Karlsruhe Institute of Technology

Tiling Rectangles with Unions of Squares

Bachelor Thesis of

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Time Period: 10th May 2021 – 10th September 2021

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Karlsruhe, September 10, 2021

Abstract

Given two squares with side lengths a and b, we can glue them together to get an (a, b)-piece. A tiling of a rectangle with side lengths C and D with (a, b)-pieces is a placement of these pieces such that every point in the rectangle is covered by exactly one piece and every piece is fully contained in the rectangle.
We show that we can decide whether a given $C \times D$-rectangle can be tiled with (a, b)-pieces and we can at least semi-decide whether, given $a, b \in \mathbb{N}^+$, there exists a $C \times D$-rectangle that can be tiled with (a, b)-pieces. Furthermore we will look at some pieces in detail and find categories that have no tilings and some that have at least one tiling. Lastly we will also prove that we can tile the whole plane with (a, b)-pieces for every combination of $a, b \in \mathbb{N}$.

Deutsche Zusammenfassung

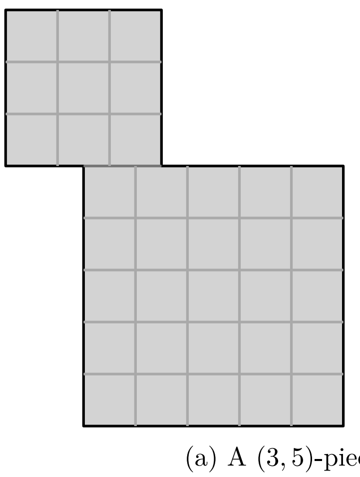
Gegeben zwei Quadrate mit Seitenlängen a und b. Verkleben wir die beiden Quadrate, so bekommen wir ein (a, b)-Teil. Eine Parkettierung eines Rechtecks mit Seitenlängen C und D mit (a, b)-Teilen ist eine Platzierung von diesen Teilen, sodass jeder Punkt in dem Rechteck von genau einem Teil überdeckt wird und jedes Teil vollständig im Rechteck enthalten ist.
Wir werden zeigen, dass man entscheiden kann, ob ein $C \times D$-Rechteck mit (a, b)-Teilen parkettiert werden kann und mindestens semi-entscheiden kann ob es überhaupt ein $C \times D$-Rechteck gibt, das mit gegebenen (a, b)-Teilen parkettiert werden kann. Weiterhin untersuchen wir einige Teile in Detail und finden Kategorien von (a, b)-Teilen die keine Parkettierungen haben und Kategorien die mindestens eine Parkettierung haben. Abschließend beweisen wir noch, dass die ganze Ebene mit (a, b)-Teilen parkettiert werden kann für jede Kombination von $a, b \in \mathbb{N}$.

Contents

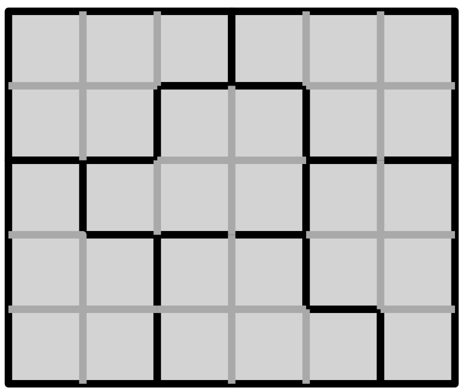
1. Introduction

Given two positive whole numbers a and b. If we take two squares, one with side length a and the other with side length b, and glue them together, we get an (a, b)-piece. A tiling of a rectangle with positive whole number side lengths C and D is an arrangement of multiple (a, b)-pieces such that no two overlap except for its edges, no piece goes out of the rectangle and every point in the rectangle is covered by at least one piece. An example for a piece is shown in Subfigure 1.1a while an example for a tiling is shown in Subfigure 1.1b.

The question:
Can we tile a rectangle with side lengths C and D with (a, b)-pieces for some $a, b, C, D \in \mathbb{N}^+$? gives way to a wide variety of aspects and theorems for which we need different tools to solve and prove. While the definition does not restrict the orientation of the pieces or gives precise restrictions on glueing squares together, we will see that there are only a finite number of orientations and ways we can glue squares together if we want to tile a rectangle. Whilst some tilings seem very simple, others with only a slightly bigger square prove to be more complicated. We can blow up tilings by a factor n to get a bigger tiling with the same pattern. Though we also find some categories of tilings that are not copies of each other but still follow a common pattern. Furthermore we can even list all rectangles that can be tiled for some specific piece and give a construction plan for each.



(a) A $(3, 5)$-piece.



(b) With $(1, 2)$-pieces you can tile a rectangle with side lengths 5 and 6.

Figure 1.1: Examples for a piece and a tiling.

1.1 Related Work

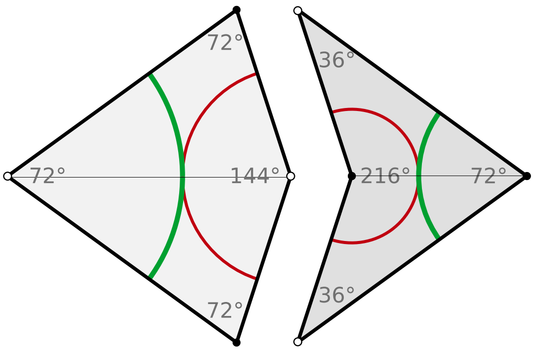
Johannes Kepler wrote about tiling the whole plane with regular polygons as early as 1619 in [Kep69]. Starting with the smallest regular polygon being the triangle, where you can form lines of triangles alternating between facing up and down, we can also tile the plane by laying squares, regular polygons with 4 sides, side by side and up and down. Lastly we can also tile the plane with hexagons yielding the three regular polygons that can tile the plane by itself.

Given a tiling of the plane, it may either be periodic, meaning there are repeating patterns of aperiodic. In 1891, Fedorov [Fed91] proved that every repeating pattern can be classified in one of seventeen groups based on its symmetries. These repeating patterns can be symmetrical in different ways generating out of the different isometries of the euclidean plane, they can just be translations of each other, rotations, reflections or so called glide reflections meaning a reflection and translation. Symmetries that arise out of combinations out of those can then be classified in only 17 different groups.

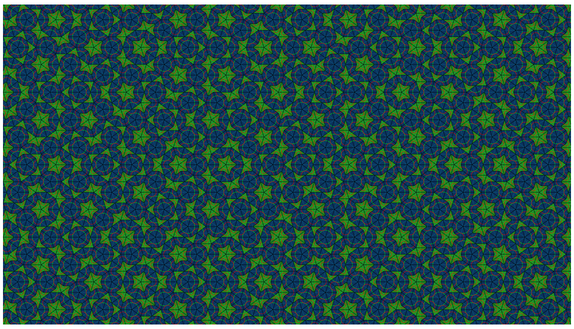
Wang tiles are squares with the same size and with colours on each sides. A Wang-Tiling is a tiling of the plane such that the squares that touch in a line, need to have the same colour on that line. Wang conjectured in 1961 [Wan65] that if there exists a finite set of Wang tiles that can tile the plane, then there must be a repeating pattern and therefore it is a periodic tiling. He added that this would imply the existence of an algorithm to decide if the whole plane can be tiled with a given set of Wang tiles. This conjecture was proven false by Berger in 1966 [Ber66]. He showed that you can translate any Turing machine into Wang tiles that can only tile the plane if the machine does not halt. Therefore he used the halting problem to show that there can not be an algorithm that decides whether we can tile the plane with a given set of Wang tiles.

Bergers initial proof used 20426 Wang tiles that can tile the plane but only aperiodically. This number was subsequently lowered. Karel Culik II found a set of 13 Wang tiles [CI96] that can again tile the whole plane but only aperiodically. A set with 11 Wang tiles was found in 2015 by Emmanuel Jeandel and Michael Rao [JR15].

Roger Penrose investigated aperiodic tilings with more than one shape. He found a set of six shapes that can only tile the plane aperiodically in 1974 [Pen74]. Using some additional restrictions, such as the colours in Wang tiles, so that two shapes have to meet additional requirements to be laid side by side other than just fitting together, he found two additional sets of shapes in the same paper, called the "Kite and Dart" and the "Rhombuses". An example for these restrictions is drawing a specific line on each piece, those lines having a colour and ending in some sides. Two pieces are then allowed to be laid next to each other if the line on one piece continues in the same colour on the other piece. The "Kite and Dart"-pieces can be seen in, a tiling with those pieces can be seen in



(a) The two pieces for Penroses second set of tiles that can only form an aperiodic tiling of the plane. Two pieces are only allowed to be placed next to each other if the green and red lines continue on edges where they meet, taken from [Gg09].



(b) Part of a tiling with "Kite and Dart"-pieces, green and red lines continue between pieces, taken from [Prz15].

1.2 Outline

We first give the definition for our problem and some basic notation we will use. We then define two main problems we want to explore. Furthermore we show that we can only use a finite number of orientations, positions and kinds of pieces to solve our problems.

In Chapter 3 we will show that one of our two problems is decidable, the other is semi-decidable. With some restrictions we will show that even the second problem can be decidable. We will also give basic algorithms that solve these problems albeit not efficient.

Following this chapter, we show that we can tile the whole plane in Chapter 4 for any parameters given.

In the next chapter we will show that we can construct tilings for multiple categories of pieces and therefore answer one of our two problems for many kinds of pieces.

In Chapter 6 we first define a construct that prevents a partial tiling from being finished and then use this to show that there are categories of pieces that can not tile any rectangle.

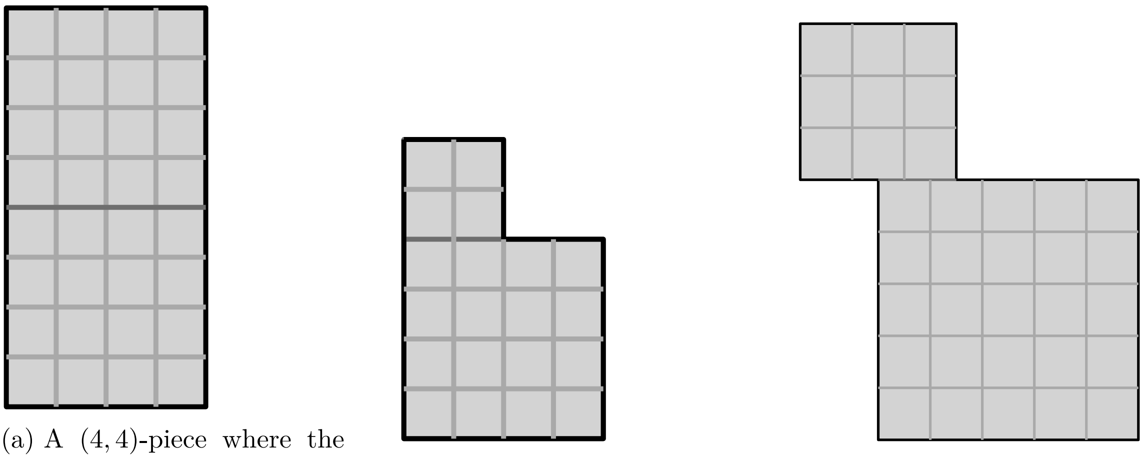
Finally we fully answer our two main problems for two kinds of pieces in Chapter 7.

2. Problem definition

In the following chapter, we define pieces and tilings to understand our problem and then prove some first observations.

Definition 2.1. *For $a, b \in \mathbb{N}^+$ an (a, b)-piece is a rectilinear polygon consisting of two squares, one with side length a and the other with side length b such that the $a \times a$-square overlaps with the $b \times b$-square in exactly one line of length greater than* 0*.*

Below are some examples for (a, b)-pieces:



(a) A $(4, 4)$-piece where the two squares meet at a line with length 4.

(b) A $(2, 4)$-piece with a joining line of length 2.

(c) A $(3, 5)$-piece where the line has length 1.5.

Figure 2.1: Some examples for pieces

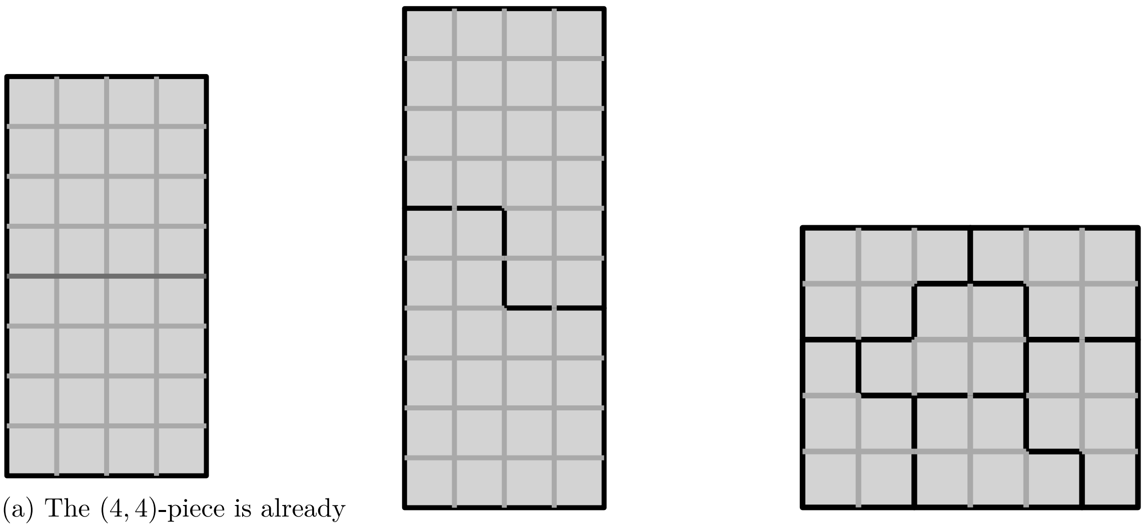
For the rest of this thesis, we will call them (a, b)*-pieces* or (a, b)*-tiles*. Since (a, b)-tiles are the same as (b, a)-tiles, we usually say (a, b)-tiles if $a \leq b$ or (b, a)-pieces if $b \geq a$, so the smaller number comes first.

Definition 2.2. *For $C, D \in \mathbb{N}^+$ a tiling for a $[C, D]$-rectangle with (a, b)-pieces is an arrangement of (a, b)-pieces, such that:*

- *In a $C \times D$-rectangle, every point is covered by at least one (a, b)-piece.*
- *Every point that is covered by more than one (a, b)-piece is only covered by edges or corners of (a, b)-pieces.*
- *No point outside of the $C \times D$-rectangle is covered.*

The second condition is of technical nature. It allows us to place (a, b)-tiles side by side, like pieces of a puzzle, while still not allowing them to overlap.

The definition and the problem in general become very clear and intuitive if we look at some examples of valid tilings:



(a) The $(4, 4)$-piece is already a valid tiling for a $[4, 8]$-rectangle.

(b) The $(2, 4)$-piece can tile a $[4, 10]$-rectangle.

(c) With $(1, 2)$-pieces you can tile a $[5, 6]$-rectangle.

Figure 2.2: Some examples for tilings

We can use infinitely many pieces of all possible configurations for a tiling, as long as they are still (a, b)-pieces for the same parameters $a, b \in \mathbb{N}^+$.

For the rest of this thesis, we will say the (a, b)-pieces tile the $[C, D]$-rectangle or there exists a tiling for the $[C, D]$-rectangle with (a, b)-pieces.

The main goal of this thesis is to find a way to answer the following question for all possible values:

$$\begin{gathered}\text{Given parameters } a, b, C, D \in \mathbb{N}^+, \\ \text{does a tiling of the } [C, D]\text{-rectangle with } (a, b)\text{-pieces exist?}\end{gathered} \tag{2.1}$$

For some a, b this question is hard to answer or almost always "No", in that case we first want to answer the weaker question:

$$\text{Given parameters } a, b \in \mathbb{N}^+, \text{ are there parameters } C, D \in \mathbb{N}^+ \tag{2.2}$$
$$\text{such that there exists a tiling of the } [C, D]\text{-rectangle with } (a, b)\text{-pieces?}$$

So we just want to know if there is some rectangle that can be tiled. In the rest of this chapter we will prove some restrictions on possible tilings.

2.1 Restriction on tilings

Looking at the examples for valid tilings in Figure 2.2, we can see that in every tiling, the edges of the pieces are all aligned to the edges of the rectangle, meaning any two edges are either parallel or orthogonal to each other. This actually holds for all tilings, meaning that each piece is placed on a grid and can't just be oriented randomly.

Theorem 2.3. *In a valid tiling for a $[C, D]$-rectangle with (a, b)-pieces, $a, b, C, D \in \mathbb{N}^+$, any two edges are either orthogonal or parallel to each other.*

Proof. We prove this by contradiction. For the rest of the proof we call edges that are either parallel or orthogonal to each other *aligned*.

Let a be an edge that is not aligned to at least one other edge b.

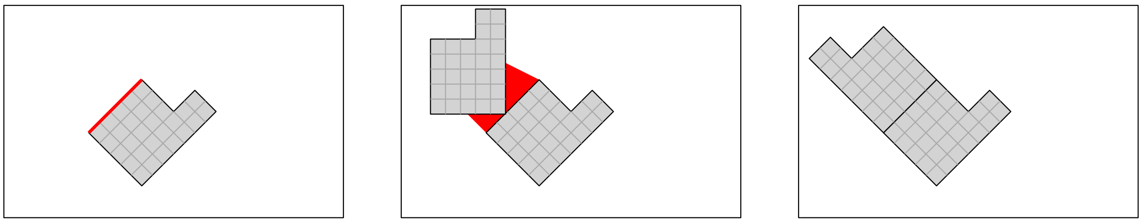
Since the edges of the $[C, D]$-rectangle are all aligned to each other, either a,b or neither aligned to the edges of the rectangle but never both, since then they would also be aligned to each other. If one of them is aligned to the edges of the rectangle, we choose the other unaligned edge.

Then that edge belongs to an (a, b)-piece X and since this is a rectilinear polygon, none of the edges of X are aligned to the edges of the rectangle.

Without loss of generality let the piece X be placed like in Subfigure 2.3a and lets look at the edge marked in red. If we place another piece that is not aligned to X on the red edge, then we create untilable pockets like the red space in Subfigure 2.3b.

Therefore the neighbours of every piece have to be aligned to the piece itself, like in Subfigure 2.3c. Since being aligned is transitive, the neighbours of the neighbours also have to be aligned and so on. Therefore every piece has to be aligned to each other in order to not leave gaps but then we are either aligned to the edges of the rectangle or we create untilable pockets with the edges of the rectangle itself.

Therefore in every gap- and overlap-free tiling, therefore valid tiling, every edge has to be aligned to every other edge. $\square$



(a) Example for an unaligned piece.

(b) Aligned and unaligned pieces create untilable pockets.

(c) Every unaligned piece implies its neighbours to be unaligned too.

Figure 2.3: Pieces have to be aligned to the edges of the outer rectangle.

With a similar argument, we will now prove that every piece has to be placed on a certain grid, meaning that every edge of every piece has a whole number distance to the edges of the rectangle. Together with Theorem 2.3 this makes this problem very discrete and easy to play with.

Definition 2.4. *We define the distance function d between two lines in two-dimensional space as the minimum length of any line between any point on the first line and any point on the second line*

In particular $d(x, y) = 0$ if x and y are touching. In general $d(x, y) \in \mathbb{R}_0$.

Theorem 2.5. *Given $a, b, C, D \in \mathbb{N}^+$ and a valid tiling for the $[C, D]$-rectangle with (a, b)-pieces, for any edge x of any (a, b)-piece and any edge y of the $[C, D]$-rectangle, we have $d(x, y) \in \mathbb{N}_0$.*

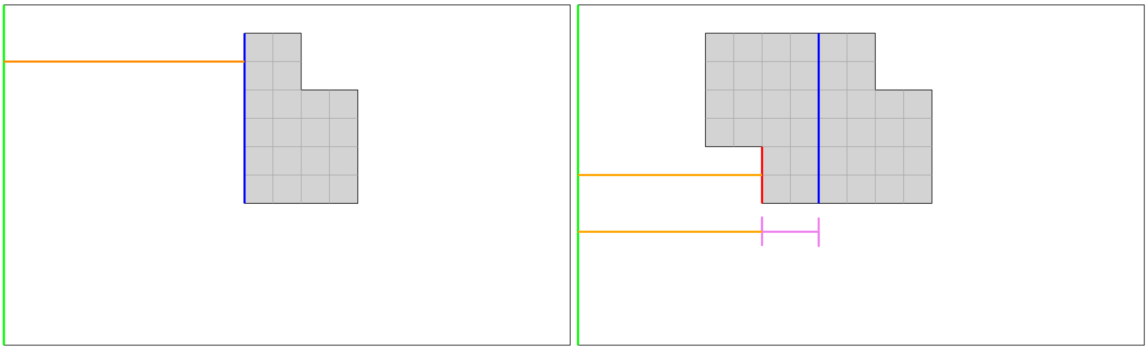
Proof. Like usual, we define the distance between two lines as the minimum distance between two points on the lines. Since all edges are aligned to each other, because of Theorem 2.3, the distance is always the length of a line between the two that is either orthogonal or parallel to every other line. For example: the distance between the blue and green edge in Subfigure 2.4a is the length of the orange line.

We also prove this by contradiction with a similar method like in Theorem 2.3 by first showing that if there is a piece that does not have the proper distance, then no piece has the proper distance and then that we cannot tile the rectangle.

Let x be an edge on a (a, b)-piece that does not have a whole number distance to an edge y of the $[C, D]$-rectangle. Lets look at one neighbouring piece touching x, this piece has a edge z parallel to x with $d(x, z) \in \{a, b, a + b\}$, then $d(y, z) = d(x, y) - d(x, z)$ like in Subfigure 2.4b and since $d(x, y) \notin \mathbb{N}$ and $d(x, z) \in \mathbb{N}$ then $d(y, z) \notin \mathbb{N}$.

Therefore the neighbours of any edge with non-integer distance to the outer edges has at least one edge with non-integer distance to the outer edges too. Furthermore this distance is smaller than the original distance. The last edge has to have distance 0 to the outer edge but we cannot reach 0 by subtracting integers from a non-integer number, therefore we can never tile a rectangle if at least one edge has non-integer distance to the outer edges.

□



(a) the distance between the blue and green edge is the length of the orange line.

(b) the distance between the red and green edge is the distance between the blue edge and green edge without the length of the pink line.

Figure 2.4: The distance between pieces and edges are either all whole numbers or never whole numbers.

With Theorem 2.5 we will continue to rule out many configurations on our journey to make this problem very discrete. In Definition 2.1 we allowed the two squares to touch each other in a line of positive length, thus allowing infinitely many configurations, now we will show that there are actually only a finite number of possibilities in order to make a valid tiling.

Corollary 2.6. *For $a, b \in \mathbb{N}$ there are only $4 \cdot (a + b - 1)$ different (a, b)-tiles to make a valid tiling.*

Proof. Without loss of generality, we assume $a \leq b$.

Every (a, b)-piece consists of an $a \times a$-square and a $b \times b$-square that touch in one line, we fix the $b \times b$-square, then the $a \times a$-square can be either on top, on the bottom or on one of the sides of the $b \times b$-square.

Without loss of generality let the $a \times a$-square touch the $b \times b$-square on the top side of the $b \times b$-square. Then the possible configurations are of one of 3 types:

- The $a \times a$-square is standing out on the left side and touches the $b \times b$-square in a line of length $l \in \{1, 2, \ldots, a - 1\}$.
- The $a \times a$-square is standing out on the right side and touches the $b \times b$-square in a line of length $l \in \{1, 2, \ldots, a - 1\}$.
- the $a \times a$-square is sitting between the left and right side of the $b \times b$-square and is touching it in a line of length a. The distance between the right edge of the $a \times a$-square and the right edge of the $b \times b$-square is always an integer.

There are $b - a + 1$ different configurations for the third category since the distance between the right sides is at maximum $b - a$, if it the $a \times a$-square still touches the $b \times b$-square in a line of length a, and decreases in whole-number steps until it reaches 0, therefore we have $b - a + 1$ possibilities.

For example, every possible configuration for $(3,5)$-pieces that can still form a tiling, are depicted in Figure 2.5.

Now we will show that every other configuration cannot be found in a valid tiling:

If an (a,b)-piece is not of one of the above configurations, then it touches the $b \times b$-square in a line of length $l \notin \mathbb{N}$, for example the orange line in Figure 2.6. If the distance between the right side of the rectangle and the right side of the $a \times a$-square and the distance between the right side of the rectangle and the right side of the $b \times b$-square are integers (due to Theorem 2.5), then the distance between the right sides of the two squares for the (a,b)-piece has to be an integer. In the case of Figure 2.6, if the green and red line have an integer length, the blue line has to have an integer length too. Lastly, the length of line l is a without the distance of the right sides and since $a \in \mathbb{N}$, it has to be an integer too. In our case, if the blue line has an integer length and the blue line plus the orange line is an integer, the orange line has to have an integer length itself.

Therefore the $a \times a$-square and the $b \times b$-square have to touch in a line with integer length. Since the length has to be above 0, this leaves only a finite number of possible configurations. To be exact, $4 \cdot ((a-1)+(a-1)+(b-a+1))$. Every configuration can only be oriented in one of four directions due to Theorem 2.3 and we have $a-1$ possibilities for the first category of configurations, also $a-1$ for the second and $b-a+1$ for the third. Therefore we have $4 \cdot ((a-1)+(a-1)+(b-a+1)) = 4 \cdot (a+b-1)$ different pieces. □

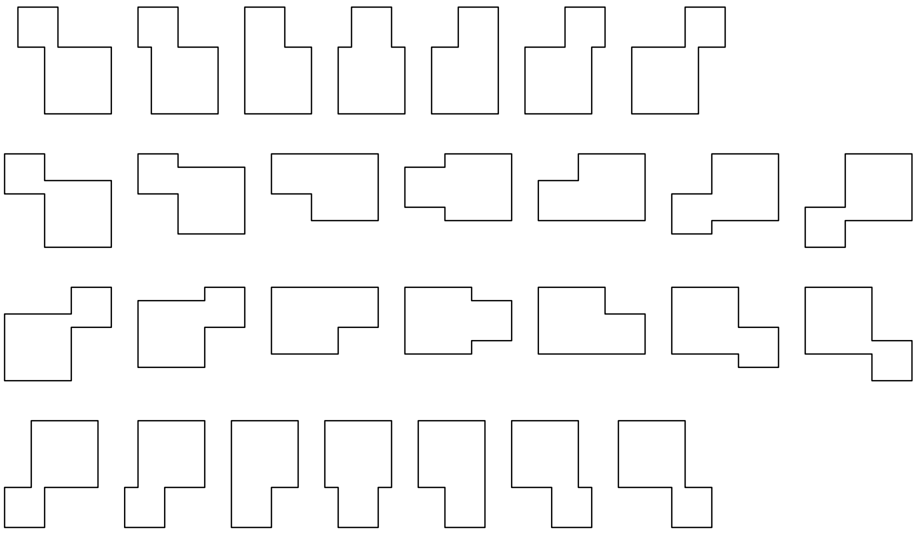


Figure 2.5: Every possible $(3,5)$-piece that can appear in a valid tiling of a rectangle.

Corollary 2.7. *In a valid tiling for a $[C,D]$-rectangle with (a,b)-pieces for some $a,b,C,D \in \mathbb{N}^+$, in a division of the $[C,D]$-rectangle in 1×1-squares, each (a,b)-piece either fully covers or does not cover a 1×1-square at all.*

Proof. Since each piece is aligned to the outer edges of the $[C,D]$-rectangle, and since each (a,b)-piece consists of only an $a \times a$-square and a $b \times b$-squares, the set of all possible valid tilings is a subset of all valid tilings with $a \times a$-squares and $b \times b$-squares.

If we prove that in all tilings with $a \times a$-squares and $b \times b$-squares, every 1×1-square is either fully or not covered at all by a square, we have proven our statement.

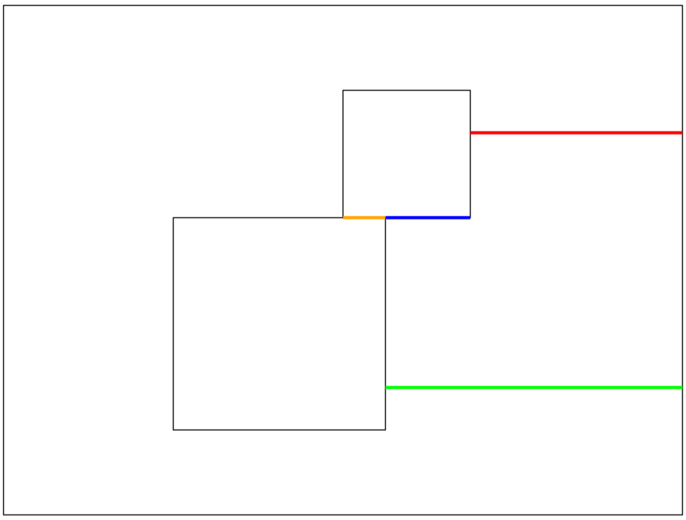


Figure 2.6: If the green and red line have an integer length, then the blue line also has to have integer length and therefore the orange line too.

If we start in the top left corner, we can either place a $a \times a$-square or a $b \times b$-square there and it covers the area between the four corner points $(0,0)$ and either $(0,a),(a,0),(a,a)$ or $(0,b),(b,0),(b,b)$ and in all cases, since $a,b \in \mathbb{N}$, this covers the 1×1-squares inside fully and none on the outside.

Likewise, if we choose the next free position and place a square, we are again starting with integer coordinates for the corner and since each square is aligned and every side length is an integer, we again only cover squares fully or not at all. This holds for every square we place and therefore also for all our (a,b)-pieces. $\square$

Corollary 2.7 helps us get some intuition for our problem. We can fully experiment with tilings by using graphing paper and let each square be our 1×1-squares. With Theorem 2.3 and Corollary 2.7, we know that each piece is drawn by only drawing on lines on the graphing paper and since our rectangle has finite size, we can draw out tilings on a big enough paper.

These results alone are enough for the next chapter about decidability but we have one more big result that helps us narrow down the number of possible tilings.

Theorem 2.8. *There is a bijection between tilings of the $[C,D]$-rectangle with (a,b)-pieces and tilings of the $[nC,nD]$-rectangle with (na,nb)-pieces. Furthermore, every rectangle that can be tiled with (na,nb)-pieces is of the form $[nC,nD]$ for some $a,b,C,D,n \in \mathbb{N}^+$.*

We will also show that every tiling with (na,nb)-pieces is a rectangle of the form $[nC,nD]$, this implies that there are exactly the same number of (na,nb)-tilings as there are (a,b)-tilings. That means that we only have to look at pairs of numbers a,b that are co-prime to each other, to find all possible tilings.

Proof. $\Rightarrow$ Every configuration of a piece can be classified by one of four orientations and without loss of generality if the $a \times a$-square is on top of the $b \times b$-square, the distance between the right side of the $b \times b$-square and the right side of the $a \times a$-square. This distance can be be any integer between $0,\ldots,b-1$.

Now we can "multiply" this tiling by n: We now use (na, nb)-pieces and try to tile the $[nC, nD]$-rectangle. We will use the tiling for the $[C, D]$-rectangle with (a, b)-pieces and "copy" it here. If we encode the tiling as an array of tuples with each tuple having the top-left-position and the classification of each piece, then the multiplied encoding has every position multiplied by n and each classification multiplied in the following way: the orientation stays the same, without loss of generality the $a \times a$-square sits on top of the $b \times b$-square, then the distance between the right sides is multiplied by n.

An example is shown in Figure 2.7. This is now a valid tiling for the $[nC, nD]$-tiling or else it wouldn't be a valid tiling of the $[C, D]$-rectangle.

First, we will show that any rectangle that can be tiled by (na, nb)-pieces is of the form $[nC, nD]$: Each side length is of the form $C = x \cdot na + y \cdot nb = n \cdot (x \cdot a + y \cdot b)$ therefore $C = n \cdot C'$ for $C', C, n \in \mathbb{N}^+$. Similarly $D = n \cdot D'$.

If we can prove, that every configuration is a multiplication of a configuration of the (a, b)-piece by n and that every starting position of the pieces is a multiple of n, we are done, since we can just revert the multiplication and get a tiling with (a, b)-pieces.

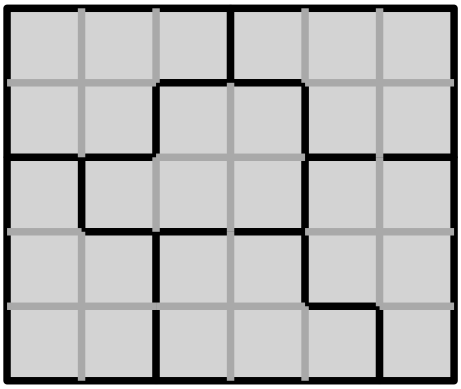
Lets first look at the starting positions, suppose we have a corner that does not have coordinates that are multiples of n. Suppose the x-coordinate is not a multiple of n, that means that the upper-left corner has a distance to the left edge of the rectangle that is not a multiple of n. If we look at the piece next to the left side of the corner, this piece has a corner that is either na, nb or $n \times (a + b)$ further left than our original corner. If our original corner did not have a multiple of n distance to the left edge of the rectangle, this corner does not either. If we continue this strategy, we find out that actually no corner has a multiple of n distance the left side and can therefore not finish the tiling since 0 is a multiple of n and has to be achieved in order to not leave holes on the left side. Therefore every corner has to have coordinates that are multiples of n.

Next, if we have a configuration that can not be reduced to a (a, b)-piece-configuration, then that means, if we suppose the $a \times a$-square is on top of the $b \times b$-square, that the right sides have a distance that is not a multiple of n. Since every edge has to have a distance that is a multiple of n to every other edge, due to the same argument as the corners above, either the $a \times a$ or the $b \times b$-square does not have a distance as a multiple of n to the right edges, if both of them have a correct distance, then the distance between them would also be a multiple of n, but we assumed otherwise.

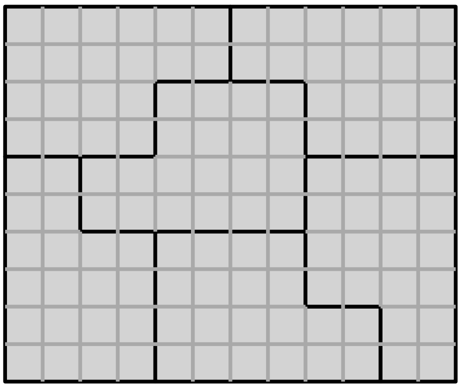
Therefore we can reduce this tiling for the $[nC, nD]$-rectangle with (na, nb)-pieces to a tiling of the $[C, D]$-rectangle with (a, b)-pieces by inversing the multiplication.

Finally we see that the number of tilings of (a, b)-pieces are the same as those of (na, nb)-pieces.

$\square$



(a) A basic tiling for the $[6, 5]$-rectangle with $(1, 2)$-pieces.



(b) The same tiling multiplied by 2, the orientation of every piece stays the same, the starting position and configuration of the pieces is multiplied by 2.

Figure 2.7: A tiling of the $[6, 5]$-rectangle with $(1, 2)$-pieces can be blown up to a tiling of the $[12, 10]$-rectangle with $(2, 4)$-pieces.

3. Decidability

In the following chapter we talk about the decidability of our problems Problem (2.1) and Problem (2.2). We give some basic algorithms to search for tilings and then explore some restrictions that improve the decidability.

3.1 Searching for tilings

Theorem 3.1. *The problem:*

$$\begin{gathered} \textit{Given parameters } a, b, C, D \in \mathbb{N}^+, \\ \textit{does a tiling of the } [C, D]\textit{-rectangle with } (a, b)\textit{-pieces exist?} \end{gathered} \tag{3.1}$$

is decidable.

Proof. With the restrictions from Chapter 2 we can try every combination and output "Yes" if we find any valid tiling and "No" otherwise. The strategy we use for our algorithm is as follows:

We first order every possible configuration for (a, b)-pieces, like in Figure 2.5. Now we choose the first configuration for the top right corner of the $[C, D]$-rectangle. We continue with choosing the first fitting configuration for the next free upper right space. We continue choosing ordered pieces to be placed in free upper right most spaces until we either have a valid tiling or no configuration fits without intersecting another or going out of the rectangle. In this case we exchange the last placed piece with the next fitting one in our ordered set of configurations.

With this method we will hit every tiling of a $[C, D]$-rectangle with (a, b)-pieces if they exist, since every tiling has to have a piece in the upper right corner and we will eventually try every one due to our backtracking. Then every tiling with a piece placed in the upper right corner is either finished or has another upper right most free space in which we try to place every configuration. Using this method we will eventually try out every combination. □

Algorithm 3.1: TILINGFINDER

Input: $a, b \in \mathbb{N}$, possibly pre filled $C \times D$-matrix A

Output: "true" if there exists a tiling of the $[C, D]$-rectangle with (a, b)-pieces, "false" otherwise

```
for i = 0 to 4 · (b + a − 1) do
    currPiece ← NEXTPIECE
    nextFree ← NEXTFREESQUARE(A)
    if FITS(currPiece, nextFree, A) then
        A ← PLACE(A, currPiece, nextFree)
        if ISTILING(A) then
            return true
        TILINGFINDER(a,b,A)
if ISEMPTY(A) then
    return false
```

We will formulate this method in the following algorithm:

In this algorithm we use a lot of helper functions with the following functionality:

- NEXTPIECE just gives out the next Piece in an ordering of all configurations for (a, b)-pieces for fixed a, b.
- NEXTFREESQUARE(A) gives out the next unoccupied space in the partially filled rectangle A, we choose to take the top most and then right most free space. In Figure 3.1 we have an exemplary priority of the $[6, 4]$-rectangle with the smallest uncovered number being chosen.
- FITS(CURRPIECE, NEXTFREE, A) gives out "true" if CURRPIECE fits in A with its top-right most position being NEXTFREE, "false" otherwise.
- PLACE(A,CURRPIECE, NEXTFREE) just places CURRPIECE on NEXTFREE in A. We have tested that this is possible in the if-clause above.
- ISTILING(A) tests if A is already a valid tiling. Our functions prevent pieces from overlapping with others or going out of the rectangle, this function therefore only checks every space if it is being covered.
- TILINGFINDER(a,b,A) is Algorithm 3.1 now with a further filled rectangle A.
- ISEMPTY(A) checks if there are any pieces placed in A. Since we use recursion, this step is an easy way to see if we can go back or if we are the first called function.

The algorithm works the following way:

In every step we have a possibly pre filled $[C, D]$-rectangle A. We pick a piece that we have not yet tried to place in the current state of A and see if it can be placed in the top-right most space.

If it can, we place it and go one recursion step down and try to place a next piece while keeping the original pieces and our just placed piece untouched in A.

6	5	4	3	2	1
12	11	10	9	8	7
18	17	16	15	14	13
24	23	22	21	20	19

Figure 3.1: NEXTFREESQUARE(A) would give out the smallest number that is not covered by a piece. This numbering is for the $[6, 4]$-rectangle

If at any step we fill A and have a valid tiling, we stop and output "true".

If we have tried every piece for a specific state of A and none fit or every piece produces a problem further down the recursion, we need to change the piece placed before, therefore going a step up in the recursion and changing the placed piece there.

Since the rectangle has to be fully covered, we can choose where to place our next pieces freely. If A has a valid tiling, there will be a piece placed in the top-right most space in every partial state of A. Therefore our preference for where to place the next piece is valid.

A very soft upper bound on the runtime is $O(c^n)$ where $c = 4 \times ((a-1)+(a-1)+(b-a+1))$ is the number of configurations and $n = \frac{C \times D}{a^2+b^2}$ is the number of pieces in a tiling of A. A has an area of $C \times D$ and every piece has an area of $a^2 + b^2$ therefore the number of pieces in a fully filled $[C, D]$-rectangle is $\frac{C \times D}{a^2+b^2}$.

3.2 Semi-decidabilty

For some a, b it is not easy to find any C, D such that the $[C, D]$-rectangle can be tiled with (a, b)-pieces.

Theorem 3.2. *The problem*

$$\text{Given parameters } a, b \in \mathbb{N}^+, \text{ are there parameters } C, D \in \mathbb{N}^+ \\ \text{such that there exists a tiling of the } [C, D]\text{-rectangle with } (a, b)\text{-pieces?} \tag{3.2}$$

is semi-decidable.

Proof. We know that we can decide whether a give rectangle can be tiled with (a, b)-pieces due to Theorem 3.1. If we order rectangles by the number of pieces we would have to use to tile it and then test each rectangle if it can be actually tiled, we will find all tilable rectangles eventually since each has a fixed number of pieces used for its tiling.

The ordering of the rectangles can be done as follows:

Given a number of pieces to be used P, we test for every rectangle with side lengths C, D between 1 and $P \cdot (a+b)$ if its area $C \cdot D = P \cdot (a^2 + b^2)$. If it is, we use algorithm Algorithm 3.1. If there is a rectangle which satisfies $C \cdot D = P \cdot (a^2 + b^2)$, with side lengths ≥ 1, one side has at most length $C, D \leq P \cdot (a^2 + b^2)$. If we place P pieces in one line, we can reach at most length $P \cdot (a+b)$, therefore we can lower the bound for each side a bit. $\square$

We will formalize one aspect we just used for later:

Theorem 3.3. *Every $[C, D]$-tiling with (a, b)-pieces satisfies $C \cdot D = p \cdot (a^2 + b^2)$ for some $p \in \mathbb{N}$.*

Proof. Every tiling has to have a whole number of pieces placed in them, let p be that number. Every piece consists of an $a \times a$-square and a $b \times b$-square therefore having an area of $a^2 + b^2$. Since there are no overlaps in the tiling and no piece is going outside of the rectangle, the area of the rectangle is the area of a single piece multiplied by the number of pieces, this gives $C \cdot D = p \cdot (a^2 + b^2)$. $\square$

Continuing with our method, we get the following algorithm:

Algorithm 3.2: FINDING ANY TILINGS

Input: $a, b \in \mathbb{N}$
Output: "true" if there exists a tiling with (a, b)-pieces. Runs infinitely if there is no tiling.

NoTilingFound $\leftarrow$ true
while NoTilingFound **do**
  for $i = 1$ **to** $P \cdot (a+b)$ **do**
    for $j = 1$ **to** $P \cdot (a+b)$ **do**
      if $i \cdot j = P \cdot (a^2 + b^2)$ **then**
        NoTilingFound $\leftarrow$ TILINGFINDER(a,b,MATRIX(i,j))
return NoTilingFound

MATRIX(i,j) produces an unfilled Matrix with i rows and j columns as a rectangle A.

3.3 Limiting the number of pieces

As we have seen, the question

$$\text{Given parameters } a, b \in \mathbb{N}^+, \text{ are there parameters } C, D \in \mathbb{N}^+ \text{ such that there exists a tiling of the } [C, D]\text{-rectangle with } (a, b)\text{-pieces?} \tag{3.3}$$

is only semi-decidable. If we restrict some aspects of the question, we can again make it decidable. The first of these restriction is the number of pieces in the tiling.

Theorem 3.4. *Given parameters $a, b, p \in \mathbb{N}^+$, the question*

$$\textit{Are there parameters } C, D \in \mathbb{N}^+ \textit{ such that there exists a tiling of the } [C, D]\textit{-rectangle with } (a, b)\textit{-pieces, such that there are no more than } p \textit{ pieces?} \tag{3.4}$$

is decidable.

Proof. As we have seen in the proof of Theorem 3.2, there are only a finite number of possible rectangles that can be tiled with a fixed number of pieces. If we place p pieces side by side, we get a maximum length of $p \cdot (a+b)$. Since every rectangle has positive whole numbers side lengths, we have a finite number of possible side lengths between 1 and $p \cdot (a+b)$. With Theorem 3.3 we can also make sure to only include rectangles that use p or less than p pieces.

Finally, given a suitable rectangle, we can decide whether there exists a tiling with (a, b)-pieces with Algorithm 3.1. □

Summarizing this gives us the following algorithm:

Algorithm 3.3: FINDING TILINGS WITH p OR LESS PIECES

Input: $a, b, p \in \mathbb{N}$

Output: "true" if there exists a tiling with at most $p \cdot (a, b)$-pieces. "false otherwise.

for $n = 1$ **to** p **do**
    for $i = 1$ **to** $n \cdot (a + b)$ **do**
        for $j = 1$ **to** $n \cdot (a + b)$ **do**
            if $i \cdot j = n \cdot (a^2 + b^2)$ **then**
                if TILINGFINDER(a,b,MATRIX(i,j)) **then**
                    return "true";
return "false"

A very soft bound on the runtime is $O((p^3a^2 + 2p^3ab + p^3b^2)c^n)$ where p is the number of pieces, and a, b are our parameters for these pieces. The parameters c, n are the same as in Algorithm 3.1. The runtime tries to find tilings for every number of pieces up to p. For each try, it goes through all values of i and j up until $n \cdot (a + b) \leq p \cdot (a + b)$. Ignoring rectangles that do not satisfy Theorem 3.3, each of those values calls Algorithm 3.1. This gives us the final runtime of $O(p \cdot (p \cdot (a+b) \cdot (p \cdot (a+b))) \cdot c^n) = O((p^3a^2 + 2p^3ab + p^3b^2)c^n)$.

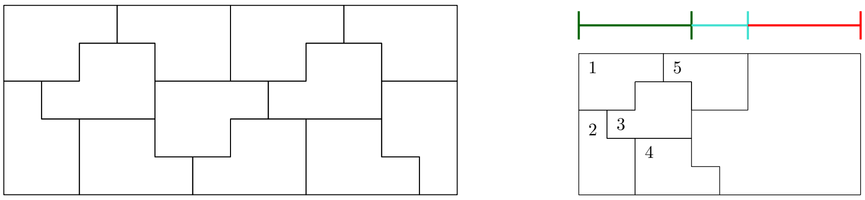
3.4 Limiting the one side length of the rectangle

Another limitation that makes Problem (2.2) decidable is the side length. If we limit just one length X of the rectangle we will prove that we can decide whether there is a rectangle with one side length being equal or less than X that can be tiled with (a, b)-pieces.

Theorem 3.5. *Given parameters $a, b, X \in \mathbb{N}^+$, if there exists a tiling of the $[X, Y]$-rectangle with (a, b)-pieces for some $Y \in \mathbb{N}^+$, then there exists a tiling of the $[X, Z]$-rectangle with (a, b)-pieces for some $Z \leq (2^{X \cdot (a+b)} \cdot (a^2 + b^2))/X$.*

Proof. Suppose there exists a tiling of a rectangle with one side length being X. Without loss of generality suppose X is the length of the left and right side of the rectangle. Then we can start from the left side and reveal pieces such that the piece that fills the left-top-most free space is revealed next. Since there are no free spaces in a valid tiling, this is always possible.

While we are revealing pieces from the left, before and after every step we have all fully tiled lines to the left, some partially filled lines in the middle and then all the unfilled lines to the right. We cannot have a partially filled line before a fully filled line because then we have not chosen the left-most free space when revealing a new piece. Likewise we also cannot have free line before a partially filled line. This method always first fully fills lines before moving on to another line. During this process, some lines get partially filled. An example can be seen in Figure 3.2.



(a) A tiling of the $[12, 5]$-rectangle with $(1, 2)$-pieces.

(b) Starting to reveal pieces like described in Theorem 3.5, results in the following order. The dark-green zone are only fully-filled lines, the turquoise zone are partially filled lines and the red zones are free lines.

Figure 3.2: Pieces can be revealed such that the tiling is only build in a specific moving zone.

The zone with partially filled lines can only be $a + b$ wide and X high. The height stems from the fact, that the tiling is only X high. If we are placing a piece, it can only fill squares up to $a + b$ to the right from its left-most position. Therefore, if our partially filled zone is wider than $a + b$, it means we placed a piece that was not on the far left side of the partially filled zone, therefore not filling the left-top-most free space.

Since this partially filled zone is of finite size, there are a finite number of settings it can have, meaning different states whether some spaces are filled and some are not. Given a partially filled zone, with height X and width w, each square can be filled or not, resulting in $2^{X \cdot w}$ settings.

If at some point during our revealing we are encountering a setting in our partially filled zone, that has been reached before, there are 2 possibilities:

Either the setting is a finished tiling, meaning there are no partially filled lines, only finished ones and empty ones. Then we have found a tiling.

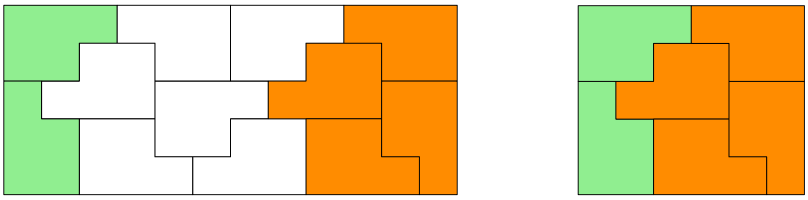
Or it is a previously encountered setting with some lines partially filled, then we can just delete all placed pieces up until the previous seen setting and continue to reveal pieces starting at that point, then we will find a smaller tiling. Since we can finish the revealing to get a tiling, starting at the later repeating setting, we can just use that continuation at the first encounter and finish the tiling without having to repeat a setting. An example of this is shown in Figure 3.3

Since every placed piece changes the current setting and we have a finite number of settings, there is a finite number of pieces that can be placed for a tiling that does not have a repeating setting. A loose bound for the remaining side length is as follows:

Since there are a maximum of $2^{X \cdot w}$ settings and each piece changes this setting, we can have at most $2^{X \cdot w}$ pieces. This gives us a maximum area for a tiling without repeating settings of $2^{X \cdot w} \cdot (a^2 + b^2)$. Since X is fixed, we can find the remaining length Z by dividing by X, since $X \cdot Z =$ area of the rectangle.

The existence of a tilable rectangle longer than this bound implies that this tiling has a repeating setting and can be reduced to a smaller tiling. □

Now that we know that if any tiling exists, we can find it in a finite bounding box, given a side length X, we can just go through every length until that and check up until our maximum length for tilings without repeating settings.



(a) We encounter the same setting after the green and before the orange zone.

(b) We can cut out everything in between and get a smaller tiling.

Figure 3.3: Encountering a setting twice, leads to a smaller tiling.

This yields the following algorithm:

Algorithm 3.4: Finding tilings with p or less pieces

Input: $a, b, X \in \mathbb{N}$

Output: "true" if there exists a tiling of a rectangle with one side length being X with (a, b)-pieces. "false otherwise.

```
for y = 1 to (2^(X·w) · (a^2 + b^2))/X do
    if X · y mod (a^2 + b^2) = 0 then
        if TilingFinder(a,b,Matrix(X,y)) then
            return "true";
return "false"
```

A soft bound on the runtime is $O(((2^{X \cdot w} \cdot (a^2 + b^2))/X \cdot) c^n)$. Parameters c, n are the same as the preceeding algorithms and the factor is the upper bound on the length of tilings with no repeating settings.

4. Tiling the whole plane

In this chapter we want to show that for every $a, b \in \mathbb{N}^+$ you can tile the whole plane with (a, b)-pieces. We will first describe the configuration and placement of the pieces and then prove that every point is covered, and no two pieces overlap.

Theorem 4.1. *Given $a, b \in \mathbb{N}^+$, we can tile the whole plane with (a, b)-pieces.*

Proof. We use the same configuration for each piece: The smaller $a \times a$-square sits on top of the $b \times b$-square such that the left side of squares align.

We place all pieces in a grid, if we describe the position as the coordinates of the left bottom corner of the piece, every position can be described with basis vectors. Choosing one piece to be placed at $\binom{0}{0}$, the left side goes up to $\binom{0}{a+b}$ and then right to $\binom{a}{a+b}$. We go down to $\binom{a}{b}$, in this corner we place the next piece, one basis vector is therefore $\binom{a}{b}$. Our piece goes to the right until $\binom{b}{b}$ and down to $\binom{b}{0}$ and finishes with the origin point.

The second basis vector is $\binom{b}{-a}$. Placing the piece at those coordinates makes the top of the $a \times a$-square aligned to the top of the $b \times b$-square of the upper left piece.

Everything described until now can be seen in Subfigure 4.1a.

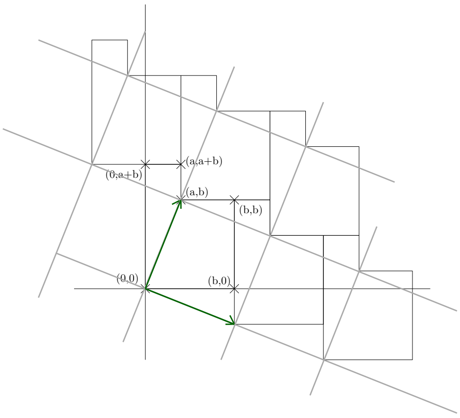
Given this placement of the pieces, we can make statements about the neighbourhood of any piece. To prove that no two pieces overlap, we only need to show that for any piece, it does not overlap with its neighbours. To show that every point is covered, we first see that if any point is not covered, therefore there is a hole, this hole has to touch the side of a piece, if we placed at least one. Therefore, to show that every point is covered, we need to show that for any piece, every point on its edges is covered.

Each piece has a starting position given by a linear combination of the two basis vectors with integer scalars: $\binom{x}{y} = i \cdot \binom{a}{b} + j \cdot \binom{b}{-a}$ with $i, j \in \mathbb{Z}$ and x, y being the starting position of a piece. With this, we can figure out the neighbourhood of any piece, an example for the $(2, 5)$-piece is given in Subfigure 4.1b. The neighbourhood of a piece with starting coordinates $\binom{x}{y}$ is given by the starting positions of the following seven pieces:

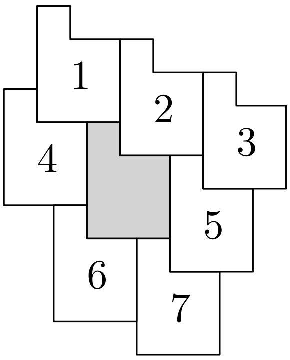
1. $\binom{x}{y} + \binom{a}{b} - \binom{b}{-a}$
2. $\binom{x}{y} + \binom{a}{b}$
3. $\binom{x}{y} + \binom{a}{b} + \binom{b}{-a}$
4. $\binom{x}{y} - \binom{b}{-a}$
5. $\binom{x}{y} + \binom{b}{-a}$
6. $\binom{x}{y} - \binom{a}{b}$
7. $\binom{x}{y} - \binom{a}{b} + \binom{b}{-a}$

We used the same numeration as in Subfigure 4.1b. Now that we know the starting positions of the lower left edge of each piece and remember that we use a configuration such that for a piece with coordinates $\binom{x}{y}$, we start a $b \times b$-square from $\binom{x}{y}$ to $\binom{x+b}{y+b}$ and the $a \times a$-square on top of that from $\binom{x}{y+b}$ to $\binom{x+a}{y+a+b}$, we can compare each square of each neighbour and the original piece if they intersect. We also see that each point on each edge of the original piece is touching another piece, therefore no hole is touching this piece. Since we choose the piece arbitrarily, this holds for any piece and therefore for the whole tiling. Therefore we can tile the whole plane without leaving holes or pieces overlapping.

□



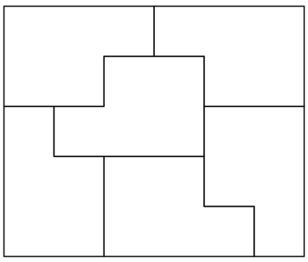
(a) A piece with starting position $\binom{0}{0}$ has the seen coordinates as corners. We can tile the whole plane by placing the same pieces on points on a grid given by the two basis vectors in green, given in Theorem 4.1.



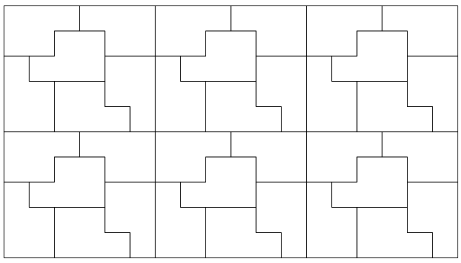
(b) For a given piece in a tiling of the whole plane, its neighbourhood always looks the same, since each piece has the same configuration and all are placed in a grid.

5. Categories of pieces with at least one tiling

As we have seen in Theorem 2.8, every tiling with (a, b)-pieces with co-prime a, b corresponds to exactly one tiling with (na, nb)-pieces with $n \in \mathbb{N}$ and vice versa. Which means if a and b are not co-prime, we divide by the common divisors until they are co-prime and can search for all tilings in the now smaller parameters a' and b'. Furthermore, if we have found a $[C, D]$-rectangle, that can be tiled with (a, b)-pieces, we can also tile the $[nC, mD]$-rectangle for $n, m \in \mathbb{N}$, since we can just put copies of the tiled rectangles in a $n \times m$-matrix like in Figure 5.1.



(a) A simple tiling of the $[6, 5]$-rectangle with $(1, 2)$-pieces



(b) Multiplying the tiling gives us another tiling with the same structure in the different "zones".

Figure 5.1: A tiling of the $[6, 5]$-rectangle with $(1, 2)$-pieces can be multiplied to tilings of the $[n6, m5]$-rectangle, in this case with $n = 3$ and $m = 2$.

With this we get the following corollary:

Corollary 5.1. *A given tiling of the $[C, D]$-rectangle with (a, b)-pieces implies the existence of tilings for any rectangle of the form $[nC, mD]$ and tilings of the $[iC, iD]$-rectangle with (ia, ib)-pieces for any $n, m, i \in \mathbb{N}^+$.*

Proof. We first show the existence of infinitely many tilings with the same (a, b)-pieces:

As seen above, we can put tiled rectangles alongside each other. Since we have one tiled rectangle, we can use the same placement of pieces to tile the same rectangle just next to it. We can place every rectangle in a line in either direction but if we choose to place one above and one right beside the original rectangle, we also have to fill in the top-right created space to make a valid filled rectangle. With this way, given a tiled $[C, D]$-rectangle, we can tile very rectangle of the form $[nC, mD]$ for $n, m \in \mathbb{N}$, therefore infinitely many.

Now we show that we also get tilings for pieces other than the original (a, b)-piece:

We already know that a tiling with (a, b)-pieces implies a tiling with (na, nb)-pieces due to Theorem 2.8. Since $n \in \mathbb{N}$, this gives us infinitely many implied tilings with pieces of the form (na, nb). We can also multiply those like above to get infinitely many tilings for fixed na, nb. □

This shows us that one tiling implies a whole category of tilings, for the rest of the chapter we will show something even stronger:
We show some categories of pieces that are each co-prime and have a tiling that is not made by multiplying existing tilings.

5.1 Pieces of the form (a, a)

The first category of pieces is simple, we look at pieces of the form (a, a) for some $a \in \mathbb{N}$. But we can already see that each piece is a tiling in itself if we align the top and bottom square. This tiles the $[2a, a]$-rectangle with (a, a)-pieces.

5.2 Pieces of the form $(a, a + 1)$

We will now show that there are tilings for every piece of the form $(a, a + 1)$. We first show that a and $a+1$ are co-prime for every $a \in \mathbb{N}$, so we can not reduce this to an easier case:

Theorem 5.2. *For every $a \in \mathbb{N}$, a and $a + 1$ are co-prime.*

Proof. Suppose they have a common divisor $b \neq 1$. Then $a = b \cdot x$ and $a + 1 = b \cdot y$. Subtracting -1 from both sides in the second equation gives $a = b \cdot y - 1$. Inserting the first equation for a gives us $b \cdot x = b \cdot y - 1 \iff b \cdot x - b \cdot y = -1 \iff b \cdot (x - y) = -1 \iff b \cdot (y - x) = 1$ which is only satisfied for $b, (x - y) = 1$ for $a, b, x, y \in \mathbb{N}$. Since $b \neq 1$, this is a contradiction. □

We will now construct a tilable rectangle for every $(a, a + 1)$-piece and therefore prove that there always exists such a rectangle:

Theorem 5.3. *For every $a \in \mathbb{N}^+$, we can tile the $[2a^2 + 2a + 1, a^2 + a]$-rectangle with $(a, a + 1)$-pieces.*

Proof. We prove this by constructing a tiling for the specified rectangle. For this proof, the $a + 1 \times a + 1$-square will be called a $b \times b$-square and the $a \times a$-square keeps its name.

We start by placing a piece in the top-left corner such that the $b \times b$-square is in the corner and the $a \times a$-square is to the right touching the top edge of the rectangle. Now there is a 1 wide gap at the bottom of the $a \times a$-square until it touches the row where the $b \times b$-square ends.

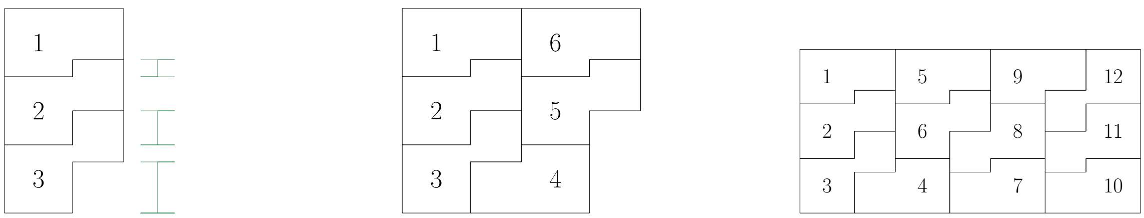
Next we place another piece directly below the first one such that the $b \times b$-square touches the first piece and the left edge of the rectangle fully and the $a \times a$-square touches the first $a \times a$-square. The mentioned gap is now 2 wide.

We place the third piece in the same way: $b \times b$-square touches the preceeding one and the left wall and the $a \times a$-square touches the preceeding one. In every step we increase the gap between the two bottom lines by 1. We can do this until the gap is a wide, then the next piece would have disconnected squares. The current progress is shown in Subfigure 5.2a for $(3, 4)$-pieces.

We fill the $a \times a$-square gap at the bottom with a piece to the right. The $b \times b$-square touches the bottom edge of the rectangle. We fill the rest of this row with pieces like before except the last piece that is already placed in the other direction. With our exemplary $(3, 4)$-piece, we are currently at Subfigure 5.2b.

The gap at the bottom of the $a \times a$-square column is now $2a$ wide. We continue our strategy of filling these gaps with pieces from the next column and then finishing these columns with pieces to the left. In every step we increase the gap to the bottom by a. If our gap is $a \cdot a$ wide, we need the whole next column of pieces to fill it. Doing that finishes the rectangle. This is done for $(3, 4)$-pieces in Subfigure 5.2c.

Since each column has $a \cdot b \times b$ squares, the whole rectangle is $a \cdot b = a \cdot (a+1) = a^2 + a$ high. Starting with all a pieces having their $a \times a$-square going out on the right, each pair of rows reduces this number by 1 until all of them are facing left. This gives us $a + 1$ columns with $b \times b$-squares and between each of them a column with $a \times a$-squares which gives us the final width of $(a+1) \cdot (a+1) + a \cdot a = 2a^2 + 2a + 1$. □



(a) We first build a a^2 high "tower". The mentioned gaps are shown in green.

(b) We fill the gap at the bottom from the other side and build the same tower for the rest of the column.

(c) We continue this strategy until the last column is just the reverse of the first column.

Figure 5.2: Construction of a tiling for $(a, a+1)$-pieces.

5.3 Pieces of the form $(a, a+2)$

We will now prove that there are tilings for for pieces of the form $(a, a+2)$ for every $a \in \mathbb{N}^+$. We first show that half of those can be reduced to the simpler $(b, b+1)$ case in **????**.

Theorem 5.4. *If* $a \bmod 2 = 0$ *then* $a, a+2$ *are not co-prime and tilings for* $(a, a+2)$*-pieces can be reduced to tilings for* $(b, b+1)$*-pieces.*

Proof. If $a \bmod 2 = 0$ then $a + 2 \bmod 2 = 0$. Therefore $a = 2b$ for some $b \in \mathbb{N}^+$. Then $a + 2 = 2b + 2 = 2 \cdot (b + 1)$ which shows that both have the common divisor 2. With Theorem 2.8 we only need to look at tilings for $(b, b+1)$-pieces since $(a, a+2) = (2b, 2\cdot(b+1))$ and we can ignore the common divisor 2. □

Therefore we only need to find tilings for $(a, a + 2)$-pieces with $a \bmod 2 = 1$.

Theorem 5.5. *For every* $a \in \mathbb{N}^+$ *with* $a \bmod 2 = 1$*, we can tile the* $[2a^2 + 6a + 4, 4a^2 + 8a]$*-rectangle with* $(a, a + 2)$*-pieces.*

Proof. We will also prove this by constructing a tiling for the rectangle. We need 3 kinds of building blocks that we will first construct and then combine to form a tiling. Again, we will say $b \times b$-square if we mean $a + 2 \times a + 2$-square.

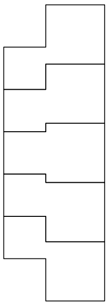
Like in the proof for Theorem 5.3, we will build towers, with alternating columns of $b \times b$-squares and columns of $a \times a$-squares. These towers should finish on the same height, which is the least common multiple of $a + 2$ and a. With Theorem 5.4 we know that they are co-prime, therefore the least common multiple is $a \cdot (a + 2)$.

For the first building block, we start by keeping a $a \times a$-square free and then placing a $b \times b$-square next it with its $a \times a$-square sitting on top of the free square. We continue stacking $b \times b$-squares on the column with the other $b \times b$-squares and its corresponding $a \times a$-squares on the column with $a \times a$-squares until there is again a hole for a $a \times a$-square left. This works because we start by having a $b \times b$-square at the bottom with its $a \times a$-square a away from the bottom. Since $b = a + 2$ this means the two squares are touching in a line of length 2. For the next piece, its $a \times a$-square is now only $a - 2$ away from the baseline of its $b \times b$-square since the underlying $a \times a$-square is 2 deep into its $b \times b$-square. Continuing, in every step, the squares of the piece touch in a length of line 2 more than the previous one. Since $a \bmod 2 = a + 2 \bmod 2 = 1$ and therefore a is not a multiple of 2 and they can touch in a line of maximum length a, there will be a piece with its squares touching in a line of length $a - 1$ and the next piece touching in a line of length a with the smaller square touching the bigger exactly in the middle. Therefore this is symmetrical and we can continue the tower, now with the touching line decreasing until we again have a free $a \times a$-square space at the top. An example for the $(5, 7)$-piece can be seen in Subfigure 5.3a. Before getting to the middle, we go through every second length of the touching line between 2 and $a - 1$ which gives us $\frac{a-1}{2}$ pieces, then 1 for the middle and since the top is the same as the bottom, we again use $\frac{a-1}{2}$ pieces, giving us a height of $a \cdot (a + 2)$ and since we only used one tower we have a width of $2a + 2$.

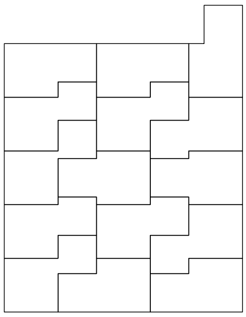
Next we build a block similar to the ones in the tiling for $(a, a + 1)$-pieces. We again start by a piece with both squares touching the top. We continue placing pieces with both squares touching the preceding one until we can not do that any more. In the first piece both squares touch in a line of length a, then since there is a 2 high gap that gets filled by the second piece, its squares only touch in a line of length $a - 2$. Continuing this we can place pieces until they touch in a line of length 1. Then we place a $a \times a$-square on top of the $a \times a$-square of the last placed piece with its connected $b \times b$-square being on the same row as the last placed piece just in the second column. Now we can continue placing pieces onto the last pieces in the first column. We are starting with a 1 high gap and therefore the next piece has a touching line of $a-1$. Since we again decrease this by 2 but are now in an even parity, we can do this until we have a touching line of 2. This last gap will also be filled by a piece from the second column. The rest of the second column gets filled with the same pieces like the first column except for the pieces that help fill the first column. Therefore in the second column, $a-2$ pieces fill the second $a \times a$-column and 2 pieces fill the first $a \times a$-column. For the second $a \times a$-column we not only need to fill the same spaces as in the first $a \times a$-column that cannot be filled by the second $b \times b$-column but also the two additional spaces left by the pieces helping to fill the preceding column. We can fill those again by pieces of the next column. Continuing this structure we reduce the number of piece pointing in the next column by 2 per column. Since we started with a pieces per column and $a \bmod 2 = 1$, we will reach the point where only 1 piece is pointing in the next column, instead we rotate this piece so its $a \times a$-square points upward. Now we have a piece that is almost a rectangle but has a $a \times a$-square standing out at one corner. An example can be seen in Subfigure 5.3b. This rectangle has height $a \cdot (a+2)$ and since each column reduces the number of pieces point to the next one by 2 starting from a and ending at 1, we have $\frac{a+1}{2}$ columns and therefore width $\frac{a+1}{2} \cdot (2a+2)$.

For the third building block we take the second building block but let the outstanding $a \times a$-square point in the next column. Here we fill the rest with the first building block. This leaves us with a block that is almost a rectangle except it has a $a \times a$-hole on either the top or bottom $a+2$ far from one corner. An example can be seen in Subfigure 5.3c. This piece has the same height as the others $a \cdot (a+2)$ and the width is the same as the second building block plus the additional column that we added, therefore $(\frac{a+1}{2}) \cdot (2a+2) + (2a+2)$.

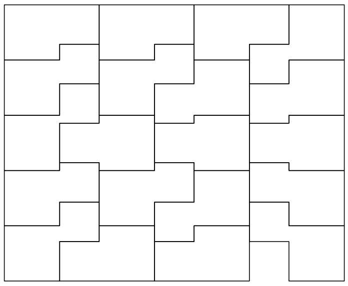
Now we finally have everything to build our tiling. We start with our third building block in the top left corner such that its hole is on the bottom right and place our second building block to the right of it, such that its outstanding $a \times a$-square is on the bottom right of it. This gives us a width of $((\frac{a+1}{2}) \cdot (2a+2)) + (\frac{a+1}{2}) \cdot (2a+2) + (2a+2) = 2a^2 + 6a + 4$. Since both building blocks are the same height we have a common baseline under them with only the hole and the outstanding piece being in the way. Next we place our first block on the right edge such that the outstanding piece fills the hole in the first block. Next we place the second piece to the left of that such that its outstanding piece fills the hole of the top left block. In the remaining space we place the second block with its outstanding piece on the left wall pointing down. The width of two blocks of the second form and one of the first form are the same as above so we are not leaving holes or standing out, the heights are also the same so we have again a common baseline. Now there is a $a \times a$-square standing out on the left edge and a $a \times a$-hole on the right side. Copying what we just made and rotating it by 180° gives us a fitting piece that fills both the hole and the outstanding piece and finishes our tiling to a rectangle. This can be seen for $(5, 7)$-pieces in Figure 5.4. Our final rectangle has the same width as before, being $2a^2 + 6a + 4$ and since we have 4 quasi rows with building blocks, we have height $4 \cdot ((a+2) \cdot a) = 4a^2 + 8a$. $\square$



(a) A simple tower of height $(a+2)\cdot a$ with an $a\times a$-square missing at the top and bottom on one side.



(b) We can build towers with $(a, a+2)$-pieces to an almost rectangle with a $a\times a$-square standing out on any corner. This has height $(a+2)\cdot a$ and width $(\frac{a+1}{2})\cdot(2a+2)$.



(c) Building the previous tower one column farther gives us an almost rectangle with a $b\times b$-square missing in one corner. Since this is just the previous one with one additional column, we have the same height and width $(\frac{a+1}{2})\cdot(2a+2)+(2a+2)$.

Figure 5.3: Building blocks for making an $(a, a+2)$-tiling.

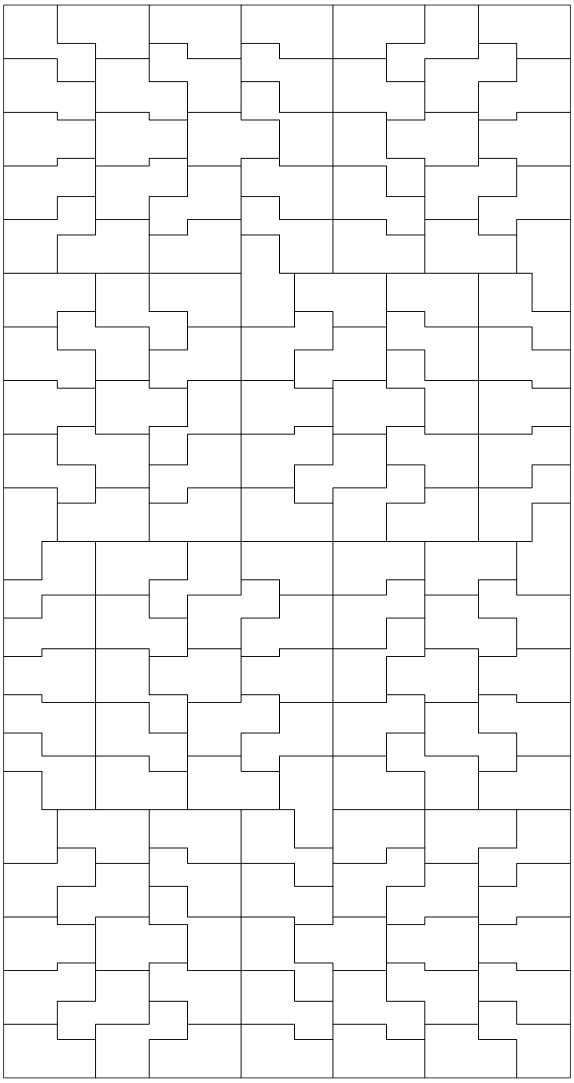


Figure 5.4: A tiling with $(5,7)$-pieces by combining our building blocks.

6. Categories of pieces with no tilings

In the following chapter, we show that (a, b)-pieces where $\frac{a}{b}$ is small, can never tile a rectangle with finite size. We first define a hole, a substructure that can not be tiled under some conditions, and then show that for some (a, b)-pieces we always encounter untilable holes.

6.1 Untilable holes

We first give the formal definition of holes and then show some examples, lastly we prove that holes are untilable under certain conditions.

Definition 6.1. *Let T be a partial placement of (a, b)-pieces in a $[C, D]$-rectangle, called a* subtiling. *A* hole *H is a $[l, w]$-rectangle in T, that satisfies $l < b$ and ($w > a$ or $l \notin \{a, 2a\}$).*

Here are some examples for holes:

The hole in Figure 6.1 satisfies all properties; the red line is the side with length $l = 1 < 4 = b$. The other side has length $w = 3 > 1 = a$, so it satisfies our definition for a hole.

In Figure 6.2 we see a hole with the other property; again, the red line is the side with length $l = 3 < 5 = b$, this time the other side has length $w = 2 = a$ but $l \notin \{2, 4\} = \{a, 2a\}$ for $a = 2$.

Both holes are untilable. We cannot tile the holes while only crossing the red line and the other sides are already confined by other pieces. We prove this property and then show, that tilings with some (a, b)-pieces always encounter these holes with only one side open.

Theorem 6.2. *A hole H with one side s with length l cannot be tiled such that sides different from s are not crossed.*

Proof. We can not tile the H with any $b \times b$-squares, since side s has length $l < b$, every $b \times b$-square would cross at least one side other than s.

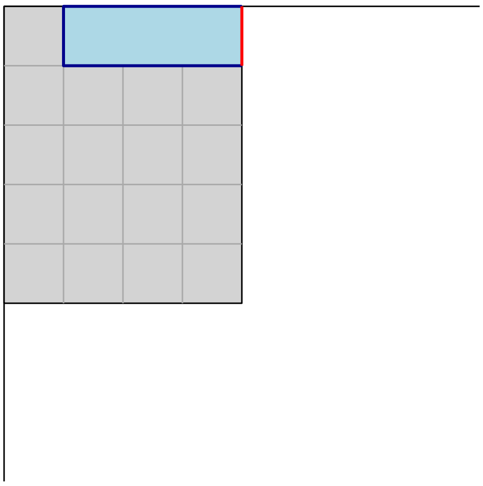


Figure 6.1: A simple hole.
The light blue area is the hole,
the red line is the side with length l

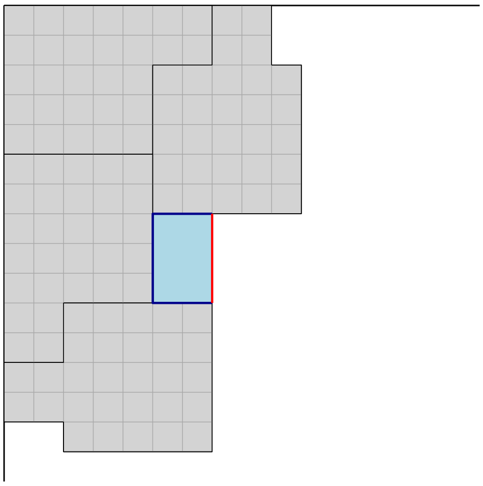


Figure 6.2: Another hole.
The light blue area is the hole,
the red line is the side with length l

Now we look at the two possible cases for holes differently:
Case 1: $w > a$

Since we cannot tile the hole with $b \times b$-squares and every $a \times a$-square has to be connected to a $b \times b$-square, we can place the $a \times a$-squares only a "deep" into the hole, by placing the $b \times b$-square at the side s on the outside and the $a \times a$ connected to it on the inside. Since the hole is deeper than $a < w$ and we can only place pieces crossing side s. We can not tile this hole.

Case 2: $l \notin \{a, 2a\}$

For $l \notin \{a, 2a\} \wedge l < 2a$ its easy to see that we can not find a combination to even tile a line of length l with only $a \times a$-squares.

For $l \notin \{a, 2a\} \wedge l > 2a$ we show that we can place at most two $a \times a$-squares in the hole. Since every $a \times a$-square has to be connected to a $b \times b$-square. We just have to show that we can place at most two $b \times b$-squares touching s.
We can place the first $b \times b$-square so that it touches s in a line of length 1. Every other placement is obviously strictly worse than this. To get the maximum amount of $b \times b$-squares touching s, we need to place them side by side. With the next square placed directly besides the first, we are already over s, since the total length touching s is $1 + b > l$. Even if we place the second piece with some distance to the first piece, the distance never gets great enough to allow one more piece to be placed: Every piece has to touch the line in at least a line of length 1 so the remaining space is: $l - 2 < b - 2 < b$ and we can not fit another piece in there.

Therefore we can only place at most two $a \times a$-squares in H and since $l > 2a$, we can not tile this hole either. □

Corollary 6.3. *Every tiling with (a, b)-pieces, with $\frac{a}{b} < \frac{1}{2}$ can not have $a \times a$-squares in any corner.*

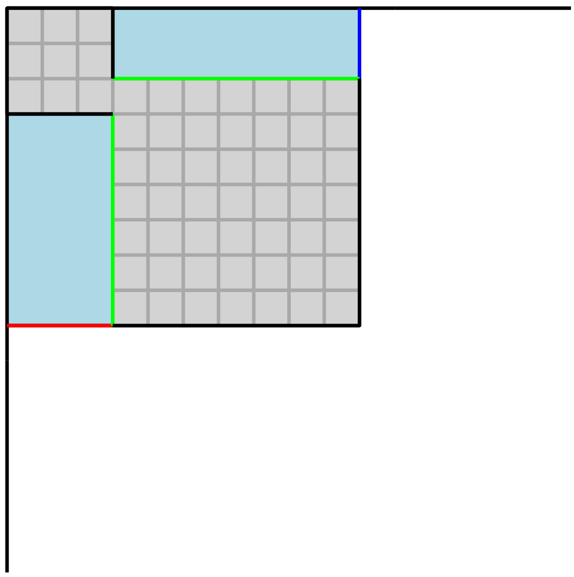


Figure 6.3: Untilable holes on both sides.
the green lines have length $w > a$, the blue line has length $l < a$ and the red line has length $l = a$

Proof. Since every $a \times a$-square has to be connected to a $b \times b$-square, we always get unavoidable untilable holes.

Without loss of generality we assume that we look at the top-left corner and place the $b \times b$-square to the right, like in Figure 6.3.

If we place the $b \times b$-square on the top side, we fill the top hole. The left hole always persists and is untilable.

Since $\frac{a}{b} < \frac{1}{2}$ we know that $2a < b$ and the green side to the left is at least $w \geq b - a > a$. If we place the $b \times b$-square on the top side, the green line is exactly $w = b - a$, every other placement just makes it longer.

The red side has length $l = a$. Since three sides, the green line, the left outer line and the bottom side of the $a \times a$-square, confine the left light-blue rectangle, we can only tile this hole with one side open, the red side s.

Since this rectangle satisfies the definition of a hole, Theorem 6.2 proves that this is untilable, therefore we can not place a $a \times a$-square in the corner. □

Now we have the necessary tools to prove some untilable categories of pieces.

Theorem 6.4. *Given $a, b, C, D \in \mathbb{N}^+$ with $\frac{a}{b} < \frac{1}{3}$, we can not tile the $[C, D]$-rectangle with (a, b)-pieces.*

Proof. Due to Corollary 6.3, we know that we have to place a $b \times b$-square in the top-left corner. Now we need to place its corresponding $a \times a$-square on either side, without loss of generality we place it on the right side.

Now we only need to look at the distance to the top edge: Placing it at any distance other than $0, a$ or $2a$ produces a hole, since the top edge, the right side of the $b \times b$-square and the top side of the $a \times a$-square confine three sides of a rectangle with width a and length l being the distance to the top edge. By Theorem 6.2, this is untilable.

Placing it with distance $2a$ to the top edge, we concentrate on filling the two spaces above the $a \times a$-square first. Since we can only place an $a \times a$-square there, we only have to look at where to place the $b \times b$-square. If the corresponding $b \times b$-square touches the top wall, we have enclosed a free space on all sides without a piece being able to fill it. If the big square does not touch the top wall, we are left with an untilable hole between the first $b \times b$-square, the top edge and the second $b \times b$-square. Therefore we cannot place the first $a \times a$-square with distance $2a$ from the top edge.

Placing it with distance a to the top edge, we must place an $a \times a$-square in the space between the first small square and the top edge. Since the corresponding $b \times b$-square must be connected to the smaller square, we will confine a hole under the first $a \times a$-square with width a and length $b - a - a$ which is too long since $b - 2a > b - \frac{2}{3}b = \frac{1}{3}b > a$. Therefore this can also not work.

The only option left is to place the $a \times a$-square touching the top edge.

Now we can choose to put a $b \times b$ or $a \times a$-square in the corner produced by the $b \times b$-square and the $a \times a$ of the first piece. Choosing the bigger square produces a hole on the top edge: Even if we place the $a \times a$-square of the second piece in this hole, we are still left with depth $b - a - a > a$ which is too long like we have shown earlier. Therefore we have to place an $a \times a$-square in this corner.

Now we need to chose where to place its corresponding $b \times b$-square. Placing it to the bottom of its $a \times a$-square produces a hole at the top edge with $l = 2a$ and $w \geq b - a$ therefore being unfillable.

Placing it at the side produces the same problem as before when we placed the first $a \times a$-square with distance a to the top edge, now with the smaller squares reversed. Since we saw that we can not tile this, we have exhausted all options and have shown that we will encounter problems no matter how we place our pieces. An example for all cases can be seen with $(1, 4)$-pieces in Figure 6.4. $\square$

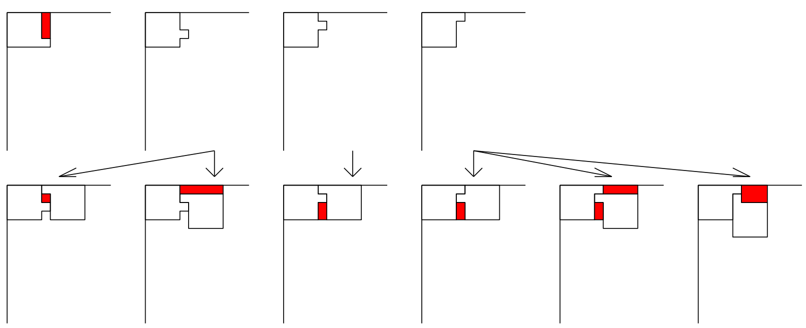


Figure 6.4: An exemplary run through all cases in Theorem 6.4 with $(1, 4)$-pieces. Everything ends in an untilable hole.

7. Fully classified pieces

In this chapter we look at some pieces for which we can answer Problem (2.1) for every $C, D \in \mathbb{N}^+$.

7.1 $(1,1)$-pieces

The first piece is the $(1,1)$-piece. We recall that for every valid tiling Theorem 3.3 has to hold. This means that any rectangle that can be tiled with $(1,1)$-pieces has to have an area as a multiple of $(1^2 + 1^2) = 2$. As it turns out, this is also a sufficient condition.

Theorem 7.1. *Given $C, D \in \mathbb{N}^+$, if $C \cdot D \bmod 2 = 0$ then the $[C, D]$-rectangle can be tiled with $(1,1)$-pieces.*

Proof. Since 2 is prime, if $C \cdot D = 2 \cdot x$ then either $C = 2 \cdot y$ or $D = 2 \cdot z$ or both. Without loss of generality, let C be the side with a side length of the form $C = 2 \cdot y$ for some $y \in \mathbb{N}^+$.

A tiling of the $[C, D]$-rectangle can be constructed as follows:
We place a $y(1,1)$-pieces on the side with length C such that the two squares that make up the piece are both touching the side. Then this line has length $y \cdot (1 + 1) = 2y = C$ and height 1. Now we can just lay D of these lines side by side. This gives us a tiling for a rectangle with length $2y = C$ and height D. An example can be seen in Figure 7.1. $\square$

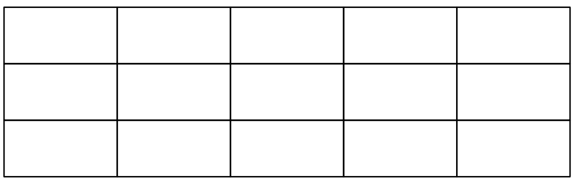


Figure 7.1: A tiling of the $[10, 3]$-rectangle with $(1,1)$-pieces. The top side has an even length

Since each rectangle that does not satisfy Theorem 3.3, can not be tiled and since we showed that each rectangle that does, can be tiled, this answers Problem (2.1) for every rectangle for $(1,1)$-pieces.

7.2 $(1,2)$-pieces

In this section we will answer Problem (2.1) for $(1,2)$-pieces. With Theorem 3.3 we see that the area of every possibly tilable rectangle has to be a multiple of $5 = (1^2 + 2^2)$. Since 5 is again prime, this means that one side has to be a multiple of 5. Without loss of generality let that side be C. We first prove some tools and then finish classifying all rectangles.

Theorem 7.2. *For every $D \in \mathbb{N}^+$, the $[1,D]$-rectangle can not be tiled with $(1,2)$-pieces.*

Proof. Since every placed piece is at least 2 wide, since it contains a 2×2-square, we can not tile a 1 wide rectangle. □

Theorem 7.3. *For every $D \in \mathbb{N}^+$, the $[3,D]$-rectangle can not be tiled with $(1,2)$-pieces.*

Proof. We look at a potentially infinite high rectangle and show that we get untilable holes very early and encounter no valid tilings in between, we start from the top row. We can not place three 1×1-squares in the first row, since then they would not be all connected to 2×2-squares and are not valid pieces. The only other option to fill the first row is therefore to use one 1×1-square and one 2×2-square. If we look at the 2×2-square, there is only one placement that does not form a hole.

If we place the piece with its 1×1-square down, then the two free spaces adjacent to the 2×2-square form a hole. Placing the 1×1-square to the side but not on the top row, leaves a 1×1-square unfilled. The last option is to place the 1×1-square on the side touching the top edge. Then there is only one option to fill row 2: a 1×1-square with its connected 2×2-square downwards forms piece 2. But then we have an untilable hole on the side of piece 2.

These cases can be seen in Figure 7.2.

Therefore we can not tile any rectangle with side length 3 with $(1,2)$-pieces. □

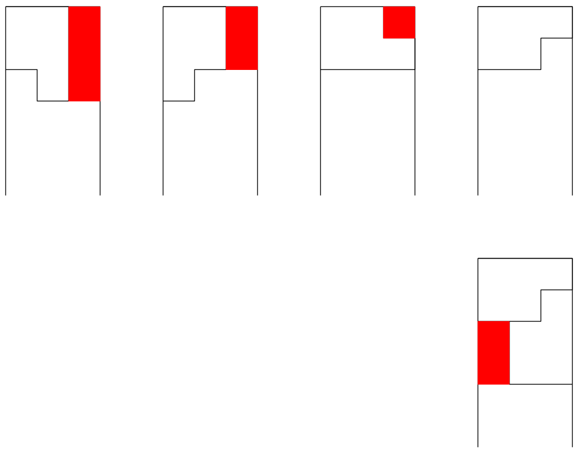


Figure 7.2: There can be no tiling of a $[3,D]$-rectangle for any $D \in \mathbb{N}^+$. The red spaces are holes like in Theorem 6.2.

Next we prove that any tiling with $(1,2)$-pieces where one side has length 5 has an even height. We can construct those by just stacking multiple tilings of the $[2,5]$-rectangle.

Theorem 7.4. *Given $D \in \mathbb{N}^+$, the $[5,D]$,rectangle can only be tiled if D is even.*

Proof. Since the area of the rectangle has to be a multiple of 5 and one side has length 5, the number of pieces in a $[5, D]$-rectangle is D. Each row can have some squares with length 2 and some with length 1 that sum up to 5. There are three possibilities: $5 = 1 \cdot 1 + 2 \cdot 2 = 3 \cdot 1 + 1 \cdot 2 = 5 \cdot 1$. Since we only have D Pieces and therefore $D \cdot (1 \times 1)$-squares and we need to tile D rows, placing more than one 1×1-square in a row results in a row having no 1×1-square. Since we cannot fill a row with 2×2-squares, this means that each row has exactly one 1×1-square.

With this we can make a big case distinction. The different cases and continuations can be seen in Figure 7.3. Each different case can end in the following way: Either it tiles a rectangle with even height, then we must start anew or it makes a hole which we can not fill or it encounters a previous configuration, in this case we can reduce it like in Theorem 3.5. If the part we cut out has even height and any reduced tiling with this setting also has even height then we can only produce tilings with even height even while repeating settings.

In every case we end in one of these possibilities, proving that there can be no tiling with one side being 5 and the other being an odd number. □

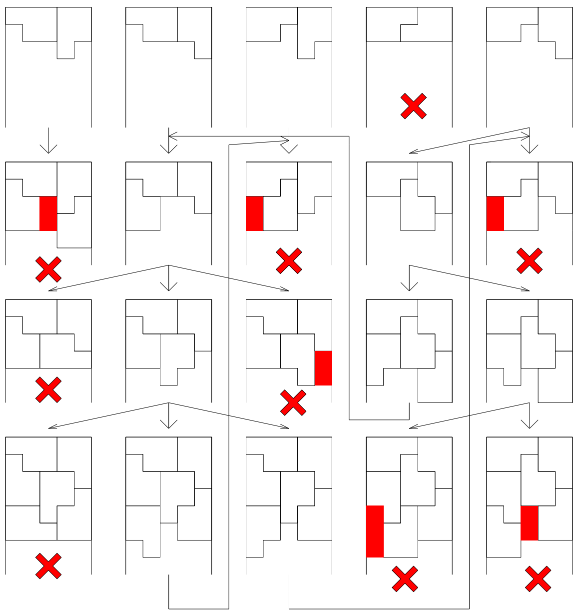
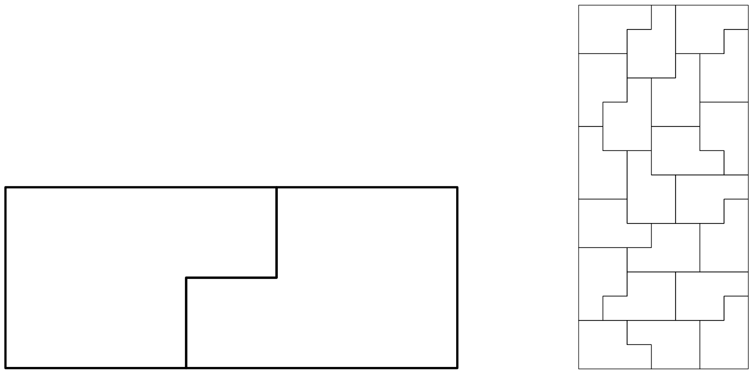


Figure 7.3: Case distinction for tilings with odd height and length 5 with $(1,2)$-pieces. Every case either ends in a rectangle with even height, produces a hole or is a setting previously seen where the reduce-able part has even length.

Now we need two more tilings from which we can build the rest, those being a tiling of the $[2, 5]$-rectangle and the $[7, 15]$-rectangle as shown in Figure 7.4. Now we can finally classify all tilings for $(1,2)$-pieces.

Theorem 7.5. *Given $C, D \in \mathbb{N}^+$, the $[C, D]$-rectangle can be tiled with $(1,2)$-pieces if C, D are of one of the following forms:*

- *$C = 5$ or $D = 5$ and the other is even.*
- *$C \bmod 10 = 0$ or $D \bmod 10 = 0$ and the other is not 1 or 3.*
- *$C \bmod 10 = 5$ or $D \bmod 10 = 5$ and the other is not 1,3 or 5.*



(a) A tiling for the $[2, 5]$-rectangle with $(1, 2)$-pieces. (b) A tiling for the $[7, 15]$-rectangle with $(1, 2)$-pieces.

Figure 7.4: Two tilings from which we can build every other tiling with $(1, 2)$-pieces.

Proof. Without loss of generality, let C mod 10 be the side that is either 0 or 5. If $C = 5$, then this category is just Theorem 7.4. A tiling is given by stacking tilings for the $[2, 5]$-rectangle together.

Rectangles of the second form can not be tiled if one side length is 1 or 3 as seen in Theorem 7.2 and Theorem 7.3. If D is even, we stack tilings of the $[2, 5]$-rectangle on top until we get a height of D. Then, since $C = 0 \bmod 10 \iff C = y \cdot 10 = 2y \cdot 5$ for some $y \in \mathbb{N}^+$. We can put $2y$ of these towers side by side to get a tiling for a rectangle with side lengths $2y \cdot 5 = C$ and D. If D then we first see, that we can tile the $[10, 5]$-rectangle by stacking $[2, 5]$-tilings. We can then grow this tiling by two rows by laying two tilings of the $[2, 5]$-rectangle on top. Therefore we can make any tiling of the form $[10, 5 + 2z]$ for some $z \in \mathbb{N}_0$. Laying multiple of those side by side gives us a tiling for any rectangle of the form $[10x, 5+2z]$ for $x \in \mathbb{N}^+$ and $z \in \mathbb{N}_0$ which are all the rectangles of the second form for odd D.

Rectangles of the third form can not be tiled if one side length is 1 or 3 as seen in Theorem 7.2 and Theorem 7.3. Since C mod $10 = 5 = 10w + 5 = 2(5w + 2) + 1$ for some $w \in \mathbb{N}^+$, C is odd. With Theorem 7.4 we therefore can not tile the $[C, 5]$-rectangle. If D is even, we can again stack $(2w + 1)$ towers of height D consisting of tilings of the $[2, 5]$-rectangle together to get a tiling for the rectangle. If D is odd, we first take the tiling for the $[15, 7]$-rectangle and lay rows of $[5, 2]$-tilings on top until we reach height D. Since we can tile any rectangle of the form $[10, x]$ for $x > 4$, we can lay as many tilings for the $[10, D]$ rectangle to the side of the $[15, D]$-rectangle until we have width C. □

8. Conclusion

In this thesis, we loosely defined (a, b)-pieces and tilings for the $[C, D]$-rectangle and showed that even with this loose definition, we have some pretty strong restrictions.

Using those restrictions we found that Problem (2.1) is decidable and Problem (2.2) at least semi-decidable. Adding some additional constraints made even the second problem decidable. It might be possible that the second problem is already decidable with a strategy similar to the restriction of one side length.

With every combination of (a, b) we can tile the whole plane using only one configuration.

We have found three categories of pieces that each have at least one tiling, the first one being (a, a)-pieces having a simple tiling since they themselves are already rectangles. For the second category we build some towers of pieces and showed that we can continuously switch the side to which pieces are connected until we are finished on the other side having build a rectangle. We used a similar concept with $(a, a + 2)$ to build 3 special building blocks, each being almost a rectangle and together being able to make one big rectangle. We have not yet found a tiling for any piece of the form $(a, a + 3)$. We do know $(1, 4)$ is not possible due to Theorem 6.4. With the algorithm in Chapter 3, we have tried every possible combination of up to 20 pieces for $(4, 7)$-pieces but have not found a tiling. For higher numbers our algorithm is too inefficient.

Next we have proven a structure that hinders a partially filled tiling from being finished. This helped us prove that we can not find a tiling for (a, b)-pieces with $\frac{a}{b} < \frac{1}{3}$. We also conjecture that there aren't any tilings for (a, b)-pieces with $\frac{1}{3} < \frac{a}{b} < \frac{1}{2}$. We know that there aren't any tilings for the $(2, 5)$-piece by searching with Algorithm 3.1 and with a huge case differentiation. We know that there are tilings for $(1, 3)$-pieces since it is of the form $(a, a+2)$.

Finally we used everything we have previously proven to fully classify two pieces, the $(1, 1)$-piece and the $(1, 2)$-piece, answering Problem (2.1) for all rectangles.

Some interesting modifications are elevating this problem to more dimensions, for example, in 3 dimensions, we can use either two or three cubes. Which cuboids can then be filled with those pieces? Another possible change could be the side lengths of the rectangle and squares, what if we want real or even imaginary numbers? Which new patterns emerge and do our previous proofs hold?

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