Karlsruhe Institute of Technology

On Rigidity of Graphs

Bachelor Thesis of

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Statement of Authorship

I hereby declare that this thesis has been composed by myself and describes my own work, unless otherwise acknowledged in the text.

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Abstract

A *rigid framework* refers to an embedding of a graph, in which each edge is represented by a straight line. Additionally, the only continuous displacements of the vertices, that maintain the lengths of all edges, are *isometries.* If the edge lengths are allowed to only be perturbed in first order, the framework is called *infinitesimally rigid.*

In this thesis the degrees of freedom of a certain, new type of rigidity, called *edge-x-rigidity*, for a given framework in the 2-dimensional Euclidean space are examined. It does not operate on the length of an edge, but on the intersection of an edge's support line and the x-axis. This may then help to determine the class of graphs where every edge intersection with the x-axis can be repositioned along the x-axis, such that there still exists a framework that satisfies these position constraints.

Deutsche Zusammenfassung

Ein *starres Fachwerk* bezeichnet eine Einbettung eines Graphen - bei der jede Kante durch ein gerade Linie repräsentiert wird - und es keine stetige *nicht-triviale* Verschiebung der Knoten gibt, welche die Längen aller Kanten gleich lässt. Dürfen sich die Kantenlängen jedoch um einen hinreichend kleinen Betrag verändern, so spricht man von einem *infinitesimal starrem Fachwerk.*

In dieser Arbeit werden die Freiheitsgrade eines neuen, speziellen Starrheitsbegriffs, genannt *Kanten-x-Starrheint*, für ein gegebenes Fachwerk im 2-dimensionalen Euklidischen Raum untersucht. Hier dienen die Schnittpunkte der Stützlinien von Kanten mit der x-Achse, statt den Kantenlängen, als Invariante. Dies soll dann genutzt werden um die Klasse der Graphen zu klassifizieren, deren Kantenschnittpunkte mit der x-Achse beliebig neupositioniert werden dürfen, sodass immer noch eine Fachwerk-Einbettung gefunden werden kann, die diese Positionsbeschränkungen erfüllt.

Contents

1. Introduction

Although Augustin Cauchy [AZ01] and James Maxwell [Max70] introduced the mathematical theory of structural rigidity in mid 19th century, this concept only gained popularity in the last fifty years after the work of Laman. He gave a combinatorial characterization for a certain class of rigid graphs, later to become known as the *Laman graphs.* Although the Austrian mathematician Hilda Geiringer gave a characterization similar to Laman's almost forty years earlier, her work has unfortunately not received enough attention. Since then, rigidity theory has been thoroughly studied. Based on this and recent work of Dujmović et al. [DFG+18], in the scope of this thesis a new type of rigidity is introduced to deduce properties of a certain class of graphs.

A *framework*, alternatively also called *realization*, consists of a graph $G = (V, E)$ and a map p, which maps every vertex of G to a point in $\mathbb{R}^d$. It is important to note, that a framework is not the same as an embedding in the usual manner. The only restrictions that a framework needs to fulfil is, that two adjacent vertices must not share the same position in $\mathbb{R}^d$. Particularly this means, that edges may cross or overlap and two vertices, that are not joined by an edge, may share the same location in $\mathbb{R}^d$.

Figure 1.1 provides a basic example of $K_{2,2}$ realized as a flexible unit square, which can be continuously transformed to a parallelogram where every edge has the same length as the corresponding edge in the original realization. Adding a diagonal edge, restricts the degrees of freedom in such a way, that only rotations and transformations, i.e. the trivial motions, may be applied to the framework as length-preserving continuous transformations. Therefore, this framework is rigid.

(a) Continuous displacement of the unit square to a parallelogram.

(b) $K_{2,2}$ with an additional diagonal edge resulting in a rigid framework.

Figure 1.1.: Basic example for rigidity.

In order to refine the term rigidity, the terms *length-preserving*, *isometry* and rigidity's complement, *flexibility*, are introduced. A map $T\colon U \to W$ is called *length-preserving*, if

$\|Tx - Ty\|_W = \|x - y\|_U$ holds for all $x, y \in U$. The *isometries* of $\mathbb{R}^d \to \mathbb{R}^d$ are exactly the length-preserving maps of $\mathbb{R}^d \to \mathbb{R}^d$. A framework is called *flexible*, if there is a continuous family of length-preserving maps $\left\{ p(i) \in \mathbb{R}^d \,\middle|\, i \in V \right\} =: p(V) \to \mathbb{R}^d$, that can not be extended to isometries. Otherwise, the framework is called *rigid*. Here, isometries refer to the space of *trivial motions* as they can be applied to any framework, maintaining the distances between any two points and in particular the length of each edge. In other words, a rigid framework admits only trivial motions as continuous length-preserving displacements of its vertices.

In this thesis, the first aim is to give a profound introduction in rigidity theory with its many kinds of different rigidity constraints and summarize important results. While the constraints for the different types of rigidity may appear unrelated, most of the concepts introduced are basically equivalent or more precisely dual. Next, an own set of constraints is introduced defining a new type of rigidity with the aim of establishing an equivalence to the well-known rigidity types.

1.1. Motivation and Application

Rigidity theory found its way into many different scientific fields with a large amount of corresponding applications. Among other examples, rigidity theory is applied within the field of chemistry, more precisely in the determination of molecular conformation, as the so called $\mathcal{NP}$-hard *molecule problem* [Hen95a]. For the molecule problem, a set of elements with unknown locations in three-dimensional Euclidean space is given and the task is to retrieve the locations of the objects relatively to each other, with only information on some subset of their pairwise distances [Hen95a].

A problem similar to the molecule problem is known as *network localization*, within the field of networking and communication. It relates to a set of nodes, some of which know their own locations while others need to calculate their locations by only knowing their distances to their neighbours. The main question is, when is such a localizability unique and, additionally, how to construct localizable networks that are unique [AEG+06].

The field of control theory presents a completely new application concept. This scenario considers *mobile agents* and its formations. The goal is to define a type of rigidity that maintains the shape of a given formation. The crucial point is that the formation must be maintained while it is in motion. For example, in the *leader-follower approach*, there is one Leader agent that dictates the trajectory of movement and every other agent (*follower agent*) must follow in such a way that the formation is preserved [Ere12].

Finally, the application of *statical rigidity* in structural engineering and statics should be mentioned, where external forces act on every vertex of the structure. Here the interest lies in those frameworks, which can resolve all these external forces by arising internal forces, i.e. at every vertex sum of internal and external forces is zero [Rot81; Whi96; Izm09a].

1.2. Scope of this thesis

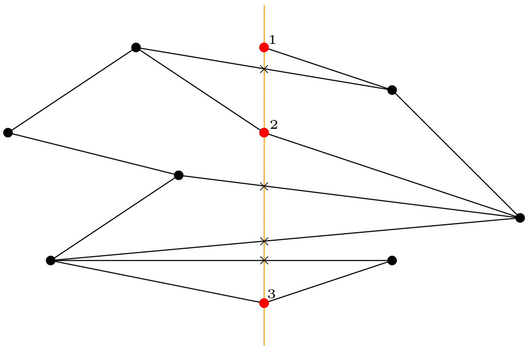
The question of this thesis is shortly introduced in the following. It is related to the concept of *free* and *(free) collinear sets*, which are described in more detail in Section 4.2, and the work of Dujmović et al. In particular, Dujmović et al. [DFG+18] have solved an open problem and in Chapter 4 it is tried to extend their statement by using the extensively studied theory of rigidity.

A *plane straight-line drawing* is an embedding of a graph $G = (V, E)$, such that no two vertices coincide and no two edges intersect. A subset $S \subseteq V$ is a *collinear set*, if there is a plane straight-line embedding, such that all vertices in S lie on one same line l. Furthermore, a subset $S \subseteq V$ is called a *free set*, if for every point set $X \subseteq \mathbb{R}^2$ of cardinality $|S|$, there exists a one-to-one map $p\colon S \to X$ and a plane straight-line embedding of G where every vertex $v \in S$ is placed at the position $p(v)$. Analogously are *free collinear sets* $S \subseteq V$ defined, where S is a collinear set and free for sets X of collinear points.

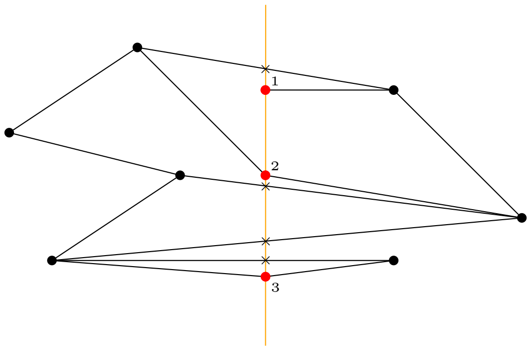
Examples for applications of free sets are, as given by Dujmović [Duj17],

- *column planarity*,
- *universal point subsets* and
- *(partial) simultaneous geometric embeddings.*

Dujmović et al. have proven, that "Every collinear set is a free collinear set". To refine this, they have shown, that, given a plane straight-line drawing with collinear $S \subseteq V$ on line l, the vertices can be arbitrarily displaced on l, not changing order along l, while the edge intersections may be prescribed *ε-precisely.* That is, every crossing point is at most ε away from its prescribed location. The example provided in Figure 1.2 describes a set of free collinear points, red vertices, with respect to the line in orange. The red vertices in Figure 1.2a were moved along the orange line and the position of the edge intersections could be prescribed absolutely, yielding the drawing in Figure 1.2b. Take note, that this hard prescription may not be possible in general.



(a) Original plane straight-line drawing.



(b) Resulting plane straight-line drawing after displacing the vertices in red.

Figure 1.2.: Visualization of free collinear sets for a basic plane straight-line drawing.

The question that forms the basis of this work: "may those crossing points be prescribed absolutely?". Hence, in this thesis a new type of rigidity is introduced in order to characterize those graphs where these intersections may not be changed, what is called *edge-x-rigidity*. Subsequently, the newly introduced concept of edge-x-rigidity is used to classify those graphs able to change the crossing point arbitrarily for every edge, called *edge-x-free* graphs. All indications are, that *infinitesimal edge-x-rigidity* is *projectively equivalent* to *infinitesimal rigidity*, which is well studied. In fact, in the theory of infinitesimal rigid frameworks, there exists a characterization of a particular class of infinitesimally rigid graphs, called *minimally infinitesimally rigid* graphs. These minimally infinitesimally rigid graphs are precisely the Laman graphs and establishing a projective equivalence between infinitesimal edge-x-rigidity and infinitesimal rigidity would yield this characterization for infinitesimal edge-x-rigidity as well. That is, the *minimally infinitesimally edge-x-rigid* graphs would exactly be the Laman graphs. Here, analogously, *infinitesimal edge-x-rigid* allows the crossing point to change in first-order. Furthermore, *minimally* means, that deleting any edge of the framework, but no vertices, would result in a flexible framework. Deleting such an edge e would then also yield a non-trivial continuous displacement Φ of the vertices. Applying Φ to the framework and reinserting the edge e afterwards, changes the crossing point of e. If the intersection can be moved arbitrarily, the minimally infinitesimal edge-x-rigid graphs are particularly edge-x-free.

1.3. Outline

In the upcoming chapter, Chapter 2, the necessary terms and notions for the subsequent chapters are defined. Here the terms (a, b)-*sparsity*, (a, b)-*tightness* and *algebraical dependence* which are frequently used in the course of this work must be highlighted.

In Chapter 3 different types of rigidity are introduced starting with *(continuous) rigidity* that arises in a somehow natural way. Second, a local version of continuous rigidity, which is called *infinitesimal rigidity*. Next, two further types of rigidity are considered that basically are different views on the model of infinitesimal rigidity. In other words, these three concepts are essentially equivalent. Last, *generic rigidity* then establishes a link between continuous and infinitesimal rigidity, which shall be applied in the final chapter, Chapter 4.

There, a new system of constraints is introduced in order to define a new type of rigidity, called *edge-x-rigidity*, regarding the intersection points of edges' supporting lines and the x-axis.

2. Preliminaries

This chapter provides the formal background required to fully understand the concept behind *rigidity* of graphs.

Throughout this thesis graphs are assumed to be finite, simple and undirected, whereas planarity is not a prerequisite. In particular, a graph G shall be denoted by an ordered pair of finite sets $V = \{1, \ldots, n\}$ and $E \subseteq \{\, \{u, v\} \mid u, v \in V \,\}$, namely *vertices* and *edges*, i.e. $G = (V, E)$. Occasionally the concept of *vertex-induced subgraphs* of a graph $G = (V, E)$, denoted by $G[V'] = (V', E[V'])$, is used. V' denotes an arbitrary subset of V and $E[V'] \coloneqq \{\, \{i, j\} \mid i, j \in V' \,\}$ is the set of edges where every endpoint lies in V'. Every subgraph $G[V']$ with $|V'| =: k$ is called a *k-subgraph* of G. If for every $2 \leq k \leq |V|$ all k-subgraphs $G[V']$ of G satisfy $|E[V']| \leq a|V'| - b$, G is said to be *(a, b)-sparse*. Additionally, if G is (a, b)-sparse and $|E| = a|V| - b$, G is called *(a, b)-tight*. There are two further graph properties, or rather classes of graphs, that are of importance for this thesis. Namely, these are the *complete graph on x-vertices* K_x and the *complete bipartite graph* $K_{x,y}$.

$$
\begin{aligned}
K_x &\coloneqq (\{1, \ldots, x\}, \{\, \{i, j\} \mid i \neq j \in \{1, \ldots, x\} \,\}), \quad x \in \mathbb{N}. \\
K_{x,y} &\coloneqq (V = V_1 \dot{\cup} V_2, \{\, \{i, j\} \mid i \in V_1, j \in V_2 \,\}), \quad x, y \in \mathbb{N}.
\end{aligned}
$$

Here $|V_1| = x$, $|V_2| = y$ and $|V| = x + y$ hold. In other words, K_x is the graph on x vertices where each vertex is connected to every other vertex. For $K_{x,y}$, V is separable into two disjoint subsets, V_1 and V_2, where each vertex of one subset is connected to the vertices of the other subset.

A *realization* or *framework* of a graph G refers to a map $p : V \to \mathbb{R}^d$ that maps every vertex of G to a point in the *Euclidean space* $\mathbb{R}^d$, such that two vertices *joined by an edge* do not lie on the same position, where $d \in \mathbb{N}$ refers to an arbitrary dimension. Another important difference between a framework and an embedding of G is, that for the former every edge is given as a straight-line segment between the incident vertices. Such a framework is denoted by (G, p) or $G(p)$. Moreover, p_i describes the position of vertex i in $\mathbb{R}^d$ given by the map p and $\|\cdot\| : \mathbb{R}^d \to [0, \infty), (x_1, \ldots, x_d) \mapsto \sqrt{\sum_{i=1}^d x_i^2}$ denotes the *Euclidean norm*.

The *Euclidean scalar product* is given by $\langle \cdot, \cdot \rangle : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d; \langle x, y \rangle \coloneqq x^\top y = \sum_{i=1}^d x_i \cdot y_i$ where x_i and y_i denote the i-th entry of x and y, respectively, for all $x, y \in \mathbb{R}^d$. Take note, that the Euclidean norm is induced by the Euclidean scalar product, i.e. $\sqrt{\langle x, x \rangle} = \|x\|$, for all $x \in \mathbb{R}^d$.

A *vector space* V over the *scalar field* $\mathbb{K}$ (also $\mathbb{K}$-*vector space*) is a set, whose elements shall be called *vectors*, with an *addition* $+ : V \times V \to V$ and *(scalar) multiplication* $\cdot : \mathbb{K} \times V \to V$ *associative*, *distributive* and satisfying $1_{\mathbb{K}} \cdot v = v$ for all $v \in V$ where $1_{\mathbb{K}}$ denotes the neutral element of the field $\mathbb{K}$ (see Beutelspacher [Beu14]). The associative and distributive property state that $\alpha \cdot (\beta \cdot v) = (\alpha \cdot \beta) \cdot v$ and $\alpha \cdot (u + v) = \alpha \cdot u + \alpha \cdot v$ and as well $(\alpha + \beta) \cdot v = \alpha \cdot v + \beta \cdot v$ for all $\alpha, \beta \in \mathbb{K}, u, v \in V$, respectively. The Euclidean space $\mathbb{R}^d$ is nothing else than the vector space $V = \mathbb{R}^d$ over the field $\mathbb{R}$, with component wise addition and scalar multiplication, equipped with the Euclidean scalar product. Sometimes $\mathbb{E}^d$ shall denote the Euclidean space of dimension d, instead of $\mathbb{R}^d$. Furthermore, take note that the scalar field $\mathbb{K}$ always induces a vector space.

The term of *dual space* or *dual vector space* V^* of a corresponding vector space V will be of need as well. Thus, the notion *linear map* or *linear transformation* is introduced for two vector spaces V and W, over the same field $\mathbb{K}$, given through a map $f : V \to W$ fulfilling the *additivity* and *homogeneity* property, i.e. $f(u+v) = f(u) + f(v)$ and $f(\alpha \cdot v) = \alpha \cdot f(u)$ for all $\alpha \in \mathbb{K}, u, v \in V$, respectively. In the case of W being equal to $\mathbb{K}$, f is said to be a *(linear) functional* of V. Then V^* is said to be the set of all functionals of V and in particularly V^* forms a vector space where the addition and multiplication means the point-wise defined addition and multiplication on functions. That is, e.g. for addition, for any two functionals f and g of V: $(f+g)(x) \coloneqq f(x)+g(x)$, for all $x \in V$ (see Beutelspacher [Beu14]).

The *kernel* $\ker(\cdot)$ of a matrix $A \in \mathbb{R}^{n \times m}$, with $n, m \in \mathbb{N}$, subsists of the set of all $x \in \mathbb{R}^m$ satisfying $A \cdot x = 0$. Furthermore, $\operatorname{rank}(\cdot)$ of a matrix $A \in \mathbb{R}^{n \times m}$ refers to the dimension of the image space of A considered as linear transformation f of $\mathbb{R}^m$ to $\mathbb{R}^n$, i.e. $\operatorname{rank}(A) = \dim(\operatorname{Im}(f))$, namely the *rank* of the matrix A.

Given a set of vectors $x_1, \ldots, x_n \in \mathbb{R}^d, n \in \mathbb{N}$ fulfilling $\sum_{i=1}^{n} \lambda_i \cdot x_i = 0$, with $\lambda_i \in \mathbb{R}$ not all zero, $x_1, \ldots, x_n$ are called *linearly dependent* and *linearly independent* otherwise. Particularly, $x_1, \ldots, x_n$ are said to be *affinely independent*, if $x_2 - x_1, \ldots, x_n - x_1$ are linearly independent.
Given a vector $y = (y_1, \ldots, y_d)^\top \in \mathbb{R}^d$, $(y)_i \coloneqq y_i$ denotes the i-th entry of y, for $i \in \{1, \ldots, d\}$.

In the following occasionally also the concept of an *isometry* appears, which is a bijective map Φ between $\mathbb{R}^d$ and $\mathbb{R}^d$ leaving distances between any two points invariant, i.e. $\|\Phi(x) - \Phi(y)\| = \|x - y\|$, for all points $x, y \in \mathbb{R}^d$. The set of all isometries of $\mathbb{R}^d$ is denoted by $\operatorname{Iso}(\mathbb{R}^d)$. One can show that every isometry $\Phi \in \operatorname{Iso}(\mathbb{R}^d)$ has the following (unique) form: $\Phi(x) = Ax + b$, for some $A \in \mathrm{O}(d) \coloneqq \{A \in \mathbb{R}^{d \times d} \mid A^\top A = I_d = AA^\top\}$ and $b \in \mathbb{R}^d$. $\mathrm{O}(d)$ is called the *orthogonal group*, see Izmestiev [Izm09a].

Two spaces V and W are said to be *canonically isomorph* or *naturally isomorph* if there exists a *canonical map* that is a bijective homomorphism, i.e. isomorphism. Written $V \simeq W$, which defines an equivalence relation. A *canonical map* is a map that arises in a somehow naturally meaning from the construction or definition. As stated by Beutelspacher in [Beu14], there is a natural isomorphism between a finite-dimensional vector space V and its double-dual space $V^{**} \coloneqq (V^*)^*$ in such a way, that no fixed bases for construction are needed:

$$\varphi : V \to V^{**},$$
$$\varphi(v) \coloneqq v^{**},$$

with $v^{**}(f) \coloneqq f(v)$.

Furthermore, a set $A = \{\alpha_1, \ldots, \alpha_m\}$ of distinct real numbers is said to be *algebraically dependent*, if there exists a polynomial $h(X_1, \ldots, X_m) \in \mathbb{Z}[X_1, \ldots, X_m]$, non-identical to zero, such that $h(\alpha_1, \dots, \alpha_m) = 0$. If A is not algebraically dependent, it is called *generic* (see Connelly [Con05], Whiteley [Whi96]). Hereby $\mathbb{Z}[X_1, \dots, X_m]$ denotes the *multivariate polynomial ring* with coefficients in $\mathbb{Z}$. Hence, its elements are exactly the polynomials in m variables with integer coefficients, i.e.

$$\mathbb{Z}[X_1, \dots, X_m] \coloneqq \left\{ \sum_{k=(k_1,\dots,k_m)\in\mathbb{N}_0^m} \lambda_k X_1^{k_1} \cdot \ldots \cdot X_m^{k_m} \;\middle|\; (\lambda_k)_{k\in\mathbb{N}_0^m} \subset \mathbb{Z}^{\mathbb{N}_0^m} \right\}.$$

Finally, $\frac{d}{dt} f(t) = f'(t)$ denotes the derivative of a function f to t, whereas the derivative of f to t at a point a is given by $\frac{d}{dt} f(t)\Big|_{t=a} \coloneqq f'(a)$.

3. Concepts of rigidity

The main interest of this chapter lies in giving an introduction to different types of rigidity, for which a strong relation shall be presented, beginning with the *length preserving* concept of rigidity.

By *length preserving* the lengths of edges of a particular framework are demanded to remain unchanged under motions of the whole framework as a "rigid" body. If the only motions that fulfil these criteria are *trivial*, i.e. isometries, such a framework is called *rigid* and otherwise *flexible*. In a next step, further interest consists of examining that concept, where edge-lengths may change in first-order. Hence, introducing a local version of rigidity so-called *infinitesimal rigidity*. For this notion there exist certain criteria to deduce whether a given realization is infinitesimally rigid or not. If a framework is not infinitesimally rigid, it is called *infinitesimally flexible*.

Another mentionable concept of rigidity form the so called *parallel designs*. Parallel means, that transformations of frameworks need to keep the slope of every edge fixed. As earlier, the *trivial* transformations, i.e. *dilation* and *translation*, of frameworks play a major role.

From there on the term *statically rigid* is treated, which is mostly relevant in statics. At this point the framework is additionally equipped with a so called *dependence* which is actually nothing else than assigning forces to every edge. A framework is then statically rigid, if it can "resolve all the permitted external loads" [Whi96]. Finally, it shall be seen that the latter three concepts (infinitesimal rigidity, parallel designs and statical rigidity) are essentially equivalent or more precisely dual.

An important result provides, that every infinitesimal rigid framework is particularly rigid. Therefore, the following question comes up: "how to treat the other direction?". This then yields the concept of *generic rigidity* in order to establish an equivalence under certain restrictions.

3.1. Rigidity

Definition 3.1 (Izmestiev [Izm09a]).
A motion[1] *of* $\mathbb{R}^d$ *is a continuous family* Φ_t *of isometries of* $\mathbb{R}^d$*, such that* Φ_0 *is the identity. Formally, a motion of* $\mathbb{R}^d$ *is a continuous map*

$$\begin{aligned}[0,1] &\to \mathrm{Iso}(\mathbb{R}^d),\\ t &\mapsto \Phi_t,\end{aligned}$$

such that $0 \mapsto \mathrm{id}$*.*

Definition 3.2 (Izmestiev [Izm09a]).
A motion[2] *of a framework* $P = G(p)$ *is a continuous family of frameworks* $P(t) = (G, p(t))$ *for* $t \in [0,1]$*, such that* $P(0) = P$ *and*

$$\|p_i(t) - p_j(t)\| = \|p_i - p_j\|\,, \tag{3.1}$$

for all $\{i,j\} \in E$ *and for all* $t \in [0,1]$*.*

As stated by Izmestiev [Izm09a] that is, each point p_i moves along a trajectory $p_i(t)$ so that the distances between points joined by an edge are preserved.

Here, P denotes the original framework and $P(t)$ its motion. The additional parameter t should diffuse any ambiguity.

Definition 3.3 (Izmestiev [Izm09a]).
A motion $\{P(t)\}_{t\in[0,1]}$ *of a framework* P *is called* trivial*, if it is induced by a motion of* $\mathbb{R}^d$*:*

$$P(t) = \Phi_t \circ P, \quad \forall t \in [0,1]\,,$$

for some motion $\{\Phi_t\}_{t\in[0,1]}$ *of* $\mathbb{R}^d$*.*

The framework given in Figure 3.1 (straight black lines) is rotated by 45° about the origin, given as the blue cross, and yields the framework given through the dashed lines. In this scenario a trivial motion is applied, since rotations are part of the group of isometries. The trajectory of this particular trivial motion is given by the red arrows. That is, moving the vertices of the given framework along these curves, yields a motion of the respective framework that is trivial in particular.

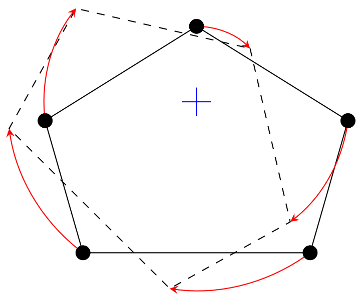


Figure 3.1.: Motion of a framework given by a rotation by 45° about the origin.

Definition 3.4 (Izmestiev [Izm09a]).
A framework is called rigid[3]*, if all of its motions are trivial. A framework is called* flexible *if it is not rigid.*

[1] or also *(continuous) flex* of $\mathbb{R}^d$.
[2] or also *(continuous) flex* of a framework P.
[3] sometimes this type of rigidity is referred to by *canonical rigidity*, *natural rigidity* or as well *continuous rigidity*, in order to distinguish more clearly between the large amount of different types of rigidity.

To conclude this section a variety of examples for rigid and flexible frameworks shall be presented. Starting with basic rigid frameworks in Figure 3.2. Those frameworks look like they were constructed, which is actually the fact, as shall be seen in Section 3.5. Those construction steps are called *Henneberg constructions* (see Theorem 3.30). Furthermore, these graphs belong to a certain class of graphs, the *Laman graphs*, which shall be examined in further detail in Section 3.5 as well.

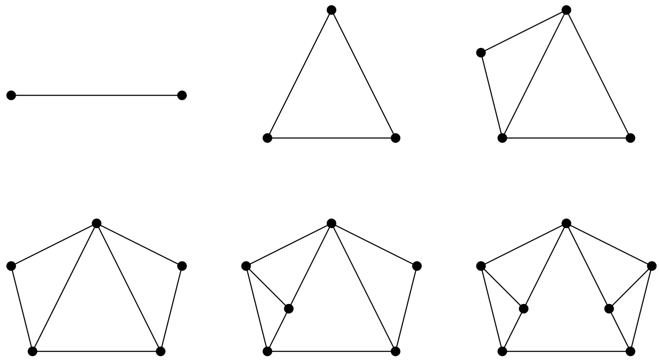


Figure 3.2.: Basic rigid frameworks.

After providing examples of basic rigid frameworks, more complex rigid frameworks are presented in a further step, see Figure 3.3. Here it is not as easy to see, that there actually does not exist any non-trivial motion, but in the upcoming sections useful criteria to analyse rigidity of certain frameworks shall be provided.

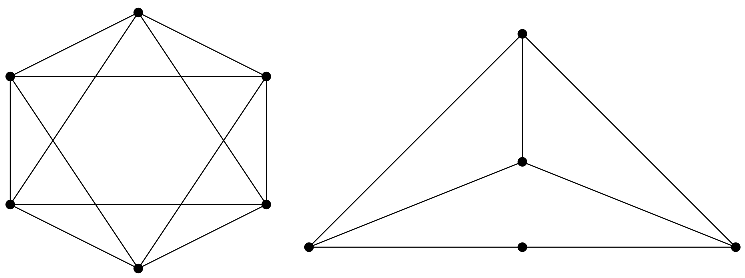


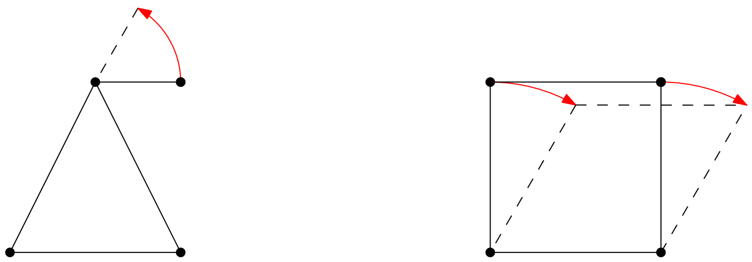
Figure 3.3.: More complex rigid frameworks [Izm09a].

Contrastly, Figure 3.4 depicts two basic flexible frameworks. On the left-hand side, see Figure 3.4a, is a triangle with a bridge, that can be arbitrarily rotated, indicated by the red arrowed curve. Moving the node continuously along this trajectory presents a non-trivial motion of the framework and therefore the framework can not be rigid.

Furthermore, on the right hand side, see Figure 3.4b, a realization as the unit square is given where the upper two nodes may be moved along the curves given in red and therefore, here again, a non-trivial motion is obtained. That is, the unit square has been transformed to a parallelogram with the length of the sides being equal to the corresponding sides of the unit square. This proves the flexibility of the unit square.

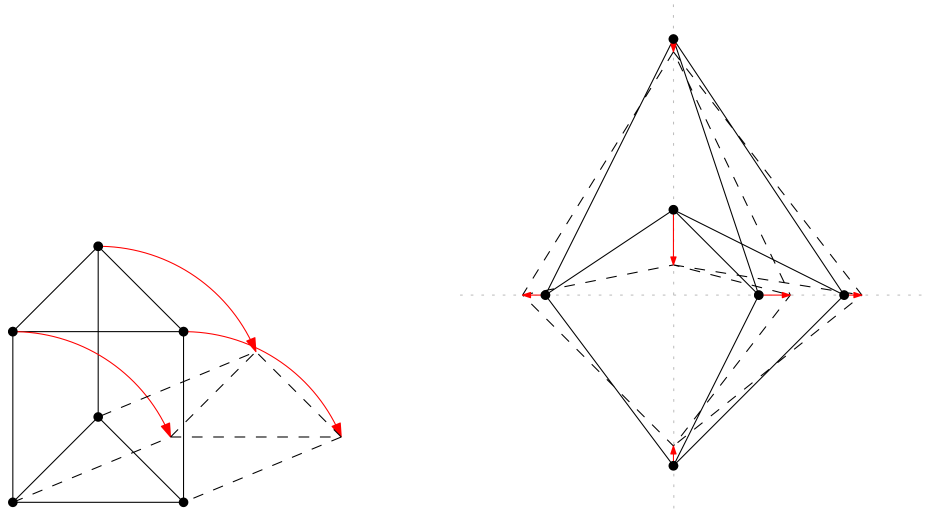
Finally, two complex flexible frameworks, as given by Laman [Lam70], are depicted in Figure 3.5. Here applies the same notation as above in the case of Figure 3.1, i.e. moving the vertices uniformly and continuously along the red trajectories yields a non-trivial motion of the framework. The transformed frameworks are given through the dashed lines and also straight lines if the respective vertices have not been transformed.

The up-following presents a concrete motion of the framework on the right-hand side, see Figure 3.5b, as given by Laman [Lam70]. Take note, that the underlying graph of the particular framework is $K_{3,3}$. Later it shall be seen, that $K_{3,3}$ belongs to a special class of



(a) Motion of a triangle with a bridge. (b) Motion of the unit square to a parallelogram.

Figure 3.4.: Basic flexible frameworks.



(a) A continuous motion similar to a rotation. (b) Stretching and squeezing of a framework.

Figure 3.5.: Complex flexible frameworks [Lam70].

graphs, where almost every realization yields a (*infinitesimally*) rigid framework. So the question that now appears is "why is this framework flexible?". This is due to the fact, that the points $1, 3, 5$ and $2, 4, 6$ are collinear - indicated by the two lightgray dotted lines - and thus form a so called *singular* realization of the respective graph. This shall be examined in further detail in Section 3.2 and Section 3.5.

The corresponding graph is then given by $G = (V, E)$ with $V = \{1, 2, 3, 4, 5, 6\}$ and $E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 6\}, \{7, 1\}, \{2, 5\}, \{3, 6\}\}$, whereas the realization p is given by

$$p\colon V \to \mathbb{R}^2, \begin{cases} i \mapsto (\xi_i, 0)^\top, & \text{for } i = 1, 3, 5. \\ j \mapsto (0, \eta_i)^\top, & \text{for } j = 2, 4, 6, \end{cases}$$

with $\xi_i, \eta_j \in \mathbb{R} \setminus \{0\}$, for $i \in \{1, 3, 5\}, j \in \{2, 4, 6\}$.

A non-trivial motion for $G(p)$ is then obtained by

$$P\colon\ [-\tau, \tau] \to \{\, G(q) \mid q \text{ is an arbitrary realization for } G\,\}; t \mapsto G(q_t),$$

where

$$\tau = \min\{\, |\eta_j| \mid j \in \{2, 4, 6\}\,\} \ \text{ and } \ q_t\colon V \to \mathbb{R}^2, \begin{cases} i \mapsto \left(\xi_i \sqrt{1 + \xi_i^{-2} \cdot t}, 0\right)^\top, & i \in \{1, 3, 5\}. \\ j \mapsto \left(0, \eta_j \sqrt{1 - \eta_j^{-2} \cdot t}\right)^\top, & j \in \{2, 4, 6\}. \end{cases}$$

3.2. Infinitesimal rigidity

As aforementioned, this section deals with a local version of rigidity where the vertices may move in a sufficiently small neighbourhood or rather the length of every edge may change about a very small amount. Frameworks that preserve this property shall then be called *infinitesimally rigid* as presented in a more formal sense in the upcoming paragraphs. Furthermore, important relations between natural rigidity and infinitesimal rigidity are treated. This yields powerful criteria for deducing rigidity properties of frameworks. More precisely infinitesimal rigidity or rather infinitesimal motions refer to derivatives of motions fulfilling the length-preserving constraint (3.1) of continuous motions evaluated at time $t = 0$. Another point of view is to consider infinitesimal motions as the *tangent space* of continuous motions, which shall not be elaborated in the scope of this thesis. Infinitesimal rigidity can be construed as a linearised version of natural rigidity, since the invariants subsist of linear equations, but still are able to preserve rigidity properties for almost all realizations. Therefore, it is much easier to consider infinitesimal rigidity instead of continuous rigidity since solely linear algebra plays a major part.

Furthermore, the idea of infinitesimal rigidity must be highlighted at this point since similar concepts form an essential role in order to deal with a new type of rigidity introduced as part of this work, see Chapter 4.

Definition 3.5 (Izmestiev [Izm09a]).
An infinitesimal motion[4] *of* $\mathbb{R}^d$ *is a vector field*

$$\begin{aligned} \xi : \mathbb{R}^d &\to \mathbb{R}^d, \\ x &\mapsto \xi(x) \end{aligned}$$

such that moving each point of $\mathbb{R}^d$ *along the vector applied at that point does not change the distances in first order:*

$$\frac{d}{dt} \left\| (x + t\xi(x)) - (y + t\xi(y)) \right\| \Big|_{t=0} = 0, \tag{3.2}$$

for all $x, y \in \mathbb{R}^d$.

Definition 3.6 (Izmestiev [Izm09a]).
An infinitesimal motion[5] *of a framework* $G(p)$ *is a map*

$$\begin{aligned} Q : V &\to \mathbb{R}^d, \\ i &\mapsto q_i \end{aligned}$$

such that

$$\frac{d}{dt} \left\| (p_i + tq_i) - (p_j + tq_j) \right\| \Big|_{t=0} = 0, \tag{3.3}$$

for all $\{i, j\} \in E$.

As in Section 3.1 the terms *trivial infinitesimal motions* and *infinitesimal rigid* frameworks are introduced analogously.

Definition 3.7 (Izmestiev [Izm09a]).
An infinitesimal motion Q *of a framework* $P = G(p)$ *is called* trivial *if it is induced by some infinitesimal motion* ξ *of* $\mathbb{R}^d$*:*

$$Q = \xi \circ P.$$

Which can also be rewritten as $q_i = \xi(p_i)$, *for all* $i \in V$.

[4] or also infinitesimal flex of $\mathbb{R}^d$.
[5] or also infinitesimal flex of a framework $G(p)$.

Definition 3.8 (Izmestiev [Izm09a]).
A framework is called infinitesimally rigid *if all of its infinitesimal motions are trivial, otherwise it is called* infinitesimally flexible.

Prior to giving concrete examples of infinitesimally rigid and flexible frameworks, the following provides a characterization for trivial infinitesimal motions analogously to the definition of isometries, as given by Izmestiev [Izm09a] and Schulze and Whiteley [SW17]. In respect thereof *skew-symmetric* matrices are defined to be of the following form:

$$S \in \mathbb{R}^{k\times k} \text{ skew-symmetric} :\Longleftrightarrow S^\top = -S$$

Lemma (Izmestiev [Izm09a], Schulze and Whiteley [SW17]).
Every infinitesimal motion ξ of $\mathbb{R}^d$ is of the form $\xi(x) = S\cdot x + b$, for $b \in \mathbb{R}^d$ and $S \in \mathbb{R}^{d\times d}$ skew-symmetric.

Since trivial infinitesimal motions are induced by infinitesimal motions of the whole space $\mathbb{R}^d$, this statement particularly applies to them as well. In this context S is also said to be an *infinitesimal rotation* and b is called an *infinitesimal translation.*

The infinitesimal rigid framework given in Figure 3.6, as in [Lam70], looks very similar to the framework given in Figure 3.5. Only the uppermost vertex has been moved in positive x-direction yielding an infinitesimal rigid framework, as proven by Laman [Lam70].

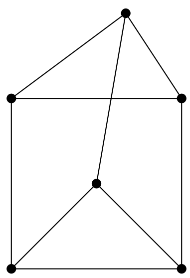


Figure 3.6.: Framework that is infinitesimal rigid [Lam70].

Lemma 3.9 (Izmestiev [Izm09a]).
A vector field ξ is an infinitesimal motion of $\mathbb{R}^d$ if and only if

$$\langle \xi(x) - \xi(y), x - y\rangle = 0, \quad \forall x, y \in \mathbb{R}^d. \tag{3.4}$$

Similarly, a map $Q\colon V \to \mathbb{R}^d, i \mapsto q_i$ is an infinitesimal motion of a framework P if and only if

$$\langle q_i - q_j, p_i - p_j\rangle = 0, \quad \forall \{i, j\} \in E. \tag{3.5}$$

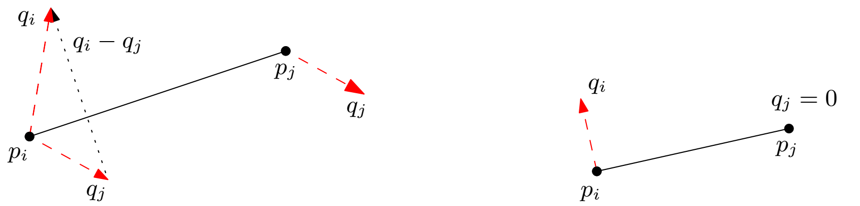
Proof. It suffices to show the equivalence between (3.3) and (3.5). The equivalence between (3.2) and (3.4) is proven analogously.

In Chapter 2 it was stated that $\sqrt{\langle x, x\rangle} = \|x\|$. Therefore, taking the derivative of the Euclidean scalar product results in $\frac{d}{dt}\langle f(t), g(t)\rangle = \langle \frac{d}{dt} f(t), g(t)\rangle + \langle f(t), \frac{d}{dt} g(t)\rangle$ and thus $\frac{d}{dt}\langle v, v\rangle = 2\langle \frac{d}{dt} v, v\rangle = \frac{d}{dt}\|v\|^2$, for any vector v that depends on t. And thus it holds:

$$\begin{aligned}
& \frac{d}{dt}\left\|(p_i + tq_i) - (p_j + tq_j)\right\|\Big|_{t=0} && = 0 \\
\Longleftrightarrow\quad & \frac{d}{dt}\left\|(p_i + tq_i) - (p_j + tq_j)\right\|^2\Big|_{t=0} && = 2\langle q_i - q_j, (p_i - p_j) + t(q_i - q_j)\rangle|_{t=0} \\
& && = 2\langle q_i - q_j, p_i - p_j\rangle = 0.
\end{aligned}$$

□

As given by Izmestiev [Izm09a], Figure 3.7 ought to clarify the condition given in (3.5), which states that the vectors $q_i - q_j$ and $p_i - p_j$ are orthogonal. Geometrically spoken, the signed lengths of the projections of q_i and q_j onto the line $p_i p_j$ match, see Figure 3.7a.



(a) General scenario. (b) Scenario with $q_j = 0$.

Figure 3.7.: A geometric meaning of the condition (3.5) as stated in [Izm09a].

Also, $q_j = 0$ results in $\langle q_i, p_i - p_j \rangle = 0$, which means that q_i is perpendicular to the edge $\{p_i, p_j\}$ or simplified $q_i \perp (p_i - p_j)$. The motion of p_i along q_i is called an infinitesimal rotation around p_j, see Figure 3.7b.

Laman [Lam70] provided a somehow different characterization for infinitesimal motions that is equivalent to the one given in this thesis, as shall be seen. There are some differences in notion which need clarification first. For instance, he names frameworks *plane skeletal structures*. In particular Laman only considers the two-dimensional case, i.e. $d = 2$, but these terms can be transferred analogously to the higher-dimensional case as well, what shall happen here silently. In his paper Laman called motions of a framework *length-preserving displacements* and introduced the term *infinitesimal displacement*, which is basically an infinitesimal motion, but without the restriction of length-preservation. That is, a vector $v \in \mathbb{R}^d$ is assigned to every vertex of a framework. As stated in [Lam70] one might think of this vector as *velocity* since every vertex of the framework is moved along its respective trajectory spanned through these velocities. Higher values in the components yield bigger steps in moving the certain vertex.

In a next step Laman introduces *small displacement* as an ordered triple (μ, α, p_t), where μ denotes an infinitesimal displacement, α is a real number greater than zero (i.e. $\alpha > 0$) and p_t is the movement of a vertex along its trajectory defined through the infinitesimal displacement μ, fulfilling

$$p_t(i) = p(i) + t \cdot \mu(i) + o(t) \quad \text{for every real } t \text{ with } |t| \leq \alpha.$$

$G(p)$ was silently assumed to be a framework with its realization p.

Here "o" refers to the *Little-o notation*, with $f(x) \in o(g(x))$ or $f(x) = o(g(x))$ meaning that $f(x)$ is *asymptotically negligible* compared to $g(x)$. In a more formal sense this means $f(x) = o(g(x)) :\Longleftrightarrow \lim_{x \to a} \left| \frac{f(x)}{g(x)} \right| = 0$, where $a \in \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$. The limit $x \to a$ is essential and can yield different statements for different a. As given by Laman [Lam70], the assignment $a = 0$ is of interest in this scenario.

In addition, Laman introduces the term *admissible* small displacement for which an equivalence to infinitesimal motion is established in the following. A small displacement (μ, α, p_t) is called *admissible*, if every edge $\{i, j\} \in E$ satisfies:

$$\|p_t(i) - p_t(j)\| - \|p(i) - p(j)\| = o(t) \quad \forall t\colon\ |t| \leq \alpha.$$

To show the equivalence, let $G(p)$ be a framework and (μ, α, p_t) a small displacement. Then it holds

$$\begin{aligned}\|p_t(i) - p_t(j)\| - \|p(i) - p(j)\| = \|(p_i + t \cdot \mu(i)) - (p_j + t \cdot \mu(j))\| - \|p_i - p_j\| = o(t)\\ \iff \lim_{t \to 0} \left| \frac{\|(p_i + t \cdot \mu(i)) - (p_j + t \cdot \mu(j))\| - \|p_i - p_j\|}{t - 0} \right| = 0\\ \iff \frac{d}{dt} \|(p_i + t \cdot \mu(i)) - (p_j + t \cdot \mu(j))\| \Big|_{t=0} = 0.\end{aligned}$$

Therefore, these terms are equivalent and it is legitimate to speak of "change of distances in first order".

Take note, that for sake of simplicity the extra "$+o(t)$" has been omitted in the equation chain for the small displacement (μ, α, p_t). ♦

Finally, two important theorems for characterizing infinitesimal rigidity and rigidity and their relation to each other shall be presented.

THEOREM 3.10 (Izmestiev [Izm09a]).
Not every rigid framework is infinitesimally rigid.

Proof. This can be seen by simply considering the so called *degenerated triangle*:

$V = \{1, 2, 3\}, E = \{\{1, 2\}, \{2, 3\}, \{1, 3\}\}$ and

$$p\colon V \to \mathbb{R}^2, \begin{cases} 1 \mapsto & (-1, 0)^\top, \\ 2 \mapsto & (0, 0)^\top, \\ 3 \mapsto & (1, 0)^\top. \end{cases} \qquad Q\colon V \to \mathbb{R}^2, \begin{cases} 1 \mapsto & 0, \\ 2 \mapsto & (0, 1)^\top, \\ 3 \mapsto & 0. \end{cases}$$

The given framework is visualized in Figure 3.8. The dashed arrow displays the infinitesimal motion of the vertex 2. Since applying any motion on the framework breaks the length-preserving invariant, the framework is rigid. But moving the framework according to the infinitesimal motion Q, changes distances only in first order and is particularly non-trivial. Therefore, the degenerated triangle is rigid but not infinitesimally rigid.

It is not easy to see that Q is non-trivial, but Theorem 3.15 provides an easier criterion to tell whether a given framework is infinitesimally rigid or not. □

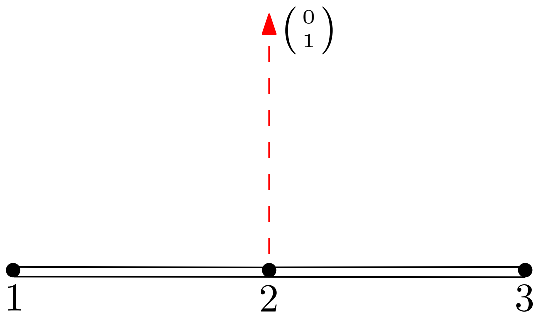


Figure 3.8.: Degenerated triangle as seen by Izmestiev [Izm09a] and Laman [Lam70] that is rigid but infinitesimally flexible.

Figure 3.9 displays another framework, with a more complex structure, that is rigid but infinitesimally flexible. Here, solely rotating the inner triangle yields an infinitesimal motion that is certainly not trivial.

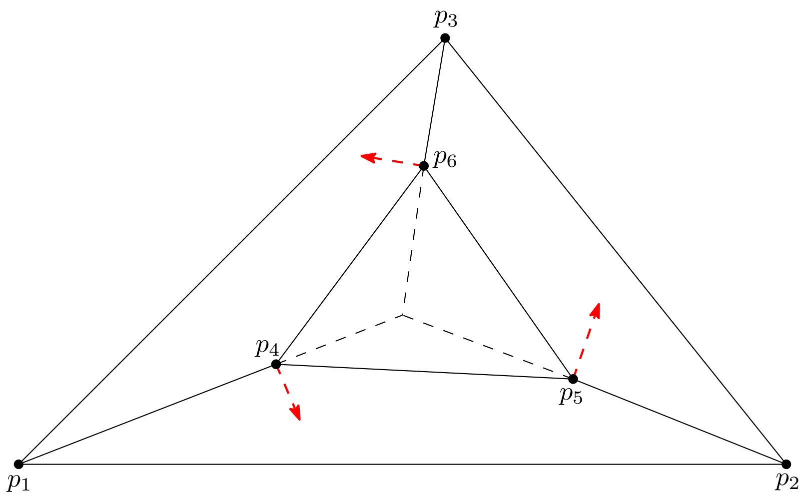


Figure 3.9.: More complex framework that is rigid but infinitesimally flexible (see Izmestiev [Izm09a]).

Theorem 3.11 (Izmestiev [Izm09a]).
Every infinitesimally rigid framework is rigid.

A further important part in the theory of infinitesimal rigid frameworks plays the *rigidity matrix* $R_G(p) \in \mathbb{R}^{m \times (d \cdot n)}$ of a framework $G(p)$, which eases deciding whether a given framework is infinitesimally flexible or infinitesimally rigid. Take note that throughout this thesis m and n refer to the amount of edges and vertices, respectively.

Each edge $\{i, j\} \in E$ corresponds to a row $r_{\{i,j\}}$ of $R_G(p)$. For $l \in \{1, \ldots, d\}$, the columns $d(i-1)+l$ and $d(j-1)+l$ hold the values $(p_i - p_j)_l$ and $(p_j - p_i)_l$, respectively, and 0 elsewhere. Therefore, the rigidity matrix is of the following structure

$$R_G(p) = \begin{array}{c} \\ \\ \{i,j\} \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccccccccc} & & & d(i-1)+1, \ldots, di & & & & d(j-1)+1, \ldots, dj & & & \end{array} \\ \begin{pmatrix} & & & & & \vdots & & & & & \\ 0 & \ldots & 0 & (p_i - p_j)^\top & 0 & \ldots & 0 & (p_j - p_i)^\top & 0 & \ldots & 0 \\ & & & & & \vdots & & & & & \end{pmatrix} \end{array}$$

Astonishingly, the rigidity matrix encodes properties of (infinitesimal) rigidity through linear equations, whereas non-linear equations formed the starting point in the case of continuous rigidity. Therefore, a deeper look into this linearity and where it comes from might be of help.

The non-linearity of $\|x\| := \sqrt{\sum_{i=1}^{d} x_i^2}$ is quite obvious. Since this is the exact usage for rigidity in (3.1) in Section 3.1 and no simplifications were made, dealing with non-linear equations in this scenario is still necessary. But in (3.3) the derivative of the invariant (3.1) to t at the point 0 has been added and therefore a much simpler characterization for infinitesimal motions, as in Lemma 3.9, could be obtained. The necessary condition in the case of infinitesimal motions of frameworks was reduced to $\langle q_i - q_j, p_i - p_j \rangle = 0$, as in (3.5), which is actually nothing else than

$$\langle q_i - q_j, p_i - p_j \rangle = \sum_{k=1}^{d} (q_i - q_j)_k \cdot (p_i - p_j)_k = 0.$$

This then provides linear equations for infinitesimal motions of frameworks and as well of $\mathbb{R}^d$, which can be seen along the same lines.

Remark 3.12.
A further mentionable powerful tool for rigidity specifies the so called *edge function* f, for a framework $G(p)$ of graph $G = (V, E)$, whose derivative yields the rigidity matrix $R_G(p)$, see also Roth [Rot81]. The edge function $f\colon \mathbb{R}^d \times \cdots \times \mathbb{R}^d \to \mathbb{R}^m$ is formally given as follows

$$f(q_1, \ldots, q_n) \coloneqq \left(\|q_i - q_j\|^2\right)_{\{i,j\}\in E}, \tag{3.6}$$

where $q_i \in \mathbb{R}^d$ represents the realization of the vertex $i \in V$. That is, $f(p_1, \ldots, p_n)$ yields a vector consisting of the squared length of every edge of $G(p)$.

The derivative of f is defined as

$$\begin{aligned} \mathrm{d}f(q) \coloneqq\ & \begin{array}{c} \\ \\ \{i,j\} \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccccccccc} & & & i & & & & j & & & \end{array} \\ \begin{pmatrix} & & & & & \vdots & & & & & \\ 0 & \cdots & 0 & \frac{\partial f}{\partial q_i} & 0 & \cdots & 0 & \frac{\partial f}{\partial q_j} & 0 & \cdots & 0 \\ & & & & & \vdots & & & & & \end{pmatrix} \end{array} \\ =\ & 2R_G(q), \end{aligned}$$

since $\|q_i - q_j\|^2 = \langle q_i - q_j, q_i - q_j\rangle = \sum_{k=1}^{d}(q_i - q_j)_k^2 =\colon \zeta(q_i, q_j)\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ and $\frac{\partial\zeta(q_i,q_j)}{\partial(q_i)_k} = 2(q_i - q_j)_k$, for $k \in \{1, \ldots, d\}$.
This yields $\frac{\partial f}{\partial q_i} \coloneqq 2(q_i - q_j)^\top$. Conclusively, it holds $\ker \mathrm{d}f(p) = \ker R_G(p)$, which will prove to be useful in the upcoming sections and chapters. ♢

Remark 3.13.
An infinitesimal motion Q of $G(p)$ can be identified with a vector $Q \in \mathbb{R}^{dn}$, where Q denotes the map $V \to \mathbb{R}^d$ and as well the corresponding vector in $\mathbb{R}^{dn}$. This can be seen by defining a map φ – that converts such an infinitesimal motion to its vectorial representation – and its inverse function φ^{-1}.

Here the map $\varphi\colon (V \to \mathbb{R}^d) \to \mathbb{R}^{dn}$ is defined as follows:

$$\varphi(Q) \coloneqq (Q(i))_{i=1}^n,$$

which is bijective with $\varphi^{-1}(v) \coloneqq \left\{\ i \mapsto (v_k)_{k=d(i-1)+1}^{di}, \quad \text{for all} \quad i \in \{1, \ldots, n\} = V\ \right.$.
Take note, that φ and φ^{-1} are higher-order functions. In the case of the map φ, the first and only argument refers to a map $V \to \mathbb{R}^d$. By contrast, φ^{-1} returns a map $V \to \mathbb{R}^d$. Furthermore, this procedure is not restricted to infinitesimal motions, but can be generalised to arbitrary maps $\{1, \ldots, k\} \to M^l$, with $k, l \in \mathbb{N}$ and any set M. ♦

Lemma 3.14.
All infinitesimal motions Q of $G(p)$ are solutions to the system of linear equations $R_G(p) \cdot x = 0$, i.e. the infinitesimal motions Q of $G(p)$ lie in $\ker R_G(p)$.

Proof. To prove the statement, let $Q\colon V \to \mathbb{R}^d, i \mapsto q_i$ be an infinitesimal motion of a framework $G(p)$.

$$R_G(p)\cdot Q = \left((p_i - p_j)^\top \cdot q_i + (p_j - p_i)^\top \cdot q_j\right)_{\{i,j\}\in E} = \left((p_i - p_j)^\top \cdot (q_i - q_j)\right)_{\{i,j\}\in E} \tag{3.7}$$

and since $(p_i - p_j)^\top \cdot (q_i - q_j) = \langle p_i - p_j, q_i - q_j\rangle$, it follows by Lemma 3.9, that $\langle p_i - p_j, q_i - q_j\rangle = 0$, for all $\{i, j\} \in E$ due to Q being an infinitesimal motion of $G(p)$.

Therefore, (3.7) is identically 0 and particularly Q in fact lies in the kernel of the rigidity matrix, i.e. $Q \in \ker R_G(p)$. □

THEOREM 3.15 (Schulze [Sch10]).
A framework $G(p)$ is infinitesimally rigid if and only if either

(1) $\operatorname{rank} R_G(p) = d\,|V| - \binom{d+1}{2}$, *or*

(2) $G = K_{|V|}$ *is the complete graph and the points* $p_i, i \in V$, *are affinely independent.*

Proof. Here, only a proof for the rank condition (1) shall be given and additionally only for the case $d = 2$. The higher-dimensional case is proven analogously under making use of the statement $\dim \mathrm{O}(d) = \frac{d(d-1)}{2}$, with $\mathrm{O}(d)$ denoting the orthogonal group, and then yielding $\dim \mathrm{Iso}(\mathbb{R}^d) = \dim \mathrm{O}(d) + d = \binom{d+1}{2}$. Here the term "$+d$" refers to the dimension of translations. The second statement (2) is necessary for the *Henneberg construction* and shall become clear in Theorem 3.30. So let $G(p)$ be a framework where G is not complete. First of all the *Rank-nullity theorem* for a linear map $f\colon X \to Y$, where X and Y denote vector spaces, needs to be mentioned. The statement reads $\dim X = \operatorname{rank} f + \dim(\ker f)$ and a proof is given by Beutelspacher [Beu14].

Every trivial motion is a linear combination of the following three linearly independent vectors:

$$
\begin{aligned}
T_x &= \begin{pmatrix} 1 & 0 & \dots & 1 & 0 & \dots & 1 & 0 \end{pmatrix}^\top \in \mathbb{R}^{2|V|} \\
T_y &= \begin{pmatrix} 0 & 1 & \dots & 0 & 1 & \dots & 0 & 1 \end{pmatrix}^\top \in \mathbb{R}^{2|V|} \\
T_r &= \begin{pmatrix} -b_1 & a_1 & \dots & -b_i & a_i & \dots & -b_{|V|} & a_{|V|} \end{pmatrix}^\top \in \mathbb{R}^{2|V|}
\end{aligned}
$$

with $p_i = (a_i, b_i)$, for $i \in V$. Here T_x and T_y denote infinitesimal x- and y-translations, respectively, whereas T_r refers to infinitesimal rotations. To be more specific, T_x, T_y and T_r define the direction of the infinitesimal motion and multiplying with a scalar changes the amount of translating for T_x and T_y or the angle of rotation for T_r.

Furthermore, $\mathcal{M} := \{\, \mu \mid \mu \text{ infinitesimal motion of } G(p) \,\} \simeq \mathbb{R}^{2|V|}$ defines the vector space of infinitesimal motions of $G(p)$. It holds $\dim \mathcal{M} = 2\,|V|$, as provided by Laman [Lam70].

In addition, $\mathcal{M}_T := \operatorname{span}\{T_x, T_y, T_r\} \subset \mathcal{M}$ refers to the vector space of trivial infinitesimal motions of $G(p)$. Here it holds $\dim \mathcal{M}_T = 3$.

According to Lemma 3.14 this yields $\dim(\ker R_G(p)) \geq 3$, in particular $\ker R_G(p) = \mathcal{M}_T$ if and only if $G(p)$ is infinitesimally rigid. Moreover, $R_G(p)$ can be construed as a linear map $\mathbb{R}^{2|V|} \to \mathbb{R}^{|E|}, x \mapsto R_G(p)x$.

Hence, the Rank-nullity theorem concludes the proof since

$$
\begin{aligned}
& G(p) \text{ infinitesimally rigid} \\
\iff & \dim(\ker R_G(p)) = 3 \\
\iff & \dim \mathcal{M} = 2\,|V| = \operatorname{rank} R_G(p) + 3 \\
\iff & \operatorname{rank} R_G(p) = 2\,|V| - 3.
\end{aligned}
$$

□

Before concluding this section, two further definitions that are commonly used in working with rigidity properties are presented. Also, the rank condition (1) of an infinitesimal rigid framework as given in Theorem 3.15, shall be emphasized at this point. This is due to the term on the right side of the equation appearing more often throughout this thesis, which is of great benefit in order to analyse rigidity properties of frameworks or even graphs (see Definition 3.26).

$$\operatorname{rank} R_G(p) = d\,|V| - \binom{d+1}{2} \tag{3.8}$$

Furthermore, it is of intuitive nature, that graphs with too few edges can not be rigid, which is actually the fact as given by Laman [Lam70] and Asimow and Roth [AR78]. Here again the rank condition (1) appears:

Corollary (Laman [Lam70], Asimow and Roth [AR78]).
Let $G(p)$ be a framework. Then, if $|E| < d\,|V| - \binom{d+1}{2}$, $G(p)$ is infinitesimally flexible.

Thus, $|E| \geq d\,|V| - \binom{d+1}{2}$, is a necessary condition for infinitesimal rigidity. This also explains why there is no realization p of $K_{2,2}$ such that $(K_{2,2}, p)$ is infinitesimally rigid, as stated by Laman [Lam70].

Definition 3.16 (Schulze and Whiteley [SW17]).
Let $G(p)$ be a realization. $G(p)$ is

(a) independent*, if the corresponding rigidity matrix has linearly independent rows.*

(b) isostatic*, if it is infinitesimally rigid and independent.*

Theorem 3.17 provides a characterization for isostatic frameworks in $\mathbb{R}^d$, which plays a major part in Section 3.5. The following statement is quoted as given by Schulze and Whiteley [SW17], but with respect to the style of notion applied in this thesis.

Theorem 3.17 (Schulze and Whiteley [SW17]).
For a framework $G(p)$ in d-space, with $|V| \geq d$, the following are equivalent:

(a) $G(p)$ is isostatic.

(b) $G(p)$ is infinitesimally rigid with $|E| = d\,|V| - \binom{d+1}{2}$.

(c) $G(p)$ is independent with $|E| = d\,|V| - \binom{d+1}{2}$.

(d) $G(p)$ is infinitesimally rigid, and removing any one bar (but no vertices) leaves an infinitesimally flexible framework.

Instead of (d), in Theorem 3.17, one could also speak of *minimally infinitesimally rigidity*, for which similar concepts are investigated in Section 3.5 and are of interest in the last chapter, Chapter 4, where a new type of rigidity is introduced.

Also, common is the term “*first-order*” instead of “infinitesimally”. This can be explained by the notion of an admissible small displacement, as given by Laman [Lam70]. There the edge lengths may change in first-order, what has been expressed with the Little-o notation. Since one could also consider “*second-order rigidity*” along the same terms, this idea is mentionable (see Connelly and Whiteley [CW96] for further details).

3.3. Parallel designs

This chapter’s purpose is to present *parallel designs* as given by Whiteley [Whi96]. Interestingly, parallel designs in fact provide a different geometrical interpretation for infinitesimal rigidity. In the scenario of parallel designs the role of the invariant is played by the slope of each edge instead of their lengths. Therefore, fitting terms would also be *directional* or *slope-preserving rigidity.*

Although this chapter only treats the two-dimensional case of parallel designs, it can be considered in broader general as well by transferring the terms analogously as presented to the higher-dimensional setting.

The rotation of a vector by 90° counter-clockwise shall be denoted by $^\perp$. In the upcoming sections and chapters $^\perp$ shall always refer to this particular rotation. More precisely $(\cdot)^\perp : \mathbb{R}^2 \to \mathbb{R}^2$ is defined as

$$\begin{pmatrix} x \\ y \end{pmatrix}^\perp := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}.$$

For a given framework $G(p)$ and a transformed framework $G(q)$ the system of constraints that needs to be satisfied in order to preserve the direction of each edge, is

$$\langle (p_i - p_j)^\perp, q_i - q_j \rangle = 0, \quad \forall \{i, j\} \in E. \tag{3.9}$$

In other words, the vector $(\overrightarrow{\boldsymbol{p_j p_i}})^\perp$ is perpendicular to the vector $\overrightarrow{\boldsymbol{q_j q_i}}$, that is obtained by the framework's transformation $G(p)$ to $G(q)$, for every edge $\{i, j\} \in E$.

Definition 3.18 (Whiteley [Whi96]).
The solutions to the homogeneous linear system (3.9) *are called* parallel designs $G(q)$ *of the original* plane design $G(p)$. *A parallel design* $G(q)$ *for* $G(p)$ *is* trivial, *if* q *is a translation or dilation of* p. *Otherwise, the parallel design is* non-trivial.

Figure 3.10 provides a simple example of non-trivial parallel designs of a cuboid.

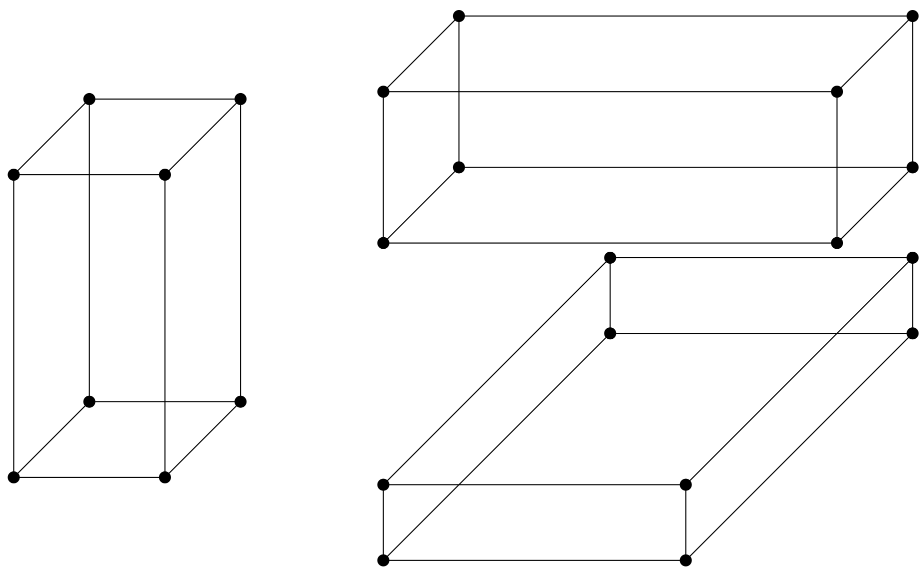


Figure 3.10.: A cuboid with two non-trivial parallel drawings.

In contrast, Figure 3.11 displays the example of a trivial parallel design for the two-dimensionally embedded unit cube. This can be obtained by stretching every edge by a factor of 3.

For parallel designs there exists a similar concept as the rigidity matrix in Section 3.2: the so called *parallel design matrix* $P_G(p)$. It is defined analogously to the rigidity matrix, but

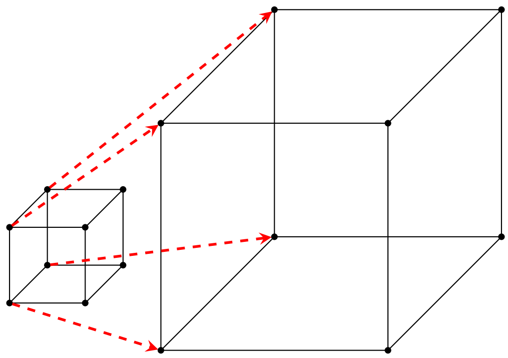


Figure 3.11.: A cube with a trivial parallel drawings

instead of $(p_i - p_j)^\top$ and $(p_j - p_i)^\top$ the entries hold $(p_i - p_j)^\perp$ and $(p_j - p_i)^\perp$ as values, respectively. That is,

$$P_G(p) = \begin{array}{c} \\ \{i,j\} \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccccccccc} & & & 2i-1,2i & & & & 2j-1,2j & & & \end{array} \\ \begin{pmatrix} & & & & & \vdots & & & & & \\ 0 & \dots & 0 & (p_i - p_j)^\perp & 0 & \dots & 0 & (p_j - p_i)^\perp & 0 & \dots & 0 \\ & & & & & \vdots & & & & & \end{pmatrix} \end{array}$$

Take note that the necessary $^\top$ has been omitted to simplify reading. The actual values of the parallel designs matrix $P_G(p)$ therefore consist of, e.g. $\left((p_i - p_j)^\perp\right)^\top$. Since the parallel design matrix and the rigidity matrix look very similar one could think that there somehow is a relation between $P_G(p)$ and $R_G(p)$, which is actually the fact as stated in the following Lemma.

Lemma 3.19 (Whiteley [Whi96]).
The terms infinitesimal motion and parallel design are equivalent.

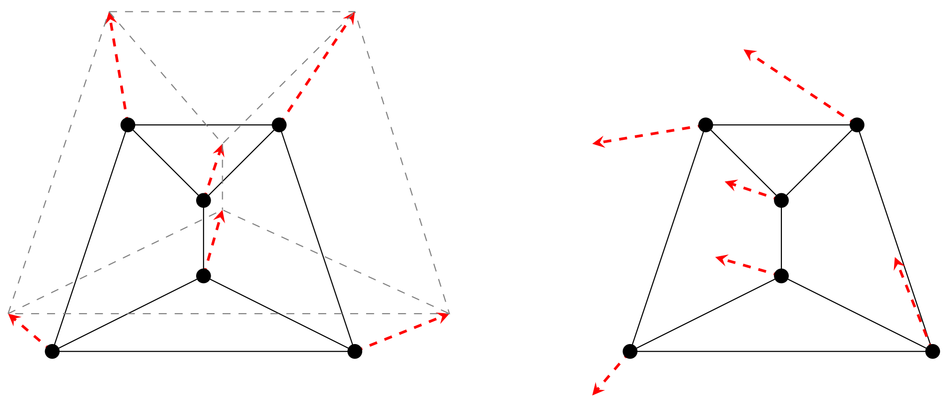
Proof. For proving the statement an equivalence between the both constraints ((3.5) and (3.9)) is established. Let $G(p)$ be a framework with an infinitesimal motion $U : V \to \mathbb{R}^2$, $i \mapsto u_i$. It then holds:

$$\begin{aligned} & \langle p_i - p_j, u_i - u_j \rangle = 0 && (3.5) \\ \iff & \langle p_i - p_j, u_i \rangle + \langle p_j - p_i, u_j \rangle = 0 \\ \iff & \langle (p_i - p_j)^\perp, u_i^\perp \rangle + \langle (p_j - p_i)^\perp, u_j^\perp \rangle = 0 \\ \iff & \langle (p_i - p_j)^\perp, q_i \rangle + \langle (p_j - p_i)^\perp, q_j \rangle = 0 && (3.9) \end{aligned}$$

Here, q_i refers to $q_i = (u_i)^\perp + p_i$, for all vertices $i \in V$. This is possible due to p itself being a (trivial) solution to the system. □

In Figure 3.12a the solid black lines refer to a framework, whereas the gray dashed lines display a non-trivial design of the same framework, which can be obtained by moving every vertex along the trajectory spanned by the vectors in dashed red. Figure 3.12b considers the same framework, but there every vector (dashed red) has been rotated by 90° clockwise, yielding an infinitesimal motion of the framework. These figures ought to clarify the relation between parallel designs and infinitesimal motions, as mentioned in Lemma 3.19, through concrete examples.

It might not be obvious to see that trivial continuous motions, e.g. rotations, can also be used to obtain a parallel design. To see this, a conversion scheme to convert infinitesimal

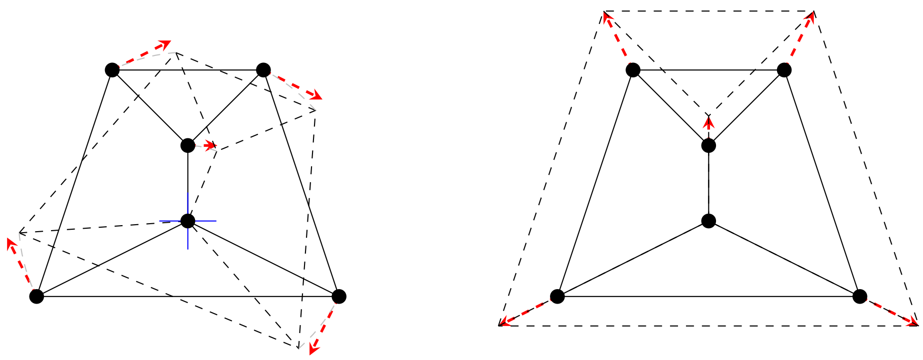


(a) Non-trivial parallel design. (b) Infinitesimal motion of the same framework.

Figure 3.12.: Framework with non-trivial parallel design that induces a non-trivial infinitesimal motion of the same framework.

rotations to dilations shall be presented. Take note, that every continuous motion induces an infinitesimal motion, where each tangent at the original vertex realization corresponds to the vertex' respective velocity.

Figure 3.13 displays such a transformation of a trivial motion for a framework. The left-hand side, Figure 3.13a, depicts a framework that has been rotated by $\frac{\pi}{4} = 45°$ clockwise about the origin (given as the blue cross). The red arrows indicate the tangent of the circle - spanned by the origin, the original vertex and the rotated vertex - at the position of the original vertex. The trajectory along which the vertices shall be moved is displayed in dotted lightgray. Rotating the tangents by 90° counter-clockwise, yields a parallel design as displayed on the right-hand side, in Figure 3.13b. Therefore, the tangents form a trivial infinitesimal motion, which then again induces a trivial parallel design, where the framework just gets uniformly dilated.



(a) Rotation of the framework by 22.5° clockwise. (b) Trivial parallel design.

Figure 3.13.: Framework with trivial motion inducing a trivial parallel design

More formally spoken, for a $\alpha \in [0, 2\pi)$, by which the framework shall be rotated, the transformation matrix is given as follows:

$$A = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix},$$

and applied on every point of the respective realization and therefore, retrieving a new realization, called q. In order to calculate the associated tangent for every vertex i, first determine the normal, which is given by $\vec{\boldsymbol{p_i}}$. Next, calculate a tangential vector by simply rotating the normal $\vec{\boldsymbol{p_i}}$ by 90° (clockwise), i.e. the tangential vector is $\vec{\boldsymbol{t_i}} := -(\vec{\boldsymbol{p_i}})^{\perp}$. It remains to adjust the tangential vector's length, such that it corresponds to the distance of the rotation: $\frac{\vec{\boldsymbol{t_i}}}{\|\vec{\boldsymbol{p_i}}\|}\|\vec{\boldsymbol{p_i}} - \vec{\boldsymbol{q_i}}\|$, which then shall be again called $\vec{\boldsymbol{t_i}}$.

Whiteley [Whi96] states the following correspondences:

1. the trivial infinitesimal motions correspond to trivial parallel designs.
2. a translation by b corresponds to a translation by $b^{\perp}$.
3. a rotation about the origin corresponds to a dilation towards the origin (see Figure 3.13).
4. a non-trivial infinitesimal motion corresponds to a non-trivial parallel design (see Figure 3.12).

3.4. Statical rigidiy

As a final reinterpretation of infinitesimal rigidity this sections deals with the *statical rigidity* of frameworks. This section is concluded by presenting the important theorem that states the duality of infinitesimal and statical rigidity.

Definition 3.20 (Izmestiev [Izm09a]).
A load *on a framework $G(p)$ is a collection of vectors, called* forces*, applied at the vertices of $G(p)$. Formally, a load is a map*

$$\begin{aligned} F : V &\to \mathbb{R}^d, \\ i &\mapsto f_i. \end{aligned}$$

A load is called an equilibrium load*, if the sum over the forces $\sum_{i \in V} f_i$ is equivalent to zero.*

Definition 3.21 (Izmestiev [Izm09a]).
A stress *on the framework $G(p)$ is a map*

$$\begin{aligned} \Omega : E &\to \mathbb{R}, \\ \{i,j\} &\mapsto \omega_{i,j}. \end{aligned}$$

The stress Ω is said to resolve *the load F, if*

$$f_i + \sum_{j \in V} \omega_{i,j}(p_i - p_j) = 0 \text{ for all } i \in V, \tag{3.10}$$

where $\omega_{i,j} = 0$ for all $\{i,j\} \notin E$.
In this situation F is also said to be resolvable *(by Ω).*

The inequality $\omega_{i,j} > 0$ means that the edge $\{i,j\}$ is under *compression*, so that it pushes the vertices i and j apart, whereas the inequality $\omega_{i,j} < 0$ means that the edge $\{i,j\}$ is under *tension*, so that it pulls the vertices i and j towards each other (as mentioned by Izmestiev [Izm09a] and Whiteley [Whi96]). Furthermore, it holds $\Omega(\{i,j\}) = \omega_{i,j} = \omega_{j,i}$, for all $\{i,j\} \in E$.

Example (Izmestiev [Izm09a]).
The following example is adopted literally as it was presented by Izmestiev [Izm09a].
Let Δ be a triangle in $\mathbb{R}^2$. A load that consists of three non-zero forces f_1, f_2 and f_3 is an equilibrium load if and only if the lines along which the forces act (dashed lines) intersect at a point and $f_1 + f_2 + f_3 = 0$, as in Figure 3.14.

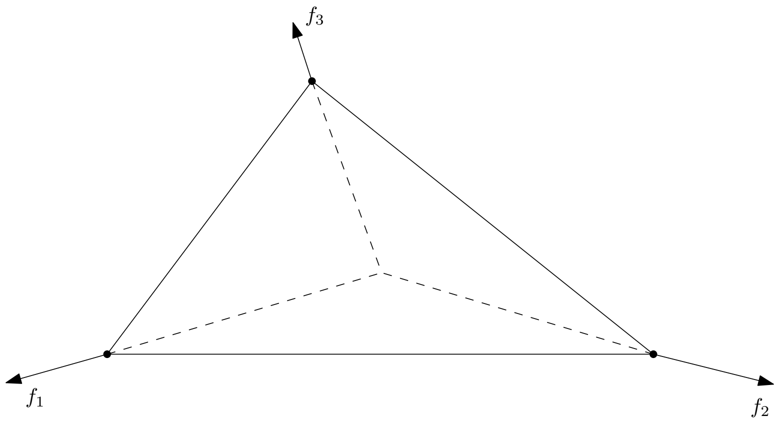


Figure 3.14.: An equilibrium load on a triangle framework in $\mathbb{R}^2$.

Lemma (Izmestiev [Izm09a]).
Every resolvable load is an equilibrium load.

Proof. Let $G(p)$ be a framework and Ω a stress resolving the load F. Then

$$\begin{aligned}\sum_{i \in V} f_i &\overset{(3.10)}{=} \sum_{i \in V} \sum_{j \in V} \omega_{i,j}\,(p_j - p_i)\\ &= \sum_{\{i,j\} \in E} \omega_{i,j}\,(p_j - p_i) + \omega_{i,j}\,(p_i - p_j) = 0\end{aligned}$$

holds and F is an equilibrium load. □

Definition 3.22 (Izmestiev [Izm09a]).
The framework $G(p)$ is called statically rigid*, if every equilibrium load on $G(p)$ can be resolved.*

Following the example of Izmestiev [Izm09a], Figure 3.15 presents a quadrilateral that is not statically rigid. Here, let f_1 and f_3 be two forces acting on p_1 and p_3, respectively, along the diagonal of the quadrilateral such that $f_1 + f_3 = 0$, i.e. f_1 and f_3 cancel each other out. Since the sum over these forces is identically zero, they specify an equilibrium load. Let Ω a stress on the quadrilateral that resolves the load, then Ω is uniquely defined and the stress Ω is given through the blue vectors in Figure 3.15. But at the other two vectors, p_2 and p_4, where zero exterior forces are applied, the stresses create non-zero interior forces. And therefore (3.10) does not hold there.

Theorem 3.23 (Izmestiev [Izm09a]).
A framework is infinitesimally rigid if and only if it is statically rigid.

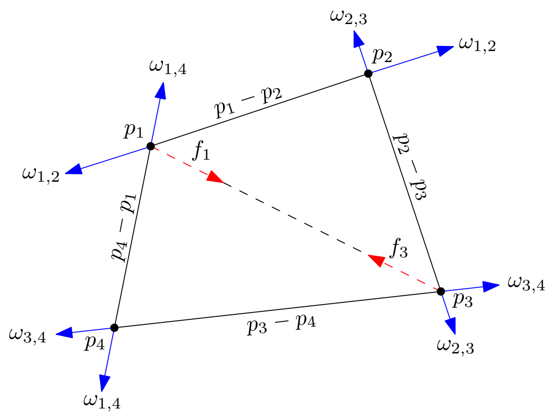


Figure 3.15.: An equilibrium non-resolvable load on a quadrilateral [Izm09a].

3.5. Generic rigidity

It is important to notice that infinitesimal rigidity as in Section 3.2 strongly depends on the given realization of frameworks. Thus, two different realizations of one fixed graph G may yield an infinitesimal rigid and an infinitesimal flexible framework, respectively.

To see this, the example of Laman [Lam70] given in his paper is considered in the upfollowing.

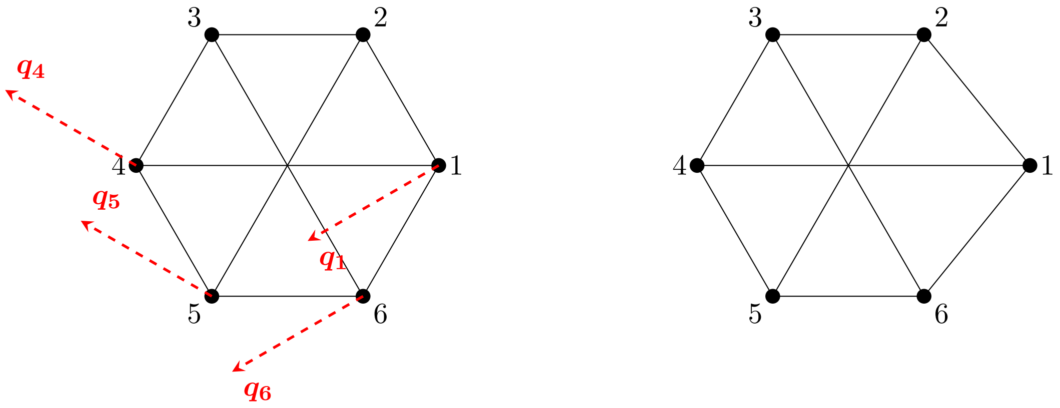
The underlying graph $G = (V, E)$ is given through $V = \{1, 2, 3, 4, 5, 6\}$ and $E = \bigcup_{i=1}^{5}\{\{i, i+1\}\} \cup \{\{1, 6\}, \{1, 4\}, \{2, 5\}, \{3, 6\}\}$. Take note, that the graph $G = (V, E)$ describes the well-known graph $K_{3,3}$, i.e. $G = K_{3,3}$. Furthermore, the edge set could be defined to consist of exactly those edges $\{i, j\}$, where i is even and j is odd, i.e. $E = \{\, \{i, j\} \mid i \in (2 \cdot \mathbb{N}) \cap V, j \in (2 \cdot \mathbb{N}_0 + 1) \cap V \,\}$. The following considers two realizations φ and ψ, whereby Q denotes a non-trivial infinitesimal motion for φ:

$$\varphi\colon V \to \mathbb{R}^2, \begin{cases} 1 \mapsto (1, 0)^\top, \\ 2 \mapsto (\frac{1}{2}, \frac{1}{2}\sqrt{3})^\top, \\ 3 \mapsto (-\frac{1}{2}, \frac{1}{2}\sqrt{3})^\top, \\ 4 \mapsto (-1, 0)^\top, \\ 5 \mapsto (-\frac{1}{2}, -\frac{1}{2}\sqrt{3})^\top, \\ 6 \mapsto (\frac{1}{2}, -\frac{1}{2}\sqrt{3})^\top. \end{cases} \qquad Q\colon V \to \mathbb{R}^2, \begin{cases} 1, 6 \mapsto q_1 := q_6 := (-\frac{1}{2}\sqrt{3}, -\frac{1}{2})^\top, \\ 4, 5 \mapsto q_4 := q_5 := (-\frac{1}{2}\sqrt{3}, \frac{1}{2})^\top, \\ i \mapsto 0, \quad \text{otherwise} \end{cases}$$

$$\psi\colon V \to \mathbb{R}^2, \begin{cases} 1 \mapsto (\delta, 0)^\top, & \text{with } \delta \neq 1, \\ i \mapsto \varphi(i), & \text{for } i \in \{2, 3, 4, 5, 6\}. \end{cases}$$

Figure 3.16a visualizes the realization φ that induces an infinitesimal flexible framework since the infinitesimal motion Q is non-trivial for the framework $G(\varphi)$. In contrast, Figure 3.16b displays the infinitesimal rigid framework $G(\psi)$.

Therefore, a goal of this chapter is to consider graphs that fulfil the property of (infinitesimal) rigidity for "almost" all realizations. These graphs shall then be called *(infinitesimally) generically rigid*. Furthermore, the class of so called *minimally rigid* or in the 2-dimensional case, i.e. $d = 2$, *Laman* graphs shall be introduced. The term Laman graph is influenced through the work of Laman, who gave a characterisation of this particular class of graphs



(a) Infinitesimal flexible realization φ of G. (b) Infinitesimal rigid realization ψ of G.

Figure 3.16.: The same graph G, but with two different realizations φ and ψ yielding infinitesimal flexible and infinitesimal rigid frameworks, respectively.

in [Lam70]. *Minimally rigid* is adequate since removing any edge would break the property of rigidity and therefore these graphs are minimal in this sense. Conclusively, an important style in notation should be mentioned: To every notion introduced so far, the prefix "d-" could be added to clarify in which space realizations take place. That is, e.g. for a d-infinitesimally rigid framework $G(p)$, the realization p is of the form $p\colon V \to \mathbb{R}^d$. This notation is important in certain situations, which Theorem 3.28 shall exemplarily clarify.

Suggestively, there was talk of these properties for *graphs*, and not realizations or rather frameworks. That is, Laman graphs are a subclass of generically rigid graphs. In the case of $d = 2$, it is even possible to state that every generically rigid graph consists of Laman graphs. One says that the Laman graphs "form the bases" for the generically rigid graphs. This is vaguely formulated due to the lack of matroidal knowledge. Nonetheless, this statement is refined by a brief excursion to the theory of rigidity matroids of frameworks and graphs in Remark 3.27. For higher-dimensional cases there has not been found such a characterization for generically rigid graphs yet and hence forms an open problem.

As in Lemma 3.14, a realization p of framework $G(p)$ identifies with a vector in $\mathbb{R}^{dn}$.

Definition 3.24 (Connelly [Con05], Whiteley [Whi96]).
A realization is generic*, if its dn coordinates are generic (as defined in Chapter 2). Otherwise, the realization is said to be* singular*. A graph G is called* generically rigid*, if $G(p)$ is rigid for every generic realization p.*

The following presents a further important theorem, which gives a connection between generic, infinitesimal and natural rigidity. Figuratively spoken, Theorem 3.25 provides the "other direction": Theorem 3.11 states that infinitesimally rigid implies rigid but Theorem 3.10 yielded that the converse is not true. But generic rigidity provides this exact direction under certain restrictions.

Theorem 3.25 (Hendrickson [Hen95b], Gluck [Glu75]).
If a graph has a single infinitesimally rigid realization, then all its generic realizations are infinitesimally rigid.

In other words Theorem 3.25 states, that infinitesimal rigidity and natural rigidity are almost always equivalent. More precisely, if p is a generic realization

$$G(p) \text{ is infinitesimally rigid} \iff G(p) \text{ is naturally rigid.}$$

See also Asimow and Roth [AR79].

As already mentioned, Laman [Lam70] presented a convenient way for characterizing the minimally rigid graphs, if $d = 2$, which shall be examined step by step.

DEFINITION 3.26.
A graph G is called a Laman graph *if G is $(2, 3)$-tight.*

REMARK 3.27 (Excursion to matroids).
Given a framework $G(p)$ of a graph $G = (V, E)$, $\mathcal{R}_G(p) = (E, \mathcal{I})$ defines the *rigidity matroid* of $G(p)$. Here, $\mathcal{I} \subseteq \mathcal{P}(E)$ denotes the so called *independent sets* and the elements are those sets of edges where the corresponding rows in the rigidity matrix $R_G(p)$ form linearly independent sets.

Given two generic configurations p and q, it holds $\mathcal{R}_G(p) = \mathcal{R}_G(q)$ and therefore one can speak of the *rigidity matroid* of G for generic realizations of G. It is simplified written as $\mathcal{R}_G$.

For a matroid $(E, \mathcal{I})$ one can define the *bases* $\mathcal{B}$ as the set of maximum elements of $\mathcal{I}$ with respect to the set inclusion $\subseteq$. As part of this work the following two results must be highlighted:

1. the Laman graphs with n vertices form the bases of the rigidity matroid of a complete graph.
2. in broader general one can even say that for every rigid framework $G(p)$ the spanning Laman subgraphs induce the bases of $\mathcal{R}_G(p)$.
   In particular, for a generically rigid graph G the bases of $\mathcal{R}_G$ are Laman graphs and G has spanning subgraphs that are Laman.

For a more precise treatment of this topic, see [Whi96]. ⧫

The term "generically" can be seen in greater general, as indicated by the upcoming theorem and sections. For further concepts of rigidity putting "generically" in front of the respective notions, which shall happen silently, refers to generic realizations as in Definition 3.24. Nevertheless, it should be inferred from the context what "generically" is actually meant to be.

THEOREM 3.28 (Laman [Lam70], Schulze [Sch10], Schulze and Whiteley [SW17], Borcea and Streinu [BS02]).
Let G be a graph. The following are equivalent:

(a) G is a Laman-graph.

(b) G is generically 2-isostatic.

(c) G is minimally 2-rigid.

Figure 3.17 depicts realizations of Laman graphs with $6 \leq n \leq 12$ vertices, starting in the upper left corner with $n = 6$ and concluding with $n = 12$ in the bottom right corner as given by Capco et al. [CGG+17].

The upcoming theorem, Theorem 3.30, implies that a Laman graph on n vertices is not unique. Nonetheless, the Laman graphs given in Figure 3.17 have a special property. In fact, those are the Laman graphs where the number of possible *complex* realizations - up to isometries - is maximal, which is examined in further detail by Capco et al. [CGG+17]. This number is also called *Laman number* and denoted by $\mathrm{Lam}(G)$ for a Laman graph G. *Complex* realization refers to a realization p, but with image space $\mathbb{C}^2$, i.e. $p\colon V \to \mathbb{C}^2$.

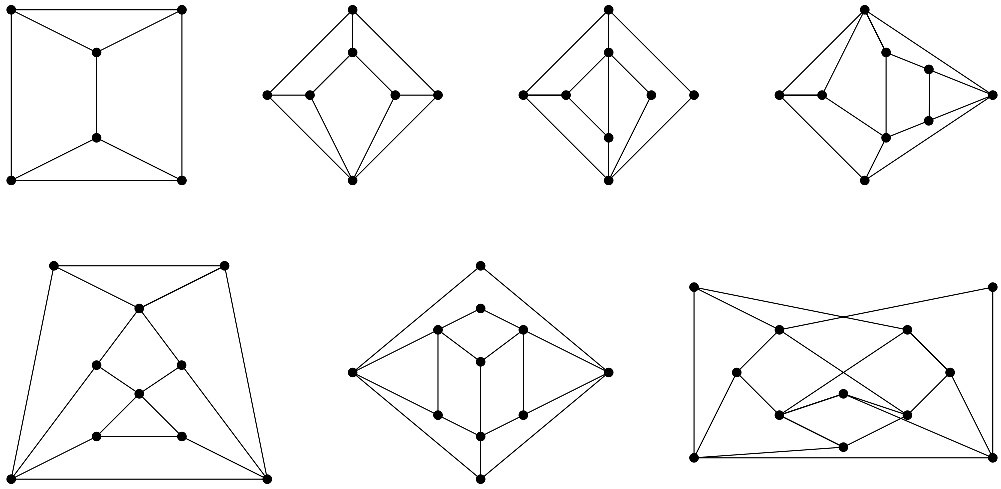


Figure 3.17.: Realizations for Laman graphs with $6 \leq n \leq 12$ vertices as in [CGG+17].

Capco et al. give a keen introduction on the complexity of calculating the Laman number $\mathrm{Lam}(G)$, which is highly non-trivial. Capco et al. were the first to provide a recursive, purely combinatorial formula calculating the exact Laman number, and thereby solved an open problem. Until then, there has only been known an upper bound of $\mathrm{Lam}(G) \leq \binom{2n-4}{n-2}$ for a graph G on n vertices, as given by Borcea and Streinu [BS02].

Finally, the so called *Henneberg constructions* are presented, which provide an inductive approach in order to determine the class of all Laman graphs, that shall be given in Theorem 3.30. But first of all the construction steps are formally defined, along the same lines as given by Schulze and Whiteley [SW17].

Definition 3.29 (Schulze and Whiteley [SW17], Capco et al. [CGG+17]).
A Henneberg d-construction*, for a graph $G = (V, E)$, is either*

(a) **vertex addition***:* $G' = (V \mathbin{\dot\cup} \{t\}, E \mathbin{\dot\cup} \{\{u_1, t\}, \ldots, \{u_d, t\}\})$
for distinct $u_1, \ldots, u_d \in V$

or

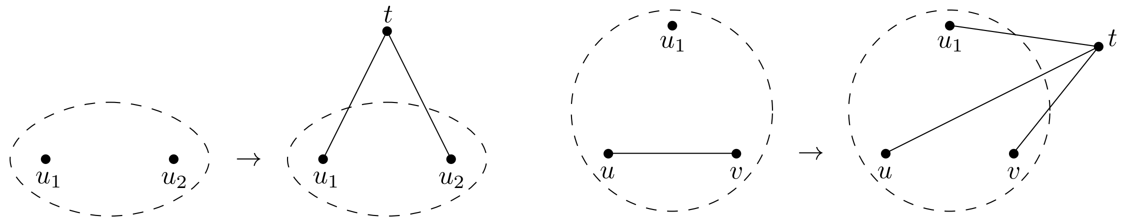
(b) **edge splitting***:* $G' = (V \mathbin{\dot\cup} \{t\}, (E \setminus \{u, v\}) \mathbin{\dot\cup} \{\{u, t\}, \{v, t\}, \{u_1, t\}, \ldots, \{u_{d-1}, t\}\})$
for distinct $u, v, u_1, \ldots, u_{d-1} \in V$ *and* $\{u, v\} \in E$*;*

and therefore yields a new graph G'*.*

Rephrased, the vertex addition firstly adds a new vertex, which shall then be connected to d further vertices (of the original graph). Hereby, it does not matter whether those d vertices are already connected through an edge or not. The edge splitting subsists of adding a new vertex and in a next step choosing an edge in the original graph. This edge is then deleted, and the new vertex is connected to both of the endpoints. Next, connecting the new vertex to $d - 1$ other vertices of the original graph concludes the construction step.

Figure 3.18 tries to visualize the just introduced concept of Henneberg 2-construction. On the left, Figure 3.18a, the transformation given by a vertex addition is depicted. The figure on the other side, Figure 3.18b, displays the construction step of an edge splitting.

Theorem 3.30 (Schulze and Whiteley [SW17]).
Let G be a graph. If there exists a sequence of Henneberg d-constructions $(V_1, E_1), \ldots, (V_k, E_k)$, such that $(V_1, E_1) = K_d$ is the complete graph on d vertices and $(V_k, E_k) = G$, then G is generically d-isostatic. In the case of $d = 2$, the equivalence is true, i.e. additionally it holds, that every generically 2-isostatic graph can be constructed by a sequence of Henneberg 2-constructions - starting with K_2.



(a) Vertex addition in the 2-dimensional case.

(b) Edge splitting in the 2-dimensional case.

Figure 3.18.: Visualization of the Henneberg construction for the case $d = 2$ [CGG+17].

3.6. Global rigidity

As final concept of rigidity, this section provides a short introduction to global rigidity where the focus solely lies in presenting a definition. The main interest is to present another equivalent definition for natural rigidity as in Section 3.1, which might provide a deeper insight to the structure of its motions. Also, this section serves as a reminder that there are much more types of rigidity not presented in this work but nonetheless just as important.

The following both terms are introduced as in Connelly [Con05]. Let $G(p)$ and $G(q)$ be two frameworks. $G(p)$ and $G(q)$ are called *equivalent*, denoted by $G(p) \equiv G(q)$, if:

$$G(p) \equiv G(q) :\Longleftrightarrow \|p_i - p_j\| = \|q_i - q_j\|, \quad \forall \{i, j\} \in E. \tag{3.11}$$

In other words, the edge lengths of the realization p match with the respective edge lengths realized by q.

Furthermore, two configurations $p = (p_1, \ldots, p_n)$ and $q = (q_1, \ldots, q_n)$ are *congruent*, written as $p \equiv q$, if:

$$p \equiv q :\Longleftrightarrow \|p_i - p_j\| = \|q_i - q_j\|, \quad \forall i, j \in V. \tag{3.12}$$

Those two conditions (3.11) and (3.12) look very similar, but the second notion is strictly stronger, i.e. $p \equiv q \implies G(p) \equiv G(q)$. This is true since (3.12) quantifies over every possible pair of vertices and (3.11) only over the edges of the graph. The inversion is generally not true, i.e. $G(p) \equiv G(q) \not\Longrightarrow p \equiv q$, for which a counterexample is given in Figure 3.19.

Furthermore, the congruence of two realizations means that not only the lengths of edges needs to be preserved but also the distance, in the case of natural rigidity: The Euclidean distance, of any two vertices.

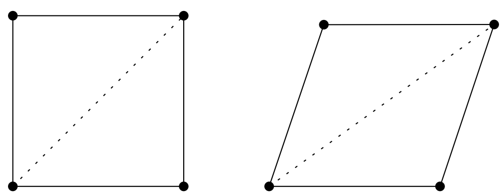


Figure 3.19.: Shifting of the unit square where the diagonal becomes larger.

Definition 3.31 (Connelly [Con05]).
A framework $G(p)$ is said to be rigid*, if:*

$$\exists \varepsilon > 0 \, \forall q \in \mathbb{R}^{dn} \colon \ \|p - q\| < \varepsilon \text{ and } G(p) \equiv G(q) \implies p \equiv q.$$

In other words Definition 3.31 states, that for a realization q sufficiently close to the original realization p, such that every edge length is the same as in the original realization, then also the distances between any two vertices are the same as for the original framework.

It might be difficult to see the link between natural rigidity as in Section 3.1 and the new characterization as in Definition 3.31, since for natural rigidity continuous motions played a major role and here just another configuration sufficiently close to the original realization. But those terms are in fact equivalent as stated in the following theorem:

THEOREM 3.32 (Schulze and Whiteley [SW17]).
For a framework $G(p)$ the following conditions are equivalent:

(a) the framework is rigid.

(b) for every motion $\{P(t)\}_{t\in[0,1]}$ of $G(p)$, $\{P(t)\}_{t\in[0,1]}$ is trivial.

(c) there is an $\varepsilon > 0$ such that, if $G(p)$ and $G(q)$ are equivalent and $\|p - q\| < \varepsilon$, then p is congruent to q.

DEFINITION 3.33 (Connelly [Con05]).
A framework $G(p)$ is called globally rigid *if*

$$G(p) \equiv G(q) \implies p \equiv q.$$

That is, the globally rigid framework are precisely those for which $G(p) \equiv G(q) \iff p \equiv q$. Figure 3.20 gives examples of frameworks that are rigid, but not globally rigid. Particularly, the framework depicted in Figure 3.20a is not globally rigid, since a non-congruent realization is obtainable by reflecting vertex 1 at the line l for which the distance of 1 and 2 has changed, but every edge length remains unchanged and hence forming an equivalent framework.

(a) Rigid framework. (b) Rigid framework with generic realization.

Figure 3.20.: Frameworks that are not globally rigid [Con05].

In contrast, Figure 3.21 provides a framework that is globally rigid. These examples are examined in further detail by Connelly [Con05].

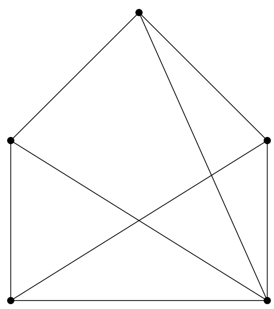


Figure 3.21.: Globally rigid framework [Con05].

As aforementioned, globally rigidity shall not be expanded at this point since it would go beyond the scope of this thesis.

3.7. Summary

3.7.1. Glossary

Natural and infinitesimal rigidity

motion of $\mathbb{R}^d$: continuous family Φ_t of isometries (for all $t \in [0,1]$), with $\Phi_0 = \mathrm{id}$.

infinitesimal motion of $\mathbb{R}^d$: derivative of a motion of $\mathbb{R}^d$ evaluated at $t = 0$, or equivalently a vector field such that moving each point $\mathbb{R}^d$ along the velocity applied at that point preserves distances in first-order.

(infinitesimal) motion of $G(p)$: a (infinitesimal) motion only applied to the vertices of $G(p)$.

trivial (infinitesimal) motion of $G(p)$: a (infinitesimal) motion $P(t)$ (or Q) of $P = G(p)$, that is induced by a (infinitesimal) motion Φ_t (or ξ), i.e. $P(t) = \Phi_t \circ P$ (or $Q = \xi \circ P$), for every $t \in [0,1]$.

(infinitesimally) rigid framework $G(p)$: every (infinitesimal) motion of $G(p)$ is trivial.

rigidity matrix of $G(p)$: the $m \times (n \cdot d)$ matrix $R_G(p)$ for the system of equations $\langle p_i - p_j, q_i - q_j \rangle = 0$, where $Q\colon V \to \mathbb{R}^d, i \mapsto q_i$ infinitesimal motion of $G(p)$.

$$R_G(p) = \begin{pmatrix} & & & \vdots & & & \\ 0 & \cdots & (p_i - p_j)^\top & \cdots & (p_j - p_i)^\top & \cdots & 0 \\ & & & \vdots & & & \end{pmatrix}$$

edge function of $G(p)$: map $f\colon \mathbb{R}^d \times \cdots \times \mathbb{R}^d \to \mathbb{R}^m, (q_1, \ldots, q_n) \mapsto (\|q_i - q_j\|^2)_{\{i,j\} \in E}$, with $\mathrm{d}f(p) = 2R_G(p)$.

Parallel designs

parallel design $G(q)$ of $G(p)$: every edge given in q is parallel to the given edge in p. That is, for every edge $\{i,j\} \in E$ it holds $\langle (p_i - p_j)^\perp, q_i - q_j \rangle = 0$.

trivial parallel design $G(q)$ of $G(p)$: q is a translation or dilation of p.

Statical rigidity

load: collection of vectors f_i, called forces, applied at the vertices of a framework.

equilibrium load: load, where sum over forces is equivalent to zero, i.e. $\sum_{i \in V} f_i = 0$.

(resolving) stress: a map of scalars $\omega_{i,j} = \omega_{j,i}$ to every edge $\{i,j\} \in E$. (It resolves the load F, if $f_i + \sum_{j \in V} \omega_{i,j}(p_i - p_j) = 0$, for all $i \in V$.)

statically rigid framework $G(p)$: every equilibrium load on $G(p)$ can be resolved.

Generic rigidity

generic realization: realization, with its dn coordinates being generic, i.e. for every dn-multivariate polynomial (non-identical to zero) the coordinates do not evaluate to zero.

generically rigid graph G: if every generic realization $G(p)$ of G is rigid.

Laman graph: graph that is $(2,3)$-tight, i.e. $|E| = 2|V| - 3$ and every k-subgraph has at most $2k - 3$ edges, where $k \geq 2$.

Henneberg d-construction: for a graph G that yields a new graph G'

(a) **vertex addition:** adding vertex t to G and connecting t to d further vertices of G.

(b) **edge splitting:** adding vertex t to G, removing edge $\{u, v\}$ of G and connecting t to the endpoints u and v, and to $d-1$ further vertices of G.

Global rigidity

equivalent frameworks $G(p)$ and $G(q)$: $G(p) \equiv G(q) :\Longleftrightarrow$ for every edge $\{i, j\} \in E$ it holds $\|p_i - p_j\| = \|q_i - q_j\|$.

congruent realizations p and q: $p \equiv q :\Longleftrightarrow$ for all $i, j \in V$ it holds $\|p_i - p_j\| = \|q_i - q_j\|$.

rigid framework $G(p)$: every realization q sufficiently close to p, with equivalent frameworks, is congruent to p.
That is, there exists an $\varepsilon > 0$ such that for every $q \in \mathbb{R}^{dn}$ with $\|p - q\| < \varepsilon$ and $G(p) \equiv G(q)$ it holds $p \equiv q$.

globally rigid framework $G(p)$: If two frameworks are equivalent, then the realizations are congruent, i.e. $G(p) \equiv G(q) \implies p \equiv q$.

3.7.2. Overview

$$
G(p)\text{ rigid} \quad \begin{array}{c} \overset{\text{Thm. 3.11}}{\Longleftarrow} \\ \underset{\text{Thm. 3.10}}{\not\Longrightarrow} \end{array} \quad \left\{ \begin{array}{l} G(p)\text{ infinitesimally rigid} \\ \overset{\text{Thm. 3.23}}{\iff} G(p)\text{ statically rigid} \\ \overset{\text{Lem. 3.19}}{\iff} G(p)\text{ trivial parallel designs only} \end{array} \right.
$$

$$\Uparrow$$

$$
\begin{array}{l} G\text{ generically rigid} \\ :\iff G(p)\text{ rigid for almost all } p \end{array} \quad \underset{\substack{\exists\text{ infinitesimal rigid} \\ \text{realization}}}{\overset{\text{Thm. 3.25}}{\iff}} \quad G(p)\text{ infinitesimally rigid for almost all } p
$$

$$\Uparrow$$

$$
\begin{array}{l} G\text{ is a Laman-graph} \\ \iff G\text{ is minimally rigid} \\ (\text{only for } d = 2) \end{array} \quad \overset{\text{Def. 3.26}}{:\iff} \quad G\text{ is } (2,3)\text{-tight} :\iff \left\{ \begin{array}{ll} & |E| = 2\,|V| - 3 \\ \text{and} & \text{every } k\text{-vertex subgraph, } k \geq 2, \\ & \text{has at most } 2k - 3\text{ edges.} \end{array} \right.
$$

4. Freedom of edge intersections

In this chapter, the main interest of this thesis is introduced. The goal is to be able to apply the knowledge of rigidity provided in the earlier chapters. Starting by defining some new terms and delivering the origin of this thesis' idea in order to fully understand the concept behind the vague title "freedom of edge intersections". This leads to a new kind of rigidity and therefore is examined in the language of rigidity theory.

4.1. Introduction

The following terms shall be presented as given by Dujmović et al. [DFG+18].

A *plane straight-line* drawing of a graph G is essentially a framework but without coinciding vertices and crossing edges. For a planar graph $G = (V, E)$, with a plane straight-line drawing, a set $S \subseteq V$ of vertices is said to be *free*, if for any set of points, $X \subseteq \mathbb{R}^2$, with $|X| = |S|$, there exists a one-to-one map $p\colon S \to X$ and a plane straight-line drawing of G, where every vertex $v \in S$ is placed at the point $p(v)$.

Furthermore, a set of vertices $S \subseteq V$ is called *collinear*, if there exists a plane straight-line drawing of G where all vertices of S lie on one line.

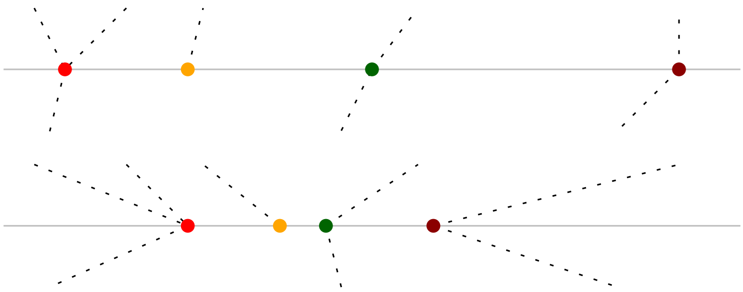
Finally, a set $S \subseteq V$ of vertices is said to be *free collinear*, if S is free but with X restricted to only collinear sets of points. Figuratively spoken this means, for a set of points that all lie on one line these points may be arbitrarily moved along this line and still a plane straight-line drawing of the graph is obtainable. Arbitrarily is not quite correct, since the order of the points must not change along that line. In contrast, the term of a free set intuitively seems to be strictly stronger than the notion of a free collinear set, since vertices may be placed arbitrarily in the whole space $\mathbb{R}^2$ and do not need to restrict on a hyperplane of $\mathbb{R}^2$. In fact, Dujmović et al. [DFG+18] provide the following astonishing result for any set $S \subseteq V$ of any planar graph $G = (V, E)$:

$$S \text{ is free set} \iff S \text{ is collinear set} \iff S \text{ is free collinear set.}$$

As the title of their paper suggests this means:

> "Every collinear set in a planar graph is free"
>
> — Dujmović, Frati, Gonçalves, Morin, and Rote

Figure 4.1 ought to indicate the concept of free collinear sets. A set of collinear points is given, which are displayed by coloured points on the gray line. The dotted lines at each vertex represent the edges incident to a certain vertex. In fact, only a truncated interesting part of an arbitrary graph is displayed, and therefore focus lies on the collinear points. The actual graph might be larger but is not of interest in this situation. This style of visualization shall occur again in Figure 4.3. The original drawing is suggested in Figure 4.1a and Figure 4.1b indicates a new drawing with the vertices displaced on the line of collinearity. Hereby also the respective edges undergo certain transformations.



(a) Initial plane straight-line drawing.

(b) New drawing with moved vertices.

Figure 4.1.: Visualization of a free collinear set.

This whole concept may be called "freedom of collinear vertices" or as well in broader general "freedom of vertices". Hopefully, thereby one gets a slight idea of what awaits behind the notion "freedom of edge intersections". Nonetheless, the subject shall be discussed step by step.

Given a plane straight-line drawing of a graph G, there possibly are edges that intersect with the x-axis. Here, not only the edge is considered but the actual line defined by the edge so-called *supporting line*. In the following, an edge and its supporting line shall be identified. In the work of Dujmović et al. [DFG+18] it was also presented that for a free collinear set $S \subseteq V$ on line l, not only the vertices $v \in S$ may positioned arbitrarily, but also the intersections of supporting lines with line l may be prescribed *ε-precisely*. That is, the crossing point is at most ε away from its prescribed position. This leads to the question: "may these positions be prescribed absolutely?". The following sections treat this question and present an idea of answering it with tools of rigidity theory. Without loss of generality and for sake of simplicity, this work only considers the x-axis as line l and not any arbitrary line. In fact, for this scenario, zeros and intersections with the x-axis of supporting lines are equivalent. Therefore, the following special situations may occur

infinitely many zeros: if the supporting line is the x-axis, i.e. the incident vertices both lie on the x-axis.

no zeros: if the supporting line is parallel to the x-axis, i.e. the y-coordinates of the incident vertices coincide.

no supporting line: in the situation of frameworks it is possible that the realization of two vertices may coincide and therefore there exist infinitely many supporting lines. But since the vertices may only coincide if they are not incident, this scenario is not relevant in this consideration.

Another question that may appear is "how can the first two special situations be eliminated?". The answer is quite simple, since every line, that is not horizontal, has exactly one zero.

Therefore, it could be useful to prohibit horizontal edges in general. Every pair of points that does not induce one of these special situations shall be called *good-natured*.

For the sake of simplicity, in this work it shall solely be spoken of *intersections* instead of "intersections of edges and the x-axis", where no ambiguity may occur. There are two questions that are of interest as part of this thesis:

- "Can those intersections be arbitrarily displaced on the x-axis such that one still obtains a plane straight-line drawing or at least a straight-line drawing?"
- "How do the trivial motions look like if one wants the intersections to stay invariant?"

Of further interest is the class of graphs where those crossing points can be arbitrarily displaced along the x-axis. Desirable would be a characterization of those graphs as given by Laman the $(2,3)$-tightness for rigidity (see Definition 3.26). That is, hope is to find an invariant such that those graphs with *free edge intersections* can be classified as rigid and therefore apply theorems of rigidity theory.

Section 4.5 presents a principle that may be used to only answer the second question and instantly retrieving a result for the first question as well. Of particular interest is answering the first question. The detour via results for the second question is only proposed since it seems to be considerably easier.

4.2. Definitions

This section turns out to be relatively technical, since all necessary terms need to be formally defined. Next, contributions to this topic as part of this work are presented. Throughout this section $G = (V, E)$ defines a graph and $G(p)$ denotes a framework of this graph. The framework $G(p)$ is restricted to only have edges which are non-horizontal. This is due to the treatment of zeros or rather the intersection of an edge's supporting line with the x-axis, which proves to be difficult for horizontal edges. For this purpose the following map is defined:

$$
\begin{aligned}
& z\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}, \\
z\left(p_i := \begin{pmatrix} a_i \\ b_i \end{pmatrix}, p_j := \begin{pmatrix} a_j \\ b_j \end{pmatrix}\right) &:= -\, b_i \cdot \frac{a_j - a_i}{b_j - b_i} + a_i \\
&= \frac{-b_i a_j + a_i b_j}{b_j - b_i} \qquad\qquad \forall i, j \in V \\
&= \frac{-\langle p_i^{\perp}, p_j \rangle}{b_j - b_i}.
\end{aligned}
$$

Essentially, z is an explicit term for the zero of the line in parameter-form given as $\overrightarrow{\boldsymbol{p_i}} + t \cdot \overrightarrow{\boldsymbol{p_i p_j}}$, for $t \in \mathbb{R}$. Take note, that z does not consider the aforementioned special situations. In this work they are assumed to not occur, i.e. only good-natured lines are evaluated.

First, the edge zeros are proposed as invariant, i.e. examining continuous motions of a framework such that the intersections stay the same. Considering a continuous motion $\{\Phi(t) := (G, p(t))\}_{t\in[0,1]}$ of $G(p)$, this motion needs to satisfy:

$$
z(p_i(t), p_j(t)) = z(p_i, p_j), \tag{4.1}
$$

for all $t \in [0, 1]$ and for all $\{i, j\} \in E$.

4.2.1. Zeros-preserving Motions

Analogously to the introduction of length-preserving maps in Section 3.1, this section provides definition for maps that preserve zeros of the line spanned by two arbitrary 2-dimensional points so-called *zeros-preserving maps*. These zeros-preserving maps are then used to introduce the term *zeros-preserving motions*, as it was the case for length-preserving motions or rather motions in the scenario of natural rigidity in Section 3.1. These terms shall be abbreviated with *motions*, but must be treated with caution due to its ambiguous character. Furthermore, a local version of these (zeros-preserving) motions shall be introduced and referred to by *infinitesimal (zeros-preserving) motions*. Particular interest lies in those motions that are *trivial*, which proves to be more difficult to examine.

DEFINITION.
Let $\Phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ *be a map.* Φ *is called* zeros-preserving*, if:*

$$z(\Phi(x), \Phi(y)) = z(x, y), \quad \forall \textit{ good-natured } x, y \in \mathbb{R}^2.$$

The terms *(zeros-preserving) motions*, *infinitesimal motions* and *trivial (infinitesimal) motions* are defined along the same lines of the definitions presented for natural rigidity and infinitesimal rigidity, see Section 3.1 and Section 3.2, respectively, but with respect to zeros-preservation instead of length-preservation. Particularly this means, that infinitesimal motions may change zeros in first-order. In order to provide a characterization for infinitesimal motions, the other way around shall be taken. That is, by defining a *zeros function*, obtaining an *edge matrix* similar to the rigidity matrix and examining its kernel. For further details see Remark 3.12 and Section 4.4.

It is not quite obvious to see how the class of trivial motions looks like but one can certainly say, that x*-shearing* and y*-stretching* of the framework are part of it. This question shall be treated more precisely in Section 4.3 and Section 4.4 deals with their infinitesimal pendants.

Before continuing an understanding for x-shearing and y-stretching shall be provided by giving concrete examples and then a definition in a formal sense. Also, *shearing* and *stretching* simplified denote x-shearing and y-stretching, respectively. Figure 4.2a visualizes the stretching of the coordinate system, given in dashed black, by a factor of two, resulting in the red thick grid. With the same notation a shearing happens in Figure 4.2b, but by factor $\frac{1}{2}$.

Both shearing and stretching are linear transformations given through matrices of certain forms which shall be elucidated hereby

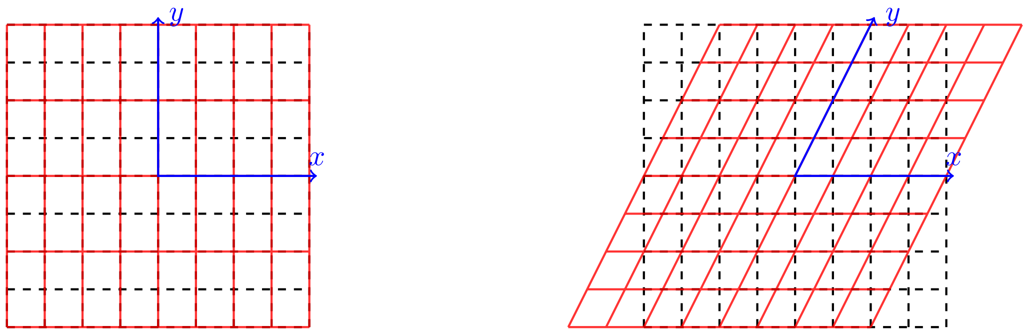
Stretching

$\mathcal{T}_{St} := \left\{ \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix} \in \mathbb{R}^{2\times 2} \right\}$ defines the set of *stretching matrices*.
A stretching is then of the form $x \mapsto A \cdot x$, for $A \in \mathcal{T}_{St}$, and k is called the *factor* of the stretching.

Shearing

$\mathcal{T}_{Sh} := \left\{ \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \in \mathbb{R}^{2\times 2} \right\}$ defines the set of *shearig matrices*.
A shearing is then of the form $x \mapsto A \cdot x$, for $A \in \mathcal{T}_{Sh}$, and k is called the *factor* of the shearing.

Concretely, the stretching matrix corresponding to Figure 4.2a is $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and the shearing matrix that corresponds to Figure 4.2b is $\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}$.



(a) Stretching of the coordinate system by the factor 2.

(b) Shearing of the coordinate system by the factor $\frac{1}{2}$.

Figure 4.2.: Trivial motions for preserving edge intersections.

4.2.2. Edge-x-rigidity and Edge-x-freedom

Finally, the definition of *edge-x-rigidity* and *edge-x-freedom* may be provided.

DEFINITION 4.1.
A framework is called edge-x-rigid *if all of its motions are trivial. A framework is called* edge-x-flexible *if it is not edge-x-rigid.*

Recent work of Dewar [Dew18] states the following propositions that may be helpful:

> "For any graph G with $|V(G)| \geq 2$, G is isostatic in the Euclidean plane if and only if G is $(2, 3)$-tight."
>
> "Let X be a non-Euclidean normed plane. Then a graph G is isostatic in X if and only if G is $(2, 2)$-tight."
>
> — Dewar

In this context *isostatic graphs* refer to "[...]rigid graphs with no proper spanning rigid subgraphs[...]", whereas *non-Euclidean normed plane* denotes "[...]a 2-dimensional space with a norm that is not induced by an inner product[...]" – Dewar. Investigating this further would go beyond the bounds of this thesis, however a short summary of this proposition might be of help. If the invariant, for edge-x-rigidity, could be defined in such a way that it deals with a norm[1] (not induced by an inner product), the *infinitesimally edge-x-rigid* frameworks, with $|E| = 2 \cdot |V| - 2$, are exactly the $(2, 2)$-tight graphs. Provided it is a norm for a non-Euclidean normed plane. Section 4.3 provides at least three degrees of freedom, which contradicts the theory of an realization in a non-Euclidean plane, as shall be seen there.

The following provides definitions for the terms *edge-x-freedom* and its complement, *edge-x-tiedness*, introduced in this thesis.

DEFINITION 4.2.
A graph $G = (V, E)$ is said to be edge-x-free *if for every $Z \in \mathbb{R}^E$ it holds*

$$\exists \text{ realization } q\colon z(q_i, q_j) = (Z)_{\{i,j\}}, \quad \forall \{i, j\} \in E.$$

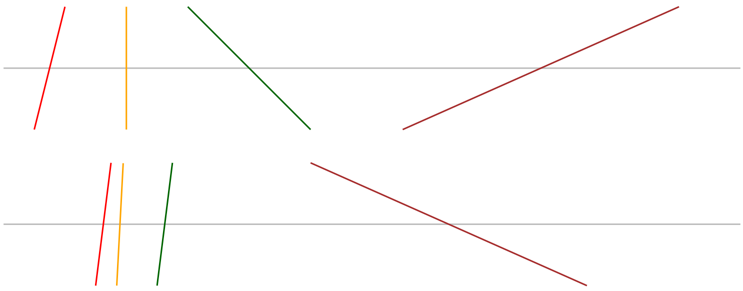
Otherwise, G is called edge-x-tied.

This definition might be too strict in that sense, that there might only be graphs that are edge-x-free for almost all configurations of zeros $Z \in \mathbb{R}^E$. That is, graphs may only be

[1] "dealing with a norm" describes the situation given in the invariant for natural rigidity

edge-x-free for generic configurations $Z \in \mathbb{R}^E$, as it is the case for Laman graphs that are rigid for almost all configurations of edge lengths.

Figure 4.3 displays an edge-x-free framework with the original situation given in 4.3a and a transformed scenario in 4.3b. Hereby, the same notation as in the description of free collinear sets take place (see Figure 4.1).



(a) Original placement of edge intersections with the x-axis.

(b) Displacement of the edge intersections along the x-axis.

Figure 4.3.: Visualization of an edge-x-free graph.

Since only edge-x-free frameworks were considered so far one could also ask oneself how it is the case with *edge-x-free sets*, i.e. only the intersections of certain edges may be displaced along the x-axis. More formally this leads to the following definition:

Definition 4.3.
For a graph $G = (V, E)$ a set $E' \subseteq E$ is called edge-x-free *if for every $Z \in \mathbb{R}^{E'}$ it holds*

$$\exists \text{ realization } q\colon z(q_i, q_j) = (Z)_{\{i,j\}}, \quad \forall \{i, j\} \in E'.$$

It is important to notice that realizations are not demanded to be free of edge crossings. The interest solely subsists of the question of the existence of a straight-line drawing (basically a framework) at all. That is, an edge-x-free framework is a graph $G = (V, E)$ with E being an edge-x-free set.

The concept of edge-x-free sets shall not be examined in further detail here, but appears as a tool for simplification of notation in Section 4.4.

4.3. Trivial motions

Finally, the degrees of freedom for any edge-x-rigid graph may be investigated in this section. In a next step, it is tried to transfer the knowledge of edge-x-rigidity acquired so far to *infinitesimal* edge-x-rigidity. The ideal case would be, that infinitesimal edge-x-rigidity is just a projective transformation of infinitesimal rigidity. This would then allow application of every statement of rigidity theory – considering infinitesimal rigidity – to infinitesimal edge-x-rigidity. In particular, one would retrieve the following statement for a graph G:

$$G \text{ is Laman graph} \overset{\text{Thm. 3.28}}{\Longleftrightarrow} G \text{ is 2-minimally rigid} \overset{?}{\Longleftrightarrow} G \text{ is minimally edge-x-rigid.}$$

A short excursion to projective geometry shall now be presented.

Definition (Beutelspacher and Rosenbaum [BR04], Izmestiev [Izm09a]).
Let $\mathbb{R}^{d+1}$ be a $d+1$-dimensional vector space and $\sim$ denotes an equivalence relation on $\mathbb{R}^{d+1} \setminus \{0\}$ given by

$$x \sim x' :\Longleftrightarrow \exists \lambda \in \mathbb{R}^\times, \text{ such that } x' = \lambda x.$$

Then the d-dimensional real-projective space $\mathbb{RP}^d$ *is defined as the* quotient space $^{\mathbb{R}^{d+1}\setminus\{0\}}/_{\sim}$. *The equivalence classes of points* $(x_1, \ldots, x_{d+1})^\top \in \mathbb{R}^{d+1}$ *or rather points of the projective space are denoted by* $[x_1 : \ldots : x_{d+1}] \in \mathbb{RP}^d$ *and called* homogeneous coordinates.

In other words, the points of $\mathbb{RP}^d$ are 1-dimensional subspaces of $\mathbb{R}^{d+1}$ without the origin.

DEFINITION (Izmestiev [Izm09a]).
Let $\tilde{\pi}$ *be a linear map of* $\mathbb{R}^{d+1}$. *A* projective transformation π *of* $\mathbb{RP}^d$ *is a map* $\mathbb{RP}^d \to \mathbb{RP}^d$ *with*

$$\pi([x]) = [\tilde{\pi}(x)], \quad \forall [x] \in \mathbb{RP}^d.$$

THEOREM 4.4 (Schulze and Whiteley [SW17], Izmestiev [Izm09b]).
For a framework $G(p)$ *and a projective transformation* T *of* $\mathbb{RP}^d$, *where no point of* p *is projected to infinity,* $G(p)$ *is infinitesimally rigid if and only if* $G(T \circ p)$ *is infinitesimally rigid.*

As stated in [Izm09a], every affine transformation of $\mathbb{R}^d$ is a projective transformation. More precisely the “Affine transformations of $\mathbb{R}^d$ are exactly those projective transformations of $\mathbb{RP}^d$ that map the line at infinity l_∞ to itself” – Izmestiev [Izm09a]. In the terms of Theorem 4.4 this does not only establish *projective invariance* for infinitesimal rigidity, but also an *affine invariance*.

As seen so far, shearing and stretching are always applicable in order to preserve edge-x-rigidity. In the following the degrees of freedom shall be visualized by considering a concrete Laman graph on 7 vertices that is obtained by only applying vertex addition of the Henneberg constructions onto K_2. The graph is then given in Figure 4.4. There the red marks denote the intersection for the corresponding edge. The dotted lines represent the supporting line of an edge, that does not directly cross the x-axis (displayed by the green thick line). This notation shall be applied to all upcoming figures of this section.

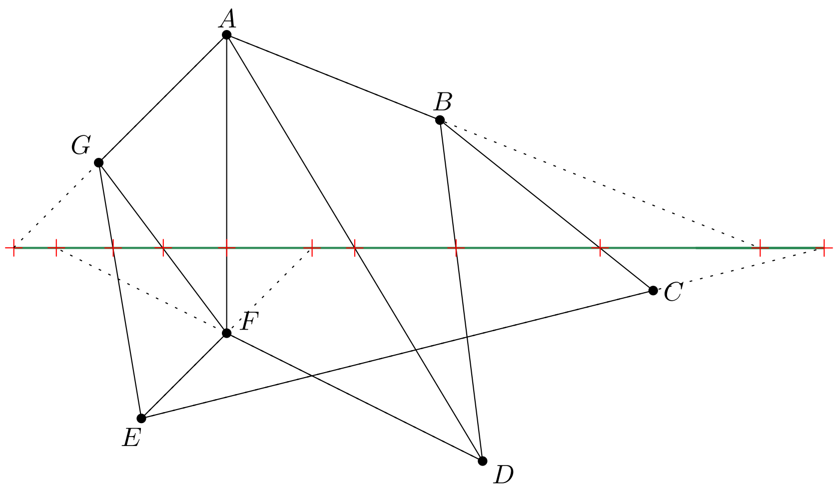
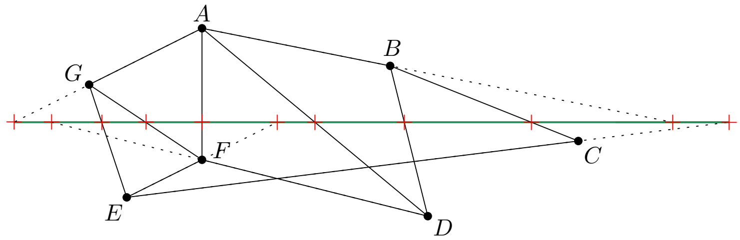


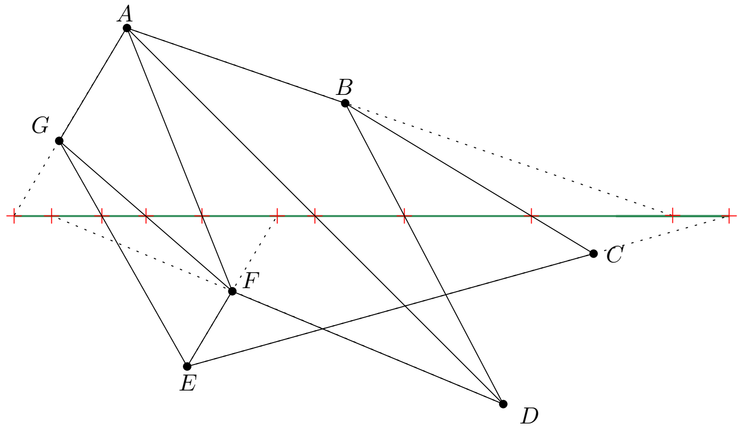
Figure 4.4.: Realization of a Laman-graph with 7 vertices, constructed by only using vertex addition as Henneberg construction.

In Figure 4.5a a simple stretching of the framework is depicted – the framework has been compressed towards the x-axis – and Figure 4.5b visualizes a simple shearing.

The following turns towards the (hopefully) last degree of freedom, where it is not quite clear what it even does, but shall be discussed more detailed. At first, a little visualization of what is happening is provided in Figure 4.6.



(a) Stretching of the framework by a factor of $\frac{1}{2}$.



(b) Shearing of the framework by a negative factor.

Figure 4.5.: Visualization of shearing and stretching as trivial motions that preserve the intersections.

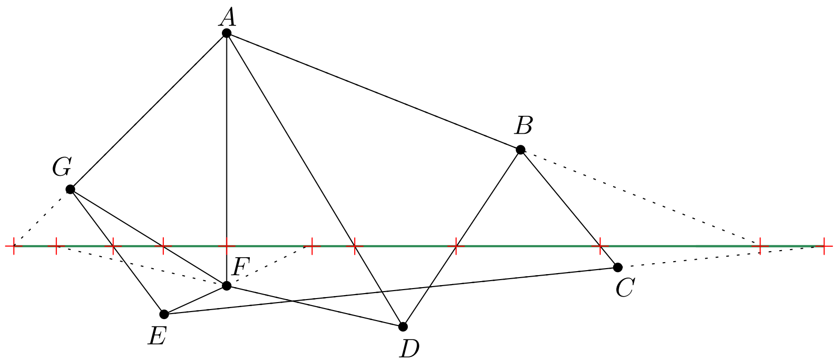


Figure 4.6.: Another – possibly trivial – motion of the framework.

Comparing it to the other figures it looks like the vertex A did not change its position, which is actually the case. Furthermore, every other vertex X moves along the line given by $\{A, X\}$ towards the intersection of the edge $\{A, X\}$. This means A is somehow the centre of this particular motion. One question that may appear is, what if the framework is realized in $\mathbb{C}_\infty \coloneqq \mathbb{C} \cup \{\infty\}$, i.e. $p\colon V \to \mathbb{C}_\infty$. Additionally, A is mapped to ∞ and every edge incident to A becomes the vertical line with the same point of intersection. Figure 4.7 gives a glimpse on this kind of realization, which in this work is called the *semi-hyperbolic* realization of a graph. Since attaching vertex A eliminates two degrees of freedom, the shearing and stretching, and only the motion of interest remains. The motion depicted in Figure 4.6 shall be referred to as a *central collineation* (with respect to centre A), which is clarified in the next step.

For the semi-hyperbolic realization the vertices may only be displaced along the corresponding orange lines, that display the edges incident to A. In fact, the only motion applicable here is again a stretching of the graph. One might think, that a graph with fewer vertices is considered and therefore also fewer edges, but this is actually not the case here. The vertex A is still part of the graph, it is just not visible any more, since it lays at infinity. The

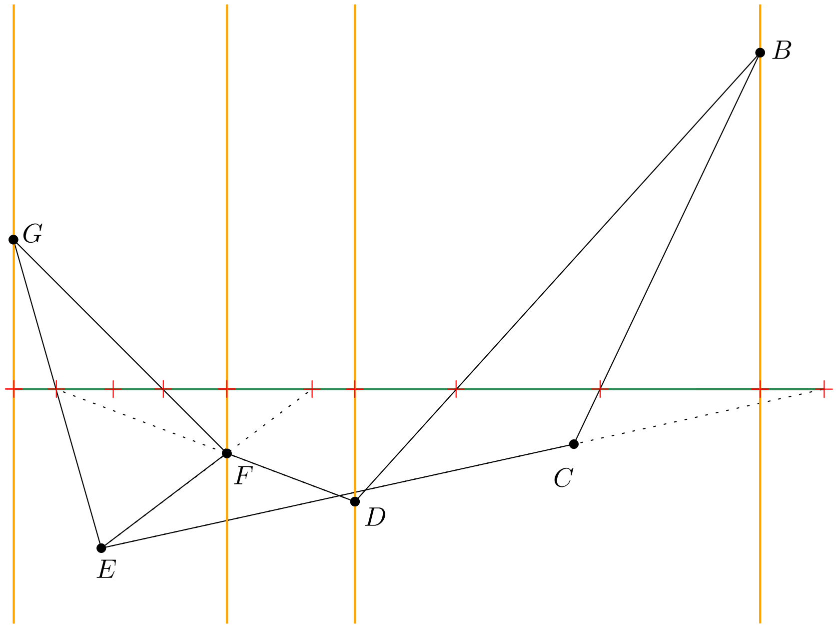


Figure 4.7.: Semi-hyperbolic realization of the graph, such that the intersections are the same as in the original embedding.

same holds for the "vertical edges" or rather orange lines, which enforce the x-coordinate or rather real part of every vertex X to be the same as the intersection of the edge $\{A, X\}$ of the realization in the classical sense. Regarding Theorem 4.4 the problem becomes clear, since A has been mapped to infinity, which does not preserve the projective invariance of infinitesimal rigidity in general. Therefore, another option to handle this certain motion is necessary, which then explains the usage of the term *central collineation*.

DEFINITION 4.5 (Beutelspacher and Rosenbaum [BR04]).
Let $\mathbb{R}\mathrm{P}^n$ *be a projective space of dimension* $n \geq 2$, *and* $\pi\colon \mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^n$ *be a collineation. Then,* π *is called a* central collineation, *if it satisfies the following two conditions:*

1. *there is a point* $Z \in \mathbb{R}\mathrm{P}^n$, *such that every line* g *through* Z *is a* fixline *of* π, *i.e.* $\pi(g) = g$.

2. *there is a hyperspace* H, *i.e. a* $(n-1)$*-dimensional subspace of* $\mathbb{R}\mathrm{P}^n$, *such that* $\pi|_H = \mathrm{id}$.

Then Z *is said to be the* centre *and* H *is said to be the* axis *of* π.

The correspondence to the unknown motion now becomes very clear. For this motion, A refers to the centre and the x-axis is the axis. Now there appears the question: "is there a reason for taking A as centre?". And the answer simply is "no, there is not". Any other vertex or even point could have been chosen as centre for this motion, as long as the axis stays the x-axis.

Finally, a useful theorem in order to handle central collineations shall be presented.

THEOREM 4.6 (Beutelspacher and Rosenbaum [BR04]).
Let $\pi\colon \mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^n$ *be a central collineation with center* Z *and axis* H.
Then for any point $X \in \mathbb{R}\mathrm{P}^n$, *with* $X \neq \pi(X)$, *the tuple* $(X, \pi(X))$ *uniquely determines* π.

4.4. Infinitesimal motions

Since the analysis of trivial motions in the scenario of edge-x-rigidity proved to be difficult, this section deals with a local version of edge-x-rigidity. Local version here refers to the equivalent of infinitesimal rigidity as presented for natural rigidity, see Section 3.2.

Therefore, the examination of the respective infinitesimal motions or rather trivial infinitesimal motions form the basis of this section. For this, obtaining a matrix similar to the rigidity matrix $R_G(p)$ in Section 3.2, that encodes the *infinitesimal edge-x-rigidity* properties plays a major part. This matrix shall be called *edge matrix* and simply referred to by $E_G(p)$, where the same notation applies as for the rigidity matrix. The subsequent task is to determine the kernel of the edge matrix $\ker E_G(p)$ and a corresponding base or rather a base for a subspace of the kernel.

First, in order to define the edge matrix, the *zeros function* e is introduced, analogously to the edge function f in Remark 3.12, which gives the zeros of every edge. That is, for a graph $G = (V, E)$, the zeros function is defined as

$$\begin{aligned} e\colon \mathbb{R}^2 \times \cdots \times \mathbb{R}^2 &\to \mathbb{R}^m \\ e(q_1, \ldots, q_n) &\coloneqq (z(q_i, q_j))_{\{i,j\} \in E} \,. \end{aligned} \tag{4.2}$$

Analogously to the edge function f, the derivative $\mathrm{d}e$ of e is determined, which then yields the edge matrix, i.e. $\mathrm{d}e(q) = E_G(q)$. For this purpose, set $q_i = (u_i, v_i) \in \mathbb{R}^2$, for $i \in V$. In respect thereof, the partial derivatives of z are given as

$$\begin{aligned} \frac{\partial z}{\partial u_i}(q_i, q_j) &= -\frac{v_j}{v_i - v_j} \\ \frac{\partial z}{\partial v_i}(q_i, q_j) &= \frac{(u_i - u_j)v_j}{(v_i - v_j)^2} = \frac{1}{v_i - v_j}\left(-z(q_j, q_i) + u_j\right) \end{aligned}$$

and hence

$$\begin{aligned} \frac{\partial z}{\partial q_i}(q_i, q_j) \coloneqq \left(\frac{\partial z}{\partial u_i}, \frac{\partial z}{\partial v_i}\right) &= \frac{1}{v_i - v_j}\left(-v_j, -z(q_j, q_i) + u_j\right) \\ &= \frac{v_j}{v_j - v_i}\left(1, \frac{u_i - u_j}{v_j - v_i}\right) \eqqcolon z'_{q_i} \in \mathbb{R}^2. \end{aligned}$$

In order to define the edge matrix, take note that z is *symmetric*, i.e. $z(q_i, q_j) = z(q_j, q_i)$. Therefore, the derivative $\mathrm{d}e(q)$ of e – and in particular the edge matrix $E_G(q)$, is given as

$$\begin{aligned} \mathrm{d}e(q_1, \ldots, q_n) = {} & \begin{array}{c c} & \begin{array}{c c c c c c c c c c c} & & & i & & & & j & & & \end{array} \\ \{i,j\} & \begin{pmatrix} & & & & & \vdots & & & & & \\ 0 & \cdots & 0 & z'_{q_i} & 0 & \cdots & 0 & z'_{q_j} & 0 & \cdots & 0 \\ & & & & & \vdots & & & & & \end{pmatrix} \end{array} \\ \coloneqq {} & E_G(q). \end{aligned} \tag{4.3}$$

Now it is possible to examine the kernel of the matrix $E_G(p)$, which is given as the solutions to $E_G(p) \cdot q = 0$ and in fact yields the following equations. Throughout this section the following notion is applied: $p_k = (a_k, b_k)$, for $k \in V$, and $b_i \neq b_j$ for any $\{i, j\} \in E$. As a reminder: $b_i \neq b_j$ denotes the non-horizontalness of every edge of the framework $G(p)$.

$$\begin{aligned}
&\langle z'_{p_i}, q_i \rangle + \langle z'_{p_j}, q_j \rangle = 0 \\
\iff &\frac{1}{b_i - b_j} \left(\langle \begin{pmatrix} -b_j \\ -z(p_j, p_i) + a_j \end{pmatrix}, q_i \rangle - \langle \begin{pmatrix} -b_i \\ -z(p_i, p_j) + a_i \end{pmatrix}, q_j \rangle \right) = 0 \\
\iff &\langle \begin{pmatrix} 0 \\ -z(p_j, p_i) \end{pmatrix} + \begin{pmatrix} -b_j \\ a_j \end{pmatrix}, q_i \rangle - \langle \begin{pmatrix} 0 \\ -z(p_j, p_i) \end{pmatrix} + \begin{pmatrix} -b_i \\ a_i \end{pmatrix}, q_j \rangle \\
= &\langle \begin{pmatrix} 0 \\ -z(p_j, p_i) \end{pmatrix}, q_i - q_j \rangle + \langle p_j^\perp, q_i \rangle - \langle p_i^\perp, q_j \rangle = 0, \quad \forall \{i, j\} \in E.
\end{aligned} \tag{4.4}$$

Given a framework $G(p)$, $T_{i,p} \in \mathbb{R}^{2|V|}$ defines an *infinitesimal central collineation* as follows

$$T_{i,p} := \left(b_j \cdot (p_j - p_i) \in \mathbb{R}^2 \right)_{j \in V}.$$

An *infinitesimal x-shearing* and *y-stretching* is given by

$$\begin{aligned}
T_x &:= \begin{pmatrix} b_1 & 0 & \dots & b_i & 0 & \dots & b_n & 0 \end{pmatrix}^\top \in \mathbb{R}^{2|V|} \quad \text{and} \\
T_y &:= \begin{pmatrix} 0 & b_1 & \dots & 0 & b_i & \dots & 0 & b_n \end{pmatrix}^\top \in \mathbb{R}^{2|V|},
\end{aligned}$$

respectively.

Theorem 4.7.
Let $G(p)$ be a framework with no horizontal edge and $i \in V$ fixed, then

$$\operatorname{span}\{T_x, T_y, T_{i,p}\} \subseteq \ker E_G(p) = \ker e(p).$$

And particularly T_x, T_y and $T_{i,p}$ are linearly independent, that is

$$\dim(\ker E_G(p)) \geq 3 = \dim(\operatorname{span}\{T_x, T_y, T_{i,p}\}).$$

A proof for Theorem 4.7 is given in Appendix A due to its length and chaotic character.

Corollary 4.8 (of Theorem 4.7).
For a framework $G(p)$, it holds $\operatorname{rank} E_G(p) \leq 2n - 3$.

Proof of Corollary 4.8. This directly follows from the Rank-nullity theorem (see Theorem 3.15), which states $\operatorname{rank} E_G(p) = 2n - \dim(\ker E_G(p))$ and since Theorem 4.7 gives $\dim(\ker E_G(p)) \geq 3$. □

Lemma 4.9.
Let $G(p)$ be framework with only non-horizontal edges and $i \in V$ fixed, then

$$T_{j,p} = (a_i - a_j) \cdot T_x + (b_i - b_j) \cdot T_y + T_{i,p} \in \operatorname{span}\{T_x, T_y, T_{i,p}\}, \quad \forall j \in V \setminus \{i\}$$

◊

In other words, Lemma 4.9 states, that the centre of any infinitesimal central collineation can be changed by linearly combining an infinitesimal central collineation, with different centre, with infinitesimal shearing and stretching.

Conjecture 4.10.
Let $G(p)$ be an edge-x-rigid framework. Fixing the position p_i of any vertex i, yields

$$\ker e(q) = \ker E_G(q) = \operatorname{span}\{T_{i,q}\}$$

Corollary 4.11 (of Conjecture 4.10).
Let $G(p)$ be a framework without horizontal edges.
$G(p)$ is infinitesimally edge-x-rigid if and only if $\operatorname{rank} E_G(p) = 2n - 3$.

4.5. Outlook

This final section outlines the concept of answering both of the questions given at the end of Section 4.1. At first, a classification is sought for the class of minimally edge-x-rigid graphs, as established in Theorem 3.28 for the minimally 2-rigid graphs. There it was proven that the minimally 2-rigid graphs are exactly the Laman graphs or rather the $(2,3)$-tight graphs. Here minimally means that the deletion of any edge yields a flexible (or edge-x-flexible) graph. The following shall illuminate this in the case of continuous rigidity and then transferring the results to minimally edge-x-rigid graphs.

Therefore, let $G = (V, E)$ be a minimally 2-rigid graph. Definition 3.31 states that rigid means, that not only the length of every edge is preserved but also the distances between any two vertices. In particular, removing any edge $e = \{u, v\} \in E$ – denoted by $G \setminus e$ – gives a flexible graph, since G is minimally 2-rigid. That is, there exists a non-trivial motion Φ_t, that leaves the edge lengths unchanged, but alters the distance between two vertices that are not adjacent. Due to the minimality of the edge e, the distance $\|u - v\|$ is modified. In a more formal sense $\|\Phi_t(u) - \Phi_t(v)\| \neq \|\Phi_0(u) - \Phi_0(v)\| = \|u - v\|$ holds, for a $0 < t \leq 1$.

Proposition 4.12.
Let $G = (V, E)$ be a minimally 2-rigid graph and $e = \{u, v\} \in E$ an edge.
Then $G \setminus e$ is flexible and in particular there exists a non-trivial motion for $G \setminus e$ that changes the distance of the nodes u and v.

Proof. Let p be a generic configuration. The flexibility of $G'(p) := G(p) \setminus e$ then follows by the definition of minimally rigid graphs.
Since $G'(p)$ is flexible, there is a non-trivial motion Φ_t of $G'(p)$ that does preserve the length of every edge of $G'(p)$. If Φ_t would also preserve the distance between any two points, the edge e would not be necessary for rigidity. This is a contradiction to the minimality of G and therefore for at least one pair of vertices the distance between them is modified.
Now it remains to show that the distance between u and v is altered. This is again due to the minimality of the edge e. If Φ_t changes the distance of, say, e.g. $a \in V$ and $b \in V$ that are not connected by an edge and $\{a, b\}$ is not the edge e, then it would be possible to apply the non-trivial motion Φ_t to the original framework $G(p)$. This could be done since Φ_t preserves the length of every edge in $G \setminus e$ and particularly also the distance between u and v, i.e. the length of the edge e. Therefore, Φ_t fulfils the condition for a non-trivial motion of $G(p)$. This, again, is a contradiction, since $G(p)$ was assumed to be rigid and therefore completes the proof. □

The same principle introduced in Proposition 4.12 shall now be applied to minimally edge-x-rigid graphs. In a short summary this means, that for a minimally edge-x-rigid graph G removing any edge e results in an edge-x-flexible graph and most importantly the intersection of e can be altered by a non-trivial motion for $G \setminus e$. Hope is to achieve the ability of moving this intersection arbitrarily along the x-axis. Is this possible, e is inserted into a set E' that is edge-x-free with respect to G. The next step is to reinsert the edge e – with changed intersection – that yields $G(p')$, which in turn is again minimally edge-x-rigid since it is a property for graphs and not only for frameworks. Repeating this procedure for every edge $e' \in E \setminus E'$ completes the construction and finally it holds $E = E'$. Therefore, G is edge-x-free.

Achieving a characterization for the minimally edge-x-rigid graphs, as Laman [Lam70] did for the minimally 2-rigid graphs ($(2, 3)$-tightness), would then give a characterization of the edge-x-free graphs. In fact, those classes of graphs would coincide. The statement in

Proposition 4.12 means for Laman graphs, that for almost all configuration of edge lengths there exists a realization. Here “almost all” is obligatory since for, e.g. a triangle, the lengths of the edges may not be prescribed to be 1, 1 and 100, respectively. As stated by Capco et al. [CGG+17], the Laman graphs are graphs for which every generic configuration of edge lengths yields a rigid realization.

5. Conclusion

Based on the concept of rigid frameworks and graphs, a new type of rigidity, concerning free and collinear sets, was introduced as edge-x-rigidity. Chapter 4 gave an introduction to edge-x-rigidity, where three degrees of freedom were deduced for any framework. These three degrees are shears, stretches and central collineations or rather their infinitesimal pendants. This type of rigidity raises further questions regarding those graphs that are minimally rigid in that sense and the class of graphs, where its edges can be placed arbitrarily along the x-axis.

QUESTION 5.1. *Do (infinitesimal) shearing, stretching and central collineation fully form the space of (infinitesimal) trivial motions?*

This question can be answered by considering the complete graph K_n on $2 \leq n \in \mathbb{N}$ vertices and its corresponding edge matrix $E_{K_n}(q)$. If the maximum rank of $E_{K_n}(q)$ is $2n-3$ the question can be answered positively. For complete graphs on small amounts of nodes $n \leq 5$, the question has been answered positively by trial and error.

In the scope of this thesis it has been proven so far, that infinitesimal shearing, stretching and central collineation are linearly independent. Furthermore, those transformations all lie in the kernel $E_G(p)$ of any framework $G(p)$ that does not have horizontal edges. Regarding infinitesimal central collineations, a proof of changing its centre by linearly combining shearing, stretching and central collineation has been given as well. Therefore, central collineations with different centre are not actual new trivial infinitesimal motions.

QUESTION 5.2. *Are the Laman graphs exactly the minimally edge-x-rigid graphs?*

QUESTION 5.3. *How can the minimally edge-x-rigid graphs be characterized?*

QUESTION 5.4. *Are the terms infinitesimal edge-x-rigid and infinitesimal rigid projectively equivalent?*

With regard to Theorem 4.4, answering Question 5.4 positively would then instantly proof Question 5.2 to be true as well. This would present an easier way in order to answer the main question Question 5.6. Nonetheless, results considering Question 5.3 would just be as useful in dealing with the main question.

In Section 4.5 the principle of relating minimally edge-x-rigid with edge-x-free graphs has been presented in the terms of natural rigidity. In the case of natural rigidity it has been stated, that one could find a realization of Laman graphs for almost all prescriptions of edge lengths. This yields the following question in terms of minimally edge-x-rigidity.

Question 5.5. *May the crossing points only be prescribed* almost *arbitrarily?*

Question 5.6. *How can the edge-x-free graphs be characterized?*

In order to treat the questions Question 5.6 and Question 5.3 a leading approach could be an examination of edge-x-rigidity in a matroidal way, as it was presented in Remark 3.27.

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Appendix

A. Proof of Theorem 4.7

Proof of Theorem 4.7. For a realization $p_k = (a_k, b_k) \in \mathbb{R}^2, k \in V$, refers to its coordinate components. Let $G(p)$ be a framework without horizontal edges, i.e. $b_k \neq b_j$ for all $k \neq j \in V$, and $i \in V$ fixed. First of all it shall be shown, that T_x, T_y and $T_{i,p}$ actually lie in the kernel of $E_G(p)$. For this, let q_k and q_j be the entries $2k-1, 2k$ and $2j-1, 2j$, respectively, of T_x, T_y and $T_{i,p}$. Then, it suffices to show, that q_k and q_j satisfy the condition given in (4.4).

I $\underline{T_x \in \ker E_G(p)}$**:** set $q_k := (T_x)_k = \begin{pmatrix} b_k \\ 0 \end{pmatrix}$ and $q_j := (T_x)_j = \begin{pmatrix} b_j \\ 0 \end{pmatrix}$.

$$\begin{aligned} &\langle \begin{pmatrix} 0 \\ -z(p_j, p_k) \end{pmatrix}, q_k - q_j \rangle + \langle p_j^{\perp}, q_k \rangle - \langle p_k^{\perp}, q_j \rangle \\ &= 0 + (-b_j) \cdot b_k - (-b_k) \cdot b_j = 0. \end{aligned}$$

II $\underline{T_y \in \ker E_G(p)}$**:** set $q_k := (T_y)_k = \begin{pmatrix} 0 \\ b_k \end{pmatrix}$ and $q_j := (T_y)_j = \begin{pmatrix} 0 \\ b_j \end{pmatrix}$.

$$\begin{aligned} &\langle \begin{pmatrix} 0 \\ -z(p_j, p_k) \end{pmatrix}, q_k - q_j \rangle + \langle p_j^{\perp}, q_k \rangle - \langle p_k^{\perp}, q_j \rangle \\ =& \left(b_j \cdot \frac{a_k - a_j}{b_k - b_j} - a_j \right) (b_k - b_j) + a_j \cdot b_k - a_k \cdot b_j \\ =& -a_j \cdot b_k + a_j \cdot b_j + b_j \cdot a_k - b_j \cdot a_j + a_j \cdot b_k - a_k \cdot b_j = 0. \end{aligned}$$

III $\underline{T_{i,p} \in \ker E_G(p)}$**:** set $q_k := (T_{i,p})_k = b_k \cdot (p_k - p_i)$ and $q_j := (T_{i,p})_j = b_j \cdot (p_j - p_i)$.

$$
\begin{aligned}
&\langle \begin{pmatrix} 0 \\ -z(p_j, p_k) \end{pmatrix}, q_k - q_j \rangle + \langle p_j^\perp, q_k \rangle - \langle p_k^\perp, q_j \rangle \\
=& -z(p_j, p_k)(b_k(b_k - b_i) - b_j(b_j - b_i)) - b_j b_k(a_k - a_i) \\
&+ a_j b_k(b_k - b_i) + b_k b_j(a_j - a_i) - a_k b_j(b_j - b_i) \\
=& -z(p_j, p_k)b_k(b_k - b_i) + z(p_k, p_j)b_j(b_j - b_i) \\
&+ b_j b_k((a_k - a_i) - (a_j - a_i)) + a_j b_k(b_k - b_i) - b_k b_j(b_j - b_i) \\
=& \, b_k(b_k - b_i)(-z(p_j, p_k) + a_j) + b_j(b_j - b_i)(z(p_k, p_j) - a_k) - b_j b_k(a_k - a_j) \\
=& \, b_k(b_k - b_i)\left(b_j \frac{a_k - a_j}{b_k - b_j}\right) - b_j(b_j - b_i)\left(b_k \frac{a_k - a_j}{b_k - b_j}\right) - b_j b_k(a_k - a_j) \\
=& \, (a_k - a_j)\left(-b_j b_k + \frac{b_j b_k((b_k - b_i) - (b_j - b_i))}{b_k - b_j}\right) \\
=& \, (a_k - a_j)(-b_j b_k + b_j b_k) = 0.
\end{aligned}
$$

It remains to show the linear independence of the three vectors, which can be seen by only considering two vertices $k \neq i$ and j whose positions do not coincide, and then considering linear combinations of their respective trivial motions q_k and q_j. Therefore, if there is no non-zero solution to the following equation proves the statement:

$$
\lambda_1 \begin{pmatrix} b_k \\ 0 \\ b_j \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ b_k \\ 0 \\ b_j \end{pmatrix} + \lambda_3 \begin{pmatrix} b_k(a_k - a_i) \\ b_k(b_k - b_i) \\ b_j(a_j - a_i) \\ b_j(b_j - b_i) \end{pmatrix} = 0
$$

In order to fulfil the constraints a necessary condition would be, e.g. $\lambda_1 = (a_k - a_i)$ and $\lambda_2 = (b_k - b_i)$. This then yields the equations

$$
\begin{aligned}
b_j(a_k - a_i) = -\lambda_3 b_j(a_j - a_i) &\overset{b_j(a_j - a_i) \neq 0}{\iff} -\lambda_3 = \frac{a_k - a_i}{a_j - a_i} \\
b_j(b_k - b_i) = -\lambda_3 b_j(b_j - b_i) &\overset{b_j(b_j - b_i) \neq 0}{\iff} -\lambda_3 = \frac{b_k - b_i}{b_j - b_i},
\end{aligned}
$$

which does not have a solution and therefore concludes the proof. $\square$