Surrounding Cops and Robbers

Bachelor's Thesis of

Samuel Schneider

at the Department of Informatics
Institute of Theoretical Informatics, Algorithmics (ITI)

Reviewer:	Torsten Ueckerdt
Second reviewer:	Thomas Bläsius
Advisor:	Torsten Ueckerdt

25. October 2021 – 25. February 2022

Karlsruher Institut für Technologie
Fakultät für Informatik
Postfach 6980
76128 Karlsruhe

I declare that I have developed and written the enclosed thesis completely by myself, and have not used sources or means without declaration in the text.

Karlsruhe, 25.02.2022

S. Schneider

(Samuel Schneider)

Abstract

The game of Cops and Robbers is a vertex pursuit game on graphs in which cops try to occupy the vertex of the robber. Both parties can move by traversing one edge each turn. We will consider the surrounding variant of the game in which the cops have to occupy the entire neighbourhood of the robber's vertex in order to win instead. The *surrounding cop number* of a graph is the least amount of cops necessary to surround a robber on that graph.

We will show a bound for the *surrounding cop number* for complete subdivisions of complete graphs and for the strong product of graphs with paths. We will furthermore prove a bound for planar graphs and the hypercube with the additional restrictions of the active cops and robbers game where the cops and the robber no longer can decide to stay on a vertex.

Additionally we will briefly consider the existing bound for the *surrounding cop number* of a graph using its treewidth and prove that this bound can be arbitrarily bad.

Zusammenfassung

Bei dem Spiel *Cops and Robbers* verfolgen mehrere *cops* einen *robber* auf den Knoten eines Graphen. Dabei können sich beide Parteien über je eine Kante pro Zug bewegen. Das Ziel der *cops* ist es, den Knoten des *robber's* zu besetzen. Wir betrachten in dieser Arbeit die *surrounding* Variante des Spiels, in welcher die *cops* stattdessen die gesamte Nachbarschaft des Knoten des *robber's* besetzen müssen, um zu gewinnen. Die *surrounding cop number* eines Graphen ist die geringste Anzahl an *cops*, die benötigt werden, um den *robber* auf diesem Graphen zu fangen.

Wir werden die *surrounding cop number* für vollständige Unterteilungen von vollständigen Graphen angeben und eine obere Schranke für die *surrounding cop number* für das *strong product* von Graphen mit Pfaden. Außerdem werden wir eine obere Schranke für die *surrounding cop number* von planaren Graphen und vom Hypercube mit der zusätzlichen Einschränkung, dass *cops* und *robber* nicht mehr auf einem Knoten verharren dürfen, angeben.

Des Weiteren werden wir kurz eine obere Schranke für die *surrounding cop number* mittels der Baumweite eines Graphen betrachten und zeigen, dass diese für bestimmte Graphen beliebig schlecht sein kann.

Contents

1 Introduction

In the game of Cops and Robbers one player, who is controlling the cops, tries to catch the robber controlled by a second player. We want to begin by giving a simple example as motivation.

For this we will look at a three by two chess board. The game starts with the cop player called Bob putting a cop on any square. Now the robber player called Alice places the robber on any square as well. Now Bob and Alice take alternating turns. In each turn they can either move their respective actor or let them stay on their square. If they decide to move their actor, they can move them to any adjacent square, including diagonal squares.

We now want to find a strategy for Bob to catch the robber in a finite amount of turns, no matter how Alice decides to move the robber. For that we let Bob place the cop c_1 on b2 at the beginning of the game (Fig. 1.1). We can observe that every square is adjacent to the cop's square. Therefore, no matter where Alice places the robber, the cop can move to that square in one move. This means we have found a winning strategy for the cops.

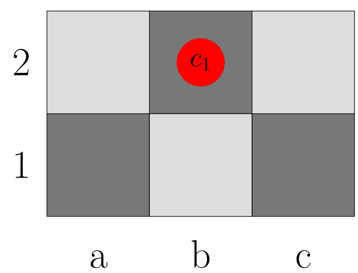

Figure 1.1: Placement of the cop

Next we want to look at what happens if we slightly alter the winning condition of the cop player. Instead of having to catch the robber by moving a cop onto the robber's square, the cops now have to surround the robber by occupying every square adjacent to the robber, once again including the diagonally adjacent squares. If a cop moves onto the square of the robber, the robber has to move to another square on their turn.

In the present example it is clear that at least three cops are needed to catch the robber as this is the least amount of squares any square is adjacent to. So instead of placing only the cop c_1, Bob now also places two additional cops c_2 and c_3 on b1 (Fig. 1.2).

Due to the symmetry of the board, Alice now has effectively two different squares where she can put the robber. She can put the robber on a1 (Fig. 1.3), which is equivalent to her putting the robber on c1, or she can put the robber on a2 (Fig. 1.4), which is equivalent to her putting the robber on c2.

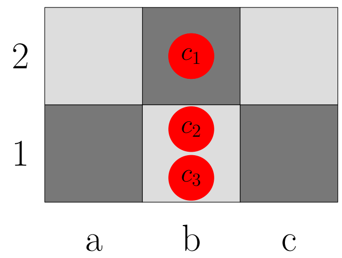

Figure 1.2: The initial placement of the cops

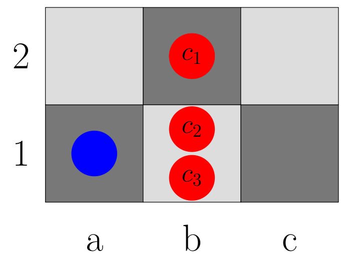

Figure 1.3: Configuration if the Robber is placed on a1

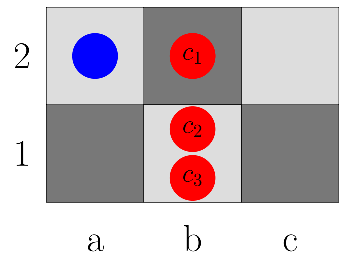

Figure 1.4: Configuration if the Robber is placed on a2

We see that the cops can surround the robber in a single step. If the robber is put on a2, they can do this by moving c_3 one square to the left (Fig. 1.5). If the robber is put on a1 instead, the cops can perform the same move (Fig. 1.6). This results in the robber having to move away from a1 and since the only unoccupied square adjacent to a1 is a2, the robber has to move there and is incidently surrounded. This means we have found a winning strategy using three cops.

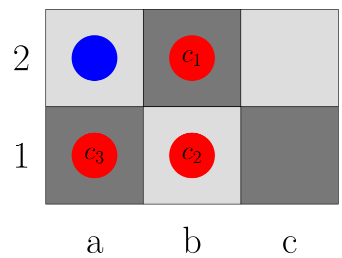

Figure 1.5: The cops move if the robber was put on a2

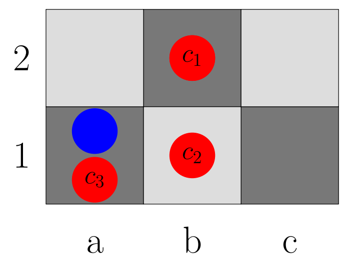

Figure 1.6: The cops move if the robber was put on a1

We now want to give a slightly more formal definition of the game and also give a short overview over the history of the game. For this we will use a graph instead of a chess board. The vertices of the graph are the possible positions for the cops and the robber. Each turn the cops and the robber can move along an edge to move from a vertex to an adjacent one. The original version of the game, independently introduced by Nowakowski and Winkler in 1983 [13] and Quilliot in 1978 [16], allows only a single cop chasing the robber. The graphs where a strategy for the cop exists so that the cop occupies the robber's vertex after a finite amount of turns, no matter how the robber moves, are called *cop win graphs* (this means the graph from our chess board example is a cop win graph). For these *cop win graphs* both the duo of Nowakowski and Winkler as well as Quilliot provided characterizations in their respective works.

The game then was generalized by Aigner and Fromme in 1984 [1] allowing $k > 0$ cops to chase a single robber changing the game significantly and introducing the *cop number* of a graph. In this version of the game k cops start the game by choosing their starting vertices on a graph G. Then the robber chooses their starting vertex and just like in the original version of the game the parties alternate taking moves. Since it is now possible to have more than one cop, it may be advantageous for a cop to stay on a vertex instead of moving to another vertex. Therefore, staying on a vertex is also a legal move. A winning strategy for the cops is a set of moves that lead to a cop occupying the vertex of the robber no matter how the robber moves. The *cop number* $c(G)$ is the least amount of cops needed to have a winning strategy for a given graph G.

While there are a lot of bounds for specific classes of graphs like $c(G) \leq 3$ if G is planar, one of the most prominent potential upper bounds known as *Meyniel's conjecture*, first mentioned in [7] as a personal note from Meyniel to Frankl, still remains unproven. Meyniel's conjecture states that $c(G) \in O(\sqrt{n})$ if G is a graph with n vertices.

This lack of a tight upper bound does not effect all variations of the game though. For example one of the most prominent variants is the *helicopter cops and robbers* variant, introduced by Seymour and Thomas in [18]. In this variant the cops can use helicopters to move to an arbitrary vertex each turn and the robber can move at an arbitrary speed. Unlike a lot of other variants this variant has a nice characterization as a robber can be caught by k cops on a graph G if and only if G has treewidth $k + 1$.

We want to look at the *surrounding cops and robbers* variant of the game introduced by Burgess et al. in 2020 [6]. This variant slightly alters the step of the robber choosing their starting position by not allowing them to choose a vertex already occupied by a cop, which is a restriction obviously not needed in the non-surrounding variant. Furthermore, the winning condition is changed to the one we looked at in our chess board example. If a cop enters the vertex of the robber in this variant, the robber has the chance to move to an adjacent vertex on their turn in order to avoid being captured. Instead, the cops winning condition is to occupy every vertex adjacent to the vertex of the robber. Similar to the original variant, the *surrounding cop number* $\sigma(G)$ denotes the least amount of cops needed to have a winning strategy for a given graph G.

While there are some upper bounds for the surrounding variant presented in [6], they are rather unpractical or can be arbitrarily bad as we will show for the bound using the treewidth of a graph in Chapter 6. Furthermore, just like for the original variant of the game, there are bounds for certain classes of graphs like planar graphs that are proven in [5]. The results presented in both those works for specific graphs suggested there might be a bound using the degeneracy of a graph. However, this is not the case as shown in Chapter 4. Therefore, instead we present some more upper bounds for specific graphs.

1.1 Assumptions and Notation

In this work we will only look at connected simple, meaning loopless and without parallel edges, graphs. We will only look at connected graphs because in non-connected graphs the surrounding cop number is simply the sum of the surrounding cop numbers of the connected components. We do not allow loops because then a vertex v could be in it's own neighbourhood $N(v)$. This would mess with the definition of our winning condition as it could require the cops to occupy the robber's vertex in order to catch them. Finally, parallel edges do not change the game whatsoever and therefore we do not need to consider them.

Since it is practical to refer to the parties with pronouns, we will use the convention used in many works regarding the game of cops and robbers and refer to the cops with male pronouns and to the robber with female pronouns.

We will from now on refer to the non-surrounding variant of the game as introduced in [1] as the *normal* variant and refer to the variant as introduced in [6] as the *surrounding* variant.

When we continuously reduce the parts of a graph accessible to the robber, we will call the parts that are accessible to her the *robber's territory*.

We will refer to the vertices of a graph G with $V(G)$ and refer to the edges with $E(G)$.

1.2 Formal Definition

We now want to give a mathematical definition of the game of cops and robbers given a graph $G = (V, E)$. While we will not use this mathematical definition when giving strategies for the cops in the following, it might aid the understanding of the game.

A strategy for the cops is a k-tuple of vertices, as starting positions of the cops, together with a function $s : V^k \times V \to V^k$. Let $s((v_1, \dots, v_k), v) = (v'_1, \dots, v'_k)$, then the following restriction must be fulfilled:

$$\text{For all } 1 \le i \le k : v'_i \in N(v_i) \cup \{v_i\} \tag{1.1}$$

The position of the cops is represented by V^k and the position of the the robber is represented by V. Restriction 1.1 ensures that the cops only move to adjacent vertices.

A strategy for the robber is a starting vertex together with a function $r : V \times V^k \to V$. Let $r(v, (v_1, \dots, v_k)) = v'$, then the following restrictions must be fulfilled:

$$v' \in N(v) \cup \{v\} \tag{1.2}$$
$$\text{For all } 1 \le i \le k : v' \ne v_i \tag{1.3}$$

Restriction 1.2 ensures that the robber only moves to adjacent vertices. Restriction 1.3 ensures that the robber does not move onto vertices occupied by a cop.

Let s and r be such functions and $(v_{1_0}, \dots, v_{k_0})$ and u_0 the respective starting positions of the cops and the robber with $u_0 \ne v_{i_0}$ for all $1 \le i \le k$. Then we construct a sequence $a_n = ((v_{1_n}, \dots, v_{k_n}), u_n)$ as follows:

$$\begin{aligned}
a_0 &= ((v_{1_0}, \dots, v_{k_0}), u_0) \\
a_{2n-1} &= (s(a_{2n-2}), u_{2n-2}) \\
a_{2n} &= ((v_{1_{2n-1}}, \dots, v_{k_{2n-1}}), r(a_{2n-1}))
\end{aligned}$$

Then $((v_{1_0}, \dots, v_{k_0}), s)$ is a winning strategy for the cops if for all robber strategies (u_0, r) there exists a $n \in \mathbb{N}_0$ such that $a_n = ((v_{1_n}, \dots, v_{k_n}), u_n)$ where for all $v \in N(u_n)$ there exists a v_{i_n} with $v = v_{i_n}$. This means there must exist a finite number n for every robber strategy such that after that many turns the robber is surrounded.

The surrounding cop number $\sigma(G)$ is the minimal k for that a winning strategy using k cops exists.

1.3 Other Variants

The game of cops and robbers has a lot of variants altering the different aspects of the game. These variants will sometimes just slightly alter the game and sometimes have huge implications. We want to give an overview of some of these variants.

1.3.1 Active, Passive and Lazy Cops and Robbers

First, we want to look at the distinction between the *passive* and *active* variants of the game. This distinction was first introduced by Aigner and Fromme [1]. In the active variant they proposed the robber has to move every single turn, while in the passive variant she may stay at a vertex. Since their variant of the game allows only a single cop, these restrictions do not apply to the cop since the cop would not have an advantage from not moving anyway.

In the generalized game where $k > 0$ cops are allowed it is possible for the cop player to get an advantage by not moving a cop for a turn. When we study the active variant together with the surrounding winning condition in Chapter 7, we will therefore use the same definition as Offner and Ojakian do in [14]. This means the *active* variant requires every cop and the robber to move every turn.

Offner and Ojakian also introduced the variant of *lazy* cops and robbers where a certain number of cops may not move each turn or a certain amount of cops may never move.

1.3.2 Speed Restrictions

Another family of variants is the introduction of different speeds for the cops or the robber. This might affect the way how both the cops and the robber move like in the *helicopter cops and robbers* variant or it might only affect one party. For example, the alteration of the robber's speed is studied in [8] where Frieze, Krivelevich, and Loh generalize Meyniel's conjecture for any finite speed the robber may move at.

1.3.3 Information

Another possibility of variation is whether the cops have perfect information. If they have no information whatsoever the problem is known under the name *graph searching*, first introduced in [15]. Some of the results from this variant also can be used for finding strategies in the game of cops and robbers. For example, the characterization of the helicopter cops and robbers uses a *monotone* search strategy referring to [10].

1.3.4 Alternate Winning Conditions

Alternating winning conditions is another source of variants. These winning conditions might make catching the robber more difficult (as the surrounding winning condition does) or instead make catching the robber easier. For example, the *cops and robbers from a distance* variant, introduced in [4] by Bonato, Chiniforooshan, and Prałat, changes the game so that the cops win as soon as a cop has distance $d \leq k$ to the robber for a certain k.

2 Simple Bounds

We now want to introduce some straightforward bounds on the surrounding cop number already introduced in [6] and then look at some graph classes in order to show some of the changes the surrounding winning condition can have on certain graphs. We will show both examples where Lemmas 2.1 to 2.3 are tight and where they are not in Sections 2.1 to 2.4.

Lemma 2.1. *Let G be a graph, then $c(G) \leq \sigma(G)$.*

This holds because any strategy that surrounds a robber implies that the cops can occupy the robber's vertex in the following turn. Note, however, that this bound only holds if the graph is connected or has no vertices of degree zero since vertices with degree zero do not need a cop guarding them in the surrounding variant but do in the normal variant.

An additional lower bound for the *surrounding cop number* of a graph is its minimal degree. This is trivially true because for the robber to be caught on any vertex this many cops are needed to surround her.

Lemma 2.2. *Let G be a graph, then $\delta(G) \leq \sigma(G)$.*

Another bound we will use later is based on a subgraph with sufficiently large minimal degree.

Lemma 2.3. *Let G be a graph and $H \subseteq G$ with $\delta(H) = k$ and for all $v \in V(H)$ $\deg(v) > k$, then $\sigma(G) > k$.*

Proof. Since $\deg(v) > k$ for all $v \in V$, the robber can never be caught by k cops as long as she stays on vertices of H. Furthermore, since $\delta(H) = k$ the robber can always move to another vertex of H if she is forced to move by a cop occupying her vertex. Therefore, the robber can always stay on vertices of H and consequently can never be caught by k cops or less. □

We now want to examine some graphs to show how the altered winning condition can drastically change the game on certain graphs.

2.1 Paths

If we look at paths P_n with n vertices, it is clear that $c(P_n) = 1$. This holds because the cop can simply start at one of the leaves and move to the other leaf and thereby catch the robber. In the surrounding variant however this is not possible, since the robber could switch positions with the cop upon him entering the robber's vertex (if the path is long enough). For $n \leq 3$ the cop can be positioned on the middle vertex and therefore $\sigma(P_n) = 1$ for all $n \leq 3$. This is an example where Lemmas 2.1 and 2.2 are tight. However, if $n > 3$ no matter how the cop is positioned the robber can always stay on non-leaf vertices and therefore never be caught. If two cops are used instead, it becomes trivial to catch the robber since both cops can start at adjacent vertices and move towards the robber, *pushing* her towards a leaf. Therefore, $\sigma(P_n) = 2$ for all $n > 3$ making this an example where Lemma 2.1 is not tight. Furthermore, this is an example where Lemma 2.2 is not tight, since $\delta(P_n) = 1$.

2.2 Cycles

Unlike paths, cycles C_n have no winning cop strategy with only one cop for sufficiently large n in the normal variant of the game. While it is trivial to catch a robber with a single cop for $n \leq 3$ (see Section 2.4), it is not possible for $n > 3$, since the robber can start at a vertex with distance of at least two to the cop and then always move in the same direction as the cop and thereby keep her distance.

If two cops are used instead, they can start at adjacent vertices and move in different directions until they catch the robber. We now slightly alter the behaviour of the cops to never move onto a vertex occupied by the robber. Doing this we can also use this strategy for the surrounding variant. This is true because the only possibility that no cop moves in a step is if both cops are already adjacent to the robber. Therefore, the game is either already won or each step the territory of the robber is reduced by at least one. This means $\sigma(C_n) = c(C_n) = 2$ for all $n > 3$.

2.3 Trees

Trees are another class of graphs with a very simple strategy for the cops. In the normal variant the cop can start at any vertex. Then he can continuously descend the sub-tree where the robber is and by that continuously reduce the territory of the robber and eventually catch her. Therefore, the cop number of a tree is one.

We can easily modify this strategy to be applicable for the surrounding variant. We do this by letting a second cop follow the cop to prevent the robber from switching vertices upon the cop entering her vertex. Now the two cops can simply descend the sub-trees where the robber is and continuously reduce the robber's territory. Therefore, the surrounding cop number of a tree is two.

2.4 Complete Graphs

An example for graphs where the surrounding and the normal variant of the game diverge severely are complete graphs K_n. Since all vertices are connected to each other, it is clear that in the normal variant one cop suffices to catch the robber. This is true because he can reach every vertex in one step and therefore trivially catch the robber in one step.

In the surrounding variant however, since $\delta(K_n) = n - 1$, we know that $\sigma(K_n) \geq n - 1$ because of Lemma 2.2. Furthermore, since all but one vertices are occupied if $n - 1$ cops are placed on pairwise different vertices, we also know that $\sigma(K_n) = n - 1$. If we interpret K_n as a subgraph of K_{n+1}, this is also an example where Lemma 2.3 is tight.

As an example where Lemma 2.3 is not tight, we want to look at the graph K_4^1, which arises from subdividing each edge of K_4 (Fig. 2.1). It is clear that the largest minimal degree any subgraph of K_4^1 can have is one. Therefore, Lemma 2.3 only tells us that $\sigma(K_4^1) > 1$. But if the robber only moves upon a cop entering her vertex, it should be clear that she cannot be caught by two cops. We will show in Chapter 4 that indeed $\sigma(K_4^1) = 3$, giving us an example where Lemma 2.3 is not tight.

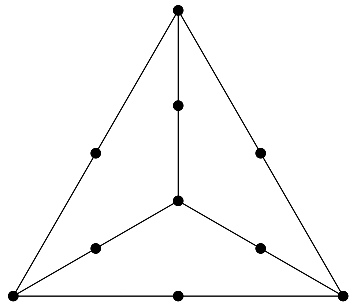

Figure 2.1: The graph K_4^1 arising from subdividing each edge in K_4.

If we look at the definition of the degeneracy of a graph, we can see that all the graphs above, with the exception of K_4^1, have a surrounding cop number that is either the same as its degeneracy or its degeneracy plus one.

Definition 2.4. *Let G be a graph and S be the set of all subgraphs of G. Then the degeneracy of G is defined as* $\deg(G) = \max\{\delta(s) \mid s \in S\}$.

This could suggest, as alluded to in the introduction, that there might exist a bound for the *surrounding cop number* of a graph involving its degeneracy. We will come back to that later in Chapter 4.

3 Guarding Paths

In order to find a winning strategy it is sometimes useful to continuously shrink the territory the robber can access. One way of achieving this is to guard paths with a certain number of cops so that the robber cannot access it without being caught. This idea was first introduced in [2] for the variant of the game introduced by Aigner and Fromme in a proof for an upper bound of graphs that exclude a certain minor. We will use a slightly different approach to the proof inspired by [5]. For this we first introduce the concept of the robber's shadow.

Definition 3.1. *Let G be a graph. Let $P = (v_0, \ldots, v_k) \subseteq G$ be a path. When the robber occupies a vertex w with $d(v_0, w) \leq k$, her shadow is on v_k and if $d(v_0, w) \geq k + 1$, her shadow is on v_k.*

Lemma 3.2. *Let G be a graph and let $P = (v_0, \ldots, v_k) \subseteq G$ be a geodesic path. Then there exists a strategy using one cop guarding it in the normal variant after a finite number of steps.*

Proof. Place one cop on v_0 and in the following turns move the cop towards the robber's shadow. Since the shadow can only move up to one step each turn and P has length $k + 1$ after at most k steps, the cop occupies the vertex with the robber's shadow. In the following turns the cop follows its movement.

Assume now that the robber wants to enter a vertex $v_i \in P$. Then she has to occupy a vertex $v_i' \in N(v_i)$. This means that $d(v_0, v_i') \leq d(v_0, v_i) + 1$. Assume now that $d(v_0, v_i') < d(v_0, v_i) + 1$. Then because P is geodesic $d(v_0, v_i') \geq d(v_0, v_i) - 1$ must hold since otherwise $|(v_0, \ldots, v_i', v_i, \ldots, v_k)| < d(v_0, v_i) - 1 + 1 + d(v_i, v_k) = |P|$ would contradict that P is geodesic. This means that upon the robber wanting to enter P, the robber's shadow and therefore the cop is either on v_{i-1}, v_i or v_{i+1}. If the cop is on v_i, it is trivial to catch the robber. If the cop instead is on one of the other vertices, he becomes adjacent to the robber upon her moving onto P. Thus, the robber can be caught trivially. □

Since a cop being adjacent to the robber is no longer sufficient in the surrounding variant of the game, it is clear this proof needs to be altered for this variant. Bradshaw did this in [5], proving that three cops suffice for a geodesic path and two cops suffice for geodesically closed paths. For the sake of completeness we want to show these proofs here, too.

Lemma 3.3. *Let G be a graph and let $P = (v_0, \ldots, v_k) \subseteq G$ be a geodesic path. Then there exists a strategy using three cops guarding it in the surrounding variant after a finite number of steps.*

Proof. Place cops c_1, c_2, c_3 on the vertices v_0, v_1, v_2. Until c_2 is occupying the vertex with the robber's shadow, the cops move towards it. Similar to the proof of Lemma 3.2 we know that this happens after a finite number of steps. Now the cops follow the movement of the robber's shadow so that both adjacent vertices on P are occupied by c_1 and c_3. The only exception to this is upon the shadow entering v_0 or v_k. In this case two cops will occupy the vertex with the robber's shadow on it. If now the shadow leaves this vertex, c_1 or c_3 respectively will not move for a turn. Thus the adjacent vertices will be occupied.

Assume now that the robber wants to enter a vertex $v_i \in P$. Then they have to occupy a vertex $v_i' \in N(v_i)$. Similar to the proof of Lemma 3.2 we know that $d(v_0, v_i) - 1 \leq d(v_0, v_i') \leq d(v_0, v_i) + 1$. We observe that for all three cases there is a cop occupying v_i. Therefore, the robber must enter a different vertex $v_j \in P$ with $v_j \in N(v_i')$. Since P is geodesic $v_j \notin N(v_i')$ must hold for all $j < i - 1$. This is true because otherwise $P' = (v_0, \ldots, v_j, v_i', v_i, \ldots, v_k)$ would be shorter than P. For the same reasons $v_j \notin N(v_i')$ holds for all $j > i + 1$. Therefore, the only vertices of P the robber could enter are v_{i-1} and v_{i+1}. In order to enter v_{i-1}, the robber's shadow would need to be on v_{i+1} since otherwise v_{i-1} would be occupied by a cop. However, for the shadow to be on v_{i+1}, $d(v_0, v_i') = i + 1$ must hold. Therefore, v_{i-1} and v_i' cannot be adjacent as that would imply $d(v_0, v_i') \leq i$. Similarly, in order to enter v_{i+1}, the robber's shadow would need to be on v_{i-1} since otherwise v_{i-1} would be occupied by a cop. This means $d(v_0, v_i') = i - 1$ would have to hold. But since $P' = (v_0, \ldots, v_i', v_{i+1}, \ldots, v_k)$ would be shorter than P, this implies that v_i' and v_{i+1} are not adjacent.

Therefore there is no way for the robber to enter any vertex of P. □

Lemma 3.4. *Let G be a graph and let $P = (v_0, \ldots, v_k) \subseteq G$ be a geodesically closed path. Then there exists a strategy using two cops guarding it in the surrounding variant after a finite number of steps.*

Proof. We place the cops c_1 and c_2 like we did in the proof of Lemma 3.2 and they move the same as long as the robber's shadow is not on v_0. If that happens, c_2 does not move onto v_0, but instead stays on v_1 until the robber's shadow leaves v_0.

We observe that just like in the proof of Lemma 3.2 c_2 will occupy the vertex with the robber's shadow on it after a finite number of steps. Assume the robber wants to enter a vertex $v_i \in V$. Then the robber needs to occupy a vertex $v_i' \in N(v_i)$. We observe that there is at most one other vertex $v_j \in P$ with $v_j \in N(v_i')$ because else P would not be geodesically closed. Furthermore, v_j must be adjacent to v_i. Assume that $v_{i-1} \in N(v_i')$. Then $d(v_0, v_i') = i$ and therefore v_{i-1} is occupied by c_1 and v_i is occupied by c_2 and the robber can't enter P. Assume now that $v_{i+1} \in N(v_i')$. Then $d(v_0, v_i') = i + 1$ and therefore v_i is occupied by c_1 and v_{i+1} is occupied by c_2 and the robber can't enter P. □

For a proof in Chapter 4 we will also show that for a very specific situation a single cop will suffice after a finite number of steps.

Lemma 3.5. *Let G be a graph. Let $P = (v_0, \ldots, v_k) \subseteq G$ be a path. If $|P'| + 2 \geq |P|$ holds for all path $P' = (v_0, \ldots, v_k)$ with $P' \neq P$ and furthermore $\deg(v_i) = 2$ for all $0 < i < k$, then there exists a strategy involving two cops for a finite number of moves, that thereafter a single cop can prevent the robber from accessing vertices of P.*

Proof. Place one cop on v_0 and move it like c_1 in the proof of Lemma 3.2. We know that after a finite number of steps the cop occupies the vertex next to the robber's shadow. It continues to behave like c_1, except if the robber moves to a vertex v with $d(v_0, v) > k$. If that happens, the cop moves to v_k until the robber enters a vertex v with $d(v_0, v) \leq k$.

Assume now that the robber is not on the path. If the robber wants to enter a vertex of P, she can either enter v_0 or v_k. If she wants to enter v_0, she has to be on a vertex w with $d(w, v_0) = 1$. This means the shadow of the robber is on v_1 and therefore the cop is on v_0 and the robber can not enter v_0. If the robber wants to enter v_k, she has to be on a vertex $w \in N(v_k, w)$. Since the shadow must not be on v_k in order to enter P, $d(w, v_0)$ must be at most k. If such a vertex $w \notin P$ exists, the path $P' = (v_0, \ldots, w, v_k)$ has length at most $k + 1$ and therefore contradicts the assumption that $|P'| + 2 \geq |P|$ holds for all paths $P' = (v_0, \ldots, v_k)$ with $P' \neq P$. This means that if the robber is not on the path P upon the cop occupying the vertex next to the robber's shadow, she can never access vertices of P.

Assume now that the robber is on a vertex of P upon the cop occupying the vertex with the robber's shadow. We now move a second cop onto P and move it to the position of the robber's shadow. We once again know that this happens after a finite number of steps. It is clear that after a finite number of steps following the robber's shadow the robber is forced to leave P. When the robber leaves P, we continue the game with the second cop and do not need the first cop anymore as the situation with the cop not on P is established. □

3.1 Planar Graphs

Bradshaw shows a bound of seven for $\sigma(G)$ if G is planar and assume that the bound is not tight and six is a bound as well. Since we want to show a bound for a variant of the game that is heavily inspired by their proof, we want to show this proof here, too. For that we first have to show how we can transform a geodesic path into a geodesically closed one. For that we will call a path $(a, \ldots, b)$ an (a, b)-path.

Lemma 3.6. *Let G be a graph with a fixed drawing in the plane. Let $P_1, P_2 \subseteq G$ be two (a, b)-paths enclosing a component $A \subseteq G \setminus (P_1 \cup P_2)$ where the robber resides on. Let furthermore P_1 be geodesic in regards to $A \cup P_1$ but not geodesically closed and let P_2 be geodesically closed in regards to $A \cup P_2$. Assume that P_1 is guarded by three cops, as described in Lemma 3.3 and P_2 is guarded by two cops, as described in Lemma 3.4. Then there exists a strategy using two additional cops for a finite amount of steps to restrict the territory of the robber to a component $B \subsetneq A$ surrounded by a geodesic and a geodesically closed path guarded by two and three cops respectively.*

Proof. Let $P_1 = (v_1, \ldots, v_n)$. Since P_1 is not geodesically closed in regards to $A \cup P_1$, we can find a path $S \not\subseteq P_1$ with $S = (v_i, v'_{i+1} \ldots, v'_{j-1}, v_j)$ such that $P_3 = (v_1, \ldots, v_i, v'_{i+1} \ldots, v'_{j-1}, v_j, \ldots, v_n)$ is geodesic in regards to $A \cup P_1$. We now choose S to be the shortest of all such paths and if multiple of those exist, we choose S so that the region A' enclosed by P_1 and N is minimized. Let $P_1(i, j) = (v_i, \ldots, v_j)$. Then $A' \cup \{v_i, v_j\}$ does not contain any (v_i, v_j)-paths that are geodesic in regards to $A' \cup S \cup P_1$.

Assume now that P_1 is guarded by three cops c_1, c_2, c_3 as described in Lemma 3.3. We now use two additional cops c_4, c_5 to guard S as described in Lemma 3.4. It is clear that if the robber is on a vertex in A', as the cops on S are in place, we are done since $A' \subsetneq A$ and it is surrounded by two geodesically closed paths.

Assume now that the cops c_4 and c_5 are in place but the robber is not on a vertex in A'. If the cops guarding P_1 are on vertices not in $P_1 \setminus P_1(i, j)$, it is clear that they can adjust their strategy to guard P_3 instead and we are done.

Assume therefore that at least one cop guarding P_1 is on a vertex in $P_1(i, j)$. This means the robber's shadow has to be on a vertex in $P_1(i, j)$ as well. Since the robber's shadow can only move up to one vertex each turn, we can move the cop on P_1 closest to v_0 to v_i before the robber's shadow arrives there. The other cops on P_1 continue their strategy. Note that since $P_1(i, j)$ is geodesically closed in regards to $A' \cup P_1(i, j)$, the two cops remaining on $P_1(i, j)$ are still guarding it and therefore if the robber at any point enters a vertex in A', we are done. Now the cop on v_i moves up on S until he enters a vertex adjacent to the cops guarding S. Now those three cops can guard P_3 if the robber's shadow is on a vertex v_k with $k \leq j$. If however the robber's shadow at any point would have entered v_{j+1}, the two cops remaining on P_1 would occupy v_{j+1} and v_{j+2} while one cop on S would occupy v_j. Therefore, those three cops can guard P_3 and we have found a strategy using three cops after a finite number of steps to guard P_3 and the robber's territory is now the region enclosed by P_3 and P_2. □

Theorem 3.7. *Let G be a planar graph. Then there exists a strategy using seven cops catching the robber.*

Proof. We want to look at a fixed drawing of G in the plane. We choose two vertices $u, v \in V(G)$ that are on the outer face of G. We use three cops guarding a geodesic path $P = (u, \dots, v)$. We call the territory available to the robber R. We now use three additional cops to guard the closest geodesic (u, v)-path to P in $R \cup \{u, v\}$. If the robber's territory is not enclosed by the two paths, we do not need the cops guarding the first path anymore. Therefore, we continuously guard paths until two guarded geodesic paths enclose R. We observe that since we always chose the closest paths, both paths P_1 and P_2 guarded by cops are geodesically closed in regards to $R \cup P_1 \cup P_2$. Therefore, two cops each suffice guarding them.

We will now continuously be in one of two cases starting in case 1:

1. The robber's territory is enclosed by two geodesically closed paths and we have three cops remaining.

2. The robber's territory is enclosed by one geodesic but not geodesically closed path and one geodesically closed path and we have two cops remaining.

If we are in case 1, we can use the three remaining cops to guard a geodesic (u, v)-path in $R \cup \{u, v\}$. We observe that this frees up two cops who formerly guarded one of the geodesically closed paths and we reduced the robber's territory. If the robber is not caught, we will now have two cops remaining and therefore are in case 2.

If we are in case 2, we can use the strategy described in Lemma 3.6 to use the two additional cops to reduce the robber's territory. If the robber is not caught, the robber's territory is now either enclosed by two guarded geodesically closed paths with three cops remaining, putting us in case 1, or the robber's territory is enclosed by one geodesic but not geodesically closed path and one geodesically closed path with two cops remaining, putting us in case 2.

Since we can continuously reduce the robber's territory this way, we catch the robber after a finite number of steps and therefore have a winning strategy using seven cops. □

Note that Bradshaw suspects that this bound is not tight.

The example Bradshaw gives for a planar graph G with $\sigma(G) \geq 6$ is the truncated icosahedron with an additional vertex v added to every face f and additional edges $\{v, u\}$ for each vertex u of f. However, they only show that five cops are not sufficient to catch a robber on that graph but not that six cops are sufficient.

Therefore, we want to give a different graph G with $\sigma(G) = 6$ which allows an easy proof for the sufficiency of six cops.

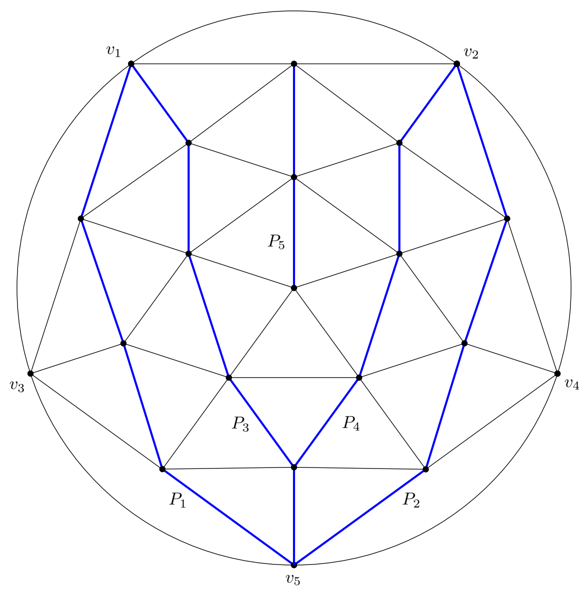

Figure 3.1: Planar graph G with geodesic paths used in the strategy to surround the robber.

Theorem 3.8. *Let $G = (V, E)$ be the graph depicted in Figure 3.1. Let G' be the graph resulting from adding a vertex v' and the edge $\{v, v'\}$ to G, for all vertices $v \in V$ with* $\deg(v) = 5$. *Then $\sigma(G') = 6$.*

Proof. First, we want to show that $\sigma(G') \geq 6$. We can show that by observing that all vertices in $G \subseteq G'$ have degree at least five. Furthermore, we observe that all vertices in G' have degree six. Therefore, Lemma 2.3 proofs that indeed $\sigma(G') \geq 6$. It remains to be shown that six cops are sufficient to catch the robber.

We start by using two cops each to guard the paths (v_1, v_3, v_5) and (v_2, v_4, v_5). Now we can use the remaining two cops to guard the path P_1. This is possible because P_1 is geodesic in regards to the subgraph the robber can inhabit. Note that the robber cannot be on a vertex with degree one adjacent to a guarded path without being captured. We observe that the cops guarding (v_1, v_3, v_5) are no longer needed, since the robber would have to pass $P_1 = (v_1, \dots, v_5)$ in order to access these vertices. This means the two cops guarding this path can now guard the path $P_2 = (v_2, \dots, v_5)$, making the cops guarding (v_2, v_4, v_5) obsolete. Since guarding $P_3 = (v_1, \dots, v_5)$ and $P_4 = (v_2, \dots, v_5)$ is making guarding P_1 and P_2 respectively obsolete, we end up with two cops we can use to guard P_5. Now the robber has no vertex left they can be on. Therefore, we have found a strategy using six cops to catch the robber on G'. $\square$

4 Subdivisions of Complete Graphs

We now want to look at subdivisions of complete graphs. Subdividing an edge $\{u, v\} \in E(G)$ in a graph G results in the new graph $G' = (V(G) \cup \{w\}, (E(G) \cup \{\{u, w\}, \{v, w\}\}) \setminus \{u, v\})$. Let G be a graph, then G^k denotes the graph resulting from k times subdividing every edge of G.

Subdividing edges in a graph can both increase and decrease the *surrounding cop number* of a graph. An example for increasing the *surrounding cop number* is the graph P_3. As shown in Section 2.1 we have $\sigma(P_3) = 1$. However, if we subdivide an edge in P_3 we get P_4 for which we know that $\sigma(P_4) = 2$.

A less trivial example is the graph G constructed in Theorem 4.3, which is obtained by subdividing certain edges of K_n. Since $\sigma(G) = 2$ it lowers the *surrounding cop number* of K_n by subdividing edges. However, we can also obtain K_n^n by subdividing edges of G for which we will prove in Theorem 4.1 that $\sigma(K_n^n) = \lceil \frac{n-1}{2} \rceil + 1$. Therefore, it is also possible to increase the *surrounding cop number* of G by subdividing it's edges.

As alluded to in Chapter 2 we want to show that the *surrounding cop number* of a graph can not be bounded with a function of its degeneracy. We show this by proving that the *surrounding cop number* of total subdivisions of K_n is dependent on n. For that we define the k-neighbourhood of a vertex v as $N_k(v) = \{u \mid d(v, u) \leq k\}$.

Theorem 4.1. *For all $n, k \in \mathbb{N}$ with $n > 2$ and $k > 0$ we have $\sigma(K_n^k) = \lceil \frac{n-1}{2} \rceil + 1$.*

Proof. First, we want to show that $\sigma(K_n^k) > \lceil \frac{n-1}{2} \rceil$ for all $n > 2$ and $k > 0$. We observe that after $\lceil \frac{n-1}{2} \rceil$ cops are placed on the graph, there must exist a vertex $v \in V(K_n^k)$ with $\deg(v) = n - 1$ with no cop occupying it. We place the robber on that vertex. Since the number of cops is less than $\deg(v)$, the robber can not be captured as long as she stays on that vertex. We also observe that if a cop is on a vertex u with $\deg(u) = n - 1$ in the k-neighbourhood $N_k(u)$ there can not be another vertex with degree $n - 1$. This holds because every vertex w with $\deg(w) > 2$ which is different from u has a distance $d(u, w) = k + 1$. If $\deg(u) = 2$ we observe that $|\{w \in N_k(u) \mid \deg(w) > 2\}| = 2$. This means that in this case exactly two vertices with degree $n - 1$ are in the k-neighbourhood of u.

If the robber only moves upon a cop moving onto her vertex, we can bound the union of all k-neighbourhoods of vertices occupied by cops upon the robber being forced to move by $2(\lceil \frac{n-1}{2} \rceil - 1) + 1 = 2\lceil \frac{n-1}{2} \rceil - 1$. If n is even, this is the same as $n - 1$ and if n is odd, it is the same as $n - 2$. This means there exists at least one vertex $v' \in G$ in whose entire k-neighbourhood $N_k(v')$ is no cop. If the robber has to move, she chooses such a vertex v' and moves there in $k + 1$ steps. Since no cop can reach v' in less steps, the robber cannot be captured on her way. We observe that upon the arrival of the robber at v' it satisfies the condition of not being occupied by a robber, that we demanded at the beginning of the proof. Therefore, the robber can indefinitely repeat the process of moving to different vertices v with $\deg(v) = n - 1$ and can never be captured.

It remains to be shown that $\lceil \frac{n-1}{2} \rceil + 1$ cops are sufficient to catch the robber. We will first prove it for even n. Consider a perfect matching of K_n. It is clear that we can get a perfect matching by using $\frac{n}{2}$ edges. For every edge (u, v) in the perfect matching there exists a shortest path $P = (u, \dots, v) \subseteq K_n^k$ that satisfies the conditions of Lemma 3.5. For each of those paths we place a cop on the vertex u. Using the strategy employed in Lemma 3.5, we can use the one cop remaining to assure that these paths cannot be accessed by the robber. We observe that the robber can not be on a vertex w with $\deg(w) = n - 1$, because since the starting positions of our cops are based on a perfect matching, each w with $\deg(w) = n - 1$ must be on a path guarded by a cop. Therefore, the robber is on a vertex w with $\deg(w) = 2$. This means she is on a shortest path $P = (v_0, \dots, v_i, w, v_{i+1}, \dots, v_k)$ with $\deg(v_0) = \deg(v_k) = n - 1$. Since the starting positions of the cops are based on a perfect matching, both v_0 and v_k must be part of paths guarded by cops. This means the robber can never leave the path P. Using the remaining cop, we now can trivially catch the robber.

We will now prove it for odd n. Consider a perfect matching of $K_n - v$ for an arbitrary vertex w with $\deg(w) = n - 1$. It is again clear that we can get a perfect matching by using $\frac{n-1}{2}$ edges. For every edge (u, v) in our perfect matching there exists a shortest path $P = (u, \dots, v) \subseteq K_n^k$ that satisfies the conditions of Lemma 3.5. For each of those paths place a cop on the vertex u. Using the strategy employed in Lemma 3.5, we can use the one cop remaining to assure that these paths cannot be accessed by the robber. We now direct the remaining cop to the unmatched vertex v. This means the robber cannot be on a vertex w with $\deg(w) = n - 1$. Therefore, the robber must be on a path between two vertices u, x with $\deg(u) = \deg(x) = n - 1$. If neither of these vertices is v, we can use the same strategy used for even n to catch the robber trivially. If w.l.o.g. $x = v$, the cop on v and the cop guarding w can move towards the robber without entering her vertex and will thereby trivially catch her after a finite number of steps. ◻

Corollary 4.2. *Let G be a graph and $\deg(G)$ its degeneracy. Then there exists no function $f : \mathbb{N}_0 \to \mathbb{N}_0$, such that $\sigma(G) \leq f(\deg(G))$.*

Proof. Since we showed that we can find a total subdivision of a complete graph K_m^1 with $\sigma(K_m^1) \geq n$ for any $n \in \mathbb{N}$, we can find graphs with an arbitrarily high surrounding cop number and degeneracy two. Therefore, we cannot find a function only dependent on the degeneracy of a graph to bound it. ◻

We now want to show that the bound presented for complete subdivisions does not hold for arbitrary subdivisions of complete graphs. We do that by showing that there exists a subdivision for every K_n that has $\sigma(K_n) = 2$.

Theorem 4.3. *For every $n \in \mathbb{N}$ with $n \geq 2$ there exists a subdivision of K_n with $\sigma(K_n) = 2$.*

Proof. First, we note that for $n = 2$ we can simply subdivide its one edge twice in order to get P_4 for which Section 2.1 tells us that $\sigma(P_4) = 2$.

For $n > 2$ let $G = K_n$ and $V(G) = \{u_1, \dots u_n\}$. Construct the graph G' with the same vertices and edges as G except that each edge which does not include u_1 is subdivided.

Since $\delta(G') = 2$, we know that $\sigma(G) \geq 2$ because of Lemma 2.2. It remains to be shown that two cops suffice to catch the robber.

For that we place two cops c_1 and c_2 on u_1. If the robber starts on a vertex v with $\deg(v) = 2$, it has two adjacent vertices u_i and u_j. Since we did not subdivide edges including u_1 they are both in $N(u_1)$. Therefore, c_1 can move to u_i and c_2 to u_j to surround the robber.

If the robber instead starts on a vertex v with $\deg(v) > 2$, we can force her to move to another vertex by moving c_1 to v. Since $v \in N(u_1)$, this is possible in one step. Now the robber has to move to a vertex v' with $\deg(v') = 2$. Therefore, c_2 can move to the remaining unoccupied vertex in $N(v')$ and surround the robber. $\square$

5 Strong Product

In this section we want to examine the surrounding cop number for graphs arising from the strong product with paths. Let G be a graph and $V(G) = \{v_1, \dots, v_n\}$. Let furthermore P_n be a path of length n and $V(P_n) = (p_1, \dots, p_n)$. We denote (v_i, p_j) the vertex v_i in the j-th copy in $G \boxtimes P_n$. Let $(v_i, p_j) \in V(G \boxtimes P_n)$. Then we observe that $N((v_i, p_j)) = \{(v_{i'}, p_{j'}) \mid v_{i'} \in N(v_i), \max\{0, j-1\} \leq j' \leq \min\{n, j+1\}\}$. We define the function $p : V \longrightarrow \mathbb{N}$ as $p((v_i, p_j)) = j$.

Theorem 5.1. *Let G be a graph. Then $\sigma(G \boxtimes P_n) \leq \sigma(G) + 2$ for all $n \in \mathbb{N}$.*

Proof. We are going to catch the robber in three phases.

Phase 1: Consider a winning strategy $(s, (v_1, \dots, v_k))$ of G using $\sigma(G) = k$ cops, where $v_1, \dots, v_k$ are the starting positions of the cops. Place cops on the vertices $(v_1, p_i), \dots, (v_k, p_i)$ for $i \in \{0, 1, 2\}$. Now the robber is placed on any other vertex on $G \boxtimes P_n$. If at any point directly after the robber moved, she is on a vertex (v_i, p_j) with $j \leq \min\{j' \mid (v_{i'}, p_{j'})$ is occupied by a cop$\}$, go to phase 2. Else on the cops turns let (v_i, p_j) be the position of a cop. This cop now moves to the vertex (v_i, p_{j+1}). It is clear that after a finite number of steps we are finished with phase 1.

Phase 2: The cops now move depending on the position of the robber and "mirror" the robber's moves along P_n as described in the following. Let r be the vertex of the robber and r' the vertex the robber started their turn:

- If $p(r) = 0 \bmod n$, let $(v_1, \dots, v_k)$ be the positions of the cops on the $p(r)$-th copy. Let furthermore $s((v_1, \dots, v_k)) = (v'_1, \dots, v'_k)$ and let c be a cop at vertex (v_i, p_j). Then the cop moves to (v'_i, p_j).
- Else if $p(r) - p(r') = -1$, , let $(v_1, \dots, v_k)$ be the positions of the cops on the $p(r)$-th copy. Let furthermore $s((v_1, \dots, v_k)) = (v'_1, \dots, v'_k)$ and let c be a cop at vertex (v_i, p_j). Then the cop moves to (v'_i, p_{j-1}).
- Else if $p(r) - p(r') = 1$, let $(v_1, \dots, v_k)$ be the positions of the cops on the $p(r)$-th copy. Let furthermore $s((v_1, \dots, v_k)) = (v'_1, \dots, v'_k)$ and let c be a cop at vertex (v_i, p_j). Then the cop moves to (v'_i, p_{j+1}).

We can easily see that the moves performed by the cops are all legal moves since $N((v_i, p_j)) = \{(v_{i'}, p_{j'}) \mid v_{i'} \in N(v_i), \max\{0, j-1\} \leq j' \leq \min\{n, j+1\}\}$. We observe that after each turn of the cops the robber is in the middle of three graph copies with cops on them. Since the cops are carrying out their turns according to a winning strategy on G, after a finite amount of steps the robber cannot move within her graph copy. This means she can only move to an adjacent graph copy. But since the cops there mirror the movements of the cops on the robber's graph copy, the only legal move for the robber is to move along P_n.

Phase 3: The cops now move only along P_n and in the direction the robber does. Thus, the robber continuously can only move along P_n, too. We now use the remaining two cops to trivially catch the robber on P_n, using the strategy described in Section 2.1 using overall $3\sigma(G) + 2$ cops. □

We now want to show that this bound cannot be generalized for arbitrary subgraphs of $G \boxtimes P_n$. For that we will show that we can find a graph with a high *surrounding cop number* as a subgraph of $K_{1,m} \boxtimes P_m$. Since we know that $\sigma(K_{1,m} \boxtimes P_m) \leq 3\sigma(K_{1,m}) + 2 = 5$, this shows that we cannot bound the *surrounding cop number* for subgraphs of a strong product of a graph with a path.

Theorem 5.2. *Let $K_{1,n}$ be the Star with $n + 1$ vertices. Then there exists $m \in \mathbb{N}$ so that $K_n^n \subseteq K_{1,m}^m \boxtimes P_n$.*

Proof. We start with the graph $(V = \emptyset, E = \emptyset)$. Let (v_i, p_j) be the vertex in the j-th copy of $K_{1,n-1}$ with degree $n - 1$. We now want to construct a path from (v_i, p_1) to (v_j, p_1). W.l.o.g. we assume that $i < j$. We now add the paths $P_1 = (u_1, \ldots, u_{n+i-j})$ and $P_2 = (u_{n+i-j}, \ldots, u_n)$ as well as the edges $\{(v_i, p_1), u_1\}, \{(v_j, p_1), u_n\}$.

This means we have connected (v_i, p_1) and (v_j, p_1) with a path of length $n + 1$. The path P_1 is a subdivision of an edge in the i-th graph copy and the path P_2 arises from the strong-product with P_n. As we did this for any (v_i, p_1) and (v_j, p_1), we have shown that we can connect all n vertices of degree $n - 1$ with disjunctive paths and therefore have constructed $K_n^n \subseteq K_{1,m}^m \boxtimes P_n$. □

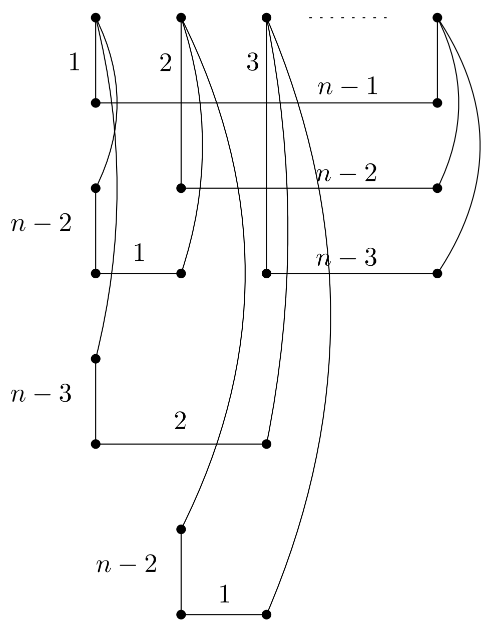

Figure 5.1: Representation of K_n^n the way it is constructed. Edge weights denote the length of the path if its longer than one.

6 Treewidth

While there are some simple lower bounds for the *surrounding cop number* of a graph (see Chapter 2), we do not have any upper bounds so far. We now want to show that an upper bound proved by Burgess et al. in [6] using the treewidth of a graph is tight.

We will use the definition used by Robertson and Seymour in [17] where the treewidth was first introduced.

Let G be a graph. A tree decomposition is a pair (X, T). $X = \{X_1, \ldots, X_n\}$ is a family of subsets of $V(G)$ called bags. T is a tree with $V(T) = X$. Furthermore, the following conditions must be satisfied:

- $V(G) = \bigcup_{i=1}^{n} X_i$,
- For every edge $\{u, v\} \in E(G)$ there is a bag X_i with $u \in X_i$ and $v \in X_i$ and
- For $X_i, X_j, X_k \in X$, if X_j is on the path from X_i to X_k in T then $X_i \cap X_k \subseteq X_j$.

We will now prove the bound already proved in [6] in order to later on show its tightness.

Theorem 6.1. *Let G be a graph, then $\sigma(G) \leq tw(G) + 1$.*

Proof. Let T be a fixed tree decomposition of G with width $\mathrm{tw}(G)$. We then place a cop on each vertex of an arbitrary bag B of T. Therefore, the robber has to start on a different bag B'. We interpret B as the root of the tree T. Let B'' be adjacent to B and in the subtree the robber resides in. It is clear that $B \cap B'' \subseteq B$ and therefore cops can occupy the entirety of B'' while ensuring that the robber can never enter B. After the cops successfully occupied B'', they can this way iteratively reduce the territory of the robber and therefore catch her after a finite amount of steps. This is basically the same strategy used to catch the robber on a tree used in Section 2.3. ◻

Note that this bound also holds if the robber can move an arbitrary amount of edges each turn. Therefore in the setting of *helicopter cops and robbers* the surrounding winning condition is equivalent to the normal winning condition.

Theorem 6.2. *The bound in Theorem 6.1 is tight.*

Proof. To show that the bound is tight we construct a graph G' for every $n \in \mathbb{N}$ with $\sigma(G') = n$ and $\mathrm{tw}(G') = n - 1$.

For that let $G = K_n$ with $V(G) = \{v_1, \ldots, v_n\}$. We now set $G' = (V(G) \cup \{v_1' \ldots, v_n'\}, E(G) \cup \{\{v_i, v_i'\} \mid v \in V(G), 1 \leq i \leq n\})$. Since K_n is a subgraph of G' and all of the vertices belonging to it have degree n in G', we know that $\sigma(G') \geq n$ because of Lemma 2.3. Furthermore, it is clear that $\mathrm{tw}(G') = n - 1$. Therefore, we have constructed a graph for an arbitrary $n \in \mathbb{N}$ for which $\sigma(G) = \mathrm{tw}(G) + 1$ holds. This proves the tightness of the bound. ◻

We now want to show that we can construct graphs for each *surrounding cop number* with an arbitrarily high treewidth.

Theorem 6.3. *For every $n, k \in \mathbb{N}$ with $2 \leq n \leq m$, there exists a graph G with $\sigma(G) = n$ and $tw(G) \geq m$.*

Proof. We start by constructing a graph G as in Theorem 4.3 for $m + 1$. It is clear that G has treewidth m since treewidth is invariant to subdivision for graphs with treewidth greater than one. This is true because for a subdivided edge $\{u, v\}$ which adds a vertex w, we can simply add the bag $\{u, v, w\}$ to our tree decomposition and add an edge to a bag that contains u and v. Since G is a subdivision of K_m it therefore mus have treewidth m. Furthermore, Theorem 4.3 tells us that $\sigma(G) = 2$. Since we constructed it for $m + 1 \geq 3$, we also know that $\delta(G) = 2$.

We now need to increase $\sigma(G)$ to n. For that we will add $n - 2$ universal vertices to G (universal vertices have edges to all other vertices of a graph) to obtain a graph G'. This means that $\delta(G') = 2 + n - 2 = n$ and therefore $\sigma(G') \geq n$. Furthermore, it is clear that $\mathrm{tw}(G') \geq m$ since we just added vertices and edges to a graph which already had sufficient treewidth.

It remains to be shown that n cops suffice to catch the robber. For that we place $n-2$ cops on the universal vertices. It is clear that the robber can never access any of these vertices. Therefore, the robber can be caught by employing the strategy used in Theorem 4.3. □

We now have shown that while the treewidth of a graph can be used as an upper bound for the *surrounding cop number* of a graph, it is not necessarily a particularly good bound as it can deviate quite heavily from the correct *surrounding cop number* of a graph.

7 Active Surrounding Cops and Robbers

We now want to look at the active variant of the game of surrounding cops and robbers. As alluded to in Section 1.3.1, we will use the same definition as in [14]. This means both cops and the robber are required to move every turn and cannot stay on a vertex any more. We want to examine this variation in combination with the surrounding winning condition. We denote the *surrounding cop number* with the active restriction of a graph G $\sigma_a(G)$. Note that in order to get a formal definition it suffices to change Restriction 1.1 to $v'_i \in N(v_i)$ for all $1 \leq i \leq k$ and Restriction 1.2 to $v' \in N(v)$.

First, we want to show that the following theorem proved in [9] for the normal variant also holds for the surrounding variant.

Theorem 7.1. *Let G be a graph. Then $\sigma(G) - 1 \leq \sigma_a(G) \leq 2\sigma(G)$.*

Proof. First, we want to show the lower bound. For that we observe that a single cop can prevent the robber from not moving by constantly following them. Therefore, any strategy in the active variant can be applied to the passive variant by adding such a cop.

Let now C be a set of cops who are used in a winning strategy in the passive variant. We now add an additional cop c'_i for every $c_i \in C$ who starts on an adjacent vertex to c_i. If now a cop c_i moves, the cop c'_i will follow its move in order to stay adjacent. If a cop c_i would stay on a vertex, he instead switches vertices with c'_i. If we now relabel the cops they both effectively stayed on their respective vertices. Therefore, we have found a strategy using twice as many cops that works in the active variant. This proves the upper bound. □

7.1 Planar Graphs with Active Restrictions

An open question that was proposed by Bradshaw in [5] is how the *surrounding cop number* on planar graphs behaves with the additional restrictions of the active variant. We will show a bound by adjusting the lemmas used for guarding paths to work with the additional restrictions.

Lemma 7.2. *Let G be a graph and let $P = (v_0, \ldots, v_k) \subseteq G$ be a geodesic path. Then there exists a strategy using four cops guarding it in the active surrounding variant after a finite number of steps.*

Proof. We will adjust the strategy used in Lemma 3.3 slightly by introducing a fourth cop c_4 who is initially placed on the same vertex as c_2 and makes the same moves as c_2. We note that the only situations the strategy used in Lemma 3.3 is not legal in the active variant is if either the cops do not move because the robber's shadow did not move or if the robber's shadow moves either to v_0 or v_k.

If the cops do not move due to the robber's shadow not moving, they are on vertices v_i, v_{i+1}, v_{i+2}. Since we added a fourth cop behaving like c_2, there are two cops occupying v_{i+1}. Therefore, we have the positions $(v_i, v_{i+1}, v_{i+1}, v_{i+2})$ and the cops can move to $(v_{i+1}, v_i, v_{i+2}, v_{i+1})$ in one step. If we now relabel the cops, they are in the same configuration as they where when the turn started. Therefore, we have found a way for the cops to effectively not move.

It remains to be shown how the cops behave upon the robber's shadow entering v_0 or v_k. Assume w.l.o.g. that the robber's shadow enters v_0. In the strategy employed in Lemma 3.3 this means that the cops move from (v_0, v_1, v_2) to (v_0, v_0, v_1). Since we added the fourth cop, we have the positions (v_0, v_1, v_1, v_2) instead. Therefore, the cops on v_0 and v_2 can move to v_1 and the cops on v_1 move to v_0, giving us the configuration (v_0, v_0, v_1, v_1). If the robber's shadow stays on v_0, the cops can just switch positions to get to an equivalent configuration. If the robber's shadow moves to v_1, the cops can just reverse the moves they made upon the robber's shadow entering v_0, in order to get back to the configuration (v_0, v_1, v_1, v_2).

Since at all times every vertex occupied in the strategy used in Lemma 3.3 is occupied by at least one cop in this strategy, this strategy guards P with four cops who move every turn. □

Lemma 7.3. *Let G be a graph and let $P = (v_0, \dots, v_k) \subseteq G$ be a geodesically closed path. Then there exists a strategy using two cops guarding it in the active surrounding variant after a finite number of steps.*

Proof. We observe that in the strategy used in Lemma 3.4 the two cops used are always at adjacent vertices. Therefore, if the cops do not move in this strategy, they can switch vertices instead, resulting in a legal strategy for the active variant. That this strategy guards P follows directly from Lemma 3.4. □

The proof of the following theorem is very similar to the proof of Lemma 3.6. But in order to make it understandable, we will repeat the complete setup instead of referencing to the proof of Lemma 3.6.

Lemma 7.4. *Let G be a graph with a fixed drawing in the plane. Let $P_1, P_2 \subseteq G$ be two (a, b)-paths enclosing a component $A \subseteq G \setminus (P_1 \cup P_2)$ where the robber resides on. Let furthermore P_1 be geodesic in regards to $A \cup P_1$ but not geodesically closed and let P_2 be geodesically closed in regards to $A \cup P_2$. Assume that P_1 is guarded by four cops, as described in Lemma 7.2, and P_2 is guarded by two cops, as described in Lemma 7.3. Then there exists a strategy using two additional cops for a finite amount of steps to restrict the territory of the robber to a component $B \subsetneq A$ surrounded by a geodesic and a geodesically closed path guarded by two and three cops respectively, which is legal under the additional restrictions of the active variant.*

Proof. Let $P_1 = (v_1, \dots, v_n)$. Since P_1 is not geodesically closed in regards to $A \cup P_1$, we can find a path $S \not\subseteq P_1$ with $S = (v_i, v'_{i+1} \dots, v'_{j-1}, v_j)$ such that $P_3 = (v_1, \dots, v_i, v'_{i+1} \dots, v'_{j-1}, v_j, \dots, v_n)$ is geodesic in regards to $A \cup P_1$. We now choose S to be the shortest of all such paths and if multiple of those exist, we choose S so that the region A' enclosed by

P_1 and N is minimized. Let $P_1(i, j) = (v_i, \ldots, v_j)$. Then $A' \cup \{v_i, v_j\}$ does not contain any (v_i, v_j)-paths that are geodesic in regards to $A' \cup S \cup P_1$.

Assume now that P_1 is guarded by four cops c_1, c_2, c_3, c_4, as described in Lemma 7.2. We now use two additional cops c_5, c_6 to guard S, as described in Lemma 7.3. It is clear that if the robber is on a vertex in A', as the cops on S are in place, we are done since $A' \subsetneq A$ and it is surrounded by two geodesically closed paths.

Assume now that the cops c_5 and c_6 are in place, but the robber is not on a vertex in A'. If the cops guarding P_1 are on vertices not in $P_1 \setminus P_1(i, j)$, it is clear that they can adjust their strategy to guard P_3 instead and we are done.

Assume therefore that at least one cop guarding P_1 is on a vertex in $P_1(i, j)$. Therefore, the robber's shadow has to be on a vertex in $P_1(i, j)$ as well. Since the robber's shadow can only move up to one vertex each turn, we can move the cops c_1 and c_2 to v_i before the robber's shadow arrives there, while the other cops on P_1 continue their strategy. Note that since $P_1(i, j)$ is geodesically closed in regards to $A' \cup P_1(i, j)$, the two cops remaining on $P_1(i, j)$ are still guarding it and therefore if the robber enters a vertex in A' at any point we are done. Now the cops c_1 and c_2 move up on S until one cop enters the vertex of another cop on S. Now those four cops can guard P_3 if the robber's shadow is on a vertex v_k with $k \leq j$. If however the robber's shadow at any point would have entered v_{j+1}, then the two cops remaining on P_1 would occupy v_{j+1} and v_{j+2} while c_6 on S would occupy v_j and the c_5 would occupy v'_{j-1}. It is clear that the first three cops can guard P_3 as long as the robber's shadow does not stay on a vertex for a turn. Let the c_5 follow the other cops. If now the robber's shadow moves towards v_0 at any point, all cops but c_5 can move a step towards v_0 while c_5 moves in the opposite direction and we obtain the correct configuration of cops to guard it according to Lemma 7.2. If at any point the robber's shadow stays on a vertex v_k for a turn, the positions of the cops are $(v_{k-2}, v_{k-1}, v_k, v_{k+1})$. We can therefore obtain the positions $(v_{k-1}, v_k, v_{k+1}, v_k)$, which is equivalent to the positions $(v_{k-1}, v_k, v_k, v_{k+1})$. Therefore, we can then carry out the strategy described in Lemma 7.2. Since the robber's shadow cannot move towards v_n indefinitely, this must happen after a finite number of steps.

Therefore, those four cops can guard P_3 and we have found a strategy using four cops after a finite number of steps to guard P_3 with the additional restrictions of the active variant and the robber's territory is now the region enclosed by P_3 and P_2. □

Theorem 7.5. *Let G be a planar graph. Then there exists a strategy using eight cops catching the robber with the additional restrictions of the active variant.*

Proof. We want to look at a fixed drawing of G in the plane. We choose two vertices $u, v \in V(G)$ that are on the outer face of G. We use four cops guarding a geodesic path $P = (u, \ldots, v)$. We call the territory available to the robber R. We now use four additional cops to guard the closest geodesic (u, v)-path to P in $R \cup \{u, v\}$. If the robber's territory is not enclosed by the two paths, we do not need the cops guarding the first path anymore. Therefore, we continuously guard paths until two guarded geodesic paths enclose R. We observe that since we always chose the closest paths, both paths P_1 and P_2 guarded by cops are geodesically closed in regards to $R \cup P_1 \cup P_2$. Therefore, two cops each suffice guarding them.

We now observe that with the four remaining cops we can use the same strategy as in the proof of Theorem 3.7 by using the equivalent strategies for the active variant proved in this section.

The additional cop is needed for the beginning when two geodesic paths have to be guarded and for the execution of the strategy in Lemma 7.4.

□

Note that just like the bound for planar graphs without the active restrictions this bound is probably not tight.

7.2 Hypercube

The cop number of the hypercube Q_n has first been determined by Maamoun and Meyniel in [11] as a special case of the Cartesian product of trees. They proved that $c(Q_n) = \lceil \frac{n+1}{2} \rceil$. Later on, the cop number for the active variant was studied by Neufeld and Nowakowski in [12] and it was shown that $c_a(Q_n) = \lceil \frac{n}{2} \rceil$ where c_a is the cop number with the restrictions of the active variant.

A generalization of the active variant where a certain number of cops have to move every turn and a certain number of cops have to not move every turn was then studied by Offner and Ojakian in [14]. The specific variant of this where only one cop may move every turn was then also studied in [3].

First, we want to examine the surrounding cop number of the hypercube without any further restrictions. For that we observe that Q_n is the Cartesian product of n paths of length one.

Theorem 7.6. *For every $n \in \mathbb{N}$ we have $\sigma(Q_n) = n$.*

Proof. First we observe that $\sigma(Q_n) \geq \delta(Q_n) = n$ by Lemma 2.2. Therefore, it remains to be shown that n cops suffice to catch the robber.

For that we use the bound $\sigma(G \square H) \leq \sigma(G) + \sigma(H)$, shown in [6] for any graphs G and H. Since P_2 has $\sigma(P_2) = 1$ and Q_n is the Cartesian product of n copies of P_2, it follows that $\sigma(Q_n) = \sigma(P_{2_1} \square \cdots \square P_{2_n}) \leq n$ for n copies of P_2. □

We now want to look at the hypercube with the restrictions of the active variant. For that let the vertices of Q_n be defined as $(x_1, \ldots, x_n) \in V(Q_n)$ where $x_i \in \{0, 1\}$. We first want to show how the parity of the distance between two vertices relates to the parity of the distance of another vertex to each of those vertices.

Lemma 7.7. *Let $v_1, v_2, v_3 \in V(Q_n)$ and $d(v_1, v_2) \bmod 2 = i$ and $d(v_1, v_3) \bmod 2 = j$ with $i, j \in \{0, 1\}$. Then $d(v_2, v_3) \bmod 2 = i + j \bmod 2$.*

Proof. Let $v_1 = (x_1^1, \ldots, x_n^1), v_2 = (x_1^2, \ldots, x_n^2), v_3 = (x_1^3, \ldots, x_n^3)$. We now want to partition the indices into four sets.

$$A_1 = \{j \in \mathbb{N} \mid 1 \leq j \leq n, x_j^1 = x_j^2 = x_j^3\}$$
$$A_2 = \{j \in \mathbb{N} \mid 1 \leq j \leq n, x_j^1 = x_j^2 \neq x_j^3\}$$
$$A_3 = \{j \in \mathbb{N} \mid 1 \leq j \leq n, x_j^1 \neq x_j^2 = x_j^3\}$$
$$A_4 = \{j \in \mathbb{N} \mid 1 \leq j \leq n, x_j^2 \neq x_j^1 = x_j^3\}$$

Note that this is indeed a partition of all indices $1 \leq j \leq n$.

We can now formulate the distances between the vertices using these sets.

$$d(v_1, v_2) = |A_3 \cup A_4| = |A_3| + |A_4|$$
$$d(v_1, v_3) = |A_2 \cup A_3| = |A_2| + |A_3|$$
$$d(v_2, v_3) = |A_2 \cup A_4| = |A_2| + |A_4|$$

Therefore, we can also write the distance between v_2 and v_3 as $d(v_2, v_3) = d(v_1, v_3) - |A_3| + d(v_1, v_2) - |A_3| = d(v_1, v_3) + d(v_1, v_2) - 2|A_3|$. Therefore, it follows directly that $d(v_2, v_3) \bmod 2 = d(v_1, v_3) + d(v_1, v_2) \bmod 2$. □

We now show that in the active variant the parity of the distance between a cop and the robber can never change. For that we denote with $d_i(c, r)$ the distance of a cop c to the robber r after the i-th move of the robber and $d'_i(c, r)$ the distance of a cop c to the robber r after the i-th move of the cops. We denote the distance after the robber is initially placed on a vertex with $d_0(c, r)$.

Lemma 7.8. *Let $d_0(c, r) \bmod 2 = i$. Then $d'_j(c, r) + 1 \bmod 2 = i$ and $d_j(c, r) \bmod 2 = i$.*

Proof. Let $d_j(c, r) \bmod 2 = i$ be the distance of the cop c to the robber r after the j-th step of the robber. If the cop moves now, he either does a move which increases or decreases the distance to the robber by one. Therefore, after he moved the parity of the distance changed and $d'_{j+1}(c, r) + 1 \bmod 2 = i$. Note that the cop does not have the option to not move due to the restrictions of the active variant.

Now let $d'_{j+1}(c, r) + 1 \bmod 2 = i$. If the robber moves now, she either increases or decreases the distance to the cop. Therefore, the parity of the distance has to change again and $d_{j+1}(c, r) \bmod 2 = i$.

Therefore, the statement follows inductively. □

We now show that the surrounding cop number of the hypercube in the active variant is significantly higher than without it. The strategy we will use to catch the robber is similar to the strategy used to catch the robber on normal Cayley graphs described in [5]. We will say that a cop or the robber does the move x_i if they move from $(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)$ to $(x_1, \ldots, x_{i-1}, x_i + 1 \bmod 2, x_i + 1, \ldots, x_n)$. We say a cop mirrors the move of the robber if he does the same movement the robber last did. We will denote the vertex resulting from doing the move x_i starting at v with $v + x_i$. We say the robber has k moves available if exactly k vertices in the neighbourhood of the robber are unoccupied.

Theorem 7.9. *Let $n \in \mathbb{N}$ with $n > 3$. Then $\sigma_a(Q_n) = 2n - 1$.*

Proof. First, we want to show a strategy using $2n - 1$ cops to catch the robber. Let C be the set of cops. We start by placing $n - 1$ cops $c_1, \dots, c_{n-1}$ on a vertex v and $n - 1$ cops $c'_1, \dots, c'_{n-1}$ on the vertex $v + x_1$. When a cop c_i is not explicitly used, he switches positions with x'_i in order to always move. In the first turn all cops c'_i will do the move x_1. In the following turns they will do the move c_i did the turn before in order to follow him. Therefore, all cops placed that way will always have another cop to switch positions with.

The remaining cop we place on an arbitrary vertex. He will move arbitrarily if not explicitly used.

We will now prove that $2n - 1$ suffice by inductively proving the following claim:

Claim 1. *If the robber has k moves available, $2k - 1$ cops suffice to catch her.*

Assume that the robber only has a single move x_i available. Then the robber alternates between the vertices $(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n)$ and $(x_1, \dots, x_{i-1}, x_i + 1 \bmod 2, x_{i+1}, \dots, x_n)$. Therefore, if the cop occupies either of these, the robber has no moves left and is caught. Therefore, the statement holds for $k = 1$.

Assume now that if the robber has k moves available, $2k - 1$ cops suffice to catch her.

We now show that the statement holds for $k + 1$. For that we will inductively prove the following claim:

Claim 2. *If the robber has $k + 1$ moves available and there is a cop with distance d to an unoccupied vertex in the neighbourhood of the robber, we can use $2k - 1$ cops for a finite number of turns, so that after that two cops can reduce the available moves of the robber by one.*

If we have a cop adjacent to an unoccupied vertex in neighbourhood of the robber, we can move the cop to that vertex. The cop now continuously mirrors the robber's moves. Therefore, the robber has k moves available.

Assume now that Claim 2 holds for a $d \in \mathbb{N}$.

Let r be the vertex occupied by the robber. Assume now that we have a cop with distance $d + 1$ to an unoccupied vertex $r + x_i$. Then the cop can reach $r + x_i$ by playing $d + 1$ moves $x_{i_1}, \dots, x_{i_{d+1}}$. If the robber plays a move x_j with $j = i_l$ with $1 \leq l \leq d + 1$ at any point, the distance of the cop to $r + x_i + x_j$ is d as it can be reached by the moves $x_{i_1}, \dots, x_{i_{l-1}}, x_{i_{l+1}}, \dots, x_{i_{d+1}}$. Therefore, the cop can switch positions with the cop following him and we can achieve a distance of d and the two cops can therefore reduce the robber's available moves by one because of Claim 2. Note that it is relevant that two cops are used here since if the distance of the cop to the robber is even after the cop moves, the cop could never be adjacent to the robber and therefore never forbid a certain move.

Assume now that the robber will never play such a move. Then her moves are effectively reduced by $d+1$ and because of Claim 1 we can use the remaining $2k-1$ cops to inductively reduce her available moves to zero. Therefore, the robber is then forced to make a move reducing the distance to a cop to d after a finite amount of steps. This proves Claim 2 and therefore we can use $2k + 1$ cops for a finite amount of steps, so that after that two cops reduce the available moves of the robber by one. Therefore, after a finite amount of steps the robber has k moves available and we have $2k + 1 - 2 = 2k - 1$ cops who are not explicitly used in restricting the moves of the robber. This proves Claim 1 and therefore the sufficiency of $2n-1$ cops as the robber has at most n moves available at the beginning of the game.

We now show that $2n-2$ cops do not suffice to catch the robber. Place $2n-2$ cops on arbitrary vertices of Q_n. Let $v \in V(Q_n)$ be a vertex that is not occupied by a cop for that a vertex $v' \in N(v)$ exists that is not occupied either. Since $\frac{1}{2}|V(Q_n)| = 2^{n-1} > 2n-1$ for $n > 3$, less than half of the vertices of Q_n are occupied by cops. Therefore, such a vertex has to exist.

Each cop has now either an even or an odd distance to v. If exactly $n-1$ cops have an odd distance and $n-1$ cops have an even distance, it is clear that at no point n cops can have distance one to the robber, as all cops change their parity simultaneously because of Lemma 7.8. Thus, the robber could never be caught. Assume therefore that k cops have an odd distance to v and $2n-2-k$ cops have an even distance to v. Since the parity is inverted for v' (Lemma 7.7), we can assume w.l.o.g. that $k > n-1 > 2n-2-k$ as we could simply relabel v and v'. We now place the robber on v. It is clear that only the cops who have an odd distance to the robber at the beginning can potentially catch her since there are less than n other cops. Furthermore it is clear that at least n of those cops have to have distance one to the robber if they want to catch her. Since the parity of the distance is only odd directly after the robber moves, she must move to a vertex already surrounded by cops in order for the cops to ever catch her.

Assume now that the robber is on the vertex $v = (x_1, \ldots, x_n)$ and it is her turn to move. Therefore, at most $2n-2-k$ cops are adjacent to her and thus she can move to $n-2n+2+k = k+2-n \geq 2$ different vertices. Assume $k = n$. Then $2n-2-k = n-2$. This means that there are at least two vertices available for the robber to move to. The union of their neighbourhoods is clearly greater than n. Therefore, the robber can move to a vertex where she is not surrounded. Assume now that $k > n$. Therefore, at most $2n-2-k$ cops are adjacent to her and she can move to $n-2n+2+k = k+2-n > 2$ different vertices. Let $v_1 = v + x_i, v_2 = v + x_j, v_3 = v + x_k$ be three such vertices. In order for the cops to catch the robber, all neighbourhoods of those vertices would have to be occupied. Since $N(v_1) = \{v + x_i + x_l \mid 1 \leq l \leq n\}, N(v_2) = \{v + x_j + x_m \mid 1 \leq m \leq n\}, N(v_3) = \{v + x_k + x_p \mid 1 \leq p \leq n\}$, we can see that $N(v_2)$ contains two vertices of $N(v_1)$ for $m \in \{j, i\}$ and $N(v_2)$ contains three vertices of $N(v_1) \cup N(v_2)$ for $p \in \{j, i, k\}$. Therefore, $|N(v_1) \cup N(v_2) \cup N(v_3)| = 3n-2-3 = 2n-2+n-3 > 2n-2$, meaning that the neighbourhoods cannot be occupied completely and consequently the robber can do a move which does not result in a capture, proving that $2n-2$ cops do not suffice to catch her.

Therefore, $\sigma_a(Q_n) = 2n-1$ for all $n > 3$. □

We now want to look at the remaining cases of the hypercube Q_1, Q_2 and Q_3. For Q_1 it is trivial that $\sigma_a(Q_1) = 1$. Since $Q_2 = C_4$, we can just simply occupy two non-adjacent vertices and trivially catch the robber. Since $\delta(Q_2) = 2$, it follows with Lemma 2.2 that $\sigma_a(Q_2) = 2$.

Theorem 7.10. *We have $\sigma_a(Q_3) = 4$.*

Proof. First, we want to show that four cops suffice to catch the robber. For that we place cops on all four blue vertices in Figure 7.1. Since all neighbourhoods of the non-blue vertices are completely occupied, the robber is caught trivially after she is placed.

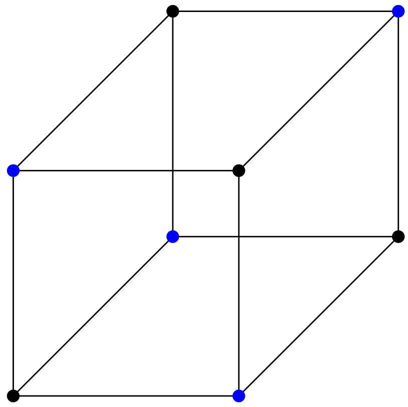

Figure 7.1: The hypercube Q_3. The blue vertices are occupied by cops.

We now show that three cops do not suffice. The arguments we use are largely the same as for $n \geq 3$. Place three cops on vertices of Q_3. We now pick any unoccupied vertex v for which an unoccupied vertex $v' \in N(v)$ exists. If the parity of the distance of the cops to v is not the same for every cop, they can never catch the robber, since at no point three cops can be adjacent to the robber. Assume therefore w.l.o.g. that all cops have odd parity to v by potentially relabelling v and v'. We now place the robber on v. Since v' is not occupied, the robber cannot be caught directly after being placed on the graph. Since furthermore the parity of the distance of the cops to the robber is only odd directly after the robber moves, in order for the robber to be caught, she must move on a vertex whose entire neighbourhood is already occupied. Since no cop is adjacent to the robber upon her having to move, she has three vertices available to move to. Since the union of the neighbourhoods of those vertices contains four vertices, she cannot be forced to move to a vertex where she gets caught. This means that three cops do not suffice to catch her.

Therefore, it follows that $\sigma_a(Q_3) = 4$. □

All together we get the following corollary.

Corollary 7.11. *We have* $\sigma_a(Q_n) = \begin{cases} 2^{n-1} & n \leq 3 \\ 2n-1 & n \geq 4 \end{cases}$ *for all* $n \in \mathbb{N}$.

8 Conclusion and Open Questions

While we were able to show tight bounds for the surrounding cop number for certain graphs, we were unable to find an upper bound using the degeneracy of graph as it was originally intended. However, we were able to prove that such a bound cannot exist. Furthermore, in combination with the results regarding the correlation of treewidth and the surrounding cop number we have to conclude that there is still a significant lack of useful upper bounds.

We conjecture however that there exists an upper bound using the maximum degree of a graph as we were not a able to construct a graph with $\sigma(G) > \Delta(G)$.

Furthermore, the question remains open under what condition the subdivision of an edge increases or decreases the surrounding cop number of a graph. The same applies to the addition or subtraction of edges or vertices.

Regarding planar graphs we assume that Bradshaw is right in the assumption that there exists no planar graph G with $\sigma(G) > 6$. However, just like them we were unable to find a proof for that bound.

Regarding the hypercube it would be interesting to generalize the restrictions of the cops to force a certain number of cops to move each turn and a certain number of cops to not move each turn, the way it is done in [14] for the normal version of cops and robbers.

Bibliography

[1] Martin S. Aigner and M. Fromme. "A game of cops and robbers". In: *Discrete Applied Mathematics* 8.1 (1984), pp. 1–12. ISSN: 0166-218X. DOI: https://doi.org/10.1016/0166-218X(84)90073-8.

[2] Thomas Andreae. "On a pursuit game played on graphs for which a minor is excluded". In: *Journal of Combinatorial Theory, Series B* 41.1 (1986), pp. 37–47. ISSN: 0095-8956. DOI: https://doi.org/10.1016/0095-8956(86)90026-2.

[3] Deepak Bal, Anthony Bonato, William B. Kinnersley, and Paweł Prałat. "Lazy Cops and Robbers on Hypercubes". In: *Combinatorics, Probability and Computing* 24.6 (2015), pp. 829–837. DOI: 10.1017/S0963548314000807.

[4] Anthony Bonato, Ehsan Chiniforooshan, and Paweł Prałat. "Cops and Robbers from a distance". In: *Theoretical Computer Science* 411.43 (2010), pp. 3834–3844. ISSN: 0304-3975. DOI: https://doi.org/10.1016/j.tcs.2010.07.003.

[5] Peter Bradshaw. "Cops and robbers on Cayley graphs and embedded graphs". MA thesis. Science: Department of Mathematics, 2020. URL: https://summit.sfu.ca/item/20916.

[6] Andrea C. Burgess, Rosalind A. Cameron, Nancy E. Clarke, Peter Danziger, Stephen Finbow, Caleb W. Jones, and David A. Pike. "Cops that surround a robber". In: *Discrete Applied Mathematics* 285 (2020), pp. 552–566. ISSN: 0166-218X. DOI: https://doi.org/10.1016/j.dam.2020.06.019.

[7] Peter Frankl. "Cops and robbers in graphs with large girth and Cayley graphs". In: *Discrete Applied Mathematics* 17.3 (1987), pp. 301–305. ISSN: 0166-218X. DOI: https://doi.org/10.1016/0166-218X(87)90033-3.

[8] Alan Frieze, Michael Krivelevich, and Po-Shen Loh. "Variations on cops and robbers". In: *Journal of Graph Theory* 69.4 (2012), pp. 383–402. DOI: https://doi.org/10.1002/jgt.20591.

[9] Ilya Gromovikov, William B. Kinnersley, and Ben Seamone. "Fully Active Cops and Robbers". In: *The Australasian Journal of Combinatorics* 59 (2020), pp. 248–265. URL: https://digitalcommons.uri.edu/math_facpubs/139.

[10] Andrea S. LaPaugh. "Recontamination Does Not Help to Search a Graph". In: *J. ACM* 40.2 (Apr. 1993), pp. 224–245. ISSN: 0004-5411. DOI: 10.1145/151261.151263.

[11] Malaz Maamoun and Henry Meyniel. "On a game of policemen and robber". In: *Discrete Applied Mathematics* 17.3 (1987), pp. 307–309.

[12] Stewart W. Neufeld and Richard Nowakowski. "A game of cops and robbers played on products of graphs". In: *Discrete Mathematics* 186.1 (1998), pp. 253–268. ISSN: 0012-365X. DOI: https://doi.org/10.1016/S0012-365X(97)00165-9.

[13] Richard Nowakowski and Peter Winkler. "Vertex-to-vertex pursuit in a graph". In: *Discrete Mathematics* 43.2 (1983), pp. 235–239. ISSN: 0012-365X. DOI: https://doi.org/10.1016/0012-365X(83)90160-7.

[14] David Offner and Kerry Ojakian. "Variations of cops and robber on the hypercube". In: *The Australasian Journal of Combinatorics* 59 (Jan. 2014), pp. 229–250.

[15] Torrence D. Parsons. "Pursuit-evasion in a graph". In: *Theory and applications of graphs*. Springer, 1978, pp. 426–441.

[16] Alain Quilliot. "Jeux et pointes fixes sur les graphes". PhD thesis. Ph. D. Dissertation, Université de Paris VI, 1978.

[17] Neil Robertson and Paul D. Seymour. "Graph minors. II. Algorithmic aspects of tree-width". In: *Journal of Algorithms* 7.3 (1986), pp. 309–322. ISSN: 0196-6774. DOI: https://doi.org/10.1016/0196-6774(86)90023-4.

[18] Paul D. Seymour and Robin Thomas. "Graph Searching and a Min-Max Theorem for Tree-Width". In: *Journal of Combinatorial Theory, Series B* 58.1 (1993), pp. 22–33. ISSN: 0095-8956. DOI: https://doi.org/10.1006/jctb.1993.1027.