Towards Product Structure in Hyperbolic Unit Disc Graphs

Bacheolor's Thesis of

Dominik Sucker

At the Department of Informatics
Institute of Theoretical Informatics (ITI)

Reviewer:	T.T.–Prof. Dr. Thomas Bläsius
Second reviewer:	PD Dr. Torsten Ueckerdt
Advisors:	T.T.–Prof. Dr. Thomas Bläsius
	Laura Merker

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Karlsruher Institut für Technologie
Fakultät für Informatik
Postfach 6980
76128 Karlsruhe

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(Dominik Sucker)

Abstract

Product structure is a method to describe the global structure of a graph class in terms of simple graphs such as paths and graphs of bounded treewidth. Dvořák et al. [Matrix Annals 2021] have shown that Euclidian unit disc graphs have product structure. Euclidian unit disc graphs are a well-known and well-studied graphs class. A graph is a Euclidian unit disc graph if there is an embedding of the vertices such that two connected vertices are at most 1 apart. Hyperbolic unit disc graphs are an extension of Euclidian unit disc graphs. They are defined by using hyperbolic instead of Euclidian geometry. Hyperbolic unit disc graphs are an interesting graph class because of their connection to hyperbolic random graphs which can be used to model real-world networks like the internet or social networks. Therefore, in this thesis we extend the approach of Dvořák et al. They partition the graph using an embedding and a tiling of the Euclidian plane. We extend this to hyperbolic tilings. We can then show that our approach does not work for proving product structure in hyperbolic unit disc graphs. Additionally, deviating from the way Dvořák et al. partition the graph does not yield product structure for some families of hyperbolic tilings as well.

Zusammenfassung

Product structure ist ein Werkzeug, um die globale Struktur von Graphklassen mit Hilfe von einfachen Graphen wie zum Beispiel Pfaden oder Graphen mit beschränkter Baumweite zu beschreiben. Dvořák et al. [Matrix Annals 2021] haben gezeigt, dass euklidische unit disc Graphen product structure haben. Euklidische unit disc Graphen sind eine bekannte und gut erforschte Graphklasse, die alle Graphen enthält, für die eine Einbettung der Knoten existiert, so dass verbundene Knoten höchstens eine Entfernung von 1 haben. Hyperbolische unit disc Graphen sind eine Erweiterung der euklidischen unit disc Graphen. Anstelle der euklidischen Geometrie nutzen sie die hyperbolische Geometrie. Hyperbolische unit disc Graphen sind von Interesse auf Grund ihrer Verwandschaft mit hyperbolischen Zufallsgraphen, die zur Modellierung von realen Netzwerken wie zum Beispiel dem Internet oder sozialen Netzwerken benutzt werden können. Wir nähern uns deshalb in dieser Arbeit der Frage der product structure von hyperbolischen unit disc Graphen an, indem wir den Ansatz von Dvořák et al. erweitern. Sie partitionieren die Graphen anhand einer Einbettung des Graphen und eines Tilings der euklidischen Ebene. Wir verwenden hyperbolische anstelle von eukldischen Tilings. Wir können zeigen dass diese Herangehensweise nicht geeignet ist, um product structure bei hyperbolischen unit disc Graphen nachzuweisen. Außerdem erbringt auch ein Abweichen von der Art der Partitionierung durch Dvořák et al. keine product structure für einige Familien von hyperbolischen Tilings.

Contents

1 Introduction

Product structure is a tool to gain structural insight into the global structure of graph classes. This is done by describing the graph class as the strong product of simple graphs, often ones with bounded treewidth. Product structure has been shown for graph classes like planar and apex-minor-free graphs [Duj+20b] and can be used to bound certain graph parameters, e.g., the non-repetitive chromatic number of planar graphs [Duj+20a]. It has also been shown for Euclidian unit disc graphs [Dvo+21]. Those graphs are a well-known and well-studied graph class. They can, for example, be used to simulate wireless networks. A graph is a Euclidian unit disc graph if there is an embedding of the set of vertices into the Euclidian plane such that the distance between two connected vertices is at most 1. In Figure 1.2 on the left-hand side, a Euclidian unit disc graph is depicted. The red circle shows the outline of the disc of the vertex marked in red. Dvořák et al.'s [Dvo+21] proof of product structure shows that fixing each vertex to the nearest gridpoint only increases the clique number by a constant factor and reveals a Euclidian grid structure. In addition, every Euclidian grid is the product of two paths. An example of this is shown in Figure 1.1.

A generalization of Euclidian unit disc graphs that has only recently been studied are hyperbolic unit disc graphs [BFKS21]. They are defined in a similar way to Euclidian unit disc graphs. The difference is that the underlying geometry is hyperbolic. We note that the radius of the discs is important for hyperbolic unit disc graphs, as there is no scaling operation in the hyperbolic plane. A special case of hyperbolic unit disc graphs are the strongly hyperbolic unit disc graphs. These are graphs where the vertices are embedded in a circle of radius R, and all discs have radius R. In Figure 1.2 on the right-hand side, a strongly hyperbolic unit disc graph is depicted. Again, the red circle shows the outline of the disc of the vertex marked in red. Strongly hyperbolic unit disc graphs exhibit a hierarchical structure in contrast to the grid structure of Euclidian unit disc graphs. This difference is shown in Figure 1.2.

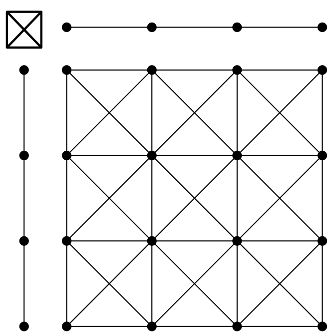

Figure 1.1: The strong product of two paths is a Euclidian grid.

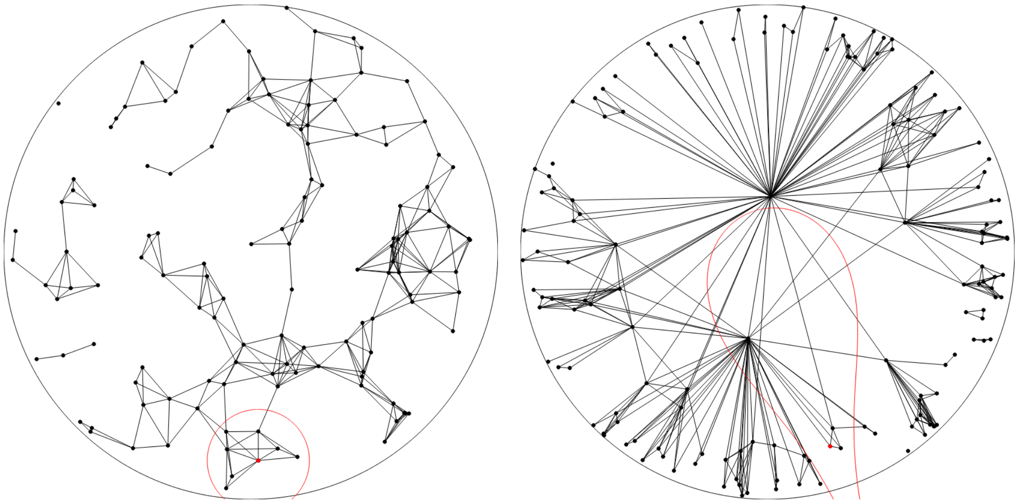

Figure 1.2: On the left side is a Euclidian unit disc graph, and on the right is a strongly hyperbolic unit disc graph. Illustration made using the interactive tool available tool at https://thobl.github.io/hyperbolic-unit-disk-graph/

Another related type of graphs are the hyperbolic random graphs. These are a random graph model first introduced by Krioukov et al. [Kri+10]. They are great at depicting real-life networks like social networks or the internet [BPK10] and can be seen as strongly hyperbolic unit disc graphs. This is because they are generated by placing n vertices in a hyperbolic circle with radius $R \approx 2\log(n)$. Two vertices are connected if the hyperbolic distance between them is at most R.

Product structure has not yet been shown for hyperbolic unit disc graphs. Its discovery would have some implications for related graph classes. First, the treewidth of strongly hyperbolic unit disc graphs would be bound by some function of the clique number because strongly hyperbolic unit disc graphs can be considered the neighborhood of one vertex in a hyperbolic unit disc graph. Additionally, product structure implies that the treewidth of the neighborhood of a vertex is bounded [Bos+22 | DMW17]. As a result, large grids would only be possible with large cliques in strongly hyperbolic unit disc graphs. Because of their definition, the same consequences would apply to hyperbolic random graphs.

Dvořák et al.[Dvo+21] use a Euclidian grid. We extend their approach to hyperbolic grids for hyperbolic unit disc graphs. These can be derived from hyperbolic tilings, the same way as grids are derived from Euclidian tilings in the Euclidian space. A regular hyperbolic (p, q)-tiling is a tiling of the hyperbolic plane using p-gons where q p-gons meet at each corner. Tilings in the hyperbolic plane work differently from hyperbolic tilings. First, there is only a limited number of regular Euclidian tilings, three to be exact. For hyperbolic tilings, there is an infinite number. Second, Euclidian tilings can be scaled. This is impossible in the hyperbolic space, as no scaling operation exists. Therefore, the size of a single tile in a (p, q)-tiling is a fixed value depending only on p and q. Figure 1.3 shows a cutout of a $(5, 4)$-tiling in the hyperbolic plane.

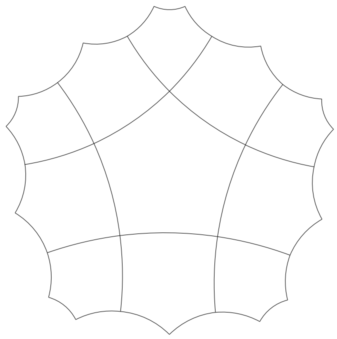

Figure 1.3: A cutout of a $(5, 4)$-tiling in the Poincaré disc model.

1.1 Contribution

We generalize Dvořák et al.'s [Dvo+21] approach by using hyperbolic tilings. For this, we define three families of graph classes as possible global structures for hyperbolic unit disc graphs. These families are using different behaviors of p and q. We show that geometric partitioning similar to Dvořák et al. is not possible for any of the three graph families. In addition, we show that product structure cannot be achieved at all using two of the families.

Hyperbolic tilings are not only a suitable choice to consider because Euclidian tilings were used to show product structure for Euclidian unit disc graphs but also because the treewidth of their duals is in $\mathcal{O}(\log n)$. The type of product structure that we conjecture would bound the treewidth of hyperbolic unit disc graph G by $\mathrm{tw}(H) \cdot f(\omega(G))$ for some graph H and some function f. Using the duals of hyperbolic tilings would therefore result in a treewidth of $\log(n) \cdot f(\omega(G))$. This matches the recent finding that hyperbolic unit disc graphs with a threshold radius greater than some constant have a treewidth in $\mathcal{O}(\log(n) \cdot \omega)$ (Thomas Bläsius, personal communication, August 8, 2023).

1.2 Related Work

The topic of this thesis touches on the areas of product structure, hyperbolic unit disc graphs, and hyperbolic tilings. We present some relevant work for each area one after the other.

1.2.1 Product Structure

Product structure was first found for planar graphs [Duj+20b]. It has since been discovered for other graph classes like k-planar graphs, graphs of Euler genus g or apex-minor-free graphs, as well [DMW22 | Dvo+21]. Product structure has also been used to optimize upper bounds for non-repetitive coloring [Duj+20a], centered coloring [DFMS] and vertex ranking [BDJM22] for planar graphs. For planar graphs there is an optimal $\mathcal{O}(n)$ algorithm to compute the factors of the strong product [BMO22]. We are interested in geometrically defined graph classes, especially Euclidian unit disc graphs, as they are defined similarly to hyperbolic unit disc graphs. Of great interest for us is that Dvořák et al. [Dvo+21] have shown product structure for Euclidian unit disc graphs. We have already discussed their approach in short and will come back to it in Chapter 3.

1.2.2 Hyperbolic unit disc graphs

Multiple graph problems using hyperbolic geometry have been studied. For example, in the hyperbolic traveling salesman problem (TSP), the vertices are placed in the hyperbolic plane, and the length between two vertices is the hyperbolic distance. Kisfaludi-Bak found an algorithm that computes a solution for hyperbolic TSP in $n^{\mathcal{O}(\log^2 n)\max\{1,1/\alpha\}}$ where α is the minimal distance between vertices [Kis20].

Hyperbolic unit disc graphs are only a recently studied graph class. Recognizing a hyperbolic unit disc graph is $\exists\mathbb{R}$-complete.[BBDJ23]. In addition, routing can be solved faster than on general graphs [BFKS21].

Multiple graph classes are closely related to hyperbolic unit disc graphs. Euclidian unit disc graphs are a subclass of hyperbolic unit disc graphs [BFKS21]. This is because the hyperbolic plane is Euclidian locally. Euclidian unit disc graphs have been studied extensively, especially in the context of wireless networks. For example, the generally NP-complete problem of finding a maximum clique can be done in polynomial time, and routing can be performed more efficiently than in general graphs [RS03 | KMRS18].

As mentioned previously, hyperbolic random graphs can be seen as strongly hyperbolic unit disc graphs. They have been found to closely resemble real-life networks such as social networks or the internet [BPK10]. This is because of their hierarchical structure, low diameter, high cluster coefficient, and power-law degree distribution [Kri+10 | FK18 | GPP12]. Multiple properties of hyperbolic random graphs have been studied like the expected treewidth, size of a separator [BFK16] and clique number[BFK18].

Geometric inhomogeneous random graphs (GIRGs) are an extension of hyperbolic random graphs. They are constructed by sampling a point x_v in $\mathbb{R}^d$ for every vertex v and assigning a weight w_v in $\mathbb{R}$. Two vertices v, u are connected if $|x_u - x_v| \leq w_u \cdot w_v$. GIRGs with points in one dimension are approximately equivalent to hyperbolic random graphs [BKL19]. Every strongly hyperbolic unit disc graph can be displayed in the GIRG model by using the mapping in [BKL19]. We use this GIRG representation for strongly hyperbolic unit disc graphs in Chapter 5.

Kisfaludi-Bak gives a different definition of hyperbolic unit disc graphs [Kis]. He defines the graph classes $\mathrm{UBG}_{\mathbb{H}^d}(p)$ that contain all hyperbolic unit disc graphs with discs of size p. For constant p some problems that are NP-hard on general graphs can be solved in quasi-polynomial time or polynomial time [Kis]. However, this cannot be transferred to the graph class defined by Bläsius et al. [BFKS21]. This is because Kisfaludi-Bak parameterizes the graph class using the radius of the discs, thus defining an infinite family of graph classes instead of one class. The difference can be visualized using stars. All stars are hyperbolic unit disc graphs as seen in Chapter 2. However, for each class $\mathrm{UBG}_{\mathbb{H}^d}(p)$ only stars S_k up to a specific size k are included because the radius p limits the number of non-connected vertices in the neighborhood of a vertex.

Noisy variants of hyperbolic unit disc graphs have also been studied [Kis]. But they go beyond the scope of this thesis.

1.2.3 Hyperbolic tilings

Throughout this thesis, we use only regular hyperbolic (p, q)-tilings, but there are further hyperbolic tilings. One example would be asymmetric or semi-regular tilings [Goo05 | DG21].

1.3 Outline

We start with a brief introduction to hyperbolic geometry and formal definitions of the used concepts. In Chapter 3, we present several notions of product structure and generalize the approach of Dvořák et al. [Dvo+21] We then introduce hyperbolic tilings in Chapter 4 and show that they are unsuitable for geometric partitioning. In Chapter 5, we construct a grid with a clique number linear in the grid size as a strongly hyperbolic unit disc graph.

2 Preliminaries

All graphs $G = (V, E)$ are undirected if not defined otherwise. We define $V(G)$ as the set of vertices of a graph G and $E(G)$ as the set of edges of G.

Treewidth has been introduced by Robertson and Seymour[RS86] as a measure of the likeness of a graph to a tree. A *tree decomposition* of a graph $G = (V, E)$ is a tuple (X, T) where $T = (X, F)$ is a tree and X contains bags of vertices such that:

1. $\bigcup_{X_i \in X} X_i = V$
2. for every edge $\{u, v\} \in E(G)$ there is a bag $X_i \in X$ such that $u, v \in X_i$.
3. for every vertex $v \in V(G)$ the subgraph of T induced by $\{X_i \in X \mid v \in X_i\}$ is a tree.

The width of a tree decomposition is defined as $\max_{X_i \in X} |X_i|$. The *treewidth* of G is the minimum width of all tree decompositions.

2.1 Hyperbolic geometry

This thesis uses the 2-dimensional hyperbolic plane $\mathbb{H}^2$. There are multiple models to depict the hyperbolic plane in the Euclidian plane. We will introduce hyperbolic geometry using the Poincaré disc model. In this model, the hyperbolic plane is mapped to the unit circle. The origin of the hyperbolic space is mapped to the Hyperbolic lines are mapped to circular arcs that meet the border of the unit circle at a right angle. This is bijective, so every circular arc that meets the border of the unit circle at a right angle is a hyperbolic line. Therefore, hyperbolic lines through the origin of the hyperbolic plane are mapped to Euclidian lines through the center of the unit circle. These can be seen as circular arcs of a circle with infinite radius. Figure 2.1 shows multiple hyperbolic lines in the Poincaré disc. We can see that for the hyperbolic line ℓ, multiple hyperbolic lines through p do not intersect ℓ. There are indeed an infinite number of hyperbolic lines through p that do not intersect ℓ. Axiomatically, this is the only difference to Euclidian geometry. This results in multiple properties that differ from the Euclidian geometry. The area of a triangle can, for example, be π at most. In addition, the hyperbolic space has exponential expansion. For comparison, the Euclidian space only expands polynomially. Besides the hyperbolic lines, Figure 2.1 shows a hyperbolic triangle and a hyperbolic circle in the Poincaré disc model. It can be seen that the Poincaré disc model maps hyperbolic circles to Euclidian circles. In the Poincaré disc, most of the hyperbolic space is mapped rather close to the border of the unit circle. This is why we use polar coordinates when displaying hyperbolic unit disc graphs. In Figure 1.2, the two graphs are depicted by embedding the polar coordinates of the vertices in the Euclidian plane. To use polar coordinates, we designate a pole O and a ray originating at O as the polar axis. Every point $p \in \mathbb{H}^2$ can then be represented by the distance to the origin $r(p)$, called radius, and the angle between the ray from O to p and the polar axis $\varphi(p)$. We write r and φ if the referred hyperbolic point is clear. The distance between two points p and q can be calculated using

$$d_{\mathbb{H}}(p, q) = \operatorname{arcosh}(\cosh(r(p))(\cosh(r(q)) - \sinh(r(p)) \sinh(r(q)) \cos(\Delta_\varphi(p, q)) \qquad (2.1)$$

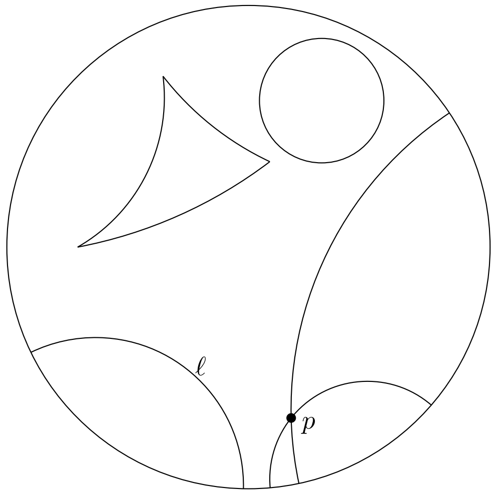

Figure 2.1: Multiple lines, a triangle and a circle in the Poincaré disc model.

where $\Delta_\varphi(p,q) = \pi - |\pi - |\varphi(p) - \varphi(q)||$ is the angular distance between p and q, $\cosh(x) = \frac{e^x+e^{-x}}{2}$, $\sinh(x) = \frac{e^x-e^{-x}}{2}$, and $\operatorname{arcosh}(x) = \log(x + \sqrt{x^2-1})$ with $\log(x)$ as the natural logarithm. By rearranging Equation (2.1) we get a formula for the maximal angular distance such that two points (r_1, φ_1) and (r_2, φ_2) are less than R apart from each other

$$\theta(r_1, r_2) = \arccos\left(\frac{\cosh(r_1)\cosh(r_2) - \cosh(R)}{\sinh(r_1)\sinh(r_2)}\right) \tag{2.2}$$

For more information on the hyperbolic space and its models, we refer to [Jam99].

2.2 Hyperbolic unit disc graphs

A graph G is a Euclidian unit disc graph if there is an embedding $\phi : V(G) \mapsto \mathbb{R}^2$ such that $\{u,v\} \in E(G)$ if and only if $|\phi(u) - \phi(v)| < 1$. We define hyperbolic unit disc graph analogously in the hyperbolic plane. A graph $G = (V, E)$ is a *hyperbolic unit disc graph* that has an embedding $\phi : V \mapsto \mathbb{H}^2$ such that $\phi(v)$ is at most r away from the origin of the hyperbolic plane for every $v \in V$ and $\{u,v\} \in E$ if and only if $d_{\mathbb{H}}(u,v) < R$. For us, a hyperbolic unit disc graph is a combinatorial object. When an embedding is needed, we say so explicitly. We call $r \in \mathbb{R}$ the radius of the ground space and $R \in \mathbb{R}$ the threshold radius. We note that R cannot be omitted as it can be in the Euclidian setting. This is because different graphs can be constructed using different threshold radii. Recall our discussion of the remarks on the alternative definition of hyperbolic unit disc graph by Kisfaludi-Bak[Kis] in the Introduction on this topic. The name hyperbolic **unit** disc graph is still appropriate because all the circles have the same radius. We define a *strongly hyperbolic unit disc graph* as a hyperbolic unit disc graph that has an embedding with $R = r$.

2.2.1 Stars are hyperbolic unit disc graphs

We use stars in multiple proofs later on. So, in this section, we show that stars of arbitrary size are hyperbolic unit disc graphs. The Star S_k is a graph with $k + 1$ vertices. It has one central vertex connected to the remaining k leaves. The graph has no other edges. We note that only stars with up to five leaves are Euclidian unit disc graphs.

We now construct an embedding for S_k. We want the central vertex to be the origin of the hyperbolic space. The leaves have distance R from the origin. Recall that $\theta_R(R,R)$ (Equation (2.2)) is the maximum angular distance so that points with distance r_1, r_2 to the origin have a distance smaller than R to each other. Using the upper bound $\pi\sqrt{e^{-R}} \geq \theta_R(R,R)$ we can see that $\theta_R(R,R)$ is monotonically decreasing [BFKS21]. This way, we know that there is an R such that $\theta_R(R,R) < 2\pi/k$. We set this R as the threshold and ground space radius of the embedding of S_k. We then place the leaves at distance R from the origin and angular distance ϕ from the neighboring leaves with $\phi < \theta_R(R,R) < 2\pi/k$. This way, neighboring leaves have no edges between each other. On the other hand, the central vertex has distance R from the leaves and is thus connected to all of them.

2.3 Strong Product

We define the *strong product* of the graph H with a graph K to be a graph $H \boxtimes K = G$ where $V(G)$ is $V(H) \times V(K)$ and $\{(u_H, u_K), (v_H, v_K)\} \in E(G)$ if one of the following is true

- $\{u_H, v_H\} \in E(H)$ and $u_K = v_K$
- $\{u_K, v_K\} \in E(K)$ and $u_H = v_H$
- $\{u_H, v_H\} \in E(H)$ and $\{u_K, v_K\} \in E(K)$

A simple, strong product has been shown in Figure 1.1 in the Introduction.

We can show that the clique number of the factors determines the clique number of a strong product.

Lemma 2.1: *Let $G = H \boxtimes Q$ for some graphs H and Q. Then $\omega(G) = \omega(H) \cdot \omega(Q)$.*

Proof. We start by showing that $\omega(G) \geq \omega(H) \cdot \omega(Q)$. Let C_H, C_Q be cliques of size $\omega(H)$ and $\omega(Q)$ in H and Q respectively. Then $C := C_H \times C_Q$ is a clique in G. We verified this by checking the three definitions for edges in strong products. Let $(u_1, v) \neq (u_2, v)$ be vertices in C. Then there is an edge between the vertices because $u_1, u_2 \in C_H$ and thus $\{u_1, u_2\} \in E(H)$. The remaining two types of edges can be the same way. It follows that $\omega(G) \geq \omega(H) \cdot \omega(Q)$.

We now show that $\omega(G) \leq \omega(H) \cdot \omega(Q)$. Let C be a clique of size $\omega(G)$ in G. Set $V_H = \{u \mid (u,v) \in C\}$ and $V_Q = \{v \mid (u,v) \in C\}$. We can see that $C \subseteq V_H \times V_Q$, so $|C| \leq |V_H| \cdot |V_Q|$. We show that V_H and V_Q are cliques, and their size is thus bounded $\omega(H)$ or $\omega(Q)$, respectively. We take two different vertices $u \neq v$ in V_H. By the definition of V_H there are $w_1, w_2 \in V_Q$ such that $(u, w_1), (u, w_2)$ are in C. Using the definition of the strong product, we can see that there is an edge between u and v in H either using case 1 ($w_1 = w_2$) or case 3 ($w_1 \neq w_2$). This means that V_H is a clique because there is an edge between any two u, v in V_H. Applying the same argument to V_Q shows that it is a clique as well. It follows that $\omega(G) = |C| \leq |V_H| \cdot |V_Q| \leq \omega(H) \cdot \omega(Q)$.

Combining both results, it follows that $\omega(G) = \omega(H) \cdot \omega(Q)$. ∎

For the treewidth of a strong prodcut $H \boxtimes Q$ we know that $\mathrm{tw}(H \boxtimes Q) \leq (\mathrm{tw}(H) - 1)|V(Q) - 1$ [HW21]. For the special case of $H \boxtimes K_n$ the treewidth is $(\mathrm{tw}(H) - 1)n - 1$ [HW21]. We are often interested in the strong product of a graph H and a complete graph K_n. The resulting graph retains the global structure of H but every vertex is blown up to a complete graph. Figure 2.2 shows an example of this product. A different view on product structure is to partition the vertices of the graph. The structure between the parts of the partition is called the quotient and is one of the factors in the strong product. Each part of the partition can then be blown

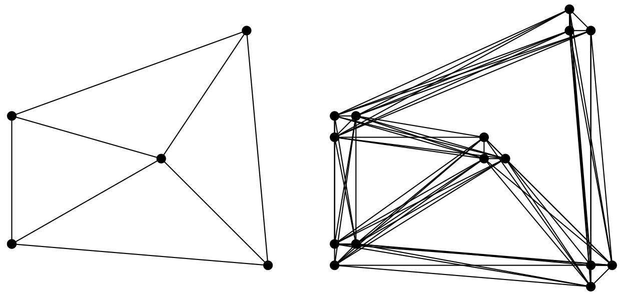

Figure 2.2: The right side shows the strong product of the graph on the left and K_3.

up to the partition width by multiplying with $K_{w_{\mathcal{P}}(G)}$ where $w_{\mathcal{P}}(G)$ to recreate the original graph. Formally, the quotient of a graph G and partition $\mathcal{P}$ of the set of vertices of G to be the graph $H = G/\mathcal{P}$ with $V(H) = \mathcal{P}$ and $E(H) = \{\{p, q\} \mid \exists u \in p \exists v \in q : \{u, v\} \in E(G)\}$. We explore the connection between product structure and partitions in more depth in Chapter 3

A graph H is a minor of a graph G if H can be received from G by

1. removing edges in G
2. removing vertices in G
3. contracting edges in G

A r-shallow minor is a minor where the radius of contracted subgraphs is at most r. We know that r-shallow minors are a subgraph of a strong product if the original graph is a strong product.

Lemma 2.2 ([HW22]): *If G is a r-shallow-minor of $H \boxtimes P \boxtimes K_l$ where H has treewidth at most t and P is a path, then $G \subseteq J \boxtimes P \boxtimes K_{l(2r+1)^2}$ where J has treewidth at most $\binom{2r+1+l}{t} - 1$.*

3 Product structure of hyperbolic unit disc graphs

In this section, we define different notions of product structure and generalize the approach used to show product structure in Euclidian unit disc graphs.

As already mentioned, Dujmović, Joret, Micek, Morin, Ueckerdt, and Wood [Duj+20b] have shown that every planar graph is a subgraph of the strong product of a path and a graph with bounded treewidth. This concept is called product structure. In general, we say a graph class $\mathcal{G}$ has *product structure* if there is a constant c such that for every $G \in \mathcal{G}$ we have $G \subseteq P \boxtimes T$ for some path P and some graph T with $\mathrm{tw}(T) \leq c$. This type of product structure is not possible for hyperbolic unit disc graphs since cliques of arbitrary size are HUDGs. An alternative definition considers this and states that product structure is present in a graph class when a graph parameter is bounded. This is equal to the factors being dependent on this graph parameter. The following definition uses the clique number as the graph parameter, as we will use the clique number going forward.

Definition 3.1 ($f(\omega)$-bounded product structure)**:** *A graph class $\mathcal{G}$ has* $f(\omega)$-bounded product structure *for some function $f : \mathbb{N} \mapsto \mathbb{N}$ if for every $G \in \mathcal{G}$ we have $G \subseteq P \boxtimes T$ for some path P and graph T with* $\mathrm{tw}\,(T) \leq f(\omega(G))$.

There is a third option for product structure that moves the dependency on the clique number to a third factor. Intuitively, the first two factors define the global structure of the graph, and the third blows the graph up to the correct size.

Definition 3.2 (strong $f(\omega)$-bounded product structure)**:** *A graph class $\mathcal{G}$ has* strong $f(\omega)$-bounded Product Structure *for some function $f : \mathbb{N} \mapsto \mathbb{N}$ if there is a constant c such that for every $G \in \mathcal{G}$ we have $G \subseteq P \boxtimes T \boxtimes K_{f(\omega(G))}$ for some path P and some graph T with* $\mathrm{tw}\,(T) \leq c$.

We can show that Definition 3.2 implies Definition 3.1

Lemma 3.3: *If a graph class $\mathcal{G}$ has strong $f(\omega)$-bounded product structure for some function $f : \mathbb{N} \mapsto \mathbb{N}$, it has $g(\omega)$-bounded product structure for some function $g \in \Theta(f)$.*

Proof. Let $\mathcal{G}$ be a graph class that has strong $f(\omega)$-bounded product structure, and $G \in \mathcal{G}$ with $G \subseteq P \boxtimes T \boxtimes K_{f(\omega(G))} = P \boxtimes T'$ using $T' = T \boxtimes K_{f(\omega(G))}$. We bound the treewidth of T' using the inequality $\mathrm{tw}(G_1 \boxtimes G_2) \leq (\mathrm{tw}(G_1) + 1)|V(G_2)| - 1$ [HW21]. Applying it to T' yields

$$\mathrm{tw}(T') = \mathrm{tw}(T \boxtimes K_{f(\omega(G))}) \leq (\mathrm{tw}(T) + 1) f(\omega(G)) - 1 \leq (c + 1) f(\omega(G)) - 1 =: g(\omega(G))$$

where $\mathrm{tw}(T) \leq c$ because of the strong $f(\omega)$-bounded product structure. Thus $G \subseteq P \boxtimes T'$ where $\mathrm{tw}(T') = g(\omega(G))$. ∎

It is unknown whether Definition 3.1 and Definition 3.2 are equivalent. The question has not been answered for product structure bounded by different graph parameters either. For k-planar graph, product structure according to Definition 3.1 has been found early on using the parameter k [DMW22]. It took a while to show that k-planar graph have product structure according to Definition 3.2. The proofs use different techniques and therefore do not promote a connection between the two types of product structure.

Using these definitions, we can now define our goal.

Conjecture 3.4: *Hyperbolic unit disc graphs have strong $f(\omega)$-bounded product structure, i.e., there is a constant c and a function f such that every hyperbolic unit disc graph G is a subgraph of $P \boxtimes T \boxtimes K_{f(\omega(G))}$ for some path P and some graph T with $\operatorname{tw}(T) \leq c$.*

Not restricting f in any way is a problem for some graph classes. When $\omega(G) \in \Theta(f(|V(G)|)$ holds for all graphs of a graph calss $\mathcal{G}$, then $\mathcal{G}$ has strong f^{-1}-bounded product structure because the clique factor alone has $|V(G)|$ vertices. For hyperbolic unit disc graphs this is not the case as there is a family of graphs with no connection between the number of vertices and the clique number. An example would be stars.

3.1 Product Structure in Euclidian unit disc graphs

Dvořák et al. [Dvo+21] have that Euclidian unit disc graph have $f(\omega)$-bounded product structure. We give the idea of the proof here in order to generalize it afterward.

Consider a Euclidian unit disc graph G with an embedding. First, tile the Euclidian plane into squares with edge length 1. We call these squares tiles. We now aim to limit the number of cliques that have to be used to cover one square and then analyze the structure of edges between squares. For the first part, we divide each square into four smaller squares with edge length $1/2$. We call these squares subtiles. The diameter of a subtile is $\sqrt{1/2}$ due to the Pythagorean theorem. Because the diameter is smaller than 1, all vertices in one subtile form a clique. Four cliques can thereby cover each tile. The remaining part is to look at the edges between tiles. Each tile has edges to only its eight neighboring tiles. We can show this by considering the minimal distance between tiles. The distance between a tile and its neighbors is 0. Recall that the edge length of a tile is 1. Therefore, the distance to a tile other than the neighbors is greater than 1. The graph that shows the possible edges between tiles is called a king's graph and can be written as $P \boxtimes P$ for some path P. Combining our observations, we can conclude that every Euclidian unit disc graph is a subgraph of $P \boxtimes P \boxtimes K_{4 \cdot \omega(G)}$ for some path P.

In short, they partition the graph using squares and analyze the edges between parts of the partition. We generalize this approach in the next definition.

Definition 3.5 ($f(\omega)$-bounded $\mathcal{H}$-partition)**:** *Let G be a graph. A H-partition of a graph G is a partition $\mathcal{P}$ of $V(G)$ such that $G/\mathcal{P}$ is H. The width of a H-partition is $w_H(G) = \max_{p \in \mathcal{P}} |p|$. Let $\mathcal{H}$ and $\mathcal{G}$ be a graph class. A $\mathcal{H}$-partition of a graph G is a partition $\mathcal{P}$ of $V(G)$ such that $H = G/\mathcal{P}$ is in $\mathcal{H}$. The $\mathcal{H}$-partition-width of a graph G is $w_{\mathcal{H}}(G) = \min_{H \in \mathcal{H}} w_H(G)$.*

We say a graph class $\mathcal{G}$ has a $f(\omega)$-bounded $\mathcal{H}$-partition if for every $G \in \mathcal{G}$ we have $w_{\mathcal{H}}(G) \leq f(\omega(G))$.

Rephrasing the result for Euclidian unit disc graphs, we say that every Euclidian unit disc graph has a $f(\omega)$-bounded $\{P_1 \boxtimes P_2 \mid P_1, P_2 \text{ are paths}\}$-partition. In this case, the graph class $\mathcal{H}$ has product structure. The next lemma shows that this is important for the equivalence to strong $f(\omega)$-bounded product structure.

Lemma 3.6: *Let $\mathcal{G}$ be a graph class. Then the following two statements are equivalent:*

1. *$\mathcal{G}$ has strong $f(\omega)$-bounded product structure.*
2. *$\mathcal{G}$ has a $f(\omega)$-bounded $\mathcal{H}$-partition for a graph class $\mathcal{H}$ that admits product structure.*

Proof. First, assume that $\mathcal{G}$ has strong $f(\omega)$-bounded product structure, i.e., every graph $G \in \mathcal{G}$ is a subgraph of a graph $P \boxtimes T \boxtimes K_{f(\omega)}$ for some path P and some graph T with $\text{tw}(T) \leq c$. For every $G \in \mathcal{G}$ we want to find a H-partition with width at most $f(\omega(G))$ such that H is part of a graph class that admits product structure. So, let G be a graph in $\mathcal{G}$. We first find a partition $\mathcal{P}$ of the vertices of G with width at most $f(\omega(G))$. For this, consider the partition $\mathcal{P}_G = \{\{(u, v, w) \mid w \in V(K_{f(\omega(G))})\} \subseteq V(G) \mid u \in V(P), v \in V(T)\}$, i.e., all vertices of a copy of $K_{f(\omega(G))}$ restricted to the vertices in G are one partition. This partition has width of at most $f(\omega(G))$ because each copy of $K_{f(\omega(G))}$ only contains $f(\omega(G))$ vertices. We set $\mathcal{H} = \{G/\mathcal{P}_G \mid G \in \mathcal{G}\}$. This way G has a $f(\omega)$-bounded $\mathcal{H}$-partition because of the definition of the quotient. It remains to show that $\mathcal{H}$ has product structure. Consider the quotient $G/\mathcal{P}_G$. Each set in $\mathcal{P}_G$ is a subset of a copy of $K_{f(\omega)}$ in $P \boxtimes T \boxtimes K_{f(\omega)}$. An edge in $G/\mathcal{P}_G$ is thereby an edge between vertices of different cliques in $P \boxtimes T \boxtimes K_{f(\omega)}$. All possible edges between copies of $K_{f(\omega(G))}$ are present in $P \boxtimes T$. It is therefor $G/\mathcal{P}_G \subseteq P \boxtimes T$. This shows that $\mathcal{H}$ has product structure as it only contains graphs of the form $G/\mathcal{P}_G$.

We now assume that every $G \in \mathcal{G}$ has a $\mathcal{H}$-partition with $w_{\mathcal{H}}(G) \leq f(\omega(G))$ for some $\mathcal{H}$ with product structue and want to prove that G is a subgraph of $P \boxtimes T \boxtimes K_{f(\omega(G))}$. For this let G be a graph in $\mathcal{G}$ with a H-partition of width $w_H(G) \leq f(\omega(G))$ for some $H \in \mathcal{H}$. Let $\mathcal{P}$ be the partition of the vertices of G. Recall that $H \subseteq P \boxtimes T$ for a path P and a graph T with constant treewidth. We can easily see that all vertices of G are present in $P \boxtimes T \boxtimes K_{f(\omega(G))}$. We have at most $|V(P)| \cdot |V(T)|$ partitions in $\mathcal{P}$ because $H \subseteq P \boxtimes T$ and each partition contains at most $f(\omega(G))$ vertices.

We now show that every edge in G is present in $P \boxtimes T \boxtimes K_{f(\omega(G))}$. For this, identify each partition of $\mathcal{P}$ by the vertex $(u, v) \subset V(P) \times V(T)$. An edge between vertices of one partition is present because each partition is represented by a copy of $K_{f(\omega(G))}$. Consider an edge between vertices of different partitions. This edge is present in $P \boxtimes T \boxtimes K_{f(\omega(G))}$ as well because all possible connections between partitions are modeled in $P \boxtimes T$.

We conclude that $G \subseteq P \boxtimes T \boxtimes K_{f(\omega(G))}$ for some path P and graph T with $\text{tw}(T) \leq c$. ∎

Using this lemma, we can rewrite Conjecture 3.4. We use this version for the remainder of the thesis.

Conjecture 3.7: *Every hyperbolic unit disc graph has a $f(\omega)$-bounded $\mathcal{H}$-partition for some $\mathcal{H}$ with product structure.*

We now explore different possibilities for $\mathcal{H}$ in Conjecture 3.7 and start with $\{P_1 \boxtimes P_2 \mid P_1, P_2 \text{ are paths}\}$. This is an obvious choice as it is the graph class used for Euclidian unit disc graphs [Dvo+21]. It is, in any case, not possible to copy the exact course of action of Dvořák et al. since regular squares of the same size with four squares meeting at one corner cannot tile the hyperbolic plane. But even considering any partition with a king's grid as the quotient does not lead to $f(\omega)$-bounded partitions.

Lemma 3.8: *For every function f there is a hyperbolic unit disc graph G such that G has no $f(\omega)$-bounded $\{P_1 \boxtimes P_2 \mid P_1, P_2 \text{ are paths}\}$-partition.*

Proof. We use stars to show the lemma. We have already shown that stars of arbitrary size are hyperbolic unit disc graphs. The high-level argument is that stars have a constant clique number, and $P \boxtimes P$ has a constant maximum degree. Stars, on the other hand, have a growing maximum degree. This cannot be hidden in the width of the partition because of the constant clique number of stars.

We choose a star graph S with at least $9 \cdot f(2)+1$ vertices. Assume that S has a $f(\omega)$-bounded $\{P_1 \boxtimes P_2 \mid P_1, P_2 \text{ are paths}\}$-partition. This means that we have a partition $\mathcal{P}$ of $V(S)$ and $H = S/\mathcal{P} = P \boxtimes P$ for some path P. Let v be the vertex in H that corresponds to the partition in $\mathcal{P}$ that contains the central vertex c of the star. All remaining vertices of the star are adjacent to c and can thus only be contained in the partitions that the neighbors of v correspond to. Each vertex in the graph $P \boxtimes P$ has at most eight neighbors. Recall that stars have a constant clique number of 2. The corresponding partitions of v and its neighbors can only contain $9 \cdot f(2)$ vertices. It is, therefore, impossible to distribute the $9 \cdot f(2) + 1$ vertices of the star to the partitions of v and its neighbors. ∎

4 Hyperbolic tilings

To explore other possible graph classes for $\mathcal{H}$ we use hyperbolic tilings. So, this chapter starts by giving an overview of hyperbolic tilings. From here on, we only consider regular tilings, i.e., tilings with regular polygons. We name tilings using the number of vertices of the polygon p and the number of polygons that meet at one vertex q. A (p, q)-tiling for us has an unique embedding in the plane. This embedding has the center of one tile at the origin of the hyperbolic plane. A (p, q)-tiling is Euclidian if and only if $1/p + 1/q = 1/2$ and a (p, q)-tiling is hyperbolic is hyperbolic if $1/p + 1/q < 1/2$ [EEK82]. Combinations with $1/p+1/q > 1/2$ would be tilings of the sphere. But these are not of interest to us. Examples for Euclidian tilings are a $(6, 3)$-tiling and a $(4, 4)$-tiling. A Euclidian $(6, 3)$-tiling and a hyperbolic $(5, 3)$-tiling are shown in Figure 4.1.

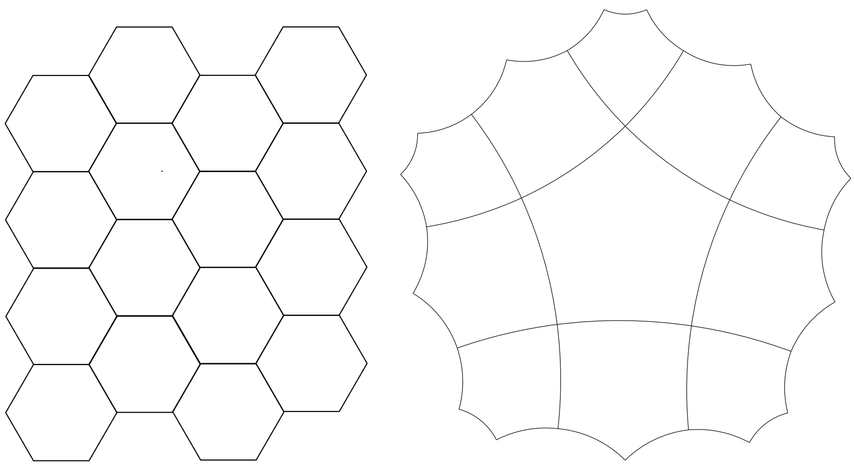

Figure 4.1: A $(6, 3)$-tiling in the Euclidian space on the left and a cutout of a $(5, 4)$-tiling of the hyperbolic space in the Poincaré disc model on the right.

We now provide equations for various lengths in hyperbolic tilings that we use later. We need the *hyperbolic laws of cosine* to prove the correctness of these equations.

Lemma 4.1 ([Jam99])**:** *Given a hyperbolic triangle with vertices A, B, C, angles α, β, γ and edges $AB = c, BC = a$ and $AC = b$, then the following hold true:*

$$\cosh a = \cosh b \cosh c - \sinh b \sinh c \cos \alpha \tag{4.1}$$

$$\cos \alpha = -\cos \beta \cos \gamma + \sin \beta \sin \gamma \cosh a \tag{4.2}$$

We can now calculate the radius of the incircle, the radius of the circumcircle, and the edge length of the polygons in a hyperbolic tiling. We denote them by $r_{\text{in}}(p, q)$, $r_{\text{out}}(p, q)$ and $\ell(p, q)$ respectively, and omit p and q if they are clear from the context.

Lemma 4.2: *Given a hyperbolic tiling (p, q) and let r_{in} be the radius of the incircle, r_{out} the radius of the circumcircle, and ℓ the edge length. Then*

$$r_{\text{in}} = \operatorname{arcosh}\left(\frac{\cos\left(\frac{\pi}{q}\right)}{\sin\left(\frac{\pi}{p}\right)}\right), r_{\text{out}} = \operatorname{arcosh}\left(\frac{\cos\left(\frac{\pi}{q}\right)\cos\left(\frac{\pi}{p}\right)}{\sin\left(\frac{\pi}{q}\right)\sin\left(\frac{\pi}{p}\right)}\right), \text{ and } \ell = 2 \cdot \operatorname{arcosh}\left(\frac{\cos\left(\frac{\pi}{p}\right)}{\sin\left(\frac{\pi}{q}\right)}\right). \tag{4.3}$$

Proof. Construct a triangle using the center of a tile C, a vertex of the tile A, and the midpoint of an edge M. Because we assume a regular tiling of p-gons, there are $2p$ congruent triangles around the center of a p-gon. One of the triangles is shown in Figure 4.2. The angle at the center of the p-gon for one of the triangles is thus $\measuredangle ACM = (2\pi)/(2p) = \pi/p$. Applying the same argument to the vertex of a p-gon yields $\measuredangle CAM = \pi/q$. The angle at M is π because $\overline{CM}$ is orthogonal to an edge in regular tilings. Applying the second rule of Lemma 4.1 to $\overline{CM}$ yields

$$\begin{aligned}
\cos\left(\frac{\pi}{q}\right) &= -\cos\left(\frac{\pi}{p}\right)\cos(\pi) + \sin\left(\frac{\pi}{p}\right)\sin(\pi)\cosh(r_{\text{in}}) \\
\cos\left(\frac{\pi}{q}\right) &= \sin\left(\frac{\pi}{p}\right)\sin(\pi)\cosh(r_{\text{in}}) \\
\frac{\cos\left(\frac{\pi}{q}\right)}{\sin\left(\frac{\pi}{p}\right)} &= \cosh(r_{\text{in}}) \\
r_{\text{in}} &= \operatorname{arcosh}\left(\frac{\cos\left(\frac{\pi}{q}\right)}{\sin\left(\frac{\pi}{p}\right)}\right)
\end{aligned}$$

Applying the second rule of Lemma 4.1 in the same fashion to the other edges of the triangle yields r_{out} and $\ell/2$. ∎

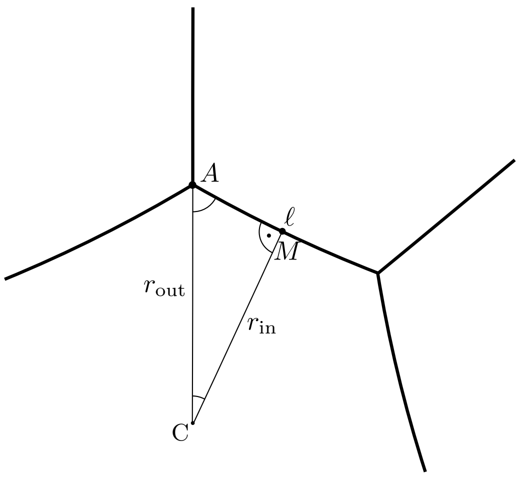

Figure 4.2: The Triangle used in proof the of Lemma 4.2

We, later on, look at different families of tilings. Therefore, the behavior of r_{in} in different tilings is of interest to us.

Lemma 4.3: *For constant q, $f(p) := \operatorname{arcosh}(\cos(\pi/q)/\sin(\pi/p))$ is $\log(p) \pm \mathcal{O}(1)$.*

Proof. It is known that $\sin(x) \approx x$ for small x and $\operatorname{arcosh}(x) \approx \log(x)$. Applying these approximations to $f(p)$ yields $f(p) \approx \log(p)$. We now show this claim more formally.

Using the Taylor series at $x = \infty$ we can see that $\sin\left(\frac{1}{x}\right) = x - \mathcal{O}\left(\frac{1}{x^3}\right)$. Applying this to $\sin\left(\frac{\pi}{p}\right)$ yields

$$\begin{aligned} \sin\left(\frac{\pi}{p}\right) &= \frac{\pi}{p} - \mathcal{O}\left(\frac{1}{p^3}\right) \\ &= \frac{\pi}{p}\left(1 - \frac{p}{\pi}\mathcal{O}\left(\frac{1}{p^3}\right)\right) \\ &= \frac{\pi}{p}\left(1 - \mathcal{O}\left(\frac{1}{p^2}\right)\right) \end{aligned} \tag{4.4}$$

Recall that $\operatorname{arcosh} = \log\left(x + \sqrt{x^2 - 1}\right)$. Using this we can rewrite arcosh as

$$\begin{aligned} \operatorname{arcosh}(x) &= \log\left(x + \sqrt{x^2 - 1}\right) \\ &= \log\left(x\left(1 + \sqrt{1 - \frac{1}{x^2}}\right)\right) \\ &= \log(x) + \log\left(1 + \sqrt{1 - \frac{1}{x^2}}\right). \end{aligned} \tag{4.5}$$

We can see that $\sqrt{1 - 1/x^2} \le 1$ because $1/x^2 > 0$. Thus

$$\log\left(1 + \sqrt{1 - \frac{1}{x^2}}\right) \le \log(2).$$

This can be used to further simplify Equation (4.5) to

$$\log(x) + \mathcal{O}(1)$$

Applying this to $f(p)$ yields

$$\begin{aligned} \operatorname{arcosh}\left(\frac{\cos\left(\frac{\pi}{q}\right)}{\sin\left(\frac{\pi}{p}\right)}\right) &= \log\left(\frac{\cos\left(\frac{\pi}{q}\right)}{\sin\left(\frac{\pi}{p}\right)}\right) + \mathcal{O}(1) \\ &= \log\left(\cos\left(\frac{\pi}{q}\right)\right) - \log\left(\sin\left(\frac{\pi}{p}\right)\right) + \mathcal{O}(1) \end{aligned}$$

As q is constant, this can be further simplified to

$$-\log\left(\sin\left(\frac{\pi}{p}\right)\right) + \mathcal{O}(1) \tag{4.6}$$

Combining Equations (4.4) and (4.6) we get

$$\begin{aligned}
\operatorname{arcosh}\left(\frac{\cos\left(\frac{\pi}{q}\right)}{\sin\left(\frac{\pi}{p}\right)}\right) &= -\log\left(\frac{\pi}{p}\left(1-\mathcal{O}\left(\frac{1}{p^2}\right)\right)\right)+\mathcal{O}\left(1\right)\\
&= -\log\left(\pi\right)+\log\left(p\right)-\log\left(1-\mathcal{O}\left(\frac{1}{p^2}\right)\right)+\mathcal{O}\left(1\right)\\
&= \log\left(p\right)\pm\mathcal{O}\left(1\right).
\end{aligned}$$

The last transformation is valid because

$$\log\left(1-\mathcal{O}\left(\frac{1}{p^2}\right)\right)\in\pm\mathcal{O}(1)$$

as $\mathcal{O}\left(1/p^2\right)$ shrinks for increasing p. We can thus conclude that

$$f(p)=\log(p)\pm\mathcal{O}(1).$$

■

4.1 Partitions using tilings

After this brief introduction to hyperbolic tilings, we use them in the context of product structure. Our goal is to use hyperbolic tilings similar to the proof for product structure in Euclidian unit disc graphs. We call this *geometric partitioning*. Given an embedding of a hyperbolic unit disc graph, we choose a tiling and then place it onto the hyperbolic plane and assign each vertex to the tile it lies in. Our goal is to cover the vertices in one tile with a bounded number of cliques. This is more difficult in the hyperbolic space than in the Euclidian space. One reason for this is that we cannot resize tilings. The size of a tile is fixed by the parameters p and q of the tiling. So, we have to consider different tilings for hyperbolic unit disc graphs of different sizes. The difficulty is in choosing the right tiling. This is a trade-off between the number of cliques required to cover a tile (i.e., the size of a tile) and the number of tiles to which edges are possible.

We now define three families of tilings and, based on them, possible graph classes $\mathcal{H}$ for Conjecture 3.7. For this, we regard a tiling as an infinite planar graph. For a planar graph G, the dual is a graph D where the set of vertices is the set of faces of G. Two faces are connected if they share an edge in G. The dual of a planar graph is itself a planar graph. We want to extend the dual to the extended dual. The extended dual has additional edges between faces that share a vertex. This means that for a (p,q)-tiling, all q tiles that share a vertex are connected in the extended dual. Note that the extended dual of a Euclidian grid, i.e., a $(4,4)$-tiling, is an infinite king's graph. This is the graph class used to show that Euclidian unit disc graphs have strong $f(\omega)$-bounded product structure. We get different extended duals for different hyperbolic (p,q)-tilings. We want to explore three options for choosing p and q. We only want to handle finite graphs. So we define all of the three following graph classes to contain all possible subgraph of the extended dual of the respective tilings. The three families of graph classes are:

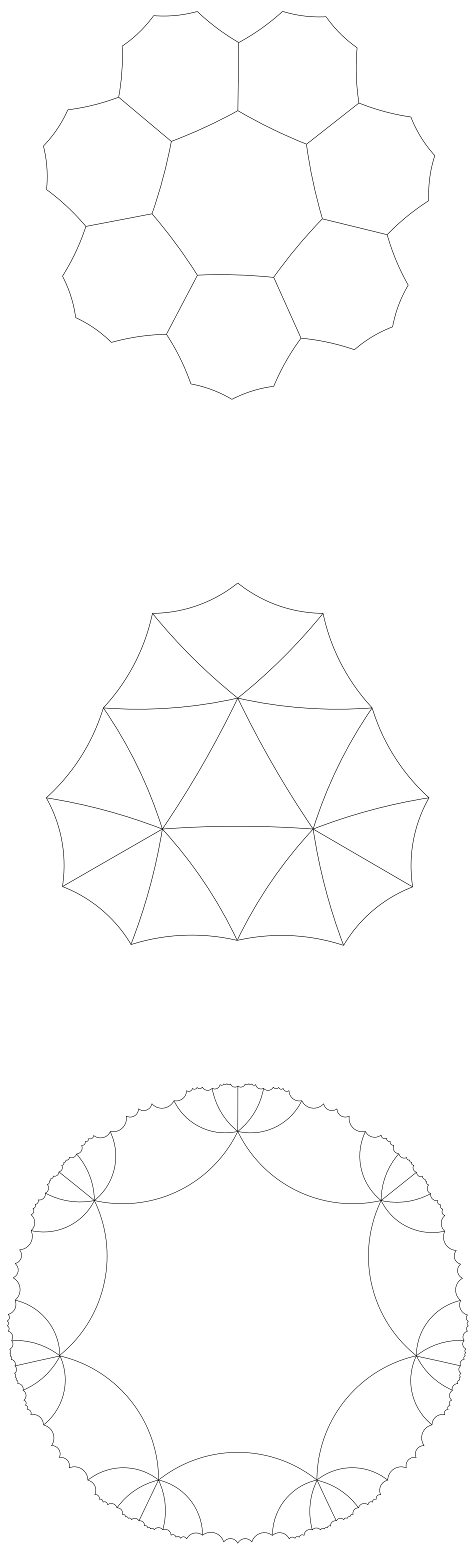

Figure 4.3: Top: Cutout of a $(7,3)$-tiling. Middle: Cutout of a $(3,7)$-tiling. Bottom: Cutout of a $(7,7)$-tiling. All tilings are shown in the Poincaré disc model.

1. p and q are constant. We call this graph class $\mathcal{H}_{p,q}$.
2. p is constant and q variable. We call this graph class $\mathcal{P}_p$.
3. q is constant and p variable. We call this graph class $\mathcal{Q}_q$.

Figure 4.3 shows three cutouts of hyperbolic tilings with different parameters to illustrate the difference between tilings with low and high choices for p and q.

We start with the case that both p and q are constant.

Lemma 4.4: *For every function f and $p, q \in \mathbb{N}$ there is a hyperbolic unit disc graph G such that G has no $f(\omega(G))$-bounded $\mathcal{H}_{p,q}$-tiling.*

Proof. We use stars similar to Lemma 3.8. The high-level argument is the same as well. We already know that stars of arbitrary size are hyperbolic unit disc graphs. The extended weak dual of a (p, q)-tiling has bounded maximum degree. Stars, on the other hand, do not have bounded maximum degree. That is, we show that graphs with unbounded degree cannot be partitioned into a graph with bounded degree while bounding the width of the partition.

We choose a star S with more than $(p \cdot (q-2)+1) \cdot f(2)$ vertices. Assume that S has a $\mathcal{H}_{p,q,}$-partition with $w_{\mathcal{H}}(S) \leq f(\omega(S))$ This means that we have a partition $\mathcal{P}$ of $V(S)$ with $H = S/\mathcal{P} \in \mathcal{H}_{p,q}$, and $w_H = (G) \leq f(\omega(G))$. Let v be the vertex in H that corresponds to the partition in $\mathcal{P}$ that contains the central vertex c of the star. All remaining vertices of the star are adjacent to c and can thus only be contained in the partitions that the neighbors of v correspond to. Each vertex in the extended weak dual of a (p, q)-tiling has at most $p \cdot (q-2)$ neighbors. This is because there are $q-1$ neighbors for every vertex of a tile. This includes the neighbors that are connected via edge-edges. We subtract 1 per vertex of the tiling to not count neighbors twice. Recall that stars have a clique number of 2. The parts of $\mathcal{P}$ corresponding to v and its neighbors can thereby contain $(p \cdot (q-2)+1) \cdot f(2)$ vertices. It is, therefore, not possible to distribute the $(p \cdot (q-2)+1) \cdot f(2)+1$ vertices of the star to the parts of $\mathcal{P}$ that correspond to v and its neighbors. ∎

Next, consider the case that p is constant and q grows. We show that the resulting graph class $\mathcal{P}_p$ of the extended duals does not have product structure. This automatically rules it out for Conjecture 3.7.

Lemma 4.5: *Let p be constant. For every c there is a $H \in \mathcal{P}_p$ such that $H \nsubseteq P \boxtimes T$ for every path P and every graph T with $\operatorname{tw}(T) \leq c$.*

Proof. Recall that $\mathcal{P}_p$ contains all subgraphs of the extended dual of (p, q)-tilings for growing q. Consider the vertex-edges in the extended dual of a (p, q)-tiling. Every tile that contains this vertex is connected with every other tile that contains the vertex. This results in a clique of size q in the extended dual. So the clique number of the extended dual of (p, q)-tilings increases with q. This is a problem for product structure. The high-level argument is that big cliques in the product can only be caused by big cliques in the factors.

We choose q larger than $2c$. Assume for the sake of contradiction that the extended dual of a (p, q)-tiling is the subgraph of $P \boxtimes T$ for some path P and some graph T with $\operatorname{tw}(T) \leq c$. Recall that the clique number of the weak extended dual of a (p, q)-tiling is at least q. Using Lemma 2.1 we see that the treewidth of T is at least $\operatorname{tw}(T) \geq \omega(G)/2 \geq q/2 > (2/2) \cdot c = c$. This is a contradiction to our assumption. ∎

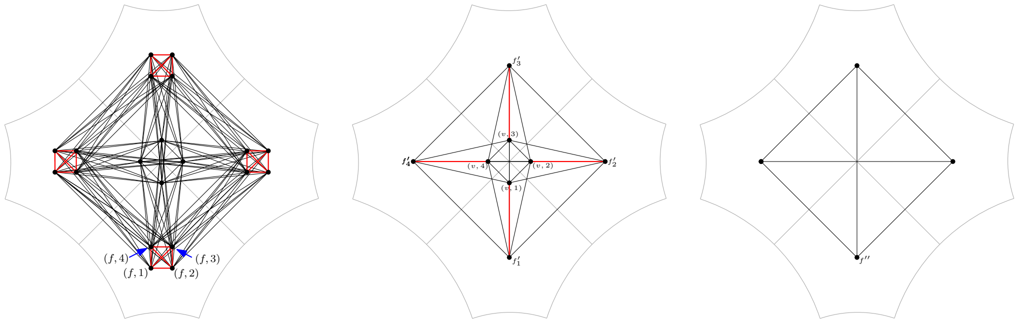

Figure 4.4: Left: F'; Middle: F' after reconstructing the face nodes; Right: extended dual (F'') of a $(5,4)$-tiling.

We have shown that a constant p and q and a constant p with a growing q are not suitable choices for Conjecture 3.7. We can rule out the third option $\mathcal{Q}_q$ only for geometric partitioning. We start by showing that $\mathcal{Q}_q$ admits product structure.

Lemma 4.6: *For every q $\mathcal{Q}_q$ admits product structure.*

Proof. Let $H \in \mathcal{Q}_q$ be the extended dual of a (p,q)-tiling. We want to show that H is a 2-shallow minor of a planar graph. As every planar graph is a subgraph of $P \boxtimes T$ for some path P and graph T with $\mathrm{tw}(T) \leq 6$ [UWY21], we can then use Lemma 2.2 to show that H is the subgraph of $P \boxtimes J \boxtimes K_{25 \cdot q}$ for some path P and some graph J with $\mathrm{tw}(J) \leq \binom{5+q}{6} - 1$. This proves that $\mathcal{Q}_q$ admit product structure as $J \boxtimes K_{25 \cdot q}$ has a constant upper bound on the treewidth that is only dependent on q.

It remains to show that H is the 2-shallow minor of a planar graph. For a (p,q)-tiling, construct the graph F where every face and every vertex of the tiling is a vertex of the graph. We call vertices that represent a face *face-nodes* and vertices that represent a vertex of the tiling *vertex-nodes*. There is an edge between a face-node and every vertex-node that it is connected to in the tiling. We can easily see that F is planar. Now, consider the graph $F' = F \boxtimes K_q$. We construct a 2-shallow minor F'' of F' and afterwards show that F'' is equal to H. We start by restoring the face-nodes. For a face-node f in F, contract the edges between the vertices $\{(f,k) \mid k \in [q]\}$. This can be done because the vertices in the set are a clique. Denote the resulting vertex with f'. We then distribute the vertices that resulted from a vertex-node. Let v be a vertex node and $f_1, \ldots, f_q$ the face-nodes that are adjacent to v. We then contract f_i' and (v,i) for $i \in \{1,\ldots,q\}$. See Figure 4.4 for an example of this step. The resulting vertex is called f_i''. We can confirm that F'' is a 2-shallow minor using Figure 4.5. It shows the subgraph that is contracted to form f''. The distance between two verties (v,i) and (u,i) is 2. The remaining vertices form a clique and have thus distance 1. After constructing F'' it remains to show that F'' is isomorphic to H. By construction, F'' contains no vertices other than f'' for every face f. The connections between tiles that share a vertex are created by contracting f' and the (v,i). This includes the connections between tiles that only share an edge because these tiles also share two vertices. No other edges are present in F''. This shows that F'' is indeed the extended dual of a (p,q)-tiling. ∎

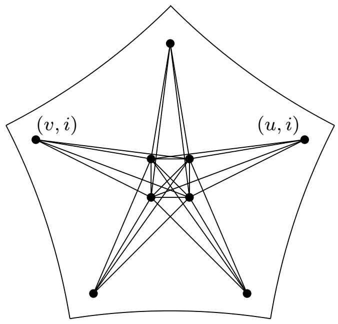

Figure 4.5: The subgraph of F' that is contracted to f'' in a $(5, 4)$-tiling.

So the graph classes $\mathcal{Q}_q$ remain a valid choice. We continue by exploring geometric partitioning for $\mathcal{Q}_q$. The trade-off between the number of cliques to cover a tile and the number of edges between tiles has already been discussed earlier. The number of cliques required to cover a tile is dependent on the threshold radius R and the size of a tile. We first explore this relation between the threshold radius and the size of a tile.

Lemma 4.7: *Let $C, q \in \mathbb{N}$ be constant. There is a $k \in \mathbb{N}$ such that for every (p, q)-tiling and every HUDG G with an embedding with a treshold radius R such that $r_{\text{in}}(p, q) \leq R/2 + C$, the vertices contained in a tile can be covered by at most k cliques.*

Proof. Recall that $r_{\text{in}} = \log(p) \pm \mathcal{O}(1)$. Let $C, q \in \mathbb{N}$. Let then $p \in \mathbb{N}$ be such that (p, q) is a hyperbolic tiling. Let G be a HUDG with an embedding such that $r_{\text{in}} \leq R/2 + C$.

We start by showing that $r_{\text{out}} = \log(p) \pm \mathcal{O}(1)$. For this, recall the transformations for arcosh from Lemma 4.3. We can use them to rewrite r_{out} as

$$\begin{aligned} r_{\text{out}} &= \log\left(\frac{\cos\left(\frac{\pi}{q}\right)\cos\left(\frac{\pi}{p}\right)}{\sin\left(\frac{\pi}{q}\right)\sin\left(\frac{\pi}{p}\right)}\right) + \mathcal{O}(1) \\ &= \log\left(\cos\left(\frac{\pi}{q}\right)\right) + \log\left(\cos\left(\frac{\pi}{p}\right)\right) - \log\left(\sin\left(\frac{\pi}{q}\right)\right) - \log\left(\sin\left(\frac{\pi}{p}\right)\right) + \mathcal{O}(1) \\ &= \log\left(\cos\left(\frac{\pi}{p}\right)\right) - \log\left(\sin\left(\frac{\pi}{p}\right)\right) \pm \mathcal{O}(1) \end{aligned}$$

The last transformation is valid because q is constant. We can then apply the transformation for $-\log(\sin(\pi/p))$ from *Lemma* 4.3 to further simply r_{out} to

$$\log\left(\cos\left(\frac{\pi}{p}\right)\right) + \log(p) \pm \mathcal{O}(1)$$

For $p \geq 3$ $\cos(\pi/p)$ is greater than $1/2$. So $\log(\cos(\pi/p))$ is between $\log(1/2)$ and 0. We can thus simply r_{out} further to $\log(p) \pm \mathcal{O}(1)$. So r_{in} and r_{out} differ at most by a constant value. We can use this to rewrite out requirement to $r_{\text{out}} \leq R/2 + C''$ for some constant C''.

We now show that a hyperbolic circle with radius $R/2 + C''$ can be covered by a constant number of cliques using the threshold radius R. For this, we use the inequality

$$\theta(r_1, r_2) \geq \sqrt{e^{R-r_1-r_2}} \tag{4.7}$$

shown by [BFKS21]. This inequality holds if (i) $R > 1$ (ii) $r_1, r_2 \in (0, R]$ (iii) $r_1 + r_2 \geq R$ (iv) $|r_1 - r_2| \leq R - 1$ We always use $r_1 = r_2 = R/2 + c$ for some $c \leq C$. We can easily see that for $R > 1$ requirements (i), (iii), and (iv) are always fulfilled. This is because $r_1 + r_2 = R/2 + c + R/2 + c = R + 2 \cdot c \geq R$ and $|r_1 - r_2| = |R/2 + c - R/2 - c| = 0 \leq R - 1$. We can also see that there is a $R' > 1$ such that every $R \geq R'$ requirement 2 is fulfilled. This is because for $4 \cdot C =: R'$ it holds that $R'/2 + c = 2 \cdot C + c \leq 3 \cdot C \leq R'$ ($c \leq C$). For any $R' > R$ this is true as well because C is a constant.

We will first handle the case that $R < R'$. There is only a finite amount of (p, q)-tilings for which this case can happen. For each (p, q)-tiling the minimum threshold radius such that an embedding of a HUDG can use this tiling is $2(r_{\text{in}}(p) - C)$. Let K_p be the number of circles with radius $2(r_{\text{in}}(p) - C)$ that are required to cover the whole tile. This is an upper limit on the number of cliques in a graph with threshold radius $2(r_{\text{in}}(p) - C)$ as every circle can be covered by one clique. Embeddings with a threshold radius larger than $(r_{\text{in}}(p) - C)$ can be covered by the same number of cliques. As we only have a finite number of (p, q)-tilings for which this happens, we can cover all these cases by at most $K = \max_p K_p$ cliques.

Next, we handle the case that $R \geq R'$. For this case, we use a similar proof to Lemma 8 in [BFKS21]. That is we cover a circle of size $r_{\text{out}} = R/2 + c'$ with circles of size R. The center of the circles with radius R will be on the boundary of the circle with radius r_{out}. We required at least $k = \pi/\theta(r_{\text{out}}, r_{\text{out}})$ circles with radius R to cover the central circle. Using Equation (4.7) we can rewrite $\theta(r_{\text{out}}, r_{\text{out}})$ to

$$\theta\left(\frac{R}{2} + c', \frac{R}{2} + c'\right) \geq \sqrt{e^{R-R/2+c'-R/2+c'}} = \sqrt{e^{-2 \cdot c'}} \tag{4.8}$$

Using this, we can approximate k by

$$k = \frac{\pi}{\theta\left(R/2 + c', R/2 + c'\right)} \leq \frac{\pi}{\sqrt{e^{-2 \cdot c'}}} = \pi e^{c'}$$

As every circle can be covered by 2 cliques [BFKS21], we can cover the circle with radius $R/2 + c'$ using $2k \leq 2\pi e^{c'} \leq 2\pi^{C'}$.

We can conclude that we can cover the vertices in a tile using $\max\{2\pi e^{C'}, K\}$ cliques. ∎

The bound in Lemma 4.7 is tight, i.e., replacing the constant with a non-constant function leads to the tiles not being coverable by a constant amount of cliques.

Lemma 4.8: *Let $q \in \mathbb{N}$ be constant. For every $C \in \mathbb{N}$ and function $f \in \omega(1)$ there is a (p, q)-tiling and a HUDG with an embedding with treshold radius R and $r_{\text{in}} = R/2 + f(R)$ such that the vertices in a tile cannot be covered by C cliques.*

Proof. We use a similar argument as Lemma 4.7. That is, we find a lower bound for the number of unconnected vertices in a tile that grows with the threshold radius R. This is also a lower bound for the number of cliques required to cover a tile. For this we use the inequality

$$\theta(r_1, r_2) \leq \pi\sqrt{e^{R-r_1-r_2}}. \tag{4.9}$$

shown by [BFKS21]. The inequality holds if $r_1, r_2 \in (0, R]$ and $r_1 + r_2 \geq R$.

Let $f \in \omega(1)$ and $r_{\text{in}} = R/2 + f(R)$. We know that there is a R' such that $f(R) \geq 0$ for every $R \geq R'$. In this case the second requirement of Equation (4.9) is fulfilled because $r_1 + r_2 = R/2 + f(R) + R/2 + f(R) = R + 2 \cdot f(R)$. If $r_{\text{in}} = R/2 + f(R)$ is greater than R we can place at least $(2\pi)/\theta(R, R)$. This grows with R because

$$\frac{2\pi}{\theta(R,R)} \geq 2\sqrt{e^R}.$$

if $R > R'$ but $R/2 + f(R) \leq R$ we can approximate the number of isolated vertices using

$$\frac{2\pi}{\theta(r_1, r_2)} \geq \frac{2}{\sqrt{e^{R-r_1-r_2}}} = \frac{2}{\sqrt{e^{-2\cdot f(R)}}} = 2e^{f(R)}.$$

∎

After exploring the coverability of a tile, we now deal with the reach of tiles. For this, we show that there are edges that can connect tiles with arbitrary distance. Recall that we called the process of partitioning a hyperbolic unit disc graph by laying a (p, q)-tiling ontop of an embedding geometric partitioning. We say the distance between two vertices u, v of a hyperbolic unit disc graph G in a (p, q)-tiling is the distance between the parts of the partition they are assigned to when using geometric partitioning.

Lemma 4.9: *Let $q \in \mathbb{N}$ and $C \in \mathbb{R}$ be constant. For every $d \in \mathbb{N}$ there is a $p \in \mathbb{N}$ such that there exists a HUDG G with an embedding with a threshold radius R with $r_{\text{in}} = R/2 - C$ such that geometric partitioning using a (p, q)-tiling leads to edges in G that have distance at least d in the extended dual of (p, q).*

Proof. We want to construct a graph G that contains only one edge. The vertices of the edge should have a distance of d

Recall that ℓ is the edge length of a (p, q)-tiling and the formular for ℓ in Equation (4.3). We show that $\lim_{p\to\infty} \ell \in \pm\mathcal{O}(1)$. Using the transformations of arcosh in Lemma 4.3, we can rewrite $\ell/2$ to

$$\log\left(\frac{\cos\left(\frac{\pi}{p}\right)}{\sin\left(\frac{\pi}{q}\right)}\right) + \mathcal{O}(1) = \log\left(\cos\left(\frac{\pi}{p}\right)\right) - \log\left(\sin\left(\frac{\pi}{q}\right)\right) + \mathcal{O}(1) = \log\left(\cos\left(\frac{\pi}{p}\right)\right) \pm \mathcal{O}(1).$$

The last transformation can be done because q is constant. We know that $\cos(\pi/p)$ approaches $\cos(0) = 1$ for growing p. This means that $\ell/2$ approaches $\log(\cos(0)) \pm \mathcal{O}(1) = \pm\mathcal{O}(1)$ for growing p. Thus $\lim_{p\to\infty} \ell = 2 \cdot (\pm\mathcal{O}(1)) = \pm\mathcal{O}(1)$. We know that ℓ is always non-neative as there is no negative edge length. So $\lim_{p\to\infty} \ell \in \mathcal{O}(1)$. Using this we know that there is a p such that $\ell(p) < 2(r_{\text{in}}(p) + C)/d$ because $\ell(p)$ approaches a constant and $r_{\text{in}}(p)$ is growing logarithmically. We now construct a graph that contains only one edge between two vertices that have distance d in the (p, q)-tiling. See Figure 4.6 for a visualization. We start by placing the first vertex v in the middle of an edge of the central tile. We then move to the second vertex using $2d$ steps. First, move to one of the corners incident to the edge. Then move to the middle of an edge that is neighboring two tiles with distance 1 to the center. Placing the second vertex there would result in a distance of 1 in the tiling. Again, move to the vertex at the opposite side and continue to an edge that is adjacent to two tiles with distance 2 to the center. We repeat this d times such that the distance is d in the tiling. We place our second

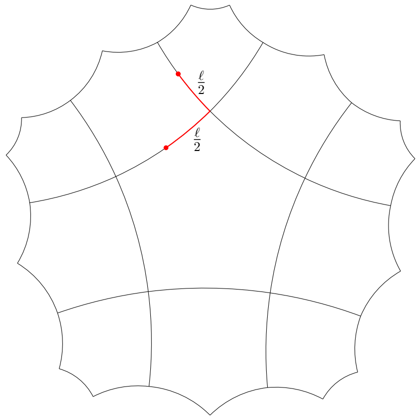

Figure 4.6: The first interation for constructing the edge of Lemma 4.9 in a $(5, 4)$-tiling.

vertex u at this place. The two vertices are connected since their hyperbolic distance is at most $d \cdot l < 2(r_{\text{in}}(p) + C) = R$, but their distance in the tiling is by construction d. We note that the following construction works for larger p as well because, again, $l(p)$ approached a constant and $r_{\text{in}}(p)$ is growing. ∎

An implication of Lemma 4.9 is that geometric partitioning using (p, q)-tilings with constant q and growing p is impossible. For any tiling choosen for geometric partitioning, we can construct an embedding of a hyperbolic unit disc graph that contains a path from Lemma 4.9. This path then connects partitions that are not connected in the extended dual of a (p, q)-tiling.

4.2 Arbitrary partitions using $\mathcal{Q}_q$

For partitions different from geometric partitions, we conjecture that $f(\omega)$-bounded $\mathcal{Q}_q$-partitions for hyperbolic unit disc graph are impossible as well. We will now outline a proof idea for the case $q = 3$. For this, we use the dual of $(p, 3)$-tilings and the paths constructed in Lemma 4.9. In order to use duals of $(p, 3)$-tilings we need to show that they are hyperbolic unit disc graphs. This is true for the duals of (p, q)-tilings in general. An example embedding would be to choose threshold radius $2 \cdot r_{\text{in}}$ and place a vertex at the center of each tile. Vertices in neighboring tiles are then connected because they have a distance of exactly $2 \cdot r_{\text{in}}$. Vertices in non-neighboring tiles are not connected as the shortest line between them covers at least r_{in} distance in both tiles and more than r_{in} in at least one of the tiles, otherwise they would be neighbors. So the distance between non-neighboring tiles in greater than $2 \cdot r_{\text{in}}$. Note that for $q = 3$, the dual and extended dual of a graph are the same graph. Recall that we defined tilings to have a unique embedding into the hyperbolic plane such that the center of one tile is at the origin. This tile is called the central tile the associated vertex in the extended dual is the central vertex. We say the dual with k layers of a (p, q)-tiling is the subgraph of the extended dual that contains all vertices with a distance up to k from the central vertex. The k-th layer in the extended dual are vertices with distance k to the central vertex.

The idea for showing that $f(\omega)$-bounded $\mathcal{Q}_q$-partitions for hyperbolic unit disc graph are impossible is as follows: We start by showing that for every f there is a dual of a $(p, 3)$-tiling G such that a partition $\mathcal{P}$ of G with $G/\mathcal{P} \in \mathcal{H}_{p',3}$ for any $p' \neq p$ has width greater than $f(\omega(G))$. This means that only a partition with a quotient in $\mathcal{H}_{p,3}$ would be possible. We then use the edges of Lemma 4.9 to shorten the distance between vertices in the extended dual of the $(p,3)$-tiling and cause the size of the parts of the partition to increase until they are greater than $f(\omega(G))$ as well.

For this, we look at the number of vertices n_k in the k-th layer. We easily see that $n_0 = 1$ and $n_1 = k$. For the third layer, we see that every tile in the second layer has the central tile and two tiles of the second layer as neighbors. This means $p-3$ neighbors are in the next layer. We thus have $p(p-4)$ vertices in the third layer when accounting for twice-counted tiles. In the third layer, we have two types of layers. The first type is the same as in the second layer. That is, they have $p-3$ neighbors. The second type is the tiles that were counted twice in the third layer. So, there are p tiles of this type. They have two neighbors in the previous layer and two neighbors in the current layer, thus only $p-4$ neighbors in the next layer. After removing twice counted tiles we can see that $n_4 = (p(p-4)-p)(p-4)+p(p-5) = p(p-4)(p-4)-p$. For the next layers, this continues. There are always n_{k-1} tiles with $p-4$ neighbors in the next layer and $n_k - n_{k-1}$ tiles with $p-3$ neighbors. Thus, the recursion for the number of vertices in a layer is $n_k = (n_{k-1} - n_{k-2})(p-4) + n_{k-2}(p-5) = n_{k-1}(p-4) - n_{k-2}$. This function grows exponentially with base depending on p because of the factor $(p-4)$ in the recursion.

For $p' < p$, the argument why for every f there is a dual of a $(p,3)$-tiling such that every partition $\mathcal{P}$ with $G/\mathcal{P} \in \mathcal{H}_{p',3}$ has width greater than $f(\omega(G))$ is that there are simply not enough vertices in the dual of a $(p',3)$-tiling. Note that the clique number of the dual of a $(p,3)$-tiling with more than 1 layer is 3. All vertices in the first k layers of the dual of a $(p,3)$-tiling need to be assigned to vertices in the first k layers of the dual of a $(p',3)$-tiling. This is because for a graph G, a partition $\mathcal{P}$ of G, two vertices u, v in $V(G)$ and the parts of the partition x, y that u and v have been assigned to, it holds that $d_{G/\mathcal{P}}(x,y) \leq d_G(u,v)$ where d is the distance in the respective graph. So let $\mathcal{P}$ be a partition of a dual of a $(p,3)$-tiling with k layers G such that $G/\mathcal{P} \in \mathcal{H}_{p',3}$. We know that if $n_k^*(p)/n_k^*(p') > c$ there is a part of the partition with a size greater than c. So if $n_k^*(p)/n_k^*(p') > f(3)$ for a k then a partition using the dual of a (p,q)-tiling is not possible for this function f. Using the approximation $n_k(p) \approx p^k$ we can approximate the number of vertices up to the k-th layer by $n_k^*(p) = (p^{k+1}-1)/(p-1)$ using a geometric series. When using the approximation $n_k(p) \approx p^k$ we can find a k for any pair p, p' with $p' < p$ and any function f such that $n_k^*(p)/n_k^*(p') > f(3)$.

For $p < p'$ we use a different argument. The k-th layer of the dual of a $(p,3)$-tiling is a circle with $n_k(p)$ vertices. The k-th layer of the dual of a $(p',3)$-tiling is, in this case, a circle with more than $n_k(p)$ vertices. So the k-th layer of the $(p,3)$-tiling cannot be embedded in the k-th layer of a $(p',3)$-tiling. Instead, it has to be embedded in a lower layer. Using our approximation this would have to be at most the $\log_{p'}(p^k)$-th layer. For sufficiently large k the first $\log_{p'}(p^k)$-th layers of the $(p',3)$-tiling should have less vertices than the k layers of the $(p,3)$-tiling. So, again, there are not enough vertices available in the $(p',3)$-tiling to accommodate the vertices of the dual of the $(p,3)$-tiling. This would show that a sufficiently large dual of a $(p,3)$-tiling can only be mapped to $(p,3)$-tiling.

Knowing that there is a dual of a $(p,3)$-tiling that can only be embedded in a $(p,3)$-tiling, we can add edges that reduce the distance in the graph between the vertices of the tiles. The edges will pull certain vertices to the center of the tiling and lead to vertices gathering near the center of the tiling. This can be done by using the edge from Lemma 4.9 with $d = 4$. It reduce the distance of a tile in the 4th layer to 3 (central vertex - first vertex of the path - second vertex of the path - vertex in 4th layer). Figure 4.7 shows a cutout of a $(p,3)$ with the added edge from Lemma 4.9 that reduces the distance between the gray vertex (4-th layer) and the central vertex (blue) to 3. By using larger d and adding multiple paths to the graph we

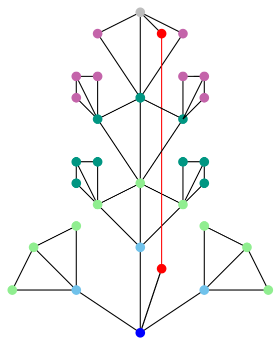

Figure 4.7: Cutout of a $(p, 3)$-tiling with the edge of Lemma 4.9 added. The blue vertex is the central vertex of the tiling. Different layers of the graph are shown in different colors. The edge of Lemma 4.9 is marked in red. The distance of the gray vertex to the center has been shortened to 3 because of the extra edge.

can move even more vertices near the center. In addition, we can iterate the process and add paths from the dth layer outwards. This causes parts of the graph that are far away from the center to be relatively close. The hope is that these paths add enough vertices to the center that do not fit into the partition. This concept does not extend to $q > 3$. This is because the dual and the extended dual differ in this case. When viewing the layers of the two graphs based on the distance, i.e., the BFS layers, we can see that the number of vertices in the dual grows slower than the number of vertices in the extended dual. This makes both arguments void. We still believe that there is only a limited range of (p', q)-tilings that a (p, q)-tiling of appropriate size can be mapped to.

5 Grids in strongly hyperbolic unit disc graphs

In the Introduction we discussed the implications product structure would have one grids in strongly hyperbolic unit disc graphs. If $f(\omega)$-bounded product structure were to be found for hyperbolic unit disc graphs large grids with constant clique number would not be hyperbolic unit disc graphs. In other words, construction a large grid with a constant clique number would disprove product structure in hyperbolic unit disc graphs. This chapter approaches the subject from this direction by constructing a grid with the clique number linear in the grid size.

For this, we use an alternative representation of strongly hyperbolic unit graphs. A weighted graph (also GIRG-representation) is a graph $G = (V, E)$ where $V \subset \mathbb{R} \times \mathbb{R}_+$ and $\{(x_u, y_u), (x_v, y_v)\} \in E$ if and only if $y_u \cdot y_v \geq |x_u - x_y|$. Every weighed graph can be embedded in the hyperbolic plane by using the mapping in [BKL19]. In terms of geometry, there is a wedge around every vertex that contains all the vertices this vertex is connected to. See Figure 5.1 for an example of these wedges.

Theorem 5.1: *There is a GIRG with a maximum clique of size ω (odd) that contains a grid of size $(\omega - 1) \times (\omega - 1)$ as a subgraph.*

Proof. We construct three columns of the grid, but more columns can be added easily. Vertices of the constructed graph are named $v_{i,j}$ where i is the row the vertex is in and j is the column. We note that the rows are not numbered from top to bottom but instead indicate order the which they are constructured.

Start the construction by placing three vertices $v_{1,k}$ such that they have the same weight, but only adjacent points have edges between them. Second place a vertex directly beneath $v_{2,2}$ at the intersection of the wedges of $v_{1,1}$ and $v_{1,3}$. See the upper row in Figure 5.1 for this step. Add vertices in the outer columns with the same weight. From now on we repeat the following steps until there are $\omega - 1$ vertices in each column.

1. Place a vertex $v_{i,2}$ above $v_{i-1,2}$ but beneath the intersection of the wedges of $v_{i-1,1}$ and $v_{i-1,3}$.
2. Place vertices $v_{i,1}$ and $v_{i,3}$ in columns one and three at the same height as $v_{i,2}$.
3. Place a vertex $v_{i+1,2}$ at the intersection of the wedges of $v_{i,1}$ and $v_{i,3}$ in column two.
4. Place vertices $v_{i+1,1}$ and $v_{i+1,3}$ in columns one and three at the same height as $v_{i+1,2}$.

(i is the number of already placed vertices in one column at the start of the iteration)
The columns of the grid are the columns we created. The rows of the grid can be seen by flipping every other column horizontally. By adding more columns we can expand the grid to the size of $\omega - 1 \times \omega - 1$. Concerning the clique number, we can easily see that all vertices in a column are a clique of size ω. Additionally, every vertex in column is connected to the first vertex in the neighboring columns. This increases the clique number to ω only as the first

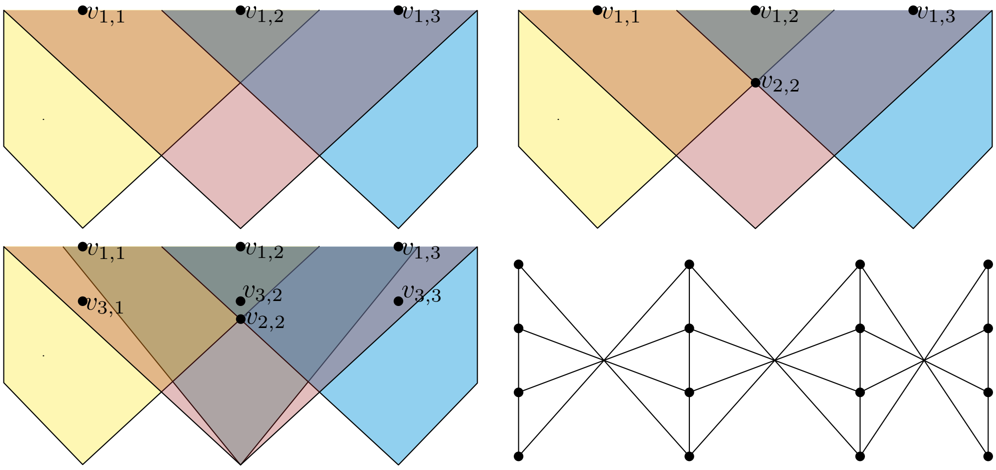

Figure 5.1: From left to right, top to bottom the first 3 steps of the construction. Note that some vertices and wedges are omitted for clarity. Bottom right: edges of the 4×4 grid.

vertices in the neighboring columns are not connected with each other. When considering only the upper $\omega - 2$ vertices of a column, we see that they form a clique with the first two vertices of the neighboring column. So this does not form a clique larger than ω. For any other set of vertices this is the same. ∎

6 Conclusion

The focus of this thesis was on hyperbolic tilings and their possible use as partitions for hyperbolic unit disc graphs. The most promising of our three graph classes based on tilings was $\mathcal{Q}_q$. This graph class contains all extended duals for (p, q)-tilings with constant q and growing p. Note that for $q = 3$ this is equivalent to the dual. For $\mathcal{Q}_q$ we showed that geometric partitioning is not possible. We did not manage to show that $\mathcal{Q}_q$ can not be used for arbitrary $f(\omega)$-bounded partitions. So this remains an interesting question. For the special case of $q = 3$, we proposed a proof idea in Chapter 4. An obvious question for further research would be if hyperbolic unit disc graphs have $f(\omega)$-bounded $\mathcal{Q}_q$-partitions for any function f and any q.

Another direction of research could be to use different hyperbolic tilings. We focused on regular (p, q)-tilings of p-gons where q p-gons meet at one corner. But there are more hyperbolic tilings that use multiple geometric figures or different patterns [Goo05 | DG21]. One could consider these tilings and their (extended) duals for (geometric) partitioning of hyperbolic unit disc graphs.

A third option would be to consider different limitations for f. Our definition of $f(\omega)$-bounded $\mathcal{H}$-partition does not limit the function f at all. A linear function f, for example, would mean that every part of the partition can be covered by a constant number of cliques. Our proofs, however, would still work with this limit. For interesting results, we would therefore have to introduce a lower bound for $\omega(G)$, e.g., $\omega(G) \in \Theta(\log(n))$. This would invalidate our proofs as we use counterexamples with constant clique number.

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