Graph Coverings: Algorithms, Complexity and Structural Results

Bachelor's Thesis of

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I declare that I have developed and written the enclosed thesis completely by myself. I have not used any other than the aids that I have mentioned. I have marked all parts of the thesis that I have included from referenced literature, either in their original wording or paraphrasing their contents. I have followed the by-laws to implement scientific integrity at KIT.

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(Lucas Schwebler)

Abstract

Covering numbers measure how well the edges of a *host graph* H can be covered with *guest graphs* from a *guest class* $\mathcal{G}$. In this thesis, we study the *global* and *local* covering numbers $c_g^{\mathcal{G}}(H)$ and $c_l^{\mathcal{G}}(H)$ introduced by Knauer and Ueckerdt [15]. The global covering number is the minimum number of guests required to cover H. For the local covering number, we want to minimize in how many guests each vertex is contained.

The main focus of this thesis is on algorithmic complexity. We consider the problems of deciding whether $c_g^{\mathcal{G}}(H) \leq k$ and $c_l^{\mathcal{G}}(H) \leq k$ for given H and k and a fixed guest class $\mathcal{G}$. There are many guest classes for which the global covering problem is known to be NP-hard while the complexity of the local covering problem has been open. To reduce this gap, we show that the local covering problem is NP-hard for many natural guest classes. In particular, this includes the guest classes of bipartite graphs, complete bipartite graphs, planar graphs and outer-planar graphs. The proofs of these results use ideas introduced by Orlin [21] as well as Lee, Liu and Tsai [17], who studied the global variants of these covering problems.

For some guest classes such as the class consisting of all cycles and the single edge K_2, we obtain polynomial time algorithms for the local covering problem. Meanwhile, we show that the corresponding global covering problem is NP-hard. This fits into the commonly observed pattern that local covering problems seem computationally easier than global covering problems. We give a counterexample to this pattern by providing a natural monotone guest class for which the global covering problem is easy while the local covering problem is NP-hard.

Apart from results on algorithmic complexity, we also obtain some structural results. We show that for every guest class $\mathcal{G}$ of fractional chromatic number at most r, the local covering number can be lower bounded by $\log_r(\chi_f(H)) \leq c_l^{\mathcal{G}}(H)$. For the guest class $\mathcal{B}$ of bipartite graphs, this yields an alternative proof of the result $c_l^{\mathcal{B}}(K_n) = \lceil \log_2(n) \rceil$, which was conjectured by Fishburn and Hammer [6] and proved by Hansel [11].

Deutsche Zusammenfassung

Überdeckungszahlen geben an, wie einfach sich die Kanten eines *Gastgebergraphen* H mit *Gästegraphen* einer gegebenen *Gästeklasse* $\mathcal{G}$ überdecken lassen. In dieser Arbeit beschäftigen wir uns mit den *globalen* und *lokalen* Überdeckungszahlen $c_g^{\mathcal{G}}(H)$ und $c_l^{\mathcal{G}}(H)$, welche von Knauer und Ueckerdt [15] eingeführt wurden. Die globale Überdeckungszahl ist die kleinste Anzahl an Gästen, die H überdecken. Bei der lokalen Überdeckungszahl minimieren wir, in wie vielen Gästen jeder Knoten enthalten ist.

In dieser Arbeit beschäftigen wir uns vor allem mit der algorithmischen Komplexität von Überdeckungsproblemen. Wir betrachten das folgende Entscheidungsproblem für eine feste Gästeklasse $\mathcal{G}$, einen gegebenen Gastgebergraphen H und eine gegebene natürliche Zahl k: ist $c_g^{\mathcal{G}}(H) \leq k$ beziehungsweise $c_l^{\mathcal{G}}(H) \leq k$? Für viele Gästeklassen ist bekannt, dass das globale Überdeckungsproblem NP-schwer ist, während die Komplexität des zugehörigen lokalen Überdeckungsproblem noch offen ist. Um diese Lücke zu schließen, zeigen wir, dass das lokale Überdeckungsproblem für viele natürliche Gästeklassen NP-schwer ist. Insbesondere erreichen wir das für die Gästeklassen der bipartiten Graphen, vollständig bipartiten Graphen, planaren Graphen und außenplanaren Graphen. Die Beweise dieser Resultate basieren auf Ideen von Orlin [21] und Lee, Liu and Tsai [17], die die globale Variante dieser Überdeckungsprobleme betrachtet haben.

Für manche Gästeklassen, wie jene bestehend aus allen Kreisen und dem Graphen K_2 auf einer einzelnen Kante, finden wir einen Polynomialzeitalgorithmus für das loka-

le Überdeckungsproblem. Andererseits zeigen wir, dass das korrespondierende globale Überdeckungsproblem NP-schwer ist. Das entspricht dem üblichen Muster, dass lokale Überdeckungsprobleme einfacher wirken als globale Überdeckungsprobleme, was die algorithmische Komplexität angeht. Wir geben ein Gegenbeispiel für dieses Muster, indem wir eine natürliche monotone Gästeklasse finden, für die das globale Überdeckungsproblem einfach ist, während das lokale Überdeckungsproblem NP-schwer ist.

Darüber hinaus erhalten wir auch strukturelle Resultate. Unter anderem zeigen wir für jede Gästeklasse $\mathcal{G}$ mit fractional chromatic number höchstens r die untere Schranke $\log_r(\chi_\mathsf{f}(H)) \leq \mathrm{c}_l^{\mathcal{G}}(H)$ für die lokale Überdeckungszahl. Als Spezialfall erhalten wir für die Gästeklasse $\mathcal{B}$ der bipartiten Graphen einen alternativen Beweis für das Resultat $\mathrm{c}_l^{\mathcal{B}}(K_n) = \lceil \log_2(n) \rceil$, welches von Fishburn und Hammer [6] vermutet und von Hansel [11] bewiesen wurde.

Contents

1 Introduction

The central idea of graph coverings is to decompose a graph into smaller pieces of a given structure. Many problems and results from the field of graph theory can be formulated in this setting. As an example, consider one of the first results in graph theory obtained by Petersen [22] in 1891. Petersen showed that every $2r$-regular graph H can be covered by r graphs $G_1, \ldots, G_r$ such that each G_i is the union of disjoint cycles. Here, we say that H is covered by $G_1, \ldots, G_r$ if every edge of H is contained in some G_i. Generally, we call H the *host* and $G_1, \ldots, G_r$ the *guests*. The class of allowed guests is called the *guest class*.

There are many graph covering problems which are studied in the literature under different names. To investigate such problems in a unified setting, Knauer and Ueckerdt [15] introduced the generalized framework of *global*, *local* and *folded* covering numbers. In this thesis, we focus on the global and local covering numbers. For global covering numbers, we want to minimize the total number of guests used to cover a graph H. In the local setting, we want to minimize in how many guests each vertex is contained.

Some examples of global covering numbers studied in the literature are the *arboricity*, *star arboricity*, *thickness* and *outerthickness*. Here, the guest classes are forests, star forests, planar graphs and outer-planar graphs, respectively. While most covering numbers have only been studied in the global setting, the local variants also appeared for some guest classes. This includes complete bipartite graphs [6] and complete graphs [23]. The study of local coverings with complete graphs was motivated by the study of *intersection graphs*. In an intersection graph, the vertices are sets and two vertices are adjacent if they intersect. For a given graph H, one can ask for the smallest k such that H is an intersection graph of sets with size k. It turns out that this k is the local covering number with complete graphs.

The main focus of this thesis is on algorithmic complexity results. In particular, we are interested in relationships between the complexity of global and local covering problems. An overview of the most relevant results can be found in Table 1.1.

When looking at some natural guest classes, it seems that the local covering problem is computationally easier than its global variant. For the guest class of star forests, the local covering number can be computed in polynomial time [15, Theorem 25] while the global covering number yields an NP-hard problem [10]. There are also examples such as the guest class of complete graphs where the global and local covering problems are NP-hard [21, 23]. However, no natural example of a guest class has been known for which the global covering problem is easy and the local variant is hard. This brings us to one of the main questions considered in this thesis:

▶ **Question 1.1.** *Is there a natural graph class for which the global covering problem can be computed in polynomial time while the local covering problem is NP-hard?*

Stumpf [28, section 6.2] already gave a partial answer by providing examples where the set of hosts is restricted. In Theorem 5.16, we answer the question by providing a natural monotone guest class for which the global covering problem is easier than its local variant.

Another goal of this thesis is to close the gap between the complexity results known for global and local covering problems. As remarked earlier, global covering numbers received more attention in the literature than local covering numbers. Thus, there are many global covering problems which are known to be NP-complete while the complexity of the corresponding local covering problem is unknown. This is the case for many natural guest classes such as bipartite graphs, complete bipartite graphs and planar graphs. We shall prove that these and some other local covering problems are NP-complete. In some rare cases, such as for cycle coverings, we are also able to provide an efficient algorithm. We again refer to Table 1.1 for an overview of the most relevant algorithmic complexity results obtained in this thesis.

	bipartite $\mathcal{B}$	cycles $\mathcal{C}$	$\mathcal{C}^* = \mathcal{C} \cup \{K_2\}$	bounded cycles $\mathcal{C}^*_{\leq m}$ for $m \geq 3$
global	[29, 8]	$\langle 4.3 \rangle$	$\langle 4.3 \rangle$	$\langle 4.11 \rangle$
local	$\langle 3.14 \rangle$	?	$\langle 4.9 \rangle$	$\langle 4.11 \rangle$

	bounded stars $\mathcal{S}_{\leq d}$ for $d \geq 3$	$\{G\}$ for connected r-regular G, $r \geq 2$	$\{K_{1,d}\}$ for $d \geq 3$	at most three edges $\mathcal{E}_3$
global	$\langle 5.3 \rangle$	$\langle 5.3 \rangle$	$\langle 5.3 \rangle$	$\langle 5.16 \rangle$
local	$\langle 5.9 \rangle$	$\langle 5.13 \rangle$	$\langle 5.15 \rangle$	$\langle 5.16 \rangle$

	planar	outer-planar	cactus	treewidth at most t for $t \geq 2$
global	[17]	[17]	[17]	[17]
local	$\langle 6.2 \rangle$	$\langle 6.2 \rangle$	$\langle 6.2 \rangle$	$\langle 6.2 \rangle$

	biclique $\mathcal{CB}$	clique $\mathcal{K}$
global	$\langle 7.10 \rangle$, [21]	$\langle 7.12 \rangle$, [21]
local	$\langle 7.1 \rangle$	$\langle 7.17 \rangle$, [23][1]

(a) in P; NP-hard for k given in the input; NP-hard for some fixed k; Unknown

Table 1.1 Overview of the main complexity results obtained in this thesis. The cells correspond to the global-$\mathcal{G}$-covering and local-$\mathcal{G}$-covering problem where $\mathcal{G}$ depends on the column. In these problems, k is given in the input. If we show NP-hardness for some fixed k, the cell is marked accordingly. Numbers $\langle X \rangle$ refer to the corresponding Theorem X in the thesis.

1.1 Outline

In **Chapter 2** we give some basic definitions and formally introduce the notion of covering numbers. We also give some simple general results on coverings.

In **Chapter 3** we obtain some structural results on local coverings. The main motivation of this chapter is to obtain lower bounds on the local bipartite covering number. Considering what is known as the fractional chromatic number, we provide a general lower bound on the

[1] They show the stronger result that the k-local-$\mathcal{K}$-covering problem is NP-hard for every fixed $k \geq 4$.

local covering number in Theorem 3.6. This bound applies to all guest classes of bounded fractional chromatic number and thus to bipartite graphs. After this, we obtain another structural result in Lemma 3.12 which relates colorings to local bipartite covers. This relationship yields NP-hardness of the local bipartite covering problem.

In **Chapter 4** we study coverings with cycles as guest classes. We show that the global covering problem is NP-hard and provide a polynomial time algorithm for its local variant. We also show that the local covering problem becomes NP-hard as well when we restrict the guest class to cycles of bounded length.

In **Chapter 5** we study covering problems with finite guest classes. This is motivated by the observation from the previous chapter that the local cycle covering problem becomes NP-hard when restricting the length of the cycles. Our main tool for proving NP-hardness in this setting is a general NP-hardness result on graph decomposition obtained by Dor and Tarsi [4]. We use this to prove that several global and local covering problems with finite guest classes are NP-complete. Finally, in Theorem 5.16 we answer Question 1.1 by providing an example of a natural finite guest class for which the global covering problem is easy while the local covering problem is NP-hard.

In **Chapter 6** we consider a recent result by Lee, Liu and Tsai [17] which proves the NP-hardness of global covering problems for many natural guest classes which are closed under taking topological minors. In particular, their result is applicable to planar and outer-planar graphs. We show that with some additional work, their result can also be applied to local covering problems.

In **Chapter 7** our main goal is to prove that the local biclique covering problem is NP-hard. We achieve this by modifying Orlin's NP-hardness proof [21, Theorem 8.1] for the corresponding global covering problem. We provide a framework which can also be used to prove NP-hardness for covering problems with other guest classes. The main idea of this framework is the concept of partial coverings. In a partial covering only some edges of the host graph need to be covered while all other edges are optional. We first show the NP-hardness of the partial covering problem and then we reduce it to the ordinary covering problem.

2 Preliminaries

For a set S and a non-negative integer $k \in \mathbb{N}_0$ we denote the set of all k-element subsets of S by $\binom{S}{k} = \{T \subseteq S \mid |T| = k\}$. For any $n \in \mathbb{N}$ we write $[n] = \{k \in \mathbb{N} \mid k \leq n\}$. Note that in this thesis $0 \notin \mathbb{N}$.

We use standard graph theory notation. All graphs considered in this thesis are finite, simple and undirected. In the following, we recall some important definitions and terminology.

Basic Graph Notation. We denote the vertex set of a graph G by $V(G)$ and its edge set by $E(G)$ and we write $G = (V(G), E(G))$. We denote an edge $e = \{u, v\}$ shortly as uv and we call u and v the *endpoints* of e. Note that $uv = vu$. Two vertices u and v are *adjacent* in G if $uv \in E(G)$ and two edges e_1 and e_2 are adjacent in G if they share an endpoint. An edge e is *incident* to a vertex v if v is an endpoint ot e. We write $H \subseteq G$ if H is a *subgraph* of G, i.e. if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. In this case, we also call G a *supergraph* of H. For a set $S \subseteq V(G)$, we denote the *induced subgraph* of G on S as $G[S] = (S, E(G) \cap \binom{S}{2})$.

For every vertex $v \in G(V)$, we denote its neighborhood as $N_G(v) = \{u \in V \mid uv \in E(G)\}$ and if G is clear from the context, we shortly write $N(v) = N_G(v)$. We denote the degree of v as $\deg_G(v) = |N_G(v)|$ or $\deg(v) = \deg_G(v)$. The *maximum degree* is denoted as $\Delta(G) = \max_{v \in V(G)} \deg_G(v)$ and the minimum degree as $\delta(G) = \min_{v \in V(G)} \deg_G(v)$. A graph is *$r$-regular* if each of its vertices has degree r.

For graphs G and H, a map $\varphi\colon V(G) \to V(H)$ is called a *homomorphism* if $uv \in E(G)$ implies $\varphi(G)\varphi(H) \in E(H)$. We also write $\varphi\colon G \to H$ in this case. If φ is bijective and its inverse map $\varphi^{-1}\colon V(H) \to V(G)$ is a homomorphism as well, we call φ an *isomorphism*, say that G and H are *isomorphic* and write $G \cong H$. If G and H are isomorphic, we also call H a *copy* of G.

Next, we define some special graphs. Let $n, m \in \mathbb{N}$ be positive integers. The *complete graph* on vertex set A is defined as $K_A = (A, \binom{A}{2})$ and we write $K_n = K_{[n]}$. For disjoint vertex sets A, B, the *complete bipartite graph* with parts A and B is $K_{A,B} = (A \cup B, \{ab \mid a \in A, b \in B\})$ and by $K_{n,m}$ we refer to a graph $K_{A,B}$ with $|A| = n, |B| = m$. The *path* P_n on n vertices is defined by $V(P_n) = \{v_1, \ldots, v_n\}$ and $E(P_n) = \{v_i v_{i+1} \mid i \in [n-1]\}$. For $n \geq 3$, the *cycle* C_n on n vertices is defined by $C_n = P_n + v_1 v_n$. Graphs without cycles are called *forests* and connected forests are *trees*.

A *clique* of G is a set of vertices $C \subseteq V(G)$ such that $G[C]$ is a complete graph. An *independent set* I of G is a set of vertices $I \subseteq V(G)$ such that $G[I]$ does not contain an edge. We write $\omega(G)$ for the size of the largest clique and $\alpha(G)$ for the size of the largest independent set of G. Further, we write $\chi(G)$ for the *chromatic number* and $\kappa(G)$ for the *clique-vertex-cover number* of G. We use the word "clique-vertex-cover" instead of the usual name "clique-cover" to stress that the parameter κ is about covering vertices and not about covering edges as in most other situations in this thesis. Recall that $\chi(G)$ ($\kappa(G)$) is the minimum number of parts k in a partition $V(G) = S_1 \cup \cdots \cup S_k$ such that each S_i is an independent set (clique). We shall mostly use another point of view on the chromatic number: A *coloring* c of G is a map $c\colon V(G) \to A$ such that $c(u) \neq c(v)$ for $uv \in E(G)$. The chromatic number $\chi(G)$ is then the minimum size of A in a coloring of G.

Graph Constructions. We say that two graphs G and H are *disjoint* or *vertex-disjoint* if $V(G) \cap V(H) = \emptyset$ and *edge-disjoint* if $E(G) \cap E(H) = \emptyset$. For two graphs G and H, we define their *union* as $V \cup H = (V(G) \cup V(H), E(G) \cup E(H))$. If G and H are vertex-disjoint, we also write $V \uplus H$ to stress this. For a graph G and an integer $k \in \mathbb{N}$, we write $k \cdot G$ for the graph $G_1 \uplus \cdots \uplus G_k$ where each G_i is a copy of G. For disjoint graphs G and H, we denote their *join* as $G \vee H$, that is

$$V(G \vee H) = V(G) \uplus V(H)$$
$$E(G \vee H) = E(G) \cup E(H) \cup \{uv \mid u \in V(G), v \in V(H)\}$$

When we use notation such as $G \vee G$, we implicitly assume that the vertices of one copy of G are renamed so that the two copies are disjoint. We also write $G \vee x$ to denote $G \vee K_1$ where the vertex of K_1 is named x. If a vertex $x \in V(G)$ is adjacent to all other vertices of G, then we also call x a *universal vertex*. For a graph G and a vertex $v \in V(G)$, we write $G - v = G[V(G) \setminus \{v\}]$ for vertex deletion. If $S \subseteq \binom{V(G)}{2}$ is a set of edges, we write $G - S = (V(G), E(G) \setminus S)$ for edge deletion and $G + S = (V(G), E(G) \cup S)$ for edge addition. If $S = \{e\}$, we shortly write $G - e$ and $G + e$. For a graph G, we define its *line graph* $L(G)$ by

$$V(L(G)) = \{L(e) \mid e \in E(G)\}$$
$$E(L(G)) = \{L(e_1)L(e_2) \mid e_1, e_2 \in E(G) \text{ and } e_1 \text{ and } e_2 \text{ are adjacent in } G\}.$$

The *kth power* of a graph G is the graph G^k on the same vertex set such that $uv \in E(G^k)$ if and only if $\mathrm{dist}_G(u, v) \leq k$. Here, the *distance* $\mathrm{dist}_G(u, v)$ is the minimum number of edges on a path in G with endpoints u and v.

For a graph G and two vertices $u, v \in V(G)$, we may *identify* u and v into a single vertex. Formally, this yields a graph H with $V(H) = V(G) \setminus \{u, v\} \uplus \{x\}$ such that

$$E(H) = E(G - u - v) \cup \{xy \mid y \in N_G(u) \cup N_G(v), y \neq x\}$$

We may also identify two edges uv and xy of G by identifying u and x as well as identifying v and y. We do not keep the parallel edge introduced by this so that the resulting graph remains simple. Note that the result of edge identification depends on the orientation with which the edges are identified. For example, the identification of vu with xy can lead to a different result. For two graphs G and H, a 1*-sum* is a graph obtained from the disjoint union $G \uplus H$ by identifying a vertex of G with a vertex of H.

Graph Classes. Let $\mathcal{G}$ be a graph class. We say that $\mathcal{G}$ is *monotone* if it is closed under taking subgraphs. We say that $\mathcal{G}$ is *induced-hereditary* if it is closed under taking induced subgraphs. When ever we talk about a graph class in this thesis, we implicitly assume that $\mathcal{G}$ is closed under isomorphisms. For example, the graph class $\mathcal{K} = \{K_n \mid n \in \mathbb{N}\}$ of complete graphs consists of all graphs G isomorphic to K_n for some $n \in \mathbb{N}$. Some important graph classes in this thesis include the class $\mathcal{CB}$ of all complete bipartite graphs, also called bicliques, the class $\mathcal{B}$ of all bipartite graphs and the class $\mathcal{C} = \{C_m \mid m \geq 3\}$ of cycles.

Next, we introduce some other graph classes which are considered in this thesis. A graph G is *planar* if it can be embedded in the plane $\mathbb{R}^2$ such that no two edges intersect. Further, G is *outer-planar* if it can be embedded in $\mathbb{R}^2$ such that no two edges intersect and such that the embedding has a face containing all vertices. A graph G is called a *cactus* if any two different cycles of G share at most one vertex.

For a graph G, a graph H is a *subdivision* of G if H can be obtained by iteratively applying the following subdivision operation: remove an edge uv from G, insert a new vertex x into G and add the edges ux and xv. We call a graph H a *topological minor* of a graph G if a subdivision of H is a subgraph of G.

A graph G is called a *k-tree* if there is a sequence of graphs $G_0, \dots, G_t$ with $t \in \mathbb{N}_0$ such that $G_0 = K_{k+1}$ and $G_t = G$ and the following holds: for each $i \in [t]$, we have $V(G_i) = V(G_{i-1}) \uplus \{v_i\}$ such that $G_{i-1}[N_{G_i}(v_i)] = K_k$. The *treewidth* $\text{tw}(G)$ of a graph G is the smallest integer $k \in \mathbb{N}$ such that G is a subgraph of a k-tree. Note that forests have treewidth 1.

2.1 Global and Local Covering Number

Let $\mathcal{G}$ be a graph class and H be a graph. A *$\mathcal{G}$-cover* φ of H is a collection of graphs $G_1, \dots, G_t$ such that the following holds: $G_i \in \mathcal{G}$ and $G_i \subseteq H$ for every $i \in [t]$ and for every $e \in E(H)$ there is an $i \in [t]$ with $e \in E(G_i)$. The graphs $G_1, \dots, G_t$ are called the *guests* of φ. For $e \in E(H)$ and $v \in V(H)$ we say that a guest G_i *covers* an edge e if $e \in E(G_i)$ and G_i *hits* vertex v if $v \in V(G_i)$. For a vertex $v \in V(H)$, its *hitcount* $\text{hit}_\varphi(v) = |\{i \in [t] \mid v \in V(G_i)\}|$ is the number of guests hitting v. In this setting, we call $\mathcal{G}$ a *guest class* and H a *host graph*. If the guests $G_1, \dots, G_t$ of φ are edge-disjoint, we call φ a *$\mathcal{G}$-decomposition*.

A $\mathcal{G}$-cover φ with t guests $G_1, \dots, G_t$ is called *t-global*. The *global $\mathcal{G}$-covering number* of H is denoted by $\text{c}_g^{\mathcal{G}}(H)$ and defined as the smallest t such that H has a t-global $\mathcal{G}$-cover. A $\mathcal{G}$-cover φ is called *k-local* if every vertex $v \in V(H)$ is hit at most k times, i.e. $\text{hit}_\varphi(v) \leq k$. The *local $\mathcal{G}$-covering number* of H is denoted by $\text{c}_l^{\mathcal{G}}(H)$ and defined as the smallest k such that H has a k-local $\mathcal{G}$-cover. Note that these definitions of the global and local covering number require that there is at least one $\mathcal{G}$-cover of H. If H has no $\mathcal{G}$-cover we set $\text{c}_g^{\mathcal{G}}(H) = \text{c}_l^{\mathcal{G}}(H) = \infty$.

▶ **Example 2.1.** Consider the example in Figure 2.1. The figure shows a $\mathcal{K}$-cover φ of a graph H where $\mathcal{K} = \{K_n \mid n \in \mathbb{N}\}$ is the guest class of complete graphs. Every guest of φ is a copy of K_3. Since φ consists of three guests, φ is 3-global. Therefore $\text{c}_g^{\mathcal{K}}(H) \leq 3$. It turns out that H has no 2-global $\mathcal{K}$-cover and thus $\text{c}_g^{\mathcal{K}}(H) = 3$. For each vertex $v \in V(H)$, its hitcount $\text{hit}_\varphi(v)$ is written next to. Since $\text{hit}_\varphi(v) \leq 2$ for each $v \in V(H)$, we obtain that φ is 2-local. It turns out that H has no 1-local $\mathcal{K}$-cover and thus $\text{c}_l^{\mathcal{K}}(H) = 2$. Finally, note that one edge is covered twice and thus φ is not a decomposition.

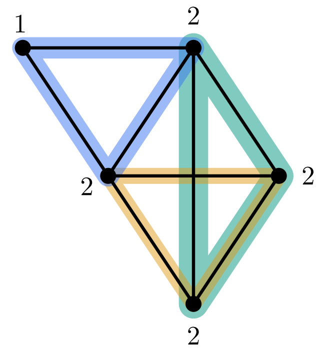


Figure 2.1 Example of a $\mathcal{K}$-cover φ of a graph H.

We are interested in the following algorithmic problems related to covering numbers:

▶ **Problem 2.2** (global-$\mathcal{G}$-covering)**.** We consider $\mathcal{G}$ to be a fixed guest class.
- **Input:** A host graph H and an integer $k \in \mathbb{N}$.
- **Question:** Does H admit a k-global $\mathcal{G}$-cover?

▶ **Problem 2.3** (local-$\mathcal{G}$-covering)**.** We consider $\mathcal{G}$ to be a fixed guest class.
- **Input:** A host graph H and an integer $k \in \mathbb{N}$.
- **Question:** Does H admit a k-local $\mathcal{G}$-cover?

It is also natural to consider versions of these problems were k is fixed instead of being given in the input. We call the corresponding problems the k-global-$\mathcal{G}$-covering and k-local-$\mathcal{G}$-covering problem. Note that NP-hardness of the k-global-$\mathcal{G}$-covering problem for some fixed $k \in \mathbb{N}$ implies NP-hardness of the global-$\mathcal{G}$-covering problem. Similarly, NP-hardness of the k-local-$\mathcal{G}$-covering problem implies NP-hardness of the local-$\mathcal{G}$-covering problem. Therefore, NP-hardness results for covering problems with fixed k can be considered to be stronger.

For a $\mathcal{G}$-cover φ of H and a set $S \subseteq V(H)$ of vertices, we can consider its *restriction* $\varphi|_S$ to S. We obtain $\varphi|_S$ from φ by replacing each guest G of φ with its induced subgraph $G[S]$. If the new guest $G[S]$ does not contain an edge, then we do not add it to $\varphi|_S$. Note that we want $\varphi|_S$ to to be a $\mathcal{G}$-cover as well, but this is only true if $G[S] \in \mathcal{G}$ for all guests G of φ. This property is satisfied for induced-hereditary guest classes $\mathcal{G}$. Thus, the restriction of a t-global k-local $\mathcal{G}$-cover φ of H yields a t-global k-local $\mathcal{G}$-cover $\varphi|_S$ of $H[S]$ if $\mathcal{G}$ is induced-hereditary.

Next, we show some basic general results on covering numbers. First of all, it is easy to see that the local covering number can always be bounded by the global covering number.

▶ **Lemma 2.4** ([15, Proposition 4])**.** *Let $\mathcal{G}$ be a guest class and let H be a host graph. Every k-global-$\mathcal{G}$-cover of H is also k-local. In particular $c_l^{\mathcal{G}}(H) \leq c_g^{\mathcal{G}}(H)$.*

We want to show that many covering problems are in NP. To achieve this, we first need to show that there is a small witness.

▶ **Lemma 2.5.** *Let $\mathcal{G}$ be a guest class and let H be a host graph. If H admits a $\mathcal{G}$-cover, then $c_l^{\mathcal{G}}(H) \leq c_g^{\mathcal{G}}(H) \leq |E(H)|$.*

Proof. Let φ be a $\mathcal{G}$-cover of H with guests $G_1, \dots, G_t$ and minimal t. For every $i \in [t]$, the guest G_i covers some edge which is not covered by any previous G_j with $j < i$. Otherwise, G_i could be removed from φ resulting in a smaller cover. Since H only has $|E(H)|$ edges, this implies $t \leq |E(H)|$. ◀

▶ **Lemma 2.6.** *Let $\mathcal{G}$ be a graph class such that the recognition problem for $\mathcal{G}$ is in NP. Then, the global-$\mathcal{G}$-covering and local-$\mathcal{G}$-covering problem are in NP.*

Proof. The used witness for $c_g^{\mathcal{G}}(H) \leq k$ or $c_l^{\mathcal{G}}(H) \leq k$ consists of a $\mathcal{G}$-cover $G_1, \dots, G_t$ of H as well as the recognition witnesses for $G_i \in \mathcal{G}$. By Lemma 2.5, there is a witness with $t \leq |E(H)|$ and thus the witness has polynomial size. Also, it can be verified in polynomial time whether the graphs $G_1, \dots, G_t$ form a $\mathcal{G}$-cover as well as whether the cover is k-global or k-local. ◀

3 Structural Results on Coverings and Colorings

In this chapter, we study relationships between coverings and colorings. For global covering numbers, the following lower bound based on the chromatic number is known:

▶ **Theorem 3.1** ([29, Proposition 4.10]). *Let $r \in \mathbb{N}_{\geq 2}$ and let $\mathcal{G}$ be a guest class with $\chi(G) \leq r$ for all $G \in \mathcal{G}$. Then, for every host graph H it holds that $\log_r(\chi(H)) \leq \mathrm{c}_g^{\mathcal{G}}(H)$.*

If the guest class consists of all graphs of chromatic number at most r, the corresponding upper bound also holds:

▶ **Theorem 3.2** ([29, Proposition 4.11]). *Let $r \in \mathbb{N}_{\geq 2}$ and let $\mathcal{G}_r = \{G \mid \chi(G) \leq r\}$ be the guest class of all graphs of chromatic number at most r. Then, for every host graph H it holds that $\mathrm{c}_g^{\mathcal{G}_r}(H) = \lceil \log_r(\chi(H)) \rceil$.*

Now, consider the guest class $\mathcal{B}$ of all bipartite graphs. Since it is NP-hard to decide whether $\chi(H) \leq t$ for a given graph H and every fixed $t \geq 3$ [8], it is easy to see that Theorem 3.2 implies that the k-global-$\mathcal{B}$-covering problem is NP-hard for every $k \geq 2$.

An important special case of Theorem 3.2 is $\mathrm{c}_g^{\mathcal{B}}(K_n) = \lceil \log_2(n) \rceil$. Fishburn and Hammer [6] conjectured that $\mathrm{c}_l^{\mathcal{B}}(K_n) = \lceil \log_2(n) \rceil$ as well. This was proved by Hansel [11] using the probabilistic method as presented in [13, Lemma 3.7]. Additionally, a very explicit proof of this result is given by Dong and Liu [3, Theorem 2.1], which characterizes the structure of optimal local $\mathcal{B}$-covers of K_n. In Corollary 3.8, we give an alternative proof of $\mathrm{c}_l^{\mathcal{B}}(K_n) = \lceil \log_2(n) \rceil$ by proving a far more general result involving a graph parameter called the fractional chromatic number.

3.1 The fractional chromatic number

In this section, we give a similar lower bound as in Theorem 3.1 for local coverings. We first need to introduce a graph parameter called the *fractional chromatic number*. Our introduction of the fractional chromatic number follows some parts of [26, Chapter 3].

A *b-fold coloring* of a graph H assigns to each vertex of H a set of b colors such that adjacent vertices receive disjoint sets of colors. More formally, a b-cold coloring of H on color set A is a map $c\colon V(H) \to \binom{A}{b}$ such that $c(u) \cap c(v) = \emptyset$ for $uv \in E(H)$. If the number of colors used is at most a, i.e. if $|A| \leq a$, we call c an *a:b-coloring* of H and we say that H is *a:b-colorable*. See Figure 3.1 for an example. The *b-fold chromatic number* of H, denoted $\chi_b(H)$, is the smallest a such that H is a:b-colorable. Note that $\chi_1(H) = \chi(H)$. We define the *fractional chromatic number* as $\chi_{\mathsf{f}}(H) = \inf_{b \in \mathbb{N}} \frac{\chi_b(H)}{b}$. From the definition, it is clear that $\chi_{\mathsf{f}}(H) \leq \chi_1(H) = \chi(H)$ and it is easy to see that χ_{f} is a monotone graph parameter. Moreover, the following can be proved:

▶ **Lemma 3.3** ([26]). *For every graph H, there are integers a and b such that H is a:b-colorable and $\chi_{\mathsf{f}}(H) = \frac{a}{b}$. In particular, $\chi_{\mathsf{f}}(H)$ is a rational number.*

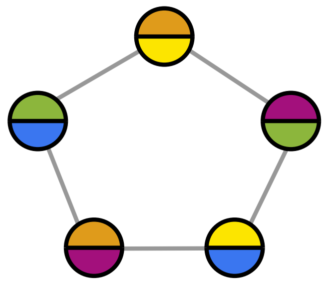


Figure 3.1 A 5:2-coloring of C_5.

To obtain the upper bound $\frac{a}{b}$ on the fractional chromatic number, it suffices to provide an a:b-coloring. For example, the 5:2-coloring of the cycle C_5 illustrated in Figure 3.1 shows that $\chi_\mathsf{f}(C_5) \leq \frac{5}{2}$. To obtain lower bounds on the fractional chromatic number, the following general result is helpful:

▶ **Lemma 3.4** ([26]). *For every graph H, it holds that $\frac{|V(H)|}{\alpha(H)} \leq \chi_\mathsf{f}(H)$.*

Proof. Consider an a:b-coloring $c\colon V(H) \to \binom{[a]}{b}$ of H. For a color $k \in [a]$, denote by $A(k) = \{v \in V(H) \mid k \in c(v)\}$ its color class, i.e. the set of vertices having color k. Since adjacent vertices have disjoint color sets, $A(k)$ is an independent set and thus $|A(k)| \leq \alpha(H)$. Note that $b \cdot |V(H)| = \sum_{k\in[a]} |A(k)| \leq a \cdot \alpha(H)$. Rearranging gives us $\frac{|V(H)|}{\alpha(H)} \leq \frac{a}{b}$. By the definition of $\chi_\mathsf{f}(H)$ as the infimum of $\frac{a}{b}$ over all a:b-colorings of H it follows that $\frac{|V(H)|}{\alpha(H)} \leq \chi_\mathsf{f}(H)$. ◀

Since $\alpha(C_5) = 2$, it follows that $\chi_\mathsf{f}(C_5) \geq \frac{5}{2}$ and combined with the previously obtained upper bound we get $\chi_\mathsf{f}(C_5) = \frac{5}{2}$. We can also use Lemma 3.4 to make another general observation:

▶ **Corollary 3.5** ([26]). *For every graph H, it holds that $\omega(H) \leq \chi_\mathsf{f}(H)$. In particular, $\chi_\mathsf{f}(K_n) = n$ for all $n \in \mathbb{N}_{\geq 1}$.*

Proof. By Lemma 3.4 and $\alpha(K_n) = 1$ we obtain $n \leq \chi_\mathsf{f}(K_n)$. Since χ_f is monotone, this gives us $\omega(H) \leq \chi_\mathsf{f}(H)$ for every graph H. Finally, $\chi_\mathsf{f}(K_n) \leq \chi(K_n) = n$ and thus $\chi_\mathsf{f}(K_n) = n$. ◀

Now, we can state a lower bound of local covering numbers similar to Theorem 3.1 based on the fractional chromatic number:

▶ **Theorem 3.6.** *Let $r \in \mathbb{R}_{>1}$ and let $\mathcal{G}$ be a guest class with $\chi_\mathsf{f}(G) \leq r$ for all $G \in \mathcal{G}$. For every host graph H it holds that $\log_r(\chi_\mathsf{f}(H)) \leq \mathrm{c}_l^{\mathcal{G}}(H)$.*

The idea of the proof is to construct an a:b-coloring of H with $\frac{a}{b} \leq r^k$ where $k = \mathrm{c}_l^{\mathcal{G}}(H)$. By definition, an a:b-coloring of H must assign exactly b colors to each vertex. However, in the construction presented in the proof, it is more natural to assign at least b colors to every vertex, some vertices receiving more colors than others. To refer to such colorings, we use the term of a *fractional coloring*. More precisely, a fractional coloring of H using colors A is a map $c\colon V(H) \to 2^A$ such that $c(u) \cap c(v) = \emptyset$ for $uv \in E(H)$. If $|c(v)| \geq b$ for all $v \in V(H)$, it is clear that a b-fold coloring can be obtained from c by replacing each $c(v)$ by a subset of size b.

Proof of Theorem 3.6. Let $k = c_l^{\mathcal{G}}(H)$ and let φ be a k-local $\mathcal{G}$-cover of H with guests $G_1, \dots, G_t$. By Lemma 3.3 there is an a_i:b_i-coloring $c_i \colon V(G_i) \to \binom{[a_i]}{b_i}$ of G_i with $\frac{a_i}{b_i} \leq r$ for all $i \in [t]$. Now, we construct a fractional coloring $\hat{c}$ of H: The set of colors used by $\hat{c}$ is $A = [a_1] \times \cdots \times [a_t]$. For a vertex $v \in V(H)$ and a color $x = (x_1, \dots, x_t) \in A$, we have $x \in \hat{c}(v)$ if and only if for every $i \in [t]$ either $v \notin V(G_i)$ or $x_i \in c_i(v)$.

We show that $\hat{c}$ is indeed a fractional coloring of H: For an edge $uv \in E(H)$, consider an arbitrary color $x \in A$. There must be some $i \in [t]$ such that uv is covered by guest G_i. This implies that $u, v \in V(G_i)$ and $x_i \notin c_i(u) \cap c_i(v) = \emptyset$. Thus, $x \notin \hat{c}(u) \cap \hat{c}(v)$. Since x was chosen arbitrarily, we get that $\hat{c}(u) \cap \hat{c}(v) = \emptyset$ and $\hat{c}$ is indeed a fractional coloring of H.

Clearly, $|A| = \prod_{i=1}^{t} a_i$. For $v \in V(H)$, consider the set $I_v = \{i \in [t] \mid v \in V(G_i)\}$. Note that by construction, we have that $|\hat{c}(v)| = \prod_{i \in [t] \setminus I_v} a_i \prod_{i \in I_v} b_i$. Since the cover φ is k-local and $\frac{a_i}{b_i} \leq r$ for every $i \in [t]$, we obtain

$$\frac{|A|}{|\hat{c}(v)|} = \prod_{i \in I_v} \frac{a_i}{b_i} \leq r^{|I_v|} \leq r^k.$$

Thus, the fractional coloring $\hat{c}$ certifies $\chi_{\mathsf{f}}(H) \leq r^k$ which implies $\log_r(\chi_{\mathsf{f}}(H)) \leq k = c_l^{\mathcal{G}}(H)$. ◀

As a special case for the guest class $\mathcal{B}$ of bipartite graphs, we obtain

► **Corollary 3.7.** *For every host graph H, it holds that $\lceil \log_2(\chi_{\mathsf{f}}(H)) \rceil \leq c_l^{\mathcal{B}}(H)$.*

Proof. Every bipartite graph G satisfies $\chi_{\mathsf{f}}(G) \leq \chi(G) \leq 2$ and the result follows from Theorem 3.6. ◀

A natural question is whether this lower bound is tight, i.e. whether $c_l^{\mathcal{B}}(H) = \lceil \log_2(\chi_{\mathsf{f}}(H)) \rceil$. It turns out that this is not the case. After developing some more theory, we shall give a counterexample in Corollary 3.13.

As another important corollary, we also obtain an alternative proof of the following result conjectured in [6] and proven in [11] and [3].

► **Corollary 3.8.** *For the guest class $\mathcal{B}$ of all bipartite graphs, we have that $c_l^{\mathcal{B}}(K_n) = \lceil \log_2(n) \rceil$.*

Proof. By Corollary 3.5, we have $\chi_{\mathsf{f}}(K_n) = n$, and thus Corollary 3.7 give us the lower bound $\lceil \log_2(n) \rceil \leq c_l^{\mathcal{B}}(K_n)$. The upper bound is provided by $c_l^{\mathcal{B}}(K_n) \leq c_g^{\mathcal{B}}(K_n) = \lceil \log_2(n) \rceil$ (see Theorem 3.2). ◀

From Corollary 3.8 it is clear that the local $\mathcal{B}$-covering number can be arbitrarily large by considering complete graphs as hosts. Our next goal is to show that the local $\mathcal{B}$-covering number can be arbitrarily large even when we only consider triangle-free graphs as hosts. More generally, in Corollary 3.11 we show that for guest classes $\mathcal{G}$ with bounded fractional chromatic number, the local $\mathcal{G}$-covering number of a triangle-free graph can be arbitrarily large. We use Mycielski's construction [20] originally introduced to construct triangle free graphs with arbitrarily large chromatic number. This construction transforms a graph G into a graph $\mathrm{Y}(G)$. The details of the construction are not important for us here and we only need some of its properties. A detailed description of the construction can be found in [26, Section 3.3].

▶ **Lemma 3.9** ([26, Theorem 3.3.3 and 3.3.4]). *Let G be a graph with at least one edge and let $\Upsilon(G)$ denote the graph obtained by applying Mycielski's construction to it. The following three properties hold:*
1. $\chi(\Upsilon(G)) = \chi(G) + 1$
2. $\omega(\Upsilon(G)) = \omega(G)$
3. $\chi_f(\Upsilon(G)) = \chi_f(G) + \frac{1}{\chi_f(G)}$

A simple calculation gives us a lower bound for the fractional chromatic number of the iterated Mycielski construction:

▶ **Corollary 3.10.** *For the sequence of graphs $(H_i)_{i\in\mathbb{N}}$ with $H_1 = K_2$ and $H_{i+1} = \Upsilon(H_i)$, we have for every $k \in \mathbb{N}$ that $\chi_f(H_k) \geq \sqrt{k}$ and H_k is triangle-free.*

Proof. Let $a_k = \chi_f(H_k)$. We show by induction that $a_k \geq \sqrt{k}$. For $k = 1$, we have $2 = \chi_f(K_2) = a_1 \geq \sqrt{1} = 1$. Now, assume that $a_k \geq \sqrt{k}$ for some $k \in \mathbb{N}$. An application of Lemma 3.9 together with the fact that the function $x \mapsto x + \frac{1}{x}$ is increasing on $[1, \infty)$ yields:

$$a_{k+1} = a_k + \frac{1}{a_k} \geq \sqrt{k} + \frac{1}{\sqrt{k}} = \sqrt{\left(\sqrt{k} + \frac{1}{\sqrt{k}}\right)^2} = \sqrt{k + 2 + \frac{1}{k}} > \sqrt{k+1}$$

By the second property of Lemma 3.9 and $\omega(K_2) = 2$ we obtain by induction that $\omega(H_k) = 2$ for every $k \in \mathbb{N}$. Thus, H_k is indeed triangle-free. ◀

▶ **Corollary 3.11.** *Let $\mathcal{G}$ be a guest class of bounded fractional chromatic number. For every $k \in \mathbb{N}$, there is a triangle free host graph H such that $c_l^{\mathcal{G}}(H) \geq k$.*

Proof. Let $\chi_f(\mathcal{G}) \leq r$. By Corollary 3.10, there exists a sequence $(H_i)_{i\in\mathbb{N}}$ of triangle-free graphs with unbounded fractional chromatic number. Thus, there exists an index j such that $\chi_f(H_j) \geq r^k$. By Theorem 3.6, we obtain $c_l^{\mathcal{G}}(H_j) \geq \log_r(r^k) = k$. ◀

3.2 Bipartite Coverings and Joins

In this section, we focus on the guest class $\mathcal{B}$ of all bipartite graphs. We show that the 2-local-$\mathcal{B}$-covering problem is NP-complete and we also show that the lower bound of Corollary 3.7 is not tight. All of this is based on the following lemma.

▶ **Lemma 3.12.** *Let H be a graph and $H' = H \vee x$ be the graph obtained from H by adding a universal vertex x connecting it to all vertices of H. In this setting, we have $c_l^{\mathcal{B}}(H') \leq 2$ if and only if $\chi(H) \leq 3$.*

Proof. We start with the easy direction: If $\chi(H) \leq 3$, then $\chi(H') \leq 4$ and thus we get $c_l^{\mathcal{B}}(H') \leq c_g^{\mathcal{B}}(H') = \lceil \log(\chi(H')) \rceil \leq 2$ by Theorem 3.2.

For the other direction, suppose that H' has 2-local $\mathcal{B}$-cover φ. We need to show that H has a 3-coloring. We may assume that the universal vertex x is hit by exactly two bipartite guests G_1 and G_2. Each vertex of H' is hit by at least one of G_1 and G_2. Thus, the graph $H' - E(G_1) - E(G_2)$ has a 1-local $\mathcal{B}$-cover and thus it is bipartite. It follows that the edges of $H' - E(G_1) - E(G_2)$ can be covered by a single guest G_3. Without loss of generality, we may thus assume that φ consists of exactly three guests G_1, G_2 and G_3 and that each vertex of H' is hit by exactly two guests.

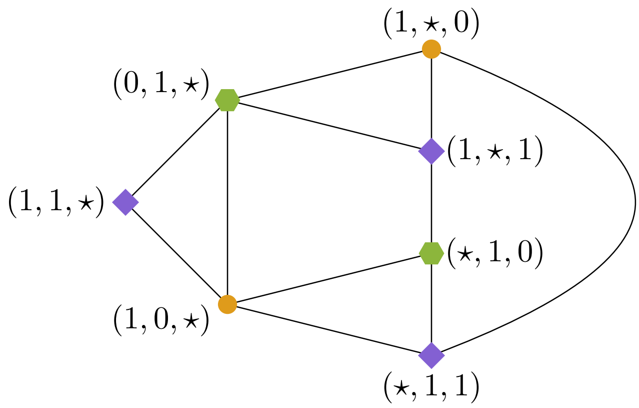


Figure 3.2 3-coloring of the graph L used in the proof of Lemma 3.12.

For each $i \in [3]$, let A_i and B_i denote the parts of G_i such that $x \in A_1$ and $x \in A_2$ (if the parts are not unique, choose them arbitrarily). To every vertex of H', we assign a label based on the membership in the parts of the guests: let $p(v) = (p_1(v), p_2(v), p_3(v))$ where

$$p_i(v) = \begin{cases} 0 & \text{if } v \in A_i \\ 1 & \text{if } v \in B_i \\ \star & \text{if } v \notin V(G_i) \end{cases}$$

We have $p(x) = (0, 0, \star)$ and for each edge $uv \in E(H')$ there is an $i \in [3]$ with $\{p_i(u), p_i(v)\} = \{0, 1\}$. Since every vertex of H is connected to x, there are seven possible labels which can be assigned to the vertices of H: $(0, 1, \star)$, $(1, 0, \star)$, $(1, 1, \star)$, $(1, \star, 0)$, $(1, \star, 1)$, $(\star, 1, 0)$ and $(\star, 1, 1)$. Consider the graph L with these seven labels as vertices and an edge between the labels (a_1, a_2, a_3) and (b_1, b_2, b_3) if there is some $i \in [3]$ with $\{a_i, b_i\} = \{0, 1\}$. This graph L has a 3-coloring $c\colon V(L) \to [3]$ as illustrated in Figure 3.2. From this we obtain a 3-coloring $c'\colon V(H) \to [3]$ of H by setting $c'(v) = c(p(v))$. ◀

▶ **Corollary 3.13.** *The lower bound of Corollary 3.7 is not tight, i.e. there exists a graph H such that $c_l^{\mathcal{B}}(H) > \lceil \log(\chi_f(H)) \rceil$.*

Proof. Let $H = Y(C_5) \vee K_1$. By Lemma 3.9, we get $\chi_f(Y(C_5)) = \frac{5}{2} + \frac{2}{5} = \frac{29}{10}$ and thus $\chi_f(H) = \frac{39}{10} < 4$. So the lower bound is $\lceil \log(\chi_f(H)) \rceil = 2$. However, $Y(C_5)$ is not 3-colorable, so by Lemma 3.12 we get that $c_l^{\mathcal{B}}(H) > 2$. ◀

As another corollary, we obtain an NP-hardness for the local covering problem with the guest class $\mathcal{B}$ of bipartite graphs.

▶ **Theorem 3.14.** *The 2-local-$\mathcal{B}$-covering problem is NP-complete.*

Proof. By Lemma 2.6, the local-$\mathcal{B}$-covering problem is in NP.

To show NP-hardness, we reduce from the 3-coloring problem which is NP-hard by [14]. Given a 3-coloring instance H, we construct the instance $H' = H \vee K_1$ of the 2-local-$\mathcal{B}$-covering problem. This construction can clearly be performed in polynomial time and by Lemma 3.12 we know that H' has a 2-local $\mathcal{B}$-cover if and only if H has a 3-coloring. ◀

We conjecture that that the k-local-$\mathcal{B}$-covering problem is also NP-hard for values of k which are larger than 2.

► **Conjecture 3.15.** *The k-local-$\mathcal{B}$-covering problem is NP-hard for every $k \geq 2$.*

One approach for proving this conjecture is to provide a construction which increases the local $\mathcal{B}$-covering number by one. A natural candidate for such construction is the join of a graph with itself. Even though this seems very natural, it turns out to be a quite difficult conjecture and we have not made considerable progress towards solving it.

► **Conjecture 3.16.** *If $\mathrm{c}_l^{\mathcal{B}}(H) = k$, then $\mathrm{c}_l^{\mathcal{B}}(H \vee H) = k + 1$.*

4 Cycle Covers

	cycles $\mathcal{C}$	$\mathcal{C}^* = \mathcal{C} \cup \{K_2\}$	bounded cycles $\mathcal{C}^*_{\leq m}$ for $m \geq 3$
global	$\langle 4.3 \rangle$	$\langle 4.3 \rangle$	$\langle 4.11 \rangle$
local	?	$\langle 4.9 \rangle$	$\langle 4.11 \rangle$

(a) in P; NP-hard for k given in the input; NP-hard for some fixed k; Unknown

Table 4.1 Overview of the complexity results obtained in this chapter. The cells correspond to the global-$\mathcal{G}$-covering and local-$\mathcal{G}$-covering problem where $\mathcal{G}$ depends on the column. Numbers $\langle X \rangle$ refer to the corresponding Theorem X in the thesis.

In this chapter, we consider coverings with cycles as guests. Let $\mathcal{C} = \{C_k \mid k \geq 3\}$ be the guest class of cycles and let $\mathcal{C}^* = \mathcal{C} \cup \{K_2\}$ also include a single edge. We remark that in some sense $\mathcal{C}^*$ is the more natural guest class to consider since including the single edge K_2 ensures that every host admits a $\mathcal{C}^*$-covering. Cycle coverings are interesting for several reasons. The guest classes $\mathcal{C}$ and $\mathcal{C}^*$ are natural and not induced-hereditary, while most other natural guest classes such as forests, bipartite graphs, planar graphs and complete graphs are all induced-hereditary. Additionally, we shall see some patterns regarding computational complexity which motivate the next chapters: The global-$\mathcal{C}^*$-covering problem is NP-complete while the local-$\mathcal{C}^*$-covering problem has a polynomial time solution. However, the local-$\mathcal{C}^*$-covering problem becomes NP-complete when restricting the guest class to cycles of a fixed maximum length. The results obtained in this chapter are summarized in Table 4.1.

As last motivation for considering cycle covers, we mention the *circuit double cover conjecture* as a big unsolved problem regarding cycle coverings (see [31] for a survey of this problem).

▶ **Conjecture 4.1** (circuit double cover). *Let H be a host graph without bridges. Then, there is a $\mathcal{C}$-cover φ of H such that each edge $e \in E(H)$ is covered by exactly two guests of φ.*

4.1 Global Cycle Cover

In this section, we show that the global-$\mathcal{C}$-covering problem and the global-$\mathcal{C}^*$-covering problem are NP-complete. Our idea is to reduce from the Hamiltonian cycle problem. To achieve this, we consider the related problem of decomposing a graph into Hamiltonian cycles.[1] We need a lemma first proved by Kotzig. For the sake of completeness, we provide a proof of it since it was only published in Slovakian.

▶ **Lemma 4.2** ([16, result 4a in the German summary]). *A 3-regular graph G has a Hamiltonian cycle if and only if its line graph $L(G)$ can be decomposed into two Hamiltonian cycles.*

[1] The idea of our proof is sketched on `https://en.wikipedia.org/wiki/Hamiltonian_decomposition`.

Proof. Let G be a 3-regular graph. First, suppose that G has a Hamiltonian cycle C containing the vertices $v_1, \ldots, v_n$ in this order. Indices are considered cyclically. Note that $M = E(G) \setminus E(C)$ is a perfect matching of G.

To obtain Hamiltonian cycles in $L(G)$, we start from the base cycle B consisting of the vertices $L(v_1v_2), L(v_2v_3), \ldots, L(v_{n-1}v_n), L(v_nv_1)$. For each matching edge $e = v_iv_j \in M$, the corresponding vertex $L(e)$ can be inserted into B at two positions corresponding to v_i or v_j: it can be inserted between $L(v_{i-1}v_i)$ and $L(v_iv_{i+1})$ or between $L(v_{j-1}v_j)$ and (v_jv_{j+1}). Choose arbitrarily between these two options and perform the insertion for every $e \in M$ to obtain a Hamiltonian cycle Y_1. By picking the other choice for each $e \in M$, we obtain another Hamiltonian cycle Y_2.

Next, we show that Y_1 and Y_2 decompose $L(G)$. Consider an arbitrary edge $L(e)L(f)$ of $L(G)$ and let $v \in V(G)$ be the endpoint shared by e and f. We consider three cases:

- If both e and f belong to C, then $L(e)$ and $L(f)$ are adjacent in the base cycle B. In one of Y_1 and Y_2 we have not inserted any vertex between $L(e)$ and $L(f)$. Thus, one of Y_1 and Y_2 contains the edge $L(e)L(f)$.
- If exactly one of e and f belongs to C, say e, then $f \in M$ is a matching edge. In one of Y_1 and Y_2 we have inserted the vertex $L(f)$ at the position corresponding to v. Thus, one of Y_1 and Y_2 contains the edge $L(e)L(f)$.
- The remaining case that e and f both belong to M is not possible since no two edges from M share an endpoint.

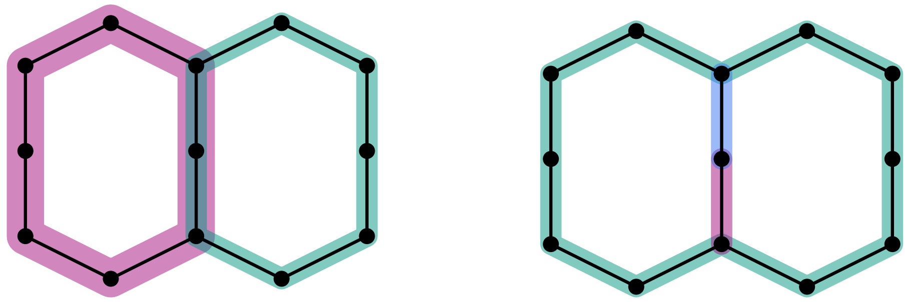
Therefore, Y_1 and Y_2 cover every edge of G and thus Y_1 and Y_2 decompose $L(G)$.

For the other direction, suppose that $L(G)$ has a decomposition into two Hamiltonian cycles Y_1 and Y_2. We iteratively perform the following operation on Y_1: if for three consecutive vertices $L(e_1), L(e_2), L(e_3)$ of Y_1, the corresponding edges e_1, e_2, e_3 of G are all incident to a common vertex, then remove $L(e_2)$ from Y_1. Let $L(e_1), \ldots, L(e_n)$ be the vertices of the resulting cycle in order. Note that by construction, e_{i+1} shares the endpoint with e_i which e_i does not share with e_{i-1}. Thus, we can write $e_i = v_iv_{i+1}$ where we identify $v_{n+1} = v_1$. Thus, $v_1, \ldots, v_n, v_1$ defines a circular walk in G, using every edge at most once. If $v_i = v_j$ for some $i \neq j$, it would follow that $\deg_G(v_i) \geq 4$, contradicting $\Delta(G) = 3$. Therefore, all v_i are distinct and $v_1, \ldots, v_n$ actually form a cycle C.

Finally, we show that C is Hamiltonian. Suppose that C does not contain some vertex $u \in V(G)$. Consider the three edges $f_1, f_2, f_3 \in E(G)$ incident to u. These edges induce a triangle $L(f_1)L(f_2)L(f_3)$ in $L(G)$. Note that if Y_1 contains an edge of this triangle, then we have $u \in V(C)$ by construction of C. Since we assumed $u \notin V(C)$, the edges of the triangle $L(f_1)L(f_2)L(f_3)$ must be fully contained in Y_2. This is impossible since Y_2 is a Hamiltonian cycle of the graph $L(G)$ with more than three vertices. Thus, we obtain that C is indeed a Hamiltonian cycle of G. ◀

▶ **Corollary 4.3.** *The 2-global-$\mathcal{C}$-covering problem and the 2-global-$\mathcal{C}^*$-covering problem are NP-complete.*

Proof. By Lemma 2.6, it is clear that these problems are in NP. To show NP-hardness, we reduce from the problem of deciding whether a given 3-regular graph G has a Hamiltonian cycle. This problem is NP-hard by [9]. We transform G into the instance $H = L(G)$ of the 2-global-$\mathcal{C}^*$-covering problem. Note that H is 4 regular. Thus, every 2-global $\mathcal{C}^*$-cover φ of H must consist of two edge-disjoint Hamiltonian cycles. By Lemma 4.2, we thus obtain that $c_g^{\mathcal{C}^*}(H) = 2$ if and only if G has a Hamiltonian cycle. Therefore, the 2-global-$\mathcal{C}^*$-covering problem is NP-hard. By the same argument, the 2-global-$\mathcal{C}$-covering problem is NP-hard. ◀



(a) (b)

Figure 4.1 Illustration of the transformation of a $\mathcal{C}^*$-cover into a $\mathcal{C}^*$-decomposition described in Lemma 4.5. (a) shows a $\mathcal{C}^*$-cover φ of a graph H with two cycles. (b) shows the corresponding $\mathcal{C}^*$-decomposition ψ with one cycle and two copies of K_2. Note that $\text{hit}_\psi(v) \leq \text{hit}_\varphi(v)$ holds for all $v \in V(H)$.

4.2 Local Cycle Covers

In this section, we describe an efficient algorithm for the local-$\mathcal{C}^*$-covering problem. The first step is to show that a graph H has a k-local $\mathcal{C}^*$-cover if and only if H has a k-local $\mathcal{C}^*$-decomposition. First, we need a well-known lemma:

▶ **Lemma 4.4.** *A graph H has a $\mathcal{C}$-decomposition if and only if each vertex of H has even degree.*

Proof. Suppose that H has a $\mathcal{C}$-decomposition $G_1, \dots, G_t$. For every vertex $v \in V(H)$, it holds that $\deg_H(v) = \sum_{i=1}^{t} \deg_{G_i}(v)$ since every edge of H is covered by exactly one guest. Since $\deg_{G_i}(v) \in \{0, 2\}$, it is clear that $\deg_H(v)$ is even.

Now, suppose that each vertex of H has even degree. We proceed by induction on the number of edges of H. If $E(H) = \emptyset$, then H trivially has a $\mathcal{C}$-decomposition. Otherwise, by elementary graph theory, H contains a cycle G'. By induction, we can find a $\mathcal{C}$-decomposition $G_1, \dots, G_t$ of $H - E(G')$, yielding the $\mathcal{C}$-decomposition $G_1, \dots, G_t, G'$ of H. ◀

Next, we show how a $\mathcal{C}^*$-cover φ can be transformed into a $\mathcal{C}^*$-partition ψ. The idea is to consider the symmetric difference of all cycles of φ. This symmetric difference can be decomposed into cycles and the remaining edges are covered by copies of K_2. See Figure 4.1 for an illustration of this idea.

▶ **Lemma 4.5.** *Let H be a graph. If H has a k-local $\mathcal{C}^*$-cover, then H also has a k-local $\mathcal{C}^*$-decomposition.*

Proof. Let φ be a k-local $\mathcal{C}^*$-cover of H with guests $G_1, \dots, G_t, F_1, \dots, F_s$ such that each G_i is a cycle and each F_j is a copy of K_2. For each edge $e \in E(H)$, let $c(e) = |\{i \in [t] \mid e \in E(G_i)\}|$ count how often it is covered by a cycle. Consider the subgraph H' of H spanned by the edges hit by an odd number of cycles: $H' = (V(H), \{e \in E(H) \mid c(e) \text{ is odd}\})$. In other words, $E(H')$ is the symmetric difference $E(G_1) \,\Delta \dots \Delta\, E(G_t)$.

Next, we show that every vertex $v \in V(H')$ has even degree in H'. For this, we introduce $c'(e) = c(e) \bmod 2$. By definition, $c'(e) = 1$ if $e \in E(H')$ and $c'(e) = 0$ otherwise. Thus, $\deg_{H'}(v) = \sum_{u \in N_H(v)} c'(uv)$. We observe that $\sum_{u \in N_H(v)} c(uv)$ is even since every cycle hitting v covers two edges incident to v. Combining these observations yields:

$$\deg_{H'}(v) = \sum_{u \in N_H(v)} c'(uv) \equiv \sum_{u \in N_H(v)} c(uv) \equiv 0 \pmod 2$$

Thus, H' has a $\mathcal{C}$-decomposition ψ' by Lemma 4.4. Covering the edges of H' according to ψ' and the edges of $E(H) \setminus E(H')$ by copies of K_2, yields a $\mathcal{C}^*$-decomposition ψ of H.

It remains to show that ψ is k-local. To achieve this, we show for the hitcount of each vertex $v \in V(H)$ that $\mathrm{hit}_\psi(v) \leq \mathrm{hit}_\varphi(v)$, i.e. v is hit at most as often by the $\mathcal{C}^*$-decomposition ψ as by the $\mathcal{C}^*$-cover φ. This is sufficient since φ is k-local.

We start by calculating $\mathrm{hit}_\psi(v)$. In ψ, the edges e with odd $c(e)$ are covered by cycles and the ones with even $c(e)$ are covered by copies of K_2. Thus, $\mathrm{hit}_\psi(v)$ can be calculated as a sum over the edges e incident to v where edges with odd $c(e)$ contribute $\frac{1}{2}$ to the sum while edges with even $c(e)$ contribute 1 to the sum. By setting $f_\psi(e) = \frac{1}{2}$ if $c(e)$ is odd and $f_\psi(e) = 1$ if $c(e)$ is even, we thus obtain $\mathrm{hit}_\psi(v) = \sum_{u \in N_H(v)} f_\psi(uv)$.

Next, we calculate $\mathrm{hit}_\varphi(v)$ in a similar fashion. In φ, the number of cycle-guests hitting v is $\sum_{u \in N_H(v)} \frac{c(uv)}{2}$. Additionally, every edge e with $c(e) = 0$ has to be covered by a copy of K_2. By setting $f_\varphi(e) = \frac{c(e)}{2}$ if $c(e) > 0$ and $f_\varphi(e) = 1$ if $c(e) = 0$, we thus obtain $\mathrm{hit}_\varphi(v) \geq \sum_{e \in N_H(v)} f_\varphi(uv)$. (In general, we do not have equality here since it is technically possible that φ contains some additional useless copies of K_2.)

Combining both calculations with the easy observation $f_\psi(e) \leq f_\varphi(e)$ finishes the proof:

$$\mathrm{hit}_\psi(v) = \sum_{u \in N_H(v)} f_\psi(uv) \leq \sum_{u \in N_H(v)} f_\varphi(uv) \leq \mathrm{hit}_\varphi(v) \leq k \qquad \blacktriangleleft$$

The idea for checking whether H has a k-local $\mathcal{C}^*$-decomposition is to first pick the edges covered by copies of K_2. We need to make sure that the remaining edges can be decomposed into cycles. To achieve this, we find edge-disjoint paths in H whose endpoints have odd degree and then we cover every edge of these paths with copies of K_2. For every vertex v of H, the number of paths containing v has to be limited to ensure that the resulting $\mathcal{C}^*$-decomposition is k-local. This allows us to reduce the local-$\mathcal{C}^*$-covering problem to the following problem:

▶ **Problem 4.6** (T-paths).
- **Input:** A graph H, a set of vertices $T \subseteq V(H)$ and a non-negative integer a_v for every vertex $v \in V(H)$.
- **Question:** Are there edge-disjoint paths $L_1, \ldots, L_t$ with both endpoints in T such that
  - every $v \in T$ is endpoint of exactly one of the paths and
  - for every $v \in V(H)$, we have $\sum_{i=1}^{t} \deg_{L_i}(v) \leq a_v$.

▶ **Lemma 4.7.** *Let H be a graph and $k \in \mathbb{N}$. We define the the T-paths instance $I = (H, T, a)$ by $T = \{v \in V(H) \mid \deg_H(v)$ is odd$\}$ and $a_v = 2k - \deg_H(v)$ for each $v \in V(H)$. In this setting, $\mathrm{c}_l^{\mathcal{C}^*}(H) \leq k$ if and only if I is a yes-instance of the T-paths problem (Problem 4.6).*

Proof. First, assume that H admits a k-local $\mathcal{C}^*$-cover φ. By Lemma 4.5, we can assume that φ is a decomposition. Let F be the subgraph of H whose edges are covered by copies of K_2. Since $H - E(F)$ can be decomposed into cycles, Lemma 4.4 gives us that each vertex has even degree in $H - E(F)$. Because of this, $\deg_F(v)$ is odd if and only if $v \in T$.

▷ Claim. There are edge disjoint paths $L_1, \ldots, L_t \subseteq F$ such that
- each L_i has both endpoints in T and
- each $x \in T$ is endpoint of exactly one L_i.

Proof. We proof the claim by induction on the size of T. The case $T = \emptyset$ is clear. When $T \neq \emptyset$, consider an arbitrary $x \in T$. Since x has odd degree in F, the connected component of F containing x must contain another vertex y of odd degree, which thus satisfies $y \in T$. Therefore, we obtain a path $L \subseteq F$ with endpoints x and y. We apply induction on $F' = F - E(L)$ and $T' = T \setminus \{x, y\}$. This is possible since $\deg_{F'}(v)$ is odd if and only $v \in T'$. Combining the paths $L_1, \ldots, L_{t-1}$ obtained by induction with L finishes the proof. $\lhd$

To show that the paths $L_1, \ldots, L_t$ from the above claim certify that I is a yes-instance of the T-paths problem, it remains to show that $\sum_{i=1}^{t} \deg_{L_i}(v) \leq a_i$ for each $v \in V(H)$. Since φ covers edges of F with copies of K_2 and the remaining edges with cycles, we have $\mathrm{hit}_\varphi(v) = \deg_F(v) + \frac{\deg_H(v) - \deg_F(v)}{2} \leq k$ and thus $\deg_F(v) \leq 2k - \deg_H(v) = a_v$. From this we get $\sum_{i=1}^{t} \deg_{L_i}(v) \leq \deg_F(v) \leq a_v$ and therefore I is indeed a yes-instance of the T-paths problem.

For the other direction, assume that the paths $L_1, \ldots, L_t$ certify that I is a yes-instance of the T-paths problem. Consider the graph $F = L_1 \cup \cdots \cup L_t$. For each vertex $v \in V(H)$, the degree $\deg_F(v)$ is odd if and only if $v \in T$. Thus, every vertex of $H - E(F)$ has even degree and by Lemma 4.4 we can decompose $H - E(F)$ into cycles. Covering the edges of F with copies of K_2 yields a $\mathcal{C}^*$-decomposition φ of H. For every vertex $v \in V(H)$, we have

$$\mathrm{hit}_\varphi(v) = \deg_F(H) + \frac{\deg_H(v) - \deg_F(v)}{2} = \frac{\deg_F(v) + \deg_H(v)}{2} \leq \frac{a_v + \deg_H(v)}{2} = k$$

and thus φ is k-local. ◀

We remark that our variant of the T-paths problem (Problem 4.6) is slightly more general than the problem which is usually referred to under the name T-paths in the literature. In the usual T-paths problem every vertex must be contained in at most one path, i.e. $a_v = 2$ in our formulation of the problem. Schrijver gave a reduction of this version with $a_v = 2$ to the linear matroid parity problem [27]. Next, we give a reduction of our problem variant to the problem of graph matching.

▶ **Theorem 4.8.** *An instance $I = (H, T, a)$ of the T-paths problem (Problem 4.6) with $|V(H)| = n$ and $|E(H)| = m$, can be transformed in $\mathcal{O}(nm)$ time into a graph G with $\mathcal{O}(m)$ vertices and $\mathcal{O}(nm)$ edges such that G has a perfect matching if and only if I is a yes-instance.*

Proof. We first show that we may assume that $a_v \leq \deg_H(v)$. Replacing each a_v with $\min(a_v, \deg_H(v))$ does not change whether I is yes- or no-instance. Indeed, the paths $L_1, \ldots, L_t$ required by the T-paths problem are edge-disjoint and thus we always have $\sum_{i=1}^{t} \deg_{L_i}(v) \leq \deg_H(v)$.

Next, we describe the construction of the graph G. See Figure 4.2 for an illustration of the construction. For every edge $e \in E(H)$, we create two vertices e_A and e_B in G and join them by an edge. For every vertex $v \in V(H)$, let $b_v = \lfloor \frac{a_v}{2} \rfloor$ if $v \notin T$ and $b_v = \lfloor \frac{a_v - 1}{2} \rfloor$ otherwise. We create $2b_v$ vertices $v_{\mathsf{A},1}, v_{\mathsf{B},1}, \ldots, v_{\mathsf{A},b_v}, v_{\mathsf{B},b_v}$ and for every $i \in [b_v]$ we join $v_{\mathsf{A},i}$ and $v_{\mathsf{B},i}$ by an edge. We refer to e_A and e_B as well as to $v_{\mathsf{A},i}$ and $v_{\mathsf{B},i}$ as *twins* and the edges connecting them are referred to as *twin-edges*. If $v \in T$, we create one additional vertex v_T. For $e \in E(H)$, let $S(e) = \{e_\mathsf{A}, e_\mathsf{B}\}$ be the set of vertices created from e. Similarly for $v \in V(H)$, let $S(v)$ be the set of vertices created from v, i.e. it consists of all vertices $v_{\mathsf{A},i}, v_{\mathsf{B},i}$ and v_T if $v \in T$. Note that by construction, $|S(v)| \leq a_v$. For every $v \in V(H)$ and $e \in E(H)$ incident to v, we add the edges $\{xy \mid x \in S(v), y \in S(e)\}$ to G.

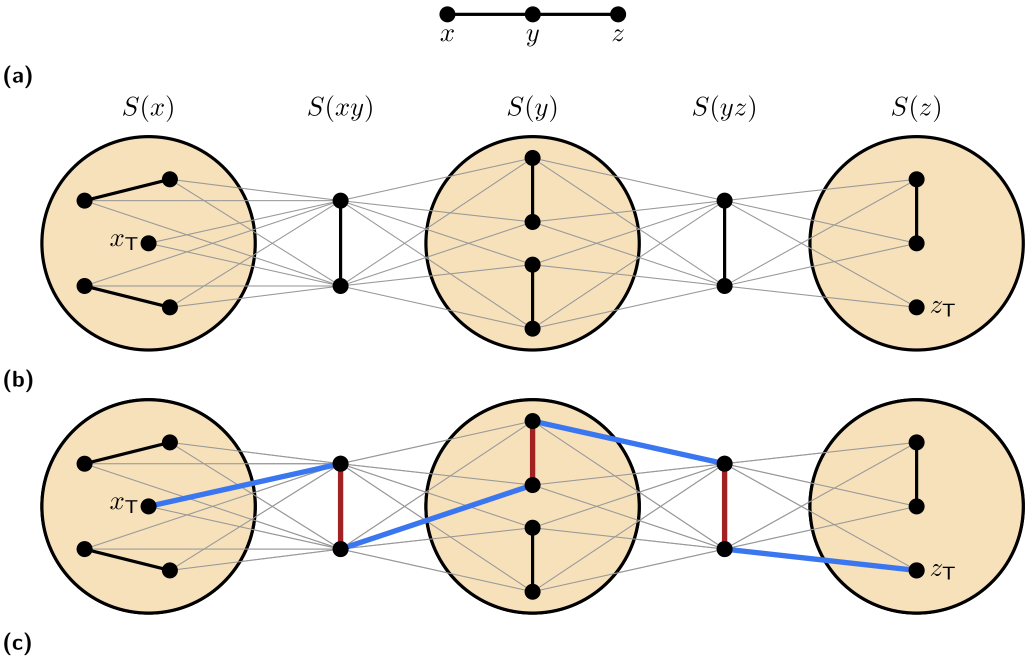


Figure 4.2 Illustration of the construction used in the proof of Theorem 4.8. In (a) we see a part of the graph H from a T-paths instance $I = (H, T, a)$. For this example, assume $T = \{x, z\}$ and $a_x = 6, a_y = 4, a_z = 3$. In (b) we see the graph G constructed from I. Twin-edges are drawn thicker. In (c) we see an example of how the path xyz in H can be transformed into a path in G which alternates between non-twin-edges (blue) and twin-edges (red). The blue edges are part of the perfect matching of G constructed in the proof of Theorem 4.8.

Next, we estimate the size of G. For the number of vertices we have

$$|V(G)| = \sum_{v \in V(H)} |S(v)| + \sum_{e \in E(H)} |S(e)| \leq \sum_{v \in V(H)} a_v + \sum_{e \in E(H)} 2 = 4m \in \mathcal{O}(m).$$

Here, we have used our assumption that $a_v \leq \deg_H(v)$ as well as $\sum_{v \in V(H)} \deg_H(v) = 2m$. For the number of edges, we have

$$|E(G)| = m + \sum_{v \in V(H)} \big(b_v + |S(v)| \cdot 2 \deg_H(v)\big) \leq 2m + 4nm \in \mathcal{O}(nm).$$

Indeed, we have m twin edges of the type $e_{\mathsf{A}} e_{\mathsf{B}}$ with $e \in E(H)$ and for each vertex $v \in V(H)$, we have b_v twin edges on $S(v)$. Additionally, each vertex in $S(v)$ is adjacent to the two vertices $e_{\mathsf{A}}, e_{\mathsf{B}}$ for every edge e incident to v. We also used that $|S(v)| \leq n$ and $\sum_{v \in V(H)} \deg_H(v) = 2m$ as well as $b_v \leq \frac{\deg_H(v)}{2}$.

It remains to show that G has a perfect matching if and only if I is a yes-instance of the T-paths problem. First, assume that the paths $L_1, \ldots, L_t$ certify that I is a yes-instance. We describe how a perfect matching M of G can be created. Consider the path L_i on the vertices $u^1, \ldots, u^k \in V(H)$ in this order. There is a natural way to transform L_i into a path $L_i' \subseteq G$ with endpoints u^1_{T} and u^k_{T} such that L_i' alternates between twin-edges and non-twin-edges. This transformation is illustrated in Figure 4.2c and has the following properties: For every $j \in [k-1]$, the twin-edge on the vertices of $S(u^j u^{j+1})$ is contained in L_i' and for every $j \in \{2, \ldots, k-1\}$, exactly one twin-edge on the vertices of $S(u^j)$ is contained in L_i'. In total,

L_i' contains $2k-3$ twin-edges and $2k-2$ non-twin-edges. We add these $2k-2$ non-twin-edges (which form a perfect matching of L_i') to M. This procedure is performed for all $i \in [t]$.

We need to make sure that the paths $L_1', \dots, L_t'$ are pairwise vertex-disjoint. For every L_i and every inner vertex v of it, we need to choose which of the b_i twin-edges on the vertices of $S(v)$ is contained in L_i'. Since $\sum_{i=1}^{t} \deg_{L_i}(v) \le a_v$, at most b_v of the paths $L_1, \dots, L_t$ contain v as an inner vertex. On $S(v)$, it is thus possible to choose different twin-edges for the paths $L_1, \dots, L_t$ containing v as an inner vertex. Further, the paths $L_1', \dots, L_t'$ do not share a vertex from $S(e)$ for $e \in E(H)$ since the paths $L_1, \dots, L_t$ are edges-disjoint. Thus, we can indeed make sure that $L_1', \dots, L_t'$ are pairwise vertex-disjoint.

For all vertices which are still unmatched after performing the above procedure, we match them with their twin. This is possible because all v_{T} are already matched and by construction, from every twin pair either both or none of its vertices are matched. So G has indeed a perfect matching.

For the other direction, suppose that G has a perfect matching M. Consider the symmetric difference of M and all twin edges of G and call the resulting graph G'. Note that $\Delta(G') \le 2$ and $\deg_{G'}(v_{\mathsf{T}}) = 1$ for $v \in T$ and $\deg_{G'}(x) \ne 1$ for every other vertex $x \in V(G) \setminus \{v_{\mathsf{T}} \mid v \in T\}$. For every $v \in T$ there is thus a unique $u \in T$ with $u \ne v$ such that there is a path P in G' with endpoints v and u. In P, consider the subsequence of vertices which belong to $S(v)$ for some $v \in V(H)$ and create the sequence $v^1, \dots, v^k \in V(H)$ from them. This sequence satisfies $v^i = v^{i+1}$ or $v^i v^{i+1} \in E(H)$ for every $i \in [k-1]$. Thus, by eliminating consecutive equal elements, we obtain a walk from v to u in H. This walk contains a path L from v to u. Performing this for all pairs of vertices $u, v \in T$ connected in G' yields paths $L_1, \dots, L_t$.

To see that these paths are pairwise edge-disjoint, let x^i, y^i be the endpoints of the path L_i. If $e \in E(L_i)$, then e_{A} is in the same connected component of G' as x^i_{T} and y^i_{T}. Since x^i and x^j are in different connected components of G' for $i \ne j$, it follows that L_i and L_j are edge-disjoint. Since we also have $\sum_{i=1}^{t} \deg_{L_i}(v) \le |S(v)| \le a_v$, these paths certify that I is a yes-instance of the T-paths problem. ◀

This gives us a polynomial time algorithm for the local-$\mathcal{C}^*$-covering problem.

▶ **Theorem 4.9.** *For a given graph H with n vertices and m edges and a given $k \in \mathbb{N}$, it can be decided in $\mathcal{O}(n \cdot m^{3/2})$ time whether $\mathrm{c}_l^{\mathcal{C}^*}(H) \le k$.*

Proof. We transform H into the instance $I = (H, T, a_v)$ of the T-paths problem as in Lemma 4.7 and apply the transformation of Theorem 4.8 to obtain an instance G of the graph matching problem. By [19], this problem can be solved in $\mathcal{O}(\sqrt{|V(G)|}|E(G)|)$ time. Together with $|V(G)| \in \mathcal{O}(m)$ and $|E(G)| \in \mathcal{O}(nm)$, this yields a $\mathcal{O}(n \cdot m^{3/2})$ time algorithm for the local-$\mathcal{C}^*$-covering problem. ◀

4.3 Cycles of Restricted Length

Interestingly, even the local covering number becomes difficult to compute if we restrict the maximum length of the cycles. More concretely, we consider the guest classes $\mathcal{C}_{\le m} = \{C_k \mid 3 \le k \le m\}$ and $\mathcal{C}^*_{\le m} = \mathcal{C}_{\le m} \cup \{K_2\}$. We derive this from a result of Holyer regarding decomposition problems. Here, the $\mathcal{G}$-decomposition problem refers to the problem of deciding whether a given host graph H admits a $\mathcal{G}$-decomposition. If $\mathcal{G} = \{G\}$ consists of a single guest, we also refer to it as the G-decomposition problem.

▶ **Lemma 4.10** (Holyer [12]). *For every $m \ge 3$, the C_m-decomposition problem and the $\mathcal{C}_{\le m}$-decomposition problem are NP-hard.*

Proof. In [12], they first show that the K_3-decomposition problem is NP-hard. In the K_3-decomposition instance H constructed by them, they notice that each edge has one of three distinct "directions" called **a**, **b** and **c**. Each triangle of H contains one edge of each direction and the total number of edges of each direction is the same. To conclude that the C_m-decomposition problem is NP-hard for $m \geq 3$, they replace each direction **a** edge of H with a path of $m-2$ edges to obtain H'.

We now argue that in every $\mathcal{C}_{\leq m}$ decomposition φ of H' every cycle must have length m. Consider a path P of length $m-2$ in H' which replaces a direction **a** edge of H. To cover P, a cycle C of length m is required. The edges of C not belonging to P are of direction **b** and **c** since each triangle of H contains one edge of each direction. Since the number of edges of each direction is the same, it follows that the cycles of length m from φ cover every edge of H'. Thus, φ is also a C_m-decomposition and it corresponds to a K_3-decomposition of H. Therefore, H' has a C_m-decomposition if and only if it has a $\mathcal{C}_{\leq m}$-decomposition which is the case if and only if H has a K_3-decomposition. It follows that the C_m-decomposition problem and the $\mathcal{C}_{\leq m}$-decomposition problem are NP-hard. ◀

▶ **Theorem 4.11.** *For every $m \geq 3$, the global-$\mathcal{C}^*_{\leq m}$-covering problem and the local-$\mathcal{C}^*_{\leq m}$-covering problem are NP-complete.*

Proof. We start with the local covering number. We reduce from the $\mathcal{C}_{\leq m}$-decomposition problem which is NP-hard by Lemma 4.10. Let H be an instance of $\mathcal{C}_{\leq m}$-decomposition. We may assume that every vertex of H has even degree, as otherwise H is a trivial no-instance. Set $k = \Delta(H)/2$ and transform H into a graph H' by attaching $(\Delta(H) - \deg(v))/2$ leaves to each vertex $v \in V(H)$. It remains to show that $c_l^{\mathcal{C}^*_{\leq m}}(H') \leq k$ if and only if H has a $\mathcal{C}_{\leq m}$-decomposition.

If H has a $\mathcal{C}_{\leq m}$-decomposition φ, we cover the edges incident to the leaves of H' separately by copies of K_2 to obtain a $\mathcal{C}^*_{\leq m}$-cover φ' of H'. For each vertex $v \in V(H)$, we have $\mathrm{hit}_{\varphi'}(v) = \deg_H(v)/2 + (\Delta(H) - \deg_H(v))/2 = k$ and for each $v \in V(H') \setminus V(H)$ we have $\mathrm{hit}_{\varphi'}(v) = 1$. Thus, φ' is k-local and we obtain $c_l^{\mathcal{C}^*_{\leq m}}(H') \leq k$.

For the other direction, suppose that H' has a $\mathcal{C}^*_{\leq m}$-cover φ'. The edges incident to the leaves of H' are covered separately by copies K_2. Thus, we can restrict φ' to H by only keeping the guests contained in H. This yields a $\mathcal{C}^*_{\leq m}$-cover φ of H. For each $v \in V(H)$ we have $\mathrm{hit}_{\varphi}(v) = k - (\Delta(H) - \deg_H(v))/2 = \deg_H(v)/2$. Since $\deg_H(v)/2$ guests must cover $\deg_H(v)$ edges incident to v, each guest of φ hitting v must be a cycle. It also follows that each edge incident to v is covered exactly once by φ. Because v was chosen arbitrarily, it follows that all guests are disjoint and the cycles form a $\mathcal{C}_{\leq m}$ decomposition of H.

Now, we come to the global covering number. We reduce from the C_m-decomposition problem which is NP-hard by Lemma 4.10. Let H be an instance of the C_m-decomposition problem. We may assume that $|E(H)|$ is a multiple of m, as otherwise H is a trivial no-instance. Clearly, H has a C_m-decomposition if and only if H has a $c_g^{\mathcal{C}^*_{\leq m}}(H) = \frac{|E(H)|}{m}$. ◀

5 Finite Guest Classes

	bounded stars $\mathcal{S}_{\leq d}$ for $d \geq 3$	$\{G\}$ for connected r-regular G, $r \geq 2$	$\{K_{1,d}\}$ for $d \geq 3$	at most three edges $\mathcal{E}_3$
global	⟨5.3⟩	⟨5.3⟩	⟨5.3⟩	⟨5.16⟩
local	⟨5.9⟩	⟨5.13⟩	⟨5.15⟩	⟨5.16⟩

(a) ■ in P ■ NP-hard for k given in the input ■ NP-hard for some fixed k ■ Unknown

Table 5.1 Overview of the complexity results obtained in this chapter. The cells correspond to the global-$\mathcal{G}$-covering and local-$\mathcal{G}$-covering problem where $\mathcal{G}$ depends on the column. Numbers $\langle X \rangle$ refer to the corresponding Theorem X in the thesis.

As we have seen in the previous chapter, covering problems seem to become more difficult when restricting the guest class to guests with a bounded number of vertices (see Theorem 4.9 and Theorem 4.11). In this chapter, we thus focus on finite guest classes and obtain some general NP-hardness results. Interestingly, this study of finite guest classes also yields an example of a natural monotone guest class for which the global covering problem is easy while the local covering problem is NP-hard (see Theorem 5.16). Table 5.1 gives an overview of the main complexity results obtained in this chapter.

5.1 Decompositions and Global Coverings

Recall that a decomposition is a cover in which each edge is covered exactly once. The $\mathcal{G}$-decomposition problem asks whether a given graph H admits a $\mathcal{G}$-decomposition. For guest classes consisting of a single graph G, this problem is well-studied in the literature. When the guest class $\mathcal{G} = \{G\}$ consists of a single graph, we also refer to a $\mathcal{G}$-decomposition shortly as a G-decomposition. One of the first NP-hardness results of a decomposition problem was obtained by Holyer [12]. They show that it is NP-hard to decide whether a given graph H admits a K_n-decomposition if $n \geq 3$. Holyer conjectured that the G-decomposition problem is NP-hard if G has at least three edges. Assuming P $\neq$ NP, this conjecture turned out to be wrong for disconnected graphs:

▶ **Theorem 5.1** ([2]). *Let G be a graph such that every connected component of G contains at most two edges, i.e. $G = sP_3 \mathbin{\dot\cup} tP_2$ for some $s, t \in \mathbb{N}_0$. There is a polynomial time algorithm which decides whether a given graph H admits a G-decomposition.*

However, the conjecture turned out to be true for connected graphs:

▶ **Theorem 5.2** (Dor, Tarsi [4]). *If a graph G has a connected component containing at least three edges, then the G-decomposition problem is NP-hard.*

By using a simple counting argument, we can conclude the following about coverings with finite guest classes:

► **Theorem 5.3.** *Let $\mathcal{G}$ be a finite guest class such that there is a unique graph $G \in \mathcal{G}$ with the maximum number of edges, i.e. G is unique with the property $|E(G)| = \max_{G' \in \mathcal{G}} |E(G')|$. If G has a connected component containing at least three edges, then the global-$\mathcal{G}$-covering problem is NP-complete.*

Proof. By Lemma 2.6, the problem is clearly in NP. To show NP-hardness, we reduce from the G-decomposition problem where $G \in \mathcal{G}$ is the unique graph with $|E(G)| = \max_{G' \in \mathcal{G}} |E(G')|$. We transform an instance H of the G-decomposition problem into the instance (H, k) of the global-$\mathcal{G}$-covering problem where $k = \frac{|E(H)|}{|E(G)|}$. We may assume that k is an integer since H is an obvious no-instance to the G-decomposition problem if $|E(H)|$ is not divisible by $|E(G)|$.

It remains to show that H has a G-decomposition if and only if $c_g^{\mathcal{G}}(H) \leq k$. Clearly, a G-decomposition of H is also a $\mathcal{G}$-cover of H and it is k-global as the number of guests in the decomposition is k. For the other direction, suppose that H has a k-global $\mathcal{G}$ cover φ with guests $G_1, \ldots, G_k$. If some guest G_i is not a copy of G, then $|E(G_i)| < |E(G)|$ and thus $\sum_{j=1}^{k} |E(G_i)| < k \cdot |E(G)| = |E(H)|$ contradicting the assumption that φ is a cover. Thus, all guests are copies of G. If two different guests G_i and G_j share an edge, then again the number of covered edges is too small. Thus, the cover φ is also a G-decomposition of H. ◄

We note that this result already applies to a lot of natural finite graph classes such as paths, cycles, stars or cliques on a bounded number of vertices. In particular, it gives an alternative proof for the result of Theorem 4.11 that the global-$\mathcal{C}^*_{\leq m}$-covering problem is NP-hard. It is a natural question whether the condition of $G \in \mathcal{G}$ being unique with the maximum number of edges is necessary in Theorem 5.3. To show that this is the case, we introduce a new graph class. For $s \in \mathbb{N}$, let $\mathcal{E}_s = \{G \mid |E(G)| \leq s, \delta(G) > 0\}$ be the guest class of graphs with at most s edges and no isolated vertices (technically, the class would become infinite if we allowed isolated vertices). For $s \geq 3$ there is a $G \in \mathcal{E}_s$ with maximum number of edges such that G has a connected component with at least three edges. Yet, it is straight forward to determine the global $\mathcal{E}_s$-cover number:

► **Proposition 5.4.** *Let $s \in \mathbb{N}$ and H be a host graph. It holds that $c_g^{\mathcal{E}_s}(H) = \left\lceil \frac{|E(H)|}{s} \right\rceil$.*

Proof. Let $k = \left\lceil \frac{|E(H)|}{s} \right\rceil$. To obtain a k-global $\mathcal{E}_s$-cover of H, decompose the edges of H arbitrarily into k subsets of size at most s. For the lower bound, the guests of a cover must have at least $|E(H)|$ edges in total, but each guest has at most s edges and thus $c_g^{\mathcal{E}_s}(H) \geq \left\lceil \frac{|E(H)|}{s} \right\rceil$. ◄

We conjecture that for connected guest classes, the assumption of a unique edge maximal graph can be dropped in Theorem 5.3:

► **Conjecture 5.5.** *Let $\mathcal{G}$ be a finite guest class such that every guest is connected and $\max_{G \in \mathcal{G}} |E(G)| \geq 3$. Then, the global-$\mathcal{G}$-covering problem is NP-hard.*

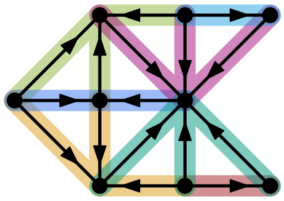


Figure 5.1 Illustration of an $\mathcal{S}_{\leq 3}$-cover and an orientation induced by it.

5.2 Local Coverings with Bounded Stars

It is also natural, to consider the statement of Theorem 5.3 for local coverings.

▶ **Question 5.6.** *Is the local-$\mathcal{G}$-covering problem NP-hard when $\mathcal{G}$ contains a unique guest G with the maximum number of edges and G has a connected component with at least three edges?*

As we show in this section, the answer is no under the assumption P $\neq$ NP. We consider the guest class $\mathcal{S}_{\leq d} = \{K_{1,n} \mid n \in \mathbb{N}_{\leq d}\}$ of stars of bounded degree. Theorem 5.3 implies that the global-$\mathcal{S}_{\leq d}$-covering problem is NP-complete for $d \geq 3$. However, we shall see that the local-$\mathcal{S}_{\leq d}$-covering problem has a polynomial time solution. Our algorithm is a small modification of the algorithm for the local-$\mathcal{S}$-covering problem given by Knauer and Ueckerdt [15] where $\mathcal{S} = \{K_{1,n} \mid n \in \mathbb{N}\}$ is the infinite guest class of all stars.

An *orientation* O of a graph H *orients* each edge $uv \in E(H)$ either towards u or towards v. For an orientation O and a vertex $v \in V(H)$, the *indegree* $\text{indeg}_O(v)$ is the number of edges oriented towards v. There is a natural correspondence between orientations and $\mathcal{S}_{\leq d}$-decompositions.

▶ **Construction 5.7.** From an $\mathcal{S}_{\leq d}$-decomposition φ of H, we obtain an orientation O of H by orienting every edge e towards the center of the star covering e. Note that for stars consisting of a single edge, there are two possible centers. In this case, we orient the edge arbitrarily. We say that the orientation O is *induced* by φ. Similarly, we can obtain an $\mathcal{S}_{\leq d}$-decomposition φ from an orientation O: for each vertex $v \in V(H)$, decompose the edges oriented towards v into the minimum possible number of stars centered at v. Note that φ contains $\lceil \text{indeg}_O(v)/d \rceil$ stars centered at v for every $v \in V(H)$. We say that the decomposition φ is *induced* by O. See Figure 5.1 for an illustration.

▶ **Lemma 5.8.** *Let H be a graph, $d \in \mathbb{N}_{\geq 2}$, $k \in \mathbb{N}$ and*

$$\xi(v) = \left\lceil \frac{d \cdot (\deg(v) - k)}{d - 1} \right\rceil .$$

We have $c_l^{\mathcal{S}_{\leq d}}(H) \leq k$ if and only if H admits an orientation O with $\text{indeg}_O(v) \geq \xi(v)$ for all $v \in V(H)$.

Proof. We need the following claim.

▷ Claim. Let O be an orientation of H and let φ be an $\mathcal{S}_{\leq d}$-decomposition induced by O. We have $\text{indeg}_O(v) \geq \xi(v)$ for all $v \in V(H)$ if and only if φ is k-local.

Proof. Consider an arbitrary vertex $v \in V(H)$. Note that v is hit by $\lceil \mathrm{indeg}_O(v)/d \rceil$ stars centered at v and by $\deg(v) - \mathrm{indeg}_O(v)$ stars containing v as leaf. A simple calculation gives us that $\mathrm{hit}_\varphi(v) \leq k$ if and only if $\xi(v) \leq \mathrm{indeg}_O(v)$:

$$\begin{aligned}
& \mathrm{hit}_\varphi(v) = \left\lceil \frac{\mathrm{indeg}_O(v)}{d} \right\rceil + (\deg(v) - \mathrm{indeg}_O(v)) \leq k \\
\iff & \frac{\mathrm{indeg}_O(v)}{d} + (\deg(v) - \mathrm{indeg}_O(v)) \leq k \\
\iff & \deg(v) - k \leq \mathrm{indeg}_O(v) \cdot \frac{d-1}{d} \\
\iff & \frac{d \cdot (\deg(v) - k)}{d-1} \leq \mathrm{indeg}_O(v) \\
\iff & \xi(v) \leq \mathrm{indeg}_O(v)
\end{aligned}$$

◁

If H admits an orientation O with $\mathrm{indeg}_O(v) \geq \xi(v)$ for all $v \in V(H)$, then a cover φ induced by O is k-local by the above claim and thus $\mathrm{c}_l^{\mathcal{S}_{\leq d}}(H) \leq k$.

For the other direction, suppose that φ admits a k-local $\mathcal{S}_{\leq d}$ cover. We can assume that φ is a decomposition since $\mathcal{S}_{\leq d}$ is a monotone guest class. Thus, φ induces an orientation O. Consider an $\mathcal{S}_{\leq d}$-decomposition φ' induced by O. Note that by construction of φ', we have $\mathrm{hit}_{\varphi'}(v) \leq \mathrm{hit}_\varphi(v) \leq k$ for all $v \in V(H)$. By the claim, this implies $\mathrm{indeg}_O(v) \geq \xi(v)$. ◀

Thus, a fast algorithm for deciding whether an orientation O with $\mathrm{indeg}_O(v) \geq \xi(v)$ exists, gives us a fast algorithm for deciding whether $\mathrm{c}_l^{\mathcal{S}_{\leq d}}(H) \leq k$.

▶ **Theorem 5.9.** *For given a given graph H and given numbers $k, d \in \mathbb{N}$, it can be decided in polynomial time whether $\mathrm{c}_l^{\mathcal{S}_{\leq d}}(H) \leq k$.*

Proof. If $d = 1$, then $\mathrm{c}_l^{\mathcal{S}_{\leq d}}(H) = \Delta(H)$. Now, assume $d > 1$. Lemma 5.8 gives us $\xi \colon V(H) \to \mathbb{Z}$ such that $\mathrm{c}_l^{\mathcal{S}_{\leq d}}(H) \leq k$ if and only if H admits an orientation O with $\mathrm{indeg}_O(v) \geq \xi(v)$ for all $v \in V(H)$. It can be decided in polynomial time by a flow algorithm whether such an orientation exists [7]. ◀

Thus, we have shown that the answer to Question 5.6 is no under the assumption P ≠ NP. However, given Theorem 5.2 it is still natural to conjecture the following:

▶ **Conjecture 5.10.** *Let $\mathcal{G} = \{G\}$ be a guest class consisting of a single graph.*
- *If $G = sP_3 \uplus tP_2$ for some $s, t \in \mathbb{N}_0$, then the* local-$\mathcal{G}$-covering *problem has a polynomial time solution.*
- *Otherwise, the* local-$\mathcal{G}$-covering *problem is* NP*-complete.*

In the next section we give some evidence for this conjecture by proving the NP-hardness result for some specific guests G. We remark that a full solution of the conjecture seems quite difficult since the proofs of Theorem 5.1 and Theorem 5.2 are very involved and the ideas might not be directly applicable to local coverings. For the global covering problem, all G-decompositions are equally good since they require the same number of copies of G. However, this is not true for the local covering problem: it matters how often individual vertices are hit. Thus, a modification of the ideas used in the proof of Theorem 5.2 presented in [4] might require rebuilding some of the machinery used in the proof. For example, they use Wilson's theorem.

▶ **Theorem 5.11** (Wilson's theorem [30]). *For every graph G there is a constant $n_0(G)$ such that for all $n \geq n_0(G)$ the complete graph K_n admits a G-decomposition if and only if both of the following conditions are satisfied:*
1. *$|E(G)|$ divides $\binom{n}{2}$.*
2. *the greatest common divisor of $\deg_G(v)$ over all $v \in V(G)$ divides $(n-1)$.*

It is easy to see that these conditions are necessary, but it is highly non-trivial that they are sufficient for sufficiently large n. In the proof of Theorem 5.2, Wilson's theorem is used to construct gadgets. Since Wilson's theorem does not guarantee the existence of a G-decomposition hitting each vertex of K_n equally often, a modification of the proof idea of Theorem 5.2 for local coverings might require a stronger version of Wilson's theorem. Therefore, a modification of the proof ideas of Theorem 5.2 or even a proof of Conjecture 5.10 seems out of reach for now. However, we can still prove NP-hardness of the local-G-covering problem for some specific guests G.

5.3 Local Coverings with one Guest

In this section, we show that the local-G-covering problem is NP-hard for some specific graphs G. We show this for regular graphs as well as for stars with at least three edges. This provides some evidence for Conjecture 5.10 as K_3 and $K_{1,3}$ intuitively seem like the easiest connected guests with three edges for the local-G-covering problem and yet the problem is NP-hard. We start with the result for regular graphs. For the proof, we need an elementary graph theory result:

▶ **Lemma 5.12.** *If H is a connected graph on at least 2 vertices, then H contains at least two vertices x_1, x_2 which are not cut-vertices of H, i.e. $H - x_1$ and $H - x_2$ are connected.*

Proof. We proceed by induction on the number of vertices. If $|V(H)| = 2$, then $H \cong K_2$ and thus both vertices of H are not cut-vertices. Now, suppose that $|V(H)| > 2$. If H does not contain a cut-vertex, we are done. Otherwise, let $v \in V(H)$ be a cut vertex and let S_1 and S_2 be the vertex sets of two different connected components of $H - v$. Consider the subgraph $G_i = G[S_i \cup \{v\}]$ for $i \in [2]$. Since $2 \leq |V(G_i)| < |V(G)|$ we can apply induction to obtain vertices $x_i, y_i \in V(G_i)$ which are not cut-vertices of G_i. We have $x_i \neq v$ or $y_i \neq v$, without loss of generality let $x_i \neq v$. Next, we show that $G - x_i$ is connected. Every vertex $u \in V(G) \setminus \{x_i\}$ has a path to v: if $u \in V(G_i)$, we find such a path in $G_i - x_i$ and otherwise we find a path in $G - S$. Thus, $x_i \in S$ is not a cut-vertex of G. Since $x_i \in S_i$ and $S_1 \cap S_2 = \emptyset$, we also have $x_1 \neq x_2$. ◀

▶ **Theorem 5.13.** *For every connected r-regular graph G with at least three edges, the local-G-covering problem is NP-complete.*

Proof. By Lemma 2.6, the local-G-covering problem is clearly in NP. To show NP-hardness, we reduce from the G-decomposition problem which is NP-hard by Theorem 5.2. Now, we describe how we transform a given G-decomposition instance H into a local-G-covering instance (H', k). First of all, we may assume that the degree of each vertex $v \in V(H)$ is a multiple of r. Otherwise, we may reject H immediately as no-instance. By Lemma 5.12, there is a vertex $x \in V(G)$ such that $G - x$ is connected. We set $k = \frac{\Delta(H)}{r}$. To create H', we start from H and for each vertex $v \in V(H)$ we create $\text{copy}(v) = k - \frac{\deg_H(v)}{r}$ copies $G'_{v,1}, \ldots, G'_{v,\text{copy}(v)}$ of G. For each such copy, we identify the vertex corresponding to the non-cut-vertex x with v. Note that $\deg_{H'}(v) = kr = \Delta(H)$ for each $v \in V(H)$. Clearly, H' can be created in polynomial time.

It remains to show that H has a G-decomposition if and only if $c_l^G(H') \leq k$. First of all, a G-decomposition φ of H can be extended trivially to a G-decomposition φ' of H' by adding one guest for each $G'_{v,i}$. For each vertex $v \in V(H)$, we have $\mathrm{hit}_{\varphi'}(v) = \frac{\deg_{H'}(v)}{r} = k$ since φ' is a decomposition and G is r-regular. For each vertex $v \in V(H') \setminus V(H)$ we have $\mathrm{hit}_{\varphi'}(v) = 1$. Thus, φ' is indeed a k-local G-cover of H.

For the other direction, let φ' be a k-local cover of H' with guests $G_1, \ldots, G_t$.

▷ Claim. Every G_i is either a subgraph of H or it is edge-disjoint from H.

Proof. If G_i is not a subgraph of H, then it contains a vertex $u \in V(H') \setminus V(H)$ belonging to some G-copy $G' = G'_{v,j}$ sharing vertex $v \in V(H)$ with H. Let $S = V(G_i) \cap V(G')$ be the set of vertices inside G' hit by guest G_i. We want to show that $S = V(G_i)$ which implies $G_i = G'$. Consider an arbitrary vertex $w \in S \setminus \{v\}$. Since G_i hits w and $\deg_{H'}(w) = \deg_{G'}(w) = r = \deg_{G_i}(w)$, it follows that every edge incident to w is covered by G_i. Thus, $N_{H'}(w) \subseteq S$. Recall that v corresponds to the vertex x of G and thus $G' - v$ is connected. Since $u \in S \setminus \{v\}$, we conclude that $(V(G') \setminus \{v\}) \subseteq S$. Additionally, some neighbor of v is in G_i and thus in S and it follows that $v \in S$ as well. Therefore, $S = V(G')$ and thus $G_i = G'$. It follows that G_i does not cover any edge of H. ◁

Because of the claim, we can restrict φ' to a G-cover φ of H by keeping the guests which are subgraphs of H. For each vertex $v \in V(H)$, we have $\deg_{H'}(v) = kr$. At most k guests of φ' hit v and each of these guests covers r edges incident to v. Therefore, every edge incident v is covered exactly once by φ'. It follows that φ is a G-decomposition of H. ◀

Our next goal is to show that the local-$K_{1,d}$-covering problem is NP-hard for $d \geq 3$. For a G-cover φ of H, it turns out to be useful to consider the total number of hits. For a vertex set $S \subseteq V(H)$, we write $\mathrm{hit}_\varphi(S) = \sum_{v \in S} \mathrm{hit}_\varphi(v)$ and we also write $\mathrm{hit}_\varphi(F) = \mathrm{hit}_\varphi(V(F))$ for a subgraph $F \subseteq H$. We start with the straight-forward observation that covers with minimum total hitcount are decompositions.

▶ **Lemma 5.14.** *For a graph H and a G-cover φ of H, we have*

$$\mathrm{hit}_\varphi(H) \geq |V(G)| \cdot \frac{|E(H)|}{|E(G)|} \tag{$\star$}$$

and equality holds if and only if φ is a G-decomposition.

Proof. Since every guest of φ covers $|E(G)|$ edges, φ consists of at least $\frac{|E(H)|}{|E(G)|}$ guests. The inequality ($\star$) follows since every guest hits $|V(G)|$ vertices. Now, suppose that equality holds in ($\star$). Then, the number of guests in the cover φ is given by $\frac{\mathrm{hit}_\varphi(H)}{|V(G)|} = \frac{|E(H)|}{|E(G)|}$. Thus, all guests of φ have $|E(G)| \cdot \frac{|E(H)|}{|E(G)|} = |E(H)|$ edges in total. Since φ covers every edge of H, it follows that each edge of H is covered exactly once and φ is indeed a G-decomposition of H. For the other direction, suppose that φ is a G-decomposition of H. Since every edge of H is covered exactly once, it follows that φ consists of $\frac{|E(H)|}{|E(G)|}$ guests. As every guest hits $|V(G)|$ vertices, equality follows in ($\star$). ◀

Now, we come to the proof that the local-$K_{1,d}$-covering problem is NP-complete for $d \geq 3$. In principle, the idea of the reduction is similar to the one used in Theorem 5.13. We also transform a graph H into a supergraph H' such that a k-local cover φ' of H' can be restricted

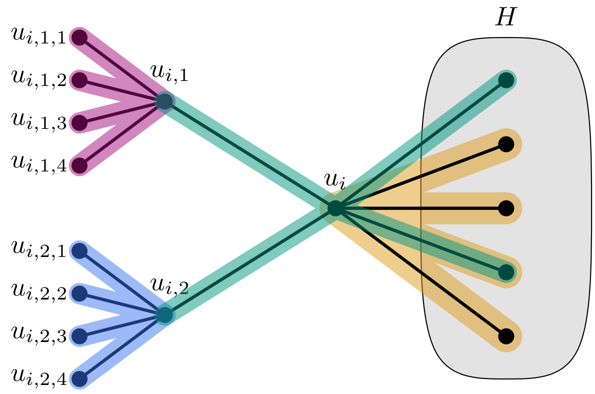


Figure 5.2 Illustration of a delocalization gadget used in the proof of Theorem 5.15 and a $K_{1,4}$ cover of it. One edge incident to a vertex of H is covered twice. In this example, we have $d = 4, k = 2, |V(H)| = 5$. (Technically, this combination of parameters cannot occur in the proof, but they are chosen to keep the figure reasonably small).

to a cover φ of H. Because of the additional edges of H' the vertices of H receive some additional hits ensuring that a φ must be a decomposition. The difference to the reduction used in Theorem 5.13 is however, that the hitcounts with respect to the decomposition φ are not known in advance since our guest $K_{1,d}$ is not regular. Therefore, our idea is to introduce *delocalization* gadgets which allow that the additional hit can be given to an arbitrary vertex of H.

▶ **Theorem 5.15.** *For $d \in \mathbb{N}_{\geq 3}$, the local-$K_{1,d}$-covering problem is NP-complete.*

Proof. By Lemma 2.6, the problem is clearly in NP. For NP-hardness, we reduce from the $K_{1,d}$-decomposition problem which is NP-hard by Theorem 5.2. We describe how we transform a $K_{1,d}$-decomposition instance H into a local-$K_{1,d}$-covering instance (H', k). First of all, we may assume that $|E(H)|$ is a multiple of d. Otherwise, we may reject H as no-instance. The idea of the reduction is that H' consists of a copy of H as well as r copies of a *delocalization gadget*. This delocalization gadget has the property that in every k-local $K_{1,d}$-cover of it, exactly one edge incident to a vertex $v \in V(H)$ is covered twice and there is such a cover for every $v \in V(H)$. By choosing r and k correctly, we achieve that H' has a k-local $K_{1,d}$-cover if and only if H has a $K_{1,d}$-cover with small hitcount. Lemma 5.14 then allows us to conclude that this cover is a $K_{1,d}$-decomposition. As we shall verify later, a good choice of r and k is given by

$$r = |V(H)|^2 - (d+1) \cdot \frac{|E(H)|}{d}$$
$$k = |V(H)| + r.$$

We have $r > 0$ since $|E(H)| \leq \binom{|V(H)|}{2}$ and $\frac{d+1}{d} < 2$.

The delocalization gadgets. See Figure 5.2 for an illustration of the following construction. For each $i \in [r]$ we perform the following construction: We start from a new vertex u_i and attach $p = dk - 1 - |V(H)|$ leaves $u_{i,1}, \dots, u_{i,p}$ to it. Afterwards we attach $q = d(k-1)$ leaves $u_{i,j,1}, \dots, u_{i,j,q}$ to $u_{i,j}$ for every $j \in [p]$. Finally, we add an edge between u_i and every vertex of H. The resulting graph is H'. Note that $p, q \geq 0$ by our choice of r and k. It is clear that this construction can be performed in polynomial time.

Correctness. We need to show that H' has a k-local $K_{1,d}$-cover if and only if H has a $K_{1,d}$-decomposition. First, assume that H' has a k-local $K_{1,d}$-cover φ'.

▷ Claim. The following holds:

- Every guest of φ' is either a subgraph of H or it does not cover any edge of H and
- for every $i \in [r]$ there is exactly one $v \in V(H)$ such that the edge $u_i v$ is covered twice by φ'.

Proof. Let $i \in [r]$. Note that $\deg_{H'}(u_i) = dk - 1$. We observe that u_i cannot be hit by a leaf of a guest from φ'. Otherwise, the other $dk - 2$ edges incident to u_i would be covered by at most $k - 1$ guests which is not possible since $d \geq 3$. Thus, u_i is only hit by centers of guests and it is hit k times. This has two important consequences. First of all, if a guest G covers an edges inside H, then it cannot hit u_i. This proves the first part of the claim. Second, exactly one edge incident to u_i is covered by two guests since the guests centered at u_i have dk edges in total and $dk - 1$ edges are incident to u_i. We need to show that the edge which is covered twice is incident to a vertex of H. Suppose for contradiction that the edge $u_i u_{i,j}$ is covered by two guests for some $j \in [p]$. The edges to the $q = d(k-1)$ neighbors $u_{i,j,1}, \ldots, u_{i,j,q}$ of $u_{i,j}$ would thus be covered by at most $k - 2$ guests, a contradiction. ◁

By the first part of the claim, we can restrict φ' to a cover φ of H by only keeping guests which are subgraphs of H. By the second part of the claim, the guests of φ' centered at u_i have a contribution of $|V(H)| + 1$ to $\mathrm{hit}_{\varphi'}(S)$ for each $i \in [r]$. Thus, in total the vertices of H are hit $r \cdot (|V(H)| + 1)$ times less by φ than by φ'. Since in φ', the vertices of H are hit at most $k \cdot |V(H)|$ times in total, we obtain

$$\begin{aligned}
\mathrm{hit}_{\varphi}(H) &= \mathrm{hit}_{\varphi'}(H) - r \cdot (|V(H)| + 1) \\
&\leq k \cdot |V(H)| - r \cdot (|V(H)| + 1) \\
&= (|V(H)| + r) \cdot |V(H)| - r \cdot |V(H)| - r \\
&= |V(H)|^2 - r \\
&= (d+1) \cdot \frac{|E(H)|}{d}
\end{aligned}$$

Since $|V(K_{1,d})| = d + 1$ and $|E(K_{1,d})| = d$, we can use Lemma 5.14 to obtain that φ is a $K_{1,d}$-decomposition of H.

For the other direction, assume that H has a $K_{1,d}$-decomposition φ. We extend φ to a k-local $K_{1,d}$-cover φ' of H': For each $i \in [r], j \in [p]$, we decompose the $d(k-1)$ edges $u_{i,j}u_{i,j,1}, \ldots, u_{i,j}u_{i,j,q}$ into $(k-1)$ guests centered at $u_{i,j}$. Before we can describe the covering of the edges incident to u_i, we need to look at the hitcounts. Since φ is a decomposition, Lemma 5.14 gives us $\mathrm{hit}_{\varphi}(H) = (d+1)\frac{|E(H)|}{d}$. Also, it is clear that $\mathrm{hit}_{\varphi}(v) \leq \deg_H(v) \leq |V(H)|$. Therefore, the integer $h_v = |V(H)| - \mathrm{hit}_{\varphi}(v)$ is non-negative and $\sum_{v \in V(H)} h_v = |V(H)|^2 - \mathrm{hit}_{\varphi}(H) = r$. Thus, we can pick a $v_i \in V(H)$ for each $i \in [r]$ such that every $v \in V(H)$ is picked h_v times. Now, we can describe how to cover the $dk - 1$ edges incident to u_i: we cover them with k guests centered at u_i such that we cover the edge $u_i v_i$ twice and the other incident edges once. Note that it is possible because $k \geq 2$. Finally, we verify that φ' is k-local. For each $i \in [t], j \in [p], \ell \in [q]$ we clearly have $\mathrm{hit}_{\varphi'}(u_{i,j,\ell}) = 1$, $\mathrm{hit}_{\varphi'}(u_{i,j}) = k$ and $\mathrm{hit}_{\varphi'}(u_i) = k$. For each $v \in V(H)$, we have $\mathrm{hit}_{\varphi'}(v) = \mathrm{hit}_{\varphi}(v) + r + h_v = |V(H)| + r = k$. Thus, φ' is indeed a k-local $K_{1,d}$-cover of H'. ◀

In this section, we have seen some evidence for the NP-hardness claims of Conjecture 5.10. In particular, we have seen that the local-$K_{1,d}$-covering problem is NP-hard for $d \geq 3$. Since $d = 1$ is trivial, only $d = 2$ is open. We conjecture that there is a polynomial time algorithm for $d = 2$ as stated in the first part of Conjecture 5.10. This problem turns out to be surprisingly difficult and we do not have a solution for it.

5.4 Local Hard, Global Easy

In this section we give a counterexample to the commonly observed pattern that the local-$\mathcal{G}$-covering problem is computationally easier than global-$\mathcal{G}$-covering problem. Stumpf [28, section 6.2] provided a general template for such examples when the host class is restricted. Now, we provide an example of a natural monotone guest class $\mathcal{G}$ for which the global-$\mathcal{G}$-covering problem can be decided in constant time while the local-$\mathcal{G}$-covering problem is NP-hard.

▶ **Theorem 5.16.** *Let $\mathcal{E}_3$ be the guest class of all graphs with at most three edges. The following holds:*

- $c_g^{\mathcal{E}_3}(H) = \left\lceil \frac{|E(H)|}{3} \right\rceil$ *for all host graphs H.*
- *The local-$\mathcal{E}_3$-covering problem is NP-complete.*

Proof. The first claim is trivial and a special case of Proposition 5.4.

For the second claim, we reduce from the K_3-decomposition problem problem which is NP-hard by Theorem 5.2. We describe how to transform an instance H of the K_3-decomposition problem into an instance of the local-$\mathcal{E}_3$-covering problem. First of all, we may assume that each vertex of H has even degree, as otherwise H is a trivial no-instance. Set $k = \frac{\Delta(H)}{2}$ and transform H into the graph H' by attaching $3 \cdot (\Delta(H) - \deg_H(v))/2$ leaves to each vertex $v \in V(H)$. It remains to show that $c_l^{\mathcal{E}_3}(H') \leq k$ if and only if H has a K_3-decomposition.

Suppose that H has a K_3-decomposition φ. It can be extended to an $\mathcal{E}_3$-cover φ' of H' by covering the additional edges by stars with three leaves. We have that φ' is k-local because each vertex $v \in V(H)$ is hit by $(\Delta(H) - \deg_H(v))/2$ stars and by $\deg_H(v)/2$ triangles and thus $\text{hit}_{\varphi'}(v) = \Delta(H)/2 = k$. Vertices of $V(H') \setminus V(H)$ are hit exactly once by φ'.

Next, we come to the harder direction of proof. Suppose that H' has a k-local $\mathcal{E}_3$-cover φ' with guests $G_1, \ldots, G_t$. We may assume that φ' is a decomposition because $\mathcal{E}_3$ is a monotone graph class. We may also assume that each guest of φ' covers at least one edge since guests without edges can be removed. Our goal is to show that all guests are either copies of K_3 or $K_{1,3}$ with the three leaves of $K_{1,3}$ not belonging to H. To every guest G we assign a *cost* $c(G)$ and a *weight* $w(G)$. The intuition is that we pay the cost $c(G)$ for using the guest G while its weight describes how "useful" it is for covering H. Later, we show that the only guests with optimal weight-to-cost-ratio are copies of K_3 or $K_{1,3}$ with the three leaves of $K_{1,3}$ not belonging to H. We then conclude that these are the only possible types of guests.

For a guest G, we define its cost as the number of vertices of H hit by it, i.e. $c(G) = |V(G) \cap V(H)|$. For an edge $e \in E(H')$, define its weight as $w(e) = 1$ if $e \in E(H)$ and $w(e) = \frac{1}{3}$ otherwise. For a guest G, we define its weight as the sum of the weights of its edges, i.e. $w(G) = \sum_{e \in E(G)} w(e)$. A simple calculation shows that

$$\begin{aligned}
\sum_{e \in E(H')} w(e) &= |E(H)| + \frac{1}{3}\left(\sum_{v \in V(H)} 3 \cdot \frac{\Delta(H) - \deg_H(v)}{2}\right) \\
&= \sum_{v \in V(H)} \frac{\deg_H(v)}{2} + \sum_{v \in V(H)} k - \frac{\deg_H(v)}{2} \\
&= k \cdot |V(H)|. \qquad (\star)
\end{aligned}$$

Next, we show that for every guest G we have $w(G) \leq c(G)$ with equality holding if and only if G is a K_3 or $K_{1,3}$ with the three leaves of $K_{1,3}$ not belonging to H. We show this by case distinction over the cost of G, i.e. number of vertices of G which belong to H:

- If $c(G) = 0$, then G does not cover any edge of H' since every edge of H' has an endpoint in H. This contradicts the assumption that every guest of φ' covers some edge of H'.
- If $c(G) = 1$, then G contains only one vertex of H and thus no edge of H. Thus, all edges of G have weight $\frac{1}{3}$. Since $|E(G)| \leq 3$ this implies $w(G) \leq 1 = c(G)$ with equality if and only if G is a copy of $K_{1,3}$ with the three leaves not belonging to H.
- If $c(G) = 2$, then G contains two vertices of H and thus at most one edge of H. This implies $w(G) \leq 1 + \frac{2}{3} < 2 = c(G)$.
- If $c(G) \geq 3$, we have $w(G) \leq c(G)$ since $w(G) \leq 3$. Equality holds if and only if $c(G) = w(G) = 3$. In this case, G is a K_3 since this is the only graph on three vertices with three edges.

We observe that $\sum_{i=1}^{t} c(G_i) \leq k \cdot |V(H)|$ since every vertex of H must be hit at most k times. Because of $(\star)$ we also have that $\sum_{i=1}^{t} w(G_i) = k \cdot |V(H)|$. Together with $w(G) \leq c(G)$ this implies that $c(G_i) = w(G_i)$ for all $i \in [t]$. In particular, all guests G_i are either copies of K_3 or $K_{1,3}$ with the three leaves of $K_{1,3}$ not belonging to H. Thus, the G_i which are copies of K_3 form a K_3-decomposition of H. ◀

6 Minor Closed Guest Classes

A recent result of Lee, Liu and Tsai [17] shows that the global-$\mathcal{G}$-covering problem is NP-hard for many natural guest classes $\mathcal{G}$. In particular, their result applies to the guest class of planar graphs, the class of outer-planar graphs, the class of cacti and the class of all graphs with treewidth at most t where $t \geq 2$. Their result requires that the guest class $\mathcal{G}$ satisfies the following three conditions:

($\mathcal{G}$1) $\mathcal{G}$ is closed under topological minors.

($\mathcal{G}$2) $\mathcal{G}$ is closed under 1-sums.

($\mathcal{G}$3) $C_3 \in \mathcal{G}$.

Recall that $\mathcal{G}$ is closed under 1-sums if for $G_1, G_2 \in \mathcal{G}$ and G' obtained from $G_1 \uplus G_2$ by identifying a vertex of G_1 and G_2, we have $G' \in \mathcal{G}$ as well.

▶ **Theorem 6.1** (Lee, Liu and Tsai [17, Theorem 3]). *For every proper graph class $\mathcal{G}$ whose membership is decidable and that satisfies properties ($\mathcal{G}$1)-($\mathcal{G}$3), the k-global-$\mathcal{G}$-covering problem is NP-hard for every fixed integer $k \geq 3$.*

The goal of this chapter is to generalize this result to the k-local-$\mathcal{G}$-covering problem. Our generalization yields the following:

▶ **Theorem 6.2.** *The k-local-$\mathcal{G}$-covering problem is NP-hard for every fixed $k \geq 3$ for the following guest classes $\mathcal{G}$:*

- *planar graphs,*
- *outer-planar graphs,*
- *cacti and*
- *graphs of treewidth at most t where $t \geq 2$ is fixed.*

6.1 General Idea

The proof of Theorem 6.1 given in [17] reduces the k-regular-edge-coloring problem to the k-global-$\mathcal{G}$-covering problem. A k-edge-coloring of a graph G is a mapping $c\colon E(G) \to [k]$ such that $c(e) \neq c(f)$ for every pair of distinct edges $e, f \in E(G)$ sharing an endpoint. The k-regular-edge-coloring problem asks if a given k-regular graph G has a k-edge-coloring. By [18], this problem is NP-hard for every fixed $k \geq 3$.

The reduction given in [17] starts from an instance G of the k-regular-edge-coloring problem and transforms it into an instance G' of the k-global-$\mathcal{G}$-covering problem. The reduction relies on the existence of a fixed auxiliary graph H which only depends on $\mathcal{G}$ and k and which satisfies both of the following properties:

(H1) H is edge-maximal with $\mathrm{c}_g^{\mathcal{G}}(H) = k$, i.e. $\mathrm{c}_g^{\mathcal{G}}(H + e) > k$ for every $e \in \binom{V(H)}{2} \setminus E(H)$.

(H2) H has an independent set $w_1, \dots, w_\alpha$ where $\alpha = \alpha(k)$ is a constant depending on k.

In this case, we shall also call H an *auxiliary graph with respect to k and $\mathcal{G}$*.

To create the graph $G' = \mathrm{comb}(G, H)$, they start from the disjoint union of G and H and then they add some edges between G and H. The exact details of the reduction are not necessary here[1], so we only summarize the properties which are important for us:

▶ **Lemma 6.3** ([17]). *Let $k \geq 3$ be an integer and let $\mathcal{G}$ be a guest class satisfying properties ($\mathcal{G}1$)-($\mathcal{G}3$). Further, let H be an auxiliary graph with respect to k and $\mathcal{G}$. For every k-regular graph G, we can build a graph $G' = \text{comb}(G, H)$ by starting from $G \cup H$ and adding some additional edges between G and H such that all of the following properties hold:*

(G'1) *Each vertex $v \in V(G)$ is adjacent to exactly k vertices $w_{v,1}, \dots, w_{v,k}$ from the independent set $w_1, \dots, w_\alpha$ of H.*

(G'2) *G' can be computed in polynomial time (with respect to the size of G).*

(G'3) *$c_g^{\mathcal{G}}(G') \leq k$ if and only if G admits a k-edge-coloring.*

Proof. Property (G'1) follows from the construction of G' and (G'2) follows from the construction as well as [17, Proposition 11] and (G'3) is proved in [17, Lemma 16]. ◀

It turns out that an auxiliary graph H with respect to k and $\mathcal{G}$ always exists under the constraints on $\mathcal{G}$:

▶ **Lemma 6.4** ([17, Lemma 13]). *Let $k \geq 3$ be an integer and let $\mathcal{G}$ be a guest class satisfying properties ($\mathcal{G}1$)-($\mathcal{G}3$). There exists a constant C such that every edge-maximal graph H with $c_g^{\mathcal{G}}(H) \leq k$ on at least C vertices is an auxiliary graph with respect to k and $\mathcal{G}$.*

Therefore, the reduction can be performed for every $\mathcal{G}$ satisfying the properties ($\mathcal{G}1$)-($\mathcal{G}3$) and Theorem 6.1 follows.

Note that the graph H can be chosen arbitrarily as long as it satisfies the properties (H1) and (H2). To apply the reduction to local coverings we make use of this by choosing H more carefully. In particular, we require an additional property from H:

(H3) Each k-local $\mathcal{G}$-cover of H is also k-global and it hits each vertex of H exactly k times.

If H satisfies properties (H1)-(H3), then we also call H a *local auxiliary graph with respect to k and $\mathcal{G}$*.

▶ **Lemma 6.5.** *Let $k \geq 3$ be an integer and let $\mathcal{G}$ be a guest class satisfying properties ($\mathcal{G}1$)-($\mathcal{G}3$). Further, let H be a local auxiliary graph with respect to k and $\mathcal{G}$. Let G be a k-regular graph and construct $G' = \text{comb}(G, H)$. In this setting, every k-local $\mathcal{G}$-cover of G' is also k-global. In particular, the local-$\mathcal{G}$-covering problem is NP-hard.*

Proof. Let φ be a k-local $\mathcal{G}$-cover of G'. Note $\mathcal{G}$ is induced-hereditary because it is closed under topological-minors. Thus, we can consider the restriction of φ to H. By property (H3), this restriction $\varphi|_H$ contains exactly k guests $H_1, \dots, H_k$ each hitting every vertex of H. Next, consider the guests $G_1, \dots, G_k$ of φ which restrict to $H_1, \dots, H_k$ in $\varphi|_H$. Recall that by (G'1), every vertex $v \in V(G)$ is adjacent to exactly k vertices $w_{v,1}, \dots, w_{v,k}$ from the independent set $w_1, \dots, w_\alpha$ of H. Each of the edges $vw_{v,j}$ is covered by one of the guests $G_1, \dots, G_k$ since the vertex $w_{v,j}$ is already hit k times by these guests. We want to show that no G_i can cover two of the edges $vw_{v,1}, \dots, vw_{v,k}$ (see also [17, Claim 18]). Assume for contradiction that G_i covers two of these edges, say $vw_{v,1}$ and $vw_{v,2}$. Consider again the restriction H_i of G_i to H. Since $\mathcal{G}$ is closed under taking topological minors by ($\mathcal{G}1$), it follows that $H_i' = H_i + w_{v,1}w_{v,2} \in \mathcal{G}$. Thus, $c_l^{\mathcal{G}}(H + w_{v,1}w_{v,2}) \leq k$ contradicting the maximality property (H1) of H. So indeed, the edges $vw_{v,1}, \dots, vw_{v,k}$ are covered by different guests.

[1] The details are described in [17] at the beginning of sections 3 and 4 as well as in the text above Lemma 16.

Therefore, every $v \in V(G)$ is hit by all of $G_1, \dots, G_k$ and it follows that φ does not contain further guests, i.e. φ is k-global.

To see that the local-$\mathcal{G}$-covering problem is NP-hard, we reduce from the NP-hard problem k-regular-edge-coloring [18]. We transform a k-regular-edge-coloring instance G into the k-local-$\mathcal{G}$-covering instance $G' = \mathrm{comb}(G, H)$. By the above, we obtain that G' has a k-local $\mathcal{G}$-cover if and only if it has a k-global $\mathcal{G}$-cover. Lemma 6.3 states that G' has a k-global $\mathcal{G}$-cover if and only if G has a k-edge-coloring. In total, G' has a k-local $\mathcal{G}$-cover if G has a k-edge-coloring. ◀

Note that by additionally requiring property (H3), the existence of the local auxiliary graph H with respect to k and $\mathcal{G}$ becomes less trivial. In the next section, we show that such H exists if we additionally require that $\mathcal{G}$ has maximal graphs with small maximum degree:

($\mathcal{G}$4) For $n \in \mathbb{N}$, there is a graph $G_n \in \mathcal{G}$ such that $|E(G_n)| = \mathrm{e}_n(\mathcal{G})$ and $\Delta(G_n) \in o(\sqrt{n})$. Here, $\mathrm{e}_n(\mathcal{G}) = \max\{|E(G)| \mid G \in \mathcal{G}, |V(G)| = n\}$ is the maximum number of edges of a graph from $\mathcal{G}$ on n vertices.

As we shall see, this property is satisfied for the guest classes of planar graphs, outer-planar graphs, cacti and graphs of bounded treewidth.

▶ **Theorem 6.6.** *For every proper graph class $\mathcal{G}$ satisfying conditions ($\mathcal{G}$1)-($\mathcal{G}$4), the k-local-$\mathcal{G}$-covering problem is NP-hard for every fixed integer $k \geq 3$.*

6.2 Constructing a local auxiliary graph

To explain the idea behind the construction of the local auxiliary graph H, consider the guest class of planar graphs as a concrete example. The idea is to construct H as the union of k edge-disjoint maximal planar graphs on the same vertex set. Since a maximal planar graph on $n \geq 3$ vertices has $3n - 6$ edges, its density is $\frac{3n-6}{n} = 3 - \frac{6}{n}$, which is increasing in n. By a simple counting argument (see Lemma 6.7), we see that every guest of a k-local cover of H must hit all vertices of H since smaller guests would have smaller density. So it only remains to find an embedding of k edge-disjoint maximal planar graphs into K_n for sufficiently large n. It is known in the literature that this is possible for $n = 6k$ [1]. However, this construction is hard to generalize and thus we shall use a more general approach for embedding edge-disjoint maximal graphs.

Now, we describe the details of the construction of the local auxiliary graph H. For a graph class $\mathcal{G}$, denote by $\mathrm{e}_n(\mathcal{G}) = \max\{|E(G)| \mid G \in \mathcal{G}, |V(G)| = n\}$ the maximum number of edges of a graph from $\mathcal{G}$ on n vertices. In the case that $\mathcal{G}$ does not contain a graph on n vertices, set $\mathrm{e}_n(\mathcal{G}) = 0$. The maximum density of a graph from $\mathcal{G}$ with n vertices is thus given by $\frac{\mathrm{e}_n(\mathcal{G})}{n}$. If this density is larger than the densities of graphs with less vertices, we can apply the following counting argument.

▶ **Lemma 6.7.** *Let $\mathcal{G}$ be a guest class, $k \in \mathbb{N}$ and H be a host graph on n vertices with $|E(H)| = k\,\mathrm{e}_n(\mathcal{G})$. Additionally, we require that $\frac{\mathrm{e}_m(\mathcal{G})}{m} < \frac{\mathrm{e}_n(\mathcal{G})}{n}$ for all $m < n$. If φ is a k-local $\mathcal{G}$-cover of H, then φ is also k-global and it hits each vertex of H exactly k-times.*

Proof. Let $d_m = \frac{e_m(\mathcal{G})}{m}$ for $m \in \mathbb{N}$. Let φ be k-local $\mathcal{G}$-cover of H with guests $G_1, \dots, G_t$. It is sufficient to show that each G_i has n vertices. Assume for contradiction that some G_j has less than n vertices. By using $d_{|V(G_j)|} < d_n$, we obtain

$$|E(H)| \leq \sum_{i=1}^{t} |E(G_i)| = \sum_{i=1}^{t} |V(G_i)| \frac{|E(G_i)|}{|V(G_i)|} \leq \sum_{i=1}^{t} |V(G_i)| \cdot d_{|V(G_i)|}$$
$$< \sum_{i=1}^{t} |V(G_i)| \cdot d_n = \mathrm{hit}_\varphi(H) \cdot d_n \leq kn \cdot d_n = k \cdot e_n(G)$$

which contradicts $|E(H)| = k \cdot e_n(\mathcal{G})$. ◀

To apply Lemma 6.7, we need an $n \in \mathbb{N}$ such that $\frac{e_m(\mathcal{G})}{m} < \frac{e_n(\mathcal{G})}{n}$ for all $m < n$. It is also important that we can find a sufficiently large n with this property.

▶ **Lemma 6.8.** *Let $\mathcal{G}$ be a graph class. If $\mathcal{G}$ is closed under 1-sums, then there are infinitely many integers $n \in \mathbb{N}$ satisfying $\frac{e_n(\mathcal{G})}{n} > \frac{e_m(\mathcal{G})}{m}$ for all $m < n$.*

Proof. Write $d_m = \frac{e_m(\mathcal{G})}{m}$ for $m \in \mathbb{N}$. Assume for contradiction that there is a maximum integer n with the property $d_n > d_m$ for all $m < n$. In particular, this implies $d_n \geq d_m$ for all $m \in \mathbb{N}$. Let G be a graph on n vertices with $e_n(\mathcal{G})$ edges. Consider a graph $G' \in \mathcal{G}$ obtained as a 1-sum of G with itself. We have $|E(G')| = 2 \cdot e_n(\mathcal{G})$ and $|V(G')| = 2n - 1$ and therefore $d_{2n-1} \geq \frac{2 \cdot e_n(\mathcal{G})}{2n-1} > d_n$ contradicting the assumption. ◀

Finally, we need to construct a graph H with $|E(H)| = k \cdot e_n(\mathcal{G})$. We use a result from the theory of graph-packing. A collection of graphs $G_1, \dots, G_t$ on n vertices is said to *pack* if there are embeddings $\sigma_i \colon G \to K_n$ such that the graphs $\sigma_i(G_i)$ and $\sigma_j(G_j)$ are edge-disjoint for $i \neq j$.

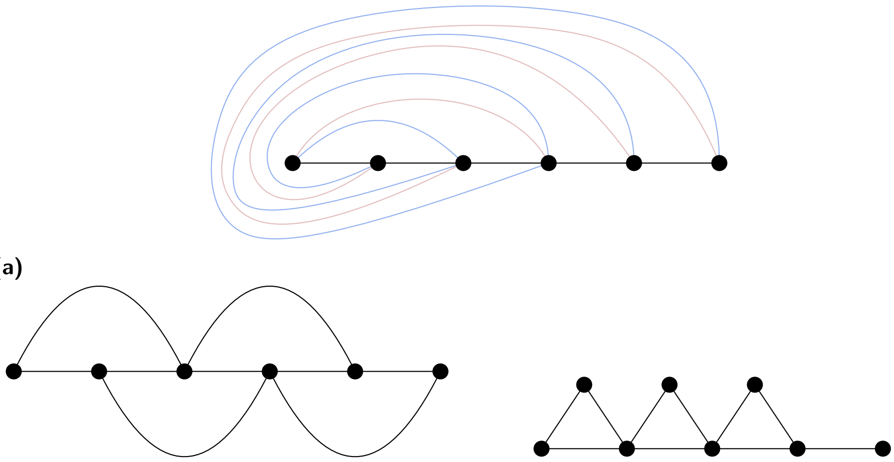
▶ **Lemma 6.9** ([25, Theorem 3])**.** *Let G_1 and G_2 be graphs on n vertices. If $2\Delta(G_1)\Delta(G_2) < n$, then G_1 and G_2 pack.*

▶ **Theorem 6.6.** *For every proper graph class $\mathcal{G}$ satisfying conditions ($\mathcal{G}$1)-($\mathcal{G}$4), the k-local-$\mathcal{G}$-covering problem is NP-hard for every fixed integer $k \geq 3$.*

Proof. Our goal is to construct a local auxiliary graph H with respect to $\mathcal{G}$ and k. Recall from Lemma 6.4 that every edge-maximal graph H with $c_g^{\mathcal{G}}(H) \leq k$ on sufficiently many vertices $|V(H)| \geq C$ satisfies (H1) and (H2). By property ($\mathcal{G}$4), there is a graph G_n for every $n \in \mathbb{N}$ such that $|E(G_n)| = e_n(\mathcal{G})$ and $\Delta(G_n) \in o(\sqrt{n})$. We can choose a sufficiently large $n_0 \geq C$ such that $2k(\Delta(G_n))^2 < n$ for all $n \geq n_0$. By Lemma 6.8, we can choose $n \geq n_0$ such that $\frac{e_n(\mathcal{G})}{n} > \frac{e_m(\mathcal{G})}{m}$ for all $m < n$.

By iteratively applying Lemma 6.9, we can pack k copies of G_n: Start from $H_1 = G_n$ and obtain H_{i+1} from H_i by packing it with another copy of G_n. For $i \leq k$, we have $\Delta(H_i) \leq i \cdot \Delta(G_n) \leq k \cdot \Delta(G_n)$ and $2k(\Delta(G_n))^2 < n$. Therefore, such packing is possible by Lemma 6.9 and we obtain the graph $H = H_k$ with $|E(H)| = k \cdot e_n(\mathcal{G})$. By Lemma 6.7, every k-local $\mathcal{G}$-cover of H is k-global and it hits each vertex of H exactly k times, i.e. H satisfies property (H3). The maximality property (H1) is satisfied by construction and property (H2) is satisfied by Lemma 6.4 since $n \geq C$ is sufficiently large.

Therefore, H is indeed a local auxiliary graph with respect to $\mathcal{G}$ and k. By Lemma 6.5, we obtain that the local-$\mathcal{G}$-covering problem is NP-hard. ◀



(a) (b) (c)

Figure 6.1 Illustrations of the edge-maximum graphs used in the proof of Theorem 6.2. (a) shows a planar embedding of P_n^3. Blue edges connect vertices with distance 2 in the original P_n while red edges connect vertices of original distance 3. (b) shows an outer-planar embedding of P_n^2. All of its vertices are on the infinitely large outer face. (c) shows a cactus on 8 vertices with the maximum possible number of edges.

Next, we apply Theorem 6.6 to prove the NP-hardness of some local covering problems.

▶ **Theorem 6.2.** *The k-local-$\mathcal{G}$-covering problem is NP-hard for every fixed $k \geq 3$ for the following guest classes $\mathcal{G}$:*

- *planar graphs,*
- *outer-planar graphs,*
- *cacti and*
- *graphs of treewidth at most t where $t \geq 2$ is fixed.*

Proof. It is well-known (and remarked in [17]) that the graph classes considered in this theorem satisfy the properties ($\mathcal{G}1$)-($\mathcal{G}3$). Thus, it only remains to verify property ($\mathcal{G}4$).

Planar Graphs: Every planar graph on $n \geq 3$ vertices has at most $3n-6$ edges. For $n \geq 3$, consider the path power $H_n = P_n^3$. It is easy to see that $|E(H_n)| = 3n-6$ and $\Delta(H_n) \leq 6$ and that H_n is planar, see Figure 6.1a. Since $6 \in o(\sqrt{n})$, this implies ($\mathcal{G}4$).

Outer-Planar Graphs: Every outer-planar graph on $n \geq 2$ vertices has at most $2n-3$ edges. For $n \geq 2$, consider the path power $H_n = P_n^2$. It is easy to see that $|E(H_n)| = 2n-3$ and $\Delta(H_n) \leq 4$ and that H_n is outer-planar, see Figure 6.1b. Since $4 \in o(\sqrt{n})$, this implies ($\mathcal{G}4$).

Cacti: A cactus on n vertices with the maximum number of edges can be obtained as the 1-sum of $\left\lfloor \frac{n-1}{2} \right\rfloor$ cycles of length 3 and a single K_2 if n is even. The resulting graph has $\left\lfloor \frac{3(n-1)}{2} \right\rfloor$ edges. The vertices identified in the 1-sums can be chosen such that the maximum degree of the resulting cactus is at most 4, see Figure 6.1c. Since $4 \in o(\sqrt{n})$, this implies ($\mathcal{G}4$).

Treewidth: Let $t \geq 2$ be a fixed integer. Consider the path power $H_n = P_n^t$. Note that H_n is a t-tree. Thus, H_n contains the maximum number of edges possible for an n-vertex graph of treewidth at most t. Since $\Delta(H_n) \leq 2t \in o(\sqrt{n})$, this implies ($\mathcal{G}4$). ◀

7 Partial Coverings

	biclique $\mathcal{CB}$	clique $\mathcal{K}$
global	$\langle 7.10 \rangle$, [21]	$\langle 7.12 \rangle$, [21]
local	$\langle 7.1 \rangle$	$\langle 7.17 \rangle$, [23][1]

(a) in P NP-hard for k given in the input NP-hard for some fixed k Unknown

Table 7.1 Overview of the complexity results obtained in this chapter. The cells correspond to the global-$\mathcal{G}$-covering and local-$\mathcal{G}$-covering problem where $\mathcal{G}$ depends on the column. Numbers $\langle X \rangle$ refer to the corresponding Theorem X in the thesis.

The main goal of this chapter is to prove that for the guest class $\mathcal{CB}$ of bicliques the local-$\mathcal{CB}$-covering problem is NP-hard which provides the answer to a question asked in [6].

▶ **Theorem 7.1.** *The local-$\mathcal{CB}$-covering problem is NP-complete.*

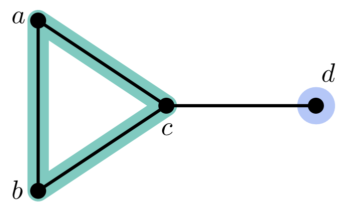
In the literature, some related results are known. Orlin [21] proved the NP-hardness of the global-$\mathcal{CB}$-covering problem and the related global-$\mathcal{K}$-covering problem where $\mathcal{K}$ is the class of all complete graphs. The NP-hardness of the local-$\mathcal{K}$-covering problem is proved in [23]. More precisely, they show that the k-local-$\mathcal{K}$-covering problem is NP-hard for $k \geq 4$ while it admits a polynomial time solution for $k = 2$. They also notice that the k-local-$\mathcal{K}$-covering problem is equivalent to deciding whether the host graph H is the intersection graph of a set-family $\mathcal{F}$ where each set from $\mathcal{F}$ has size at most k.

▶ **Theorem 7.2** ([23, Theorem 2.3])**.** *The 2-local-$\mathcal{K}$-covering problem has a polynomial time solution and the k-local-$\mathcal{K}$-covering problem is NP-hard for every $k \geq 4$.*

Proving the NP-hardness of the local-$\mathcal{CB}$-covering problem turns out to be surprisingly difficult. The essence of our proof idea is essentially the same as in Orlin's proof [21] for the NP-hardness of the global-$\mathcal{CB}$-covering problem. However, a lot more technical details and some additional ideas are required. We start by introducing the notion of a *partial cover*. After this, we show that partial covering problems can be reduced to ordinary covering problems if gadgets of a certain type can be constructed for the considered guest class. The final step is the construction of these gadgets. Before actually considering the local-$\mathcal{CB}$-covering problem, we give a proof of the NP-hardness of the global-$\mathcal{CB}$-covering problem since this explains the most important ideas in an easier setting. As a byproduct we also obtain a more general reduction framework which also applies to other guest classes as well. See Table 7.1 for an overview of the complexity results obtained in this chapter.

Now, we introduce the notion of a *partial cover*. Let $\mathcal{G}$ be a guest class, H be a host graph and $S \subseteq E(H)$ be a subset of its edges. While a $\mathcal{G}$-cover of H has to cover all edges of

[1] They show the stronger result that the k-local-$\mathcal{K}$-covering problem is NP-hard for every fixed $k \geq 4$.



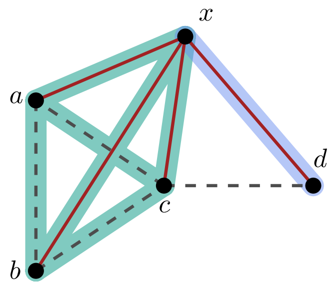


Figure 7.1 Illustration of the reduction described in Lemma 7.3. (a) shows a graph G with clique-vertex-cover number $\kappa(G) = 2$. (b) shows the corresponding graph $H = G \vee x$ with global and local S-partial $\mathcal{K}$-covering number 2. Edges from S are red and optional edges are dashed. The clique $\{a, b, c\}$ of G corresponds to the clique $\{a, b, c, x\}$ of H.

H, an *S-partial $\mathcal{G}$-cover* of H only has to cover edges from S. More formally, an S-partial $\mathcal{G}$-cover φ of H is a collection of graphs $G_1, \ldots, G_t$ such that the following holds: $G_i \in \mathcal{G}$ and $G_i \subseteq H$ for every $i \in [t]$ and for every $e \in S$ there is an $i \in [t]$ with $e \in E(G_i)$. We say that the edges of $E(H) \setminus S$ are *optional*. The definition of a k-global (k-local) S-partial cover is similar as for an ordinary k-global (k-local) cover.

We are interested in the computational problem of deciding whether a host graph H admits a k-global (k-local) S-partial $\mathcal{G}$-cover. We refer to this problem as the partial-global-$\mathcal{G}$-covering (partial-local-$\mathcal{G}$-covering) problem. Note that H, k and S are given in the input.

The main idea is that it is often easier to show that such a partial covering problem is NP-hard. We then reduce this problem to an ordinary covering problem by attaching a gadget to every optional edge $e \in E(H) \setminus S$. This gadget ensures that a valid cover contains a guest G such that e is covered by G, but no other edge of H is covered G.

7.1 NP-hardness results for partial cover problems

In this section, we show that the partial covering problems are NP-hard for the guest classes of cliques and bicliques. We reduce from the clique-vertex-covering problem of deciding whether $\kappa(H) \leq k$ for given H and k. Recall that $\kappa(H)$ is the smallest number of cliques needed to hit all vertices of H, i.e. $\kappa(H)$ is the minimum t such that there are cliques $A_1, \ldots, A_t$ of H with $V(H) = A_1 \cup \cdots \cup A_t$. We start with the guest class $\mathcal{K}$ of complete graphs.

▶ **Lemma 7.3.** *Let G be a graph and $H = G \vee x$ be obtained by joining a universal vertex x to G. Let $S = \{xv \mid v \in V(G)\}$ be the set of edges incident to the universal vertex x and let $k \in \mathbb{N}$ be a positive integer. The following statements are equivalent:*

1. *$\kappa(G) \leq k$*
2. *H has a k-local S-partial $\mathcal{K}$-cover*
3. *H has a k-global S-partial $\mathcal{K}$-cover*

Proof. The construction is illustrated in Figure 7.1.

We first show that (1) implies (2): Let $V(G) = A_1 \cup \cdots \cup A_k$ be a clique-vertex-cover of G. Adding x to each of these cliques yields k cliques $B_1, \ldots, B_k$ where $B_i = A_i \cup \{x\}$. These cliques form an S-partial $\mathcal{K}$-cover φ: for every edge $xv \in S$ there is some $i \in [k]$ with $v \in A_i$ and thus xv is covered by the clique B_i. Clearly, φ is k-global and thus k-local.

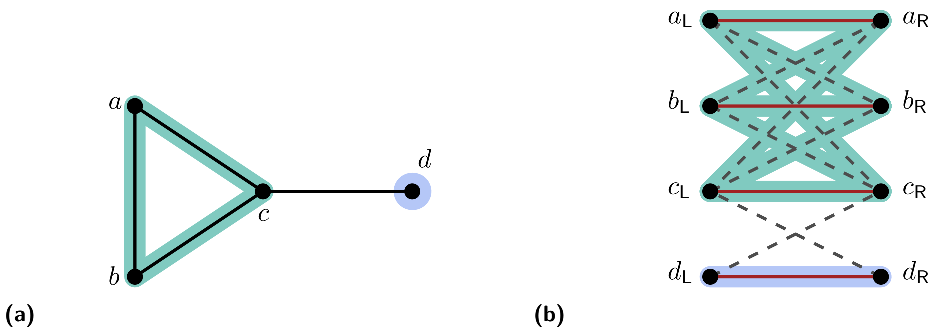


Figure 7.2 Illustration of the reduction described in the proof of Lemma 7.5. (a) shows a graph G with vertex-clique-cover number $\kappa(G) \leq 2$. (b) shows the corresponding graph H with global S-partial $\mathcal{CB}$-covering number 2. Edges from S are red and optional edges are dashed. The clique $\{a, b, c\}$ of G corresponds to the biclique of H which has parts $\{a_\mathsf{L}, b_\mathsf{L}, c_\mathsf{L}\}$ and $\{a_\mathsf{R}, b_\mathsf{R}, c_\mathsf{R}\}$.

Next we show that (2) implies (3): Remove all guests from the local cover which do not contain x. At most k guests remain and they cover S since all edges of S are incident to x.

Finally, we show that (3) implies (1): Let φ be a k-global S-partial $\mathcal{K}$-cover of H. Take all guests $G_1, \ldots, G_s$ in φ hitting x and remove x from them. As a result, we obtain cliques $A_1, \ldots, A_s$ where $A_i = V(G_i) \setminus \{x\}$. Every vertex $v \in V(G)$ is contained in some clique A_i since the edge $vx \in S$ is covered by some guest G_i of φ. Thus, G has a clique-vertex-cover of size at most k and thus $\kappa(G) \leq k$. ◀

▶ **Corollary 7.4.** *The partial-global-$\mathcal{K}$-covering problem and the partial-local-$\mathcal{K}$-covering problem are NP-hard.*

Proof. We reduce from the clique-vertex-covering-problem, which NP-hard [14]. Given an instance (G, k) of the clique-vertex-covering-problem, we define the instance (H, S, k) of the partial-global-$\mathcal{K}$-covering problem (partial-local-$\mathcal{K}$-covering problem) similarly to the situation of Lemma 7.3, i.e. $H = G \vee x$ and $S = \{xv \mid v \in V(G)\}$. This reduction can obviously be performed in polynomial time and the correctness follows immediately from Lemma 7.3. ◀

Next, we move on to the class $\mathcal{CB}$ of bicliques. Again, it turns out to be useful to reduce from the clique-vertex-covering-problem.

▶ **Lemma 7.5** ([21, Theorem 8.1])**.** *The partial-global-$\mathcal{CB}$-covering problem is NP-hard.*

Proof. We again reduce from the clique-vertex-covering-problem, which is NP-hard [14]. Next, we describe how an instance (G, k) of the clique-vertex-covering-problem is transformed into an instance (H, S, k) of the partial-global-$\mathcal{CB}$-covering problem. We create the bipartite graph H from G by creating two copies $v_\mathsf{L}, v_\mathsf{R}$ of each vertex v of G, i.e. $V(H) = \{v_\mathsf{L}, v_\mathsf{R} \mid v \in V(G)\}$. The vertices u_L and v_R are joined by an edge if $uv \in E(G)$ or if $u = v$. Let $S = \{v_\mathsf{L} v_\mathsf{R} \mid v \in V(G)\}$ be the edges between the two copies of each vertex. See Figure 7.2 for an illustration. It is obvious that this reduction can be performed in polynomial time.

It remains to show that $\kappa(G) \leq k$ if and only if H has a k-global S-partial $\mathcal{CB}$-cover. First, assume that there is a clique-vertex-cover $V(G) = A_1 \cup \cdots \cup A_k$. From this, we obtain the k-global S-partial $\mathcal{CB}$-cover φ whose guests are the bicliques with parts $\{v_\mathsf{L} \mid v \in A_i\}$ and $\{v_\mathsf{R} \mid v \in A_i\}$. Each edge $v_\mathsf{L} v_\mathsf{R} \in S$ is covered by φ since $v \in A_i$ for some $i \in [k]$.

(a) H **(b)** H'

Figure 7.3 Illustration of the reduction described in the proof of Lemma 7.6. (a) shows an instance H of the local-$\mathcal{B}$-covering problem. H can be covered by a single bipartite guest and the parts of this guest are highlighted in blue and orange. (b) shows the corresponding instance H' of the partial-local-$\mathcal{CB}$-covering problem obtained by adding the optional edges to make the graph complete. Optional edges are dashed while the edges from the set S are red. The bipartite guest from H can be transformed into a biclique of H' by adding an optional edge to it.

For the other direction, we start with a k-global S-partial $\mathcal{CB}$-cover φ of H with guests $G_1, \dots, G_k$. For each $i \in [k]$ we define $A_i = \{v \in V(G) \mid v_\mathsf{L}, v_\mathsf{R} \in V(G_i)\}$ as the set of vertices of G such that both of its copies $v_\mathsf{L}, v_\mathsf{R}$ are hit by the i-th guest. We observe that A_i is a clique: for distinct $u, v \in A_i$ we have $u_\mathsf{L}, v_\mathsf{R} \in V(G_i)$, thus $u_\mathsf{L} v_\mathsf{R} \in E(H)$ and therefore $uv \in E(G)$. Finally, observe that every vertex $v \in V(G)$ is contained in some A_i. Indeed, $v_\mathsf{L} v_\mathsf{R} \in S$ is covered by some guest G_j and thus $v \in A_j$. Therefore, $V(G) = A_1 \cup \dots \cup A_k$ is a clique-vertex-cover of G and we obtain $\kappa(G) \leq k$. ◀

Now, we come to the partial-local-$\mathcal{CB}$-covering problem covering problem. This time, we do not reduce from the clique-vertex-covering-problem, but from the local-$\mathcal{B}$-covering problem. Here, $\mathcal{B}$ is the class of all bipartite graphs.

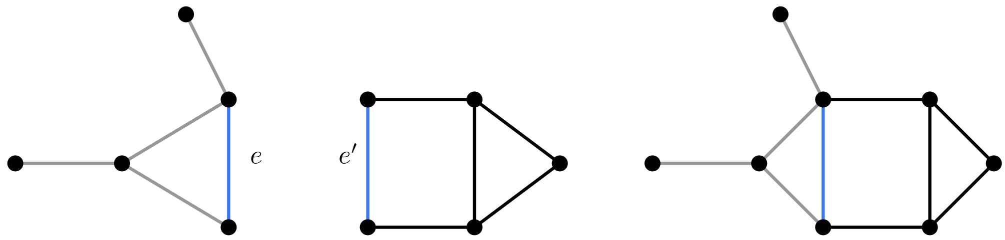
▶ **Lemma 7.6.** *The partial-local-$\mathcal{CB}$-covering problem is NP-hard.*

Proof. We reduce from the 2-local-$\mathcal{B}$-covering problem which is NP-hard by Theorem 3.14. We transform an instance H of the 2-local-$\mathcal{B}$-covering problem into the instance (H', S, k) of the partial-local-$\mathcal{CB}$-covering problem by setting $H' = K_{V(H)}$, $S = E(H)$ and $k = 2$, i.e. we turn H into a complete graph by adding optional edges. See Figure 7.3 for an illustration. This can be clearly done in polynomial time, so it only remains to show the correctness.

Suppose that H has a 2-local $\mathcal{B}$-cover φ with guests $G_1, \dots, G_t$. It is clear that φ is also a 2-local S-partial $\mathcal{B}$-cover of H', but it might not be an S-partial $\mathcal{CB}$-cover of H' since the guests G_i might not be complete bipartite. Note that every G_i is a subgraph of a biclique G_i' on the vertex set of G_i. Now, $G_1', \dots, G_t'$ forms a 2-local S-partial $\mathcal{CB}$-cover of H.

For the other direction, a 2-local S-partial $\mathcal{CB}$-cover of H' can be transformed into a 2-local $\mathcal{B}$-cover of H by intersecting the edge sets of the guests with S. ◀

Now, we know that the global and local partial covering problems for the guest class $\mathcal{K}$ of complete graphs and for the guest class $\mathcal{CB}$ of bicliques are NP-hard. In the next sections we build the general framework to reduce the partial covering problems to ordinary covering problems and we construct the gadgets required by the framework with respect to $\mathcal{K}$ and $\mathcal{CB}$. This then gives us that the global and local covering problems with respect to the guest classes $\mathcal{K}$ and $\mathcal{CB}$ are NP-hard.



(a) Gadget (F, e) **(b)** Graph H **(c)** Gadget attached to H

Figure 7.4 Example of attaching a gadget (F, e) to the edge e' of a graph H.

7.2 Global Covers

The gadgets which we use are described as pairs of a graph F and one of its edges $e \in E(F)$. *Attaching* the gadget (F, e) to an edge e' of a graph H means that we take the disjoint union of F and H and identify the edges e and e'. This operation is illustrated in Figure 7.4. If a graph H can be obtained by attaching a gadget (F, e) to an edge of some graph H', then H is called an *extension* of F. In the case of global covering numbers, we consider gadgets of the following type:

▶ **Definition 7.7.** *Let $\mathcal{G}$ be a guest class. A gadget (F, e) is a* global-partial-reduction gadget with respect to $\mathcal{G}$ *if it satisfies the following properties:*

(Glob-1) $F \in \mathcal{G}$

(Glob-2) *If H is an extension of (F, e) and φ is a $\mathcal{G}$-cover of H, then φ contains a guest G which is a subgraph of F.*

To illustrate this definition, we give a global-partial-reduction gadget with respect to the class $\mathcal{CB}$ of bicliques:

▶ **Lemma 7.8.** *Let $F = C_4$ be a four-cycle with vertices labeled a, b, u, v (in order) and let $e = ab$. Then, (F, e) is a global-partial-reduction gadget with respect to $\mathcal{CB}$.*

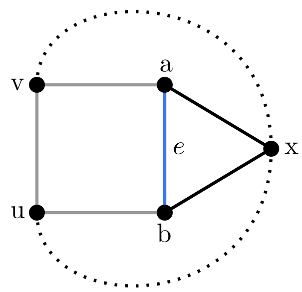
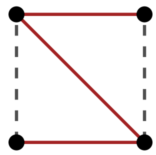
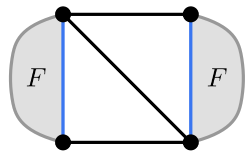


Figure 7.5 An extension of the global-partial-reduction gadget with respect to $\mathcal{CB}$ given in Lemma 7.8. Note that x is not adjacent to u or v.

Proof. An extension of the gadget is illustrated in Figure 7.5. We need to verify the two properties of Definition 7.7. The first property (Glob-1) is satisfied since $F = C_4 \in \mathcal{CB}$ is a biclique. For the second property (Glob-2), we consider an arbitrary extension H of (F, e) and a $\mathcal{CB}$-cover φ of H. Let G be a guest of φ covering the edge uv. Consider a vertex $x \in V(H) \setminus V(F)$. By construction, x is not adjacent to u or v in H. Since G is a biclique containing the edge uv, it follows that G does not contain x. Thus, G is a subgraph of F. ◀



(a)



(b)

Figure 7.6 Illustration of the reduction described in the proof of Theorem 7.9. (a) Shows an instance (H, S, k) of the partial-global-$\mathcal{G}$-covering problem. Edges from S are red while optional edges are dashed. (b) shows that corresponding instance (H', k') of the global-$\mathcal{G}$-covering problem obtained by attaching a global-partial-reduction gadget to each optional edge of H. In this example, $k' = k+2$.

Next, we show that a global-partial-reduction gadget with respect to $\mathcal{G}$ can be used to reduce the partial-global-$\mathcal{G}$-covering problem to the global-$\mathcal{G}$-covering problem by attaching the gadget to every optional edge.

▶ **Theorem 7.9.** *Let $\mathcal{G}$ be an induced-hereditary guest class such that the partial-global-$\mathcal{G}$-covering problem is NP-hard. If there is a global-partial-reduction-gadget (F, e) with respect to $\mathcal{G}$, then the global-$\mathcal{G}$-covering problem is NP-hard.*

Proof. We reduce from the partial-global-$\mathcal{G}$-covering problem, which is NP-hard by assumption. We describe how we transform an instance (H, S, k) of the partial-global-$\mathcal{G}$-covering problem into the instance (H', k') of the global-$\mathcal{G}$-covering problem. Construct H' by attaching the global-partial-reduction gadget (F, e) to every optional edge, i.e. to every edge of $E(H) \setminus S$. Further, set $k' = k + |E(H)| - |S|$. See Figure 7.6 for an illustration of the reduction. It is clear that this construction can be performed in polynomial time, so it only remains to prove the correctness of the reduction.

First, suppose that H has a k-global S-partial $\mathcal{G}$-cover φ. From φ, we obtain a k'-global $\mathcal{G}$-cover φ' of H' by covering each gadget attached to H by a single guest isomorphic to F. This way, $|E(H)| - |S|$ additional guests are added and they cover all edges of $E(H') \setminus S$. The edges of S are covered by the k guests of φ.

Next, suppose that H' has a k'-global $\mathcal{G}$-cover φ'. We obtain $\hat{\varphi}$ from φ' by removing all guests from φ' which are subgraphs of some gadget attached to H. By property (Glob-2) of a global-partial-reduction gadget, we remove at least one guest for every gadget attached to H. Thus, $\hat{\varphi}$ has size at most $k' - (|E(H)| - |S|) \leq k$. Since the removed guests do not cover any edge of S, we see that $\hat{\varphi}$ is a k-global S-partial $\mathcal{G}$-cover of H'. By restricting $\hat{\varphi}$ to H, we obtain the k-global S-partial $\mathcal{G}$-cover $\varphi = \hat{\varphi}|_H$ of H. Note that this restriction is possible since $\mathcal{G}$ is induced-hereditary. ◀

With, this we can prove that the global covering problems for the guest class $\mathcal{CB}$ of bicliques and the guest class $\mathcal{K}$ of complete graphs are NP-complete.

▶ **Theorem 7.10** ([21] Theorem 8.1)**.** *The global-$\mathcal{CB}$-covering problem is NP-complete.*

Proof. By Lemma 2.6, the problem is clearly in NP.

By Lemma 7.5 the partial-global-$\mathcal{CB}$-covering problem is NP-hard and by Lemma 7.8 there is a global-partial-reduction gadget with respect to $\mathcal{CB}$. Thus, the requirements of Theorem 7.9 are satisfied and we obtain that the global-$\mathcal{CB}$-covering problem is NP-hard. ◀

Before we can apply Theorem 7.9 to the guest class $\mathcal{K}$ of complete graphs, we first need to a new global-partial-reduction gadget. This gadget is very similar to the one constructed in Lemma 7.8 for the class of bicliques.

▶ **Lemma 7.11.** *Let $F = K_3$ be a triangle with vertices labeled a, b, c and let $e = ab$. Then, (F, e) is a global-partial-reduction gadget with respect to the class $\mathcal{K}$ of complete graphs.*

Proof. We need to verify the requirements of a global-partial-reduction gadget. The first property (Glob-1) is satisfied since $F = K_3 \in \mathcal{K}$. For the second property (Glob-2), we consider an arbitrary extension H of (F, e) and a $\mathcal{K}$-cover φ of H. Let G be a guest of φ covering the edge bc. Consider a vertex $x \in V(H) \setminus V(F)$. By construction, x is not adjacent to c in H. Since G is a clique containing c, it follows that G does not contain x. Thus, G is a subgraph of F. ◀

▶ **Theorem 7.12** ([21] Theorem 8.1)**.** *The global-$\mathcal{K}$-covering problem is NP-complete.*

Proof. By Lemma 2.6, the problem is clearly in NP.

By Corollary 7.4, the global-$\mathcal{K}$-covering problem is NP-hard and by Lemma 7.11 there is a global-partial-reduction gadget with respect to $\mathcal{K}$. Thus, the requirements of Theorem 7.9 are satisfied and we obtain that the global-$\mathcal{K}$-covering problem is NP-hard. ◀

7.3 Local Covers

In the case of local covering numbers, we need a slightly different type of gadget:

▶ **Definition 7.13.** *Let $\mathcal{G}$ be a guest class. A gadget (F, e) is a k-*local-partial-reduction gadget *with respect to $\mathcal{G}$ if it satisfies the following properties:*

(Loc-1) *There is a k-local $\mathcal{G}$-cover of F which hits the endpoints of e only once.*

(Loc-2) *If H is an extension of (F, e) and φ is a k-local $\mathcal{G}$-cover of H, then φ contains a guest G covering e such that $G \subseteq F$.*

To illustrate this definition, we give a local-partial-reduction gadget with respect to the class $\mathcal{K}$ of complete graphs:

▶ **Lemma 7.14.** *For $k \in \mathbb{N}$, let F_k be a star with center c and $k+1$ leaves $v_1, \dots, v_{k+1}$ and one additional edge $e_k = v_1v_2$ connecting two leaves, i.e. $E(F) = \{cv_i \mid i \in [k+1]\} \cup \{v_1v_2\}$. For every $k \in \mathbb{N}$, it holds that (F_k, e_k) is a k-local-partial-reduction gadget with respect to $\mathcal{K}$.*

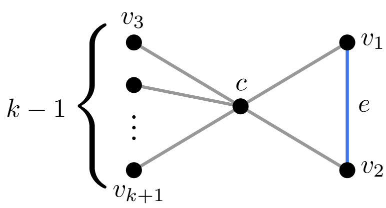


Figure 7.7 Illustration of the k-local-partial-reduction gadget with respect to the class $\mathcal{K}$ of complete graphs described in Lemma 7.14.

Proof. See Figure 7.7 for an illustration of the gadget. We need to verify the two properties of Definition 7.13. To verify (Loc-1), observe that we can cover the triangle cv_1v_2 with a copy of K_3 and all other edges of F_k by $k-1$ copies of K_2. This yields a k-local cover of F_k which hits v_1, v_2 only once. Thus, (Loc-1) is indeed satisfied.

For the second property (Loc-2), let H be an extension of (F_k, e_k) and φ be a k-local $\mathcal{K}$-cover of H. Let $G_1, \dots, G_\ell$ be the guests hitting c where $\ell \leq k$. Since c has degree $k+1$, there must be some guest G_i which covers at least two edges incident to c. Since v_1 and v_2 are the only neighbors of c joined by an edge, it follows that G_i covers the triangle cv_1v_2 and in particular G_i covers e_k. Since G_i is a clique containing c and $N_H(c) = V(F)$, it follows that $G_i \subseteq F_k$. Thus, the second property is also satisfied. ◀

For the class $\mathcal{CB}$ of bicliques the construction of a k-local-partial-reduction gadget turns out to be more complicated. In fact, the construction presented later in the chapter is algebraic and only works for certain k. Thus, we shall not require the existence of a k-local-partial-reduction gadget for every k in Theorem 7.16. However, we still need to ensure that a k-local-partial-reduction gadget exists for sufficiently many k.

▶ **Definition 7.15.** *Let $\mathcal{G}$ be a guest class. For a set $A \subseteq \mathbb{N}$, we call $\{(F_k, e_k)\}_{k \in A}$ a* local-partial-reduction family with respect to $\mathcal{G}$ *if the following properties are satisfied for some $\beta \in \mathbb{N}$:*

(LF-1) *(F_k, e_k) is a k-local-partial-reduction gadget with respect to $\mathcal{G}$ for every $k \in A$.*
(LF-2) *Given $k \in A$, it is possible to construct (F_k, e_k) in $\mathcal{O}(k^\beta)$ time.*
(LF-3) *Given $\ell \in \mathbb{N}$, it is possible to find a $k \geq \ell$ in $\mathcal{O}(\ell^\beta)$ time which satisfies $k \in A$ and $k \in \mathcal{O}(\ell^\beta)$.*

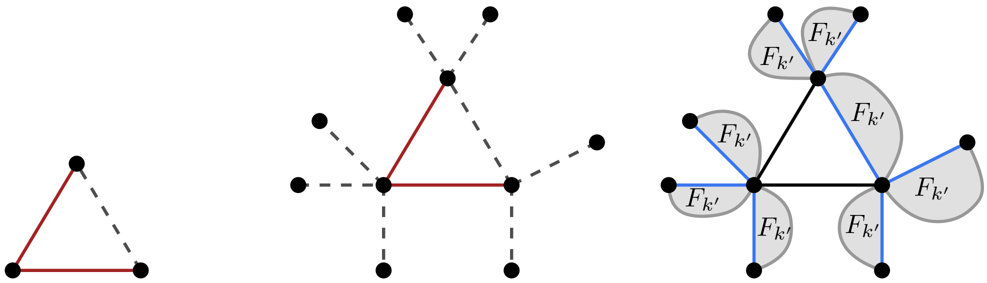
The goal of the next theorem is to use a local-partial-reduction family to reduce the partial-local-$\mathcal{G}$-covering problem to the local-$\mathcal{G}$-covering problem. For an instance (H, S, k) of the partial-local-$\mathcal{G}$-covering problem, it seems natural to attach a k'-local-partial-reduction gadget to each optional edge of H similarly to the proof of Theorem 7.9. However, we need to handle two more technical details here: First of all, we need to properly pick k'. Second, we need to make sure that each vertex of H is incident to the same number of optional edges. This ensures that each vertex of H is hit by the same number of guests covering optional edges.

▶ **Theorem 7.16.** *Let $\mathcal{G}$ be an induced-hereditary guest class such that the partial-local-$\mathcal{G}$-covering problem is NP-hard. If there is a local-partial-reduction family $\{(F_k, e_k)\}_{k \in A}$ with respect to $\mathcal{G}$, then the local-$\mathcal{G}$-covering problem is NP-hard.*

Proof. We reduce from the partial-local-$\mathcal{G}$-covering problem which is NP-hard by assumption. We start by describing the basic idea of how we transform an instance (H, S, k) of the partial-local-$\mathcal{G}$-covering problem into the instance (H', k') of the local-$\mathcal{G}$-covering problem. First, we create an intermediate graph $\hat{H} \supseteq H$ which ensures that every vertex $v \in V(H)$ is adjacent to exactly $k' - k$ optional edges by attaching leaves to v. After this, we attach one copy of a k'-local-partial-reduction gadget to each optional edge of $\hat{H}$ yielding H'. See Figure 7.8 for an illustration.

Reduction: Now, we describe how we transform (H, S, k) into (H', k'). First of all, we may assume $k \leq |E(H)|$ (see Lemma 2.5).

We start the construction, by picking an integer $k' \geq k + |V(H)|$: let $\tilde{k} = k + |V(H)|$ and use the property (LF-3) of a local-partial-reduction-family to pick $k' \geq \tilde{k}$ in polynomial time such



(a) H **(b)** $\hat{H}$ **(c)** H'

Figure 7.8 Illustration of the reduction used in the proof of Theorem 7.16. We start from an instance (H, S, k) of the partial-global-$\mathcal{G}$-covering problem in (a). Edges from S are red while optional edges are dashed. From this, we create an intermediate graph $\hat{H}$ in (b) which ensures that each vertex of H is incident to the same number of optional edges. Finally in (c), a gadget is attached to every optional edge of $\hat{H}$ resulting in H'.

that $k' \in A$ and the value of k' is polynomial in $\tilde{k}$. Next, we make sure that every vertex of H is incident to the same number of optional edges. For every vertex v of H count the number of optional edges $\text{opt}(v)$ incident to v. More formally, $\text{opt}(v) \coloneqq |\{u \in V(H) \mid uv \in E(H) \setminus S\}|$. We construct the graph $\hat{H}$ from H by creating $k' - k - \text{opt}(v)$ leaves and connecting them with optional edges to v for every vertex $v \in V(H)$. As $k' - k - \text{opt}(v) \geq |V(H)| - \text{opt}(v) \geq 0$, this operation is well-defined. Note that every vertex $v \in V(H)$ of H is incident to exactly $k' - k$ optional edges in $\hat{H}$. Finally, construct H' by attaching the k'-local-partial-reduction gadget $(F_{k'}, e_{k'})$ to every optional edge of $\hat{H}$, i.e. to every edge of $E(\hat{H}) \setminus S$.

Now, we show that the reduction can be performed in polynomial time. First of all, we have that $\tilde{k} = k + |V(H)|$ is polynomial in $|V(H)|$ since we assumed $k \leq |E(H)|$. Therefore, k' is polynomial in k and it can be computed in polynomial time by (LF-3). Thus, $\hat{H}$ has polynomial size. Since $F_{k'}$ has polynomial size by (LF-2), we obtain that H' has polynomial size as well. So it only remains to prove the correctness of the reduction.

Correctness: First suppose that H has a k-local S-partial $\mathcal{G}$-cover φ. Then a k'-local $\mathcal{G}$-cover φ' of H' can be obtained by first covering the gadgets separately. By property (Loc-1) of a k'-local-partial-reduction gadget, we can cover every copy of the gadget $(F_{k'}, e_{k'})$ with a k'-local cover hitting the endpoints of $e_{k'}$ only once. After this, only the edges of S need to be covered and these can be covered using φ. Recall that every vertex of H is incident to $k' - k$ optional edges in $\hat{H}$. Thus, every vertex of $\hat{H}$ is hit at most $k' - k$ times when covering the gadgets and at most k times by φ and thus at most k' times in total. Every other vertex $v \in V(H') \setminus V(\hat{H})$ belongs to a copy G' of a gadget and thus v is only hit by guests of the k'-local cover of G'. So φ' is indeed k-local.

Next, suppose that H' has a k'-local $\mathcal{G}$-cover φ'. Let $\tilde{\varphi}$ be obtained from φ' by removing every guest G' which is a subgraph of a copy of the gadget $(F_{k'}, e_{k'})$. Note that $\tilde{\varphi}$ is an S-partial $\mathcal{G}$-cover of H' since the removed guests do not cover any edges from S. From property (Loc-2) of a k'-local-partial-reduction-gadget, we can conclude that every optional edge $e \in E(\hat{H}) \setminus S$ is covered by some guest which does not cover any other edge of $\hat{H}$. Thus, for each vertex $v \in V(H)$, we remove at least $k - k'$ guests hitting v when creating $\tilde{\varphi}$ from φ'. Thus, $\tilde{\varphi}$ hits every vertex of H at most k times. Therefore, the restriction $\varphi = \tilde{\varphi}|_H$ is a k-local S-partial $\mathcal{G}$-cover of H. ◀

Now, we can apply Theorem 7.16 to the guest class $\mathcal{K}$ of complete graphs to obtain an NP-hardness result for the local covering problem. Note however, that our result is weaker than the result of [23], since we only prove NP-hardness for k given in the input.

▶ **Theorem 7.17.** *The local-$\mathcal{K}$-covering problem is NP-hard.*

Proof. By Corollary 7.4, the partial-local-$\mathcal{K}$-covering problem is NP-hard and by Lemma 7.14 there is a k-local-partial-reduction gadget (F_k, e_k) with respect to $\mathcal{K}$ for every $k \in \mathbb{N}$. Since the gadget given in Lemma 7.14 has polynomial size in k, there is a partial-reduction-family with respect to $\mathcal{K}$. Therefore, the requirements of Theorem 7.16 are satisfied and we obtain that the local-$\mathcal{K}$-covering problem is NP-hard. ◀

7.4 Local biclique coverings

The goal of this section is to construct a k-local-partial-reduction gadget for the class $\mathcal{CB}$ of bicliques. We first need a construction for graphs H with known local covering number $c_l^{\mathcal{CB}}(H) = k$ such that $|V(H)|$ is polynomial in k. The idea is to use an algebraic construction of C_4-free graphs with n vertices and $\Omega(n^{3/2})$ edges introduced by Erdős, Rényi and Sós [5][2]. A $\mathcal{CB}$-cover of such a graph is a star-cover and the large number of edges thus ensures a large covering number.

For the construction, let p be a prime and $\mathbb{F}_p$ be the finite field with p elements. For an element $x = \binom{x_1}{x_2} \in \mathbb{F}_p^2$ of the two dimensional vector space $\mathbb{F}_p^2$ over $\mathbb{F}_p$, we denote its coordinates as x_1 and x_2 and its transpose as $x^\mathsf{T} = (x_1, x_2)$. We construct the graph $G^{(p)}$ on the vertices $V(G^{(p)}) = \mathbb{F}_p^2 \setminus \{\binom{0}{0}\}$ with edges $E(G^{(p)}) = \{\{x, y\} \in \binom{V(G_p)}{2} \mid x^\mathsf{T} y = 1\}$. We first need to prove some known properties of this graph.

▶ **Lemma 7.18** (Erdős, Rényi, Sós [5]). *For every prime p, the graph $G^{(p)}$ has both of the following properties:*

(i) *$G^{(p)}$ does not contain a C_4.*

(ii) *Every vertex $v \in V(G^{(p)})$ has degree $\deg(v) \in \{p-1, p\}$ and there is at least one vertex of degree p.*

Proof. We first show that $G^{(p)}$ does not contain a C_4. Assume for contradiction that a, b, c, d (in order) form a C_4 in $G^{(p)}$. We can see that a and c are linearly independent because if $a = \lambda c$ for some $\lambda \in \mathbb{F}_p$, then $1 = a^\mathsf{T} b = \lambda c^\mathsf{T} b = \lambda$ and thus $a = c$ which is a contradiction. Thus, the matrix $A = \begin{pmatrix} a_1 & a_2 \\ c_1 & c_2 \end{pmatrix}$ is invertible. Therefore, the equation $Ax = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ has exactly one solution x. Since b and d are solutions of this equation, it follows that $b = d$, a contradiction.

Now, consider an arbitrary vertex $v \in V(G^{(p)})$. We want to show $\deg(v) \in \{p-1, p\}$. A vertex x is adjacent to v if and only if $v_1 x_1 + v_2 x_2 = 1$. At least one of v_1 and v_2 is not 0. Without loss of generality, assume that $v_2 \neq 0$. For every possible choice of x_1, there is a unique choice of $x_2 = v_2^{-1}(1 - v_1 x_1)$ which satisfies the equation. Thus there are exactly p values x which satisfy the equation. Note however, that it is possible that $v^\mathsf{T} v = 1$ and we do not connect v to itself. So vertices with $v^\mathsf{T} v = 1$ only have degree $p - 1$. Since $u = (1, 1)^\mathsf{T}$ satisfies $u^\mathsf{T} u = 2 \not\equiv 1 \pmod{p}$, we have that u is vertex of degree p. ◀

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[2] Erdős, Rényi and Sós originally used a slightly different construction. However, they introduced the idea of constructing C_4-free graphs using finite fields. The construction used by us can found in https://www.its.caltech.edu/~dconlon/EGT8.pdf.

Therefore, every $\mathcal{CB}$-cover of $G^{(p)}$ is a star-cover. To determine the exact value of the local $\mathcal{CB}$-covering number of $G^{(p)}$, we first need a result on coverings with the guest class $\mathcal{S}$ of stars.

▶ **Lemma 7.19** ([15, Lemma 7, Theorem 9]). *For every graph H, we have $c_l^{\mathcal{S}}(H) \leq \left\lceil \frac{\Delta(H)}{2} \right\rceil + 1$*

Proof. By [15, Theorem 9], we have $c_l^{\mathcal{S}}(H) \leq p(H) + 1$ where $p(H)$ is a graph parameter known as the *pseudoarboricity*. By [15, Lemma 7], we have $p(H) = \max_{\emptyset \neq S \subseteq V(H)} \left\lceil \frac{|E(H[S])|}{|S|} \right\rceil$. Since $|E(H[S])| \leq \frac{\Delta(H) \cdot |S|}{2}$ for every nonempty set $S \subseteq V(H)$, we have $p(H) \leq \left\lceil \frac{\Delta(H)}{2} \right\rceil$. ◀

▶ **Lemma 7.20.** *If $p > 3$ is prime, then $c_l^{\mathcal{CB}}(G^{(p)}) = \frac{p+3}{2}$.*

Proof. We start with the upper bound. Lemma 7.19 together with the fact $\Delta(G^{(p)}) = p$ from Lemma 7.18 (i) yields $c_l^{\mathcal{S}}(G^{(p)}) \leq \left\lceil \frac{\Delta(H)}{2} \right\rceil + 1 = \left\lceil \frac{p}{2} \right\rceil + 1 = \frac{p+3}{2}$. Since every star is a biclique, we obtain $c_l^{\mathcal{CB}}(G^{(p)}) \leq \frac{p+3}{2}$.

For the lower bound, assume for contradiction that φ is a k-local $\mathcal{CB}$-cover of $G^{(p)}$ with $k = \frac{p+1}{2}$ and with guests $G_1, \ldots, G_t$. Note that every guest of φ is a star because $G^{(p)}$ is C_4-free by Lemma 7.18 (i). By Lemma 7.18 (ii), every vertex v of $G^{(p)}$ has degree at least $p - 1 > k$ since $p > 3$. For every vertex v, there is thus a guest centered at v and the total number of guests is at least $t \geq \left|V(G^{(p)})\right|$. By Lemma 7.18 (ii), we have $\left|E(G^{(p)})\right| > \frac{p-1}{2}\left|V(G^{(p)})\right|$. With this, we obtain for the total hitcount of φ:

$$\mathrm{hit}_{\varphi}(G^{(p)}) = \sum_{i=1}^{t} |V(G_i)| = \sum_{i=1}^{t} (|E(G_i)| + 1) \geq \left|E(G^{(p)})\right| + t > \frac{p+1}{2}\left|V(G^{(p)})\right|.$$

This is a contradiction to the assumption that φ is $(\frac{p+1}{2})$-local. ◀

To describe the next constructions, we introduce some notation. For graphs $H_1, \ldots, H_t$, we define their *cyclic blowup* $F = \mathrm{cbl}(H_1, \ldots, H_t)$ by

$$V(F) = V(H_1) \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} V(H_t)$$
$$E(F) = \bigcup_{i \in [t]} E(H_i) \cup \{uv \mid u \in V(H_i), v \in V(H_{i+1})\}$$

where we consider indices cyclically, i.e. $H_{t+1} = H_1$.

In the special case of $F = \mathrm{cbl}(K_1, tH)$, we call F the *t-star-blowup of H*. Further, we call the vertex corresponding to K_1 the *center* of F and we refer to each of the t copies of H as a *leaf-block* of F. See Figure 7.9 for an illustration. An easy application of the pigeonhole principle gives us an important property of star-blowups.

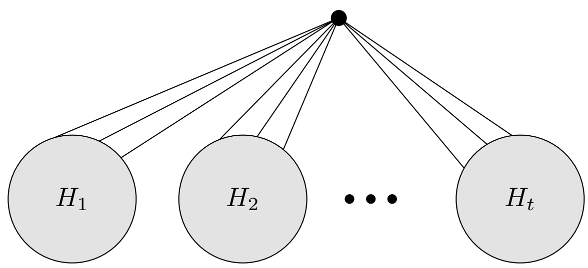

Figure 7.9 Illustration of the t-star-blowup construction.

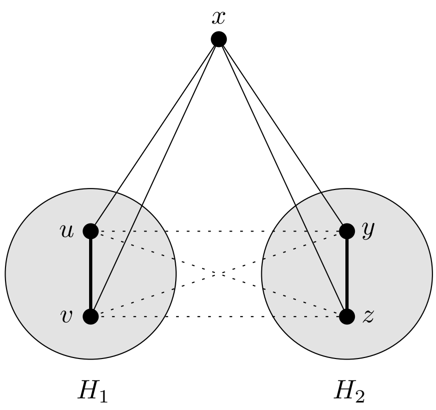

Figure 7.10 Illustration of the proof of Lemma 7.21. Since there are no edges between leaf-blocks (illustrated by dotted lines), no biclique can cover edges from multiple leaf-blocks.

► **Lemma 7.21.** *Let F be a graph and let S be an induced subgraph of F. Further assume that S is a t-star-blowup of H with center x. If φ is a k-local $\mathcal{CB}$-cover of F, then there are at least $t-k$ leaf-blocks of S which are edge-disjoint from the guests hitting x.*

Proof. Consider an arbitrary guest G from φ. It is not possible that G covers edges inside two different leaf-blocks of S. Assume that there are two different leaf-blocks H_1, H_2 such that G covers an edge uv of H_1 and an edge yz of H_2 (see Figure 7.10). Since G is a biclique containing these edges, it also contains one of the edges uy and uz. However, none of these edges exists in F, which gives us a contradiction. Thus, each guest can cover edges inside at most one leaf-block. Since there are at most k guests hitting x and t leaf-blocks, it follows that there are at least $t-k$ leaf-blocks which are edge-disjoint from the guests hitting x. ◄

This observation gives us a construction to increase the local $\mathcal{CB}$-covering numbering of an arbitrary host by one. While this is not necessary for the construction of the gadget, it is interesting for two reasons. First of all, the proof illustrates the ideas used later in a simplified setup. Second, if it turns out that the k-local-$\mathcal{CB}$-covering problem is NP-hard for some $k \in \mathbb{N}$, then it allows to conclude that it is also NP-hard for all larger k.

► **Lemma 7.22.** *Let H be a graph with $\mathrm{c}_l^{\mathcal{CB}}(H) = k$ and let H' be the $(k+1)$-star-blowup of H. It holds that $\mathrm{c}_l^{\mathcal{CB}}(H') = k+1$.*

Proof. It is clear that $\mathrm{c}_l^{\mathcal{CB}}(H) \leq k+1$: We can cover all edges incident to x by a single star and then cover each F_i individually.

To show the lower bound, assume for contradiction that there is a k-local $\mathcal{CB}$-cover φ. By Lemma 7.21, there is a leaf-block $\hat{H}$ of H' which is edge-disjoint from all guests hitting x. Since each vertex of $\hat{H}$ is hit at least once by a guest hitting x, it follows that the restriction of φ to $\hat{H}$ is $(k-1)$-local and thus $\mathrm{c}_l^{\mathcal{CB}}(H) \leq k-1$ which contradicts the assumption $\mathrm{c}_l^{\mathcal{CB}}(H) = k$. ◄

Now, we come to the actual construction of the local-partial-reduction gadgets. Our construction is based on a cyclic blowup of copies of the graph $G^{(p)}$ from the above algebraic construction. Since our reduction requires p to be prime, we will not obtain a k-local-partial-reduction gadget for every $k \in \mathbb{N}$. However, the gaps between two consecutive are sufficiently small, so we still obtain a local-partial-reduction family.

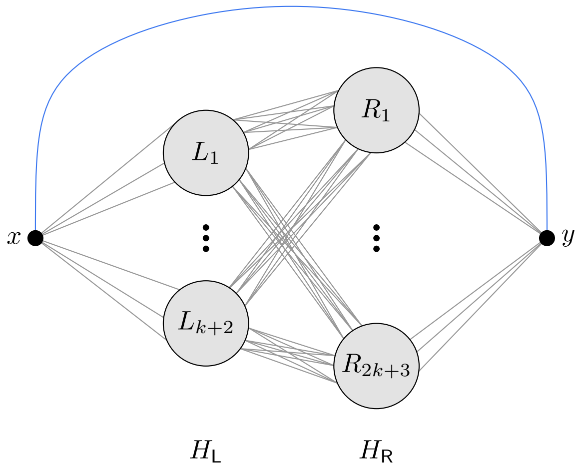

Figure 7.11 Illustration of the $(k+1)$-local-partial-reduction gadget F with respect to $\mathcal{CB}$ constructed in the proof Lemma 7.23.

▶ **Lemma 7.23.** *There is a local-partial-reduction family with respect to the guest class $\mathcal{CB}$ of bicliques.*

Proof. The main part of the proof is the transformation of a given graph H with $c_l^{\mathcal{CB}}(H) = k$ into a $(k+1)$-local-partial-reduction gadget. After proving the correctness of the construction, we use Lemma 7.20 together with a result about the distribution of primes to obtain a local-partial-reduction family.

The gadget. Let H be a graph with $c_l^{\mathcal{CB}}(H) = k$. We create the graph H_{L} as the union of $k+2$ vertex-disjoint copies $L_1, \ldots, L_{k+2}$ of H and the graph H_{R} as the union of $2k+3$ vertex-disjoint copies $R_1, \ldots, R_{2k+3}$ of H. Further, we create the cyclic blowup $F = \mathrm{cbl}(K_1, H_{\mathsf{L}}, H_{\mathsf{R}}, K_1)$ and we refer to the vertex of the first K_1 as x and to the one of the last K_1 as y. Our goal is to prove that (F, xy) is a $(k+1)$-local-partial-reduction gadget with respect to $\mathcal{CB}$. The gadget is illustrated in Figure 7.11.

We start by verifying property (Loc-1). We need to show that F has a $(k+1)$-local $\mathcal{CB}$-cover hitting the vertices x and y only once. To obtain such a cover, we cover each L_i and each R_j separately with a k-local cover of H. The remaining edges can be covered by a single biclique with parts $\{x\} \cup V(H_{\mathsf{R}})$ and $\{y\} \cup V(H_{\mathsf{L}})$. Clearly, the obtained cover is $(k+1)$-local and hits the vertices x and y only once and thus we have verified (Loc-1).

Verifying property (Loc-2) is more difficult. The key idea is to show that the local $\mathcal{CB}$-covering number of F increases when we remove the edge xy.

▷ Claim. $c_l^{\mathcal{CB}}(F - xy) \geq k+2$.

Proof. Assume for contradiction that there is a $(k+1)$-local $\mathcal{CB}$-cover φ of $F' = F - xy$. Note that $S_{\mathsf{L}} = F[\{x\} \cup V(H_{\mathsf{L}})]$ is a $(k+2)$-star-blowup and $S_{\mathsf{R}} = F[\{y\} \cup V(H_{\mathsf{R}})]$ is a $(2k+3)$-star-blowup of H. Applying Lemma 7.21 to S_{L} yields an index i such that L_i is edge-disjoint from all guests hitting x. Applying it to S_{R} yields a set J of at least $k+2$ indices such that R_j is edge-disjoint from all guests hitting y for all $j \in J$.

Since $c_l^{\mathcal{CB}}(L_i) = c_l^{\mathcal{CB}}(H) = k$, there is a vertex $z \in V(L_i)$ which is hit at least k times by the guests of φ which cover an edge of L_i. Note that the subgraph S_{M} of F' induced by $\{z\} \cup \bigcup_{j \in J} V(R_j)$ is a $(k+2)$-star-blowup of H with center z. By Lemma 7.21 there is thus

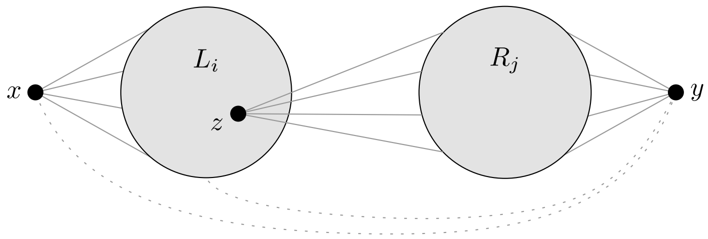

Figure 7.12 Illustration of the situation described in the proof of Lemma 7.23: No guest hitting x covers an edge inside L_i and no guest hitting y covers an edge inside R_j. Additionally, no guest hitting z covers an edge inside R_j. It is important to note that there is no edge between y and x, nor between y and any vertex of L_i as illustrated by the dotted edges.

an index $j \in J$ such that R_j is edge-disjoint from all guests hitting z. See Figure 7.12 for an illustration of the current situation.

Next, consider a guest G of φ hitting y. We want to show that G does not hit z. First of all, y is not adjacent to any vertex of $V(V_i) \cup \{x\}$. Since G is a biclique containing y, this implies that y does not cover any edge inside L_i and G does not hit x. If G hits z, then z is hit at least $k+2$ times: k times by guests which cover edges inside L_i, once by a guest hitting x and once by G. Therefore, no guest hits both z and y.

Now, consider the guests which cover an edge inside R_j. Since $c_l^{\mathcal{CB}}(R_j) = k$, there is some vertex $w \in V(R_j)$ which is hit at least k times by these guests. We see that w is hit by a guest G_z hitting z and by a guest G_y hitting y. By construction, G_z and G_y do not cover any edges of R_j. Since no guest hits both of y and z, we also have $G_z \neq G_y$. However, this implies that w is hit at least $k+2$ times, a contradiction. ◁

This claim allows us now to verify property (Loc-2). Consider an extension Q of (F, xy) and a $(k+1)$-local-$\mathcal{CB}$-cover φ of Q. We need to show that φ has a guest G covering xy such that $G \subseteq F$. Observe that the restriction $\varphi|_F$ must contain a guest G_F such that $G_F - xy$ is not a biclique. Otherwise, the edge xy can be removed from all guests of $\varphi|_F$ giving us a $(k+1)$-local cover of $F - xy$ which is a contradiction to the above claim. Since G_F is a biclique and $G_F - xy$ is not, this means that G_F contains a four-cycle $xyuv$. Consider the guest G from φ which restricts to G_F in $\varphi|_F$. If G contains a vertex $z \in V(Q) \setminus V(F)$, then G contains one of the edges zu and zv, but these edges are not in Q. Thus, $G \subseteq F$ and property (Loc-2) holds.

Reduction Family. Let $p > 3$ be a prime number. By Lemma 7.20, there is a graph $G^{(p)}$ on $\mathcal{O}(p^2)$ vertices with $c_l^{\mathcal{CB}}(G^{(p)}) = \frac{p+3}{2}$. Performing the gadget construction described above with $H = G^{(p)}$ thus yields a k-local-partial-reduction gadget (F_k, e_k) with respect to $\mathcal{CB}$ where $k = \frac{p+5}{2}$. Clearly, this gadget has polynomial size in k and the construction can also be performed in polynomial time. It remains to verify property (LF-3) to show that $\{(F_k, e_k)\}_{k \in A}$ with $A = \{\frac{p+5}{2} \mid p > 3 \text{ and } p \text{ is prime}\}$ is indeed a local-partial-reduction family with respect to $\mathcal{CB}$. The property follows from Betrand's postulate [24] which states that for each positive integer $n \in \mathbb{N}$, there is a prime in the interval $[n, 2n]$. Thus, for a given $\ell \in \mathbb{N}$, we can find $k \geq \ell$ with $k \in A$ and $k \in \mathcal{O}(\ell)$. A valid value of k can be found in polynomial time by iterating over all candidates k and checking for each of them if is $2k - 5$ is prime. Note that this does not require the existence of efficient primality-testing algorithms since the numbers to check are small (trial division is sufficient). ◀

Now that we have the local-partial-reduction family, we can finally apply Theorem 7.16 to prove Theorem 7.1.

▶ **Theorem 7.1.** *The local-$\mathcal{CB}$-covering problem is NP-complete.*

Proof. By Lemma 2.6, the local-$\mathcal{CB}$-covering problem is clearly in NP. By Lemma 7.6 the partial-local-$\mathcal{CB}$-covering problem is NP-hard and by Lemma 7.23 there is a local-partial-reduction family with respect to $\mathcal{CB}$. Thus, the requirements of Theorem 7.16 are satisfied and we can conclude that the local-$\mathcal{CB}$-covering problem is NP-hard. ◀

8 Conclusion

In this thesis, we investigated global and local covering numbers. First, we obtained a lower bound on local covering numbers with guest classes of bounded fractional chromatic number. After this, we have obtained NP-completeness results for covering problems with several specific guest classes and in some cases we also obtained fast algorithms. See Table 1.1 for an overview of the main complexity results obtained. In particular, we have proved the NP-completeness of the local-$\mathcal{G}$-covering problem for many natural guest classes $\mathcal{G}$ including bipartite graphs, bicliques, cliques, planar graphs, outer-planar graphs and graphs of treewidth at most t for $t \geq 2$. We have also seen a nontrivial polynomial time algorithm for the local-$\mathcal{C}^*$-covering problem where the guest class $\mathcal{C}^*$ consists of all cycles and the single edge K_2. Apart from these specific results (and the frameworks used to prove them), we also considered the following natural question related to Question 1.1: Is the local-$\mathcal{G}$-covering problem computationally easier than the global-$\mathcal{G}$-covering problem if $\mathcal{G}$ satisfies some set of natural requirements? We have seen that for the very natural monotone guest class $\mathcal{E}_3$ of all graphs with at most three edges the global-$\mathcal{E}_3$-covering problem is easy while the local-$\mathcal{E}_3$-covering problem is NP-hard (see Theorem 5.16). Because of this, it seems less likely that there is a general complexity hierarchy for global and local covering problems.

In the remaining part of this section, we discuss some open questions and ideas for future work.

8.1 Structural Results

In Chapter 3, we have seen the following general lower bound based on the fractional chromatic number:

▶ **Theorem 3.6.** *Let $r \in \mathbb{R}_{>1}$ and let $\mathcal{G}$ be a guest class with $\chi_{\mathsf{f}}(G) \leq r$ for all $G \in \mathcal{G}$. For every host graph H it holds that $\log_r(\chi_{\mathsf{f}}(H)) \leq \mathrm{c}_l^{\mathcal{G}}(H)$.*

In particular, for the class $\mathcal{B}$ of bipartite graphs we have seen that $\lceil \log_2(\chi_{\mathsf{f}}(H)) \rceil \leq \mathrm{c}_l^{\mathcal{B}}(H)$. We have also seen in Corollary 3.13 that there is a host graph H for which this bound is not tight. Still, the following question remains open:

▶ **Question 8.1.** *Is there a function $f\colon \mathbb{R}_{\geq 1} \to \mathbb{R}$ such that $\mathrm{c}_l^{\mathcal{B}}(H) \leq f(\chi_{\mathsf{f}}(H))$?*

For answering this question it seems sufficient to look at a graph family known as Kneser graphs. The reason is that every graph H admits a homomorphism $\varphi\colon H \to K_{a:b}$ to some Kneser graph of the same fractional chromatic number as H. Further, a k-local $\mathcal{B}$-cover of $K_{a:b}$ can be transformed into a k-local $\mathcal{B}$-cover of H by applying φ^{-1}.

Another interesting conjecture related to our structural results is about the complexity of the local bipartite covering problem.

▶ **Conjecture 3.15.** *The k-local-$\mathcal{B}$-covering problem is NP-hard for every $k \geq 2$.*

Recall that we have proved the case $k = 2$ in Theorem 3.14. One very natural approach for solving this conjecture is to provide a construction to increase the local bipartite covering number. A natural choice is to consider the join of a graph with itself.

▶ **Conjecture 3.16.** *If* $c_l^{\mathcal{B}}(H) = k$*, then* $c_l^{\mathcal{B}}(H \vee H) = k + 1$*.*

Even though this conjecture seems very natural, it is surprisingly difficult. Some evidence for the conjecture is given by the fact the lower bound $\log_2(\chi_f(H))$ of Theorem 3.6 increases by one when joining H with itself since $\chi_f(H \vee H) = 2\chi_f(H)$.

As a natural generalization of our results on bipartite graphs, it would be interesting to obtain similar results for r-partite graphs.

▶ **Conjecture 8.2.** *The* local-$\mathcal{G}_r$-covering *problem is* NP*-hard where* $\mathcal{G}_r$ *is the class of* r*-partite graphs and* $r \geq 2$*.*

8.2 Finite Guest Classes

We have given some separate attention to finite guest classes after noticing that covering problems tend to become more difficult when restricting guest classes to finite subsets. For example, we have observed this when considering the local-$\mathcal{C}^*$-covering and local-$\mathcal{C}^*_{\leq m}$-covering problem. Thus, a variant of the general NP-completeness results of Theorem 5.2 on the decomposition problem might hold for coverings with finite guest classes.

▶ **Conjecture 5.5.** *Let* $\mathcal{G}$ *be a finite guest class such that every guest is connected and* $\max_{G \in \mathcal{G}} |E(G)| \geq 3$*. Then, the* global-$\mathcal{G}$-covering *problem is* NP*-hard.*

▶ **Conjecture 5.10.** *Let* $\mathcal{G} = \{G\}$ *be a guest class consisting of a single graph.*

- *If* $G = sP_3 \cup tP_2$ *for some* $s, t \in \mathbb{N}_0$*, then the* local-$\mathcal{G}$-covering *problem has a polynomial time solution.*
- *Otherwise, the* local-$\mathcal{G}$-covering *problem is* NP*-complete.*

We also noticed that some tools from the literature work best for finite guest classes while others work better for infinite guest classes. For finite guest classes, we have seen that the general NP-hardness result by Dor and Tarsi (Theorem 5.2, [4]) for decomposition problems turns out to be useful. Meanwhile, we have considered a general result in Theorem 6.1 which only works for infinite guest classes $\mathcal{G}$ since finite guest classes are not closed under 1-sums. However, some tools might work for finite and infinite guest classes equally well. This seems to be the case for the partial-covering framework introduced in Chapter 7.

8.3 Simplifying and Generalizing Results

In some cases, we believe that our results and proofs can be simplified. For example, in Theorem 6.6 we have introduced the additional technical requirement ($\mathcal{G}4$) on the graph class $\mathcal{G}$. For all specific guest classes which we considered in Chapter 6, this property was satisfied.

▶ **Question 8.3.** *Does every graph class* $\mathcal{G}$ *satisfying properties* ($\mathcal{G}1$)-($\mathcal{G}3$) *also satisfy* ($\mathcal{G}4$)*?*

Another result which might be simplified is our proof of the NP-completeness of the local-$\mathcal{CB}$-covering problem. First of all, the gadget construction in the proof of Lemma 7.23 is quite technical and we believe that there is an easier construction. Second, it would be interesting to obtain a non-algebraic construction for a k-local-partial-reduction gadget which works for all $k \in \mathbb{N}$.

As we have seen for the class $\mathcal{CB}$ of bicliques, it can be quite difficult to construct local-partial-reduction gadgets for specific guest classes. Thus, it would be useful to obtain some general constructions which work for many guest classes.

▶ **Question 8.4.** *Is there a natural property P such that for every guest class $\mathcal{G}$ satisfying P there exists a local-partial-reduction family?*

A natural first step for obtaining more general NP-hardness results via the partial covering framework is to first apply it to some other specific guest classes. We remark that it seems possible to apply the framework to prove the NP-hardness of the local covering problem with respect to the guest class of all complete $n \times m$ grid graphs.

Finally, one limitation of our partial covering framework is that it does not yield NP-hardness proofs of covering problems with fixed k. As shown in [23], the 4-local-$\mathcal{K}$-covering problem is NP-hard, while our proof only shows that the corresponding problem is NP-hard for k given in the input.

▶ **Question 8.5.** *Is the k-local-$\mathcal{CB}$-covering problem NP-hard for some fixed $k \in \mathbb{N}$?*

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