9.9 state our results for paths, cycles, stars and matchings, respectively.

Our other main contribution is considering and comparing offline and online strategies for both Builder and Painter. After defining the concepts, we mainly focus on the question of whether online strategies can prove strictly better bounds than offline ones. For the game $\mathcal{G}(2K_2,n,1,3)$ we prove in Theorems 10.5 and 10.10 that for both Builder and Painter there exist online strategies proving a better bound than any offline one. We conjecture, that the same holds for all or almost all cases, i.e. in (almost) every game setup, the optimal strategies for both Builder and Painter are online.

The thesis is structured as follows. In Section 2 we present the preliminaries of graph and game theory. In Section 3 we first present Ramsey variants related to the Ramsey turnaround number. Then we show further similar games played on graphs. In Section 4 we define the Ramsey turnaround game and also formalize it game-theoretically. In Section 5 we study the game with $2K_2$ and P_3 as simple examples of G and with different values for the forbiddance number and the number of colors. Section 6 presents results for the two-color scenario. In Section 7 we first study the monotonicity properties of Ramsey turnaround numbers and then prove lower and upper bounds for general G. In Section 8 we study G as complete graph. First we give constructive lower bound proofs using matchings and balanced colorings, and then prove a stronger bound via the probabilistic method. We also present an upper bound proof that builds on multicolor Ramsey number results. In Section 9 we consider G belonging to other graph families and show lower and upper bound results for G as paths, cycles, stars and matchings. In Section 10 we compare online and offline strategies for both Builder and Painter. Finally, in Section 11 we conclude the thesis by listing open questions.

2 Preliminaries

We use the notation $[q] = \{1, \ldots, q\}$. We use the abbreviations *LB* for lower bound and *UB* for upper bound. We use the notation #{something} for *the number of* something.

2.1 Basics of graph theory

Most graph theory definitions and notations introduced in this and later sections coincide with Diestel's book [16]. For further graph-theoretic notations we also refer the reader to his book.

A *graph* G is a pair $G = (V, E)$ where V is a set and $E \subseteq \binom{V}{2}$. We call V as *vertex set* and E as *edge set*. The *order* of G is the size of the vertex set, denoted by $|G|$ or $|V(G)|$. The *size* of G is the size of the edge set, denoted by $||G||$ or $|E(G)|$. In this thesis we only consider simple graphs, i.e. finite graphs with undirected edges and with no multi-edges or loops. We say that an edge e is *incident* to a vertex x if $e = xy$ for some vertex y. Vertices x and y are *adjacent* if they are connected by an edge.

The *complement* of a graph G, denoted by $\overline{G}$, is defined as $V(\overline{G}) := V(G)$ and $\forall e : e \in E(\overline{G}) \Leftrightarrow e \notin E(G)$. Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *isomorphic*, denoted by $G_1 \simeq G_2$, if there exists a bijection $f : V_1 \to V_2$, so that

$$\forall x, y \in V_1 : xy \in E_1 \Leftrightarrow f(x)f(y) \in E_2.$$

For graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$ we say that H is a *subgraph* of G if $V_H \subseteq V_G$ and $E_H \subseteq E_G$. A vertex set S is a *vertex cover set* if $\forall xy \in E : x \in S$ or $y \in S$. A set of vertices is called *independent* if no two vertices are connected by an edge. A set of edges is called *independent* if no two edges share a vertex. A set of independent edges is also called a *matching*. A *perfect matching* is a matching of size $\frac{n}{2}$. We denote the matching of size t by tK_2.

A *complete graph* is $(V, \binom{V}{2})$ for a vertex set V. We denote the n-vertex complete graph by K_n. A graph is *r-partite* if its vertex set can be divided into r parts such that each edge has its ends in different parts. A *bipartite graph* is a graph $G = (V, E)$ whose vertex set can be divided into two independent vertex sets A and B such that $V = A \cup B$ and $A \cap B = \emptyset$. For such bipartite graphs we can also write $G = (A \cup B, E)$. A *complete bipartite graph* is a bipartite graph $G = (A \cup B, E)$ where $\forall a \in A, \forall b \in B : ab \in E(G)$. A complete bipartite graph $G = (A \cup B, E)$ with $|A| = a, |B| = b$ is denoted by $K_{a,b}$. A *star* on n vertices is the graph $K_{1,n-1}$. A *path* on n vertices is a graph $G = (V, E)$ where $V = \{v_0, \ldots, v_{n-1}\}$ and $E = \{v_0v_1, \ldots, v_{n-2}v_{n-1}\}$ with no repeated vertices, i.e. $v_i \neq v_j$ for $i \neq j$. We denote a path on n vertices by P_n. A *cycle* of length n is a graph $G = (V, E)$ with $V = \{v_0, \ldots, v_{n-1}\}$ and $E = \{v_0v_1, \ldots, v_{n-2}v_{n-1}, v_{n-1}v_0\}$ with no repeated vertices, i.e. $v_i \neq v_j$ for $i \neq j$. We denote a cycle on n vertices by C_n. A *Hamiltonian path* of a graph G is a path of length $|G|$. A *Hamiltonian cycle* of a graph G is a cycle of length $|G|$.

The *degree* of a vertex v, denoted by $\deg(v)$ is the number of edges incident to v. The maximum degree of a graph G is denoted by $\Delta(G)$. A graph is called *k-regular* if all vertices have degree k. A *tree* is a connected graph without cycles. A *leaf* of a tree is a vertex of degree 1.

A proper *vertex coloring* of a graph G with k colors is a function $c : V(G) \to [k]$ so that if $xy \in E(G)$ then $c(x) \neq c(y)$. The *chromatic number* of a graph G is denoted by $\chi(G)$ and is defined as

$$\chi(G) := \min\{k : \exists \text{ proper coloring } c : V(G) \to [k]\}.$$

A proper *edge coloring* of a graph G with k colors is a function $c : E(G) \to [k]$ so that for any vertex $v \in V(G)$ all incident edges of v have a different color. The *edge-chromatic number* of a graph G is denoted by $\chi'(G)$ and is defined as

$$\chi'(G) := \min\{k : \exists \text{ proper coloring } c : E(G) \to [k]\}.$$

The *extremal number* of a graph H is denoted by $\mathrm{ex}(n, H)$ and defined as

$$\mathrm{ex}(n, H) := \max\{|E(G)| : |G| = n, H \not\subset G\}.$$

The set $\mathrm{EX}(n, H)$ is the set of *extremal graphs* of H, defined as

$$\mathrm{EX}(n, H) := \{G : |G| = n, ||G|| = \mathrm{ex}(n, H), H \not\subset G\}.$$

The *Turán-graph* on n vertices, denoted by $T_r(n)$, is a complete r-partite graph with r parts of almost equal sizes, i.e. differing in their size by at most 1. We refer to the size of the Turán-graph as *Turán-number*, and use the notation $||T_r(n)||$. In 1941 Turán proved the following theorem connecting extremal and Turán numbers:

Theorem 2.1 (Turán [45]). *Let $n, k \in \mathbb{N}$ with $r \geq 2$ and $n \geq 1$. Then we have*

$$\mathrm{ex}(n, K_{r+1}) = ||T_r(n)|| = \left(1 - \frac{1}{r}\right)\binom{n}{2}.$$

In addition, $\mathrm{EX}(n, K_{r+1}) = \{T_r(n)\}$.

For G graph the classical *Ramsey number* is denoted by $r(G)$ and defined as

$$r(G) := \min\{n : \text{every 2-edge-coloring of } K_n \text{ contains a monochromatic copy of } G\}.$$

Figure 1 shows examples of some of the previously introduced graph theory concepts.

2.2 Basics of game theory

Most game theory definitions introduced in this and later sections build on the books of Neumann and Morgenstern [36] and of Fudenberg and Tirole [26]. As the listings here are non-exhaustive, we refer the reader for further game-theoretic notations to these books.

A *game* consists of three elements:

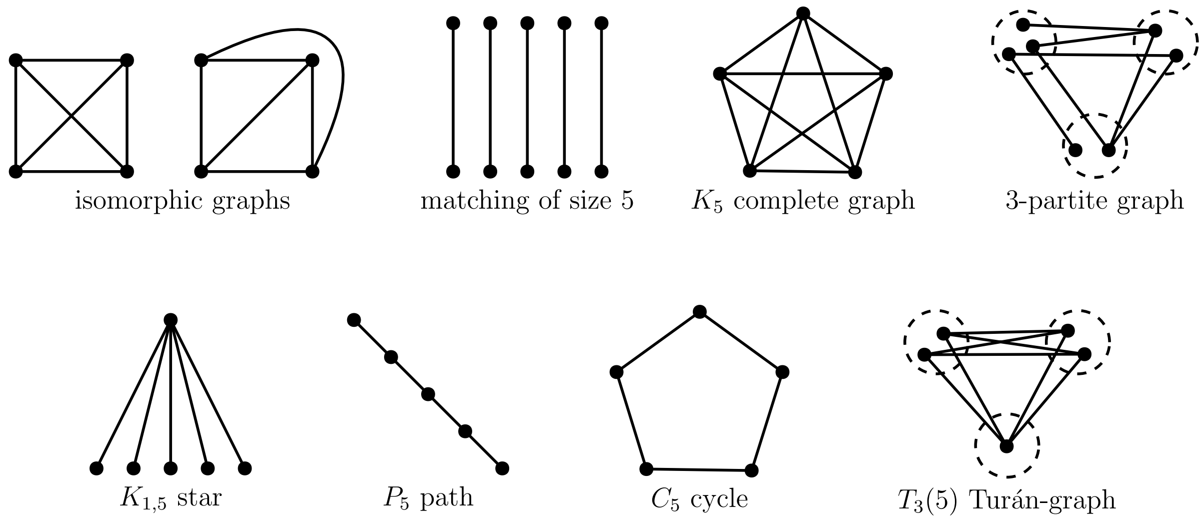


Figure 1: Examples of graph families.

- the set of players $\mathcal{P}$,
- the pure-strategy space $\mathcal{S}_i$ for each player $i \in \mathcal{P}$,
- the payoff function $u_i(s)$ for each player $i \in \mathcal{P}$ and strategy s.

A game consists of one or more rounds, and each player chooses his move from his possible choices. Imagine that each player, instead of choosing the next move as the necessity for it arises, makes a complete plan in advance for all possible situations, i.e. as the player begins to play, the plan specifies what choices to choose for every possible actual information that he may have at a certain point of the game. Such a plan is called a *pure strategy*. The set of all possible pure strategies is the *pure-strategy space*. The *payoff function* is designated to evaluate the players' possible strategies. The players' aim is to maximize their own payoff function, which may help or hinder the other players.

A *two-player-zero-sum game* is a game such that for all strategy s holds, that

$$\sum_{i=1}^{2} u_i(s) = 0.$$

As a consequence, in a two-player-zero-sum game the gains of one player equal the losses of the other player. A *mixed strategy* is a probability distribution over pure strategies. A *Nash-equilibrium* in a game with $\mathcal{P} = \{1, 2\}$ is some $V \in \mathbb{R}$ so that player 1 can choose a mixed strategy to guarantee himself expected gains of at least V and player 2 can choose a mixed strategy to guarantee himself expected losses of at most V. This common value V is also called the *game payoff*. In 1928 Neumann proved the existence of such equilibrium, building the basis of modern game theory:

Theorem 2.2 (Neumann's classic Minimax Theorem [35]). *In every two-player-zero-sum game a Nash-equilibrium exists, i.e. for $\mathcal{P} = \{1, 2\}$ there is some $V \in \mathbb{R}$ so that* *player 1 can choose a mixed strategy to guarantee himself expected gains of at least V and player 2 can choose a mixed strategy to guarantee himself expected losses of at most V.*

3 Background

3.1 Preliminaries on Ramsey variants

This section first introduces the main results of Ramsey numbers and then presents several Ramsey variants. Ramsey's theorem from 1930, stated in Theorem 3.1, builds the base of the whole Ramsey theory [29].

Theorem 3.1 (Ramsey [39]). *For $k, l \in \mathbb{N}$ there exists an $n \in \mathbb{N}$ such that any red-blue-edge-coloring of K_n contains either a red copy of K_k or a blue copy of K_l.*

In this thesis we only consider the symmetric case, i.e. the case $k = l$. For graphs G, H we write $G \to H$ if any red-blue-edge-coloring of G contains a red or a blue copy of H.

Definition 3.2 (Ramsey number). *For a graph G we define the Ramsey number*

$$r(G) := \min\{n : K_n \to G\}.$$

We use the notation $r(K_t) = r(t)$ for complete graphs. Despite that the problem of proving bounds on Ramsey numbers has garnered significant interest among mathematicians during the last century, the rate of advancement has been extremely slow. The first lower bound proof on Ramsey numbers is due to Erdős [17], who gave in 1947 a probabilistic proof on the lower bound of

$$r(t) \geq 2^{k/2}.$$

This non-constructive proof initiated the development of *the probabilistic method* [1] in combinatorial mathematics. In 1975 Spencer [42] asymptotically improved the lower bound, proving the currently known best lower bound:

$$r(t) \geq (1 - o(1)) \frac{\sqrt{2}t}{e} 2^{\frac{t}{2}}.$$

The first upper bound on Ramsey number was given by Erdős and Szekeres [22] in 1935, proving

$$r(t) \leq 4^t.$$

This bound was improved in 1988 by Thomason [44] by a polynomial factor, who showed that there is some positive constant A such that

$$r(t) \leq t^{-1/2 + A/\sqrt{\log t}} \binom{2t}{t}.$$

In 2009, Conlon [13] extended Thomason's method and improved the bound by a super-polynomial factor. He proved that there is a positive constant C such that

$$r(t) \leq t^{-C \log t / \log \log t} \binom{2t}{t}.$$

In 2023 Sah [40] further extended their method and proved that there is a positive constant C such that

$$r(t) \le 4^{t - C(\log t)^2}.$$

Recently, Campos et al. [12] published the currently known best upper bound,

$$r(t) \le (4-\epsilon)^t$$

for the constant $\epsilon = 2^{-7}$ and sufficiently large t. They also suspect that the value of ϵ could be improved further with somewhat technical optimization. Theorem 3.3 states the currently known best bounds on Ramsey numbers.

Theorem 3.3 (best Ramsey bounds). *For sufficiently large $t \in \mathbb{N}$ and $\epsilon = 2^{-7}$ holds that*

$$(1 - o(1))\frac{\sqrt{2}t}{e}2^{\frac{t}{2}} \le r(t) \le (4-\epsilon)^t.$$

The concept of Ramsey numbers has been generalized to q colors:

Definition 3.4 (q-color Ramsey number). *For G graph and $q \in \mathbb{N}, 2 \le q$ the q-color Ramsey number $r(G,q)$ is defined as*

$$r(G,q) := \min\{n : \textit{every } q\textit{-edge-coloring of } K_n \textit{ contains a monochromatic copy of } G\}.$$

For complete graphs we use the notation of $r(K_t, q) = r(t,q)$. The currently known best lower and upper bounds on q-color Ramsey number for complete graphs are due to Sawin [41] and Erdős and Szekeres [22] respectively. Theorem 3.5 shows these bounds.

Theorem 3.5 (bounds on q-color Ramsey numbers). *For $t, q \in \mathbb{N}$ and $q \ge 3$ we have*

$$2^{0.383796(q-2)t + \frac{t}{2} + o(t)} \le r(K_t, q) \le 2^{tq\log q}.$$

Definition 3.6 (size Ramsey number). *For a graph G we define the size Ramsey number $\hat{r}(G)$ as*

$$\hat{r}(G) := \min\{m : \exists H \textit{ with } |E(H)| = m \textit{ and } H \to G\}.$$

In other words, the *size Ramsey number* is the smallest integer m such that a graph H with $|E(H)| = m$ exists for which each two-coloring of the m edges contains a monochromatic copy of G. The notation was first considered by Erdős et al. in [18] in 1978. From the Ramsey number and size Ramsey number definitions follow, that for any graph G we have

$$\hat{r}(G) \le \binom{r(G)}{2}.$$

For complete graphs we use the notation $\hat{r}(K_t) = \hat{r}(t)$. Erdős et al. showed in [18], that

$$\hat{r}(t) = \binom{r(t)}{2}.$$

Definition 3.7 (q-color size Ramsey number)**.** *For a graph G we define the q-color size Ramsey number $\hat{r}(G, q)$ as the smallest integer m such that a graph H with $|E(H)| = m$ exists, where each q-coloring of the m edges contains a monochromatic copy of G.*

For complete graphs we use the notation $\hat{r}(K_t, q) = \hat{r}(t, q)$.

Definition 3.8 (Ramsey game)**.** *The original Ramsey game is played between two players, Builder and Painter. Builder exposes edges one at a time and Painter colors them in either red or blue. Builder's goal is to force Painter to create a monochromatic copy of a fixed graph G, while Painter tries to avoid it as long as possible.*

The original *Ramsey game* was introduced independently by Beck [6] and Kurek and Rucinski [33].

Definition 3.9 (online Ramsey number)**.** *The minimum number of edges the Builder must expose to achieve her goal in the Ramsey game is called the online Ramsey number, denoted by $\tilde{r}(G)$.*

For complete graphs we use the simplified notation $\tilde{r}(K_t) = \tilde{r}(t)$. We can easily show the following connection between Ramsey numbers and online Ramsey numbers:

Lemma 3.10. *For $t \in \mathbb{N}$ we have*

$$\frac{1}{2}r(t) \leq \tilde{r}(t) \leq \binom{r(t)}{2}.$$

Proof. <u>LB:</u> For the sake of contradiction assume $\frac{1}{2}r(t) > \tilde{r}(t)$. By definition of online Ramsey numbers, Builder can expose $\tilde{r}(t)$ edges so that she forces a monochromatic K_t regardless of how Painter plays. We know that $\tilde{r}(t)$ edges cover at most $2\tilde{r}(t)$ vertices. So if Builder can force a monochromatic K_t by exposing $\tilde{r}(t)$ edges, then she can also force it by exposing all edges of $K_{2\tilde{r}(t)}$, regardless of how Painter plays. Thus any two-edge-coloring of $K_{2\tilde{r}(t)}$ contains a monochromatic copy of K_t. As $2\tilde{r}(t) < r(t)$, this contradicts the Ramsey number definition. Thus our assumption was false, proving the lower bound.

<u>UB:</u> Builder can play by exposing all edges of a $K_{r(t)}$. By definition of Ramsey numbers, every two-edge-coloring of $K_{r(t)}$ contains a monochromatic copy of K_t. Thus Builder can force a monochromatic copy of K_t in $|E(K_{r(t)})| = \binom{r(t)}{2}$ rounds. □

Rödl [33] conjectured the following connection between the size Ramsey number and the online Ramsey number:

$$\lim_{t\to\infty} \frac{\tilde{r}(t)}{\hat{r}(t)} = 0, \text{ i.e. } \tilde{r}(t) = o\left(\binom{r(t)}{2}\right).$$

In 2009 Conlon [14] approached this conjecture by proving that for infinitely many values of t the following holds:

$$\tilde{r}(t) \leq 1.001^{-t}\binom{r(t)}{2}.$$

In the same paper [14], he proved the currently known best upper bound, i.e. there exists a constant $c > 0$ such that

$$\tilde{r}(t) \leq t^{-c\frac{\log t}{\log\log t}} 4^t.$$

The best lower bound known for the online Ramsey number is from 2020, when Conlon et al. [15] used the probabilistic method to prove that

$$\tilde{r}(t) \geq 2^{(2-\sqrt{2})t+O(1)}.$$

Theorem 3.11 (best online Ramsey bounds)**.** *For $t \in \mathbb{N}$ and some positive constant c holds that*

$$2^{(2-\sqrt{2})t+O(1)} \leq \tilde{r}(t) \leq t^{-c\frac{\log t}{\log\log t}} 4^t.$$

The Ramsey game can be extended to use not only red and blue but possibly more colors, naturally leading to the following definition:

Definition 3.12 (q-color online Ramsey number)**.** *The minimum number of edges the Builder must expose to achieve her goal in the q-color Ramsey game is called the q-color online Ramsey number, denoted by $\tilde{r}(G, q)$.*

3.2 Graph coloring games

This section presents various examples of graph coloring games. For further games we refer the reader to the books of Beck on combinatorial games [7], and of Hefetz, Krivelevich, Stojakovic and Szabó on positional games [31].

Definition 3.13 (Maker-Breaker game)**.** *A Maker-Breaker game is a game played by two players, Maker and Breaker. A set of elements X and a family of winning subsets of X, denoted by $\mathcal{F}$, is given. The two players alternately occupy elements of X. Maker wins if he manages to occupy each element of a winning subset set $F \in \mathcal{F}$, while Breaker wins if he prevents this by occupying one element of each $F \in \mathcal{F}$.*

In a *Maker-Breaker graph coloring game* on graph G, the set X is the set of some elements of G, Maker wants to color all elements of X following the rules, and Breaker wants to make this impossible for him.

The *vertex coloring game* was first proposed in 1981 by Brems [28] and then rediscovered by Bodlaender [9] in 1991. The game is played on a graph $G = (V, E)$ with two players alternately coloring the vertices of the graph from the color set C. The order of the coloring is defined by a given vertex order $(v_1, \ldots, v_{|G|})$. They color so that any two adjacent vertices have different colors. The first player wins if and only if in the end all vertices are colored. The *game chromatic number* of a graph is the smallest number of colors, for which the first player has a winning strategy in the vertex coloring game. Faigle et al. [24] started investigating the parameter.

The *edge coloring game* was introduced by Cai and Zhu in [11] in 2001. This game is very similar to the vertex coloring game with the only difference being that the two players color the edges of the graph instead of the vertices. Hereby the *game chromatic index* of a graph is the smallest number of colors, for which the first player has a winning strategy in the edge coloring game.

The (a, b)*-vertex coloring game* was introduced by Kierstead [32] in 2004. This is an asymmetric version of the vertex coloring game, where in each turn the first player colors a vertices, and the second player colors b vertices. Note that the $(1, 1)$-vertex coloring game is just the vertex coloring game. The corresponding parameter is the (a, b)*-game chromatic number*, defined as the smallest number of colors needed for the first player to have a winning strategy.

Definition 3.14 (online F-avoidance game)**.** *The online F-avoidance game for a graph F is a single-player game played on a board of n vertices. The player receives a random sequence of edges of the underlying complete graph K_n, and colors each edge as it comes into one of two colors. His goal is to color as many edges as possible without creating a monochromatic copy of F. The game ends as soon as a monochromatic copy of F is created.*

First Friedgut et al. [25] considered the game in 2003 for $F = K_3$. Then Balogh and Butterfield [5] introduced the game for general F in 2010. The online F-avoidance game also may be considered as a one-player variant of the Ramsey game introduced in Definition 3.8.

4 The Ramsey turnaround game of graph G

4.1 General definition

The *Ramsey turnaround game* and *number* were first introduced by Mirbach in 2017 in her Bachelor's thesis [34]. The new concepts were inspired by the original Ramsey game introduced in Definition 3.8. The turnaround game is the result of switching Builder's and Painter's goals. The game setup remains the same, but the players play on a board of n vertices and Builder wants to *avoid*, and Painter wants to *achieve* a monochromatic copy of a fixed graph G. Without further restrictions the trivially best Painter strategy would be to always choose the same color. This motivates the introduction of the *forbiddance number*, which makes the question of Painter's strategy more interesting by allowing Builder to forbid the usage of a fixed number of colors for Painter in each round. Although the two-color turnaround game is still not really a game (see Section 6), increasing the number of colors brings the turnaround game to life.

Definition 4.1 (Ramsey turnaround game [34])**.** *Let $n, f, q \in \mathbb{N}$ with $f < q$, let G be a graph and let $n \geq r(G, q)$. We define $\mathcal{G}(G, n, f, q)$ as the Ramsey turnaround game (RTA). It is a game between Builder and Painter on a board of n vertices. The goal of Painter is to force a monochromatic copy of graph G and Builder tries to avoid this as long as possible. In each round of the game, Builder exposes one new edge and is allowed to forbid the usage of f colors for the Painter to color this currently exposed edge. Painter colors the edge according to these restrictions. The total set of colors is $[q]$. The game is over as soon as Painter manages to achieve a monochromatic copy of G.*

Note that the game with r rounds can be looked at as a list of pairs (b_i, p_i) for $i = 1, \ldots, r$, where

- b_i is the i-th step of Builder and p_i is the i-th step of Painter,
- $b_i = (e_i, C_i)$, where e_i is the exposed edge and C_i with $|C_i| = f$ is the set of forbidden colors,
- p_i is the color chosen by Painter for the exposed edge.

For the sake of simplicity, we refer to Builder as *she* and to Painter as *he*. Also note that we chose $n \geq r(G, q)$, thus by definition of multicolor Ramsey numbers Painter can always achieve his goal and end the game in at most $\binom{n}{2}$ rounds. Note that during the thesis we also present Builder strategies, where she sometimes forbids less than f colors. These can be considered as if she would choose the remaining forbidden colors arbitrarily as they are irrelevant to the strategy.

Definition 4.2 (Ramsey turnaround number [34])**.** *The Ramsey turnaround number $\mathfrak{R}_f(G, n, q)$ equals the minimum number of exposed edges in the RTA game $\mathcal{G}(G, n, f, q)$, that Painter needs to reach his goal, a monochromatic copy of G, no matter what Builder's strategy is.*

Definition 4.3 (forbiddance number [34]). *The forbiddance number f is the number of colors the Builder is allowed to forbid for an exposed edge.*

Definition 4.4 (forcing). *A player is forced to make a move or one of a set of moves if any other moves are either invalid or lead to the other player winning the game. If a player can force an edge-colored graph G, he has a strategy for achieving a copy of G regardless of the other player's moves.*

Example 4.5. *In the game $\mathcal{G}(G, n, 1, 3)$ and step $b_i = (e, \{3\})$, Painter is forced to use either color 1 or 2 on edge e.*

Let $\mathcal{G}(G, n, f, q)$ be a game and $\mathfrak{R}_f(G, n, q)$ be the corresponding RTA number. Then from the game theoretical background of RTA numbers follows, that

- there exists a Builder strategy assuring that the game $\mathcal{G}(G, n, f, q)$ does not end in $\mathfrak{R}_f(G, n, q) - 1$ rounds,
- there exists a Painter strategy assuring that the game $\mathcal{G}(G, n, f, q)$ ends in at most $\mathfrak{R}_f(G, n, q)$ rounds.

4.2 Game-theoretical formalization

The Ramsey turnaround game $\mathcal{G}(G, n, f, q)$ is a two-player-zero-sum game with

- set of players $\mathcal{P} = \{\text{Builder, Painter}\}$,
- strategies defining for each game situation the exposed edge and the forbidden colors in the case of Builder, and the chosen color in the case of Painter,
- payoff function is the total number of exposed edges, i.e.

$$u_{\text{Builder}}(s) = \#\{\text{edges exposed during the game when playing strategy } s\},$$
$$u_{\text{Painter}}(s) = -\#\{\text{edges exposed during the game when playing strategy } s\}.$$

Both Builder and Painter want to maximize their payoff functions, thus Builder wants the game to last longer, and Painter wants it to end sooner. This leads to the Nash equilibrium of the game, defined as the Ramsey turnaround number $\mathfrak{R}_f(G, n, q)$.

The Ramsey turnaround game has, inter alia, the following game-theoretical properties [26][36]:

- Sequential game: The two players make their choices after each other, knowing and possibly reacting to each other's choice.
- Complete information game: Both players possess full information about the payoffs, strategies, the current game stands, and the other player's previous moves.
- Repeated game: The game consists of several repetitions of the same round, i.e. Builder exposes an edge and forbids colors, and Painter chooses a color.

The game can be visualized with a *game tree*, a rooted tree where

- a vertex corresponds to a decision point of a player (either Builder or Painter),
- incident edges correspond to the possible choices of the player,
- a root-leaf path corresponds to a complete run of a game,
- the half of the length of the root-leaf path (i.e. the length of the corresponding game run) is assigned to each leaf.

See an example of a game tree for game $\mathcal{G}(P_3, 3, 1, 3)$ in Figure 2. Note that we visualized the case $n = 3$ for the sake of simplicity, even though it does not meet the condition $n \geq r(P_3, 3)$ of the Ramsey turnaround game definition.

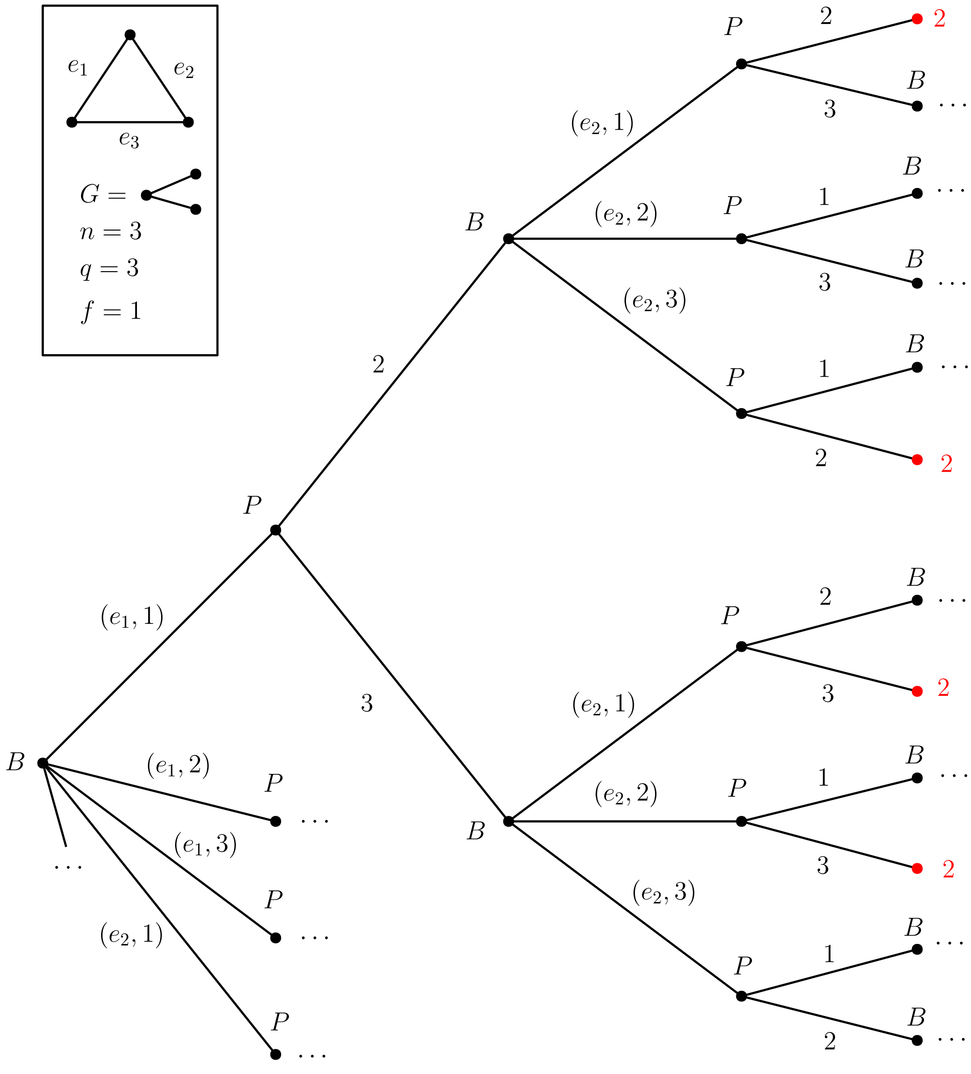


Figure 2: Part of the game tree of game $\mathcal{G}(P_3, 3, 1, 3)$. Leaves with their assigned values are shown in red.

5 The game for simple examples of G

In this section we consider the scenarios of G as $2K_2$ or P_3. We investigate the Ramsey turnaround numbers in different cases regarding the number of colors and the forbiddance number.

5.1 G as $2K_2$

5.1.1 2 colors, 1 forbidden

First note that the two-color case is not really a game: by forbidding a color, Builder forces the Painter to use the other color and thus Painter has no choice, Builder determines the entire course of the game. Also note that in the case of $f = 1$ and $q = 2$, if edges of $3K_2$ are exposed, by Pigeonhole principle a monochromatic copy of $2K_2$ is forced, as shown in Figure 3. Thus Builder wants to avoid the exposure of a copy of $3K_2$.

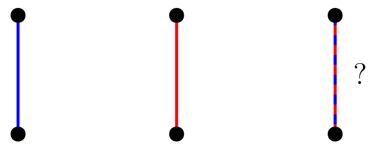


Figure 3: With 3 independent edges Painter can force a monochromatic $2K_2$.

Theorem 5.1. *For any $n \in \mathbb{N}, n \geq 7$ holds, that*

$$\mathfrak{R}_1(2K_2, n, 2) = 2n - 2.$$

Proof. LB: We show $2n - 2 \leq \mathfrak{R}_1(2K_2, n, 2)$ with a strategy for Builder. Let color 1 be blue and color 2 be red. First expose 2 independent edges e_1, e_2. For e_1 forbid color 2 and for e_2 forbid color 1. So Painter colors e_1 in color 1 and e_2 in color 2. Expose all edges of a star $K_{1,n-1}$ containing e_1, and forbid color 2 for each, such that the whole star is colored in color 1. Now expose all edges of a star $K_{1,n-1}$ containing e_2, and forbid color 1 for each, thus the star is colored in 2 (except for one edge being part of both stars, which already has color 1). See the resulting graph in Figure 4. Hence $2n - 3$ edges are exposed with no monochromatic $2K_2$, proving the lower bound $2n - 2 \leq \mathfrak{R}_1(2K_2, n, 2)$.

UB: Now we show $\mathfrak{R}_1(2K_2, n, 2) \leq 2n - 2$. Erdős and Gallai [19] proved

$$\text{ex}(n, tK_2) = \max\left\{\binom{2t-1}{2}, (t-1)(n-t+1) + \binom{t-1}{2}\right\}.$$

Thus we have

$$\text{ex}(n, 3K_2) = \max\left\{\binom{5}{2}, (3-1)(n-3+1) + \binom{2}{1}\right\} = 2n - 3.$$

Hence by definition of extremal numbers, every 2-edge-coloring of $2n - 2$ edges on n vertices contains a monochromatic $2K_2$. Thus regardless of how Builder plays, the game ends after $2n - 2$ rounds, proving the upper bound. □

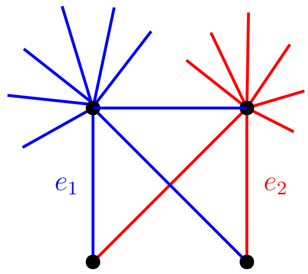


Figure 4: Builder strategy without monochromatic $2K_2$.

5.1.2 3 colors, 1 forbidden

First note that if Builder exposes 3 independent edges, then Painter can reach his goal by either painting the first two edges of this matching in the same color, or otherwise choosing an appropriate color on the third edge.

Lemma 5.2. *Let $n \in \mathbb{N}$ with $n \geq r(2K_2, 3)$ and consider the game $\mathcal{G}(2K_2, n, 1, 3)$. Let Builder play so that at any point of the game, the exposed edges have a vertex cover set of size 2. Then Painter has a strategy that guarantees a game length of at most $n + 2$.*

Proof. We prove this by showing a strategy for Painter. During the whole game, let Painter create a monochromatic $2K_2$ if possible and follow the strategy rules otherwise. Let G be a graph on n vertices with vertex cover set $\{v_1, v_2\}$ and

$$E(G) := \{xy : x = v_1 \text{ or } x = v_2\},$$

as shown in Figure 5.

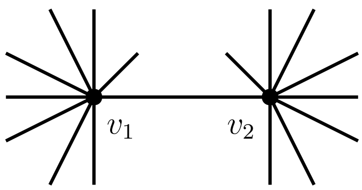


Figure 5: Graph G with main vertices v_1 and v_2.

During the game the size of the minimum vertex cover is at most 2, thus the exposed graph is a subgraph of G at any point of the game. Call the vertices v_1, v_2 in G as *main* vertices. Note that in the exposed graph any vertex w with $\deg(w) \geq 3$ must correspond to a main vertex of G. Thus throughout the whole game, call vertices with at least 3 incident exposed edges as *main* vertices. Note that in the beginning there are no main vertices and after $n + 2$ rounds, there are 2 of them, because by Pigeonhole principle there must be two vertices with a degree of at least 3. Call the first vertex with 3 exposed incident edges as v_1 and the second as v_2. When both v_1 and v_2 are identified, then all exposed edges must be incident to at least one of them. Thus v_1 and v_2 in the exposed graph correspond to v_1 and v_2 in G. When both v_1 and v_2 are identified, we call each edge e with $e \neq v_1v_2$ as *good*. Note that the main vertices v_1, v_2 are not predetermined, neither Builder nor Painter knows them at the beginning of the game. As they only can be identified when 3 incident edges of a vertex are exposed, the goodness of edges can be decided only when both main vertices are discovered. However, if a vertex has multiple incident edges, then we may say that at least all but one of these edges is good, even though it is not decidable yet, which ones.

Painter does not know the main vertices in advance, but he can identify a main vertex when 3 of its incident edges are exposed. Let the Painter paint the exposed edges arbitrarily as long as there is no vertex with 3 incident edges exposed. When Builder exposes the third incident edge of a vertex, this vertex can be identified as a main vertex, say v_1. It is also possible, that this newly exposed edge reveals both v_1 and v_2, i.e. two vertices with degree 3 appear. Then let Painter only focus on v_1 and its 3 incident edges. Let Painter color this newly exposed edge so that the 3 incident edges of v_1 use exactly 2 colors. W.l.o.g. say that it results in edges v_1w_1, v_1w_2, v_1w_3 with colors $1, 1, 2$ respectively. Note that at this point, if v_2 is not identified yet, at most 2 arbitrarily colored other edges may be already exposed, both incident to the future v_2. See the current incident edges of v_1 in Figure 6.

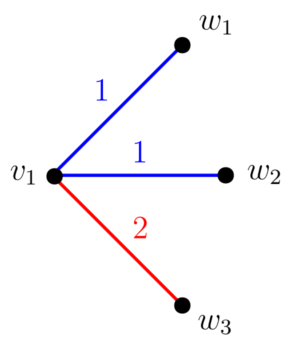


Figure 6: Vertex v_1 has 3 incident edges, using exactly 2 colors. Vertex v_2 may be not yet identified.

Recall that we are defining a Painter strategy for a game of length at most $n + 2$. After the exposure of $n + 2$ edges, the exposed edges build a graph with a vertex cover of size at most 2. With such $n + 2$ edges, there must be two vertices with a degree of at least 4. Thus after $n + 2$ rounds, both of the main vertices are identified and have at least 3 incident good edges exposed. Let Painter continue painting the newly exposed edges arbitrarily as long as one of the following cases occurs: (1) a fourth edge incident to v_1 is exposed, or (2) a third edge incident to another vertex (identifiable as v_2) is exposed. Let us distinguish the two cases:

Case (1): Color the newly exposed edge v_1w_4 so that v_1 has no 3 incident edges of the same color, i.e. color it in either color 2 or 3. At this point, v_2 is unidentified but each vertex except for v_1 has at most 2 incident edges. Continue coloring the exposed edges arbitrarily as long as there is no vertex other than v_1 with 3 incident edges. When the third incident edge e of some other vertex is exposed, this vertex can be identified as v_2. Again, we can distinguish two cases: (1.1) $v_2 \in \{w_1, w_2, w_3, w_4\}$, or

(1.2) $v_2 \notin \{w_1, w_2, w_3, w_4\}$.

Case (1.1): As edge v_1v_2 is already exposed, v_2 has only 1 incident good edge already colored. In a game of length $n+2$, at least 3 incident good edges of v_2 are exposed. Painter can color at least e or some later good edge e' so that it builds a monochromatic $2K_2$ together with v_1w_i for some $i \in [4]$.

Case (1.2): As edge v_1v_2 is not exposed, e is the third incident good edge of v_2. Painter can color e so that together with v_1w_i for some $i \in [4]$ it builds a monochromatic $2K_2$.

Case (2): At this point, Painter identifies v_2, because the newly exposed edge e is the third incident edge of v_2. Color the e so that the 3 edges incident to v_2 use exactly 2 colors. Both v_1 and v_2 have exactly 3 incident edges using 2 colors, possibly not the same 2. Again, we can distinguish the following two cases: (2.1) v_1v_2 is already exposed, or (2.2) v_1v_2 is not exposed yet.

Case (2.1): In this case, both main vertices are identified and have exactly 2 good incident edges exposed. The next exposed edge is incident to either v_1 or v_2, but not both. W.l.o.g. say it is incident to v_1. Color it so that v_1 has 3 incident good edges using exactly 2 colors. The next exposed edge e^* is either incident to v_1 or v_2:

- If $v_1 \in e^*$: Color e^* so that v_1 has no 3 incident good edges the same color. The vertex v_2 has 2 incident good edges exposed. When the third one is exposed, Painter can color it such that a monochromatic $2K_2$ appears.
- If $v_2 \in e^*$: Color e^* so that v_2 has 3 incident good edges using exactly 2 colors. Vertex v_1 has 3 incident good edges exposed using exactly 2 colors as well. W.l.o.g. say v_1 has incident good edges in colors $(1,1,2)$, then v_2 must have its incident good edges in colors $(2,3,3)$, as shown in Figure 7. Otherwise a monochromatic $2K_2$ would already exist. But then Painter can color the next edge so that a monochromatic $2K_2$ appears.

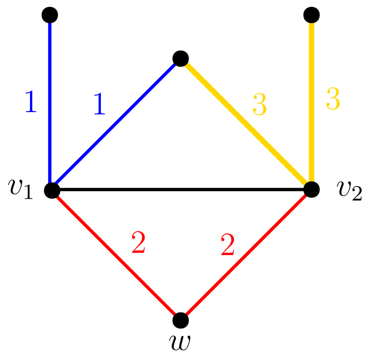


Figure 7: Both v_1 and v_2 have an incident edge in color 2, sharing a vertex w.

Case (2.2): Similarly to Case (2.1), both main vertices have 3 incident good edges exposed. Say v_1 has incident good edges in colors $(1,1,2)$, then v_2 has incident good edges in colors $(2,3,3)$, otherwise a monochromatic $2K_2$ already exists. Then Painter can color the next exposed edge such that a monochromatic $2K_2$ appears.

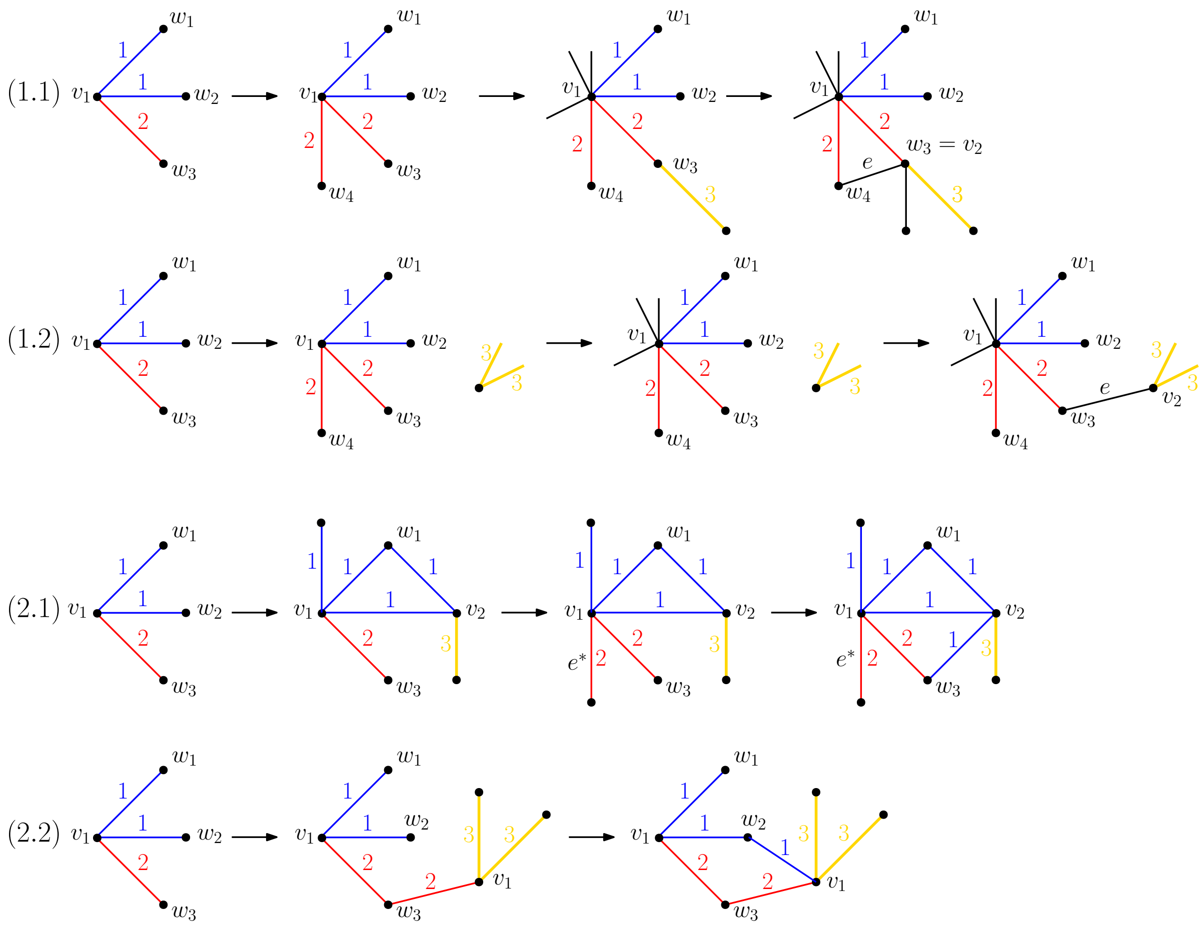


Figure 8: Examples of the scenarios (1.1) – (2.2).

See examples of the four cases (1.1) – (2.2) in Figure 8.

□

Definition 5.3 ($N(X)$)**.** *For G graph and $X \subseteq V(G)$ let*

$$N(X) := \{v : v \in V(G) \setminus X \text{ and } \exists x \in X \text{ such that } vx \in E(G)\}.$$

Theorem 5.4 (Hall [38])**.** *Let $G = (A \cup B, E)$ a bipartite graph with parts A and B. Then the following holds:*

$$\exists \text{ matching covering } A \Leftrightarrow \forall X \subseteq A : |X| \leq |N(X)|.$$

Lemma 5.5. *Let $n \in \mathbb{N}$ with $n \geq r(2K_2, 3)$ and consider the game $\mathcal{G}(2K_2, n, 1, 3)$. If at some point during the first $n+2$ rounds of the game there are at least 3 vertices $\{v_1, v_2, v_3\}$ with $\deg(v_i) \geq 3$, then Painter can force the game to end by round $n+3$.*

Proof. Note, that if Builder exposes 3 independent edges, then Painter can reach his goal by either painting the first two edges of this matching in the same color, or otherwise choosing an appropriate color on the third edge. During the whole game let Painter create a monochromatic $2K_2$ if possible and color the exposed edge arbitrarily otherwise. Let G graph on n vertices and with vertices $\{v_1, v_2, v_3\}$ having $\deg(v_i) \geq 3$. It suffices to show that G has at most $n+2$ edges or contains a copy of $3K_2$ to conclude the proof.

Let $A := \{v_1, v_2, v_3\}$ with $v_i \in V(G)$ and $\deg(v_i) \geq 3$. In the following, we refer to the condition

$$\forall X \subseteq A : |X| \leq |N(X)|$$

from Theorem 5.4 as Hall's condition. There are 3 independent edges covering A if and only if Hall's condition is fulfilled. We distinguish cases based on the number of edges induced by A, as shown in Figure 9.

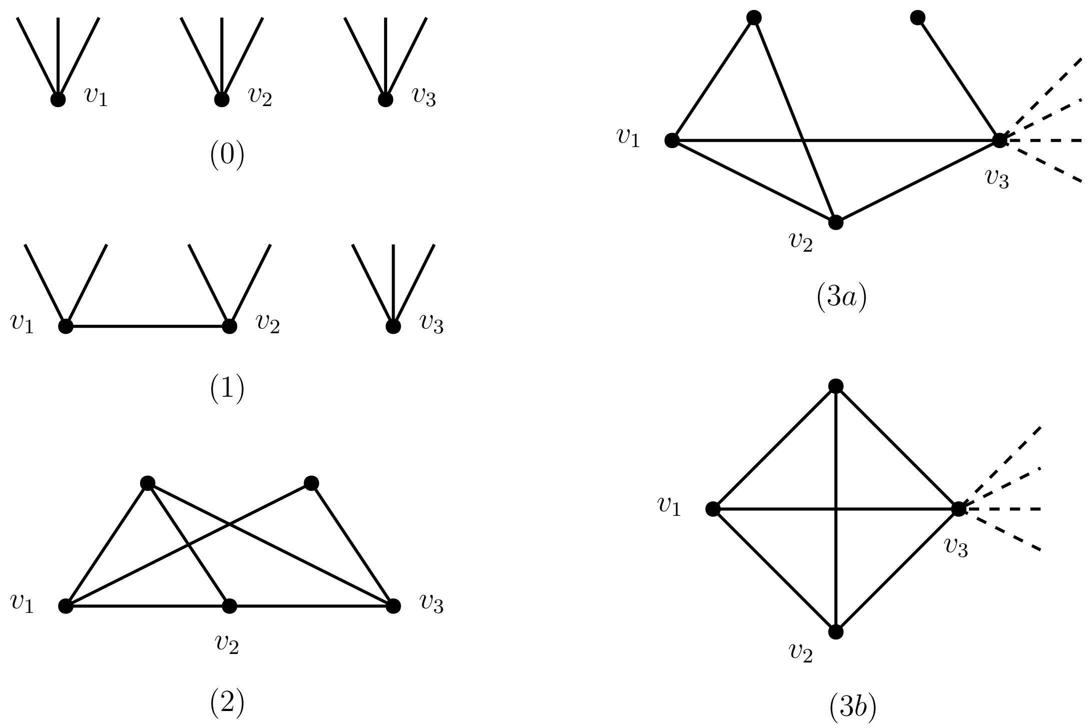


Figure 9: Case distinction based on the number of edges induced by A.

Case 0: If A induces no edges, $|N(v_i)| \geq 3$ for each $i \in [3]$. Thus Hall's condition is fulfilled, G contains a $3K_2$.

Case 1: W.l.o.g. say A only induces the edge v_1v_2. Then $|N(v_1)| \geq 2$, $|N(v_2)| \geq 2$ and $|N(v_3)| \geq 3$. Thus Hall's condition is fulfilled, G contains a $3K_2$.

Case 2: In this scenario, the only G violating Hall's condition, and thus not containing a $3K_2$ is as shown in Figure 9 (2). We have $n > 4$, so this G has at most $n+2$ edges.

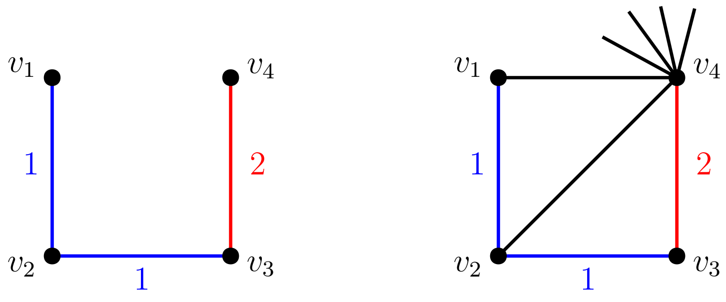
Case 3: In this case, there are two scenarios where the incident edges of v_1, v_2, v_3 violate Hall's condition. These two scenarios are shown in Figure 9 (3a) and (3b). In the case of (3a), adding incident edges to v_3 keeps the graph $3K_2$-free. However, any additional edge not incident to v_3 creates a $3K_2$. Thus a $3K_2$-free G has at most $n+2$ edges in this case. In the case of (3b) we have a K_4. Adding additional edges incident to the same vertex, w.l.o.g. say v_3, does not create a $3K_2$. This allows up to $n+2$ vertices. However, adding additional edges incident to multiple vertices of K_4 keeps G to be $3K_2$-free only as long as $G \subseteq K_5$. So graph G has at most $n+2$ edges or contains a $3K_2$ in case (3) too, concluding our proof.

□

Theorem 5.6. *For any $n \in \mathbb{N}$ with $n \geq r(2K_2, 3)$ we have*

$$n + 2 \leq \mathfrak{R}_1(2K_2, n, 3) \leq n + 3.$$

Proof. LB: We prove $n+2 \leq \mathfrak{R}_1(2K_2, n, 3)$ by showing a strategy for Builder. First, Builder wants to force a P_4 colored as shown in Figure 10 (a). She starts by exposing two independent edges v_1v_2 and v_3v_4 forcing them to have different colors by forbidding the color of v_1v_2 when exposing v_3v_4. W.l.o.g. say that Painter colors them in colors 1 and 2, respectively. Then Builder exposes edge v_2v_3 and forbids color 3, so the Painter must use one of the previous colors. Say he chooses color 1. Then Builder exposes all edges incident to v_4 forbidding color 1, resulting in the graph shown in Figure 10 (b). This construction has $n+1$ edges and contains no monochromatic copy of $2K_2$, no matter how Painter colors edges incident to v_4 in 2 or 3. Thus the bound $n+2 \leq \mathfrak{R}_1(2K_2, n, 3)$ is proven.



(a) (b)

Figure 10: Builder strategy without monochromatic copy of $2K_2$.

UB: We prove $\mathfrak{R}_1(2K_2, n, 3) \leq n+3$ by combining the proofs of Lemmas 5.2 and 5.5 to create a Painter strategy. Let Painter always create a monochromatic $2K_2$ if possible. Hence he ends the game as soon as 3 independent edges are exposed, either by painting the first two edges of this matching in the same color, or otherwise by choosing an appropriate color on the third edge. Now let Painter follow the strategy presented in the proof of Lemma 5.2. If during the whole game the exposed graph has a minimum vertex cover of size 2, then by Lemma 5.2 we know, that Painter's strategy ends the game in at most $n+2$ rounds. Otherwise at some point during the first $n+2$ rounds of the game, the exposed graph has a minimum vertex cover of size at least 3. At this point, if the exposed graph has more than one component, then it must contain 3 independent edges and thus Painter can end the game. Otherwise the exposed graph is connected and has a minimum vertex cover of size at least 3. The same holds after round $n+2$, the exposed graph is connected, has $n+2$ edges, and has a minimum vertex cover of size at least 3. This graph either contains 3 independent edges, letting Painter end the game, or 3 vertices $\{v_1, v_2, v_3\}$ with $\deg(v_i) \geq 3$. In this latter case, by Lemma 5.5 we know that Painter ends the game by round $n+3$. Thus regardless of how Builder plays, Painter can end the game in at most $n+3$ rounds, concluding the proof. □

5.1.3 q colors, 1 forbidden

Theorem 5.7. *For any $n, q \in \mathbb{R}, n \geq r(2K_2, q), q \geq 4$ we have*

$$n + 1 \leq \mathfrak{R}_1(2K_2, n, q) \leq n + 3.$$

Proof. LB: We prove $n+1 \leq \mathfrak{R}_1(2K_2, n, q)$ by showing a strategy for Builder. Say Painter colors the first exposed edge v_1v_2 in color 1. Let w be a vertex with $w \notin \{v_1, v_2\}$. Expose all edges incident to w and forbid color 1. We get a graph with n edges not containing a monochromatic copy of $2K_2$. Hence the bound $n+1 \leq \mathfrak{R}_1(2K_2, n, q)$ is proven.

UB: Theorem 5.6 showed a Painter strategy proving $\mathfrak{R}_1(2K_2, n, 3) \leq n+3$. Let Painter use only colors $\{1, 2, 3\}$ and play according to the strategy shown in Theorem 5.6. As this strategy guarantees a game length of at most $n+3$, the upper bound is proven. □

5.1.4 q colors, f forbidden with f>1

Theorem 5.8. *For any $n, q, f \in \mathbb{R}$ with $n \geq r(2K_2, q)$ and $f < q$ we have*

$$f + n \leq \mathfrak{R}_f(2K_2, n, q) \leq (f+1)\left(n - 1 - \frac{f}{2}\right) + 1.$$

Proof. LB: We prove the lower bound with a strategy for Builder. First, expose $f+1$ independent edges $e_1, \ldots, e_{f+1}$ and always forbid every previously used color to force Painter to color the $f+1$ edges in different colors, $c_1, \ldots, c_{f+1}$ respectively. Now expose edges of a full star $K_{1,n-1}$ containing the first edge e_1, forbidding colors $c_2, \ldots, c_{f+1}$. Builder exposed $n+f-1$ edges without having a monochromatic copy of $2K_2$, thus the lower bound is proved.

UB: If Builder exposes $f+2$ independent edges $e_1, \ldots, e_{f+2}$, Painter can always force a monochromatic copy of $2K_2$: either the first $f+1$ edges already contain two of the same color, or the last edge can be colored in one of the previous $f+1$ colors. Hence we have

$$\mathfrak{R}_f(2K_2, n, q) < ex(n, (f+2)K_2) + 1.$$

By Erdős and Gallai [19] we have

$$\text{ex}(n, tK_2) = \max\left\{\binom{2t-1}{2}, (t-1)(n-t+1) + \binom{t-1}{2}\right\}.$$

Thus we get

$$\text{ex}(n, (f+2)K_2) = (f+1)\left(n - \frac{f}{2} - 1\right),$$

proving the upper bound.

□

5.2 G as P_3

The case of P_3 as G was considered in Mirbach's thesis [34], and most theorems in this section are derived from her thesis.

5.2.1 2 colors, 1 forbidden

Note that the two-color version is not really a game, as by forbidding one color, Builder leaves no choice for Painter but forces him to use the other color.

Theorem 5.9 ([34])**.** *For any $n \in \mathbb{N}, n \geq r(P_3, 2)$ holds, that*

$$\mathfrak{R}_1(P_3, n, 2) = \begin{cases} n+1 & n \text{ even,} \\ n & n \text{ odd.} \end{cases}$$

5.2.2 3 colors, 1 forbidden

Theorem 5.10 ([34])**.** *For any $n \in \mathbb{N}, n \geq r(P_3, 3)$ holds, that*

$$\mathfrak{R}_1(P_3, n, 3) = n.$$

Proof. LB: We prove $n \leq \mathfrak{R}_1(P_3, n, 3)$ by giving a strategy for Builder assuring that after the exposure of $n-1$ edges, there is no monochromatic P_3. Let P be a path on n vertices, i.e. $P = (v_1, \cdots, v_n)$. Builder first exposes edge v_1v_2. In rounds $2 \leq i \leq n-1$ Builder exposes edge v_iv_{i+1} and forbids the color of edge $v_{i-1}v_i$. After $n-1$ rounds there is no monochromatic copy of P_3, proving $n \leq \mathfrak{R}_1(P_3, n, 3)$.

UB: We prove $n \geq \mathfrak{R}_1(P_3, n, 3)$ by giving a strategy for Painter assuring that he creates a monochromatic P_3 in at most n steps. We call an isolated edge *lonely* if the graph has no other isolated edge in its color. We call an edge-colored graph H *good* for Painter if

1. there is a lonely edge or
2. considering the set S of edges incident to vertices with degree 1 (and S containing two copies of each independent edge), there is a color i for which S contains an odd number of edges in color i or
3. there is a monochromatic copy of P_3.

See examples of good graphs fulfilling conditions 1. and 2. in Figure 11 (a) and (b), respectively.

The Painter strategy is to create a monochromatic P_3 if possible, otherwise color the newly exposed edge so that the resulting graph is good. Note that if 3 edges incident to

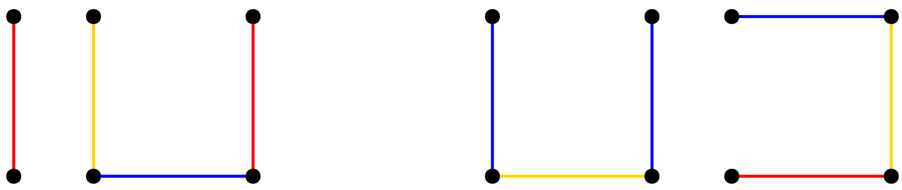

(a) (b)

Figure 11: Good graphs fulfilling conditions 1.(a) and 2.(b).

the same vertex are exposed, Painter can achieve his goal. For the sake of contradiction assume that Builder can play so that regardless of Painter's strategy, after n rounds there is no monochromatic copy of P_3 (i.e. there is a Builder strategy proving a lower bound of $n+1$). Painter's strategy ensures that after n rounds the colored graph is still good. As all vertices have a degree of at most 2, the graph consists of cycles. Then conditions 1. and 2. of a good graph are not fulfilled as there is no vertex of degree 1. So the good graph fulfills the 3. condition and contains a monochromatic copy of P_3. This is a contradiction, so the assumption was wrong and after n rounds Painter can force a monochromatic P_3.

The only thing left is to show that Painter can always color the newly exposed edge so that the resulting graph is good. Call the firstly exposed edge starter, and say that Painter colors it red. Starter is lonely so the graph is good. As long as the starter is lonely, Painter can keep the graph good by using the other two colors, keeping the starter lonely. When the first edge incident to the starter is exposed, after coloring it the resulting graph contains only one vertex with degree 1 and red incident edge, so by the 2. condition, it is good. As there is an even number of vertices with degree 1, there are two colors fulfilling the 2. condition. From this point on, Painter can always choose a color for the new edge so that the 2. condition stays fulfilled. This way Painter can keep the graph good until n edges are exposed, proving $\mathfrak{R}_1(P_3, n, 3) \leq n$.

□

5.2.3 q colors, 1 forbidden

Theorem 5.11 ([34]). *For any $n \in \mathbb{N}$ with $n \geq r(P_3, q)$ and $q \geq 3$ we have*

$$\mathfrak{R}_1(P_3, n, 3) = n.$$

Proof. The Builder strategy shown in the proof of Theorem 5.10 proves the lower bound $n \leq \mathfrak{R}_1(P_3, n, 3)$ in the case of $q \geq 3$ as well. To prove the upper bound $n \geq \mathfrak{R}_1(P_3, n, 3)$, let Painter use only the colors $\{1, 2, 3\}$. Then he can use the Painter strategy presented in the proof of Theorem 5.10 to create a monochromatic P_3 in at most n rounds. □

5.2.4 q colors, f forbidden with f>1

Theorem 5.12. *For any $n, q, f \in \mathbb{N}$ with $n \geq r(P_3, q)$ and $f < q$ we have*

$$\frac{f(n-f)}{4} < \mathfrak{R}_f(P_3, n, q) \leq \frac{n(f+1)}{2} + 1.$$

Proof. This is actually a special case of Theorem 9.6, which provides general bounds for stars.

LB: We prove the lower bound by giving a strategy for Builder. First consider the case when f is even with $f = 2k$. For each edge xy of K_{k+2} holds, that from both x and y there are k other incident edges. They use at most $2k = f$ colors. By forbidding these at most f colors for each edge xy, Builder can prevent an occurrence of a monochromatic copy of P_3.

Hence for even f, Builder can expose $\left\lfloor \frac{n}{\frac{f}{2}+2} \right\rfloor = \left\lfloor \frac{2n}{f+4} \right\rfloor > \frac{2n-f-4}{f+4}$ disjoint $K_{\frac{f}{2}+2}$ without a monochromatic P_3. This gives a lower bound of

$$\frac{2n-f-4}{f+4} \cdot \binom{\frac{f}{2}+2}{2} = \frac{(2n-f-4)(f+2)}{8} > \frac{f(n-f)}{4}.$$

For odd f she can act forbidding at most $f-1$ colors in each step and thus use the even-case strategy, exposing edges of several $K_{\frac{f+3}{2}}$ instances. She can expose the edges of $\left\lfloor \frac{n}{\frac{f+3}{2}} \right\rfloor = \left\lfloor \frac{2n}{f+3} \right\rfloor > \frac{2n-f-3}{f+3}$ disjoint $K_{\frac{f+3}{2}}$ instances without a monochromatic P_3, proving a lower bound of

$$\frac{2n-f-3}{f+3} \cdot \binom{\frac{f+3}{2}}{2} = \frac{(2n-f-3)(f+1)}{8} > \frac{f(n-f)}{4}.$$

Note that this lower-bound construction can be slightly increased by exposing a clique of the remaining vertices, see Figure 12.

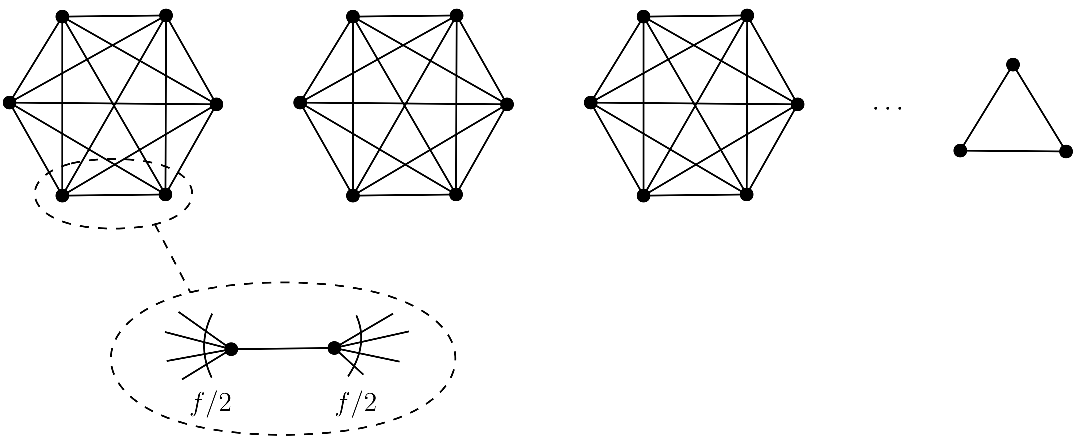


Figure 12: Builder strategy construction for $\mathfrak{R}_f(P_3, n, q)$.

UB: If Builder exposes all edges of a star $K_{1,f+2}$, then Painter can force a monochromatic P_3. If he is forced to color the first $f+1$ edges in different colors, then Builder cannot forbid all $f+1$ colors for the last exposed edge. By choosing one of the previously used colors, Painter creates a monochromatic P_3. Hence Builder can expose at most $ex(n, K_{1,f+2})$ edges without a $K_{1,f+2}$. A graph being $K_{1,f+2}$-free means that each vertex has the degree of at most $f+1$. Thus $ex(n, K_{1,f+2}) \leq \frac{n(f+1)}{2}$ proves the upper bound.

$\square$

6 The two-color case: connection to chromatic Ramsey numbers

The two-color case is actually not a really game itself, as with one color forbidden for each edge, Painter has one single color left to choose. The same holds for any $f = q - 1$ setup. In this part we consider the two-color case of the Ramsey turnaround game. Although it is not interesting in the game-theoretical sense, it has a strong connection to the chromatic Ramsey number, which was introduced by Burr, Erdős and Lovász in [10]. The content of this section builds on Simon Gaa's Bachelor's thesis [27], who reiterated and complemented the results of Erdős et al. [21] and Hancock et al. [30].

Definition 6.1 (chromatic Ramsey number)**.** *The chromatic Ramsey number of a graph H, denoted by $R_\chi(H)$, is defined as the minimum chromatic number of a graph G containing a monochromatic copy of H in any 2-edge-coloring, i.e.*

$$R_\chi(H) := \min\{k : \exists G \text{ graph with } \chi(G) = k \text{ and } G \to H\}.$$

The function *f(n,H)* was first introduced by Bialostocki et al. [8] in 1990, and also considered by Erdős et al. [21].

Definition 6.2 ($f(n, H)$)**.** *Let H be a graph and $n \in \mathbb{N}$. We define the function*

$$f(n, H) := \max\{||G|| : |G| = n, G \not\to H\},$$

i.e. the maximum number of edges in a graph G on n vertices which can be edge-partitioned into two H-free subgraphs.

Theorem 6.3. *Let $n \in \mathbb{N}$ and G graph. Then we have*

$$\mathfrak{R}_1(G, n, 2) = f(n, G) + 1.$$

Proof. By forbidding one of the two colors, Builder determines the color for each exposed edge. Thus her best strategy is to expose and color with two colors as many edges as she can without creating a monochromatic copy of G. By definition, this means exposing $f(n, G)$ edges. Exposing one more edges creates a monochromatic copy of G. □

Theorem 6.4 ([27])**.** *Let H be a bipartite graph and let $n \in N$. Then we have*

$$\mathfrak{R}_1(H, n, 2) = f(n, H) + 1 = 2\text{ex}(n, H)(1 - o(1)) + 1.$$

Definition 6.5 (homomorphism)**.** *For H and H' graphs, a homomorphism from H to H' is a function $f : V(H) \to V(H')$ such that for $\forall uv \in E(H) : f(u)f(v) \in E(H')$. We say that H' is a homomorphic image of H if such f exists. The set of all homomorphic images of H is denoted by $\mathcal{H}(H)$.*

Example 6.6 ($\mathcal{H}(C_5)$)**.** *See Figure 13 showing the all homomorphic images of C_5.*

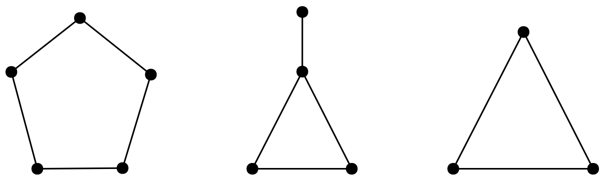


Figure 13: $\mathcal{H}(C_5)$.

Theorem 6.7 (Theorem 3.17 in [27]). *Let H be a graph with $\chi(H) = 3$ and let $n \in \mathbb{N}$. Then we have*

$$\mathfrak{R}_1(H, n, 2) = f(n, H) + 1 = \begin{cases} T_4(n)(1 + o(1)) + 1 & C_5 \in \mathcal{H}(H), \\ T_5(n)(1 + o(1)) + 1 & \textit{else}. \end{cases}$$

Theorem 6.8 (Theorem 3.18 in [27]). *Let H be a graph with $\chi(H) \geq 3$ and let $n \in \mathbb{N}$. Then we have*

$$\mathfrak{R}_1(H, n, 2) = f(n, H) + 1 = T_{R_\chi(H)-1}(n)(1 + o(1)) + 1 = \left(1 - \frac{1}{R_\chi(H) - 1}\right)\binom{n}{2}(1 + o(1)) + 1.$$

7 The game for general G

In this chapter we first examine monotonicity properties of Ramsey turnaround numbers. Then we provide several lower and upper bounds, some given by Mirbach [34] and some using various graph theory concepts like applying extremal numbers, maximum degrees or polychromatic numbers.

7.1 Monotonicity of Ramsey turnaround numbers

Theorem 7.1 (monotonicity of f [34])**.** *Let G graph and $f, q, n \in \mathbb{N}$ with $f < q$ and $n \geq r(G, q)$. Let $f_0 \in \mathbb{N}$ such that $f < f_0 < q$, then we have*

$$\mathfrak{R}_f(G, n, q) \leq \mathfrak{R}_{f_0}(G, n, q).$$

Theorem 7.2 (monotonicity of q [34])**.** *Let G graph and $f, q, n \in \mathbb{N}$ with $f < q$ and $n \geq r(G, q)$. Let $q_0 \in \mathbb{N}$ such that $q < q_0$, then we have*

$$\mathfrak{R}_f(G, n, q) \geq \mathfrak{R}_f(G, n, q_0).$$

Theorem 7.3 (monotonicity of f and q)**.** *Let G graph and $q, n \in \mathbb{N}$ with $q > 1$ and $n \geq r(G, q)$. Let $f, q_0 \in \mathbb{N}$ with $1 \leq f < q_0 \leq qf$. Then we have*

$$\mathfrak{R}_1(G, n, q) \leq \mathfrak{R}_f(G, n, q_0).$$

Proof. Let $B := \mathfrak{R}_1(G, n, q)$. There exists a Builder strategy for the game $\mathcal{G}(G, n, 1, q)$ exposing at least $B - 1$ edges so that in the end there is no monochromatic copy of G. Call this strategy the *original* strategy. Let $S_1 \cup \cdots \cup S_q = [q_0]$ with S_i's being pairwise disjoint sets of size at most f. Then we can modify the original strategy so that for each $i \in [q]$, instead of forbidding color i, Builder forbids all colors of the set S_i. As the original strategy guarantees for each $i \in [q]$ that no monochromatic copy of G in color i occurs, the modified strategy ensures that no monochromatic copy of G in any color of S_i occurs. Thus $B \leq \mathfrak{R}_f(G, n, q_0)$ proves the statement. □

Remark. Note that Theorems 7.1 and 7.2 show, that increasing the forbiddance number leads to a larger or equal RTA number, while increasing the number of colors leads to a smaller or equal RTA number. However, Theorem 7.3 shows cases where increasing both q and f leads to a larger or equal RTA number.

Theorem 7.2 raises the question if strict inequality can be achieved, i.e. if increasing the number of colors can lead to a smaller RTA number. Example 7.4 addresses this question.

Example 7.4 (strict inequality in Theorem 7.2)**.** *Mirbach's thesis [34] provides an example of strict inequality in Theorem 7.2. We have*

$$\mathfrak{R}_1(P_3, 6, 2) = 7 > 6 = \mathfrak{R}_1(P_3, 6, 3).$$

For proofs see Theorems 5.5 and 7.10 in [34].

7.2 General upper bounds

Observation 7.5 (trivial upper bound). *Let G graph, $f, q, n \in \mathbb{N}$ with $n \geq r(G, q)$ and $f < q$. Then we have*

$$\mathfrak{R}_f(G, n, q) \leq \binom{n}{2}.$$

Theorem 7.6 (easy upper bound [34]). *Let G graph, $f, q, n \in \mathbb{N}$ with $n \geq r(G, q)$ and $f < q$. Then we have*

$$\mathfrak{R}_f(G, n, q) \leq \text{ex}(n, K_{r(G,q)}) + 1.$$

Theorem 7.7 (general upper bound). *Let G graph, $f, q, n \in \mathbb{N}$ with $n \geq r(G, q)$ and $f < q$. Then we have*

$$\mathfrak{R}_f(G, n, q) \leq \min\Big\{(f+1)\text{ex}(n, G); \text{ex}(n, K_{r(G,f+1)})\Big\} + 1.$$

Proof. We prove this by showing two strategies for Painter. First we prove

$$\mathfrak{R}_f(G, n, q) \leq (f+1)\text{ex}(n, G) + 1.$$

Let Painter use only colors $[f+1]$ for coloring. For every exposed edge he may choose a color from this set arbitrarily. As Builder may forbid at most f colors, there is always at least one available for Painter to choose from. After $(f+1)\text{ex}(n, G) + 1$ edges are exposed, by Pigeonhole principle there exists a color class containing at least $\text{ex}(n, G)+1$ colors. By definition of extremal numbers, a monochromatic copy of G is present in this color.

Now we show

$$\mathfrak{R}_f(G, n, q) \leq \text{ex}(n, K_{r(G,f+1)}) + 1.$$

We can prove it by combining the previous idea with Theorem 7.6. Let Painter use only the color set $[f+1]$ for coloring. For every exposed edge he may choose a color from this set arbitrarily. By definition of the multicolor Ramsey number, every $(f+1)$-edge-colored graph on n vertices and $ex(n, K_{r(G,f+1)}) + 1$ edges contains a monochromatic copy of G, concluding the proof. □

Remark. Note that the tightness of the upper bound of $(f+1)\text{ex}(n, G)+1$ in Theorem 7.7 varies for different choices of G. For example, in the case of complete graphs we have $\binom{n}{2} < (f+1)\text{ex}(n, K_t)$, thus in this case the bound is useless. However, in other cases it can lead to stronger results, like in the case of paths in Theorem 9.2. Also, the bound of $\text{ex}(n, K_{r(G,f+1)}) + 1$ might be the stronger one, however it is more difficult to apply due to our limited knowledge of multicolor Ramsey numbers.

7.3 General lower bounds

Observation 7.8 (trivial lower bound). *Let G graph and $f, q, n \in \mathbb{N}$ with $n \geq r(G, q)$ and $f < q$. Then we have*

$$\text{ex}(n, G) < \mathfrak{R}_f(G, n, q).$$

As a corollary of the lower bound proof of Theorem 9.6, we present another general lower bound:

Theorem 7.9 (lower bound with $\Delta(G)$)**.** *Let G be a graph, $f, q, n \in \mathbb{N}$ and $\Delta := \Delta(G)$ with $f < q$ and $n \geq r(G, q)$. Then we have*

$$\frac{n((f+1)(\Delta-1)+1)}{4} < \mathfrak{R}_f(G, n, q).$$

Proof. First, note that graph G has a vertex $v \in V(G)$ with $\deg(v) = \Delta$. Thus the star $K_{1,\Delta}$ is a subgraph of G. During the game, if there is no monochromatic star $K_{1,\Delta}$, there is also no monochromatic copy of G. Thus we have

$$\mathfrak{R}_f(K_{1,\Delta}, n, q) \leq \mathfrak{R}_f(G, n, q).$$

In Theorem 9.6 we proved for stars the lower bound of

$$\frac{n((f+1)(t-1)+1)}{4} < \mathfrak{R}_f(K_{1,t}, n, q).$$

Applying this result for the star $K_{1,\Delta}$ proves our statement. □

Definition 7.10 (H-polychromatic number)**.** *Let H be a graph and $|H| < n$. An edge-coloring of K_n with k colors is H-polychromatic if all subgraphs in K_n isomorphic to H contain all k colors of the coloring. The H-polychromatic number of K_n, denoted by $poly_H(K_n)$, is the largest k such that a H-polychromatic edge-coloring of K_n with k colors exists.*

Theorem 7.11 (general lower bound)**.** *Let G graph and $t, f, q, n \in \mathbb{N}$ with $n \geq r(G, q)$ and $f < q \leq f \cdot poly_G(K_t)$ and $|G| \leq t$. Then we have*

$$\left\lfloor \frac{n}{t} \right\rfloor \binom{t}{2} < \mathfrak{R}_f(G, n, q).$$

Proof. We show a strategy for Builder to prove the lower bound. First consider the case of $q = poly_G(K_t)$ and $f = 1$. Let $c : E(K_t) \to [q]$ be a G-polychromatic coloring with q colors. Let Builder expose all edges $e \in E(K_t)$ and forbid color $c(e)$ for them, then for each copy of G in K_t every color is forbidden for at least one of its edges. Thus there is no monochromatic copy of G. She continues by exposing all edges of other disjoint K_t's similarly. Builder can expose $\left\lfloor \frac{n}{t} \right\rfloor$ copies of K_t without any monochromatic copy of G, proving the lower bound.

In the case of fewer colors, she can do the same but not forbid any colors for edges $e \in E(K_t)$ where $c(e) > q$. For larger forbiddance number f, she can apply the strategy shown in Theorem 7.3. Let the set of colors be $[q] = S_1 \cup \cdots \cup S_{poly_G(K_t)}$, where S_i's are pairwise disjoint sets of size at most f. Builder forbids S_i on each edge $e \in E(K_t)$ where $c(e) = i$. This strategy guarantees that there is no monochromatic copy of K_t and allows up to $f \cdot poly_G(K_t)$ colors altogether. □

Remark. The presented lower bound results are applicable in different cases. The trivial bound of $\mathrm{ex}(n, G)$ is a weak bound but applicable without further knowledge. The bound shown in Theorem 7.9 requires only the additional knowledge of the maximum degree of G. The bound shown in Theorem 7.11 is more difficult to apply due to our limited knowledge of G-polychromatic numbers.

8 G as complete graphs

This section presents the main results on RTA numbers for complete graphs. Theorem 8.3 gives a lower bound using a construction via matchings. Theorem 8.10 improves the lower bound by extending the same idea to balanced colorings. Theorem 8.12 achieves an even greater improvement for large enough t. We also tried improving the previous idea via the probabilistic method in Theorem 8.13, resulting in a stronger bound for very large t. Finally, we give an upper bound proof in 8.15. We have compiled Table 2 presenting RTA results on complete graphs for easier comparison.

Theorem	Bound	Method	Type	Comment
7.8	$\left(1-\frac{1}{t-1}\right)\binom{n}{2}$	extremal numbers	LB	
8.3	$\left(1-\frac{1}{2t-3}\right)\binom{n}{2}$	matchings	LB	$q \leq 2t-3$
8.10	$\left(1-\frac{1}{(\frac{t}{2})^2}\right)\binom{n}{2}$	balanced colorings	LB	$q \leq \frac{t}{2}$
8.12	$\left(1-\frac{1}{(t-t^{0.525})^2}\right)\binom{n}{2}$	balanced colorings	LB	$q \leq t-t^{0.525}-1$, t large enough
8.13	$\left(1-\frac{1}{t^{\frac{t^{1-\epsilon}}{\ln t}}}\right)\binom{n}{2}$	probabilistic	LB	$t > t_0$ for some t_0 with $q \in o(t_0^\epsilon)$
8.15	$\left(1-\frac{1}{q^{qt}}\right)\binom{n}{2}$	q-color Ramsey	UB	

Table 2: Bounds on $\mathfrak{R}_1(K_t, n, q)$.

Remark. Note that our best lower bound is in Theorem 8.13, proving

$$\left(1-\frac{1}{t^{\frac{t^{1-\epsilon}}{\ln t}}}\right)\binom{n}{2} \leq \mathfrak{R}_1(K_t, n, q)$$

with $t > t_0$ for some t_0 with $q \in o(t_0^\epsilon)$. The best upper bound is in Theorem 8.15, proving

$$\mathfrak{R}_1(K_t, n, q) \leq \left(1-\frac{1}{q^{qt}}\right)\binom{n}{2}.$$

For the sake of easier comparison of our best lower and upper bounds, we can rewrite the upper bound by substituting $q := t^\epsilon$ (acquired from the lower bound), resulting in

$$\mathfrak{R}_1(K_t, n, q) \leq \left(1-\frac{1}{t^{\epsilon t^\epsilon t}}\right)\binom{n}{2} = \left(1-\frac{1}{t^{t^{1+\epsilon}\epsilon}}\right)\binom{n}{2}.$$

8.1 Lower bound via matchings

Definition 8.1 (maximal matching sequencibility)**.** *Let the maximal matching sequencibility of a graph G, denoted by $ms(G)$, be the largest integer s for which there is an ordering of the edges of G such that every s consecutive edges form a matching.*

Lemma 8.2 (Alspach [2]). *Let $n \in \mathbb{N}$. Then we have*

$$ms(K_n) = \left\lfloor \frac{n-1}{2} \right\rfloor.$$

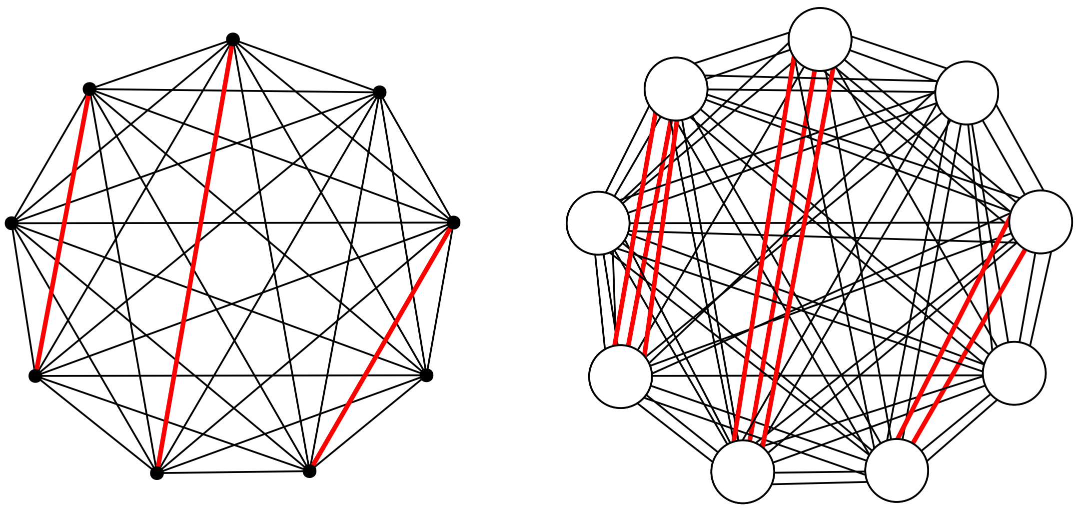
Theorem 8.3. *Let $f, q, n, t \in \mathbb{N}$ with $t > 2, n \geq r(K_t, q)$ and $f < q \leq (2t-3)f$. Then*

$$||T_{2t-3}(n)|| < \mathfrak{R}_f(K_t, n, q).$$

Example 8.4 (proof idea for Theorem 8.3). *Before proving the theorem, we show the proof idea through an example, see Figure 14. This example shows*

$$||T_9(n)|| < \mathfrak{R}_1(K_7, n, 5)$$

by proposing a strategy for Builder. First consider K_9 and fix 3 edges forming a matching, shown in red. Note that any copy of K_7 contains at least one of the three red edges on Figure 14(a). By forbidding color 1 for these three red edges, Builder can make sure that no copy of K_7 appears in color 1 after exposing all edges of K_9. Similarly, in the case of n vertices, she can expose all edges of the Turán-graph $T_9(n)$, and forbid color 1 for all edges corresponding to the red edges, as shown in Figure 14(b). To prevent monochromatic copies of K_7 in other colors as well, she can do for each color $i \in [5]$ the following: pick any matching of size 3 and forbid color i for the 3 matching edges.



(a) K_9 (b) $T_9(n)$

Figure 14: Builder strategy for K_9 and $T_9(n)$ to prevent a monochromatic K_7.

Proof. Let $u := 2t - 3$ and $f = 1$, we discuss larger forbiddance numbers later. We first show a Builder strategy for exposing all edges of K_u, and then expand it to work for exposing all edges of $T_u(n)$, just as in Example 8.4. By forbidding color i for all edges of a matching of size $u-t+1$, Builder prevents a monochromatic K_t in color i. Let $\mathcal{O} = (e_1, \cdots, e_{\binom{V}{2}})$ be an ordering of $E(K_u)$ so that any $\frac{u-1}{2}$ consecutive edges form a matching. Such ordering exists by Lemma 8.2. Note that for $u = 2t-3$ we have

$$u - t + 1 = \frac{u-1}{2},$$

so let $s := u - t + 1$. Let $M_1, \cdots, M_u$ be pairwise disjoint sets of edges with $\forall i \in [u] : |M_i| = s$, so that any edge in M_i appears before any edge in M_j for $i < j$ in the edge-ordering above. In other words, $M_1 = \{e_1, \ldots, e_s\}, M_2 = \{e_{s+1}, \ldots, e_{2s}\}, \ldots$ All edges of K_{2t-3} are ordered, and we have

$$\frac{|E(K_{2t-3})|}{t-2} = \frac{(2t-3)(t-2)}{t-2} = 2t-3,$$

thus we can define $2t-3$ such matchings. For $i \in [2t-3]$, Builder forbids color i for each edge of M_i. Note that each matching M_i spans a set of $u-1 = 2t-4$ vertices, thus any t vertices induce at least one edge of M_i. Thus Builder guarantees that there is no monochromatic K_t in color i for each $i \in [2t-3]$. To expand the strategy to n vertices, take the Turán-graph $T_{2t-3}(n)$ and let the $2t-3$ parts correspond to the vertices of K_{2t-3}. For each edge forbid the same color as which was forbidden for the corresponding edge in K_{2t-3}.

For larger forbiddance number f we can adjust the strategy as in the proof idea of Theorem 7.3. We can define pairwise disjoint color sets $S_1, \ldots, S_{2t-3}$ with

$$\forall i \in [2t-3] : |S_i| \leq f \text{ and } S_1 \cup \cdots \cup S_{2t-3} = [f(2t-3)].$$

We can adjust the Builder strategy by instead of forbidding color i for edge e, forbidding all colors from set S_i for e. So this modified strategy allows $(2t-3)f$ colors altogether. □

Remark. Note that we used Alspach's Theorem 8.2 to show that the whole edge set of K_u can be decomposed into disjoint matchings. In this specific case of $u = 2t-3$, we need complete matchings to prevent a monochromatic copy of K_t. It is well-known that the edge set of K_u can be decomposed into disjoint complete matchings, and thus Alspach's theorem is not strictly necessary in this case. However, the advantage of this proof is that the edge set of K_u can be decomposed into disjoint matchings of any fixed size. In Turán graphs with fewer parts, smaller matchings also suffice to prevent a monochromatic copy of K_t, just as shown in Example 8.4. Thus by reducing the number of Turán parts, the analogous method allows more colors, i.e. it proves slightly weaker lower bounds but for larger q.

8.2 Lower bound via balanced colorings

Definition 8.5 (balanced (r,n)-coloring). *We call an edge r-coloring of K_N a balanced (r,n)-coloring if any set of $\lceil N/r \rceil$ vertices for any $i \in [r]$ contains a monochromatic K_n in color i.*

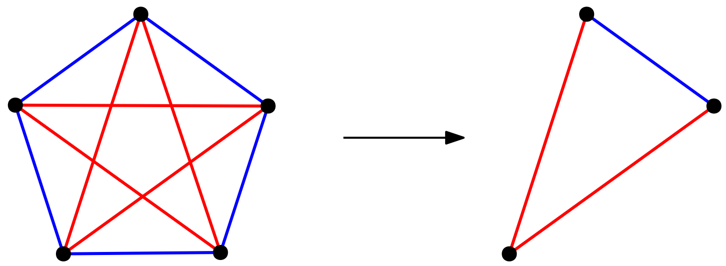


Figure 15: Balanced $(2,2)$-coloring.

Example 8.6 (balanced $(2,2)$-coloring)**.** *An example of a balanced $(2,2)$-coloring of K_5 is shown in Figure 15. Any $\lceil 5/2 \rceil = 3$ vertices induce an edge (a K_2) in both colors.*

Lemma 8.7 (Erdős-Gyárfás, Theorem 5 in [20])**.** *If a finite projective plane of order $r+1$ exists, then K_{r^2+r+1} has a balanced $(r,2)$-coloring. In other words, there is an r-edge-coloring of K_{r^2+r+1} so that for any $i \in [r]$ any $r+2$ vertices induce an edge in color i.*

Remark. Note that a projective plane is a $2-(q^2+q+1, q+1, 1)$-design. Also note that a projective plane of order q can be constructed for any prime power q. The question of whether a projective plane of order k exists is open for other integers k, but the answer is conjectured to be no.

Theorem 8.8. *Let $f, q, n, t \in \mathbb{N}$ with $n \geq r(K_{t+2}, q)$, $f < q \leq t \cdot f$ and with t such that a finite projective plane of order $t+1$ exists. Then*

$$||T_{t^2+t+1}(n)|| < \mathfrak{R}_f(K_{t+2}, n, q).$$

Proof. First we show a strategy for Builder in the case of $q = t$ and $f = 1$. Let $c : E(K_{t^2+t+1}) \to [t]$ be an edge-coloring so that for any $i \in [t]$, any $t+2$ vertices induce an edge in color i. Such a coloring exists by Lemma 8.7. By exposing all $e \in E(K_{t^2+t+1})$ with forbidding color $c(e)$, Builder can prevent the occurrence of a monochromatic K_{t+2}. She can have an analogous strategy on n vertices using the Turán-graph $T_{t^2+t+1}(n)$. Let parts of the Turán-graph correspond to vertices of K_{t^2+t+1}. Let $b : V(T_{t^2+t+1}(n) \to V(K_{t^2+t+1})$ be a function mapping the Turán-graph vertices to their Turán-graph parts. Let each Turán-edge xy correspond to edge $b(x)b(y)$ of K_{t^2+t+1}. Thus we can define $c' : E(T_{t^2+t+1}(n)) \to [t]$ so that for each $e' \in E(T_{t^2+t+1}(n))$ with corresponding edge $e \in E(K_{t^2+t+1})$ we have $c'(e') := c(e)$. By exposing each edge $e' \in E(T_{t^2+t+1}(n))$ and forbidding color $c'(e')$, Builder can prevent the occurrence of a monochromatic K_{t+2}. Thus $||T_{t^2+t+1}(n)|| < \mathfrak{R}_1(K_{t+2}, n, t)$.

In the case of fewer colors, i.e. $q < t$, the strategy works with the adjustment of not forbidding anything on edges $\{e : c'(e) > q\}$. For larger forbiddance number f we can adjust the strategy to work with more colors, just as in proof of Theorem 7.3 by defining sets of colors instead of each color to forbid. This strategy allows up to $t \cdot f$ colors altogether. □

Remark. The previous theorem works for prime power $(t+1)$ only. In the following, we generalize it using Ortlieb's proof idea of Theorem 4.25 in [37].

Lemma 8.9 (Bertrand-Chebyshev Theorem [43])**.** *For all $n \in \mathbb{N}$ there exists a prime p_n such that $p_n \in [n, 2n]$.*

Theorem 8.10. *Let $f, q, n, t \in \mathbb{N}$ with $n \geq r(K_{t+1}, q)$ and $f < q \leq \frac{tf}{2}$. Then*

$$||T_{(\frac{t}{2})^2+\frac{t}{2}+1}(n)|| < \mathfrak{R}_f(K_{t+1}, n, q).$$

Proof. If t is a prime, Theorem 8.8 proves an even stronger bound. Let p_t be a prime such that $p_t \in [\frac{t}{2}, t]$. Such p_t exists by Lemma 8.9. By Lemma 8.7, there exists a (p_t-1)-edge-coloring of $K_{(p_t-1)^2+(p_t-1)+1}$ so that for any color $i \in [p_t-1]$ any (p_t+1) vertices induce an edge in color i. By applying Theorem 8.8 we get

$$||T_{(p_t-1)^2+(p_t-1)+1}(n)|| < \mathfrak{R}_f(K_{p_t+1}, n, (p_t-1)f).$$

As $p_t+1 < t+1$, in the same (p_t-1)-edge-coloring of $K_{(p_t-1)^2+(p_t-1)+1}$ any $t+1$ vertices also induce an edge in each color. As $\frac{t}{2} \leq p_t - 1$, a (p_t-1)-edge-coloring of $K_{(\frac{t}{2})^2+\frac{t}{2}+1}$ with the same property also exists, so

$$||T_{(\frac{t}{2})^2+\frac{t}{2}+1}(n)|| < \mathfrak{R}_f(K_{t+1}, n, (p_t-1)f).$$

As $\frac{t}{2} \leq p_t - 1$, there also exists a $\frac{t}{2}$-edge-coloring of $K_{(\frac{t}{2})^2+\frac{t}{2}+1}$ where any $t+1$ vertices induce an edge in each color as well. Hence

$$||T_{(\frac{t}{2})^2+\frac{t}{2}+1}(n)|| < \mathfrak{R}_f(K_{t+1}, n, \frac{tf}{2}).$$

□

For large enough t the bound can be bettered using the following lemma.

Lemma 8.11 (Baker [4])**.** *There exists $x_0 \in \mathbb{R}$ such that for all $x > x_0$, the interval*

$$[x - x^{0.525}, x]$$

contains at least one prime number.

Theorem 8.12. *Let $f, q, n, t \in \mathbb{N}$ with $n \geq r(K_{t+2}, q)$ and $f < q \leq (t - t^{0.525} - 1) \cdot f$ and t large enough (let $t > x_0$ from Lemma 8.11). Then we have*

$$||T_{(t-t^{0.525}-1)^2+(t-t^{0.525}-1)+1}(n)|| < \mathfrak{R}_f(K_{t+2}, n, q).$$

Proof. Let t' be such that $(t'+1)$ is prime and $(t'+1) \in [t - t^{0.525}, t]$. Such t' exists by Lemma 8.11. By applying Theorem 8.8 we have

$$||T_{t'^2+t'+1}(n)|| < \mathfrak{R}_f(K_{t'+2}, n, q).$$

As $t > t'$ we also have

$$||T_{t'^2+t'+1}(n)|| < \mathfrak{R}_f(K_{t+2}, n, q).$$

As $t - t^{0.525} - 1 \leq t'$ we have

$$||T_{(t-t^{0.525}-1)^2+(t-t^{0.525}-1)+1}(n)|| < \mathfrak{R}_f(K_{t+2}, n, q).$$

□

8.3 Lower bound via the probabilistic method

In this section we give a better lower bound via the probabilistic method. The idea is the same as in Subsections 8.1 and 8.2: we aim to color all edges of a complete graph on possibly many vertices such that each copy of K_t induces edges in all of the q colors.

Theorem 8.13. *Let $\epsilon > 0$ and $q \in \mathbb{N}$. There exists $t_0 \in \mathbb{N}$ with $q \in o(t_0^\epsilon)$ such that $\forall t > t_0$ with $t \in \mathbb{N}$ and $n > \max(r(K_t, q), t^{\frac{t^{1-\epsilon}}{\ln t}})$ we have*

$$||T_{t^{\frac{t^{1-\epsilon}}{\ln t}}}(n)|| < \mathfrak{R}_1(K_t, n, q).$$

Proof. Let $\xi := t^{\frac{t^{1-\epsilon}}{\ln t}}$. We will show that a q-edge-coloring of K_ξ exists with all copies of K_t inducing edges in all q colors. Then Builder can expose all edges of K_ξ in an arbitrary order and forbid the corresponding color for each, while Painter cannot create a monochromatic K_t. We show the existence of such coloring via the probabilistic method. Consider a random edge-coloring of K_ξ, where for each edge $e \in E(K_\xi)$ and for each color $i \in [q]$

$$\mathbb{P}(e \text{ has color } i) = \frac{1}{q}.$$

Then for any $S \subset V(K_\xi)$ with $|S| = t$ we have

$$\mathbb{P}(\text{each edge of the } K_t \text{ induced by } S \text{ has color from } \{1, \cdots, q-1\}) = \Big(1 - \frac{1}{q}\Big)^{\binom{t}{2}}.$$

By defining the bad event A as

$$A := \{\text{there exists a } K_t \text{ colored into at most } q-1 \text{ colors}\},$$

we get

$$\begin{aligned}
\mathbb{P}(A) &= \mathbb{P}\Bigg(\bigcup_{\substack{S\subset V(K_\xi)\\|S|=t}} \Big(\bigcup_{i\in[q]} \{S \text{ does not induce an edge in color } i\}\Big)\Bigg) \\
&\le \sum_{\substack{S\subset V(K_\xi)\\|S|=t}} \Bigg(\sum_{i\in[q]} \mathbb{P}(S \text{ does not induce an edge in color } i)\Bigg) \\
&\le \sum_{\substack{S\subset V(K_\xi)\\|S|=t}} \Bigg(\sum_{i\in[q]} \Big(1-\frac{1}{q}\Big)^{\binom{t}{2}}\Bigg) = \binom{\xi}{t} q \Big(1-\frac{1}{q}\Big)^{\binom{t}{2}} < \xi^t \cdot q \cdot e^{-\frac{\binom{t}{2}}{q}}.
\end{aligned}$$

We want $\mathbb{P}(A) < 1$, so that by definition of A a coloring of K_ξ exists such that every copy of K_t contains all q colors. So far we have

$$\mathbb{P}(A) < \xi^t \cdot q \cdot e^{-\frac{\binom{t}{2}}{q}}.$$

We can derive the following through equivalent transformations:

$$\xi^t \cdot q \cdot e^{-\frac{\binom{t}{2}}{q}} < 1$$

$$\Updownarrow$$

$$e^{t\ln\xi + \ln q - \frac{\binom{t}{2}}{q}} < 1$$

$$\Updownarrow$$

$$t\ln\xi + \ln q - \frac{\binom{t}{2}}{q} < 0$$

$$\Updownarrow$$

$$t\ln\left(t^{\frac{t^{1-\epsilon}}{\ln t}}\right) + \ln q - \frac{\binom{t}{2}}{q} < 0$$

$$\Updownarrow$$

$$t^{2-\epsilon} + \ln q - \frac{\binom{t}{2}}{q} < 0$$

As $q \in o(t_0^\epsilon)$ for some $t_0 < t$ and we chose t to be large enough, the last inequality holds. Thus there exists an edge-coloring of K_ξ where every copy of K_t induces edges in all q colors. Now blow up K_ξ to $T_\xi(n)$ and extend the edge-coloring such that for every $v_1, v_2 \in V(K_\xi)$ with v_1, v_2 being blown up into blobs B_1, B_2, every edge running between B_1 and B_2 is colored into the color of v_1v_2. By exposing all edges of $T_\xi(n)$ and forbidding the assigned edge color, Builder can prevent Painter from creating a monochromatic K_t. □

Remark. Note that although this bound is stronger than the previous ones, it works only for very large t compared to q.

We also tried another probabilistic approach to improve the current bounds. Though the attempt was unsuccessful, we describe the method in the following, as it may contain ideas useful for further work. In Subsections 8.1-8.3 the main idea was to find a coloring of a possibly large complete graph where each copy of K_t induces all colors, and then blow it up to the Turán-graph on n vertices. Now we want to color possibly many (but not necessarily all) edges of K_n such that each copy of K_t induces all colors. Note that coloring all edges is not possible due to $n \geq r(K_t, q)$.

Consider a random graph G, where for each $e \in E(K_n)$,

- $\mathbb{P}(e \notin E(G)) = 1 - p$
- $\mathbb{P}(e \text{ has color } i) = \frac{p}{q}$ for each $i \in [q]$.

Note that each edge e is present in G with probability p. Let $p = \left(1 - \frac{1}{\xi}\right)$, then $\mathbb{E}(|E(G)|) = p\binom{n}{2} = \left(1 - \frac{1}{\xi}\right)\binom{n}{2}$. For $S \subseteq V(K_n)$ with $|S| = t$ we have

$$\mathbb{P}(S \text{ induces } K_t \text{ and each edge has color from } \{1, \cdots, q-1\}) = \left(\left(1 - \frac{1}{q}\right)p\right)^{\binom{t}{2}}.$$

Defining the event

$$A := \{\text{there exists a } K_t \text{ colored into at most } q-1 \text{ colors}\},$$

we get

$$\begin{aligned}
\mathbb{P}(A) &= \mathbb{P}\Bigg(\bigcup_{\substack{S\subseteq V(K_n)\\|S|=t}} \Big(\bigcup_{i\in[q]} \{S \text{ induces } K_t \text{ with no edge in color } i\}\Big)\Bigg) \\
&\leq \sum_{\substack{S\subseteq V(K_n)\\|S|=t}} \Big(\sum_{i\in[q]} \mathbb{P}(S \text{ induces } K_t \text{ with no edge in color } i)\Big) \\
&= \sum_{\substack{S\subseteq V(K_n)\\|S|=t}} \left(\sum_{\substack{C\subseteq[q]\\|C|\leq q-1}} \left(\left(1 - \frac{1}{q}\right)p\right)^{\binom{t}{2}}\right) = \binom{n}{t} \cdot q\left(\left(1 - \frac{1}{q}\right)p\right)^{\binom{t}{2}} \\
&\leq n^t \left(q\left(1 - \frac{1}{q}\right)^{\binom{t}{2}}\right)p^{\binom{t}{2}} = n^t \cdot f(t,q) \cdot p^{\binom{t}{2}}.
\end{aligned}$$

After leaving the constant part and substituting $p = \left(1 - \frac{1}{\xi}\right)$ we have

$$\mathbb{P}(A) \leq n^t \left(1 - \frac{1}{\xi}\right)^{\binom{t}{2}} \leq n^t \cdot e^{-\frac{\binom{t}{2}}{\xi}}.$$

We want $\mathbb{P}(A) < 1$ so that by definition of A a coloring of G exists such that every copy of K_t contains all q colors. So we can derive the following:

$$\begin{aligned}
\mathbb{P}(A) < n^t \cdot e^{-\frac{\binom{t}{2}}{\xi}} &< 1 \\
e^{t\log n - \frac{\binom{t}{2}}{\xi}} &< 1 \\
t \log n - \frac{\binom{t}{2}}{\xi} &< 0 \\
\xi &< \frac{t-1}{2\log n}
\end{aligned}$$

So let $\xi := \frac{t-1-\epsilon}{2\log n}$ and then $p = 1 - \frac{1}{\xi} = 1 - \frac{2\log n}{t-1-\epsilon}$. However, as $n \gg t$, this would mean a negative probability, which is a contradiction.

Remark. Note that this does not prove the non-existence of such graphs with many edges. This also does not prove that the existence of such graphs cannot be proven via the probabilistic method. It could be that this specific probabilistic approach did not work because such graphs with good edge coloring are not "random" enough but are structured in some way.

8.4 Upper bound

Lemma 8.14 states the current best upper bound on multicolor Ramsey numbers, which is proved through a slight modification of the neighborhood-chasing argument of Erdős and Szekeres in [23].

Lemma 8.14 (upper bound on multicolor Ramsey numbers)**.** *For $t, q \in \mathbb{N}$ and $q \geq 3$ we have*

$$r(K_t, q) \leq q^{qt}.$$

Theorem 8.15. *Let $f, q, n, t \in \mathbb{N}$ with $n \geq r(K_t, q)$ and $3 \leq q$ and $f < q$. Then*

$$\mathfrak{R}_f(K_t, n, q) \leq \left(1 - \frac{1}{q^{qt}}\right)\binom{n}{2} + 1.$$

Proof. Theorem 7.6 states

$$\mathfrak{R}_f(K_t, n, q) \leq ex(n, K_{r(K_t,q)}) + 1.$$

By applying Lemma 8.14 we get

$$\mathfrak{R}_f(K_t, n, q) \leq ex(n, K_{r(K_t,q)}) + 1 = \left(1 - \frac{1}{r(K_t, q)}\right)\binom{n}{2} + 1 \leq \left(1 - \frac{1}{q^{qt}}\right)\binom{n}{2} + 1.$$

□

9 G belongs to other graph families

In this section we present our results where G belongs to different graph families, including paths, cycles, stars and matchings.

9.1 Paths

Lemma 9.1 (Erdős-Gallai [19]). *Let $t, n \in N$ with $k := \lfloor \frac{t-1}{2} \rfloor$. Then we have*

$$\mathrm{ex}(n, P_t) \geq \binom{k}{2} + k(n-k).$$

Proof. We prove this by constructing a P_t-free graph on n vertices and $\binom{k}{2} + k(n-k)$ edges. Let G graph on n vertices. Let $T \subseteq V(G)$ with $|T| = k$. Let

$$E(G) := \{xy : x \in T \text{ and } y \in T\} \cup \{xy : x \in T \text{ and } y \notin T\}.$$

Then we have $|E(G)| = \binom{k}{2} + k(n-k)$ and G contains no copy of P_t, proving the inequality. □

Theorem 9.2 (bounds for paths). *Let $t, f, q, n \in \mathbb{N}$ with $f < q$ and $n \geq r(P_t, q)$. Let $k := \lfloor \frac{t-1}{2} \rfloor$. Then we have*

$$\binom{k}{2} + k(n-k) + 1 \leq \mathfrak{R}_f(P_t, n, q) \leq \frac{(f+1)(t-2)n}{2} + 1.$$

Proof. LB: By definition of extremal numbers we have

$$\mathrm{ex}(n, P_t) + 1 \leq \mathfrak{R}_f(P_t, n, q).$$

Thus by Lemma 9.1 we have the following:

$$\binom{k}{2} + k(n-k) + 1 \leq \mathrm{ex}(n, P_t) + 1 \leq \mathfrak{R}_f(P_t, n, q).$$

UB: Theorem 7.7 states

$$\mathfrak{R}_f(P_t, n, q) \leq (f+1)\mathrm{ex}(n, P_t) + 1.$$

Erdős and Gallai [19] proved the following upper bound on extremal number of paths:

$$ex(n, P_{t+1}) \leq \frac{1}{2}(t-1)n.$$

Using their result we get

$$\mathfrak{R}_f(P_t, n, q) \leq (f+1)\mathrm{ex}(n, P_t) \leq \frac{(f+1)(t-2)n}{2} + 1,$$

proving the upper bound. □

9.2 Cycles

Recall that $poly_H(K_n)$ denotes the largest integer k such that a k-edge-coloring of K_n exists with all subgraphs in K_n isomorphic to H containing all k colors of the coloring.

Theorem 9.3 (Axenovich et al. [3]). *Let $t \in \mathbb{N}$, then*

$$\left\lfloor \log_2 \frac{8(t-1)}{3} \right\rfloor \leq poly_{C_t}(K_t).$$

Theorem 9.4 (bounds for cycles). *Let $t, f, q, n \in \mathbb{N}$ with $f < q \leq f \cdot \left\lfloor \log_2 \frac{8(t-1)}{3} \right\rfloor$ and $n \geq r(C_t, q)$. Then we have*

$$\left\lfloor \frac{n}{t} \right\rfloor \binom{t}{2} \leq \mathfrak{R}_f(C_t, n, q) \leq \frac{1}{2}(f+1)(n-1)t + 1.$$

Proof. LB: We can apply the result of Axenovich et al. in Theorem 9.3 to the general lower bound result of Theorem 7.11 to get the lower bound.

UB: Theorem 7.7 states

$$\mathfrak{R}_f(C_t, n, q) \leq (f+1)\mathrm{ex}(n, C_t) + 1.$$

Erdős and Gallai [19] proved the following upper bound on extremal number of cycles:

$$\mathrm{ex}(n, C_t) \leq \frac{1}{2}t(n-1).$$

Using their result we get

$$\mathfrak{R}_f(C_t, n, q) \leq (f+1)\mathrm{ex}(n, C_t) + 1 \leq \frac{1}{2}(f+1)(n-1)t + 1.$$

□

9.3 Stars

Lemma 9.5 (Alspach on Hamiltonian decomposition [2]). *Let $k \in \mathbb{N}$. K_{2k+1} can be decomposed into k Hamiltonian cycles. K_{2k} can be decomposed into $k-1$ Hamiltonian cycles and a 1-factor.*

Theorem 9.6. *Let $t, f, q, n \in \mathbb{N}$ with $f < q$ and $n \geq r(K_{1,t}, q)$. Then we have*

$$\frac{n((f+1)(t-1)+1)}{4} < \mathfrak{R}_f(K_{1,t}, n, q) \leq \frac{n(f+1)(t-1)}{2} + 1.$$

Proof. Note that a graph not containing a monochromatic $K_{1,t}$ means that there is no vertex having t incident edges of the same color.

LB: We prove this with a strategy for Builder. By Lemma 9.5, the graph K_n can be decomposed into Hamiltonian cycles and possibly one 1-factor, depending on the parity of n. We can construct a k-regular graph on n vertices for any $k < n$ simply by taking the union of some Hamiltonian cycles and, if k is odd, the 1-factor as well. Construct such a regular graph G with degree $\lfloor \frac{(f+1)(t-1)+1}{2} \rfloor$. Builder exposes all edges of G in an arbitrary order. When exposing edge xy, both x and y have at most $\frac{(f+1)(t-1)-1}{2}$ incident edges which are already colored, so altogether at most $(f+1)(t-1)-1$ colored edges. See an illustration in Figure 16. By Pigeonhole principle, there are at most f colors, for which the color is assigned to at least $t-1$ of these edges. Builder forbids these at most f colors, making sure that no color class of size t appears. Now she repeats these steps for all other edges of G. Hence she guarantees that no vertex has t incident edges in the same color, and no monochromatic $K_{1,t}$ occurs.

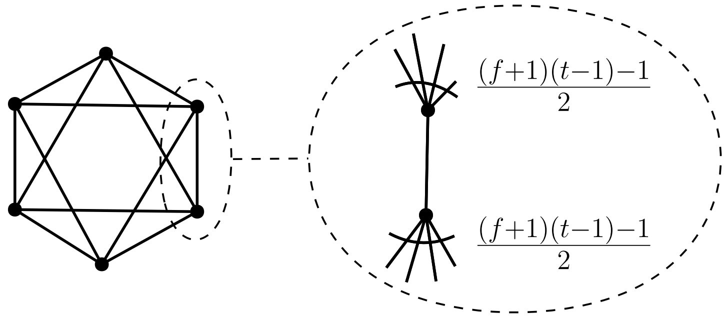


Figure 16: Builder strategy construction for $\mathfrak{R}_f(K_{1,t}, n, q)$.

UB: We prove this with a strategy for Painter. Say he uses only colors $[f+1]$. If Builder exposed $\frac{n(t-1)(f+1)}{2} + 1$ edges, by Pigeonhole principle there must be a vertex v with $\deg(v) \geq (t-1)(f+1)+1$. As Painter uses only $f+1$ colors, again by Pigeonhole principle there must be t edges of the same color incident to v. These t edges together build a monochromatic $K_{1,t}$, proving the upper bound. □

9.4 Matchings

Recall that tK_2 denotes a matching of size t.

Theorem 9.7 (Erdős-Gallai [19])**.** *Let $n, t \in \mathbb{N}$. Then we have*

$$\mathrm{ex}(n, tK_2) = \max\left\{ \binom{2t-1}{2}, (t-1)(n-t+1) + \binom{t-1}{2} \right\}.$$

Corollary 9.8. *For $n \geq \frac{5t-2}{2}$ we have*

$$\mathrm{ex}(n, tK_2) = (t-1)(n-t+1) + \binom{t-1}{2} = (t-1)\left(n - \frac{t}{2}\right).$$

Theorem 9.9. *Let $t, n, q, f \in \mathbb{N}$ with $f < q$ and $n \geq r(tK_2, q)$. Then we have*

$$(t-1)\left(n - \frac{t}{2}\right) + 1 \leq \mathfrak{R}_f(tK_2, n, q) \leq (f+1)(t-1)\left(n - \frac{t}{2} - \frac{f(t-1)}{2}\right) + 1.$$

Proof. LB: By Theorems 7.8 and 9.7 we have

$$\mathfrak{R}_f(tK_2, n, q) > \mathrm{ex}(n, tK_2) = (t-1)\left(n - \frac{t}{2}\right),$$

proving the lower bound.

UB: Let Painter use only colors $[f+1]$ for coloring. For every exposed edge he may choose a color from this set arbitrarily. If $(t-1)(f+1)+1$ independent edges are exposed, then by Pigeonhole principle there are t independent edges of the same color, inducing a monochromatic copy of tK_2. So this Painter strategy proves an upper bound of

$$\mathfrak{R}_f(tK_2, n, q) \leq \mathrm{ex}(n, ((t-1)(f+1)+1)K_2) + 1.$$

By Theorem 9.7 we have

$$\mathrm{ex}(n, ((t-1)(f+1)+1)K_2) = (t-1)(f+1)\left(n - \frac{(t-1)(f+1)+1}{2}\right),$$

proving the upper bound.

□

10 Online vs. offline strategies

10.1 Builder strategies for 1 forbidden color

We say that a Builder strategy is *offline* if the Builder's moves are independent of the Painter's choices, i.e. Builder moves are prescribed. Thus an *offline Builder strategy* for the game $\mathcal{G}(G, n, 1, q)$ consists of a graph H, a q-edge-coloring of H and an ordering of $E(H)$, defining the exposed edge and the forbidden color for each step. However, the order of edge exposure does not matter, as after exposing all edges of H there is no monochromatic copy of G. An offline Builder strategy that guarantees that the game does not end in m steps, guarantees a game length of at least $m + 1$.

Definition 10.1 (online Builder strategy for $\mathcal{G}(G, n, 1, q)$)**.** *In the game $\mathcal{G}(G, n, 1, q)$, every Builder strategy that is not offline is called online.*

Definition 10.2 (offline Builder strategy for $\mathcal{G}(G, n, 1, q)$)**.** *In the game $\mathcal{G}(G, n, 1, q)$, an offline Builder strategy that guarantees a game length of at least $m + 1$ is defined as a pair (H, c) with H graph, $|H| = n$, $||H|| = m$ and $c : E(H) \to [q]$ edge-coloring, such that each copy of G in H induces an edge in all q colors.*

Definition 10.3 ($F(G, n, q)$)**.** *Let $F(G, n, q)$ be the largest number of edges in a graph H on n vertices such that there is a q-edge-coloring of H where each copy of G in H induces all q colors.*

Remark. Equivalently, $F(G, n, q)$ equals the largest m such that there is an offline Builder strategy guaranteeing a game length of at least $m + 1$ in the game $\mathcal{G}(G, n, 1, q)$. Thus by definition,

$$F(G, n, q) < \mathfrak{R}_1(G, n, q).$$

Sections 7, 8 and 9 all described offline Builder strategies only, i.e. graphs of order n with each copy of G inducing all q colors. This raises the question of whether offline strategies are always optimal. Do online strategies exist that lead to even better bounds? More formally, we know $F(G, n, q) < \mathfrak{R}_1(G, n, q)$. Is this a strict bound with $F(G, n, q) = \mathfrak{R}_1(G, n, q) - 1$, or does $F(G, n, q) \ll \mathfrak{R}_1(G, n, q)$ hold, meaning that some better online strategies exist? Online Builder strategies are not easy to construct for large graph families like complete graphs. However, we have results for concrete graphs showing an online Builder strategy better than any offline ones. Theorems 10.4 and 10.5 show an example for the game $\mathcal{G}(2K_2, n, 1, 3)$.

Theorem 10.4. *In game $\mathcal{G}(2K_2, n, 1, 3)$ the best offline Builder strategy guarantees a game length of at least n, i.e. proves the lower bound*

$$n \leq \mathfrak{R}_1(2K_2, n, 3).$$

Proof. We want an offline Builder strategy proving a lower bound of m, i.e. guaranteeing a game length of at least m, for the largest possible m. Such a strategy defines a graph H with $|H| = n$ and $|H| = m$, where a 3-edge-coloring of H exists so that each copy of

$2K_2$ contains all 3 colors. Thus Builder can expose all edges of H and forbid the color corresponding to the coloring to prevent a monochromatic copy of $2K_2$. As we have 3 colors but $2K_2$ has only 2 edges, H may not contain any copy of $2K_2$. regardless of the coloring. The graph H is edge-maximal if $H \simeq K_{1,n-1}$, meaning $|E(H)| = n - 1$. So

$$F(2K_2, n, 3) = ex(2K_2, n) = n - 1,$$

proving that the best offline Builder strategy gives the lower bound of $n \leq \mathfrak{R}_1(2K_2, n, 3)$ by guaranteeing that the game cannot end in $n - 1$ steps. □

Theorem 10.5. *There exists an online Builder strategy proving a better lower bound than any offline Builder strategy for the game* $\mathcal{G}(2K_2, n, 1, 3)$.

Proof. Recall that Theorem 5.6 defines the following online Builder strategy for the game $\mathcal{G}(2K_2, n, 1, 3)$, proving $\mathfrak{R}_1(2K_2, n, 3) \geq n + 2$. For easier understanding, see Figure 17. First, Builder exposes edge v_1v_2 and forbids no color, and w.l.o.g. Painter paints it in color 1. Then Builder exposes an independent edge v_3v_4, and forbids the previously used color 1, w.l.o.g. say Painter colors it into color 2. Note that Builder's move is online, as the forbidden color depends on Painter's previous decision. Next, Builder exposes the edge v_2v_3, and forces the usage of a previously used color by preventing color 3. This is an online move again, as the forbidden color depends on the previous Painter steps. Say Painter chooses color 1. Now we have an edge with a unique color, edge v_3v_4. Builder chooses its free endpoint, v_4, and exposes all of its outgoing edges while forbidding color 1. Note that these moves are online again, as both the edge and the forbidden color choice depend on Painter's previous moves. So Builder can expose $n + 1$ edges while preventing a monochromatic copy of $2K_2$, guaranteeing a game length of at least $n + 2$.

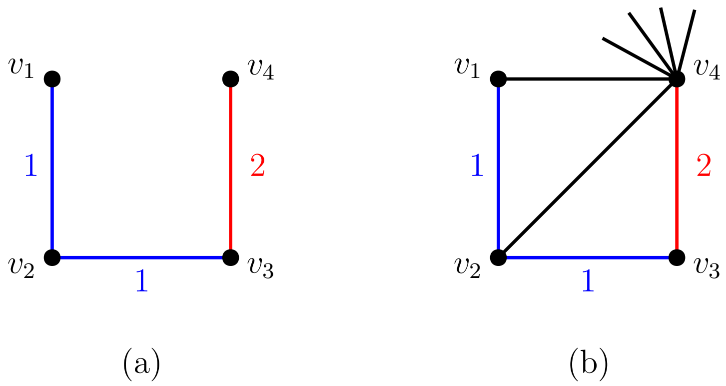


Figure 17: Online Builder strategy for game $\mathcal{G}(2K_2, n, 1, 3)$.

From Theorem 10.4 we know, that the best offline Builder strategy possible proves the lower bound $n \leq \mathfrak{R}_1(2K_2, n, 3)$. Thus the previously defined online strategy is better than any offline ones, concluding our proof. □

10.2 Painter strategies for 1 forbidden color

We can distinguish offline and online strategies in the case of Painter as well. We say that a Painter strategy is offline if the Painter's moves are independent of the Builder's choices, i.e. Painter moves are prescribed. In the case of offline Painter strategies, a prescription of moves is slightly more complicated than in the case of Builder, as the possible choices of Painter depend on which colors are forbidden by Builder.

Definition 10.6 (online Painter strategy for $\mathcal{G}(G, n, 1, q)$)**.** *In the game $\mathcal{G}(G, n, 1, q)$, every Painter strategy that is not offline is called online.*

Definition 10.7 (offline Painter strategy for $\mathcal{G}(G, n, 1, q)$)**.** *In the game $\mathcal{G}(G, n, 1, q)$, an offline Painter strategy is defined as a list of pairs $(a_1, b_1), \ldots, (a_m, b_m)$ for some $m \leq \binom{n}{2}$, where each pair (a_i, b_i) defines the priorities for the possibly chosen 2 colors. I.e., in round i Painter chooses b_i, if the forbidden color is a_i, and chooses a_i otherwise.*

Definition 10.8 ($N(G, n, q)$)**.** *Let $N(G, n, q)$ be the smallest m such that Painter has an offline strategy guaranteeing a game length of at most m in the game $\mathcal{G}(G, n, 1, q)$.*

Remark. Thus by definition,

$$\mathfrak{R}_1(G, n, q) \leq N(G, n, q).$$

Remark. Note that we chose an arbitrary definition of an offline Painter strategy, but other definitions could be suitable as well. As an example, consider the following Painter strategy. Painter defines for all edges of K_n the first and second preferred colors in advance. More formally, we could define the prescription as $E(K_n) \to [m] \times [m]$. instead of $[m] \to [m] \times [m]$. Whenever an edge e is exposed, he colors it according to these predefined preferences (a_e, b_e). Definition 10.7 considers this as an online Painter strategy, as the chosen color does depend on the Builder's edge choice. However one could say that the preferences for each edge are prescribed and do not depend on the current game situation or the Builder's move. Nevertheless, in the following we only consider offline strategies meeting our Definition 10.7.

Sections 7, 8 and 9 all described offline Painter strategies only, i.e. prescribed strategies with Painter's choice independent of Builder's moves. This raises the question of whether offline strategies are always optimal, or if online Painter strategies exist that lead to even better bounds. More formally, which does hold,

$$\mathfrak{R}_1(G, n, q) = N(G, n, q) \text{ or } \mathfrak{R}_1(G, n, q) < N(G, n, q),$$

meaning that some better online strategies exist? Such online strategies are not easy to construct for large graph families like complete graphs. However, we have results for concrete graphs showing an online Painter strategy better than any best offline ones. Theorems 10.9 and 10.10 show an example for the game $\mathcal{G}(2K_2, n, 1, 3)$.

Theorem 10.9. *In the game $\mathcal{G}(2K_2, n, 1, 3)$ the best offline Painter strategy guarantees a game length of at most $2n - 2$, i.e. proves the upper bound*

$$\mathfrak{R}_1(2K_2, n, 3) \leq 2n - 2.$$

Proof. An offline Painter strategy proving the upper bound $\mathfrak{R}_1(2K_2, n, 3) \leq 2n - 2$ means, that regardless of Builder's strategy, Painter can create a monochromatic $2K_2$ in at most $2n - 2$ steps. As an offline Painter strategy is a prescribed strategy, we could think of it as if Builder also knew this prescription, i.e. Builder knows for each $i \in [2n - 2]$ Painter's prescribed pair of color choices for step i. To prove that there is no offline Painter strategy proving an upper bound of $2n - 3$ or less, it suffices to show a Builder strategy, which, given this prescription of Painter's moves, can force the game to last longer than $2n - 3$ moves. Note that Builder can basically choose a color for the edge exposed in step i from colors $\{a_i, b_i\}$ by either forbidding a_i or not. Let H be the 2-edge-colored union of two stars $K_{1,n-1}$ with $|H| = n$ as shown in Figure 18. If Builder could force a colored copy H, she could force a game length of at least $2n - 2$.

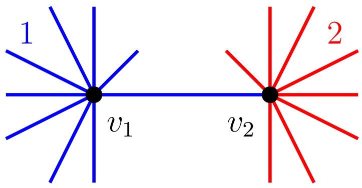


Figure 18: Builder aims to force graph H in the first $2n - 3$ steps.

Let Painter's moves be $p : (a_1, b_1), \ldots, (a_{2n-2}, b_{2n-2})$, where $\forall i : a_i, b_i \in [3]$ holds and $a_i \neq b_i$. Consider only the first $2n - 3$ pairs. For $i \in [3]$ let

$$s(i) := \#\{(a_j, b_j) : i \in (a_j, b_j), j \in [2n - 3]\},$$

i.e. the number of pairs containing color i. Note that we have

$$s(1) + s(2) + s(3) = 2(2n - 3).$$

W.l.o.g. say that $s(3) \leq s(1)$ and $s(3) \leq s(2)$, i.e.

$$s(1) \geq \frac{2}{3}(2n - 3) \text{ and } s(2) \geq \frac{2}{3}(2n - 3).$$

As Builder can choose color 1 and 2 from at least $\frac{2}{3}(2n - 3)$ pairs each, she can force a copy of H by exposing the edges in the right order. So Builder can force a game length of at least $2n - 2$ and thus the best offline Painter strategy can prove no better upper bound than $\mathfrak{R}_1(2K_2, n, 3) \leq 2n - 2$. The offline Painter strategy $\forall i : (a_i, b_i) := (1, 2)$ is a strategy proving this bound, as any 2-coloring of $2n - 2$ edges on n vertices contains a monochromatic $2K_2$. Thus the best offline Painter strategy proves the upper bound $\mathfrak{R}_1(2K_2, n, 3) \leq 2n - 2$. □

Theorem 10.10. *There exists an online Painter strategy proving a better upper bound than any offline Painter strategy for the game $\mathcal{G}(2K_2, n, 1, q)$.*

Proof. We know by proof of Theorem 5.6, that there exists a Painter strategy proving $\mathfrak{R}_1(2K_2, n, 3) \leq n+3$ in the game $\mathcal{G}(2K_2, n, 1, q)$. See the theorem proof for the strategy details. By Theorem 10.9 we know, that the best possible offline Painter strategy proves the upper bound $\mathfrak{R}_1(2K_2, n, 3) \leq 2n-2$. Thus the online Painter strategy shown in Theorem 5.6 proves a better upper bound than the best offline Painter strategy, concluding our proof. □

Remark. The proof of Theorem 10.9 on offline strategies raises the question of how well offline Painter strategies perform in general. Thinking of the situation as if Builder knew the prescribed offline Painter strategy may help prove stronger bounds even for larger graph classes.

10.3 Larger forbiddance number

We can generalize the definitions of offline and online strategies for larger forbiddance numbers as well. This section presents the generalized definitions of offline Builder and Painter strategies. For some set S and some $k \in \mathbb{N}$ let $\binom{S}{k}$ denote the set of all k-element subsets of S.

Definition 10.11 (online Builder strategy for $\mathcal{G}(G, n, f, q)$)**.** *In the game $\mathcal{G}(G, n, f, q)$, every Builder strategy that is not offline is called online.*

Definition 10.12 (offline Builder strategy for $\mathcal{G}(G, n, f, q)$)**.** *Let (H, c) be a pair with H graph, $|H| = n$, $||H|| = m$ and $c : E(H) \to \binom{[q]}{f}$, i.e. c assigns a set of f colors to each edge. In the game $\mathcal{G}(G, n, f, q)$, the pair (H, c) is an offline Builder strategy that guarantees a game length of at least $m + 1$ if each copy of G in H for each color $i \in [q]$ induces an edge e so that $i \in c(e)$.*

Definition 10.13 ($F_f(G, n, q)$)**.** *Let $F_f(G, n, q)$ be the largest number of edges in a graph H on n vertices such that there is a function $c : E(H) \to \binom{[q]}{f}$, so that each copy of G in H induces an edge e for each color $i \in [q]$ so that $i \in c(e)$.*

Remark. Equivalently, $F_f(G, n, q)$ equals the largest m such that there is an offline Builder strategy guaranteeing a game length of at least $m + 1$ in the game $\mathcal{G}(G, n, f, q)$. Thus by definition,

$$F_f(G, n, q) < \mathfrak{R}_f(G, n, q).$$

Remark. Recall that Theorem 7.3 suggests a strategy, how to extend an offline Builder strategy of game $\mathcal{G}(G, n, 1, q)$ to a strategy in game $\mathcal{G}(G, n, f, q)$ by assigning fixed color sets to the edges. However, note that in an offline strategy, the assigned color sets are not necessarily fixed. See an example in Figure 19.

Definition 10.14 (online Painter strategy for $\mathcal{G}(G, n, f, q)$)**.** *In the game $\mathcal{G}(G, n, f, q)$, every Painter strategy that is not offline is called online.*

Definition 10.15 (offline Painter strategy for $\mathcal{G}(G, n, f, q)$)**.** *In the game $\mathcal{G}(G, n, f, q)$, an offline Painter strategy is defined as a list of $(f + 1)$-tuples*

$$(a_{1,1}, \ldots, a_{1,f+1}), \ldots, (a_{m,1}, \ldots, a_{m,f+1})$$

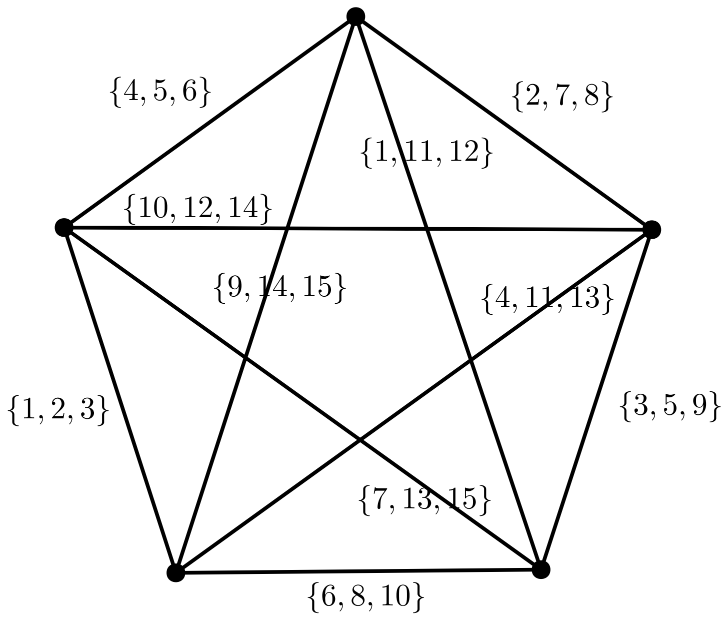


Figure 19: Offline Builder strategy for game $\mathcal{G}(K_4, n, 3, 15)$ assuring a game length of at least 11.

for some $m \leq \binom{n}{2}$*, where each* $(f+1)$*-tuple* $(a_{i,1}, \ldots, a_{i,f+1})$ *defines the priorities for the possibly chosen* f *colors. I.e., in round* i *Painter chooses the first element of the tuple* $(a_{i,1}, \ldots, a_{i,f+1})$*, which is not forbidden.*

Definition 10.16 ($N_f(G, n, q)$)**.** *Let* $N_f(G, n, q)$ *be the smallest* m *such that Painter has an offline strategy guaranteeing a game length of at most* m *in the game* $\mathcal{G}(G, n, f, q)$*.*

Remark. Thus by definition,

$$\mathfrak{R}_f(G, n, q) \leq N_f(G, n, q).$$

11 Open questions

We conclude this thesis by listing problems that are closely related to the presented results and are open for further exploration.

- In Theorem 5.6 we consider $\mathfrak{R}_1(2K_2, n, 3)$ and prove

$$n + 2 \leq \mathfrak{R}_1(2K_2, n, 3) \leq n + 3.$$

  Which does hold,

$$\mathfrak{R}_1(2K_2, n, 3) = n + 2 \text{ or } \mathfrak{R}_1(2K_2, n, 3) = n + 3?$$

- In Section 8 we consider lower bounds for G as complete graph, i.e. $\mathfrak{R}_f(K_t, n, q)$. In Theorem 8.12 we give a constructive lower bound proof of $\left(1 - \frac{1}{(t-t^{0.525})^2}\right)\binom{n}{2}$, while in Theorem 8.13 we prove the lower bound of $\left(1 - \frac{1}{t^{\frac{t^{1-\epsilon}}{\ln t}}}\right)\binom{n}{2}$ probabilistically. How could one improve the constructive bounds? And can the probabilistic lower bound be improved?
- In Section 9 we prove results for paths, cycles, stars and matchings. Can these bounds be improved? How about other classes like complete bipartite graphs or k-regular graphs?
- Is there an offline strategy for some game $\mathcal{G}(G, n, f, q)$ that proves the best possible upper or lower bound? I.e. is there a scenario with

$$F_f(G, n, q) + 1 = \mathfrak{R}_f(G, n, q) \text{ or } \mathfrak{R}_f(G, n, q) = N_f(G, n, q)?$$

- In Section 10 Theorems 10.5 and 10.10 show examples of online strategies for the game $\mathcal{G}(2K_2, n, 1, 3)$, where some online strategy proves a stronger bound than any offline one. Are there other examples of games and online strategies, where the same holds? Or can we go even further and say that for (almost) all game scenarios, the optimal strategies of the players are online, outperforming all offline ones?

List of Theorems

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