The Planar Graph Grabbing Game

Bachelor thesis
of

Benedikt Hahn

At the Department of Informatics
Institute of Theoretical Informatics

Reviewers: PD Dr. Torsten Ueckerdt
T.T.-Prof. Dr. Thomas Bläsius

Advisors: PD Dr. Torsten Ueckerdt

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Karlsruhe, March 31, 2022

Abstract

In this work we introduce a new game on graphs called “The planar graph grabbing game”. The game is played on a plane graph with vertices weighted 0 and 1. Vertices with weight 1 are called *cherries*. Two players – Alice and Bob starting with Alice – take turns removing single vertices (and in doing so their incident edges) from the outer face of the remaining graph. The game ends when no vertices are left and the player who obtained the most weight wins.

We call an instance *Bob-dominant* when Bob is able to obtain all cherries on the instance, no matter which strategy Alice follows. First, we show that there are Bob-dominant instances with arbitrarily many cherries. We also prove that there are even 4-connected triangulated Bob-dominant instances with arbitrarily many cherries. We then give examples for 4-connected triangulated Bob-dominant instances of odd size with up to six cherries and prove that no such graphs can exist for seven or more cherries.

Finally, we briefly pursue the question what share of cherries Alice is guaranteed to get on odd 4-connected triangulated instances with “many” cherries.

Deutsche Zusammenfassung

In dieser Arbeit führen wir ein neues Spiel auf Graphen mit dem Namen „Das planare Graph Grabbing Game“ ein. Das Spiel wird auf einem planaren Graph mit fester Einbettung, dessen Knoten entweder Gewicht 0 oder 1 haben, gespielt. Die Knoten mit Gewicht 1 nennen wir *Kirschen.* Zwei Spieler – Alice und Bob beginnend mit Alice – entfernen abwechselnd einzelne Knoten (und deren anliegende Kanten) von der äußeren Facette des verbleibenden Graphen. Das Spiel endet, wenn alle Knoten entfernt wurden; der Spieler, der am meisten Gewicht gesammelt hat, gewinnt.

Wir nennen eine Instanz *Bob-dominant*, wenn Bob unabhängig von Alice’s Strategie in der Lage ist alle Kirschen zu erhalten. Zuerst zeigen wir, dass es Bob-dominante Instanzen mit beliebigen vielen Kirschen gibt. Außerdem beweisen wir, dass es sogar 4-fach zusammenhängende triangulierte Bob-dominante Instanzen mit beliebig vielen Kirschen gibt. Wir geben anschließend Beispiele für 4-fach zusammenhängende triangulierte Bob-dominante Instanzen mit einer ungeraden Anzahl Knoten und bis zu sechs Kirschen und zeigen, dass solche Graphen mit sieben oder mehr Kirschen nicht existieren können.

Zum Schluss betrachten wir kurz die Frage, welchen Anteil der Kirschen Alice auf ungeraden 4-fach zusammenhängenden triangulierten Instanzen mit “vielen” Kirschen mindestens erhält.

Contents

1. Introduction

In 2003, Peter Winkler introduced the first graph grabbing game in his book about mathematical puzzles [1]. In a graph grabbing game two players take turns removing vertices from a graph with weighted vertices. The game ends when there are no vertices left. The player who collects the larger amount of weight wins. It can easily be proven that, in this simple setup, it is best for both players to employ a greedy strategy. What makes this interesting, however, is adding restrictions on which vertices can be removed throughout the game. Such restrictions have proven to make the resulting games very hard to analyze. Specifically, even determining which player can win is often $\mathcal{PSPACE}$-complete [2]. So in this work, we take a look at a new graph grabbing game: The planar graph grabbing game.

An instance of the game is a planar graph $G = (V, E)$ with a fixed plane embedding and a weight function $c : V \to \mathbb{R}_{\geq 0}$. In the game, two players (Alice and Bob) take turns removing vertices from the outer face of the remaining graph until no vertices are left. Alice begins. The players' goal is to maximize the weight of the vertices they obtain.
Since this is the first work dealing with this particular game, we only consider a simplified version of the problem in which the weight function is restricted to weights in $\{0, 1\}$. This makes our questions easier to analyze and discuss. We define $\mathcal{C}(G) := \{v \in V \mid c(v) = 1\}$ and call the vertices in $\mathcal{C}(G)$ cherries.

Since Alice always makes the first move, she usually has an advantage. So an interesting question to ask is:

"How good can the situation get for Bob?"

Answering questions of this form is the focus of this thesis. The best situation for Bob would be an instance on which he can obtain all cherries – no matter which strategy Alice pursues. We call such an instance *Bob-dominant*.

Structure

After introducing basic definitions in Chapter 2, we concretize the question posed above in Chapter 3:

"For any $n \in \mathbb{N}$, is there a Bob-dominant graph with n cherries?"

In Section 3.1, we show that such games do indeed exist. Then, in Section 3.2 we restrict the problem to a subclass of planar graphs (4-connected and triangulated planar graphs). This makes our previous construction for Bob-dominant graphs impossible. On this subclass we find a new construction for Bob-dominant games which also leads to a positive answer to the question.
On the even more restricted subclass of 4-connected, triangulated planar graphs with an odd number of vertices, which we analyze in Section 3.3, we show that the above is not true anymore. In particular, we find in Section 3.3.2 that for $n \leq 6$ there are Bob-dominant games with $|\mathcal{C}(G)| = n$ but for any instance with $|\mathcal{C}(G)| \geq 7$, Alice can always obtain at least one cherry. We prove this statement in Section 3.3.1.
After having learned that Alice can always obtain some share of the weight for graphs with "many" cherries, we investigate *what* share of the weight Alice can always get when playing optimally in Chapter 4. We prove a trivial upper bound of $\frac{1}{2}$ and a lower bound of $\frac{1}{8}$ for graphs with "nice" structure.
The work concludes in Chapter 5 with a discussion of the results and gives ideas on what other questions could be asked about our game for future work.

Related Work

A vast amount of games have already been considered on graphs before. One of the most extensively covered such game is *the game of cops and robbers*, with an entire book dedicated to it [3]. In this game, a set of cops and one robber take turns moving on a graph. The cops win if they catch the robber at some point in the game, i.e., land on the same vertex as the robber. The robber wins if he gets never caught. On planar graphs for example, three cops are always sufficient to catch the robber [4].
As mentioned in the introduction, grabbing (or take-away) games in particular were first studied after being introduced in Peter Winkler's book about mathematical puzzles [1]. The grabbing game he presented is played by two players on weighted connected graphs with the restriction that only non-cut vertices can be removed. The first results that were proven on this game were on cycles (often thought of as pizza slices). In particular, it was shown by two independent teams that the first player can always obtain $\frac{1}{2}$ of the weight on even cycles and $\frac{4}{9}$ of the weight on odd cycles [5, 6]. The game has since then also been considered on other classes of graphs, such as trees [7, 8], graphs not containing certain subgraphs [9] and more [10, 11]. Another variation that has been considered is restricting the weight function to weights in $\{0, 1\}$ [12].
The *graph sharing game* is very similar to this original game. Here, the restriction is, that the subgraph formed by the removed vertices needs to stay connected, instead of the subgraph formed by the remaining vertices. This game has also been considered in multiple works [13–15] Another grabbing game similar to the planar grabbing game is the *convex grabbing game* [16], in which two players take turns removing weighted points in the plane. The restriction here is that only points in the convex hull of the remaining point set can be removed. The motivation for our work and the approach we take is motivated by a paper by Dvorak and Nicholson [17] which aims to find good configurations for Bob in the convex grabbing game.

2. Preliminaries

In this chapter we will first give a few general definitions which are already well-established in the field of graph theory and then introduce new terms which we will then use throughout this work.

2.1 General Definitions

The following definitions can be found for example in [18]. We will assume that all graphs we consider in this work are simple, undirected graphs. Unless specified otherwise, we are in the context of some graph G.

A *path* P in a graph is a sequence of vertices $(v_1, \dots, v_n)$ such that $v_i v_{i+1} \in E(G)$ for $1 \leq i \leq n-1$. A graph is called *connected* if there exists a path $P_{u,v} = (u, \dots, v)$ for every $u, v \in V(G)$.

For a graph G and a subset of vertices $S \subseteq V(G)$, the *induced subgraph* of S is the graph $G[S] \coloneqq (S, \{ab \mid a, b \in S,\ ab \in E(G)\})$.

A graph G with more than k vertices is called *k-connected* if for any subset of vertices $S \subseteq V(G)$ with $|S| = k-1$, $G[V(G) \setminus S]$ is connected.

A graph is called *plane* if it consists not only of an abstract graph but also an embedding of the graph into the Euclidean plane where

- vertices are points in the plane,
- edges are curves between the points of their incident vertices,
- and edges only intersect on the points of their incident vertices.

A *face* of a plane graph is a connected component of the Euclidean plane where the plane embedding of the graph has been removed.
A plane graph is called *triangulated* if there are exactly three vertex points on the boundary of every face of the graph. An abstract graph is called *planar* if there is a plane graph with the same underlying graph. A planar graph is called *maximal planar* if adding any non-existent edge would result in a non-planar graph.

2.2 The Graph Grabbing Game

An *instance* of the planar graph grabbing game is a plane graph $G = (V, E)$ with a weight function

$$w : V \to \{0, 1\}.$$

We usually assume implicitly that graphs and structures we are talking about are contained in such an instance G. A *cherry* is a vertex v with $w(v) = 1$.
The *weight of an instance* is

$$w(G) := \sum_{v \in V(G)} w(v).$$

We define the *set of cherries* on a graph G as

$$\mathcal{C}(G) := \{v \in V(G) \mid w(v) = 1\}.$$

We call an instance G *even* if $|V(G)| \equiv 0 \pmod 2$. Otherwise, we call G *odd*.
For an instance of the game G, we call $C \subseteq V(G)$ a *configuration*.

$$w(C) := \sum_{v \in C} w(v)$$

$$\operatorname{out}(C) := \{v \in C \mid v \text{ is incident to the outer face of } G[C]\}$$
$$\operatorname{Follow}(C) := \{C \setminus \{v\} \mid v \in \operatorname{out}(C)\}$$

A *game* on an instance G is a sequence of configurations

$$(V = C_0, C_1, C_2, \ldots, C_{|V|} = \emptyset) \in (2^V)^{|V|+1}$$

for which $C_i \in \operatorname{Follow}(C_{i-1})$ for all $1 \leq i \leq |V|$. We define $\mathcal{G}(G)$ as the set of all games on the instance G. A configuration of G is *valid* if it is part of a game $\mathfrak{G} \in \mathcal{G}(G)$.
For a set of vertices U we define the minimal set of vertices contained in any valid configuration which contains U as

$$M(U) := \bigcap_{\substack{D \text{ valid config.} \\ U \subseteq D}} D.$$

Furthermore, we define the set of vertices which are *hidden* by U as

$$H(U) := M(U) \setminus U$$

and the set of vertices on the inside of U

$$\operatorname{in}(U) := M(U) \setminus \operatorname{out}(U).$$

For a given game $\mathfrak{G} = (C_0, \ldots, C_{|V|}) \in \mathcal{G}(G)$ we define

$$\mathcal{A}(\mathfrak{G}) := \sum_{\substack{1 \leq i \leq |V| \\ i \equiv 1 \pmod 2}} w(C_{i-1} \setminus C_i)$$

$$\mathcal{B}(\mathfrak{G}) := \sum_{\substack{1 \leq i \leq |V| \\ i \equiv 0 \pmod 2}} w(C_{i-1} \setminus C_i)$$

as the weight Alice and Bob obtain in that game.
Furthermore, we define $A(G)$ as the weight Alice obtains if both players play optimally.

For $V(G) = \emptyset$ we have $A(G) = 0$ and for $V(G) = \{v\}$ we get $A(G) = w(v)$. Otherwise we define

$$A(G) := \max_{v_1 \in \mathrm{out}(V)} \left(\min_{v_2 \in \mathrm{out}(V \setminus \{v_1\})} \Big(w(v_1) + A(G - \{v_1, v_2\}) \Big) \right).$$

The weight Bob obtains in this case is

$$B(G) := w(G) - A(G).$$

We call G *Bob-dominant* if $B(G) = w(G)$.

We furthermore define the following names for graphs:
A wheel graph of size $k \geq 4$ where the single vertex of degree $k-1$ is a cherry and all the others are not is called a *cherry wheel* of size k. Cherry wheels are examples of Bob-dominant graphs. We will often refer to cherry wheels by the vertex set forming them. If two cherry wheels are not vertex disjoint we call them *joint*.
Two cherry wheels U, W which are not joint but have two (not necessarily vertex-disjoint) edges $e_1, e_2 \in U \times W$ (here and at some other points in our work we abuse notation and mean $\{ab \mid a \in U,\ b \in W\}$ by $U \times W$) incident to the outer face of $G[U \cup W]$ are said to *span a corridor*. The edges e_1, e_2 are called the *spanning edges*. The *corridor* in this case is made up from the cycle containing e_1, e_2 as well as the vertices in $U \cup W$ which are not incident to the outer face on $G[U \cup W]$ and everything contained within this cycle. We call a corridor with $e_1 \cap e_2 \neq \emptyset$ *narrow* and all others *wide*. Two cherry wheels with a wide corridor are shown in Figure 2.1. In such drawings, cherries are always red squares and non-cherries black circles.

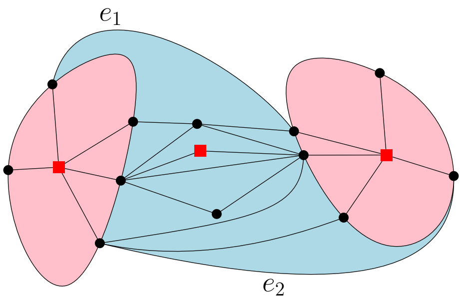


Figure 2.1: An example of two cherry wheels with a wide corridor colored blue.

For a corridor C in a plane graph we define $\overline{C}$ as the set of points in the plane which lie in the interior region and on the boundary of the cycle bounding C. Two corridors C, D spanned by cherry wheels C_1, C_2 and D_1, D_2 respectively are *crossing* if for any two simple plane curves $c : [0,1] \longrightarrow \overline{C}$ with $c(0) \in C_1$ and $c(1) \in C_2$ and $d : [0,1] \longrightarrow \overline{D}$ with $d(0) \in D_1$ and $d(1) \in D_2$ there are some $a, b \in [0,1]$ with $c(a) = d(b)$. The configuration of two crossing corridors is called a *corridor crossing*.

For three cherry wheels U, V, W which are not contained within each other we call V *hidden by U and W* if $V \subseteq M(U \cup W)$. Then $M(U \cup W)$ is a *hiding* which is *spanned by U and W*. A hiding is a *corridor hiding* if U and W span a corridor and V is contained within it. A corridor hiding is shown in Figure 2.2. If U and W are joint, the hiding is called an *edge hiding* as there is one edge $e \in U \times W$ on $\mathrm{out}(U \cup W)$ which together with edges on $(U \cup W) \setminus \mathrm{out}(U \cup W)$ forms a cycle containing V on its inside in the plane drawing. As in corridors we call e the *spanning edge*. An edge hiding is shown in Figure 2.3.

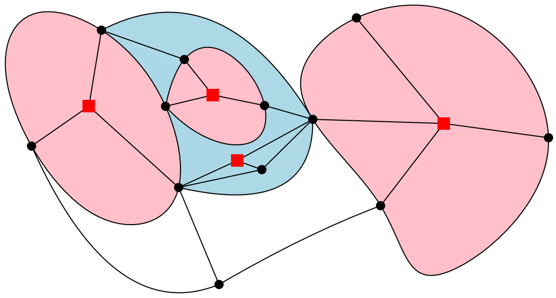


Figure 2.2: This graph contains a corridor hiding with a narrow corridor colored in blue and the three cherry wheels colored pink.

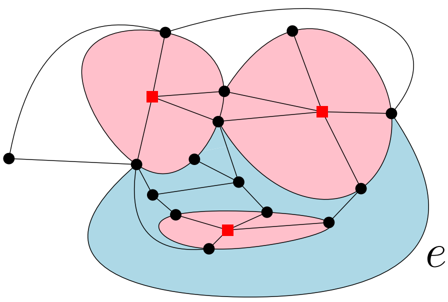


Figure 2.3: This graph contains an edge hiding where e is the spanning edge.

3. Bob-dominant Games

In this section, we will focus on the existence and non-existence of Bob-dominant graphs with a fixed amount of cherries for different subclasses of planar graphs. In Section 3.1, we first look at arbitrary plane graphs and then consider narrower subclasses in the following subsections. Before we start with our main theorems however, we will first introduce two lemmas which will be helpful for proving Bob-dominance of graphs made up of smaller Bob-dominant graphs:

Lemma 1. *Let G and H be even plane graphs. Let K be a plane graph which consists only of G and H (drawn separately) and edges connecting G and H such that* $\mathrm{out}(V(K)) = \mathrm{out}(V(G)) \cup \mathrm{out}(V(H))$*.*
Then $B(K) \geq B(G) + B(H)$.

Note that this implies that such a K is Bob-dominant when G and H are Bob-dominant. Note also that the inequality is strict in some cases. If $G = H = Q$ where Q is the graph shown in Figure 3.1 for example, then $B(K) = 1 > 0 = 2B(Q)$.

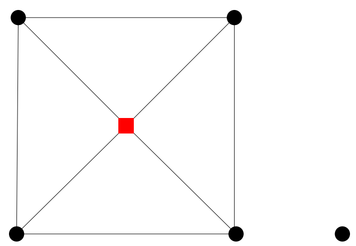


Figure 3.1: A graph Q with $B(Q) = 0$

Proof. Let G and H be arbitrary plane even graphs and K the corresponding graph containing G, H and some edges between them. A sketch of such a plane graph K can be seen in Figure 3.2. For any configuration C of K and some vertex $v \in \mathrm{out}(C) \cap V(G) \subseteq V(K)$, we get a new configuration $C' = C \setminus \{v\}$ by removing v from K. This removal uncovers the exact same vertices as removing the corresponding vertex in G would.
More formally: $\mathrm{out}(C') = \mathrm{out}((V(H) \cap C)) \cup \mathrm{out}((V(G) \cap C) \setminus \{v\})$. This holds because G and H are only connected by edges on their outer faces. By symmetry, the same argument also works for vertices in H of course.

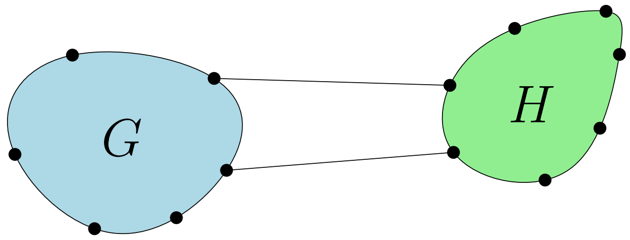


Figure 3.2: Sketch of a graph K as described in Lemma 1

We can give a strategy $\mathcal{S}_K$ with which Bob can obtain at least as much weight on K as he can on G and H combined by using optimal strategies $\mathcal{S}_G$, $\mathcal{S}_H$ for Bob on G and H:

- If Alice removes a vertex from G, then Bob will always remove the vertex Bob would have removed according to $\mathcal{S}_G$ on the next turn. This is always possible because

  1. Alice's and Bob's moves on K uncover the same vertices as on G (which we have seen above).
  2. there is always at least one vertex remaining in G. This holds because if Bob follows $\mathcal{S}_K$, it is only Alice's turn when there is an even number of vertices in G left. So after Alice removes a vertex from G, there must be at least one vertex left.

- If Alice removes a vertex from H, then Bob will always remove the vertex Bob would have removed following $\mathcal{S}_H$ on the next turn. This is always possible for the same reasons given above.

Strategy $\mathcal{S}_K$ copies $\mathcal{S}_G$ and $\mathcal{S}_H$ and therefore only ever encounters configurations C of K in which $V(G) \cap C$ would also occur with $\mathcal{S}_G$ on G and $V(H) \cap C$ would also occur with $\mathcal{S}_H$ on H. Therefore, $\mathcal{S}_G$ and $\mathcal{S}_H$ will always give a valid next move. Because Bob obtains the same vertices on K with $\mathcal{S}_K$ as he does on G and H with $\mathcal{S}_G$ and $\mathcal{S}_H$. We get $B(K) \geq B(G) + B(H)$ which is our intended result. $\square$

Lemma 2. *Let G be an arbitrary plane graph and H a plane even graph. Let K be a plane graph which consists only of G and H (drawn separately) and edges connecting G and H such that* $\mathrm{out}(V(G)) \subset \mathrm{out}(V(K))$.
Then $B(K) \geq B(G)$.

Proof. The proof is very similar to the proof of Lemma 1. We can give a strategy $\mathcal{S}_K$ with which Bob can obtain all cherries by using an optimal strategy $\mathcal{S}_G$ for Bob on G:

- If there are no vertices left in G after Alice's turn and the game is not over yet, Bob will just take any vertex on the outer face of the remaining subgraph of H.
- If Alice removes a vertex from G which was not the last, Bob will always remove the vertex Bob would have removed in $\mathcal{S}_G$ on the next turn. This can always be accomplished because of the argument about uncovered vertices from the proof of Lemma 1.
- If Alice removes a vertex from H while there are still vertices left in G, Bob removes another vertex from H. This is always possible because when there are still vertices left in G and Bob followed $\mathcal{S}_K$ up to this point, it is only Alice's turn when there is an even number of vertices left in H. So after Alice removes a vertex from H, there must be at least one vertex left.

Strategy $\mathcal{S}_K$ copies $\mathcal{S}_G$ and therefore only ever encounters configurations C of K in which $V(G) \cap C$ would also occur with $\mathcal{S}_G$ on G (except when $V(G) \cap C = \emptyset$). So $\mathcal{S}_G$ will always give the next possible move while there are vertices left in G.
By following $\mathcal{S}_K$ on K, Bob obtains the same vertices of $V(K) \cap V(G)$ as he does by following $\mathcal{S}_G$ on G. Therefore, we get $B(K) \geq B(G)$ which is what we wanted to prove. □

3.1 General Graphs

Now, we will prove our first main theorem: There are Bob-dominant graphs with arbitrarily many cherries.

Theorem 1. *For every $n \in \mathbb{N}$, there exists a plane Bob-dominant graph G with n cherries, i.e., $|\mathcal{C}(G)| = n$.*

Proof. We use induction:
Base case $n = 1$. We see that the graph Δ_1 shown in Figure 3.3 is Bob-dominant. Alice can only remove either v_1, v_2 or v_3, uncovering c in the process. Bob can then always take c and we have $B(\Delta_1) = 1 = w(\Delta_1)$.

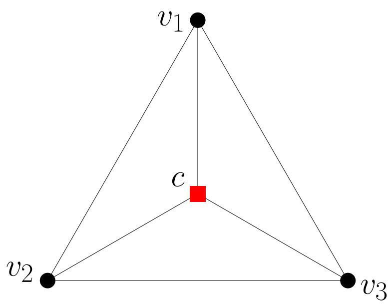
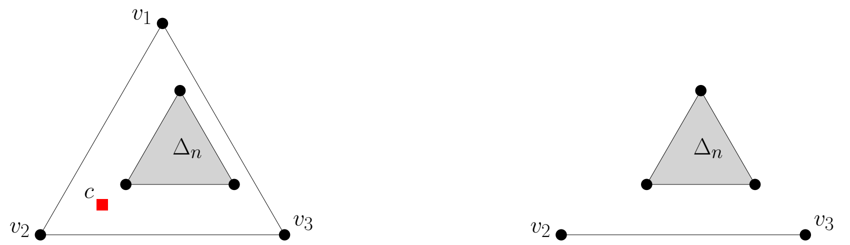


Figure 3.3: The smallest Bob-dominant graph with non-zero weight Δ_1, a cherry wheel of size 4

Inductive step $n \rightsquigarrow n+1$. Suppose we have a plane Bob-dominant graph Δ_n with $|\mathcal{C}(G)| = n$. We obtain Δ_{n+1} by embedding Δ_n into a triangle formed by three new vertices v_1, v_2, v_3 $(w(v_i) = 0)$ and adding a cherry c next to Δ_n in the triangle. This construction can be seen in Figure 3.4a.
Then, Alice must remove some $u_1 \in \{v_1, v_2, v_3\}$. If Bob then removes c, we are left with a configuration similar to what is depicted in Figure 3.4b. With Lemma 2 we get:

$$\begin{aligned}
A(\Delta_{n+1}) &= \max_{u_1 \in \{v_1,v_2,v_3\}} \left(\min_{u_2 \in \operatorname{out}(V(\Delta_{n+1}) \setminus \{u_1\})} \Big(w(u_1) + A(\Delta_{n+1} - \{u_1, u_2\}) \Big) \right) \\
&\leq \max_{u_1 \in \{v_1,v_2,v_3\}} \Big(0 + A(\Delta_{n+1} - \{u_1, c\}) \Big) \\
&= \max_{u_1 \in \{v_1,v_2,v_3\}} \left(A\left(\Delta_n \cup \binom{\{v_1, v_2, v_3\} \setminus \{u_1\}}{2} \right) \right) \\
&\overset{\text{Lemma 2}}{\leq} \max_{u_1 \in \{v_1,v_2,v_3\}} \left(A(\Delta_n) + A\left(\binom{\{v_1, v_2, v_3\} \setminus \{u_1\}}{2} \right) \right) \\
&= \max_{u_1 \in \{v_1,v_2,v_3\}} (0 + 0) = 0 \\
&\Rightarrow B(\Delta_{n+1}) = w(\Delta_{n+1}) = n + 1.
\end{aligned}$$

This proves that Δ_{n+1} is Bob-dominant. □



(a) Construction of Δ_{n+1} from Δ_n. (b) A configuration of Δ_{n+1} after two moves.

Figure 3.4: The construction of Δ_{n+1} from Δ_n.

We have now shown that Bob-dominant instances with arbitrarily many cherries do exist. However, we had to resort to repeatedly nesting Bob-dominant graphs to accomplish that. So a natural question to ask is whether we can also get arbitrarily large Bob-dominant graphs which are not nested like this. We tackle this in the following section.

3.2 4-connected Triangulated Graphs

We now add two restrictions to our instances to make the search for Bob-dominant graphs harder:

- G must be 4-connected, i.e., there exists no cut set of size 3.
- G must be triangulated, i.e., every face of G is incident to three vertices. This is equivalent to G being maximal planar.

With these restrictions, the construction from the previous chapter no longer works, as our graphs cannot contain separating triangles.

Lemma 3. *In any plane triangulation G with $|V(G)| \geq 5$, the following equivalence holds:*

$$G \text{ is 4-connected} \Longleftrightarrow G \text{ does not contain a separating triangle}$$

Proof.
$\Longrightarrow$: Follows directly from the definition of k-connectedness.

$\Longleftarrow$: We use contraposition. Suppose G is a plane triangulation and not 4-connected. We first show that G must be 3-connected. Suppose we had a cut set $\{x, y\}$ of size 2 where $G - \{x, y\} = A \cup B$ such that A and B are non-empty and G contains no edges in $A \times B$. Since x and y cannot form a cycle, an edge $e \in A \times B$ could be added to G without destroying planarity. So G is not maximal planar, a contradiction to G being a plane triangulation.
So we have a cut set $\{x, y, z\} \subset V(G)$ such that $G - \{x, y, z\} = A \cup B$ where A and B are again non-empty and G contains no edges in $A \times B$. Vertices x, y and z must have neighbors in both A and B. Otherwise, we would have a smaller cut set which is a contradiction to 3-connectedness of G. Since G is a plane triangulation, it is also maximal planar. Using this and the fact that there is no edge in $A \times B$, the edges xy, xz and yz must be present in G which gives us our separating triangle. □

From this we get the following corollary.

Corollary 1. *In any 4-connected triangulated instance of the graph grabbing game G with a vertex $v \notin \mathrm{out}(V(G))$, the subgraph induced by the neighbors of v and v itself $G[N(v) \cup \{v\}]$ is a wheel. Furthermore, there are no vertices hidden by v and its neighbors, i.e., $H(N(v) \cup \{v\}) = \emptyset$.*

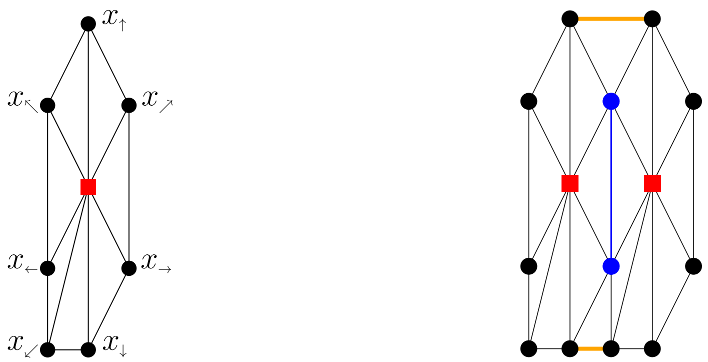
Proof. Let $v_1, \ldots, v_n$ be the neighbors of v ordered clockwise by the outgoing edges of v. All following operations are implicitly mod n. Because G is a plane triangulation, $v_i v_{i+1} \in E(G)$ for $1 \leq i \leq n-1$ and $v_n v_1 \in E(G)$. Suppose there was another edge $v_i v_j \in E(G)$ with $|i - j| \neq 1$. Then, we get a separating triangle by v_i, v_j and v. This is a contradiction to Lemma 3. So $G[N(v)]$ is a cycle and since $v \notin \mathrm{out}(V(G))$, the cycle must contain v on its inside. Therefore, $G[N(v) \cup \{v\}]$ is a wheel.
Suppose there was some $u \in H(N(v) \cup \{u\})$. Then u would be contained in one of the faces formed by the wheel. The vertices forming the face would separate u from the rest of the graph. Since every face in the wheel is a triangle, this is again a contradiction to Lemma 3. This concludes the proof. $\square$

We now proceed with the main theorem of this section.

Theorem 2. *For every $n \in \mathbb{N}$ there exists a 4-connected triangulated Bob-dominant instance of the planar grabbing game G with $|\mathcal{C}(G)| = n$.*

To prove Theorem 2 we construct such instances and show that they are Bob-dominant. This would be simple when using Theorem 4 which we will prove at a later point in the thesis. However, we will restrict ourselves to the tools we already obtained, making the proof a bit more complicated.

Proof. We start by explicitly constructing Bob-dominant graphs. Then, we will prove inductively that subgraphs of our construction which do not fulfill the new conditions are Bob-dominant. Finally, we show Bob-dominance of our actual graphs.
We will call the basic part of our construction the $\mathcal{O}$-tile (shown in Figure 3.5a). It is isomorphic to a cherry wheel of size 8. We now define $\mathcal{O}_n$ as n $\mathcal{O}$-tiles put next to each other where two adjacent $\mathcal{O}$-tiles are merged as shown in Figure 3.5b. Note that merged $\mathcal{O}$-tiles share two vertices. We call the cherries in $\mathcal{O}_n$ $c_1, \ldots, c_n$ from left to right.



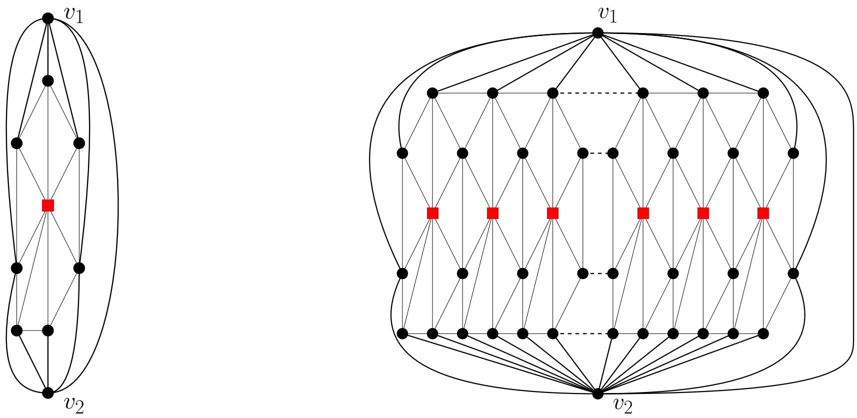
(a) The $\mathcal{O}$-tile

(b) Two merged $\mathcal{O}$-tiles share two vertices and an edge (blue) and form two new edges (orange)

Figure 3.5: The basic building blocks of our 4-connected triangulated Bob-dominant graphs

Note that $\mathcal{O}_n$ is even because $\mathcal{O}_1$ contains 8 vertices and every merged $\mathcal{O}$-tile adds 6 vertices. Our graphs $\mathcal{O}_n$ are neither triangulated – the outer face is not a triangle – nor

4-connected (even though it does not contain any separating triangles). We fix this by adding two vertices v_1 and v_2 to $\mathcal{O}_n$ as shown in Figure 3.6b. We call this final graph on n cherries $\overline{\mathcal{O}}_n$.



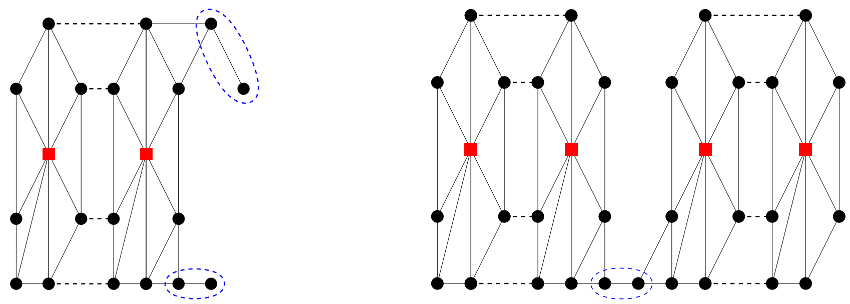
(a) $n = 1$ (b) The general construction

Figure 3.6: The final 4-connected triangulated Bob-dominant graph $\overline{\mathcal{O}}_n$

By checking that every face is incident to three vertices, we see that $\overline{\mathcal{O}}_n$ is triangulated. Using Lemma 3 and knowing that there is no separating triangle in $\mathcal{O}_n$, we only have to check that $\overline{\mathcal{O}}_n$ has no separating triangles containing v_1 and v_2 for 4-connectedness. This can be done by verifying that $N(v_1)$ and $N(v_2)$ induce cycles. It is left to show that our constructed graphs are Bob-dominant. We start by proving that the $\mathcal{O}_n$ are Bob-dominant. We do this by induction:

Base case $n = 1$. First, $\mathcal{O}_1$ is Bob-dominant because Alice always uncovers the cherry in her first turn which Bob can then just take.
Inductive step $1, \dots, n-1 \rightsquigarrow n$. Suppose that $\mathcal{O}_1 \dots \mathcal{O}_{n-1}$ are Bob-dominant. Whatever vertex Alice removes in her first turn, she will uncover a cherry which Bob will then take. Suppose Bob took c_j.
Case 1: $j = 1$ or $j = n$. After Bob's turn we are left with a configuration C which consists of an $\mathcal{O}_{n-1}$ and four more vertices a, b, c, d. Such a scenario is depicted in Figure 3.7a. These four vertices do not obstruct any of the vertices in $\text{out}(\mathcal{O}_{n-1})$. By Bob-dominance of $\mathcal{O}_{n-1}$ and Bob-dominance of the subgraph containing only a, b, c and d we get from Lemma 1 that the graph in C is Bob-dominant. Since Bob also obtains the first cherry, Bob can get all $n - 1 + 1 = n$ cherries in this scenario.
Case 2: $j \neq 1$ and $j \neq n$. After Bob's turn we are left with a configuration C, which consists of an $\mathcal{O}_{j-1}$, an $\mathcal{O}_{n-j}$ and two more vertices a, b. Such a configuration is shown in Figure 3.7b. Since none of these three subgraphs obstruct visibility of the other's outer vertices in any way and all three are even and Bob-dominant, we can once again apply Lemma 1 twice and reach the result that the graph in C is Bob-dominant. So Bob can obtain all $(j - 1) + (n - j) + 1 = n$ cherries in this case as well.
Since either case 1 or 2 must occur and Bob can obtain all n cherries in both cases, $\mathcal{O}_n$ is Bob-dominant.
We now consider our graphs $\overline{\mathcal{O}}_n$ and derive their Bob-dominance from the Bob-dominance of the $\mathcal{O}_n$. We give an optimal strategy $\mathcal{S}_{\overline{\mathcal{O}}_n}$ for Bob until Alice removes either v_1 or v_2. After that, Bob will use an optimal strategy for the remaining graph which we will obtain later.

1. When Alice removes either v_1 or v_2, Bob will remove the other one.
2. When there is an uncovered cherry, Bob will take it.



(a) Case 1: There are an $\mathcal{O}_{n-1}$ and four more vertices left.

(b) Case 2: There are an $\mathcal{O}_{j-1}$, an $\mathcal{O}_{j-n}$ and two more vertices left.

Figure 3.7: Possible configurations of $\mathcal{O}_n$ after two moves

3. Otherwise, Bob removes some vertex $v \notin \{v_1, v_2\}$ without uncovering a cherry.

We will show later that removing such a vertex in the third case is always possible.
Let $\mathcal{G} = (C_0, \dots, C_{2k}, \dots, C_{6n+4})$ be a game on $\overline{\mathcal{O}}_n$ in which we follow our strategy as long as possible (until v_1 and v_2 have been removed). Let C_{2k} be the configuration after Bob made the last move with $\mathcal{S}_{\overline{\mathcal{O}}_n}$. Thereafter, Bob uses an optimal strategy. As mentioned above, we will show later how to obtain one. Furthermore, let $r := w(\overline{\mathcal{O}}_n \setminus C_{2k})$ be the weight that was removed from $\overline{\mathcal{O}}_n$ until C_{2k}.
We will now prove three things about such a game $\mathcal{G}$:

(i) All cherries uncovered until C_{2k} will be uncovered in order from left to right.

(ii) In any configuration until C_{2k} in which it is Bob's turn and neither case 1 or 2 of $\mathcal{S}_{\overline{\mathcal{O}}_n}$ applies, Bob can make a valid move according to case 3. Furthermore, Bob will obtain all r cherries which are removed until C_{2k}.

(iii) In C_{2k}, the remaining graph will only consist of an $\mathcal{O}_{n-r}$ and an even amount of vertices without weight which do not obstruct outer vertices of $\mathcal{O}_{n-r}$.

(i): Since v_1 and v_2 have not been removed yet, v_1 and v_2 enclose the embedded $\mathcal{O}_n$ in such a way that cherries can only be uncovered by a path from the left: For any $s \in \{1, \dots r\}$, let $P_s = (u_1, \dots, u_n)$ be a path with

- $v_1, v_2 \notin P_s$.
- $u_1 = x^1_{\leftarrow}$, i.e., $x_{\leftarrow}$ of the leftmost $\mathcal{O}$-tile: The only vertex on the outer face of $\mathcal{O}_n$ which is neither v_1 nor v_2.
- u_n is the first and only vertex adjacent to c_s.

The path P_s must at some point cross the cycle induced by any c_j with $j < s$, the $x^j_{\downarrow}, x^j_{\uparrow}$ of the respective j-th $\mathcal{O}$-tile and v_1, v_2. Since $v_1, v_2 \notin P_s$, only $c_j, x^j_{\downarrow}, x^j_{\uparrow}$ are possible options for crossing that cycle in P_s. All three of these vertices are either adjacent to c_j or require c_j to be uncovered before. Because P_s was arbitrary, c_j is uncovered before c_s for all $j < s$.

(ii): Let C_{2l-1} ($l < k$) be some configuration of $\mathcal{G}$ in which c_j is the leftmost cherry which is neither uncovered nor taken. So $N(c_j) \subset C_{2l-1}$. Since we know from (i) that cherries are uncovered from left to right, we get $N(c_m) \subset C_{2l-1}$ $\forall m \geq j$. Therefore, the $N(c_m)$ form an $\mathcal{O}_{n-(j-1)}$. Because it is Bob's turn and $\overline{\mathcal{O}}_n$ is even, there is an odd number of vertices left. We know that $H := C_{2l-1} \setminus V(\mathcal{O}_{n-(j-1)})$ is odd because $\mathcal{O}_{n-(j-1)}$ is even. Since $l < k$, we get $\{v_1, v_2\} \subseteq H$. Therefore, H must contain another vertex $v \notin \{v_1, v_2\}$ which does not uncover the next cherry c_j. So case 3 of $\mathcal{S}_{\overline{\mathcal{O}}_n}$ is always applicable.

From this we directly get that Bob will never uncover a cherry using this strategy. The only vertices adjacent to multiple cherries are the shared $x^l_{\nearrow}$ and $x^{l+1}_{\nwarrow}$ or $x^l_{\rightarrow}$ and $x^{l+1}_{\leftarrow}$ of two neighboring $\mathcal{O}$-tiles. As in (i), a path to reach such a vertex before C_{2k} must at some point cross the cycle $v_1, v_2, c_l, x^l_{\downarrow}, x^l_{\uparrow}$ without stepping through v_1 or v_2. As in (i), this implies that c_l is uncovered before such a vertex is reached. Therefore, removing a vertex adjacent to multiple cherries will only ever uncover one cherry until C_{2k}. So Alice will only ever uncover one cherry at once which Bob will then take in $\mathcal{G}$. So Bob will obtain all r cherries until C_{2k}.

(iii): If $n = r$, all cherries have been removed before C_{2k} so the remaining graph consists only of an even number of vertices without weight. In this case, the statement is therefore true.
Otherwise, similar arguments as in (ii) imply that C_{2k-2} must consist of an $\mathcal{O}_{n-r}$, v_1, v_2 and an even amount of vertices without weight which all lie to the left of the $\mathcal{O}_{n-r}$. Since v_1 and v_2 are removed in the following two moves, $\operatorname{out}(C_{2k})$ contains all the $x_{\uparrow}, x_{\downarrow}, x_{\swarrow}$ of the $\mathcal{O}$-tiles in $\mathcal{O}_{n-r}$. We also have $x^n_{\nearrow}, x^n_{\rightarrow} \in \operatorname{out}(C_{2k})$ because they were only hidden by the edge between v_1 and v_2. Finally, $x^{r+1}_{\nwarrow}, x^{r+1}_{\leftarrow}$ are in $\operatorname{out}(C_{2k})$ because $c_r \notin \operatorname{out}(C_{2k})$. So we get $\operatorname{out}(\mathcal{O}_{n-r}) \subset \operatorname{out}(C_{2k})$.

From Bob-dominance of the $\mathcal{O}_{n-r}$, (iii) and Lemma 2, we get that $n - r \leq B(C_{2k}) \leq B(\mathcal{O}_{n-r}) = n - r$. So C_{2k} is Bob-dominant which gives us our optimal strategy for the moves after C_{2k}. Since Bob also obtained all the other r cherries of $\overline{\mathcal{O}}_n$ until C_{2k}, $B(\overline{\mathcal{O}}_n) = r + n - r = n = w(\overline{\mathcal{O}}_n)$. So $\overline{\mathcal{O}}_n$ is Bob-dominant. Since $\overline{\mathcal{O}}_n$ is 4-connected and triangulated, this concludes the proof. $\square$

3.3 Odd 4-connected Triangulated Graphs

We add another restriction to the graphs to make finding Bob-dominant instances even harder. The graphs $\overline{\mathcal{O}}_n$ in Theorem 2 are even and the proof of Bob-dominance relied on this fact at multiple points. So a natural question for us to ask is whether we can also find *odd* Bob-dominant 4-connected triangulated plane graphs with an arbitrary amount of cherries. Theorem 3 tells us that this is not the case.

Theorem 3.

1. *For every $n \leq 6$, there exists a 4-connected triangulated odd plane graph G with $|\mathcal{C}(G)| = n$ which is Bob-dominant.*
2. *Let G be a 4-connected triangulated odd plane graph with $|\mathcal{C}(G)| = n \geq 7$. Then, G is not Bob-dominant.*

In order to prove Theorem 3, we need a few lemmas and corollaries which we will provide throughout the rest of Section 3.3. In Section 3.3.1, we will then give a proof for part 2 of Theorem 3. Part 1 will be proven afterwards in Section 3.3.2.

Lemma 4. *If an instance G of the game is Bob-dominant, $\mathcal{C}(G)$ forms an independent set.*

Proof. Suppose $c_1, c_2 \in \mathcal{C}(G)$ were adjacent and C the first configuration in a game $\mathcal{G}$ on G in which either c_1 or c_2 is in $\operatorname{out}(C)$. Without loss of generality, $c_1 \in \operatorname{out}(C)$. If it is Alice's turn in C, she can just take c_1. Otherwise, Bob can either take c_1 or leave it. In the first case, Alice grabs c_2 and in the second case she takes c_1. Therefore, Alice can always receive at least one cherry, so $A(G) \geq 1$ and the game is not Bob-dominant. $\square$

Because cherries can not be in $\mathrm{out}(V(G))$ in a Bob-dominant graph G and using Lemma 4 and Corollary 1 we get that in any Bob-dominant 4-connected triangulated instance G, every cherry $c \in \mathcal{C}(G)$ forms a cherry wheel with its neighborhood $N(c)$.

Theorem 4. *Let G be a 4-connected triangulated instance of the planar graph grabbing game with $|V(G)| \equiv b$ (mod 2) and $\mathcal{W}$ its set of cherry wheels.*
The graph G is Bob-dominant if and only if the following three conditions hold:

- *Every cherry induces a cherry wheel with its neighbors.*

For any subset of cherry wheels $\mathcal{V} \subseteq \mathcal{W}$ in G where $M(\bigcup_{W \in \mathcal{V}} W)$ contains only the cherry wheels from $\mathcal{V}$

- $|M(\bigcup_{W \in \mathcal{V}} W)| \equiv b$ *(mod 2).*
- $\mathrm{out}(\bigcup_{W \in \mathcal{V}} W)$ *contains no common vertices of two cherry wheels in $\mathcal{V}$.*

Proof. We first show "$\Longrightarrow$":
The first statement is true as every cherry induces a cherry wheel with its neighbors by Lemma 4 and Corollary 1.
Assume now that either the second or third statement were false for some subset of cherry wheels $\mathcal{V} \subseteq \mathcal{W}$ which hide no other cherry wheels. So either $M(\bigcup_{W \in \mathcal{V}} W) \equiv 1 - b$ (mod 2) or $\mathrm{out}(\bigcup_{W \in \mathcal{V}} W)$ contains some vertex $v \in A \cap B$ for some $A, B \subseteq \mathcal{W}$. We give $\mathcal{S}_A$, a strategy for Alice for all configurations C where Alice did not yet obtain a cherry:

- If there is a cherry in $\mathrm{out}(C)$, Alice takes it.
- Otherwise, if there is some vertex $u \notin M(\bigcup_{W \in \mathcal{V}} W)$, she takes that.
- Else, she takes a common vertex of two cherry wheels in V.

We now have to prove that this strategy is always applicable. If Bob removes the first vertex w in $\mathrm{out}(\bigcup_{W \in \mathcal{V}} W)$, he uncovers a cherry which Alice can then take. Alice will only be the first to remove a vertex from $\mathrm{out}(\bigcup_{W \in \mathcal{V}} W)$ when the current configuration contains only vertices from $M(\bigcup_{W \in \mathcal{V}} W)$. Since this can only happen when the parity of this set's cardinality is b, there must be some vertex $v \in A \cap B$. Using $\mathcal{S}_A$, Alice removes v which reveals two cherries. One of these is still left when it is Alice's turn again so she takes it. Therefore, Alice will always obtain a cherry using $\mathcal{S}_A$ which implies that G is not Bob-dominant.

For "$\Longleftarrow$" we now assume that the three conditions hold. We give a strategy $\mathcal{S}_B$ for Bob with which he will always obtain all cherries. Let C be a configuration and $\mathcal{V} \subseteq \mathcal{W}$ be the set of cherry wheels left in C. Strategy $\mathcal{S}_B$ is defined for C as follows.

- If there is a cherry in $\mathrm{out}(C)$, Bob takes it.
- Otherwise, there is some vertex $u \notin M(\bigcup_{W \in \mathcal{V}} W)$ which Bob takes.

By assumption $|M(\bigcup_{W \in \mathcal{V}} W)| \equiv b$ (mod 2), so Bob will always be able to follow $\mathcal{S}_B$ as it is only his turn on configurations with parity $1 - b$. We have to show that Bob will obtain all cherries if he follows $\mathcal{S}_B$. As Bob will never take vertices in $M(\bigcup_{W \in \mathcal{V}} W)$, he will never uncover a cherry. Since there are no vertices in $\mathrm{out}(\bigcup_{W \in \mathcal{V}} W)$ which are adjacent to multiple cherries in $\mathcal{V}$, Alice will only uncover exactly one cherry when removing a vertex from $\mathrm{out}(\bigcup_{W \in \mathcal{V}} W)$. Therefore, Alice will uncover every cherry, one at a time. Following $\mathcal{S}_B$, Bob will obtain all these cherries which implies that G is Bob-dominant. $\square$

Corollary 2. *An odd 4-connected triangulated instance of the planar graph grabbing game with an even-sized cherry wheel W is not Bob-dominant.*

Proof. This follows directly from Theorem 4 when we use $\mathcal{V} = \{W\}$, since $|M(W)| = |W| \equiv 0 \not\equiv 1 \pmod 2$ which implies that the graph is not Bob-dominant. □

Corollary 3. *Let G be an instance of the planar graph grabbing game. If G contains two cherry wheels W and U and a shared vertex $v \in U \cap W$ with $v \in \mathrm{out}(U \cup W)$, then G is not Bob-dominant.*

Proof. This also follows directly from Theorem 4 with $\mathcal{V} = \{U, W\}$ because $v \in \mathrm{out}(U \cup W)$, implying that G is not Bob-dominant. □

Note that this does not imply that cherry wheels cannot share vertices in a Bob-dominant graph. An example is given in Figure 3.8.

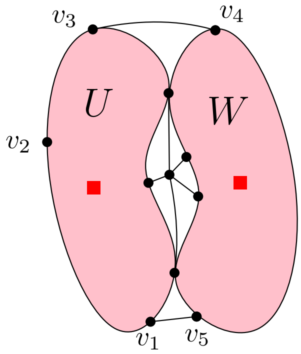


Figure 3.8: A Bob-dominant graph where two cherry wheels share vertices. For visual clarity, edges inside the cherry wheels are not drawn.

Corollary 4. *In a Bob-dominant 4-connected triangulated instance of the planar graph grabbing game G with two cherry wheels U, W which are joint, $\mathrm{out}(U \cup W)$ must induce a cycle not containing any shared vertices. This cycle contains exactly two edges connecting U and W.*

Proof. There is no shared vertex on $\mathrm{out}(U \cup W)$ by Corollary 3. Let $v_1, \dots v_n$ be an ordered vertex sequence of $\mathrm{out}(U \cup W)$ such that $v_1, \dots v_k \in U$ and $v_{k+1}, \dots, v_n \in W$ as shown in Figure 3.8 with $k = 3, n = 5$. All further arithmetic operations are assumed to be mod n. Vertices of consecutive indices are adjacent ($v_i v_{i+1} \in E(G)$) because U and W are connected. There are no recurring vertices in the sequence because the vertex sequence is made up of two disjoint parts $v_1, \dots v_k \in U \setminus W$ and $v_{k+1}, \dots v_n \in W \setminus U$ and $\mathrm{out}(V)$ and $\mathrm{out}(W)$ do not have recurring vertices. Non-consecutive vertices are not adjacent ($v_i v_{i+j} \notin E(G)$ for $j \geq 2$) because of three observations:

- Any such adjacency between two vertices in the same cherry wheel results in a separating triangle.
- An adjacency between the cherry wheels would imply a different vertex sequence.
- $v_1 v_k,\ v_{k+1} v_n \notin E(G)$ as there is a shared vertex in the cherry wheels which prevents these adjacencies.

Therefore, $\mathrm{out}(U \cup W)$ induces a cycle. The two edges connecting U and W are then $v_k v_{k+1},\ v_n v_1$. □

Corollary 5. *Let G be an instance of the planar graph grabbing game. If G is Bob-dominant then for any three cherry wheels U, V, W, we have $U \cap V \cap W = \emptyset$.*

Proof. Suppose there is some vertex $v \in V \cap U \cap W$. Let $u \in \text{out}(U \cup V \cup W)$ w.l.o.g. $u \in U$. If u is also in V or W, apply Corollary 3. Otherwise, we see that removing u and the cherry c_U in U uncovers v since $vc_U \in E(G)$. Therefore, $v \in \text{out}(V \cup W)$ so we can again apply Corollary 3 and get that G is not Bob-dominant. □

Lemma 5. *Let G be an odd 4-connected triangulated instance of the planar graph grabbing game. If G contains two cherry wheels W and U such that $(W \cap U) \cup H(W \cup U)$ is even, G is not Bob-dominant.*

Proof. If either U or W is even, we can apply Corollary 2 and are done.
Otherwise

$$\begin{aligned} |M(W \cap U)| &= |W| + |U| - |U \cap W| + |H(U \cup W)| \\ &\equiv |W| + |U| + |U \cap W| + |H(U \cup W)| \pmod 2 \\ &\equiv 1 + 1 + 0 \pmod 2 \\ &\equiv 0 \pmod 2 \end{aligned}$$

where we use $x \equiv -x \pmod 2$ in the first equivalence. We get from Theorem 4 that G is not Bob-dominant. □

Recall from Chapter 2 that two cherry wheels U, W span a corridor if they are not joint and there are two edges in $U \times W$.

Corollary 6. *Let G be an odd 4-connected triangulated instance of the planar graph grabbing game. If G contains two cherry wheels U and W which are neither joint nor span a corridor, then G is not Bob-dominant.*

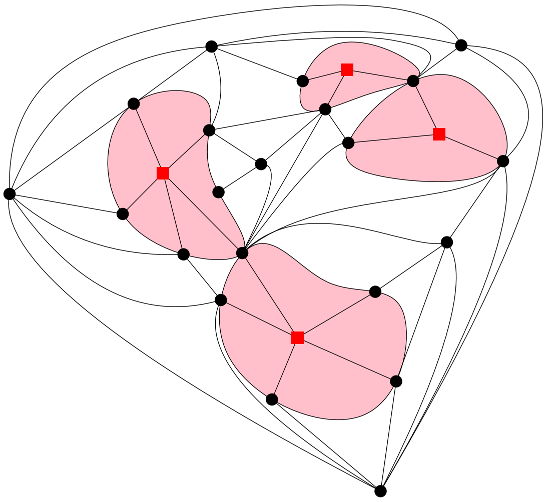
Proof. By definition, U and W are vertex-disjoint and there is at most one connecting edge between the two. Therefore, $(U \cap W) \cup H(U \cup W) = \emptyset$. From Lemma 5 we get non-Bob-dominance. □

Corollary 6 is very helpful for proving the main theorem in Section 3.3.1 and Section 3.3.2.

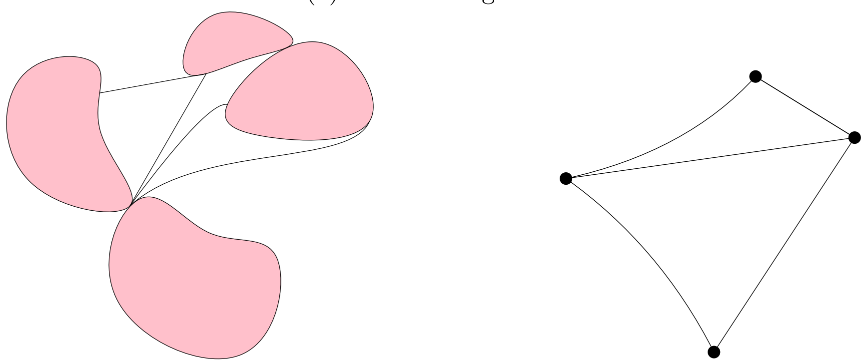
3.3.1 Non-existence of Bob-dominant graphs with seven or more cherries.

First we give a rough idea of the proof for Theorem 3.2.:
We construct an auxiliary graph on the cherry wheels which needs to be complete for the graph to be Bob-dominant. This auxiliary graph is "mostly planar". By the non-planarity of the complete graph on five vertices K_5, we then get that Bob-dominant instances whose auxiliary graphs do not use the non-planarity can only contain up to four cherries. We show by an extensive case analysis that the possible non-planarity can barely be utilized in Bob-dominant instances. More specifically, the bar can only be raised by two cherries, leaving us with the result that a Bob-dominant instance with seven ore more cherries is impossible.

Proof. We start with the construction of the auxiliary graph $X(G)$ for any odd 4-connected triangulated instance of the planar graph grabbing game G where any cherry induces a cherry wheel with its neighborhood and two cherry wheels can only intersect on their boundaries. These properties hold for Bob-dominant graphs by Corollary 1 and Lemma 4. An examplary construction of the auxiliary graph is shown in Figure 3.9.



(a) The drawing of an instance G



(b) The modified drawing containing the four cherry wheels and two corridors

(c) Since all but one pair of cherry wheels are either joint or span a corridor, $X(G)$ is K_4 with one edge missing

Figure 3.9: An exemplary construction of the auxiliary graph $X(G)$.

First, we remove unnecessary detail from the drawing of G to make talking about important structures easier: We do not care about actual vertices in G so we remove them from the drawing. Furthermore, we only leave edges of G which are either on the outer cycle of cherry wheels or one of the spanning edges of a corridor. We end up with a drawing of cherry wheels as closed non-self-intersecting curves and the spanning edges connecting them. This can be seen in Figure 3.9b. In coming figures, pink will also be used as a fill color for cherry wheels.
For $X(G)$, we interpret the cherry wheels as our vertices and two of these cherry wheels as connected by an edge if their bounding curves intersect or they are connected by a corridor. This is depicted in Figure 3.9c. By Corollary 6, any pair of cherry wheels in a Bob-dominant graph G are either joint or share a corridor. This is the case if and only if $X(G)$ is complete.

We now first assume that our Bob-dominant G contains no corridors. Therefore, any pair of cherry wheels must be joint. From Corollary 5, we know that no three cherry wheels can share the same vertex. We get a planar embedding of $X(G)$ by using the positions of the cherries in cherry wheels as the vertex positions. For the edge between two vertices v_1, v_2 in $X(G)$ with the respective cherries c_1, c_2 and a shared neighbor $w \in N(c_1) \cap N(c_2)$, we draw the edge from c_1 over w to c_2. Since the shared vertex is different for every pair of cherry wheels, the embedding is planar.
So $X(G)$ must be planar and complete. This is only possible for four vertices or fewer.

Therefore, Bob-dominant G without corridors can not contain five or more cherries. Unfortunately, this planarity argument cannot be applied anymore when working with corridors. There are two configurations which can destroy planarity in $X(G)$:

- Corridor crossings (Two corridors which cross)
- Corridor hidings (A cherry wheel contained in a corridor)

Recall their definitions from Chapter 2. We will show that all graphs containing these configurations are either not Bob-dominant or contain fewer than seven cherries. We will take care of these two cases not one after the other but in a mixed manner since they can occur in combination. To give an overview, we present a list of statements which we will prove for Bob-dominant graphs in the provided order.

(i) Any two spanning edges of two crossing corridors share a vertex.

(ii) Two wide corridors cannot cross.

(iii) A wide and a narrow corridor cannot cross.

(iv) Two narrow corridors can only cross by having the same one-vertex ending.

(v) Any corridor crossing is also an edge hiding.

From this point on, we will not consider crossings anymore but only edge hiding and corridor hiding.

(vi) A hidden cherry wheel cannot share vertices with the spanning edges of its hiding.

(vii) Two hidden cherry wheels must be in a common hiding.

($viii$) A hiding cannot contain another hiding when the inner and outer hiding are spanned by different cherry wheels.

(ix) A corridor hiding cannot contain another hiding.

(x) There cannot be more than two cherry wheels in a corridor hiding.

(xi) In a graph with a corridor hiding there cannot be more than six cherries[1].

(xii) An edge hiding can only have up to one cherry wheel on its outside.

($xiii$) In a graph with an edge hiding there cannot be more than six cherries.

In the following proofs we always assume that the configuration is contained in an odd Bob-dominant 4-connected triangulated graph G.

(i) By definition, any two curves going through the two corridors of a corridor crossing which connect the cherry wheels of their corridor intersect at some point. In particular this implies that the spanning edges intersect in our plane drawing. Therefore, they share a vertex.

(ii) Let L, K be crossing wide corridors, $l_1 = ab, l_2 = cd$ and k_1, k_2 their spanning edges. By (i), any l_i and k_j have a common vertex and since L and K are wide corridors, k_1 and k_2 also cover the set $\{a, b, c, d\}$ w.l.o.g. $k_1 = ac$, $k_2 = bd$. So the four edges form a C_4 which enclose the full inner parts of the corridors L and K.
Corridors must contain vertices not belonging to the cherry wheels which span the corridor (Lemma 5). Therefore, $\{a, b, c, d\}$ must induce the C_4 since adding another edge would introduce a separating triangle in the corridor. Now a is incident to both a cherry wheel

[1] It is even possible to show that a Bob-dominant graph with a corridor hiding can only contain five or fewer cherry wheels but we will not need this result for our purposes.

from L and K. We call them A_L and A_K. By Corollary 4, a is not in $\text{out}(A_L \cup A_K)$. The two necessary edges needed to hide a cannot be in the corridor. Thus, the cycle $\text{out}(A_L \cup A_K)$ must enclose the corridor and all cherry wheels involved.
This argument also implies that the same must be true for the cherry wheels D_L and D_K sharing d. These two conditions contradict each other in a plane graph.
Therefore, no two wide corridors can cross. See Figure 3.10 for a visual explanation.

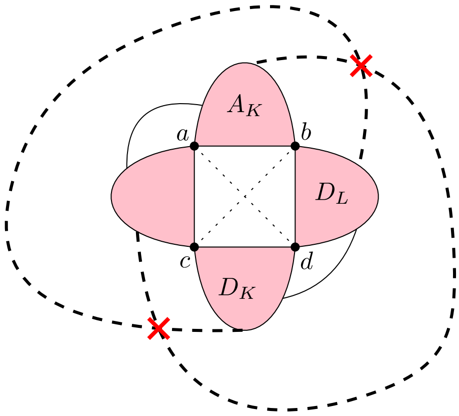


Figure 3.10: Two crossing wide corridors would defy planarity

(iii) Let W be the wide and N the narrow corridor with the spanning edges $w_1 = ab, w_2 = cd, n_1, n_2$. Suppose they are crossing. From (i) we again get that n_1, n_2 are edges in $\{a, b, c, d\}$. The edges n_1, n_2 have a common vertex because N is narrow. Let this common vertex be a w.l.o.g. Then $\{n_1, n_2\} = \{ac, ad\}$. Since N must contain a vertex not belonging to its spanning cherry wheels, a, c, d form a separating triangle.
Therefore, a wide corridor and a narrow corridor cannot cross. A picture is given in Figure 3.11.

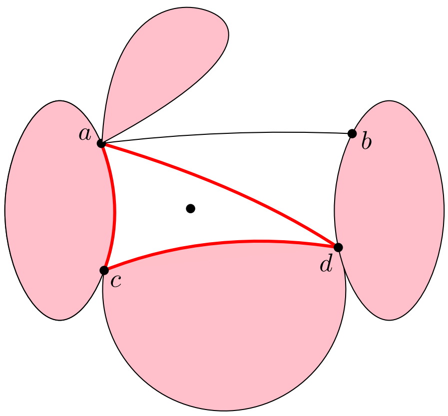
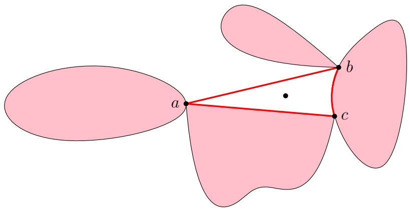
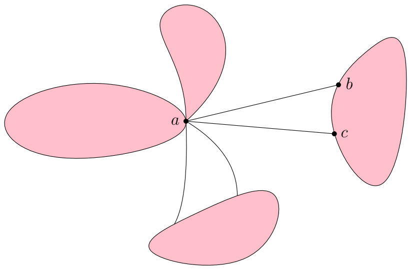


Figure 3.11: A wide and a narrow corridor which cross imply a separating triangle

(iv) We know already that two crossing corridors can only be narrow. So let N, O be two narrow corridors with spanning edges $n_1 = ab, n_2 = ac, o_1, o_2$. Suppose now that the shared vertex of o_1, o_2 is not a. Then the shared vertex is either b or c. W.l.o.g. we can assume b to be the shared vertex. Now either o_1 or o_2 must be the edge bc. This implies that N is bounded by a triangle. Since the corridor also has a non-empty inside, a, b, c form a separating triangle. This configuration can be seen in Figure 3.12a. So for two narrow corridors to cross, the shared vertex of their crossing edges must be the same.
Since we already ruled out every other case of corridor crossing, we are left with only this possibility. This configuration is depicted in Figure 3.12b



(a) Two narrow corridors with different shared vertices of their spanning edges always imply a separating triangle



(b) Two narrow corridors with the same shared vertex of their spanning edges

Figure 3.12: Crossings between two narrow corridors

(v) Let N, O be two crossing corridors with cherry wheels N_1, N_2, O_1, O_2. From (iv) we know that N and O are narrow with a shared vertex $v \in N_1 \cap O_1$. By Corollary 4 $v \notin \text{out}(N_1 \cup O_1)$. So there must be two edges $e_1, e_2 \in N_1 \times O_1$ which are on $\text{out}(N_1 \cup O_1)$ and hide v. Any such edge in $N_1 \times O_1$ cannot lie on N_2 or O_2 since these two cherry wheels do not share vertices with one either N_1 or O_1 by the definition of corridor spanning. So we get that N_2, O_2 are hidden within $M(N_1 \cup O_1)$ as they are directly connected to v. This gives us our edge hiding spanned by N_1 and O_1 with a spanning edge $e \in \{e_1, e_2\}$. This configuration is depicted in Figure 3.13.
Therefore, we only have to consider hidings from now on.

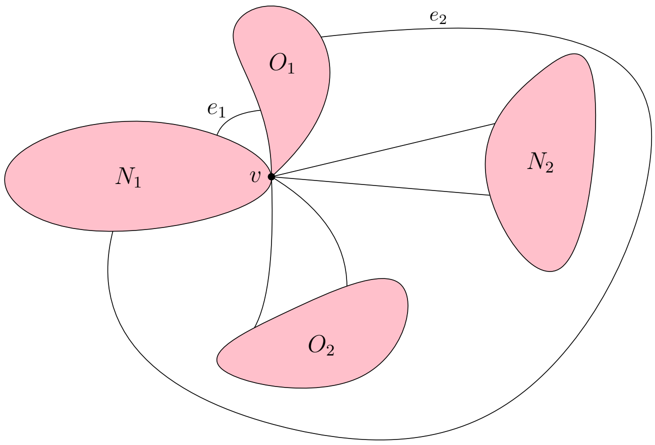


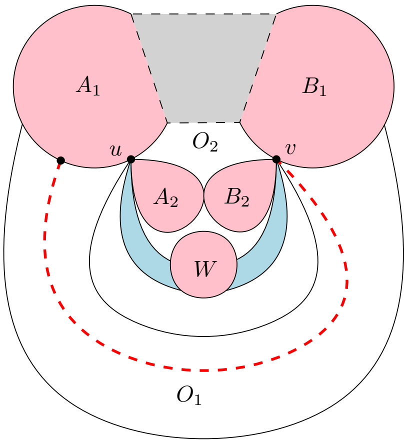
Figure 3.13: Two crossing corridors in a Bob-dominant graph always form an edge hiding.

(vi) Let U and W be two cherry wheels spanning a hiding H and a cherry wheel V hidden in H. Suppose W shared a vertex v with one of the spanning edges. W.l.o.g. $v \in U$. Then by definition of the spanning edges $v \in \text{out}(U \cup W)$ and therefore $v \in \text{out}(U \cup V)$ which is a contradiction to Corollary 4. We therefore have that $V \subseteq \text{in}(M(U \cup W))$ in any Bob-dominant graph where V is contained in a hiding spanned by U and W.

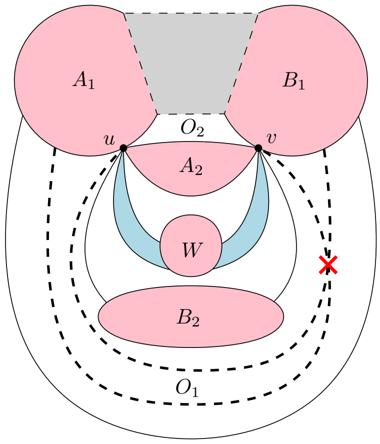
(vii) Suppose there are two cherry wheels U, W hidden in hidings K, L such that $\text{in}(L) \cap \text{in}(K) = \emptyset$. By ($vi$), U, W are not incident to any of the spanning edges of their hidings. Therefore, any path from U to W has a length of at least two. So U and W are neither joint nor span a corridor which implies that such a configuration cannot be contained in a Bob-dominant graph.

($viii$) Let A_1, B_1 span the outer hiding O and A_2, B_2 the inner hiding I. We will leave it ambiguous at first of which kind the two hidings are. Let W be a cherry wheel hidden in the inner corridor, E the set of spanning edges of the inner corridor and U their set of

vertices.
The cherry wheel W needs to span a corridor with A_1 and B_1, which implies that there need to be two vertices $u, v \in U \cap (A_1 \cup B_1)$ with two narrow corridors from u, v to W. Vertex u being equal to v would only be possible if it was a shared vertex of A_1 and B_1. But since u is also in A_2 or B_2 this would imply a vertex shared by three cherry wheels, a contradiction to Bob-dominance. Thus $u \neq v$.
It is helpful to think about the boundary of I as a cycle which touches A_1 and B_1, therefore splitting O in two parts O_1 and O_2. Consider u w.l.o.g. $u \in A_1 \cup A_2$ and the part of O (w.l.o.g. O_1) which is incident to a spanning edge containing u. There needs to be another vertex from A_2 on O_1 to hide u from this side with an edge between A_1 and A_2. The only vertex which can fulfill these properties is v, as having any other vertex would imply that $v \notin U$. Therefore, $u, v \in A_2$. This is impossible if I is an edge hiding because there is only one spanning edge and $u \neq v$. This setup can be seen in Figure 3.14a. If I is a corridor



(a) If the inner hiding is an edge hiding, the shared vertices u, v of the inner and outer hiding cannot be hidden on the component O_1 incident to the spanning edge uv.



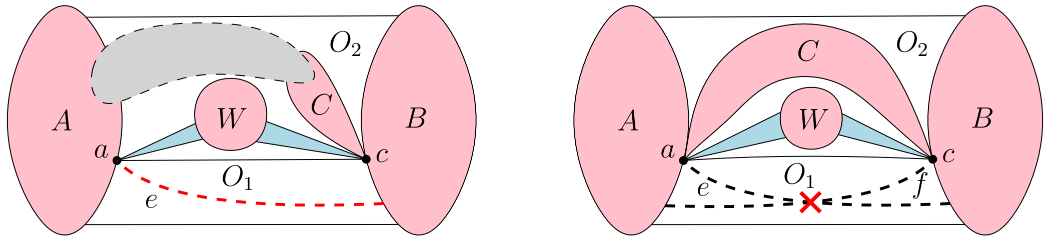
(b) If the inner hiding is a corridor hiding, the edges hiding u and v on O_1 would intersect.

Figure 3.14: Problems with a hiding containing another hiding spanned by different cherry wheels. The gray area on the outer hiding represents ambiguity about the kind of hiding. The corridors between the cherry wheel in the edge hiding and the outer cherry wheels are marked in green.

hiding we also get a contradiction from the fact that the above argument for u must also be true for v on O_1. This implies that there is an edge between A_1 and u as well as an edge between A_2 and v through O_1. These two edges intersect, a contradiction to planarity. This problem is depicted in Figure 3.14b. So we get that a hiding cannot contain another hiding with different cherry wheels.

(*ix*) In (*viii*) we already dealt with the case that the outer and inner hidings do not have common spanning cherry wheels. If they have both cherry wheels in common, then we are talking about the same hiding. Therefore, we now assume that the inner and outer hidings have one cherry wheel in common. The outer hiding is a corridor hiding, the inner one either a corridor or edge hiding. As in (*viii*), we can deal with these two cases in one go: Let A and B span the outer hiding O, A and C the inner hiding and let W be a cherry wheel in the inner corridor. Cherry wheels W and B cannot be joint because of (*vi*). Therefore, they have to span a corridor which is only possible if a spanning edge $ac \in A \times C$ of the inner corridor is incident to B. Since A and B are not joint this implies that the shared vertex must be c. Now as in (*viii*) O is split into two parts O_1, O_2 by C and the spanning edges of the inner corridor. Let O_1 be the part incident to ac. In order to hide c

on O_1, there needs to be an edge $e \in B \times C$ on O_1 not containing c. The only vertex in C incident to O_1 other than c is possibly a. If a is not in C we are done as e cannot contain c itself. This situation is illustrated in Figure 3.15a.



(a) When $a \notin C$, c cannot be hidden on O_1. The gray area represents ambiguity whether the inner hiding is a corridor or edge hiding.

(b) When $a \in C$, the two edges hiding a and c on O_1 must intersect, a contradiction to planarity.

Figure 3.15: Problems which emerge in a corridor hiding containing another hiding with a shared cherry wheel. The corridors between the inner hidden cherry wheel and the outer cherry wheels are colored green.

Otherwise a is a shared vertex of A and C, implying that there needs to be an edge $f \in A \times C$ on O_1 not containing a. Since A and B are not joint, e and f have to intersect – a contradiction to planarity depicted in Figure 3.15b.
With $(viii)$ we get that a corridor hiding cannot contain another hiding inside it.

(x) Suppose there are three cherry wheels U, V, W in a corridor spanned by A and B. By (ix) there cannot be another hiding in our corridor, disallowing corridor hidings and corridor crossings in the outer corridor. Therefore, we can directly draw plane $X(G) - AB$ by drawing edges through spanned corridors or shared vertices. Since the drawing is contained within the corridor, A, B are on the outside of $X(G) - AB$. By connecting A and B using the outer face we get a plane drawing of $X(G) = K_5$ which is impossible. Thus, a corridor can only contain up to two cherry wheels.

(xi) We show that if we have a corridor spanned by cherry wheels A, B with a hidden cherry wheel in it, there can not be more than two cherry wheels on the outside of the corridor. Suppose there are three cherry wheels O_1, O_2, O_3 outside of the corridor. All of them must be incident to at least one of the vertices on the corridor spanning edges to span a corridor with the inner cherry wheel. Let v_i be such a vertex for O_i. For these vertices we have $v_i \neq v_j$ for $i \neq j$ because otherwise we would have a vertex shared by three cherry wheels. By the pigeonhole principle two of the v_i must be on the same spanning edge. W.l.o.g. $e = v_1 v_2 \in A \times B$ is a spanning edge. Since O_1 and A must hide v_1, the cycle on $\text{out}(O_1 \cup A)$ has to contain the corridor and any cherry wheels directly attached to it: In particular also O_2 and B. But since the same is true for O_2 and the cycles only contain vertices from their respective cherry wheels we get a contradiction to planarity. This configuration can be seen in Figure 3.16.
This insight, paired with (x), gives us that if our graph contains a hidden cherry wheel, then we can only have up to

$$\underbrace{2}_{\substack{\text{C.w. spanning} \\ \text{the corridor}}} + \underbrace{2}_{\substack{\text{Max. amount of c.w.} \\ \text{in the corridor}}} + \underbrace{2}_{\substack{\text{Max. amount of c.w.} \\ \text{outside of corridor}}} = 6$$

cherry wheels.

(xii) Suppose there are two cherry wheels O_1, O_2 on the outside of an edge hiding spanned by A and B with the spanning edge $e = ab \in A \times B$. The cherry wheels O_1 and O_2 must

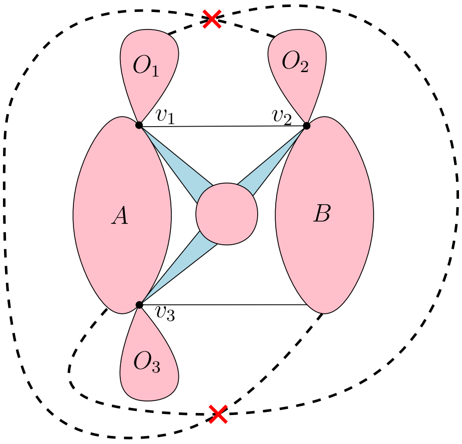


Figure 3.16: For three cherry wheels to be on the outside of a corridor hiding, two edges would need to intersect.

be incident to either a or b. However, both cannot be incident to the same vertex as we would then get a vertex shared by three cherry wheels. So w.l.o.g. $a \in O_1 \setminus O_2, b \in O_2 \setminus O_1$. Now $a \notin \text{out}(A \cup O_1)$ since a is a shared vertex. We get that the cycle on $\text{out}(A \cup O_1)$ must contain a and with it also O_2 and B on its inside. The same argument for b gives us that O_1 and A must be contained within the cycle on $\text{out}(O_2 \cup B)$. Because the intersection of the two cycles must be empty (otherwise we would have a shared vertex on the outside once again) this is a contradiction. This can be seen in Figure 3.17.

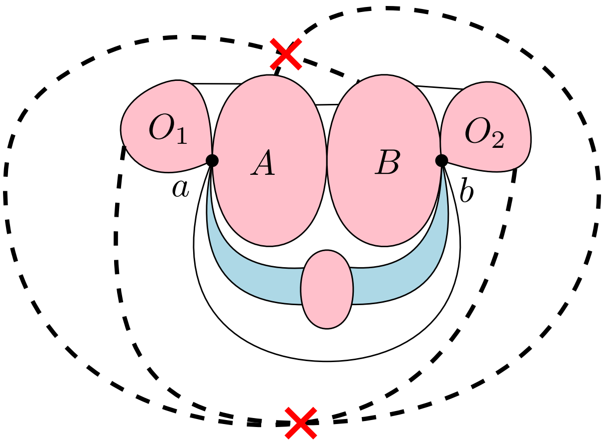
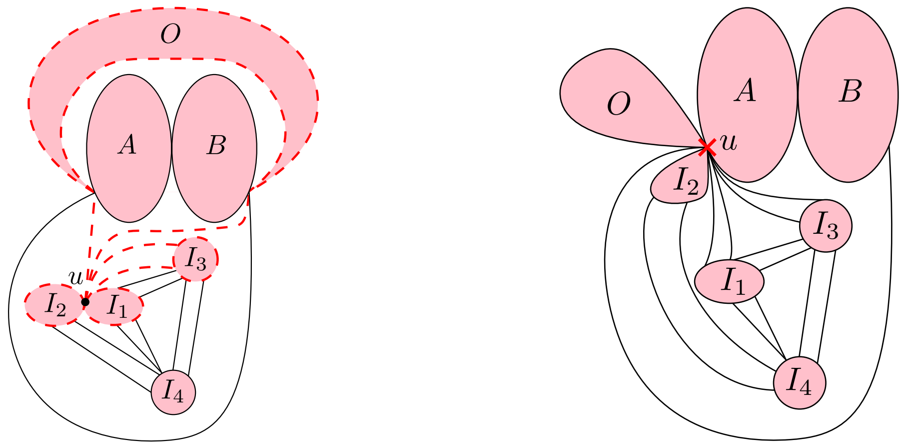


Figure 3.17: Two cherry wheels on the outside of an edge hiding give us two intersecting edges – a contradiction to planarity.

(*xiii*) Let A, B be cherry wheels spanning an edge hiding. Suppose first there are five cherry wheels $I_1, \dots I_5$ in the edge hiding. By (*viii*), none of these five inner cherry wheels can span a hiding disallowing any kind of non-planarity (corridor hidings or corridor crossings) within them. So we get a plane drawing of $X(G[I_1 \cup \dots \cup I_5]) = K_5$ by drawing the edges through the corridors and shared vertices which is impossible.
Suppose now that there where four cherry wheels I_1, I_2, I_3, I_4 in the edge hiding and one (it can not be more than that by (*xii*)) cherry wheel O on the outside of our edge hiding. Again by (*viii*), none of these four inner cherry wheels can span a hiding, implying that $X(G[I_1 \cup I_2 \cup I_3 \cup I_4]) = K_4$ can be drawn plane directly. This means that one of the inner cherry wheels – w.l.o.g. I_1 – is enclosed by the other three by corridors and shared vertices. By (*vi*), O has to be incident to the spanning edge of the edge hiding in order to span a corridor with the hidden cherry wheels. A corridor between O and I_1 has to

cross the barrier formed by I_2, I_3, I_4. This corridor cannot contain any I_i, since we know that any graph with a corridor hiding cannot have more than six cherry wheels from (*xi*). Therefore, we have a corridor crossing between the corridors spanned by O, I_1 and w.l.o.g. I_2, I_3. The shared vertex (see (*iv*)) u of the two corridors cannot be on I_1 as this would imply an edge hiding spanned by I_1 and either I_2 or I_3 (by (*v*)) which again is impossible by (*viii*). This problem is depicted in Figure 3.18a. Therefore, $v \in O$ which is, however, also not possible as neither I_2 nor I_3 can share vertices with O (again by (*vi*)) as can be seen in Figure 3.18b. We have shown that the amount of cherry wheels on the outside and inside of an edge hiding cannot exceed four. We therefore get that a graph with an edge hiding can only contain up to six cherry wheels in total.



(a) If v is on the inner cherry I_1, we get a corridor crossing in a hiding which is impossible.

(b) When v is on the outer cherry wheel O, one of the inner cherry wheels must be incident to the spanning edge of the outer corridor. This is impossible.

Figure 3.18: Problems which emerge when a edge hiding contains four cherry wheels on its inside and there is one cherry wheel on its outside. In both cases we get a corridor crossing with shared vertex v.

We have thus shown that any Bob-dominant graph contains only up to six cherry wheels, thereby completing the proof. □

3.3.2 Existence of Bob-dominant graphs with less than seven cherries

We will now give the proof for part one of Theorem 3 by giving examples of such Bob-dominant graphs.

Proof. We call the graphs we give as examples for odd 4-connected triangulated Bob-dominant graphs with n cherries G_n.
For G_0, any odd 4-connected triangulated graph G with $\mathcal{C}(G) = \emptyset$ will qualify by the definition of Bob-dominance. As G_1 we can just take a cherry wheel of size 5 with two more vertices on its outside to get a triangulated graph (see Figure 3.19).
A valid G_2 can be constructed by just taking two odd cherry wheels, letting them have one shared vertex (to fulfill Lemma 5), making sure that the shared vertex is hidden by the cherry wheels and again adding two vertices for the graph to be triangulated. The resulting graph is depicted in Figure 3.20.
From now on we will actually prove Bob-dominance of our graphs by utilizing Theorem 4. The same can of course be done for the previous graphs but we do not consider it necessary. Our G_3 has three pairwise joint cherry wheels which hide their shared vertices. There is

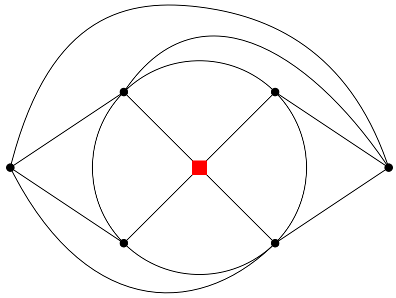


Figure 3.19: G_1: An odd 4-connected triangulated Bob-dominant graph with one cherry

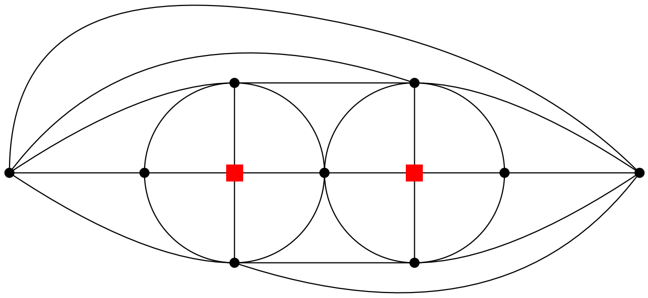


Figure 3.20: G_2: An odd 4-connected triangulated Bob-dominant graph with two cherries

also one additional vertex hidden by the three cherry wheels as a whole which is revealed when one of the cherry wheels is opened. The graph is depicted in Figure 3.21.
We have to check two conditions in order to prove Bob-dominance using Theorem 4:

1. Any shared vertex of two cherry wheels A, B is not in $\mathrm{out}(A \cup B)$.
2. For any subset of cherry wheels V which do not hide any other cherry wheels, $M(\bigcup_{W \in V} W)$ is odd.

There are three shared vertices in G_3 which are all hidden by the two cherry wheels they are shared by. Let U, V, W be the three cherry wheels in G_3.

$$|M(U \cup V \cup W)| = 19$$

$$|M(U \cup V)| = |M(U \cup W)| = |M(V \cup W)| = 13$$
$$|M(U)| = |M(V)| = |M(W)| = 7$$

This proves Bob-dominance of G_3.
For G_4, we take three cherry wheels arranged in a triangle with one cherry wheel in the middle, such that any pair of cherry wheels share one vertex and any set of three cherry wheels which contain the middle cherry wheel hide one vertex, as in the G_3 case. See Figure 3.22 for an image.
It can be easily confirmed that shared vertices are hidden in G_4. Now for the parity of cherry wheel subsets: Let $O = \{A, B, C\}$ be the set of outer cherry wheels and I be the inner cherry wheel. For any $X, Y \in O, X \neq Y$ we have:

$$|M(A \cup B \cup C \cup I)| = 31.$$

Since I can not be removed first, we only have to check subsets of size three with I.

$$|M(X \cup Y \cup I)| = 25$$

For subsets of size two we have the two cases that I is either contained or not.

$$|M(X \cup Y)| = 13 \qquad |M(X \cup I)| = 19$$

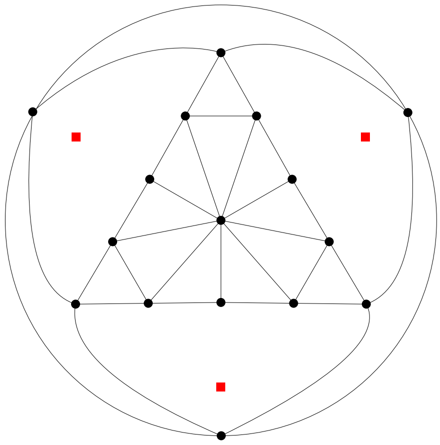


Figure 3.21: G_3: An odd 4-connected triangulated Bob-dominant graph with three cherries. The edges incident to cherries are left out for visual clarity.

For single cherries, we have to again make a distinction between I and the other cherry wheels:

$$|M(X)| = 7 \qquad |M(I)| = 13$$

Therefore, G_4 is Bob-dominant.

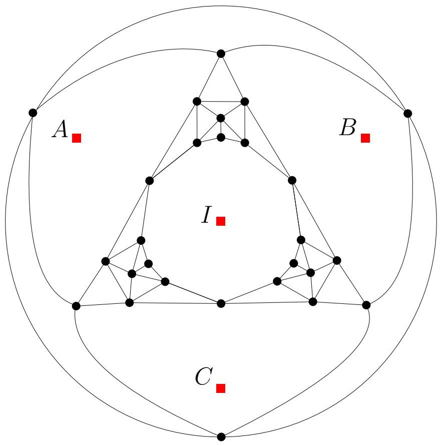


Figure 3.22: G_4: An odd 4-connected triangulated Bob-dominant graph with four cherries. The edges incident to cherries are left out for visual clarity.

As we have seen in Section 3.3.1, four cherries is the best we can do without using corridors. For G_5, we have a corridor hiding spanned by A_1 and B with one cherry wheel on the outside (A_2) and two on the inside (C_1 and C_2). For a picture, see Figure 3.23. The only shared vertex is between A_1 and A_2 and is not on $\mathrm{out}(A_1 \cup A_2)$. Now for parity:

$$|M(A_1 \cup A_2 \cup B \cup C_1 \cup C_2)| = 43$$

Four-cherry wheel subsets can only have either A_1 or A_2 missing. In any case, we get (for $A \in \{A_1, A_2\}$)

$$|M(A \cup B \cup C_1 \cup C_2)| = 37.$$

There are four possible three-cherry wheel subsets which can be left. Let $A \in \{A_1, A_2\}$. Then we have:

$$|M(B \cup C_1 \cup C_2)| = 29 \quad |M(A_2 \cup B \cup C_2)| = 25 \quad |M(A \cup C_1 \cup C_2)| = 23$$

Single cherry wheels are odd, so we can conclude this case.
For subsets of size two we see that any pair X, Y of cherry wheels which do not hide cherry wheels, either share exactly one vertex or hide one vertex in their corridor. Therefore, we get an odd amount of vertices in total:

$$|M(X \cup Y)| \equiv |X| + |Y| + |(X \cap Y) \cup H(X \cup Y)| \equiv 1 + 1 + 1 \equiv 1 \pmod 2$$

Therefore, G_5 is also Bob-dominant.

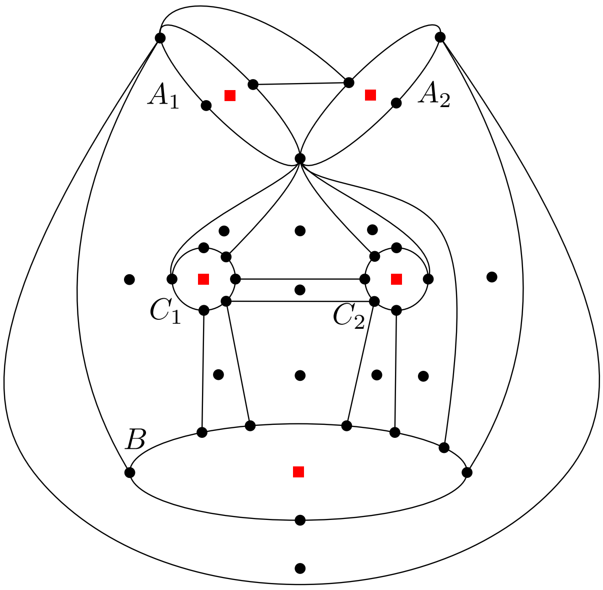


Figure 3.23: G_5: An odd 4-connected triangulated Bob-dominant graph with five cherries. The edges incident to vertices which seem not to be connected to anything are left out for visual clarity. Implicitly ,such vertices are connected to every vertex on the boundary of its face.

For our last graph G_6 we cannot use any corridor hidings as already mentioned in a footnote in Section 3.3.2. The idea here is to have three outer cherry wheels A_1, A_2, A_3 such that any two of those span an edge hiding. All of these three edge hidings overlap in such a way that they contain three inner cherry wheels B_1, B_2, B_3. Additionally, one of the edge hidings – in our case the one spanned between A_1 and A_3 – also contains the other outer cherry wheel. This needs to be the case to avoid a separating triangle in the middle. Now the three cherry wheels in the middle need to span corridors with all the outer cherry wheels. To accomplish this, we heavily rely on corridor crossings: The shared vertices of the outer cherry wheels are used to connect an inner cherry wheel to two cherry wheels on the outside using only one corridor. So each of the inner cherry wheels spans two corridors with two different shared vertices of the outer cherry wheels which is enough to have them fully connected. A picture of this construction is given in Figure 3.24.
The three shared vertices are hidden by their respective edge hidings. For subset parity we get the following:

$$|M(A_1 \cup A_2 \cup A_3 \cup B_1 \cup B_2 \cup B_3)| = 61$$

Only A_1 or A_3 can be opened at first. We get

$$|M(A \cup A_2 \cup B_1 \cup B_2 \cup B_3)| = 57 \quad A \in \{A_1, A_3\}.$$

Removing any of the other outer cherry wheels next leaves us with

$$|M(A \cup B_1 \cup B_2 \cup B_3)| = 49 \quad A \in \{A_1, A_2, A_3\}.$$

If we instead remove one of the inner cherry wheels, we are left with

$$|M(A \cup A_2 \cup B \cup B_2)| = 39 \quad A \in \{A_1, A_3\},\ B \in \{B_1, B_3\}.$$

As for the previous graphs, three-, two- and one-cherry wheel subsets also have an odd amount of vertices since any two cherry wheels hide or share a vertex and any three-cherry wheel subset also hides one vertex. This grants us Bob-dominance for our final graph, thereby completing the proof. □

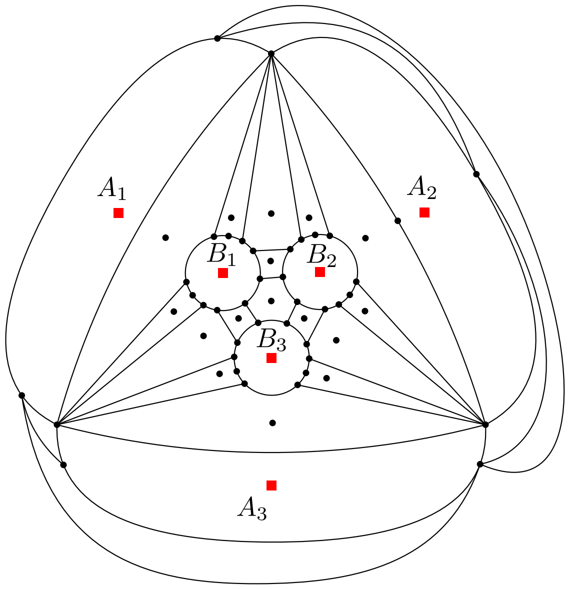


Figure 3.24: G_6: An odd 4-connected triangulated Bob-dominant graph with six cherries. The edges incident to vertices which seem not to be connected to anything are left out for visual clarity. Implicitly, such vertices are connected to every vertex on the boundary of its face.

4. Asymptotic Bounds on Optimal Strategies for Alice

In previous sections we only considered games which were optimal for Bob. But as we learned in Section 3.3 there are no Bob-dominant games with more than six cherries in the class of odd 4-connected triangulated graphs. Naturally, the following question arises: "What is the maximum share of cherries that Bob can obtain for graphs with n cherries?". Phrasing it more formally from the viewpoint of Alice where G_n denotes the set of odd 4-connected triangulated graphs with n cherries, we get:

"What share of cherries $s_n = \inf_{G \in G_n} \frac{A(G)}{n}$ can Alice obtain on every odd 4-connected triangulated graphs with n cherries if both players play optimally?"

However, we do not want an answer for a specific n but instead look at this question asymptotically: We want to know the value of $s = \lim_{n\to\infty} s_n$. We will not be able to achieve that in this thesis. Instead, we will only give rough upper and lower bounds for s.
Proving an upper bound u for s involves finding instances for arbitrarily large n in which Alice can never obtain more than $u \cdot n$ cherries *independent of the strategy she uses.*
A trivial upper bound is $\frac{1}{2}$: For even n, we can just take graphs G_n with n cherries and one non-cherry. For every configuration C of G_n with $|C| \geq 2$ we have $|\text{out}(C)| \geq 2$. Therefore, both players can obtain one cherry in every move except for Alice's very last move. Since Alice does $\frac{n}{2} + 1$ moves on G_n, $A(G_n) = \frac{n}{2}$ and $\lim_{n\to\infty} \frac{A(G_n)}{n} = \frac{1}{2}$. Thus, we have proven the upper bound.
We also believe that it is feasible to prove the upper bound $\frac{1}{3}$ using graphs such as the one pictured in Figure 4.1. However, this is outside of the scope of this thesis.
To prove a lower bound l, it is necessary to prove that Alice can obtain at least $l \cdot n$ cherries on *any* graph with n cherries for large n. Giving a lower bound is tricky since we cannot assume anything about the structure of our graphs. We still want to give at least a minor result for which we need a lot of assumptions.

Theorem 5. *Let X_n be the set of all odd 4-connected triangulated plane graphs with n cherries where cherries only occur in cherry wheels and there are no corridor crossings, no hidden cherry wheels and no vertices shared by three or more cherry wheels.*
Then $A(G) \geq \frac{n}{8}$ for any $G \in X_n$ when $n \equiv 0$ (mod 8).

Proof. Let $G \in X_n$. Then, $X(G)$ admits a plane drawing since edges drawn between cherry wheels through shared vertices and corridors do not cross by our assumptions about G. By

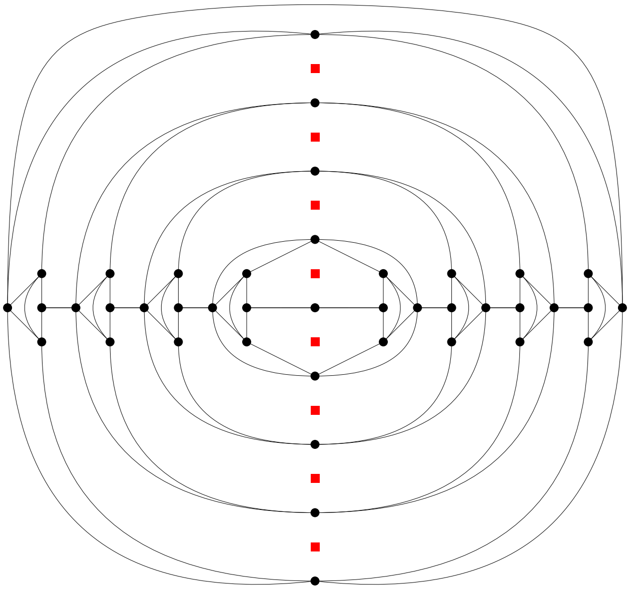


Figure 4.1: An instance on which Alice can (probably) only obtain up to approximately $\frac{1}{3}$ of the weight if she plays optimally. To build larger graphs with this property it is sufficient to add cherry wheels from above and below in the same manner as depicted. Edges in cherry wheels are left out for visual clarity.

the four color theorem, $X(G)$ has an independent set W of size at least $\frac{n}{4}$. Let V be the set of all vertices in G which belong to cherry wheels in W. We now give a strategy $\mathcal{S}_A$ for Alice with which she obtains an eighth of the cherry wheels. Let C be a configuration on G which can occur when Alice uses $\mathcal{S}_A$:

- If there is some cherry $c \in \text{out}(C)$, Alice takes it.
- If there is some vertex $v \in \text{out}(C) \cup V$, Alice takes it.
- If there is a vertex in $\text{out}(C)$ which does not uncover any cherries when removed, Alice takes it.
- Otherwise, Alice takes a vertex belonging to a cherry wheel in W which is odd.

Such a move as in the fourth case is always possible because if none of the first three cases apply, there are only untouched cherry wheels from W left. These do not share any vertices and contain an odd amount of vertices as a whole since it is only Alice's turn when there is an odd number of vertices left. Therefore, one of the remaining cherry wheels must be odd which Alice then opens.
It is now only left to prove that $\mathcal{S}_A$ actually gains Alice at least $\frac{n}{8}$ vertices: In the first part of the game – until there are only vertices from V left – she obtains all cherries from W which Bob uncovers at some point. From then on, whenever Alice has to uncover a cherry in V she "changes the parity" of the game by opening an odd cherry wheel. So Bob has to open the next cherry wheel, as they contain an even amount of vertices. Therefore, for any cherry Bob receives in W, Alice also receives one cherry.
Since W contains $\frac{n}{4}$ cherries, Alice obtains $\frac{n}{8}$ of them which finishes the proof. $\square$

5. Conclusion and Outlook

In this work, we mostly focused on the question in which cases in the planar graph grabbing game Bob can obtain all cherries. We could prove that in the general case and on the subclass of 4-connected triangulated graphs, there is a graph for every n with n cherries such that Bob can obtain all of its weight. With the additional condition that the graphs must be odd, we then got the surprising result that Alice can always obtain at least one cherry on any graph with more than six cherries. In the last chapter, we continued dealing with this class of graphs and briefly considered the question asymptotically, i.e., we were asking what share of the cherries Alice can obtain on graphs with "a lot of" weight. In this domain there are probably more results that could be achieved. Other approaches that could be interesting to pursue in the future include the following:

- The restriction we posed in the beginning for our weight function $w : V \to \{0, 1\}$ could be removed to allow *arbitrary positive or* even *negative weight*.
- Future work could ask questions about the *computational complexity* of the planar graph grabbing game on general graphs or subclasses of graphs. As is true with many such games, it might be possible to show that the game is in $\mathcal{PSPACE}$.
- On simple subclasses of graphs it might be feasible to *find optimal strategies* for the planar graph grabbing game given arbitrary weight assignment to the vertices. A simple example is the class of outerplanar graphs on which the optimal strategy for both players is to be greedy. When finding optimal strategies is too hard, it might still be possible to get bounds on the optimal gain for both players.
- It might be possible to show that the planar graph grabbing game is *equivalent to other grabbing games*.
- Varying the amount of moves a player can make in one turn or increasing the amount of players could also lead to interesting results. In particular, one might find that the parity of the graph is important not by mod 2 but mod some other k – perhaps the amount of players.

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