On Covering Numbers of Different Kinds

Bachelor Thesis of

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Statement of Authorship

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Abstract

A *covering number* measures how "difficult" it is to cover all edges of a *host graph* with *guest graphs* of a given *guest class*. E.g., the global covering number, which has received the most attention, is the smallest number k such that the host graph is the union of k guest graphs from the guest class. In this thesis, we consider the *global*, the *local* and the *folded covering number* and compare them for same pairs of host and guest classes.

We investigate by how much global, local and folded covering number can differ. We give an example of a hereditary union-closed guest class where the folded covering number is at most 2, whereas the local covering number can be arbitrarily large. In contrast we show that within every host class the global covering number with regards to a topological minor-closed union-closed guest class is bounded if and only if its corresponding folded covering number is bounded.

In the context of computational complexity, we construct a host class within which computing the local covering number with regards to interval graphs is $\mathcal{NP}$-hard, whereas computing the global covering number is possible in constant time. Further, we give an example of a union-closed guest class and a host class within which the global covering number is easily computable and the local covering number is not computable at all.

Moreover, we spend major attention to the guest class of *linear forests* (collections of paths). We prove the *Local Linear Arboricity Conjecture*, and show that the folded, the local and the global linear arboricity of a graph can differ and that deciding whether the linear arboricity is at most k is $\mathcal{NP}$-complete for every $k \geq 2$.

Finally, we consider the *boxicity* of a graph H as global covering number of the complement H^c and introduce the corresponding union-closed covering number as *union boxicity* and the local covering number (which in this setting coincides with the folded covering number) as *local boxicity*. We present geometric interpretations for these parameters and show that the boxicity may be arbitrarily large for host graphs with local and union boxicity 1.

This thesis answers thereby several questions raised by Knauer and Ueckerdt [KU12].

Deutsche Zusammenfassung

Eine *Überdeckungszahl* misst wie "schwer" es ist alle Kanten eines *Gastgebergraphen* mit *Gastgraphen* einer gegebenen *Gastklasse* zu überdecken. Die globale Überdeckungszahl ist z.B. die kleinste Zahl k für die der Gastgebergraph die Vereinigung von k Gastgraphen ist. Sie hat bisher am meisten Aufmerksamkeit erhalten. In dieser Arbeit betrachten wir die *globale*, die *lokale* und die *gefaltete Überdeckungszahl* und vergleichen sie bezüglich gleicher Paare von Gastgebergraph und Gastklasse.

Wir untersuchen wie stark die globale, die lokale und die gefaltete Überdeckungszahl voneinander abweichen können. Wir geben ein Beispiel einer hereditären Gastgeberklasse an, bezüglich der die gefaltete Überdeckungszahl höchstens 2 ist, während die entsprechende lokale Überdeckungszahl beliebig groß werden kann. Im Gegensatz dazu zeigen wir, dass in jeder Klasse von Gastgebern die globale Überdeckungszahl bezüglich einer Gastklasse, die für jeden enthaltenen Graph auch all seine topologischen Minoren enthält und unter disjunkter Vereinigung abgeschlossen ist, genau

dann beschränkt ist, wenn die entsprechende gefaltete Überdeckungszahl beschränkt ist.

Im Kontext der Berechnungskomplexität konstruieren wir eine Gastgeberklasse für deren Gastgebergraphen die Berechnung der lokalen Überdeckungszahl bezüglich der Gastklasse der Intervalgraphen $\mathcal{NP}$-schwer ist, während die globale Überdeckungszahl in konstanter Zeit berechnet werden kann. Wir geben weiter ein Beispiel einer Gastklasse, die unter disjunkter Vereinigung abgeschlossen ist, und eine Gastgeberklasse, in der die globale Überdeckungszahl einfach berechenbar ist, die lokale Überdeckungszahl jedoch überhaupt nicht berechenbar ist.

Darüber hinaus schenken wir der Gastklasse der linearen Wälder (disjunkter Vereinigungen von Pfaden) größere Aufmerksamkeit. Wir beweisen die *Local Linear Arboricity Conjecture*, und zeigen, dass die globale, die lokale und die gefaltete Überdeckungszahl voneinander abweichen können und, dass es für jedes $k \geq 2$ $\mathcal{NP}$-schwer ist zu entscheiden, ob die lineare Arboricity eines gegebenen Gastgebergraphen höchstens k beträgt.

Schließlich betrachten wir die *Boxicity* eines Graphen H als eine globale Überdeckungszahl seines Komplements H^c und führen die entsprechende Überdeckungszahl bezüglich dem Abschluss der Gastklasse unter disjunkter Vereinigung als die *union Boxicity* sowie die entsprechende lokale Überdeckungszahl (die in diesem Fall mit der gefalteten übereinstimmt) als *lokale Boxicity* ein. Wir geben für diese geometrische Interpretationen an und zeigen, dass die Boxicity für Gastgeberklassen mit union und lokaler Boxicity 1 beliebig groß werden kann.

Diese Arbeit beantwortet damit verschiedene Fragen, die Knauer und Ueckerdt aufgestellt haben [KU12].

Contents

1. Introduction

The *(global) covering number* of a *host graph* H with regards to a *guest class* $\mathcal{G}$ of *guest graphs* is the smallest number d such that H has d subgraphs $G_1, \ldots, G_d \in \mathcal{G}$ that form a *cover* of H, i.e., every edge of H is contained in at least one of them. The first survey addressing the general problem of covering graphs was written by Beineke in 1969 [Bei69]. A similar problem is the problem of *decomposing* graphs, which asks for the minimum number d such that H has d *edge-disjoint* subgraphs $G_1, \ldots, G_d \in \mathcal{G}$ that form a cover of H. Actually, these two problems coincide sometimes, especially for guest classes that are *hereditary* (closed under taking subgraphs). Already in 1891 Petersen presented a decomposition of $2n$ regular graphs into n 2-factors, which coincides with covers using n collections of cycles [Pet91].

Another example is the covering number with regards to forests, called *arboricity*. Nash-Williams proved for the arboricity $c_g^{\mathcal{F}}(H)$ of every host graph H that we need only as many forests as guests, as needed to provide enough edges to cover every subgraph, i.e., that $c_g^{\mathcal{F}}(H) = \max_{H' \subseteq H} \lceil |H|/(||H|| - 1) \rceil$ [NW64]. In 1970, Akiyama, Exoo and Harary stated the *Linear Arboricity Conjecture (LAC)* with a similar assertion for the covering number with regards to linear forests (collections of paths), introduced by Harary [Har70] as *linear arboricity*. It states for every host graph H with maximum degree Δ that its linear arboricity, denoted by $c_g^{\overline{\mathcal{P}}}(H)$, is either $\lceil \Delta/2 \rceil$ or $\lceil (\Delta + 1)/2 \rceil$ [AEH80]. Since linear forests have maximum degree 2, we need at least $\lceil \Delta/2 \rceil$ guests to cover all edges incident to a vertex v of degree Δ and, if a path of a guest ends in v, we need $\lceil (\Delta+1)/2 \rceil$, correspondingly. The LAC has received much attention, but it is still open today. We give an overview of related work in Chapter 4.

Covering numbers we consider in this thesis beneath the arboricity and the linear arboricity are the *star arboricity* (w.r.t. star forests) [AA89], the *caterpillar arboricity* (w.r.t. caterpillar forests) and the *track-number* (w.r.t. interval graphs) [GW95] as well as the *chromatic index* (w.r.t. matchings).

There are several applications for covering numbers. E.g., in VLSI layout we use multiple layers to realize graphs. While every vertex is present in every layer, edges are realized in single layers. However, if two edges would cross in the same layer, the layer of one of them must be changed to avoid this by a “cross cut” instead. Since too many cross cuts cause high costs, it is of interest to avoid them. Further, a high number of layers increases the costs, too. We want therefore to reduce the number of used layers. The covering number with regards to planar graphs (called *thickness*) gives the smallest number of layers such that no cross cut is needed [AKL+85].

Another example is secure broadcasting [SLY06]. We can think of messages and receivers as the two partition sets of a bipartite graph H where a message is connected to those receivers who shall receive this message. To avoid that other receivers can read the message we use keys to encrypt the messages. As those keys are costly, we want to use a minimum number of keys in total. Every message encrypted with the same key is received by all receivers knowing that key. Therefore, the distribution of one key covers a complete bipartite subgraph of H and the number of keys needed in total is given by the covering number with regards to complete bipartite graphs called *bipartite dimension*.

Knauer and Ueckerdt introduced the local and folded covering number [KU12]. Let H be a host graph and $\mathcal{G}$ be a guest class. The *local covering number* considers, instead of the total number of guest graphs, the maximum number of guest graphs containing a common vertex. I.e., the local covering number of H with regards to $\mathcal{G}$, denoted by $c_l^{\mathcal{G}}(H)$, is defined as the minimum number d such that H has some subgraphs $G_1, \ldots, G_m \in \mathcal{G}$ such that every edge is contained in at least one of them and every vertex is contained in at most d of them. The *folded covering number* is based on the concept of *folding*. Folding describes the process of identifying two non-adjacent vertices v and w giving a new vertex u that is connected to every vertex that has been connected to v or w before. The folded covering number of H with regards to $\mathcal{G}$ is denoted by $c_f^{\mathcal{G}}(H)$ and is the smallest number d such that there is a guest graph $G \in \mathcal{G}$ such that H can be received by foldings in G such that every vertex v in H is the result of folding at most d vertices of G.

Two local covering numbers have already been investigated: The *bipartite degree* was introduced by Fishburn and Hammer as local covering number with regards to complete bipartite graphs [FH96] and the local covering number with regards to complete graphs has been introduced by Javadi, Maleki and Omoomi [JMO12] under the name *local clique cover number*, while Skums, Suzdal and Tyshkevich considered it before [SST09].

The *interval number* is the folded covering number with regards to interval graphs and has been extensively studied. It was introduced by Trotter and Harary [TH79] and, e.g., has applications in DNA sequence comparison [JMT92], RNA secondary structure prediction [Jia10] as well as scheduling and resource allocation as already stated by Trotter and Harary and explained by Butman et al. [BHLR10].

One of their examples is loss minimization. We can view the run-time of a process as a set of intervals on the time line. If the run-times of two processes overlap, this can be expressed by an edge between them in the corresponding intersection graph H. A maximum independent set on H yields therefore an optimal choice of processes that can run without interference. Instead of run-times of processes we can also consider e.g. requests of clients. The problem of serving a maximal number of clients without overlapping requests is equivalent. Clients with requests that fill multiple intervals in the time line are a natural scenario in, e.g., remote education where clients make breaks during video programs [BYHN$^+$06].

Further, more abstract problems can be formulated in terms of covering numbers. E.g., *vertex covers* can be formulated as covers with regards to the guest class of stars and the *intersection number* equals the covering number with regards to complete graphs [Rob85].

Since it is possible to consider other problems as covering problems, covering numbers are equally interesting for mathematicians and computer scientists.

Let us consider *union-closed* guest classes, i.e., those that are closed under taking disjoint unions. For the folded covering number we can consider the disjoint union of guest graphs as one guest graph and then identify (fold) vertices of different components, which allows all covers allowed for the local covering number. Additionally, we allow foldings within single components. Thereby, the folded covering number is a relaxation of the local covering number. Since every vertex is covered by at most all guest graphs of a cover, the local covering number is itself a relaxation of the global covering number.

Since the folded and local covering number are relaxations for union-closed guest classes, we have an inequality stating that in this case the folded covering number is at most the local covering number and this again is at most the global covering number [KU12].

It appears that it is also computationally easier to deal with folded and local covering number rather than with the global one. I.e., Knauer and Ueckerdt observed cases in which determining the global, the local and the folded covering number is $\mathcal{NP}$-hard and cases where determining the global covering number is $\mathcal{NP}$-hard, whereas the local and folded covering number can be computed in polynomial time. However, we are not aware of any case where determining the folded or the local covering number is $\mathcal{NP}$-hard, whereas the global covering number is computable in polynomial time [KU12].

As stated by Beineke, the usual approach to learn about global covering numbers is first to prove upper bounds and then find host graphs for which these are sharp [Bei69]. Now we have another approach considering folded and local covering number. They give (for union-closed guest classes) lower bounds and sometimes the corresponding covers can be transformed to covers yielding also upper bounds. Additionally, it can be seen as indication for the upper bound of a global covering number to be correct if this upper bound holds for the folded or local covering number. In this context it is interesting in which cases the differences between folded, local and global covering number are bounded and by how much they may differ. Knauer and Ueckerdt have given examples of pairs of guest and host class where the local covering number is constant, whereas the global covering number can be arbitrarily large, and asked for a similar result for the comparison of the folded and the local covering number [KU12].

This thesis is organized as follows.

In **Chapter 2** we give basic definitions considered in this thesis.

In **Chapter 3** we state alternative definitions of the global, the local and the folded covering number in terms of edge-surjective graph homomorphisms and give related definitions. Further, we introduce some of the guest classes we use, especially interval graphs, and state general conclusions, thereby identifying conditions on guest classes under which a reasonable comparison of the different covering numbers is possible.

In **Chapter 4** we treat the *Linear Arboricity Conjecture* and give related work. Its main goal is to prove the corresponding Local Linear Arboricity Conjecture stated by Knauer and Ueckerdt [KU12]. It states for every host graph H with maximum degree Δ that its local covering number with regards to linear forests (its *local linear arboricity*) is either $\lceil \Delta/2 \rceil$ or $\lceil (\Delta+1)/2 \rceil$. This conjecture is a consequence of following the approach of considering folded and local covering number to attack a problem for the global covering number.

In **Chapter 5** we deal with separations of different covering numbers. I.e., we consider the possible difference between two covering numbers for the same guest class and host class/graph taking different restrictions to the guest class into account. First, we state results of Knauer and Ueckerdt. Then we prove for the hereditary union-closed guest class of bipartite graphs that the folded covering number is at most 2, whereas the local covering number can be arbitrarily large. Further, we prove that the folded, the local and the global linear arboricity may differ. Finally, we discuss results for even more restricted guest classes and prove that for topological minor-closed union-closed guest classes the global covering number in every host class is bounded if and only if the corresponding folded covering number is bounded.

In **Chapter 6** we discuss the computational complexity of determining a covering number of a given host graph especially considering whether it can be computationally harder to compute a folded or local covering number than the corresponding global covering number. We prove that deciding whether the *local track-number*, the covering number with regards to interval graphs, is at most k is $\mathcal{NP}$-complete for every $k \geq 2$ as stated by Knauer and Ueckerdt. Further, we present an induced-hereditary union-closed guest class and a union-closed host class such that determining the local covering number for a given host graph is $\mathcal{NP}$-complete, whereas the global covering number is easily computable. We even prove that for a union-closed guest class and a union-closed host class the local covering number may be not computable at all, whereas the global covering number is easily computable. Finally, we prove that deciding whether the linear arboricity of a given host graph is at most k is $\mathcal{NP}$-complete for every $k \geq 2$, which is a new result for $k \geq 3$.

In **Chapter 7** we consider the *boxicity* of graphs in terms of covering numbers. It was introduced by Roberts and considers graphs as intersection graphs of multidimensional intervals, called *boxes* [Rob69]. As Cozzens and Roberts already observed, the boxicity of a graph H equals the global covering number of its complement H^c with regards to $\mathcal{I}^c$, the class of complements of interval graphs [CR83]. Correspondingly, we introduce the local and global covering number of H^c with regards to $\overline{\mathcal{I}^c}$, the union-closure of $\mathcal{I}^c$, as the local boxicity and the union boxicity of H. For these parameters we give geometric interpretations: A graph with union boxicity d is the intersection graph of boxes whose dimensions can be partitioned into d sets such that every box is the Cartesian product of intervals that equal $\mathbb{R}$ in all but at most one dimension of every set. A graph with local boxicity d, however, is the intersection graph of boxes that are the Cartesian product of intervals that equal $\mathbb{R}$ in all but at most d dimensions. Further, we give examples of graphs with local and union boxicity 1 that have arbitrarily large boxicity.

2. Preliminaries

For any set S we define $\binom{S}{2} = \{\{a,b\} : a,b \in S, a \neq b\}$. For any mapping $\phi : A \to B$ and any subset $C \subseteq A$ we write $\phi(C)$ for $\{\phi(c) : c \in C\}$. Further, we define $\phi_{|C} : C \to B$ as $x \mapsto \phi(x)$. For any $k \in \mathbb{N}_{>0}$ we write $[k]$ for $\{i \in \mathbb{N}_{>0} : i \leq k\}$.

In this thesis a *graph* G is a tuple (V, E) with the finite vertex set V and the edge set $E \subseteq \binom{V}{2}$ and $\forall e \in E : \exists u, v \in V : u \neq v, e = \{u, v\}$. The graph G is said to be a graph *on* V. Elements of V are called *vertices* and elements of E are called *edges*. An edge $\{v, u\}$ is shortly denoted as vu. By $V(G)$ we denote the vertex set of G and by $E(G)$ its edge set. We sometimes write $v \in G$ for $v \in V(G)$. We call $|V|$ the *order* of G and it is denoted by $|G|$. We call $|E|$ the *size* of G and it is denoted by $||G||$.

For two edges $vu, wv \in E$ we call the vertices v and u *adjacent*, we call the edges uv and vw *adjacent* and we call the vertex v and the edge uv incident. We say vu connects v and u.

A graph $G' = (V', E')$ is called *subgraph* of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. The graph G is then called a *supergraph* of G' and said to *contain* G' and we write in short $G' \subseteq G$. The graph G' is called *induced subgraph* of G if $V' \subseteq V$ and $E' = E \cap \binom{V'}{2}$. We say V' *induces* G' in G. The graph G' is called *spanning* in G if $G' \subseteq G$ and $V' = V$. A subset $W \subseteq V$ is called *independent* if it induces a graph without edges. The graph induced by W is then also called *independent set*. The *chromatic number* of G is the smallest number k such that $V(G)$ can be partitioned into k independent sets. We write $\chi(G)$ for the chromatic number of G. If $\chi(G) \leq 2$ we call G a *bipartite* graph. If its two independent sets are fully connected, the graph G is called *complete bipartite*. We say a subset $F \subset E$ induces (U, F) in G where U is the set of all vertices incident to an edge of F in G ($U = \bigcup F$). A set of pairwise non-adjacent edges is called *matching*. We also call a graph a *matching* if its induced by a matching. A spanning matching is called *perfect*. A k-regular spanning subgraph is called *k-factor*.

The *intersection* of two graphs G_1 and G_2 is defined as $(V(G_1) \cap V(G_2), E(G_1) \cap E(G_2))$ and denoted by $G_1 \cap G_2$.

The union of disjoint sets is itself called *disjoint*. If we speak of the *disjoint union of graphs* we assume their vertex sets to be disjoint (Formally, for a family $\{G_i : i \in I\}$ of graphs with index set I we replace for every $i \in I$ every vertex v in the vertex set V_i (and all edges) of G_i by (v, i) and speak of (v, i) $(V(G_i) \times \{i\})$ as v in G_i $(V(G_i))$.

Let G and H be graphs. The *disjoint union* of G and H is denoted by $G \cupdot H$ and is defined as the graph with vertex set $V(G) \cupdot V(H)$ and edge set $E(G) \cupdot E(H)$.

For any vertex $v \in V$ its *neighbourhood* is defined as $\{u \in V : vu \in E\}$ and denoted by $\mathrm{N}(v)$. Its elements are called *neighbours* of v. The *degree* of v is defined as $|\mathrm{N}(v)|$ and is denoted by $\deg_G(v)$ or simply $\deg(v)$. If $\deg(v) = 0$, then v is called *isolated.* If $\deg(v) = 1$, then v is called a *leaf.* If $\deg(v)$ is even, then v is called *even.* Otherwise v is called *odd.* The *maximum degree* of G is defined as $\Delta(G) := \max_{v\in V} \deg(v)$ and the *minimum degree* of G is defined as $\delta(G) := \min_{v\in V} \deg(v)$. If $\delta(G) = \Delta(G) =: r$, then G is called *r-regular* and has *regularity* r. The *average degree* $\mathrm{avd}(G)$ of a graph G is $\frac{\Sigma_{v\in V(G)} \deg(v)}{|V(G)|} = \frac{2|E(G)|}{|V(G)|}$. By the *maximum average degree* $\mathrm{mad}(G)$ of a graph G we denote the maximum average degree of all induces subgraphs of G,i.e., $\mathrm{mad}(G) = \max_{H\subseteq G} \mathrm{avd}(H)$.

The *complement* $\overline{G}$ of G is defines as the graph $(V, \binom{V}{2} \setminus E)$.

If $E = \binom{V}{2}$ then G is called *complete*, a *clique* and to be a K_n with $n = |G|$. We call the graph $(\emptyset, \emptyset)$ the *empty graph.* The graph G is called a *path* if up to relabeling of vertices $V = \{v_0, \dots, v_{n-1}\}$ and $E = \{v_iv_{i+1} : 0 \le i < n\}$. Graph G is then denoted by P_n and its *ends* are v_0 and v_{n-1}. Note that for $n > 1$ a path P_n has two ends and $\Delta(P_n) \le 2$. In a graph G we say two vertices are *connected by a path* if they are the ends of a subgraph of G that is a path. The *length* of P_n is $n - 1$ for $n \in \mathbb{N}_{>0}$ and denoted by $||P_n||$. A cycle C_n is a graph received from a path P_{n+1} with $n \ge 3$ by identifying its ends. Note that a cycle is 2-regular. The *girth* of G is the smallest size of a cycle that is a subgraph of G. We denote the girth by $\mathrm{g}(G)$. A graph is called *forest* if it has girth ∞, i.e., it has no cycle as subgraph. The graph G is called *triangle-free* if $\mathrm{g}(G) \ge 4$. A spanning path is called *Hamilton path.* A spanning cycle is called *Hamiltonian cycle.*

We say G is *connected* if any two vertices of G are connected by a path. If G' is an inclusion-maximal connected subgraph of G, then it is called a *(connected) component* of G. Connected forests are called *trees.*

The *line graph* $L(G)$ of G is defined as the graph (E, F) with edge set

$$F = \{\{e_1, e_2\} : e_1, e_2 \in E, e_1 \text{ and } e_2 \text{ are adjacent in } G\}.$$

A planar graph is a graph G with an embedding into the plane. An embedding into the plane is an injective mapping from the vertices of G to elements of $\mathbb{R}^2$ and a mapping from the edges of G to Jordan curves such that the ends of the Jordan curves are the images of the ends of the corresponding edges and the Jordan curves do neither intersect otherwise nor contain images of vertices otherwise.

Let G and H be graphs. A function $\phi : V(G) \to V(H)$ is called a *homomorphism* if $vu \in E(G)$ implies $\phi(v)\phi(u) \in E(H)$. We also write $\phi : G \to H$. Note that two adjacent vertices $u, v \in G$ may not be mapped onto the same vertex $x \in H$, since this would imply an edge from x to itself, which is not possible in graphs in this thesis. If for every $xy \in E(H)$ there is an edge $vu \in E(G)$ such that $\phi(\{v, u\}) = \{x, y\}$ ($\phi(\{u, v\}$ is the image $\{\phi(u), \phi(v)\}$), then ϕ is called *edge-surjective.* If this edge $vu \in E(G)$ is unique for every edge $xy \in E(H)$, then ϕ is called *edge-bijective.* Let $G' \subseteq G$. We write $\phi(G')$ for the graph $(\phi(V(G')), F)$ where $F = \{\phi(e) : e \in E(G')\}$.

An *Eulerian tour* of G is an edge-bijective homomorphism from a cycle to G.

An bijective, edge-bijective homomorphism is called an *isomorphism.* A graph G is called *isomorphic* to a graph H if there is an isomorphism $\phi : G \to H$. In this case we write $G \simeq H$. Note that $\simeq$ is an equivalence relation.

In this paper we assume all graph classes to be closed under taking isomorphic graphs.

3. Covering Numbers

3.1 Motivation

Knauer and Ueckerdt have considered the classical covering number and introduced some of its generalizations [KU12]. They defined *a*) the local covering number as a generalization of the bipartite degree introduced by Fishburn and Hammer [FH96] and the local clique covering number introduced by Javadi, Malexi and Omoomi [JMO12] and *b*) the folded covering number as a generalization of the interval number introduced by Trotter and Harary [TH79]. They called the classical covering number itself the global covering number.

We show that the local covering number is a relaxation of the global covering number, and for *union-closed* graph classes the folded covering number is also a relaxation of the local covering number. It further appears to be easier to deal with the folded covering number than to deal with the local one, and that it is easier to deal with the local one than with the global one. This encourages considering the new covering numbers whenever they are sufficient. Further it motivates the approach to consider them in order to learn more about global covering numbers. To support this approach, it is of interest to analyse how the covering numbers are related.

3.2 Global, Local and Folded Covering Number

Let $\mathcal{G}$ be a class of graphs. A *cover of H with regards to $\mathcal{G}$* is a finite multiset $S = \{G_1, \dots, G_n\}$ with $G_i \in \mathcal{G}$ for $i \in [n]$ and an edge-surjective homomorphism c from $G_1 \mathbin{\dot\cup} \dots \mathbin{\dot\cup} G_n$ to H. The *size* of c is defines as $|S|$. The elements of S are called *guests* of c. We say that a graph $R \subseteq G_1 \mathbin{\dot\cup} \dots \mathbin{\dot\cup} G_n$ covers a vertex $v \in H$ if $v \in c(V(R))$ and it covers an edge $uv \in H$ if there is an edge $xy \in E(R)$ with $c(x) = u$ and $c(y) = v$. We say that a vertex $v \in G_1 \mathbin{\dot\cup} \dots \mathbin{\dot\cup} G_n$ covers $c(v)$ and an edge $vw \in G_1 \mathbin{\dot\cup} \dots \mathbin{\dot\cup} G_n$ covers $c(v)c(w)$ if this is an edge in H. We call $\mathcal{G}$ the *guest class*, its elements *guest graphs* and H the *host graph* of c. The cover c is *injective* if for all $i \in [n]$ the restriction $c_{|V(G_i)}$ is injective. An injective edge-bijective cover is called a *decomposition*. Figure 3.1 shows two examples of covers using paths as guests.

Let H be a graph and $\mathcal{G}$ be a graph class. The *global covering number* of H with regards to $\mathcal{G}$ is denoted by $c_g^{\mathcal{G}}(H)$ and defined as

$$c_g^{\mathcal{G}}(H) = \min\{\text{size of } c : c \text{ is injective cover of } H \text{ with regards to } \mathcal{G}\}.$$

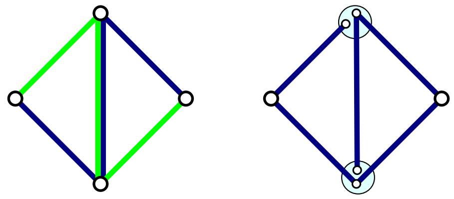

Figure 3.1: On the left: Injective cover of size 2 with regards to $\mathcal{P}$. On the right: non-injective cover of size 1 with regards to $\mathcal{P}$. Here $\mathcal{P}$ denotes the class of all paths.

We can imagine it as the smallest number of colours needed to colour all edges of H (allowing multiple colours on the same edge) such that the edges of every colour induce a guest graph (a graph in $\mathcal{G}$).

The *local covering number* of H with regards to $\mathcal{G}$ is denoted by $c_l^{\mathcal{G}}(H)$ and defined as

$$c_l^{\mathcal{G}}(H) = \min\{\max_{v \in H} |c^{-1}(v)| : c \text{ is injective cover of } H \text{ with regards to } \mathcal{G}\}.$$

We can imagine it as a colouring number as we do for the global covering number, but here we do not count the total number of colours but rather the maximum number of colours used on the edges incident to any single vertex.

The *folded covering number* of H with regards to $\mathcal{G}$ is denoted by $c_f^{\mathcal{G}}(H)$ and defined as

$$c_f^{\mathcal{G}}(H) = \min\{\max_{v \in H} |c^{-1}(v)| : c \text{ is cover of } H \text{ with regards to } \mathcal{G} \text{ of size } 1\}.$$

We call a cover of size 1 that is not necessarily injective a *folded cover*. For a folded cover $c : G \to H$ with $G \in \mathcal{G}$ and any vertex $v \in H$ we say that the vertices in $c^{-1}(v)$, which is an independent set in G, are *folded* into v by c. We can imagine the folded covering number with regards to $\mathcal{G}$ as the smallest $k \geq 0$ such that we can partition the vertex set of a guest graph into independent sets of size at most k such that identifying the vertices in each partition class into a single vertex yields H. (By definition the empty graph has folded covering number $-\infty$, but we do not consider it. For independent sets it is 0 and otherwise it is at least 1.)

Example 3.1
Figure 3.2 and Figure 3.3 give two example of the covering numbers of two different graphs with regards to $\overline{\mathcal{C}_4}$, the class of all disjoint unions of C_4 cycles. (We introduce the line over a class as notation for the closure under taking disjoint unions later.)

Consider the injective cover of the graph H_1 in Figure 3.2. Since all three C_4 cycles are edge disjoint and the only C_4 cycles in H_1, every injective cover of H_1 with regards to $\overline{\mathcal{C}_4}$ must contain these three cycles to cover all edges. Therefore there are at least three guests in an injective cover of H_1. Thus, we have $c_g^{\overline{\mathcal{C}_4}}(H_1) = 3$. Further, there are at most two guests covering the same vertex in the given injective cover. Thus, we have $c_l^{\overline{\mathcal{C}_4}}(H_1) = 2$. Now consider the non-injective cover of H_1 of size 1. The maximum number of vertices covering the same vertex is 2 (the vertices covered twice are the vertices of the triangle). Hence, we have $c_f^{\overline{\mathcal{C}_4}}(H_1) = 2$. Note that a folded or global covering number of 1 means the host graph is in the guest class. The same holds for local covering numbers if the guest class is closed under taking disjoint unions.

Now Consider the injective cover of the graph H_2 in Figure 3.3 using four C_4 cycles as guests. For each cycle C of them there is an edge only covered by C. Further, they are the only C_4 cycles in H_2, therefore every injective cover of H_2 with regards to $\overline{\mathcal{C}_4}$ must *contain these four cycles to cover all edges. Since they cover all edges, we have* $c_g^{\overline{\mathcal{C}_4}}(H_2) = 4$. *Further, they cover all the same vertex* x. *Therefore we have also* $c_l^{\overline{\mathcal{C}_4}}(H_2) = 4$. *On the other hand, consider the given non-injective cover on the right of Figure 3.3. By this mapping every vertex is covered at most twice and therefore we have* $c_f^{\overline{\mathcal{C}_4}}(H_2) = 2$.

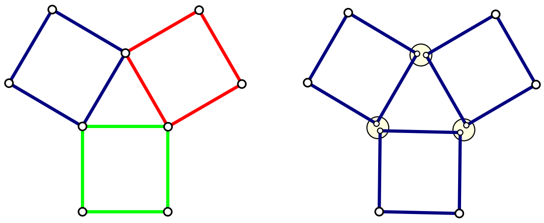

Figure 3.2: On the left is a graph H_1 injectivly covered by three C_4 cycles. On the right is the same graph H_1 covered by the corresponding non-injective cover of size 1, which has the disjoint union of the three C_4 cycles as guest.

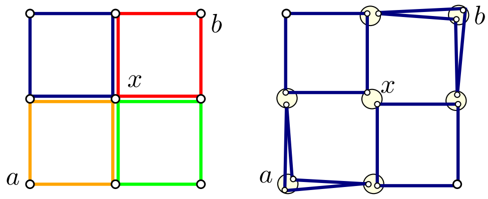

Figure 3.3: On the left is a graph H_2 injectivly covered by four C_4 cycles. On the right is a non-injective cover, which can be obtained by using the disjoint union of the four guests of the cover on the left as the guest, while using the same mapping except that the vertices of the top-right and bottom-left C_4 that are originally mapped to x are mapped instead to a and b, respectively.

Let $k \in \mathbb{N}_0$. We call an injective cover a *k-global ($\mathcal{G}$-)cover* if it has a size of at most k. We call it *k-local ($\mathcal{G}$-)cover* if it does not map more than k vertices to the same vertex. And we call a cover of size 1 a *k-folded ($\mathcal{G}$-)cover* if it maps at most k vertices to the same vertex. We say k-($\mathcal{G}$-)cover in short for any of these terms and may replace the word cover by decomposition if the cover is a decomposition. In particular, the global covering number $c_g^{\mathcal{G}}(H)$ is the smallest k for which there is a k-global $\mathcal{G}$-cover of H. We call a k-cover *optimal* if the corresponding covering number equals k.

3.3 Considered Graph Classes

In this section we introduce some of the graph classes that we consider as guest classes, and that may be unknown to the reader.

A *star* is a tree in that every vertex, except at most one, is a leaf. A *caterpillar* is a tree that contains a path such that all vertices are leaves or elements of the path. We denote the class of all stars by $\mathcal{S}t$ and the class of all caterpillars by $\mathcal{C}p$.

To introduce interval graphs we first consider the more general definition of intersection graphs.

An *intersection graph* of a family f of sets $s_1, \ldots, s_n$ is a graph G with vertex set $V(G)$, a bijection $b : \{1, \ldots, n\} \to V(G)$ and edge set $E(G) = \{b(i)b(j) : s_i \cap s_j \neq \emptyset\}$. I.e., two vertices are connected by an edge if and only if their corresponding sets in f have a common element. The family f is then called a *representation* of G.

An *interval graph* is an intersection graph with a representation containing only closed intervals in $\mathbb{R}$. Such a representation is called an *interval representation*. (Actually, it is equivalent to allow any intervals or only open intervals). Interval graphs were firstly examined by Lekkerkerker and Boland in 1962 [LB62]. Note that stars and caterpillars are interval graphs as shown in Figure 3.4. It gives an example for stars, caterpillars and for interval graphs. We denote the class of interval graphs by $\mathcal{I}$.

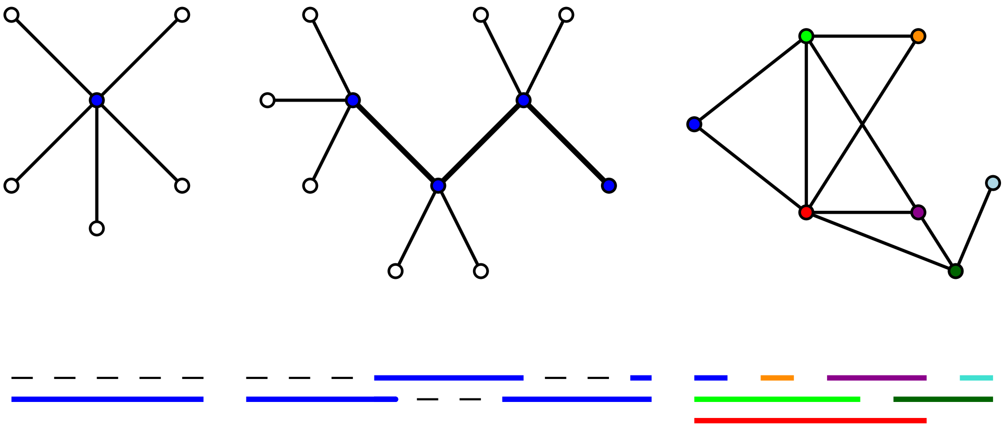

Figure 3.4: From left to right: a star, a caterpillar and an interval graph. Below are corresponding interval representations.

We call a graph class $\mathcal{G}$ *union-closed* if it is closed under taking disjoint unions, i.e., if $G_1, G_2 \in \mathcal{G}$, then $G_1 \cup G_2 \in \mathcal{G}$. We show later that if a guest class has this property, then the corresponding covering numbers are better comparable. Let $\mathcal{G}$ be a graph class. Then $\overline{\mathcal{G}}$ denotes the *union-closure* of $\mathcal{G}$ that is defined as the class of all finite disjoint unions of graphs in $\mathcal{G}$. Note that we have generally $\overline{\overline{\mathcal{G}}} = \overline{\mathcal{G}}$. We denote the class of paths by $\mathcal{P}$. We call the graphs in $\overline{\mathcal{P}}$ *linear forests*, the graphs in $\overline{\mathcal{S}t}$ are called *star forests*, the graphs in $\overline{\mathcal{C}p}$ are called *caterpillar forests*. The classes of forests (denoted by $\mathcal{F}$), planar graphs and interval graphs are other examples for union-closed graph classes. Note that by a result of Eckhof [Eck93] the caterpillar forests are exactly the triangle-free interval graphs (no point is shared by more than two intervals, therefore the interval representation of every component is that of a caterpillar. On the other side every caterpillar has an interval representation (see Figure 3.4)).

Further, we call $\mathcal{G}$ *induced-hereditary* if it is closed under taking induced subgraphs, i.e., if $G \in \mathcal{G}$, then all induced subgraphs of G are also contained in $\mathcal{G}$. And $\mathcal{G}$ is *hereditary* if it is closed under taking subgraphs. Observed that $\overline{\mathcal{C}_4}$ is not induced-hereditary, the graph in Figure 3.3 gives an example where the global and local covering number are lower for the closure of the guest class under taking induced subgraphs. All forests mentioned before and planar graphs are hereditary. We denote the class of all complete graphs by $\mathcal{K}$. The classes $\overline{\mathcal{K}}$ and $\mathcal{I}$ are induced-hereditary, but not hereditary.

The following proposition shows properties making it interesting to consider induced-hereditary and hereditary guest classes.

Proposition 3.2
Let $\mathcal{G}$ be an induced-hereditary guest class and let H and $H' \subseteq H$ be host graphs.

(i) If there is a k-folded/local/global $\mathcal{G}$-cover of H and H' is an induced subgraph of H, then there is a k-folded/local/global $\mathcal{G}$-cover of H'.

(ii) If there is a k-folded/local/global $\mathcal{G}$-cover of H and $\mathcal{G}$ is hereditary, then there is a k-folded/local/global $\mathcal{G}$-cover of H'.

(iii) If there is a k-folded/local/global $\mathcal{G}$-cover of H and $\mathcal{G}$ is hereditary, then there is an edge-bijective k-folded/local/global $\mathcal{G}$-cover of H

Proof. For item (i) and (ii) remove all vertices and edges from the guests that are not mapped into H' by c. For item (iii) just remove all but one edge from the guests of c mapped to the same edge in the host graph for every edge in H. □

In Chapter 5 we consider host graphs with an arbitrarily large difference between folded and local covering number. I.e., we show that this is easily achieved by a 'dirty trick' using non-induced-hereditary guest classes. Demanding an induced-hereditary guest class makes the result much more interesting. Further, we argue in the next section that it is more fruitful to consider union-closed graph classes in terms of covering numbers.

We call a graph class $\mathcal{G}$ *closed under taking folded components* if for any $G \in \mathcal{G}$, any homomorphism $\phi : G \to H$ into some graph H and every component C of G the class contains the graph $\phi(C)$. We show that folded and local covering number with regards to guest classes closed under taking folded components are equal. Examples for such graph classes are stars, star forests, complete graphs and the union-closure of complete graphs. Later we consider complements of interval graphs, another class with this property.

The covering number of a *host class* $\mathcal{H}$ with regards to a guest class $\mathcal{G}$ is the maximum covering number of a host graph in $\mathcal{H}$, i.e.,

$$c_i^{\mathcal{G}}(\mathcal{H}) = \sup_{H \in \mathcal{H}} c_i^{\mathcal{G}}(H)$$

for $i = f, l, g$.

3.4 General Conclusions

In this thesis we make often use of the following proposition about covering numbers by Knauer and Ueckerdt.

Proposition 3.3 (Knauer, Ueckerdt [KU12])
Let $\mathcal{G}$ and $\mathcal{G}'$ be guest classes, let $\mathcal{H}$ and $\mathcal{H}'$ be host classes and let H be a host graph. Let $k \in \mathbb{N}_0$. Then each of the following holds:

(i) For H any k-global $\mathcal{G}$-cover is also a k-local $\mathcal{G}$-cover. Especially $c_l^{\mathcal{G}}(H) \leq c_g^{\mathcal{G}}(H)$.

(ii) If $\mathcal{G}$ is union-closed, then any k-local $\mathcal{G}$-cover of H yields a k-folded $\mathcal{G}$-cover. Especially $c_f^{\mathcal{G}}(H) \leq c_l^{\mathcal{G}}(H)$.

(iii) If $\mathcal{G}$ is closed under taking folded components, then $c_f^{\mathcal{G}}(H) \geq c_l^{\mathcal{G}}(H)$.

(iv) If $\mathcal{H} \subseteq \mathcal{H}'$, then $c_i^{\mathcal{G}}(\mathcal{H}) \leq c_i^{\mathcal{G}}(\mathcal{H}')$ for $i = f, l, g$.

(v) If $\mathcal{G} \subseteq \mathcal{G}'$, then $c_i^{\mathcal{G}'}(\mathcal{H}) \leq c_i^{\mathcal{G}}(\mathcal{H})$ for $i = f, l, g$.

Proof. Since a k-global $\mathcal{G}$-cover c of H has size at most k and is injective, at most k vertices are mapped onto any vertex of H. Therefore c is also a k-local $\mathcal{G}$-cover. By definition there is a $c_g^{\mathcal{G}}(H)$-global $\mathcal{G}$-cover of H. This is also $c_g^{\mathcal{G}}(H)$-local and therefore proves $c_l^{\mathcal{G}}(H) \leq c_g^{\mathcal{G}}(H)$ (i).

Let $\mathcal{G}$ be union-closed and c be a k-local $\mathcal{G}$-cover of H. Then we receive a k-folded $\mathcal{G}$-cover of H from c as follows: Let G be the disjoint union of all guests of c. Then map every vertex $v \in G$ to $c(v)$. This proves (ii).

Let $\mathcal{G}$ be closed under taking folded components. Let $k = c_f^{\mathcal{G}}(H)$ and $c : G \to H$ be a k-folded $\mathcal{G}$-cover. Then we receive a k-local cover from c as follows: Let $C_1, \ldots, C_n$ be the components of G. We know $c(C_i) \in \mathcal{G}$ for $i \in [n]$, since $\mathcal{G}$ is closed under component folding. We define $c' : c(C_1) \uplus \ldots \uplus c(C_n) \to H$ with $c'_{|c(C_i)}$ being the identity function for $i \in [n]$. Since c is a cover, so is c'. Obviously, c' is injective. For any $v \in H$ we have $k \geq c^{-1}(v) = c^{-1}(c'^{-1}(v)) \geq c'^{-1}(v)$. This proves that c' is a k-local cover and therefore (iii).

Items (iv) and (v) follow from the definition of covering numbers. □

So, if $\mathcal{G}$ is a union-closed guest class, we have

$$c_f^{\mathcal{G}}(H) \leq c_l^{\mathcal{G}}(H) \leq c_g^{\mathcal{G}}(H). \qquad (*)$$

I.e., by determining folded or local covering number we receive lower bounds for the global covering number. In Chapter 5 we prove, however, that an upper bound for the global covering number cannot be deduced from folded or local covering number in general. Inequality (*) shows that it is much more interesting to consider union-closed guest classes when comparing different covering numbers. An extreme example for a non-union-closed class is the guest class $\mathcal{K}_2$ containing only K_2. While $c_l^{\mathcal{K}_2}(H) = \Delta(H)$, we have $c_g^{\mathcal{K}_2}(H) = ||H||$, and only K_2 graphs have a folded cover at all. If, however, you consider the class of matchings, the union-closure of $\mathcal{K}_2$, we have $c_f^{\overline{\mathcal{K}_2}}(H) = c_l^{\overline{\mathcal{K}_2}}(H) = \Delta(H)$ and $c_g^{\overline{\mathcal{K}_2}}(H) \in \{\Delta(H), \Delta(H)+1\}$.

The next Lemma is used to prove that certain guest graphs cannot be used to cover a certain host graph, which is part of the proof of the upcoming proposition.

Lemma 3.4
Let G and H be graphs and let $c : G \to H$ be a homomorphism. Then $\chi(G) \leq \chi(H)$.

Proof. Let $V(H)$ be partitioned into $k = \chi(H)$ independent sets $S_1, \ldots, S_k$. Then also $c^{-1}(S_i)$ is an independent set for $i \in [k]$, since an edge induced by $c^{-1}(S_i)$ would induce an edge in S_i by definition of a homomorphism. Since the sets $c^{-1}(S_1), \ldots, c^{-1}(S_k)$ form a partition of $V(G)$ into independent sets, we conclude $\chi(G) \leq k = \chi(H)$. □

The following proposition gives general upper bounds for the covering numbers of any host graph H in terms of $||H||$ or $\Delta(H)$. Due to the fact that folded covers use only one guest graph, there is no general upper bound of the folded covering number for non-union-closed graph classes (see item (iv) of the following proposition).

Proposition 3.5
Let $\mathcal{G}$ be a guest class, $\mathcal{H}$ be a host class and H be a host graph. Then each of the following holds:

(i) $c_i^{\mathcal{G}}(H) = \infty$ *or* $c_i^{\mathcal{G}}(H) \leq ||H||$ *for* $i = l, g$.

(ii) If $\mathcal{G}$ is union-closed then $c_f^{\mathcal{G}}(H) = \infty$ *or* $c_f^{\mathcal{G}}(H) \leq ||H||$.

(iii) If $\mathcal{G}$ is induced-hereditary and contains a graph G with $||G|| > 0$, then $c_l^{\mathcal{G}}(H) \leq \Delta(H)$.

(iv) Let $f : \mathbb{N}_0 \to \mathbb{N}_0$ be any monotonically increasing function. Then there are guest and host classes $\mathcal{G}$ and $\mathcal{H}$ such that $\max\{c_f^{\mathcal{G}}(H) : |H| = n, H \in \mathcal{H}\} \in \Omega(f(n))$, *while every graph $H \in \mathcal{H}$ still has a folded $\mathcal{G}$-cover.*

Proof. If there is an injective $\mathcal{G}$-cover c of H, then we can choose for every edge of H a guest $G(e)$ covering this edge. By restricting c to those guests its size is at most $||H||$ and it is still edge-surjective. Therefore $c_g^{\mathcal{G}}(H) \leq ||H||$, and by Proposition 3.3(i) follows the statement (i) for $i = l$.

Item (ii) follows from (i) and Proposition 3.3(ii).

If $\mathcal{G}$ is induced-hereditary and contains a graph G with $||G|| > 0$, then it contains a matching of size 1, i.e., a single edge with its ends. By covering each edge of H with another edge, all edges are covered and every vertex $v \in H$ is covered by $\deg(v)$ guests. This proves (iii).

Let $f : \mathbb{N}_0 \to \mathbb{N}_0$ be a monotonously increasing function. By a result of Erdős [Erd59] we know that for arbitrarily large n and k there is a graph G with $\mathrm{g}(G) \geq n$ and $\chi(G) \geq k$.

We define a sequence of graphs $(G_i)_{i \in \mathbb{N}_0}$ as follows: G_0 is a K_3. For $i > 0$ let G_i be a graph with $\chi(G) \geq 3$ and $\mathrm{g}(G_i) \geq |G_{i-1}|$. Now we define the host class $\mathcal{H}$ as the graphs in this sequence. We define the guest class $\mathcal{G}$ also as the graphs of the sequence, but we add to every G_n $f(n)|G_n|^2$ isolated vertices resulting in guest graph G'_n for $n \in \mathbb{N}_0$.

Every host graph G_i has a folded cover: Use G'_i as guest and let the cover be a bijection from the G_i of G'_i to the host G_i, in which the isolated vertices are mapped arbitrarily. Since there are at least $f(i)|G_i|$ vertices in G'_n per vertex in the host graph for $n \geq i$, using one of those as guest does not provide a $(f(i)|G_i| - 1)$-folded cover of G_i.

Assume there is a folded cover c of G_i using a G'_n with $n < i$ as guest. Then $c(V(G_n))$ would induce a forest F in G_i: Since $\mathrm{g}(G_i) > |G_n|$ (follows from induction on i), the graph induced by $c(V(G_n))$ is smaller than every cycle in G_i and therefore no cycle. Since $\chi(F) \leq 2$ we conclude by Lemma 3.4 that $\chi(G_n) \leq 2$, which is a contradiction. Therefore we have $f(i)|G_i| \leq c_f^{\mathcal{G}}(G_i)$. That concludes the proof of (iv). □

4. Local Linear Arboricity Conjecture and Related Work

4.1 Linear Arboricity Conjecture

A quite natural way to cover a graph is using forests as guests. The global covering number with regards to forests is called *arboricity*. Since the class of all forests is hereditary, the global covering number of a graph H is the size of a smallest partition of $E(H)$ into forests.

Nash-Williams proved that every graph can be covered using only as many forests as needed to provide enough edges in every subgraph (a forest F has at most $|F| - 1$ edges).

Theorem 4.1 (Nash-Williams [NW64])
For any graph H we have $c_g^{\mathcal{F}}(H) = \max_{H' \subseteq H} \left\lceil \frac{||H'||}{|H'|-1} \right\rceil$.

Folded or local covering number with regards to forests (called *folded* and *local arboricity*) cannot be smaller (pigeonhole principle): Let $H' \subseteq H$. In every guest of a cover of H' there are at least $\left\lceil \frac{|H'|}{|H'|-1} \right\rceil$ vertices per edge. That makes at least $\left\lceil \frac{||H'||}{|H'|-1} \right\rceil$ guest vertices per host vertex in a cover. With Proposition 3.3 follows:

Corollary 4.2 (Knauer, Ueckerdt [KU12])
For every graph H we have $c_f^{\mathcal{F}}(H) = c_l^{\mathcal{F}}(H) = c_g^{\mathcal{F}}(H)$.

Further, by a result of Edmonds [Edm65] an optimal $\mathcal{F}$-cover can be computed in polynomial time.

For restricted kinds of forests (see Section 3.3 in the last chapter) there are no such strong results. Instead it has been shown to be $\mathcal{NP}$-complete to decide $c_g^{\mathcal{G}}(H) \leq k$ for $k = 2$ and any graph H for the guest class of star forests ($\overline{\mathcal{S}t}$) by Hakimi, Mitchem and Schmeichel [HMS96], for the class of caterpillar forests ($\overline{\mathcal{C}p}$) by Shermer [She96] and for the class of linear forests ($\overline{\mathcal{P}}$) by Péroche [Pé84]. In Chapter 6 we proof $\mathcal{NP}$-completeness for any $k \geq 2$ and $\overline{\mathcal{P}}$. This discourages to find a simple characterization of the corresponding global covering numbers.

In this chapter we consider linear forests only. The corresponding covering numbers are called *(local/folded) linear arboricity.* The linear arboricity was introduced by Harary [Har70] in 1970. Figure 4.1 gives two examples of optimal $\overline{\mathcal{P}}$-covers.

The following lemma gives a lower bound on the (local/folded) linear arboricity based on the maximum degree.

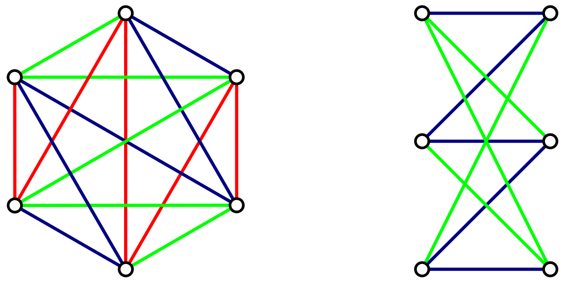

Figure 4.1: Optimal $\overline{\mathcal{P}}$-covers for K_6 and $K_{3,3}$.

Lemma 4.3 (Knauer, Ueckerdt [KU12])
For every graph H with maximum degree Δ holds $c_g^{\overline{\mathcal{P}}}(H) \geq c_l^{\overline{\mathcal{P}}}(H) \geq c_f^{\overline{\mathcal{P}}}(H) \geq \left\lceil \frac{\Delta}{2} \right\rceil$. If Δ is even and H is regular, we have even $c_g^{\overline{\mathcal{P}}}(H) \geq c_l^{\overline{\mathcal{P}}}(H) \geq c_f^{\overline{\mathcal{P}}}(H) \geq \left\lceil \frac{\Delta+1}{2} \right\rceil$

Proof. Since the maximum degree of any linear forest is at most 2, at least $\left\lceil \frac{\Delta}{2} \right\rceil$ vertices of linear forests must be mapped to a vertex v of degree Δ to cover all edges incident to v. Therefore we have $c_f^{\overline{\mathcal{P}}} \geq \left\lceil \frac{\Delta}{2} \right\rceil$. By Proposition 3.3 follows the statement, since $\overline{\mathcal{P}}$ is union-closed.

Let H be a Δ-regular graph with Δ even. Consider a folded $\overline{\mathcal{P}}$-cover of H. The paths in the guest have to end in some vertices, enforcing that those vertices are covered at least $\left\lceil \frac{\Delta+1}{2} \right\rceil$ times. □

Similar to Nash-Williams' Theorem 4.1, Akiyama, Exoo and Harary stated 1980 the *Linear Arboricity Conjecture* (LAC) that states all graphs can be optimally covered by linear forests considering these bounds. Despite much attention, this conjecture is still open today.

Conjecture 4.4 (Linear Arboricity Conjecture (LAC); Akiyama, Exoo, Harary [AEH80])
The linear arboricity of any graph H with maximum degree Δ is either $\left\lceil \frac{\Delta}{2} \right\rceil$ or $\left\lceil \frac{\Delta+1}{2} \right\rceil$.

So far, the conjecture has been proven for $\Delta = 3, 4, 5, 6, 8$ and 10: Akiyama, Exoo and Harary proved the conjecture for $\Delta = 3, 4$ in the paper stating the conjecture [AEH80][AEH81]. Enomoto and Péroche were able to prove it for $\Delta = 5, 6, 8$ [EP84] and by a result of Guldan the conjecture holds also for $\Delta = 10$ [Gul86a]. It was also investigated for certain host graphs: Wu and Wu proved it for planar graphs (using one paper of 11 pages filled with case-distinctions solely for the case of maximum degree 7) [Wu99][WW08]. An easy observation is truth for the case of complete graphs (follows e.g. from Theorem 4.10, but you can state optimal covers directly, see Figure 4.1). Complete bipartite graphs where also covered by Akiyama, Exoo and Harary in the paper stating the conjecture [AEH80]. Alon proved the conjecture for all graphs of even regularity with girth at least 50 [Alo88]. He used this result to prove LAC to be asymptotically true [Alo88]. Moreover, Guldan proved the best-known upper bound $c_g^{\overline{\mathcal{P}}}(H) \leq \left\lceil \frac{3\Delta+2}{5} \right\rceil$ [Gul86b]. Note that, further, there is research on *linear k-arboricity* using only linear forests as guests in which all paths have length at most k [AW98]. Another associated parameter is the *vertex linear arboricity* of a graph H that is the smallest d such that the vertex set can be partitioned into d sets, each inducing a linear forest. Matsumoto has proven upper bounds for it reminding to the LAC [Mat90].

Following the approach of considering folded and local covering number, Knauer and Ueckerdt introduced the folded and local linear arboricity mentioned before [KU12]. They proved for any graph H with maximum degree Δ that $\left\lceil \frac{\Delta}{2} \right\rceil \leq c_f^{\overline{\mathcal{P}}}(H) \leq \left\lceil \frac{\Delta+1}{2} \right\rceil$, which proves the folded variant of LAC, and stated the corresponding conjecture for the local linear arboricity:

Conjecture 4.5 (Local Linear Arboricity Conjecture (LLAC); Knauer, Ueckerdt [KU12])
The local linear arboricity of any graph H with maximum degree Δ is either $\left\lceil \frac{\Delta}{2} \right\rceil$ or $\left\lceil \frac{\Delta+1}{2} \right\rceil$.

4.2 Reduction to Graphs of Odd Regularity

The main goal of this chapter is to prove this conjecture. The first step to reach this goal is the reduction to graphs of odd regularity. This can be achieved using the following lemma.

Lemma 4.6
Every graph H with maximum degree Δ is an induced subgraph of a Δ-regular graph.

Proof. If the graph H is not regular itself, then its minimum degree δ is smaller than Δ. We can obtain a graph H' that contains H as induced subgraph, has the same maximum degree Δ and has a minimum degree that is by one greater than δ. To this end, we copy the graph H and connect every vertex with degree δ with its copy. By repeating this step we finally obtain a Δ-regular graph containing H as induced subgraph. □

The next theorem reduces LAC and LLAC to regular graphs.

Theorem 4.7 (N. Alon [Alo88]; local version by Knauer, Ueckerdt [KU12])
For any $\Delta \in \mathbb{N}_0$ the following two statements are equivalent:
(i) The local/classical linear arboricity of any Δ-regular graph H is $\left\lceil \frac{\Delta+1}{2} \right\rceil$.
(ii) The local/classical linear arboricity of any graph with maximum degree Δ is either $\left\lceil \frac{\Delta}{2} \right\rceil$ or $\left\lceil \frac{\Delta+1}{2} \right\rceil$.

Proof. "$(i) \Rightarrow (ii)$": By Lemma 4.6 every graph H with maximum degree Δ is a subgraph of a Δ-regular graph S. If S has local/classical linear arboricity $\left\lceil \frac{\Delta+1}{2} \right\rceil$, then we know by Proposition 3.2(ii) that the local/classical linear arboricity of its subgraph H is at most $\left\lceil \frac{\Delta+1}{2} \right\rceil$. By Lemma 4.3 follows statement (ii).

"$(i) \Leftarrow (ii)$": Since in any Δ-regular graph H that is covered by linear forests there is a vertex v in which a linear forest F ends (and therefore has degree 1 in v) the cover must contain at least $\left\lceil \frac{\Delta-1}{2} \right\rceil$ more linear forests covering v to cover all edges incident to v. Together with F that are at least $\left\lceil \frac{\Delta+1}{2} \right\rceil$ linear forests covering v. This proves $c_g^{\overline{\mathcal{P}}}(H) \geq c_l^{\overline{\mathcal{P}}}(H) \geq \left\lceil \frac{\Delta+1}{2} \right\rceil$. This shows that statement (ii), which gives $\left\lceil \frac{\Delta+1}{2} \right\rceil$ as upper bound of the local/classical linear arboricity, induces statement (i). □

With the following lemma we finally reduce LAC and LLAC to graphs of odd regularity.

Lemma 4.8 (Knauer, Ueckerdt [KU12])
To prove LAC or LLAC it suffices to prove for every odd Δ that any Δ-regular graph has classical or local linear arboricity $\left\lceil \frac{\Delta+1}{2} \right\rceil$.

Proof. Let H be a $(\Delta + 1)$-regular graph with Δ odd. There is a spanning linear forest F in H [Gul86b](every k-regular graph with k even has a 2-factor, by deleting one edge of every circuit of the 2-factor we obtain a spanning linear forest.). By removing the edges of F we receive a graph H' with maximum degree Δ.

By Lemma 4.6 and Proposition 3.2 we know, if all Δ-regular graphs have classical/local linear arboricity at most $\left\lceil \frac{\Delta+1}{2} \right\rceil$, then the classical/local linear arboricity of H' is at most $\left\lceil \frac{\Delta+1}{2} \right\rceil = \frac{\Delta+1}{2}$. Since every cover of H' can be extended to a cover of H by adding F as linear forest, the classical/local linear arboricity of H is then at most $\frac{\Delta+1}{2} + 1 = \left\lceil \frac{(\Delta+1)+1}{2} \right\rceil$. By Lemma 4.3 we get the lower bound. This completes the proof with Theorem 4.7. □

4.3 Path Decomposing Graphs of Odd Regularity

Now, for graphs of odd regularity LLAC follows from a theorem of Lovasz, which we state here. Roughly speaking, the proof of that theorem transforms a graph of odd regularity into one that contains exactly one even vertex, and then uses the following lemma for induction. Note that we say that a vertex is even/odd if its degree is even/odd and that we denote the class of paths by $\mathcal{P}$.

Lemma 4.9 (László Lovász[Lov68])
Let H be a graph with a vertex x adjacent to exactly one even vertex y and let d be the minimum number such that $H - xy$ has a d-global $\mathcal{P}$-decomposition. Then H has a d-global $\mathcal{P}$-decomposition.

Proof. Let d be the minimum number, such that $H - xy$ has a d-global $\mathcal{P}$-decomposition. Let c be such a d-decomposition of $H - xy$. In every odd vertex u there has to end at least one path $\mathrm{p}(u)$ (for the graphs for which we apply this lemma later it is actually exactly one path). The idea is to add edge xy to the path $\mathrm{p}(y)$. This causes a cycle in $\mathrm{p}(y)$ if it already contains an edge vx at x. But this edge can again be added to the path $\mathrm{p}(v)$.

We claim that repeating this step terminates if we add an edge to a path that does not contain x and therefore stays a path with the new edge. To prove this, we introduce an order on $\mathrm{N}(x)$ such that the problematic edge is passed only in one direction. This enforces termination. Figure 4.2 shows an example for such an exchange of path edges and is used for further explanation.

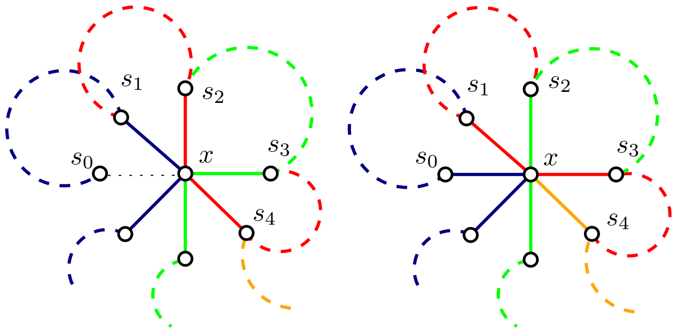

Figure 4.2: On the left: Cover of $H - xy$ ($s_0 = y$). On the right: Induced cover of H.

Let $v \in \mathrm{N}(x)$ be a neighbour of x in H. Then v is of odd degree in $H - xy$ and therefore we have the path $\mathrm{p}(v)$ of c ending in v. Starting in v and following $\mathrm{p}(v)$ we call the last vertex before x the successor $\mathrm{succ}(v)$ of v. If x is not contained in $\mathrm{p}(v)$, then there is no successor of v. In Figure 4.2 vertex s_5 has no successor.

If v has a vertex w with $\mathrm{succ}(w) = v$, called predecessor, then it is unique, since $\mathrm{p}(w)$ must contain vx and v separates w and x in $\mathrm{p}(x)$. Since $y \notin \mathrm{N}(x)$ it has no predecessor.

Define the sequence s by $s_0 = y$ and $s_{i+1} = \mathrm{succ}(s_i)$ (if $\mathrm{succ}(s_i)$ does not exist, then the sequence terminates) for $i \in \mathbb{N}_0$. The sequence s contains no element twice (otherwise the first repeated had two different predecessors). Therefore sequence s has a last element s_m. Now we can cover H as follows: For every $v \in s \setminus \{s_m\}$ add vx and remove $\mathrm{succ}(v)x$ from $\mathrm{p}(v)$. Thereby every path becomes a new path (see e.g. the path starting in s_0 in Figure 4.2, the paths can be modified on both sides of x (see the path starting in s_1), but modifying one side does not affect the order of the other side) and still every edge in $H - xy$ is covered by exactly one path. Finally $s_m x$ can be covered by adding it to $\mathrm{p}(s_m)$, which, per definition of s_m, does not contain x.

That way every edge is covered by exactly one path: If an edge does not connect x to an element of s, then it is covered by the same path as in $H - xy$, and every edge xs_i $(0 \leq i \leq m)$ is covered by $\mathrm{p}(s_i)$. Since we did not introduce new paths their number did not increase. □

Note that this lemma allows multiple even vertices in H, and actually this can be used for generalizations of the next theorem if we can order the edges correspondingly for reduction.

Note that the number of odd vertices in a graph is always even.

Theorem 4.10 (László Lovász[Lov68])
Let H be a graph that contains n odd vertices and at most one non-isolated even vertex. Then H has a $\frac{n}{2}$-global $\mathcal{P}$-decomposition.

Proof. We use Lemma 4.9 for induction on the number of edges. If H is a K_2, then it can be covered by at most $\frac{2}{2}$ edge-disjoint paths, using the edge as one path. Now let H be a graph with at most one non-isolated even vertex and assume the statement holds for all such graphs with less edges. Without loss of generality we may assume H to be connected (otherwise we can separately consider every component).

Case 1: If H contains an even vertex y, then there is a neighbour x of y. Since y is the only even vertex, x has y as its only even neighbour vertex. In $H - xy$ the parity of x and y is switched, while all other vertices have the same parity as in H. Thus, x is the only even vertex in $H - xy$. Since it has one edge less than H, the induction hypothesis holds for $H - xy$. The Theorem follows for H by Lemma 4.9, since the number of odd vertices in both graphs is the same.

Case 2: Let H contain no even vertex. Then it contains an edge xz with $\deg(x) > 1$. (Otherwise, since H is connected, it is trivial or a K_2. That is the base case of the induction.)

Subdivide the edge xz to xy and yz with $y \notin V(H)$ and call the resulting graph H' (see Figure 4.3). In H' the vertex y is the only even vertex. Hence, in $H' - xy$ only x is even and since $\deg_{H'-xy}(x) = \deg_H(x) - 1 > 0$ it is non-isolated. Therefore there is a neighbour v of x in $H' - xy$. Finally consider graph $H' - xy - xv$. In $H' - xy - xv$ only v is even. Since it has one edge less than H, we can use the induction hypothesis on $H' - xy - xv$. Since H', $H' - xy$ and $H' - xy - xv$ have the same number of odd vertices, the induction hypothesis follows for H' by applying Lemma 4.9 first for $H' - xy$ and $H' - xy - xv$ and second for H' and $H' - xy$.

The odd vertices of H and H' are the same. Considering H' to be covered with $\frac{n}{2}$ edge-disjoint paths, in every vertex but y there has to end a path, so in y there cannot end a path. Therefore, since H' can be covered with at most $\frac{n}{2}$ edge-disjoint paths, this induces a way to cover H with the same number of edge-disjoint paths (replace xy and yz in the same path by xz). This concludes the proof. □

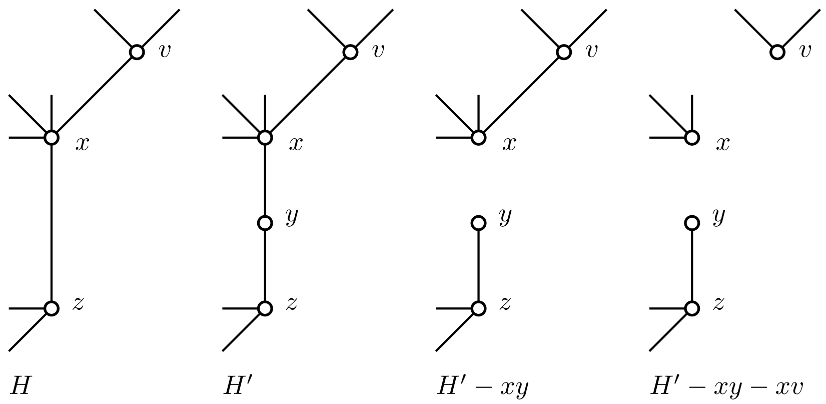

Figure 4.3: The different graphs for Case 2 in the proof of Theorem 4.10.

Actually László Lovász proved that any graph H can be covered by $\left\lceil \frac{|H|}{2} \right\rceil$ edge-disjoint cycles and paths. Theorem 4.10 follows directly if you consider that in every vertex of odd degree at least one path ends (to achieve this number of paths all guests must be paths). The proof given here is an adaption of László Lovász's proof considering only such graphs.

We can now finally conclude that LLAC is true.

Corollary 4.11
The Local Linear Arboricity Conjecture (Conjecture 4.5) holds.

Proof. Due to Lemma 4.7 it is sufficient to consider graphs of odd regularity. Let H be a Δ-regular graph with Δ odd and $|H| = n$. According to Theorem 4.10, graph H can be decomposed into at most $\frac{n}{2}$ paths. Since in every odd vertex there has to end a path, and there are at most $\frac{n}{2} \cdot 2 = n$ path endings, at every vertex there ends exactly one path. Therefore every vertex is covered by $1 + \frac{\Delta-1}{2} = \frac{\Delta+1}{2}$ paths and cannot be covered by less. Since paths are linear forests, we get $c_l^{\overline{\mathcal{P}}}(H) = \lceil \frac{\Delta+1}{2} \rceil$. □

The proofs yield an algorithm to compute an optimal local $\overline{\mathcal{P}}$-cover of a graph G in $O(|H| + ||H||^2)$.

The constructed cover is a $\mathcal{P}$-cover. To receive a k-global $\overline{\mathcal{P}}$-cover with small k it is necessary to partition the guest paths into a small number of linear forests. But it appears difficult to realize that. The base case of the induction is a matching that is already a linear forest. Ideally $k = \lceil \frac{\Delta+1}{2} \rceil$, which would prove LAC. During the induction steps the guest paths start sharing common vertices and thus must be put into distinct linear forests. This and the choices which edge to remove next for the induction step must be made with a global foresight avoiding future collisions. Since every induction step potentially changes the ends of multiple paths it seems hard to find structural properties giving such a foresight. Another approach to use LLAC to prove LAC is the question whether local and global arboricity are always equal, as it is the case for the arboricity (see Corollary 4.2). Unfortunately, that statement does not hold, as we prove in the next chapter in Theorem 5.4.

5. Separability

In this chapter we deal with the question by how much global, local and folded covering numbers can differ for the same host and guest classes, i.e., if the difference between two covering numbers can be arbitrarily large.

Recall from Proposition 3.3 that for any host graph H and any guest class $\mathcal{G}$ we have $c_f^{\mathcal{G}}(H) \leq c_l^{\mathcal{G}}(H) \leq c_g^{\mathcal{G}}(H)$, provided $\mathcal{G}$ is union-closed (closed under taking disjoint unions). We are mainly interested in such guest classes (We already argued in Section 3.3 of Chapter 3 that it is more interesting to compare covering numbers with regards to union-closed guest classes.). Therefore, we consider separations of global and local covering number as well as separations of local and folded covering number. Clearly both separations imply a separation of global and folded covering number.

Considering only union-closed guest graphs makes it considerably more difficult to find such a separation. It excludes, for example, guest classes with bounded number of edges or components. Such guest classes are troublesome for the total number of guest graphs of a cover, whereas it is not necessarily a problem for the number of graphs covering a single vertex.

Knauer and Ueckerdt proved that the guest class $\overline{\mathcal{K}}$ of collections of complete graphs and the host class $\mathcal{L}$ of line graphs provide $c_g^{\overline{\mathcal{K}}}(\mathcal{L}) = \infty$ and $c_l^{\overline{\mathcal{K}}}(\mathcal{L}) = 2$ [KU12, Theorem 10]. They also indicated the following separation of local and global covering number with respect to interval graphs.

Theorem 5.1 (Milan [MSW12], Whitney [Whi32])
For the guest class $\mathcal{I}$ of interval graphs and the host class $\mathcal{L}$ of line graphs, we have $c_g^{\mathcal{I}}(\mathcal{L}) = \infty$ and $c_l^{\mathcal{I}}(\mathcal{L}) = 2$.

Proof. Milans, Stolee and West [MSW12] proved $c_g^{\mathcal{I}}(\mathrm{L}(K_n)) \in \Omega(\log^*(n))$, whereas by a result of Whitney [Whi32] for every line graph L and the guest class $\mathcal{K}$ of complete graphs we have $2 \geq c_l^{\mathcal{K}}(L) \geq c_l^{\mathcal{I}}(L)$, since complete graphs are interval graphs. This inequality is best-possible. □

5.1 Separability of Folded and Local Covering Number

Also for comparison of the folded and local covering number with respect to the same guest and host classes we consider only union-closed guest classes.

One trick to separate folded and local covering number for a union-closed guest class $\mathcal{G}$ and a host class $\mathcal{H}$ is to choose $\mathcal{G}$ and $\mathcal{H}$ such that 'essential' guest graphs of $\mathcal{G}$ are not subgraphs of host graphs in $\mathcal{H}$. Thereby, those guest graphs cannot be guests in injective covers and therefore do not affect the local covering number, whereas folding them once can be enough to make them subgraphs of host graphs and, thereby, provide a small folded covering number. Knauer and Ueckerdt provided an example using the guest class of collections of cycles and the host class of paths [KU12, Observation 11], in which the local covering number is ∞, since no injective covering exists. On the other hand, the folded covering number is 2, since every path can be covered by a cycle such that every vertex is covered at most twice.

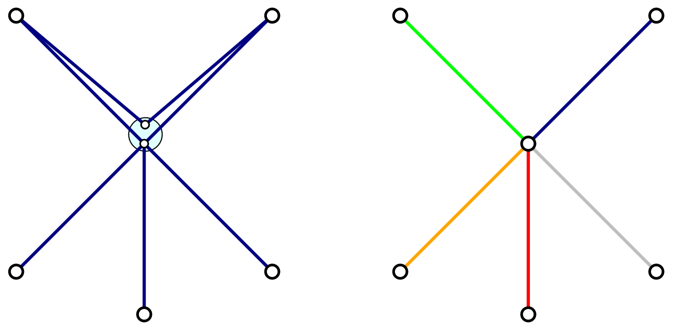

Figure 5.1: On the left: a 2-folded $\mathcal{G}$-cover of a star. On the right: an optimal 5-global $\mathcal{G}$-cover of the same star.

An example in which local covers exist uses the guest class $\mathcal{G}$ of all matchings and all supergraphs of C_4, i.e., $\mathcal{G} := \{G : G \text{ is a matching or } C_4 \subset G\}$ and the host class $\overline{\mathcal{S}t}$ of star forests. Here the supergraphs of C_4 are not subgraph of any star forest, since star forests do not contain C_4. Therefore only matchings can be used for injective covers. This, however, means the local covering number with regards to $\mathcal{G}$ of a star forest F equals its maximum degree $\Delta(F)$ (see Figure 5.1). This can be arbitrarily high. Hence, we have $c_l^{\mathcal{G}}(\overline{\mathcal{S}t}) = \infty$. On the other side, if a star forest H is no matching, then it contains a path P_3 of two edges. By connecting the ends of this path with a new vertex v we receive a graph containing a C_4, which is therefore a guest graph. By folding the vertex in the middle of the path with v we obtain H. This proves $c_f^{\mathcal{G}}(H) = 2$ and therefore $c_f^{\mathcal{G}}(\overline{\mathcal{S}t}) = 2$.

To avoid this trick we can consider only induced-hereditary guest graphs (closed under taking induces subgraphs). This excludes a desired part of a guest graph to be excluded because it has another part that is not useful and prevents the graph from being used.

The class $\mathcal{B}$ of bipartite graphs is a union-closed guest class and is even hereditary (closed under taking any subgraphs). However, $\mathcal{B}$ still provides an arbitrarily large separation of folded and local covering number.

Theorem 5.2
For the guest class $\mathcal{B}$ of bipartite graphs and the host class $\mathcal{G}$ of all graphs, we have $c_l^{\mathcal{B}}(\mathcal{G}) = \infty$ and $c_f^{\mathcal{B}}(\mathcal{G}) = 2$.

Proof. First, we prove $c_f^{\mathcal{B}}(H) \leq 2$ for any graph H. Therefore, we define a bipartite graph B with vertex set $V = V(H) \times \{1, 2\}$ and edge set $E = \{(v, 1)(w, 2) : vw \in E(H)\}$. You can see that B is obtained from H by splitting every vertex into a first and a second vertex and keeping the edges such that no edge connects two first or two second vertices. An example for a 2-folded $\mathcal{B}$-cover of K_4 can be seen in Figure 5.2.

The graph B is bipartite with partition classes $V(H) \times \{1\}$ and $V(H) \times \{2\}$. Now consider the function $\phi : V(B) \to V(H), (v, i) \mapsto v$. It is an edge-surjective homomorphism from B

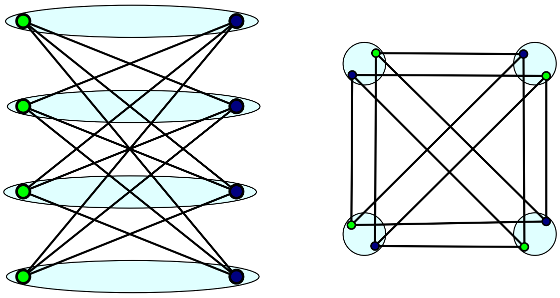

Figure 5.2: Two drawings of the same 2-folded $\mathcal{B}$-cover of K_4.

to H, since for every edge vw in H there are the edges $(v,1)(w,2)$ and $(w,1)(v,2)$ in B with $\phi((v,1)) = \phi((v,2)) = v$ and $\phi((w,1)) = \phi((w,2)) = w$ and only such edges exist in B. It is therefore a cover and has size 1. For every $v \in H$ we have $\phi^{-1}(v) = \{(v,1),(v,2)\}$, which is a set of size 2. Thus, we have $c_f^{\mathcal{B}}(H) \leq 2$.

To prove $c_l^{\mathcal{B}}(\mathcal{G}) = \infty$, we prove $c_l^{\mathcal{B}}(\mathcal{K}) = \infty$ for the class of complete graphs $\mathcal{K}$. We use the same argument as used by Fishburn and Hammer [FH96, Theorem 5], which states the same allowing only complete bipartite graphs as guest graphs: Assume $c_l^{\mathcal{B}}(\mathcal{K}) = k < \infty$. Let n be a number with $c_l^{\mathcal{B}}(K_n) = k$.

Let $N = (n-1)\cdot k+2$. Then any cover of K_N includes k bipartite graphs $B_i \in \mathcal{B}$ $(1 \leq i \leq k)$ that cover a common vertex v, by definition of k. Since the graphs B_i must cover $(n-1)\cdot k+1$ edges incident to v, there is a bipartite graph B_i in which v has at least n neighbours between whom no edge is in B_i, since it is bipartite (see Figure 5.3).

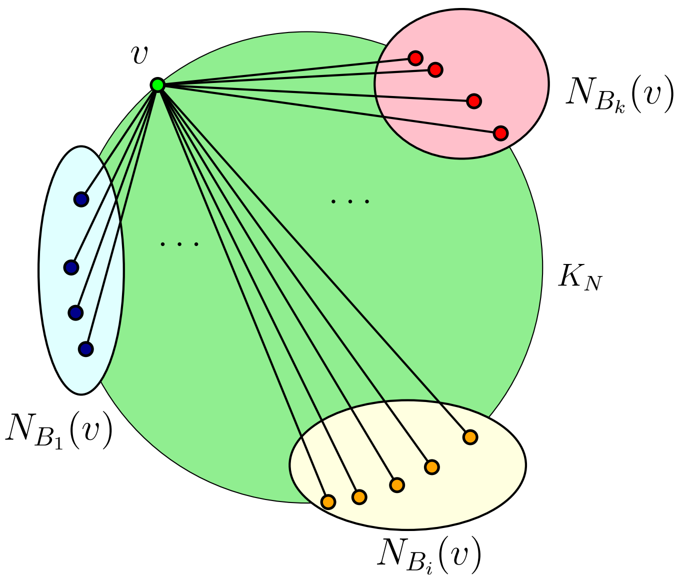

Figure 5.3: View of v in K_n. For $1 \leq i \leq k$ the neighbourhood $N_{B_i}(v)$ of v within B_i is independent. As $\deg(v) = (n-1)+k+1$, one of them contains n vertices.

Since these have to be covered, at least $c_l^{\mathcal{B}}(K_n) = k$ more bipartite graphs cover a single vertex of those neighbours and therefore $c_l^{\mathcal{B}}(K_N) \geq k+1$. This contradicts our assumption and concludes the proof. □

5.2 Separation of Linear Arboricities

In Chapter 4 we argued that an equivalence of $c_g^{\overline{\mathcal{P}}}$ and $c_l^{\overline{\mathcal{P}}}$ would imply the Linear Arboricity Conjecture (LAC). Unfortunately this equivalence does not hold in general. We show here that this two covering numbers differ at least as much as possible without contradicting LAC. A byproduct of this discovery is a proof of $\mathcal{NP}$-completeness of deciding $c_g^{\overline{\mathcal{P}}}(H) \leq k$ for any graph H and $k \geq 2$, which was unknown for $k \geq 3$ before. It is part of Chapter 6. For the folded linear arboricity we prove a characterisation allowing a separation of folded and local linear arboricity.

For separation of $c_g^{\overline{\mathcal{P}}}$ and $c_l^{\overline{\mathcal{P}}}$, we transform graphs to achieve a relation between coverings with matchings and coverings with linear forests. Therefore, we define the graph E_n (E for 'every end enforcer') for $n \in \mathbb{N}_0$ as the union of K_{2n} and a vertex v that is connected to n vertices of K_{2n} (see Figure 5.4 for an E_2 graph as example). We denote the class of all matchings by $\overline{\mathcal{K}_2}$.

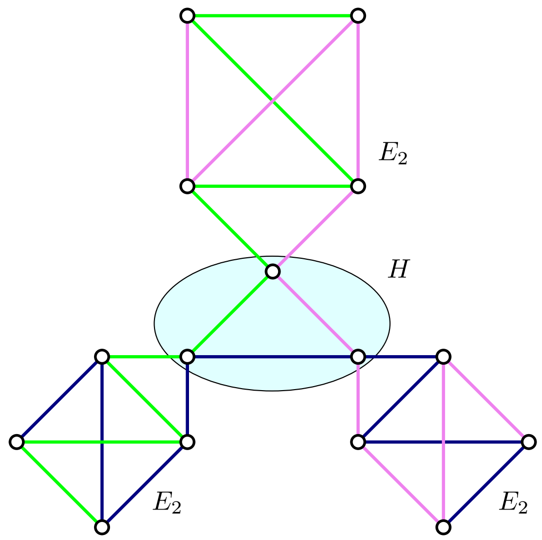

Figure 5.4: Graph H' for $H = K_3$ in Lemma5.3 with a 2-local 3-global $\overline{\mathcal{P}}$-cover.

Lemma 5.3
Let H be a n-regular graph. Let H' be the graph obtained by identifying every vertex of H with vertex v in another copy of E_n. Then holds:

$$c_g^{\overline{\mathcal{K}_2}}(H) \leq n \Leftrightarrow c_g^{\overline{\mathcal{P}}}(H') \leq n$$

Proof. For both directions we first consider E_n and then show the actual statement. Figure 5.4 gives an example for the construction of H'.

"$\Leftarrow$": Let c be a n-global $\overline{\mathcal{P}}$-cover of E_n. Then no linear forest may end in $\mathrm{N}(v)$, since those vertices have all degree $2n$. In every vertex $w \in H \setminus (\mathrm{N}(v) \cup \{v\})$ at most one linear forest may end, since it has degree $2n-1$. And since every linear forest has at least 2 ends, that are in total at least $2n$ ends, leaving n ends that have to be in v. As those have to be n ends of different forests, in every of the n vertices of degree $2n-1$ there ends another of the n linear forests. Since these have no other ends, they have exactly two ends. Therefore they are paths.

Let x be a vertex in H. Now suppose there is a n-global $\overline{\mathcal{P}}$ cover of H'. Since E_n^x, the copy of E_n whose vertex v is identified with x, is subgraph of H', all edges of E_n^x incident to $v = x$ belong to different linear forests. And since there are only n linear forests, all n edges incident to x in H belong to different linear forests. Since x was chosen freely, this means that the n linear forests partition all edges of H into n disjoint sets of non-adjacent edges, i.e., n matchings.

"⇒": We can construct a n-global $\mathcal{P}$-cover of E_n as follows: We consider a n-global $\mathcal{P}$-cover of K_{2n} (by Theorem 4.10 follows that such a cover exists, since all vertices have odd degree). Since in every vertex there ends exactly one path, we can choose n vertices such that every path ends in exactly one of them. Now we connect those n vertices to a new vertex v and add every new edge xv to the path ending in x. Since every path is expanded exactly once to v, this forms a n-global $\mathcal{P}$-cover of E_n.

Suppose c is a n-global $\overline{\mathcal{K}_2}$-cover of H. Then c can be extended to H' by identifying each of n paths of a n-global $\mathcal{P}$-cover of every E_n with another matching covering H. This yields a n-global $\overline{\mathcal{P}}$-cover of H' (as paths are only linked in their ends, the maximum degree is for every guest 2. Any cycle $C \subseteq H'$ is subgraph of H or a copy of E_n, but in those graphs no paths are linked). □

Note that $c_g^{\overline{\mathcal{K}_2}}(H) \geq n$ (all edges incident to a vertex v with $\deg(v)$ belong to different guests) and $c_g^{\overline{\mathcal{P}}}(H') \geq n$ (see Lemma 4.3). Knowing there are n-regular graphs without a n-global $\overline{\mathcal{K}_2}$-cover, the following theorem is straightforward.

Theorem 5.4
For every $n \geq 2$ exists a host graph H' with maximum degree $\Delta = 2n$ such that:

$$c_g^{\overline{\mathcal{P}}}(H') > n = c_l^{\overline{\mathcal{P}}}(H').$$

Proof. For every $n \geq 2$ there is a n-regular graph H that does not contain a perfect matching (standard exercise [Wes01][3.3.7]) and hence, has no n-global $\overline{\mathcal{K}_2}$-cover. By Lemma 5.3 follows for the graph H' obtained by identifying every vertex of H with vertex v in another copy of E_n that $c_g^{\overline{\mathcal{P}}}(H') > n$.

However, a n-local $\overline{\mathcal{P}}$-cover can be constructed as follows (see Figure 5.4 for an example): Let c be a n-local $\mathcal{K}_2$-cover of H (e.g., use every edge as another guest). Then c can be extended to H' by identifying for every vertex $x \in H$ each of the n paths of a n-global $\mathcal{P}$-cover of E_n^x, the copy of E_n containing x as v, with another edge incident to H. This yields a n-local $\overline{\mathcal{P}}$-cover of H' (every guest has now degree 2 in every $x \in H$. As paths are only linked in their ends, the maximum degree is for every guest 2. Any cycle $C \subseteq H'$ is subgraph of H or a copy of E_n, but in those graphs no paths are linked). □

For the folded linear arboricity we have a characterization that allows it easily to separate it from the local linear arboricity. Knauer and Ueckerdt already proved

$$\lceil \Delta/2 \rceil \leq c_f^{\overline{\mathcal{P}}}(H) \leq \lceil (\Delta+1)/2 \rceil \qquad (**)$$

for any graph H with maximum degree Δ using Euler tours [KU12] and its easily extended to a full characterization.

Theorem 5.5
Let H be a graph with maximum degree $\Delta > 0$. Then holds:

$$c_f^{\overline{\mathcal{P}}}(H) > \left\lceil \frac{\Delta}{2} \right\rceil \Leftrightarrow \Delta \textit{ is even and there is a } \Delta\textit{-regular connected component } C \textit{ of } H.$$

Proof. "⇐": If Δ is even and there is a Δ-regular connected component, then the left statement follows directly from Lemma 4.3.

"⇒": Assume that the right statement is false. By $(**)$ follows for odd Δ directly that the left statement is false. Hence, Δ is even and there is no Δ-regular component. Note that since $\Sigma_{v \in G} \deg(v) = 2||G||$, the number of odd vertices is even in every graph G.

Let C be a component of H. Define H' as the graph obtained by adding a vertex y and connecting it to all vertices of H of odd degree. By definition all vertices of H' are even. Therefore there is an Euler tour on H', i.e., there is an edge-bijective $\frac{\Delta}{2}$-folded cover c of H' using a cycle C_m as guest.

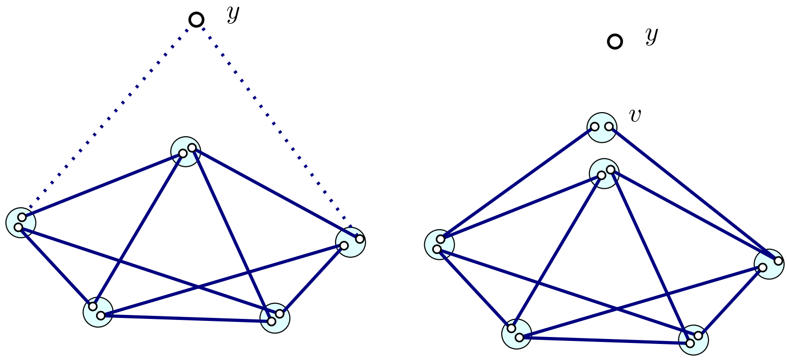

Figure 5.5: Two 2-folded $\overline{\mathcal{P}}$-covers of two different graphs induced by Euler tours. On the left deleting y splits the cycle, on the right the cycle is split in v.

If $\deg(y) > 0$, then this cover induces an edge-bijective $\frac{\Delta}{2}$-folded cover of C using a linear forest as guest (at least one vertex of C_m was mapped to y and can be removed). Otherwise C contains no odd vertex and therefore a vertex v with $\deg(v) \leq \Delta - 2$. By splitting a vertex $u \in c^{-1}(v)$ such that the guest becomes a path, only the number of vertices covering v is increased by one, but was at most $\lceil \deg(v)/2 \rceil \leq (\Delta - 2)/2$ before (see Figure 5.5 for two examples). Therefore, the obtained cover is also a $\frac{\Delta}{2}$-folded cover of C but uses a path, which is a linear forest, as guest. Therefore also the left statement is false. This concludes the proof. □

Note that, since $c_f^{\overline{\mathcal{P}}}(H) \in \{\lceil \Delta/2 \rceil, \lceil (\Delta+1)/2 \rceil\}$, this theorem allows to determine $c_f^{\overline{\mathcal{P}}}(H)$ easily by evaluating the right statement. Further, it allows easily to separate local and folded linear arboricity.

Corollary 5.6
For every $n \geq 2$ exists a graph H with maximum degree $\Delta = 2n$ and $c_l^{\overline{\mathcal{P}}}(H) > n = c_f^{\overline{\mathcal{P}}}(H)$.

Proof. It suffices to consider a connected graph H in which every vertex has degree $2n$, except of two vertices x and y that have degree $2n - 1$. An example of such a graph is K_{2n+1} minus an edge. By Theorem 5.5 follows $c_f^{\overline{\mathcal{P}}}(H) = n$. On the other hand, assume $c_l^{\overline{\mathcal{P}}}(H) \leq n$. Then there is a n-local $\overline{\mathcal{P}}$-cover c of H. Then in x and y there may end one linear forest and in all other vertices there may end no linear forest. This allows at most 2 ends of linear forests and therefore at most one linear forest. This implies $2n = \Delta(H) \leq 2$ and therefore contradicts $n \geq 2$. Hence, we have $c_l^{\overline{\mathcal{P}}}(H) > n$. This concludes the proof. □

5.3 Separations with Stronger Restrictions

We have already seen that there are strong separations for local and global covering number using induced-hereditary union-closed guest classes and for folded and local covering number even hereditary union-closed guest classes. Now we can further ask, whether there are separations for even stronger restrictions.

A stronger restriction than being hereditary is being *topological minor-closed*, i.e., closed under taking topological minors. Recall that a graph G is *topological minor* of a graph H if G can be derived from H by removing edges and vertices and smoothing. *Smoothing* means removing a vertex of degree 2 and connecting its 2 neighbours by an edge. See Figure 5.6 for an example.

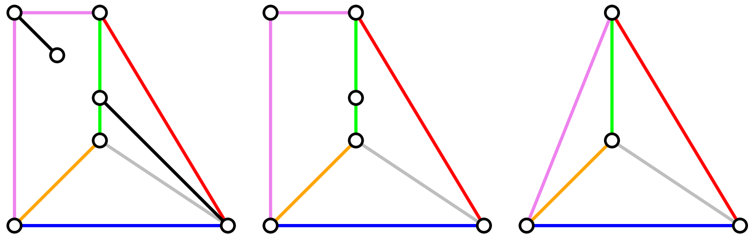

Figure 5.6: On the left: a graph G containing K_4 as topological minor. In the middle: G' after removing vertices and edges not in the minor. On the right: the topological-minor K_4 after smoothing.

Examples for union-closed, topological minor-closed graph classes are forests, star forests, caterpillar forests, linear forests, planar graphs and graphs of bounded degree. The class of all bipartite graphs is hereditary but not topological minor-closed (e.g., C_4 is bipartite and C_3 is not). It turns out this restriction restricts separations in a way that a host class cannot have a finite folded covering number, while the corresponding global covering number is ∞. To prove this we use a theorem from extremal graph theory.

Recall that the *average degree* $\mathrm{avd}(G)$ of a graph G is $\frac{\Sigma_{v\in V(G)}\deg(v)}{|G|} = \frac{2||G||}{|G|}$. And we define the *maximum average degree* of a graph G denoted by $\mathrm{mad}(G)$ as the maximum average degree of all induced subgraphs of G, i.e., $\mathrm{mad}(G) = \max_{H\subseteq G} \mathrm{avd}(H)$.

Theorem 5.7 (Bollobàs, Thomason [BT98])
There is a constant $c \in \mathbb{R}$ such that, for every $r \in \mathbb{N}_0$, every graph G of average degree $\mathrm{avd}(G) \geq cr^2$ contains K_r as a topological minor.

Further we consider an upper bound for the star arboricity of graphs ($c_g^{\overline{\mathcal{S}t}}$, where $\overline{\mathcal{S}t}$ denotes the class of star forests) in terms of the maximum average degree. We denote the class of forests by $\mathcal{F}$.

Lemma 5.8
Let H be a graph. Then $c_g^{\overline{\mathcal{S}t}}(H) \leq 2\,\mathrm{mad}(H)$.

Proof. First note for all subgraphs H' of H we have

$$\frac{||H'||}{|H'|-1} = \frac{||H||}{|H|} \cdot \frac{|H|}{|H|-1} = \frac{\mathrm{avd}(H)}{2} \cdot (1 + \frac{1}{|H|-1}) \leq \mathrm{avd}(H).$$

By Theorem 4.1 follows $c_g^{\mathcal{F}} \leq \mathrm{avd}(H)$.

By Alon et al. we know $c_g^{\overline{\mathcal{S}t}}(H) \leq 2c_g^{\mathcal{F}}(H)$ [AMR92], since $c_g^{\overline{\mathcal{S}t}}(\mathcal{F}) = 2$. This implies $c_g^{\overline{\mathcal{S}t}}(H) \leq 2c_g^{\mathcal{F}}(H) \leq 2\,\mathrm{mad}(H)$. □

Note that keeping the factor $\frac{|H|}{2(|H|-1)}$ gives significantly better upper bounds for a high order of H.

Theorem 5.9
Let $\mathcal{G}$ be a union-closed topological minor-closed class of graphs. Let $\mathcal{H}$ be a class of graphs such that $c_f^{\mathcal{G}}(\mathcal{H}) = c < \infty$. Than there is a constant $d = d(\mathcal{G}, c)$ with $c_g^{\mathcal{G}}(\mathcal{H}) \leq d$.

Proof. Since $\mathcal{G}$ is hereditary we can assume $\mathcal{H}$ to be also hereditary, since we can apply Proposition 3.2(ii) for all subgraphs.

We shall consider two properties of the guest graphs: Maximum average degree and maximum degree. We shall show that if $\mathrm{mad}(H)$ is unbounded for $H \in \mathcal{H}$, then $\mathcal{G}$ must be the class of all graphs. Otherwise there is an $a \in \mathbb{N}_0$ such that $\mathrm{mad}(H) \leq a$ for all $H \in \mathcal{H}$ and therefore $c_g^{\overline{\mathcal{S}t}}(\mathcal{H}) < \infty$. In that case, if $\Delta(G)$ is unbounded for $G \in \mathcal{G}$, then $c_g^{\mathcal{G}}(\mathcal{H}) \leq \infty$. Otherwise there is a $b \in \mathbb{N}_0$ with $\Delta(H) \leq b$ for $H \in \mathcal{H}$ and this allows coverings with a bounded number of matchings.

Case 1: For every $n \in \mathbb{N}_0$ there is a guest graph $G \in \mathcal{G}$ with $\mathrm{avd}(G) \geq n$. By Theorem 5.7 follows every K_i ($i \in \mathbb{N}_0$) is contained in a guest graph of $\mathcal{G}$ as topological minor. Since $\mathcal{G}$ is topological minor-closed, it therefore contains all complete graphs. Since $\mathcal{G}$ is hereditary, it follows that $\mathcal{G}$ is the class of all graphs. Thus, we have $c_g^{\mathcal{G}}(H) \leq 1$ for any graph H and especially for all host graphs.

Case 2: There is a constant a such that $\sup_{G\in\mathcal{G}} \mathrm{avd}(G) \leq a$. Let H be a host graph in $\mathcal{H}$. Since $c_f^{\mathcal{G}}(H) \leq c$, there is a guest graph G and an edge-surjective homomorphism ϕ from G to H with $\max_{v\in H} |\phi^{-1}(v)| \leq c$. Since every edge in H is covered by at least one edge of G, we have $||H|| \leq ||G||$. Since $\max_{v\in V(H)} |\phi^{-1}(v)| \leq c$, the number of vertices in G is at most $c|H|$. Therefore we have $\mathrm{avd}(H) = \frac{2||H||}{|H|} \leq \frac{2||G||}{\frac{1}{c}|G|} \leq ca$. And therefore we have $\forall H \in \mathcal{H} : \mathrm{mad}(H) \leq ca$.

By Lemma 5.8 follows $c_g^{\overline{\mathcal{S}t}}(H) \leq 2ca$ for any $H \in \mathcal{H}$. If $\mathcal{G}$ contains all star forests, then this concludes the proof with $d = 2ca$. Otherwise there exists a constant $\Delta_0 \in \mathbb{N}_0$ with $\Delta_0 = \sup_{G\in\mathcal{G}} \Delta(G)$, since a graph with a vertex of degree k has the star S_k with k leaves as subgraph.

Let H be a host graph in $\mathcal{H}$. Since $c_f^{\mathcal{G}}(H) \leq c$, we have $\deg(v) \leq c\Delta_0$. As we already know, H can be covered by $2ca$ star forests. Since H has maximum degree at most $c\Delta_0$ this is also an upper bound for the sizes of the stars in the covering star forests. Therefore every star forest can be covered by at most $c\Delta_0$ matchings. By replacing every star forest by at most $c\Delta_0$ matchings we get therefore a cover of H using at most $2ac^2\Delta_0$ matchings and therefore $d \leq 2ac^2\Delta_0$.

If $\mathcal{H}$ contains only independent sets, then the global covering number is 0. Otherwise $\mathcal{G}$ must contain a graph containing an edge. Since it is hereditary it then contains a K_2. Since it is union-closed, it therefore contains all matchings. Therefore we have $c_g^{\mathcal{G}}(\mathcal{H}) \leq 2ac^2\Delta_0$. This concludes the proof. □

Note that Theorem 5.9 does not exclude that folded and local or local and global covering number differ arbitrarily for union-closed topological minor-closed guest graphs, since the covering numbers could grow differently fast.

By a result of Knauer and Ueckerdt we know that we can find graphs such that the difference between local and global covering number is arbitrarily large for certain union-closed guest classes that are even *minor-closed* (star and caterpillar forests). A graph class $\mathcal{G}$ is called *minor-closed* if it is hereditary and it contains for every $G \in \mathcal{G}$ and each edge $vw \in G$ also graph G', which is obtained by removing edge vw and identifying the vertices v and w.

Theorem 5.10 (Knauer, Ueckerdt [KU12])
Let $k \geq 1$. Then there is a bipartite graph H such that

$$c_l^{\mathcal{I}}(H) = c_l^{\overline{\mathcal{C}p}}(H) \leq c_l^{\overline{\mathcal{S}t}}(H) \leq k+1 \leq 2k \leq c_g^{\mathcal{I}}(H) = c_g^{\overline{\mathcal{C}p}}(H) \leq c_g^{\overline{\mathcal{S}t}}(H).$$

On the other hand, we have only separations of local and global covering number, such that the local covering number is finite and the global one is infinite, for induced-hereditary non-hereditary guest classes. Especially, both investigated separations use guest classes containing all complete graphs. This would imply being the class of all graphs if closed under taking subgraphs. Such a separation is at least not possible for the host class of all graphs.

Theorem 5.11
Let $\mathcal{K}$ denote the class of all complete graphs and let $\mathcal{G}$ be a class of graphs and r the smallest natural number with $K_r \notin \mathcal{G}$. Let $r < \infty$. Then $c_l^{\mathcal{G}}(\mathcal{K}) = \infty$.

Proof. We use an argument similar to the one used in the proof of Theorem 5.2. We enforce a large enough gap in a guest graph such that the number of guest graphs covering the vertices at the gap is increased, whereas the incident edges of the host graph in the guest graph's gap are still to be covered.

Assume for contradiction that $c_l^{\mathcal{G}}(\mathcal{K}) = d < \infty$. Let s be the smallest number such that $c_l^{\mathcal{G}}(K_s) = d$. By Ramsey's Theorem we know that for every $n \in \mathbb{N}_0$ there exists an $t \in \mathbb{N}_0$ such that every graph of order at least t contains either K_n or K_n^c as subgraph.

Let $n = \max(r, s)$ and t be the order enforcing K_n or K_n^c to be induced subgraph of a graph by Ramsey's Theorem.

Let $l = d(t-1) + 2$. Consider the graph K_l. Since $c_l^{\mathcal{G}}(K_l) = d$ there is a cover c of K_l with regards to $\mathcal{G}$ with $\max_{v \in V(K_l)} |c^{-1}(v)| = d$. Let v denote a vertex in K_l. Since $\deg(v) = l - 1 = d(t-1) + 1$, there is at least one guest graph component G of order at least t covering v. Since G does not contain K_r as a subgraph, it must have K_s^c as a subgraph by the definition of t. Consider the cover $c_{|V(K_s)}$, that is the cover c restricted to the vertices of the copy of K_s^c in K_l. By definition of s there is a vertex $w \in K_s$ with $c_{|V(K_s)}^{-1}(w) \geq d$. But since w is also covered by G we have $c^{-1}(w) \geq d + 1$ in contradiction to its definition. □

Note that by Proposition 3.3 (i) this theorem can also be used to prove several classical covering numbers to be unbounded.

6. Computational Complexity

In the last chapter we compared the three different kinds of covering numbers by their value for common guest and host classes. In this chapter we compare the different kinds of covering numbers by the computational complexity of determining them.

Knauer and Ueckerdt observed that for the class of interval graphs $\mathcal{I}$ as guest class and $\mathcal{G}$ as host class computing any of the three covering numbers is $\mathcal{NP}$-hard. This was proved for the folded covering number called interval number by Shmyos and West [WS84] and for the global covering number called track number by Jiang [Jia13]. Knauer and Ueckerdt claimed Jiang's proof is adoptable for the local covering number with regards to $\mathcal{I}$. A detailed proof is given in this chapter.

Further, they observed cases in which the computational complexity of determining the global covering number is $\mathcal{NP}$-hard, while determining the local and folded covering number is possible in polynomial time for certain guest and host classes. In particular the problems of determining $c_g^{\mathcal{S}t}(G)$ and determining $c_g^{\overline{\mathcal{K}_2}}(G)$ for a graph G with regards to the class of all star forests $\mathcal{S}t$ and the class of all matchings $\overline{\mathcal{K}_2}$ are $\mathcal{NP}$-complete, while the folded and local covering numbers $c_f^{\mathcal{S}t}(G) = c_l^{\mathcal{S}t}(G)$ and $c_f^{\overline{\mathcal{K}_2}}(G) = c_l^{\overline{\mathcal{K}_2}}(G)$ can be determined in polynomial time [KU12].

On the other hand, we know no case in which the global covering number can be determined in polynomial time, while determining the folded or local covering number is $\mathcal{NP}$-hard. Therefore, Knauer and Ueckerdt raised the question whether there is such a case [KU12, Question 2]. In this chapter we give a general construction of such cases and use the guest class of interval graphs as example, therefore we need the $\mathcal{NP}$-hardness of determining the corresponding local covering number.

Part of a problem's $\mathcal{NP}$-completeness is its membership in the class $\mathcal{NP}$. This property is usually given for covering problems, as the following Lemma shows. Let $\mathbb{L}_i^{\mathcal{G}}(\mathcal{H})$ denote the problem of determining $c_i^{\mathcal{G}}(H)$ for a given $H \in \mathcal{H}$ for $i = g, l, f$.

Lemma 6.1
Let $\mathcal{G}$ and $\mathcal{H}$ be graph classes such that the corresponding recognition problems are in $\mathcal{NP}$ and $\mathcal{G}$ is union-closed and induced-hereditary. Then for $i = f, l, g$ the problem $\mathbb{L}_i^{\mathcal{G}}(\mathcal{H})$ is in $\mathcal{NP}$.

Proof. If $\mathcal{G}$ contains only graphs without edges, the covering numbers can be determined trivially. Otherwise we know by Lemma 3.5(i) and (iii) that the covering numbers are bounded by $||H||$. Therefore, a witness consisting of all guests and corresponding recognition-witnesses has a polynomial size and allows to verify $c_i^{\mathcal{G}}(\mathcal{H}) \leq k$ for any k such that a k-$\mathcal{G}$-cover of H exists. The problem of deciding $c_i^{\mathcal{G}}(\mathcal{H}) \leq k$ is therefore in $\mathcal{NP}$. And since we can apply binary search, the statement holds. □

6.1 Complexity of the Local Track Number Determination

We prepare now for proving that determining $c_l^{\mathcal{I}}(H)$ given any host graph H is $\mathcal{NP}$-hard. To this end we rely on the next theorem and enhance parts of its proof. Thereby, we can transform graphs for a reduction on determination of the local covering number.

Theorem 6.2 (Jiang [Jia13])
Deciding $c_g^{\mathcal{I}}(H) \leq k$ is $\mathcal{NP}$-complete for every $k \geq 2$.

Recall that an *intersection graph* of a family f of sets $s_1, \ldots, s_n$ is a graph G with vertex set $V(G)$, a bijection $b : \{1, \ldots, n\} \to V(G)$ and edge set $E(G) = \{b(i)b(j) : s_i \cap s_j \neq \emptyset\}$. I.e. two vertices are connected by an edge if and only if their corresponding sets in f have a common element. The family f is then called a *representation* of G.

A *d-track interval* is the union of d disjoint intervals on d disjoint parallel lines called *tracks*, one interval on each track.

A *d-local track interval graph G* is an intersection graph of a family f of d-track intervals that not necessarily share the same d tracks. On the other side, a *d-track interval graph H* is an intersection graph of a family f_H of d-track intervals that share the same d tracks. The representation f of G is called *d-local track interval representation* of G. The representation f_H of H is called *d-track interval representation*. See Figure 6.1 for an example.

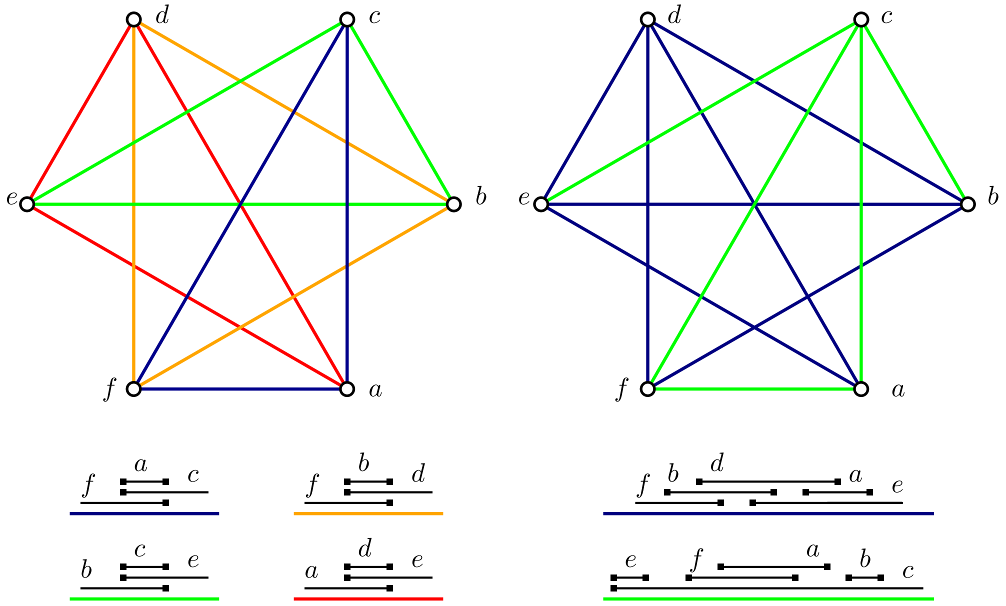

Figure 6.1: On the left: A 2-local track interval representation of $L(K_3)$ using four tracks. On the right: A 2-track interval representation of $L(K_3)$. Blocked ends are indicated by squares.

We say that a track t *realizes* an edge uv of G with regards to f if u and v are assigned to two d-track intervals that share a point on t. Let I be a d-track interval in a d-(local) track interval representation r of a graph G. Let $v \in G$ denote the vertex assigned to I. Let I_t be the interval of I in a track t. Then an end of I_t is called *blocked* end (of I and v in r) if it is not contained in another d-track interval in r. Otherwise it is called *free* end (of I and v in r). The representation r has a *free end* in a track t if t contains an interval I_t of I such that I_t has an end that is neither end of nor contained in another d-interval of r.

For later use we state a slightly stronger version of Jiang's Theorem 6.2, which easily follows from his proof.

Theorem 6.3
Deciding $c_g^{\mathcal{I}}(H) \leq d$ for H being a graph with $c_g^{\mathcal{I}}(H) \leq d+1$ is $\mathcal{NP}$-complete for every $d \geq 2$.

Proof. Directly from Jiang's proof of Theorem 6.2 follows, that the graph H for which $c_g^{\mathcal{I}}(H)$ is to determine can be restricted such that the following holds: There is a subgraph S of H with maximum degree 3 and a d-track interval representation r of $H' = (V(H), E(H) \setminus E(S))$ such that there is a track t in which all vertices in $V(S)$ have no blocked ends [Jia13]. Recall that Akyama showed that graphs of maximum degree 3 have a linear arboricity of 2 [AEH80]. Since linear forest are interval graphs, this means the track number of S is at most 2. Therefore there is a 2-track interval representation s of S. By arranging the intervals of the vertices of S in t in r as in the first track of s and adding the second track of s as a new track to r we receive a $d+1$-track interval representation of H (see Figure 6.2). Therefore we have $c_g^{\mathcal{I}}(H) \leq d+1$. □

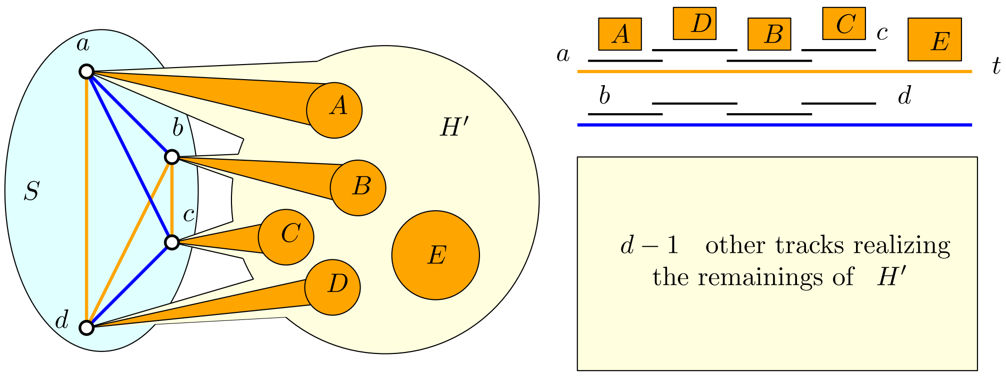

Figure 6.2: Sketch of a $d+1$-track interval representation of H. The parts of H' realized in track t are presented as A, B, C, D and E.

The following lemma delivers a component for which in every d-local track interval representation no vertex has two free ends in the same track.

Lemma 6.4
For every number $d \in \mathbb{N}_{>0}$ holds $c_g^{\mathcal{I}}(K_{2d,2d-1}) = d$ and every d-local track interval representation of $K_{2d,2d-1}$ is a d-track interval representation.

Proof. By a result of Akiyama, Exoo and Harary [AEH80], we know $c_g^{\overline{\mathcal{P}}}(K_{2d,2d-1}) = d$. Since linear forests are interval graphs, we get $c_g^{\mathcal{I}}(K_{2d,2d-1}) \leq d$.

Let r be a d-local track interval representation of $K_{2d,2d-1}$. Since $K_{2d,2d-1}$ is a bipartite graph and therefore triangle-free, every point is contained in at most two intervals of r. So, when we read a track of the representation from left to right, we obtain at most one new edge at the left endpoint of each interval except of the first (triangle-free interval graphs are caterpillar forests). Therefore every track realizes at most $|K_{2d,2d-1}| - 1 = 4d - 2$ edges.

Note that $||K_{2d,2d-1}|| = 2d(2d-1) = (4d-2)d$. Thus, if the intervals of all vertices are distributed on the same d tracks, then in r every track must realize $4d-2$ edges. Therefore, in every track each but the first interval intersects the previous interval. In particular, all d tracks must be used.

Using more tracks in total (but only d for each vertex) means more first intervals, and results thereby in a lower number of realizable edges. Thus, realizing all edges of $K_{2d,2d-1}$ in a d-local track interval representation is not possible using more than d tracks in total (whereas every vertex uses only d tracks). In Figure 6.3 you can see a representation of $K_{4,3}$ as part of a graph we define next. □

We define X_d as follows: Take $2d$ copies $K^1, \ldots, K^{2d}$ of $K_{2d,2d-1}$ and connect one vertex v^i of degree $2d-1$ of every copy K^i by an edge to the same new vertex x, which then has degree $2d$. See Figure 6.3 for an example. The next lemma allows us to use it to enforce usage of d certain tracks.

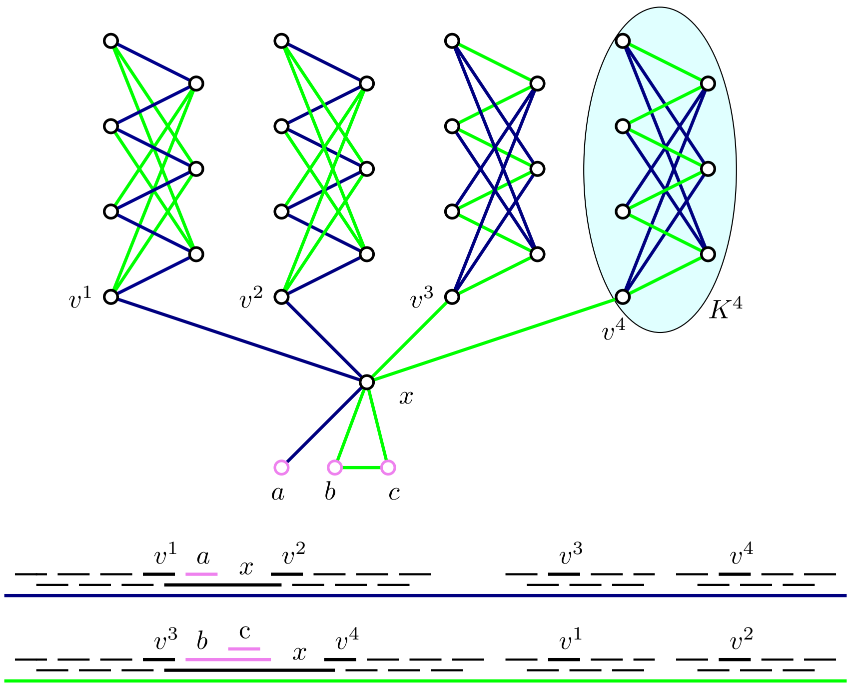

Figure 6.3: Graph X_2 with three added vertices a, b and c and a 2-track interval representation.

Lemma 6.5
Let $d \geq 1$. Then holds:

(i) Let H be the graph that is received by adding a new vertex u to $K_{2d,2d-1}$ and connecting it by an edge with a vertex v with $\deg(v) = 2d$. Let r' be a d-local track interval representation of H. Then v blocks an end of u in r'.

(ii) There is a d-track interval representation of X_d and a d-track interval L that is in each track proper, a subset of the d-track interval J of x and disjoint to the d-track intervals of all other vertices of X_d.

(iii) Let r be a d-local track interval representation of X_d. Then all ends of vertex x are blocked in r.

Proof. (i): There is no track t such that the interval of v can be contained in the interval of u in t, since within $K_{2d,2d-1}$ and t another vertex w must block an end of v and would therefore be connected with u. Therefore vertex v must block an end of u.

(ii): We prove that X_d has a d-track interval representation by constructing a corresponding cover: By Akiyama, Exoo and Harary [AEH80] we know there is a global cover of $K_{2d,2d-1}$ using d linear forests. Since in every of the $2d$ vertices of degree $2d-1$ ends at most one and in vertices of degree $2d$ ends no linear forest, in every vertex of degree $2d-1$ ends a path. Cover each copy of $K_{2d,2d-1}$ with d linear forests (which actually are paths) and extend the path p^i ending in v^i to x. Identify for every $1 \leq i \leq d$ the paths p^{2i-1} and p^{2i}. Finally identify all other paths arbitrarily such that there remain d linear forests as guest graphs. Thereby, we receive a cover of X_d using d linear forests. This proves that there are d-track interval representations of X_d that have in each track t a proper interval L_t contained in the interval J_t of x and disjoint to all other intervals in the track. (There are two intervals intersecting J_t in t. These may not intersect and have to block both ends of J_t allowing space in between.)

(iii): By (i) follows that every copy of $K_{2d,2d-1}$ blocks another end of x in each d-local track interval representation r of X_d. Therefore vertex x has no free end in r. □

Now we can prove the following announced theorem.

Theorem 6.6
Deciding $c_l^{\mathcal{I}}(H) \leq d$ for a given host graph H with $c_g^{\mathcal{I}}(H) \leq d+1$ is $\mathcal{NP}$-complete for every $d \geq 2$.

Proof. Interval graphs can be recognized in linear time, as shown by Booth and Lueker [BL76]. By Lemma 6.1 follows the problem of deciding $c_l^{\mathcal{I}}(H) \leq d$ is in $\mathcal{NP}$.

For every graph H and every number $d \geq 2$ we construct a graph $G(H,d)$ in polynomial time with

$$c_g^{\mathcal{I}}(H) \leq d \Leftrightarrow c_l^{\mathcal{I}}(G(H,d)) \leq d.$$

Thereby we reduce the $\mathcal{NP}$-complete problem of determining $c_g^{\mathcal{I}}(H) \leq d$ to the problem of determining $c_l^{\mathcal{I}}(H) \leq d$. To solve the problem for the global covering number, it suffices to solve the local problem on the constructed graph. This proves $\mathcal{NP}$-hardness.

Let $d \geq 2$ and H be a graph with $c_g^{\mathcal{I}}(H) \leq d+1$. Without loss of generality, graph H is connected (otherwise separately decide for every component).

We take H and d copies $X^1, \ldots, X^d$ of X_d. Let x_i denote the vertex x of X^i for $1 \leq i \leq d$. Then we connect every vertex x_i to every vertex of H. Call the resulting graph $G(H,d)$ (see Figure 6.4 for a sketch).

We have $c_g^{\mathcal{I}}(G(H,d)) \leq d+1$, since we can extend a $d+1$-global $\mathcal{I}$-cover of H to $G(H,d)$: First, represent $X^1, \ldots, X^d$ on the same d tracks $t_1, \ldots, t_d$. By Lemma 6.5(ii) follows that we may demand for every $1 \leq i \leq d$ an interval J_i in track t_i that is subset of the interval of x_i and does not intersect another interval. Now embed d tracks of a $d+1$-track interval representation of H in d different intervals J_i (add disjoint intervals for vertices that are not on the original track) and add the remaining track as new track to our representation. Thereby, we receive a $d+1$-track interval representation of $G(H,d)$.

"⇒:" If $c_g^{\mathcal{I}}(H) \leq d$, then there is a corresponding cover and a corresponding d-track interval representation r with d tracks $t_1, \ldots, t_d$. Without loss of generality, we may assume that every vertex has an interval in every track. Now consider a d-track interval representation s with d tracks $s_1, \ldots, s_d$ for the (disjoint) union of $X^1 \ldots, X^d$ provided by Lemma 6.5(ii). We can embed track t_i in the interval L_i of X^i in s_i such that all intervals of t_i are contained in L_i. This provides the desired representation.

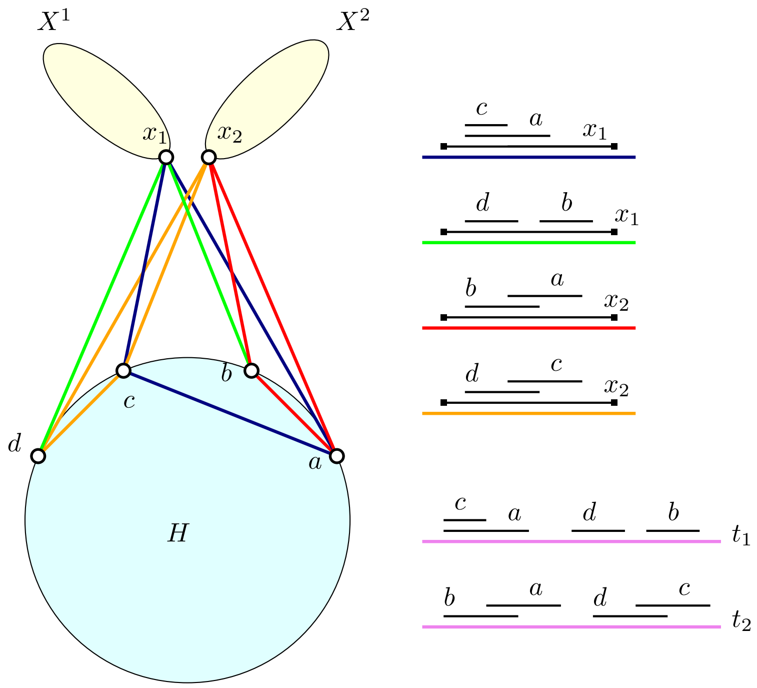

Figure 6.4: Graph $G(H, 2)$ and a 2-local track interval representation for four vertices. The part of the representation for X^1 and X^2 has been omitted, but they block the ends of x_1 and x_2 as indicated by squares. The tracks t_1 and t_2 yield a 2-track interval representation of H.

"$\Leftarrow$:" If $c_l^{\mathcal{I}}(G(H, d)) \leq d$, then there is a corresponding cover and a corresponding d-local track interval representation. By Lemma 6.5(iii) it follows that every vertex x_i has all interval ends blocked. Let $v \in H$. The d-track interval of v intersects x_i for $1 \leq i \leq d$, and since the intervals of those vertices are disjoint, every interval of v is subset of the interior of an interval of another x_i. Now, for every $1 \leq i \leq d$ we can embed all d intervals of x_i disjoint into the same new track t_i (we are no longer interested in representing $G(H, d)$ completely.). Thereby v has one interval in every of the d tracks t_i (see Figure 6.4). Further, the intersections with other intervals of vertices in H stay the same. Hence, we have found a d-track interval representation of H. In other words, it is possible to identify all interval graphs covering an edge to the same vertex x_i to one new guest. This guest is then itself an interval graph.

If d is larger than the number of vertices, then the local and global covering number are smaller than d and no computation is needed. Otherwise adding the copies of $K_{2d,2d-1}$ is possible in $O(|H|d(4d - 1 + 4d^2 - 2d + 1)) = O(|H|d^2) \subseteq O(|H|^3)$.

Therefore the given problem is $\mathcal{NP}$-complete. □

6.2 Computationally Easier Determination of the Global Covering Number

The following theorem gives a construction plan for host classes such that determining the local covering number is $\mathcal{NP}$-hard, while determining the global covering number is possible in constant time.

Theorem 6.7
Let $\mathcal{G}$ be a union-closed guest class and $\mathcal{H}'$ be a host class, such that:

(i) $c_g^{\mathcal{G}}(\mathcal{H}') = d < \infty$.

(ii) Determining $c_l^{\mathcal{G}}(H) < d$ for $H \in \mathcal{H}'$ is $\mathcal{NP}$-hard.

(iii) There is a graph $G \in \mathcal{H}'$ such that $c_l^{\mathcal{G}}(G) < d = c_g^{\mathcal{G}}(G)$.

Let $\mathcal{H} = \{H \cup G : H \in \mathcal{H}'\}$. Then determining $c_l^{\mathcal{G}}(J)$ for $J \in \mathcal{H}$ is $\mathcal{NP}$-hard, while determining $c_g^{\mathcal{G}}(J)$ for $J \in \mathcal{H}$ is possible in constant time.

Proof. By the first and third item follows that the global covering number with regards to $\mathcal{G}$ of any graph in $\mathcal{H}$ is d. Therefore determining $c_g^{\mathcal{G}}(H)$ for $H \in \mathcal{H}$ can be done by just returning d. This is possible in constant time.

To prove that determining $c_l^{\mathcal{G}}(J) < d$ for $J \in \mathcal{H}$ is $\mathcal{NP}$-hard, we reduce the problem of determining $c_l^{\mathcal{G}}(H) < d$ for $H \in \mathcal{H}'$ to it: Given any graph $H \in \mathcal{H}'$ we can construct in polynomial time $J := H \cup G$. Since $c_l^{\mathcal{G}}(G) < d$, we have $c_l^{\mathcal{G}}(J) < d \Leftrightarrow c_l^{\mathcal{G}}(H) < d$. Therefore we can now decide $c_l^{\mathcal{G}}(H) < d$ by deciding $c_l^{\mathcal{G}}(J) < d$. This concludes the proof. □

Note that we can replace the local by the folded covering number in this theorem. Further, we may replace $\mathcal{H}$ by $\overline{\mathcal{H}}$.

Now, with previous results this theorem provides a family of host classes to the guest class of interval graphs with the desired property. The following lemma provides graphs matching the description of G in Theorem 6.7(iii) for many different numbers d.

Lemma 6.8
For any $n \geq 3$ we have $c_g^{\mathcal{I}}(L(K_{n+1})) \leq c_g^{\mathcal{I}}(L(K_n)) + 2$.

Proof. K_{n+1} can be seen as $M := K_n$ plus another vertex v connected to all other vertices of M by an edge. Consider the line graph $L(K_{n+1})$. It contains the line graph J of M. There are n edges incident to v in K_{n+1}. Therefore the line graph $L(K_{n+1})$ contains, additionally to J, a disjoint copy of K_n. We refer to it by K. Every edge in M is incident to exactly two vertices in M and therefore to two edges incident to v. Therefore every vertex of J in $L(K_{n+1})$ is connected to exactly two vertices of K. Those edges are together with the edges of M all edges of $L(K_{n+1})$. See Figure 6.5 for an example for $n = 3$.

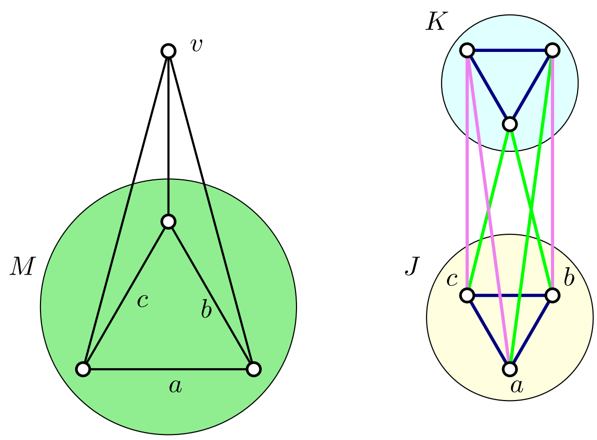

Figure 6.5: On the left: The graph K_4. On the right: The line graph $L(K_4)$ with a cover constructed as in the proof of Lemma 6.8.

Let $d := c_g^{\mathcal{I}}(J)$. There is an optimal d-global $\mathcal{I}$-cover of J. Since $n \geq 3$ we know $d \geq 1$. Therefore, we can take the disjoint union of K and G, one of the guest graphs, as a new guest graph replacing G. We create two new guest graphs G_1 and G_2 as follows: For every vertex u of J we know $\deg(u) = 2$. Therefore we can add one of the incident edges to G_1

and the other one to G_2. Thereby G_1 and G_2 are star forests and therefore interval graphs. By adding these two guest graphs to the cover, we receive an injective cover of $L(K_{n+1})$ with regards to $\mathcal{I}$ of size $d+2$. That concludes the proof. □

Let $n(d) = \min\{n \in \mathbb{N}_0 : c_g^{\mathcal{I}}(L(K_n)) = d\}$, $\mathcal{J}_{\leq d} := \{J : c_g^{\mathcal{I}}(J) \leq d\}$ and $\mathcal{J}_d := \{J \cup L(K_{n(d)}) : J \in \mathcal{J}_{\leq d}\}$.

Corollary 6.9
Let $m \geq 2$. Then there is a $d \in \{m, m+1\}$ such that determining $c_l^{\mathcal{I}}(H)$ for $H \in \mathcal{J}_d$ is $\mathcal{NP}$-complete, while determining $c_g^{\mathcal{I}}(H)$ for $H \in \mathcal{J}_d$ is possible in constant time.

Proof. Let $d \geq 2$ such that there is a $n \in \mathbb{N}_0 : c_g^{\mathcal{I}}(L(K_n)) = d$. By Lemma 6.1 follows the problems of deciding $c_l^{\mathcal{I}}(H) < d$ and $c_g^{\mathcal{I}}(H) \leq d$ are in $\mathcal{NP}$, since interval graphs can be recognized in polynomial time as proved by Booth and Lueker [BL76]. Then for the guest class $\mathcal{I}$ and the host class $\mathcal{J}_d$ we have: By construction of $\mathcal{J}_{\leq d}$ follows $c_g^{\mathcal{G}}(\mathcal{J}_{\leq d}) = d < \infty$ corresponding to item (i) of Theorem 6.7. By Theorem 6.6 follows item (ii) and by definition of d and by Theorem 5.1 follows item (iii). This proves the statement for the given d.

Since $L(K_3) = K_3 \in \mathcal{I}$ we have $c_g^{\mathcal{I}}(L(K_3)) = 1$. Let $m \geq 2$ and $k \in \{m, m+1\}$ with $n_0 \in \mathbb{N}_0$ such that $c_g^{\mathcal{I}}(L(K_{n_0})) = k$. Then let $n = \max\{n : c_g^{\mathcal{I}}(L(K_n)) = k.\}$. If $k = m+1$, then $k \in \{m+1, m+2\}$. Otherwise we have $k = m$ and by Lemma 6.8 follows $c_g^{\mathcal{I}}(L(K_{n+1})) \in \{k+1, k+2\} = \{m+1, m+2\}$. Therefore, there is a $k' \in \{m+1, m+2\}$ such that $c_g^{\mathcal{I}}(L(K_{n+1})) = k'$. By induction follows the statement. □

Note that the guest class $\mathcal{I}$ is union-closed and induced-hereditary, while the host class is artificial. Actually, if the host class is not restricted, then even worse results are possible, as the following theorem shows. Recall that $\overline{\mathcal{P}}$ denotes the class of linear forests. Let $S \subset \mathbb{N}_0$. We define $\overline{\mathcal{P}}_S$ as the closure of $\overline{\mathcal{P}} \cup \{C_k : k \in S\}$ under taking disjoint unions. By choosing S such that deciding $k \in S$ for $k \in \mathbb{N}_0$ is undecidable, we get a pair of guest and host class such that determining the local covering number is not possible at all, while the global covering number is constant. Also weaker results, where the local covering number can be computed with high complexity, are possible, but therefore the complexity of deciding $k \in S$ must be at least exponential.

Theorem 6.10
There is a class of graphs $\mathcal{H}$ such that: For all $S \subset \mathbb{N}_0$ such that $3, 4 \in S$ and $5, 6 \notin S$ let $\mathcal{IN}_S$ be the problem of deciding $k \in S$ for $k \in \mathbb{N}_0$. Then $\mathcal{IN}_S$ can be reduced in $\mathrm{O}(k)$ time to the problem of determining $c_l^{\overline{\mathcal{P}}_S}(H)$ for $H \in \mathcal{H}$, while determining $c_g^{\overline{\mathcal{P}}_S}(H)$ for $H \in \mathcal{H}$ is possible in constant time.

Proof. We firstly construct a graph G with $c_l^{\overline{\mathcal{P}}_S}(G) = 2$ and $c_g^{\overline{\mathcal{P}}_S}(G) = 3$ that additionally ensures 4-regularity. We use it to construct the graphs of a graph class comparable to the host class in Theorem 6.7. A difference is that the complexity is not delivered by the host class, but by the guest class.

We start the construction of G with the complete bipartite graph $K_{2,3}$ with partition class A of 2 vertices and partition class B of 3 vertices. We add 3 edges between the vertices of B, such that B induces a K_3. Finally we add a new vertex v and connect it to both vertices of A by an edge. Thereby all vertices of G have degree 4 except of v that has degree 2. See Figure 6.6.

Since the maximum degree of G is 4 and the maximum degree of any graph in $\overline{\mathcal{P}}$ is at most 2, the local covering number of G with regards to $\overline{\mathcal{P}}_S$ is at least 2. On the other hand,

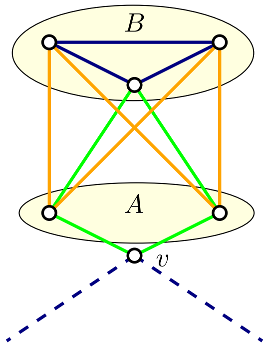

Figure 6.6: Graph G with an optimal 2-local 3-global $\overline{\mathcal{P}}_S$-cover

there is a cover proving $c_l^{\overline{\mathcal{P}}_S}(G) = 2$: The first guest graph is the K_3 induced by set B. The second guest graph is the C_4 on v, A and any vertex $u \in B$. The third guest graph is the C_4 on the remaining 4 edges. Since the guest graphs are edge-disjoint, every vertex in G is contained in at most 2 guest graphs. That proves $c_l^{\overline{\mathcal{P}}_S}(G) = 2$. Note that the cover also proves $c_g^{\overline{\mathcal{P}}_S}(G) \leq 3$.

Assume for sake of contradiction that $c_g^{\overline{\mathcal{P}}_S}(G) \leq 2$. In that case there is a cover of G using only two guests. Since every vertex but v in G has degree 4, no path may end in another vertex but v. Therefore, no guest may contain a path component. Since G contains only 6 vertices and $\overline{\mathcal{P}}$ does not contain C_5 or C_6, all components in the guest graphs have to be C_3 or C_4. Since v is not contained in an induced K_3 of G and there are only 5 other vertices, there cannot be 2 disjoint copies of K_3 in G. Therefore, both guest graphs have to be a C_3 or a C_4 covering at most $2 \cdot 4 = 8 < 10 = |E(G)|$ edges of G in contradiction to every edge being covered. We have therefore $c_g^{\overline{\mathcal{P}}_S}(G) > 2$. With our previous note follows $c_g^{\overline{\mathcal{P}}_S}(G) = 3$.

We can now construct the graphs of class $\mathcal{H}$. Let $k \geq 3$. We construct H_k as follows: Consider a C_k and for each vertex of C_k a copy of G. Finally identify each vertex of C_k with vertex v of the corresponding copy of G. Now we can define $\mathcal{H} = \{H_k : k \geq 3\}$.

Let $k \geq 3$. Since H_k contains G as induced subgraph, we know $c_g^{\overline{\mathcal{P}}_S}(H_k) \geq 3$. By construction H_k is 4-regular. As a result of Akiyama, Exoo and Harary we know 4-regular graphs have linear arboricity 3 [AEH81]. Since $\overline{\mathcal{P}} \subset \overline{\mathcal{P}}_S$ this shows $c_g^{\overline{\mathcal{P}}_S}(H_k) = 3$.

On the other hand, assume $c_l^{\overline{\mathcal{P}}_S}(H_k) = 2$. Then there is an injective cover c of H_k such that every vertex is contained in at most 2 guest graphs. Since the guest graphs have maximum degree at most 2, every guest graph has to have in every vertex degree 2 and therefore must be a disjoint union of cycles. Since the copies of G are biconnected components of H_k, all guest graph components covering them cover no edge of the C_k in H_k. Further every vertex of the C_k is contained in a guest graph component covering v in the corresponding copy of G. Therefore, every vertex of C_k may be covered by at most one guest graph component covering an edge of C_k. Therefore, cycle C_k has to be a guest graph component i.e., $C_k \in \overline{\mathcal{P}}_S$, as $\overline{\mathcal{P}}_S$ is closed under taking components (and even taking subgraphs).

If, however, $C_k \in \overline{\mathcal{P}}_S$ then we can cover every copy of G as stated before and consider C_k as another guest graph proving $c_l^{\overline{\mathcal{P}}_S}(H_k) = 2$. This concludes to $c_l^{\overline{\mathcal{P}}_S}(H_k) = 2 \Leftrightarrow C_k \in \overline{\mathcal{P}}_S$.

Since H_k can be constructed in $O(k)$ we can therefore reduce the problem of deciding $k \in S$ to the problem of determining $c_l^{\overline{\mathcal{P}}_S}(H)$ for $H \in \mathcal{H}$. Since for every graph $H \in \mathcal{H}$ holds $c_g^{\overline{\mathcal{P}}_S}(H) = 3$, the global covering number can be determined in constant time by just returning 3. This closes the proof. □

Note that k as input of a problem has size $\log(k)$. Therefore, the reduction in Theorem 6.10 has exponential complexity, and provides only interesting results for problems with a higher complexity. Especially, if the problem of deciding $k \in S$ is not decidable at all, e.g., for S being the set of encodings of halting Turing machines, then neither is the problem of determining the local covering number of graphs in $\mathcal{H}$. Since $H \in \mathcal{H}$ is equivalent to $c_g^{\overline{\mathcal{P}}_S}(H) = 1$, a pair of host and guest class for that determining the local covering number is harder than determining the global covering number due to complexity of the guest class is only possible if the host class is not hereditary. Therefore it is interesting to consider only hereditary host graphs or at least guest graphs that can be recognized in polynomial time.

6.3 Computational Complexity of the Linear Arboricities

Péroche proved that deciding $c_g^{\overline{\mathcal{P}}}(G) \leq 2$ is $\mathcal{NP}$-complete. We can extend our result on differing local linear arboricity and linear arboricity to prove $\mathcal{NP}$-completeness for $c_g^{\overline{\mathcal{P}}}(G) \leq k$ with any $k \geq 2$ and thereby generalize his result.

But first we extend Péroche's result to the local and folded linear arboricity.

Let $\mathcal{H} = \{H : \Delta(H) = 4,\, \delta(H) = 3 \text{ and } |\{v \in V(H) : \deg(v) = 3\}| = 4\}$.

Theorem 6.11 (Péroche [Pé84])
Deciding $c_g^{\overline{\mathcal{P}}}(H) \leq 2$ for $H \in \mathcal{H}$ is $\mathcal{NP}$-complete.

Corollary 6.12
Deciding $c_l^{\overline{\mathcal{P}}}(H) \leq 2$ for $H \in \mathcal{H}$ is $\mathcal{NP}$-complete.

Proof. It suffices to prove for all $H \in \mathcal{H}$ that $c_g^{\overline{\mathcal{P}}}(H) \leq 2 \Leftrightarrow c_l^{\overline{\mathcal{P}}}(H) \leq 2$. By Proposition 3.3(i) we have $c_g^{\overline{\mathcal{P}}}(H) \leq 2 \Rightarrow c_l^{\overline{\mathcal{P}}}(H) \leq 2$. On the other hand, if there is an injective cover c of H with $\forall v \in V(H) : c^{-1}(v) \leq 2$, then in every vertex at most one path ends, and in every vertex of degree 4 no path ends. This allows at most 4 path ends and hence at most 2 paths in c. In particular, c is an injective cover of H with regards to $\overline{\mathcal{P}}$ of size 2. This proves $c_g^{\overline{\mathcal{P}}}(H) \leq 2 \Leftrightarrow c_l^{\overline{\mathcal{P}}}(H) \leq 2$. This concludes the proof. □

We claim that, if the recognition of compositions of k directed Hamilton cycles is $\mathcal{NP}$-complete for any $k \geq 2$, then it is $\mathcal{NP}$-complete to decide $c_l^{\overline{\mathcal{P}}}(H) \leq k$ for a given graph H (Péroche's proof is based on the case $k = 2$).

Theorem 6.13
Let $k \geq 2$. Then the problem $\mathbb{L}_g^k$ of deciding $c_g^{\overline{\mathcal{P}}}(H) \leq k$ for a given graph H is $\mathcal{NP}$-complete.

Proof. Theorem 6.11 proves the statement for $k = 2$. Therefore we can assume $k \geq 3$. Since linear forest can be recognized in polynomial time, we know by Lemma 6.1 that $\mathbb{L}_g^k$ is in $\mathcal{NP}$.

We reduce the problem of deciding whether a k-regular graph has a 1-factorization (a decomposition into r perfect matchings), which is $\mathcal{NP}$-complete (for $k \geq 3$) as proven by Leven and Galil [LG83], in polynomial time to $\mathbb{L}_g^k$. This proves $\mathcal{NP}$-hardness and therefore concludes the proof.

Let H be a k-regular graph. Then identify every vertex of H with v in another copy of E_n (see Section 5.2 for a definition). Since $|V(E_n)| \in \mathrm{O}(n^2)$, this is possible in polynomial time. Call the resulting graph H'. By Lemma 5.3 we know $c_g^{\overline{\mathcal{K}_2}}(H) \leq n \Leftrightarrow c_g^{\overline{\mathcal{P}}}(H') \leq n$ and since the left statement just means there is a 1-factorization of H our reduction is complete. □

Note that Theorem 5.5 yields a linear time algorithm to determine the folded linear arboricity. Therefore, the class of linear forests is a first example of a guest class where it is $\mathcal{NP}$-complete to determine the local covering number, whereas the folded covering number can be computed in polynomial time.

7. Boxicity as Covering Number

The *boxicity* is a parameter generalizing the concept of interval graphs and was introduced by Fred S. Roberts [Rob69]. In this chapter we consider it as a global covering number of complement graphs, as Cozzens and Roberts already did [CR83], and introduce the corresponding *local* and *folded boxicity*, as well as the global covering number with regards to the corresponding union-closed guest class, called *union boxicity*. We argue why we choose the union-closed guest class also for local and folded boxicity and that those coincide. Further, we give geometric interpretations of these parameters and present graphs for that the local and union boxicity are 1, while the classical boxicity can be arbitrarily large.

7.1 Introduction of Union Boxicity and Local Boxicity

Recall from Section 3.3 that an *intersection graph* of a family f of sets $s_1, \ldots, s_n$ is a graph G with vertex set $V(G)$, a bijection $b : \{1, \ldots, n\} \to V(G)$ and edge set $E(G) = \{b(i)b(j) : s_i \cap s_j \neq \emptyset\}$ meaning two vertices are connected by an edge if and only if their corresponding sets in f have a common element. The family f is then called an *intersection representation* of G.

We already considered the class $\mathcal{I}$ of interval graphs that are the intersection graphs of intervals in $\mathbb{R}$. An intersection representation of a graph G containing only intervals is called an *interval representation* of G.

A natural generalization is to consider intervals of higher dimension that can be described as Cartesian products of intervals in $\mathbb{R}$. We call them *boxes*, and the *dimension* of a box b is the number of intervals whose Cartesian product is b. Actually, every graph G is an intersection graph of d-dimensional boxes for some $d \leq |G|/2$[Rob69]. Correspondingly, the *boxicity* is the smallest $d \in \mathbb{N}_0$ such that G is an intersection graph of d-dimensional boxes.

The following theorem characterizes the boxicity of a graph H as the smallest number of interval graphs whose intersection is H. Figure 7.1 gives an (suboptimal) example for such an intersection.

Theorem 7.1 (Roberts [Rob69])
For any graph G we have $\operatorname{box}(G) \leq d \Leftrightarrow \exists I_1, \ldots, I_d \in \mathcal{I} : G = I_1 \cap \ldots \cap I_d$.

Proof. Consider an intersection graph G of d dimensional boxes. For every dimension $1 \leq i \leq d$ we can consider the family F_i of all intervals $I_i^{(v)} \subset \mathbb{R}$ corresponding to any vertex

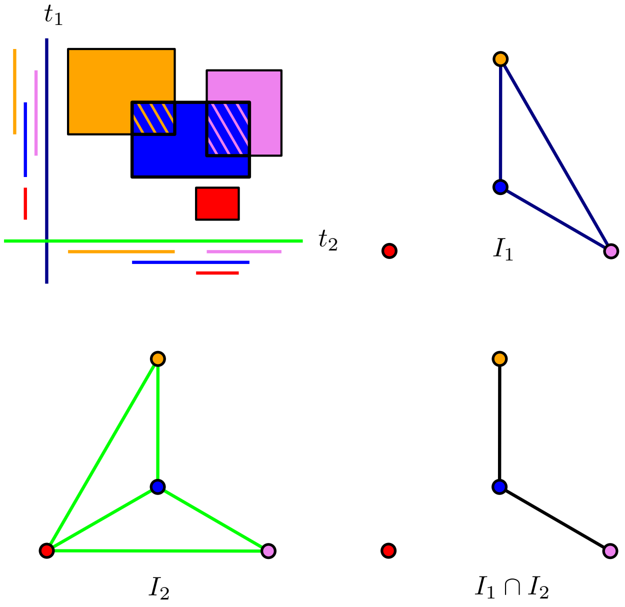

Figure 7.1: Top left: Representation of a graph H as intersection graph of 2-dimensional boxes with the projections of the boxes into the both tracks t_1 and t_2. Top right: Interval graph I_1 represented on track t_1. Bottom left: Interval graph I_2 represented on track t_2. Bottom right: Graph $G = I_1 \cap I_2$.

v of G and its corresponding box $I^{(v)} = I_1^{(v)} \times \cdots \times I_d^{(v)}$. Then we have the intersection graph I_i of F_i that is an interval graph and has the same vertex set as G. We realize two boxes $I^{(a)} = I_1^{(a)} \times \cdots \times I_d^{(a)}$ and $I^{(b)} = I_1^{(b)} \times \cdots \times I_d^{(b)}$ intersect if and only if for every dimension $1 \leq i \leq d$ the intervals $I_i^{(a)}$ and $I_i^{(b)}$ intersect. This again is equivalent to a and b being neighbours in I_i. Therefore we have $G = I_1 \cap \ldots \cap I_d$. That means G is intersection of d interval graphs.

If on the other side there is a graph H that is intersection of d interval graphs $I_1, \ldots, I_d$ with fixed interval representations, then for every vertex v of H there is the box $I^{(v)} = I_1^{(v)} \times \cdots \times I_d^{(v)}$ in which $I_i^{(v)}$ describes the interval in the interval representation of I_i corresponding to vertex v for $1 \leq i \leq d$. We just saw, two of these boxes intersect if and only if the corresponding vertices intersect in each of the interval graphs, and therefore if and only if the corresponding vertices are connected by an edge in H. This shows H to be an intersection graph of d dimensional boxes.

Hence, the boxicity $\operatorname{box}(G)$ of a graph G can also be characterized as the minimum number d such that G is intersection of d interval graphs $G = I_1 \cap \ldots \cap I_d$. □

Recall that De Morgan's Law states for any sets $S_1, \ldots, S_n$ that $S_1^c \cap \ldots \cap S_n^c = (S_1 \cup \ldots \cup S_n)^c$. Therefore, the equation $G = I_1 \cap \ldots \cap I_d$ is equivalent to $G^c = I_1^c \cup \ldots \cup I_d^c$. That is just another way to say that the global covering number of G with regards to the class $\mathcal{I}^c$ of complements of interval graphs equals d. Therefore, we can view the boxicity of a graph as a covering number of its complement [CR83]. We call a graph that is the complement I^c of an interval graph $I \in \mathcal{I}$ also *cointerval graph*.

Since we usually are interested in union-closed guest classes, we define the *union boxicity* of a graph G, denoted by $\overline{\operatorname{box}}(G)$, as the global covering number of G^c with regards to $\overline{\mathcal{I}^c}$, the class of collections of cointerval graphs. Correspondingly, we define the *local boxicity* of G, denoted by $\operatorname{box}_l(G)$, as the local covering number and the *folded boxicity* of G, denoted by $\text{box}_f(G)$, as the folded covering number of G^c with regards to the class $\overline{\mathcal{I}^c}$. I.e., we define $\overline{\text{box}}(G) := c_g^{\overline{\mathcal{I}^c}}(G^c)$, $\text{box}_l(G) := c_l^{\overline{\mathcal{I}^c}}(G^c)$ and $\text{box}_f(G) := c_f^{\overline{\mathcal{I}^c}}(G^c)$.

Considering the closure with regards to disjoint union of the guest class $\mathcal{I}^c$ is more interesting for the folded covering number, since, as we show below, it is always 1 or ∞ with regards to $\mathcal{I}^c$ (with trivial exceptions where it is 0). For the local covering number this has no effect, since we can consider in every d-local cover every component as another guest and receive still a d-local cover.

Recall that a homomorphism ϕ from a graph G to a graph H is a function $\phi : V(G) \to V(H)$ such that if two vertices v and w are connected by an edge in G, then their images in H are also connected. Recall that we call a graph class $\mathcal{G}$ closed under taking folded components if for any $G \in \mathcal{G}$, any homomorphism $\phi : G \to H$ into some graph H and every component C of G the class contains the graph $\phi(C)$. By *contracting* two vertices v and w in a graph G we mean the process of removing the vertices v and w from G and adding a new vertex u that is adjacent to all remaining vertices that were adjacent to v or w.

Lemma 7.2
If a guest class $\mathcal{G}$ is closed under contracting non-adjacent vertices, then for every host graph H except independent sets we have $c_f^{\mathcal{G}}(H) < \infty \Leftrightarrow H \in \mathcal{G} \Leftrightarrow c_f^{\mathcal{G}}(H) = 1$.

Proof. The right equivalence follows by definition. The induction from right to left in the left equivalence is thereby obvious, and it is left to show that a finite folded covering number equals always 1. If and only if there is a folded cover c of H, then the folded covering number is finite. We use induction over the number of folded vertices that equals $|D| - |I|$ for a cover $c : D \to I$.

Start of induction: If no vertices are folded in a folded $\mathcal{G}$-cover c, then it is injective and therefore $c_f^{\mathcal{G}}(\text{Img}(c)) = 1$.

Induction Hypothesis: Let $n \in \mathbb{N}_0$ and every folded $\mathcal{G}$-cover c folding n vertices fulfills $c_f^{\mathcal{G}}(\text{Img}(c)) = 1$.

Induction Step: Then let $c' : G \to H$ be a folded $\mathcal{G}$-cover of H folding $n+1$ vertices. There are two folded vertices v and w with $c(v) = c(w)$. Consider graph G' that we receive by contracting v and w in G. Since two vertices can only be folded if they are non-adjacent, and $\mathcal{G}$ is closed under contracting non-adjacent vertices we know $G' \in \mathcal{G}$. The cover c' induces a folded $\mathcal{G}$-cover $c : G' \to H$ in which only n vertices are folded (map every vertex as in c' and map the new vertex as v and w). Therefore we can apply the Induction Hypothesis, which proves $c_f^{\mathcal{G}}(\text{Img}(c')) = 1$. This concludes the proof. □

The following lemma allows us to apply this proposition to cointerval graphs and allows us to prove that folded and local boxicity coincide.

Lemma 7.3 *(i) Let $I^c \in \mathcal{I}^c$ be a cointerval graph. Then I^c has at most one non-trivial connected component.*

(ii) The class of cointerval graphs $\mathcal{I}^c$ is closed under contracting non-adjacent vertices.

(iii) The class of cointerval graphs $\mathcal{I}^c$ is closed under taking folded components.

Proof. Let $I^c \in \mathcal{I}^c$. We know that C_4 is not induced subgraph of any interval graph [LB62]. Hence, we have $C_4 \not\subseteq I$ and therefore $C_4^c \not\subseteq I^c$. Since the complement of C_4 is a matching of two non-adjacent edges, all edges of I^c are adjacent and therefore in the same component. Therefore item (i) holds.

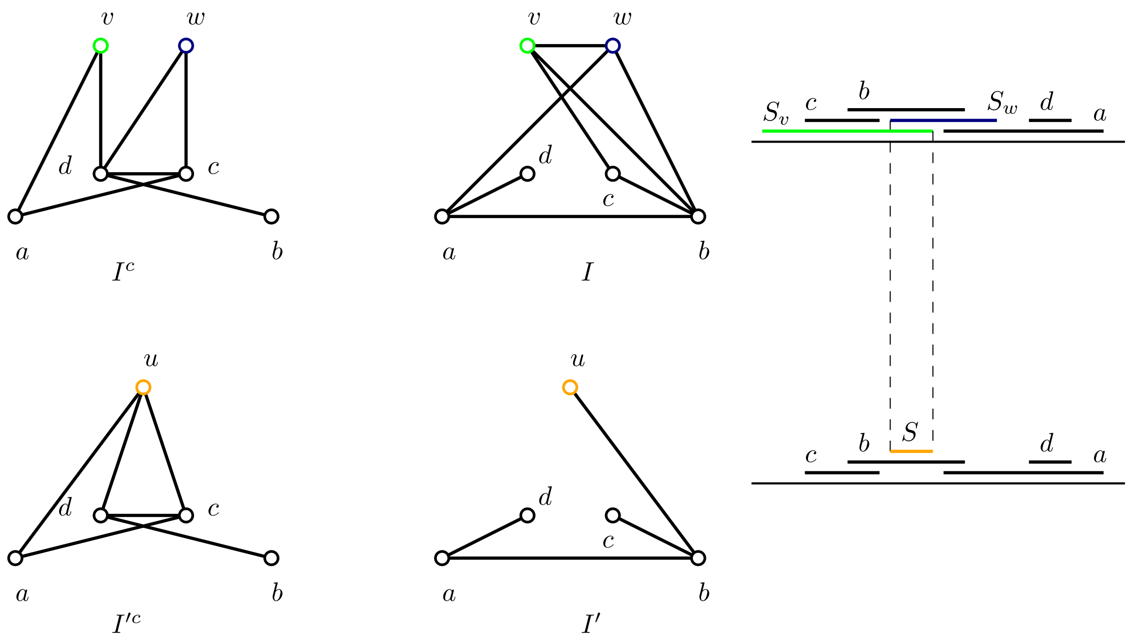

Figure 7.2: Top: Cointerval graph I^c, interval graph I and its interval representation. Bottom: Cointerval graph I'^c resulting from contraction of v and w in I^c, interval graph I' and its interval representation.

Let v and w be non-adjacent vertices in a cointerval graph I^c and r be an interval representation of interval graph I with the intervals S_v and S_w corresponding the vertices v and w (see Figure 7.2). Then the intersection $S = S_v \cap S_w$ is non-empty. The interval of any vertex z intersects S if and only if it intersects S_v and S_w. This, however, is equivalent to z being neither adjacent to v nor adjacent to w in I^c. Let u be the vertex and I'^c be the graph resulting from contracting v and w in I^c. We receive an interval representation of I' by replacing S_v and S_w in r by S and associating S with u. As we have seen, an interval of a vertex z intersects S if and only if z is neither adjacent to v nor to w and is therefore not adjacent to u in I. The intersections of other intervals and the adjacency of other vertices did not change. Therefore, we have an interval representation for I' proving $I'^c \in \mathcal{I}^c$. Therefore $\mathcal{I}^c$ is closed under contracting non-adjacent vertices and item (ii) holds.

Let $J^c \in \mathcal{I}^c$ and $\phi : J^c \to H$ be a graph homomorphism. Since $\mathcal{I}^c$ is induced-hereditary, as is $\mathcal{I}$, every component I^c of J^c is also a cointerval. To show item (iii) it suffices to show $\phi(I^c) \in \mathcal{I}^c$. Since $\phi_{|I^c} : I^c \to \phi(I^c)$ is edge-surjective, it is a folded $\mathcal{I}^c$ cover of $\phi(I^c)$. By Lemma 7.2 follows $\phi(I^c) \in \mathcal{I}^c$ and thereby item (iii). □

As a consequence of Lemma 7.2 and Lemma 7.3 (ii), the folded covering number of host graphs that are not independent sets with regards to cointerval graphs is always 1 or ∞. Note that this is due to the definition of homomorphisms that allows only non-adjacent vertices to be folded. One may ask instead for the smallest number d for a graph H such that it is the result of contractions (of adjacent vertices) in a cointerval graph I^c where every vertex is the result of at most d contractions.

As seen in Chapter 5 the folded and local covering number with regards to the same pairs of guest and host class can differ arbitrarily. But the following theorem shows, as a consequence of Lemma 7.3 (iii), that they coincide with regards to $\overline{\mathcal{I}^c}$.

Theorem 7.4
Let G be any graph. Then $\mathrm{box}_f(G) = \mathrm{box}_l(G)$.

Proof. Since $\overline{\mathcal{I}^c}$ is union-closed by definition and closed under taking folded components by Lemma 7.3 (iii), we can apply Proposition 3.3 (ii) and (iii) and have therefore

$$c_f^{\overline{\mathcal{I}^c}}(G^c) \leq c_l^{\overline{\mathcal{I}^c}}(G^c) \text{ and } c_f^{\overline{\mathcal{I}^c}}(G^c) \geq c_l^{\overline{\mathcal{I}^c}}(G^c).$$

Therefore $\text{box}_f(G) = c_f^{\overline{\mathcal{I}^c}}(G^c) = c_l^{\overline{\mathcal{I}^c}}(G^c) = \text{box}_l(G)$. □

7.2 Geometric Interpretation

The next Theorem provides a geometric interpretation of the local boxicity in terms of intersection graphs. While for a boxicity of d we may use only d-dimensional boxes, for a local boxicity of d we may choose boxes of arbitrary dimension but restrict every box $I = I_1 \times \cdots \times I_m$ by demanding $I_i = \mathbb{R}$ for all but at most d numbers $i \in \{1, \ldots, m\}$. See Figure 7.3 for an example.

Theorem 7.5
Let H be a graph on n vertices. Then $\text{box}_l(H) \leq d \Leftrightarrow H$ is intersection graph of boxes in $\mathbb{R}^m (m \in \mathbb{N}_0)$ each being the Cartesian product of m non-empty intervals in $\mathbb{R}$ of which at most d do not equal $\mathbb{R}$.

Proof. The left statement $\text{box}_l(H) \leq d$ is equivalent to $c_l^{\overline{\mathcal{I}^c}}(H^c) \leq d$ and means there is a disjoint union of cointerval graphs and an injective homomorphism from each of them to H^c such that every edge of H^c is covered by one of these homomorphisms and every vertex of H^c is covered at most d times.

Equivalently, you can colour every edge of H^c allowing multiple colours on each edge such that for every vertex the incident edges have at most d different colours and the edges of any colour and all vertices of H^c form a cointerval graph ($\overline{\mathcal{I}^c}$ is closed under adding isolated vertices). This means for every colour we have an interval representation in which without loss of generality every vertex that is isolated with regards to this colour has interval $\mathbb{R}$.

Consider one representation r_c for every colour c and for every vertex v the Cartesian product of the intervals of v of r_c for every colour c. Two of the boxes retained that way intersect exactly if the intervals of the corresponding vertices x and y intersect in every representation. This, however, is equivalent to xy being not contained in any colour set and therefore not contained in $E(H^c)$. This in turn is equivalent to $xy \in E(H)$. Every such box representation induces interval representations of cointerval graphs that cover H^c such that for every vertex the number of its intervals that do not equal $\mathbb{R}$ is equivalent to the number of cointerval graphs in which it is non-isolated. □

There is a corresponding interpretation for the union boxicity: For a union boxicity of d of a graph H we may again use boxes of arbitrary dimension, but we have to choose a partition of the dimensions into d sets $D_1, \ldots, D_d$ such that every box equals $\mathbb{R}$ in all dimensions except of at most one in each set D_i $(1 \leq i \leq d)$ completely:

The sets $D_1, \ldots, D_d$ correspond to the d guests $G_1, \ldots, G_d$ in a d-global $\overline{\mathcal{I}^c}$-cover of H^c. Let $C_{i,1}, \ldots, C_{i,m_i}$ be the components of G_i for every $1 \leq i \leq d$.

We can associate each guest component C with an interval representation of its complement and thereby with another dimension. This representation contains an interval for every vertex v in that component. While each guest can have arbitrarily many components, every vertex is covered by at most one component of each guest and has therefore at most one interval $I_{i,j_{v,i}}^{(v)}$ in a representation of a component $C_{i,j_{v,i}}$ of G_i $(1 \leq i \leq d)$. By associating it with interval $I_{i,k}^{(v)} := \mathbb{R}$ in the representations of each other component $C_{i,k}$ of G_i, we do not

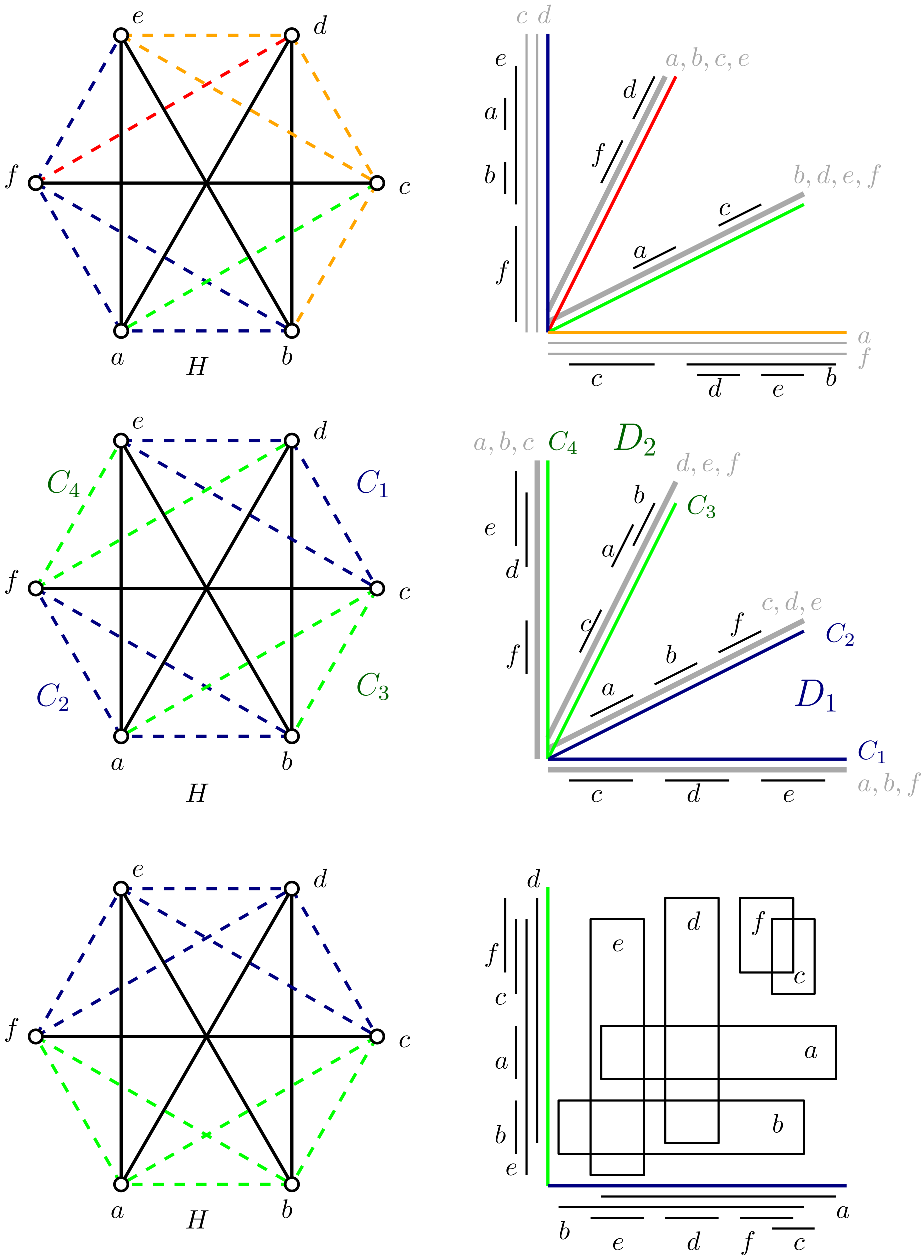

Figure 7.3: Top: Graph H and a representation for $\mathrm{box}_l(H) \leq 2$. Center: Graph H and a representation for $\overline{\mathrm{box}}(H) \leq 2$. Bottom: Graph H and a representation for $\mathrm{box}(H) \leq 2$. In all three cases the dashed edges belong to the complement H^c of H and are covered by cointerval graphs and unions of cointerval graphs, respectively. Note that in the top case every vertex has at most 2 intervals unequal $\mathbb{R}$, in the center case every vertex has at most one such interval in every dimension set D_1 and D_2 and in the bottom case we use only two dimensions at all. In the first and second case H is an intersection graph of 4-dimensional boxes that are not presented directly, but are for every vertex the Cartesian product of its intervals.

change the realization of edges in H^c. But we can now associate every vertex v with box $(I_{1,1}^{(v)} \times \cdots \times I_{1,j_{v,1}}^{(v)} \times \cdots \times I_{1,m_1}^{(v)}) \times \cdots \times (I_{d,1}^{(v)} \times \cdots \times I_{d,j_{v,d}}^{(v)} \times \cdots \times I_{d,m_d}^{(v)})$. The intersection graph of those boxes is H, since in every dimension one guest component is realized.

The classical boxicity allows only one component per guest and therefore only one dimension in every set D_i $(1 \leq i \leq d)$. And this is its known interpretation.

7.3 Separation of Local and Union, and Classical Boxicity

By Proposition 3.3 (i) and (v) we know for any graph G that $c_l^{\overline{\mathcal{I}^c}}(G^c) \leq c_g^{\overline{\mathcal{I}^c}}(G^c) \leq c_g^{\mathcal{I}^c}(G^c)$, since $\overline{\mathcal{I}^c}$ is union-closed and superclass of $\mathcal{I}^c$. Therefore we have

$$\mathrm{box}_l(G) \leq \overline{\mathrm{box}}(G) \leq \mathrm{box}(G).$$

It is now interesting by how much these numbers may differ. Actually, it is quite easy to find graphs where union boxicity and classical boxicity differ arbitrarily, as we can see in the next theorem.

Theorem 7.6
Let $n \in \mathbb{N}_0$ and M_n be the matching consisting of n edges. Then $\overline{\mathrm{box}}(M_n^c) = 1$, whereas $\mathrm{box}(M_n^c) = n$.

Proof. By Lemma 7.3 (i) we know every cointerval graph has at most one component containing an edge. Hence, a global $\mathcal{I}^c$-cover of M_n contains at least n guests to cover all n components. Since K_2 is a cointerval graph, there actually is a n-global $\mathcal{I}^c$-cover of M_n. Thus, we have $\mathrm{box}(M_n^c) = c_g^{\mathcal{I}^c}(M_n) = n$. On the other hand, the class $\overline{\mathcal{I}^c}$ is union-closed and, since K_2 is a cointerval graph, it contains all matchings. Therefore we have $\overline{\mathrm{box}}(M_n^c) = 1$. □

Note that the complements of matchings where already used by Roberts to show that the boxicity is unbounded [Rob69].

8. Conclusion

In this thesis, we investigated the *global*, the *local* and the *folded covering number* of host graphs (and classes) with regards to different guest classes. Specifically, we investigated the linear arboricity and boxicity in their global, local and folded variants. Further, we compared the different kinds of covering numbers for same pairs of host and guest classes in terms of separation and computational complexity. Several interesting questions remain open, which we summarize below.

8.1 Separation of Folded, Local and Global Covering Number

We have presented the bipartite graphs as union-closed hereditary guest class with regards to which all folded covering numbers are bounded by a constant k (which in this case equals 2), whereas the corresponding local covering numbers get arbitrarily large. On the other hand, we have proven that this is not possible for topological-minor closed union-closed guest classes on any host class, i.e., with regards to such guest classes a host class with bounded folded covering number has also a bounded global covering number.

However, Knauer and Ueckerdt showed that there are host graphs where the global covering number is asymptotically twice the local covering number for increasing local covering numbers with regards to certain minor-closed union-closed guest glasses, i.e., star forests and caterpillar forests [KU12]. But we do not know whether the global covering number can be larger than twice the corresponding local covering number. If there is a bound on this factor, it may be useful for bounding global covering numbers.

Question 8.1
Is there for every $r > 0$ a minor-closed union-closed guest class $\mathcal{G}$ and a host graph H such that $c_g^{\mathcal{G}}(H) \geq r \cdot c_l^{\mathcal{G}}(H)$?

Further, we know that folded and local covering number with regards to linear forests, a minor-closed union-closed guest class, may differ by 1, but we do not know whether we have larger differences for such guest classes. The original question stays in this case open.

Question 8.2
Let $\mathcal{G}$ be a minor-closed union-closed guest class and H be a host graph. By how much can $c_f^{\mathcal{G}}(H)$ and $c_l^{\mathcal{G}}(H)$ differ?

As part of our results, we proved that with regards to the guest class of bipartite graphs the folded covering number is bounded on all graphs. Further, there are several pairs of host classes and guest classes such that global, local or folded covering number are bounded (e.g., the guest class of star forests with every kind of forest we mentioned in this thesis as host class, see Lemma 5.8). One may therefore ask for which guest classes the local or global covering number of all graphs is absolutely bounded. By Theorem 5.11 we know such a guest class must contain all complete graphs and coincides therefore with the class of all graphs if it is hereditary.

Question 8.3
Let $i = g, l$. Is there an induced-hereditary guest class $\mathcal{G}$ that is not the class of all graphs and a number $d \in \mathbb{N}_0$ such that for every graph H we have $c_i^{\mathcal{G}}(H) \leq d$?

In the end we are still interested in restrictions that strongly bound the difference between global and local covering number or make them even coincide, as it is the case for global, local and folded arboricity.

8.2 Computational Complexity

We have presented a union-closed host class for which determining the folded and local covering number with regards to interval graphs is $\mathcal{NP}$-complete, whereas the corresponding global covering number is computable in polynomial time, since it is constant. Since the host class is artificial, it is interesting whether this is also possible for a more natural host class. Especially an induced-hereditary host class $\mathcal{H}$ is interesting, since it does not allow to add a component to every host graph H to achieve a constant global covering number, since the graph without that component is also contained in $\mathcal{H}$.

Question 8.4
Is there a union-closed guest class $\mathcal{G}$ and an induced-hereditary host class $\mathcal{H}$ such that determining the local or folded covering number for host graphs in $\mathcal{H}$ is $\mathcal{NP}$-complete, whereas the global covering number can be computed in polynomial time?

We are especially interested in such results for the host class of all graphs, as these would be more general.

Question 8.5
Is there a union-closed guest class $\mathcal{G}$ such that determining the local or folded covering number for any given graph is $\mathcal{NP}$-complete, whereas the global covering number can be computed in polynomial time?

In the end we are interested in general properties for guest and host class such that it is not possible. This would strengthen the approach of considering folded and local covering number, as it would allow proving $\mathcal{NP}$-hardness of determining the global covering number by proving it for the folded or local covering number.

Further, we have reduced the problem of determining the track number to the problem of determining the local track number. One may ask for general properties of host and guest class such that $\mathcal{NP}$-hardness of determination of the global covering number induces that it is also $\mathcal{NP}$-hard to determine the local or folded covering number. This would allow proving that the global covering number is computable in polynomial time by proving so for the folded or local covering number. Since this appears to be usually not true for “small” guest classes like the classes of matchings and the class of star forests [KU12], considering hereditary or minor-closed guest classes does not seem to be helpful.

Question 8.6
Which properties of a guest class $\mathcal{G}$ enforce that if determining $c_g^{\mathcal{G}}(H)$ for any graph H is $\mathcal{NP}$-complete, then determining $c_l^{\mathcal{G}}(H)$ for any graph H is also $\mathcal{NP}$-complete?

In Theorem 5.2 we showed that the folded covering number of any graph H with regards to the class $\mathcal{B}$ of bipartite graphs is at most 2. Since bipartite graphs can be recognized in polynomial time, we can decide whether $c_f^{\mathcal{B}}(H) = 1$ in polynomial time. Therefore, we can determine $c_f^{\mathcal{B}}(H)$ in polynomial time.

On the other hand, it is open whether the problems of determining the corresponding local covering number are $\mathcal{NP}$-complete, which is probably the case. Note that the problem of determining the corresponding global covering number is known to be $\mathcal{NP}$-complete by a result of Orlin [Orl77].

We already know cases in which determining the global, the local and the folded covering number are three $\mathcal{NP}$-complete problems, and cases in which determining folded and local covering number is possible in polynomial time, whereas determining the global covering number is $\mathcal{NP}$-complete [KU12]. If determining the local covering number with regards to bipartite graphs is $\mathcal{NP}$-complete, this yields an example where determining the folded covering number is possible in polynomial time, whereas determining the local covering number is $\mathcal{NP}$-complete. The linear arboricity is another guest class with this property, as discussed in Section 6.3.

Question 8.7
Is the problem of determining $c_l^{\mathcal{B}}(H)$ for a given graph H $\mathcal{NP}$-complete?

8.3 Folded, Local and Global Linear Arboricity

We have proven the Local Linear Arboricity Conjecture (LLAC) stated by Knauer and Ueckerdt [KU12] as a weakening of the Linear Arboricity Conjecture (LAC). However, following the approach of considering folded and local covering number to attack a problem for the global covering number is not straightforward.

While global, local and folded arboricity coincide, we could present examples where folded and local linear arboricity differ and examples where local and global linear arboricity differ. As a byproduct we could show that deciding whether the linear arboricity of a given graph is at most k is $\mathcal{NP}$-complete for a fixed $k \geq 2$, which is a new result for $k \geq 3$.

The LAC remains open and some weaker conjectures, as stated by Knauer and Ueckerdt [KU12], remain open, too. Our proof of LLAC is another indicator of its trueness and possibly it can be used to attack LAC further. The resulting algorithm to compute an optimal local cover has a runtime in $\mathrm{O}(|V| + |E|^2)$ and one may ask for a faster algorithm.

Question 8.8
Is there an algorithm that computes for a given graph $H = (V, E)$ an optimal local $\overline{\mathcal{P}}$-cover with a time complexity less than $\mathrm{O}(|V| + |E|^2)$?

8.4 Boxicity

Finally, We have considered the boxicity of a graph H as $c_g^{\mathcal{I}^c}(H^c)$, the global covering number of its complement H^c with regards to $\mathcal{I}^c$, the class of all complements of interval graphs. We have introduced the corresponding union boxicity as the covering number $c_g^{\overline{\mathcal{I}^c}}(H^c)$ with the union-closed guest class variant and the local boxicity, which is the corresponding local covering number $c_l^{\overline{\mathcal{I}^c}}(H^c)$, and coincides with the folded one. We have found geometric interpretations for both parameters. A graph H with $\operatorname{box}_l(H) = d$ is the intersection graph of boxes that equal $\mathbb{R}$ in all but d dimensions, and a graph H with $\overline{\operatorname{box}}(H) = d$ is the intersection graph of boxes with d sets of dimensions such that every box equals $\mathbb{R}$ in every but at most one dimension of every set. Finally, we gave an example of graphs with local and union boxicity 1 but arbitrarily large classical boxicity.

One may ask several general questions in terms of covering numbers on union boxicity and local boxicity. The classical boxicity is an upper bound on both these parameters, but there are probably better upper bounds. We are generally interested in sharp lower and upper bounds of the local and the union boxicity. There may also be properties of host classes such that union boxicity and classical boxicity coincide or such that their difference is bounded.

Question 8.9
What are tight bounds for union or local boxicity?

Further, we know that computing the classical boxicity is $\mathcal{NP}$-hard by a result of Cozzens [Coz81]. Perhaps their proof can be modified to show that computing the union boxicity is $\mathcal{NP}$-hard, too. A corresponding result for the local boxicity would be also interesting.

Question 8.10
Is the problem of determining $\operatorname{box}_l(H)$ *or* $\overline{\operatorname{box}}(H)$ *for any graph* H $\mathcal{NP}$*-complete?*

While intersection graphs of boxes induce interval graphs in each dimension, other intersection graphs lack a corresponding property. But maybe some of them can still be treated in a similar way.

Question 8.11
Are there other kinds of intersection graphs that can be considered in terms of covering numbers?

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