Local Page Numbers

Bachelor Thesis of

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Abstract

A *k-local book embedding* consists of a linear ordering of the vertices of a graph and a partition of its edges into sets of edges, called *pages*, such that any two edges on the same page do not cross and every vertex has incident edges on at most k pages. Here, two edges cross if their endpoints alternate in the linear ordering. The *local page number* $p_l(G)$ of a graph G is the minimum k such that there exists a k-local book embedding for G.

Given a graph G and a vertex ordering, we prove that it is $\mathcal{NP}$-complete to decide whether there exists a k-local book embedding for G with respect to the given vertex ordering for any fixed $k \geq 3$. Additionally, we show that there is a planar graph with local page number 3. For every $k \geq 1$ there exists a k-tree with local page number k. For complete graphs, we prove that $\lceil (n-1)/4 \rceil \leq p_l(K_n) \leq \lceil n/2 \rceil - 1$ for every $n \geq 5$.

Deutsche Zusammenfassung

Eine *k-lokale Bucheinbettung* besteht aus einer totalen Ordnung der Knoten eines Graphen und einer Partition seiner Kanten in Mengen von Kanten, die *Seiten* genannt werden, sodass sich je zwei Kanten auf der gleichen Seite nicht kreuzen, und die zu einem Knoten inzidenten Kanten auf maximal k Seiten eingebettet sind. Dabei kreuzen sich zwei Kanten, wenn ihre Endpunkte in der Knotenordnung alternieren. Die *lokale Seitenanzahl* $p_l(G)$ eines Graphen G ist das kleinste k, sodass eine k-lokale Bucheinbettung für G existiert.

Für einen gegebenen Graphen mit fester Knotenordnung und jedes feste $k \geq 3$ zeigen wir die $\mathcal{NP}$-Vollständigkeit des Entscheidungsproblems, ob eine k-lokale Bucheinbettung mit der gegebenen Knotenordnung existiert. Zusätzlich zeigen wir die Existenz von planaren Graphen mit lokaler Seitenanzahl 3 und von k-Bäumen mit lokaler Seitenanzahl k für jedes $k \geq 1$. Für vollständige Graphen zeigen wir für jedes $n \geq 5$ die Ungleichung $\lceil (n-1)/4 \rceil \leq p_l(K_n) \leq \lceil n/2 \rceil - 1$.

Contents

1. Introduction

Since Ollmann and Taylor [OT73] have introduced the concept of book embeddings this concept has been studied extensively. Based on this concept, we propose a local version of book embeddings in this thesis.

A *book* consists of half-planes, called *pages*, which have a common boundary, called *spine*. In a *book embedding*, the vertices of a graph are embedded on the spine, and each edge is embedded in exactly one page, where two edges on the same page may not cross. While the *(global) page number* of a graph G is the smallest number of pages needed for a book embedding of G, the number of pages is not restricted for the local page number. The *local page number* of a graph G is the minimum k such that there is a book embedding of G and every vertex has incident edges on at most k pages. In this thesis we study the local page number for special graph classes and its computational complexity.

1.1 Motivation and Application

Book embeddings have lots of motivations and applications including Very Large-Scale Integration (VLSI) design, permutations, and biology. In VLSI design, processing elements of an electrical circuit are modeled as vertices and connections between them as edges [Yan89]. One approach is to place the vertices on a (conceptual) line and the connecting wires on tracks that are parallel to this line. This setting can be interpreted as an embedding of a graph. Consequently, optimization problems like minimizing the number of tracks (and therefore the area needed for the wiring) can be seen as optimizing the properties of an embedding of the respective graph.

Rosenberg [Ros83] described the Diogenes approach, which is a strategy for designing testable fault-tolerant array of processors. Again, processing elements are modeled as vertices of a graph which lie on a line. The processing elements are connected with bundles of wires which function as a stack. The line can be interpreted as spine of a book embedding and each bundle as a page. With this, the number of required bundles is the page number of the given graph.

Furthermore, book embeddings can be used to permute elements using disjoint, parallel stacks [CLR87]. Here, we have an initial sequence $1, \dots, n$ which is pushed to stacks in ascending order. After that, the elements are popped forming a permutation. Tarjan [Tar72] asked which permutations are possible with a fixed number of stacks. For this, a bipartite graph G with vertices $a_1, \dots, a_n, b_1, \dots, b_n$ and edges a_ib_i for $i \in \{1, \dots, n\}$ is

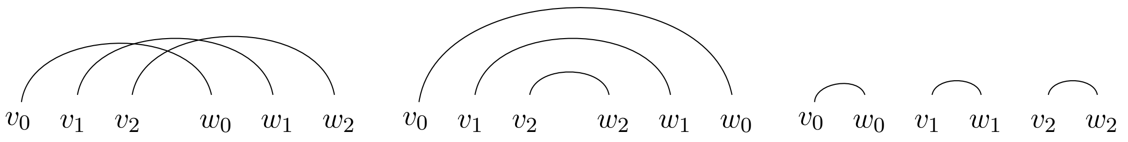


Figure 1.1: Forbidden edge sets in book embeddings, queue layouts, and arch layouts, respectively

constructed. Given a permutation π, the vertices are embedded on the spine in the ordering $a_1, \ldots, a_n, b_{\pi(1)}, \ldots, b_{\pi(n)}$. Forming π with k parallel stacks is equivalent to embedding G with that fixed ordering in k pages [CLR87].

Finally, book embeddings are used to investigate RNA structures. RNA structure can be visualized with vertices (base pairs) on a spine and each page contains a secondary structure [CDD$^+$12].

1.2 Related Work

The idea of local page numbers is based on the concept of (global) page numbers. Related to this, we discuss queue numbers and arch numbers in the following section. Finally, we consider the local covering number, which is the basis for the definition of the local page number in the next chapter.

We start by giving an overview of some results on book embeddings which have been studied extensively in the past. For instance, Bernhart and Kainen [BK79] give a characterization of graphs with small page number. A graph has page number at most 1 if and only if it is outerplanar. Additionally, a graph has page number at most 2 if and only if it is a subgraph of a planar Hamiltonian graph. Yannakakis [Yan89] proved that every planar graph can be embedded in a 4-page book. On the other hand, Bernhart and Kainen [BK79] showed that there exist planar graphs with page number 3. Heath [Hea84] considered a subclass of planar graphs and proved that stellations of a triangle can be embedded in three pages. For a stellation of a triangle, we start with a triangle. In each iteration step we place a vertex in every face and triangulate the resulting graph (see Definition 2.8).

Additionally, Ganley and Heath [GH01] showed that every k-tree (maximal graphs with treewidth k, see Definition 2.7) can be embedded in a book using at most $k+1$ pages. Vandenbussche, West, and Yu [VWY09] proved that this bound is best possible by constructing a k-tree that does not embed in k pages for $k \geq 3$. Moreover, Rengarajan and Veni Madhavan [RVM95] proved that every 2-tree has page number at most 2. For complete graphs on n vertices, Bernhart and Kainen [BK79] proved that the page number equals $\lceil n/2 \rceil$ for $n \geq 4$.

Regarding $\mathcal{NP}$-completeness, we distinguish whether the ordering of the vertices on the spine is fixed or not. If the vertex ordering is fixed, then determining the number of pages needed for embedding a graph is equivalent to the circle graph coloring problem which is $\mathcal{NP}$-complete [GJMP80]. On the other hand, Wigderson [Wig82] proved that the Hamiltonian circuit problem is $\mathcal{NP}$-complete for maximal planar graphs. Hence, deciding whether a graph without fixed vertex ordering has page number at most 2 is $\mathcal{NP}$-complete.

Given a linear ordering $\prec$ of a vertex set, a *k-twist* is a set E of edges such that $E = \{v_i w_i \colon v_i, w_i \in V,\ i \in \{0, \ldots, k-1\}\}$ and $v_0 \prec \cdots \prec v_{k-1} \prec w_0 \prec \cdots \prec w_{k-1}$ (see Figure 1.1a). Since two edges on the same page may not cross in any book embedding, no two edges embedded on the same page form a twist. Hence, a graph with a fixed vertex ordering that has a k-twist cannot be embedded in less than k pages. However, Kostochka [Kos88] (see also [KK97]) proved that there are vertex orderings with no $(k+1)$-twist which cannot be embedded in less than $\Omega(k \log k)$ pages.

Book embeddings are also called stack layouts [HR92]. Here, each page is seen as a stack. When scanning the vertices on the spine from left to right, an edge is pushed to the stack when its left endpoint is scanned. Similarly, an edge is popped from the stack when its right endpoint is scanned. If a vertex is the left endpoint of more than one edge, then the edges are pushed in descending order of their right endpoints. Note that edges are pushed and popped in first-in-last-out order since edges on the same page do not cross.

In comparison to the concept of twists and book embeddings, Heath and Rosenberg [HR92] introduced rainbows and queue layouts. For a linear ordering $\prec$ of a vertex set V, a *k-rainbow* is a set E of edges such that $E = \{v_i w_i \colon v_i, w_i \in V,\ i \in \{0, \dots, k-1\}\}$ and $v_0 \prec \cdots \prec v_{k-1} \prec w_{k-1} \prec \cdots \prec w_0$. See Figure 1.1b for an illustration of a 3-rainbow. Given a linear ordering of vertices, a *queue* is a set of edges such that no two edges form a 2-rainbow. When scanning the vertices from left to right, we say an edge is enqueued when its left endpoint is scanned. Similarly, an edge is dequeued when its right endpoint is scanned. If a vertex is the left endpoint of more than one edge, then the edges are pushed in ascending order of their right endpoints. Observe that edges in a queue appear in first-in-first-out order. A *k-queue layout* of a graph G consists of a linear ordering of the vertex set $V(G)$ and a partition of $E(G)$ into k queues.

While there are vertex orderings of a graph with no $(k+1)$-twist which do not admit a k-page book embedding [Kos88], queue layouts can be characterized using rainbows. Heath and Rosenberg [HR92] proved that there is a k-queue layout for a vertex ordering of a graph if and only if it has no $(k+1)$-rainbow. Given a graph G and a linear ordering of its vertices, they also showed that such a k-queue layout can be found in $\mathcal{O}(|E(G)| \log \log |V(G)|)$. However, deciding whether a graph without fixed vertex ordering has queue number at most 1 is $\mathcal{NP}$-compete [HR92].

Heath and Rosenberg [HR92] conjectured that the queue number of planar graphs can be bounded by a constant. Although the best lower bound on the queue number of planar graphs is a constant, their conjecture remains open. However, Dujmović [Duj15] proved that there exists a queue layout with $\mathcal{O}(\log n)$ queues for every n-vertex planar graph.

Similar to book embeddings and queue layouts, Dujmović and Wood [DW04] discussed the concept of arch layouts. For a linear ordering $\prec$ of a vertex set V, a *k-necklace* is a set E of edges such that $E = \{v_i w_i \colon v_i, w_i \in V, i \in \{0, \dots, k-1\}\}$ and $v_0 \prec w_0 \prec v_1 \prec w_1 \prec \cdots \prec v_{k-1} \prec w_{k-1}$ as shown in Figure 1.1c. Given a linear ordering of vertices, an *arch* is a set of edges such that no two edges form a 2-necklace. A *k-arch layout* of a graph G consists of a linear ordering of the vertex set $V(G)$ and a partition of $E(G)$ into k arches. The *arch number* of a graph G is the minimum k such that there exists a k-arch layout of G.

There are special graph classes for which the arch number is known [DW04]. For instance, a complete graph on n vertices has arch number $\lfloor n/2 \rfloor$. Additionally, planar graphs require at most three arches, and this bound is best possible. Like queue layouts can be characterized using rainbows, a similar result can be stated for arch layouts. Dujmović and Wood [DW04] proved that there is a k-arch layout for a vertex ordering of a graph if and only if it has no $(k+1)$-necklace. In contrast to book embeddings, the minimum number of arches required for an arch layout and an assignment of edges to arches can be computed in $\mathcal{O}(|V(G)| + |E(G)|)$ if the vertex ordering is given [DW04]. However, determining whether a graph has arch number at most k is $\mathcal{NP}$-compete for $k \geq 2$ [DW04].

For the graph parameters discussed above, local versions can be defined. Knauer and Ueckerdt [KU16] introduced the local covering number. In a covering problem, an input graph H and a covering class $\mathcal{G}$ is given. The graph H is covered by a set of graphs from $\mathcal{G}$ if every covering graph is a subgraph of H and every edge of H is contained in some covering graph. Covers are defined more formally in Section 2.1.

The (global) covering number is the minimum k such that H can be covered with k graphs from $\mathcal{G}$. In contrast, the number of covering graphs from $\mathcal{G}$ is not restricted for the local covering number. The local covering number is the minimum k such that every vertex in $V(H)$ is contained in at most k covering graphs from $\mathcal{G}$. There are different input graphs and covering classes for which the local covering number has been studied. For instance, complete bipartite graphs were considered by Fishburn and Hammer [FH96], and complete graphs by Skums, Suzdal, and Tyshkevich [SST09]. Additionally, Knauer and Ueckerdt [KU16] studied linear, star, and caterpillar forests and interval graphs as covering classes. Based on the local covering number, we propose a local version of book embeddings in Chapter 2.

1.3 Outline

In the second chapter we introduce notions that are necessary for the subsequent chapters. For instance, we define the global and local page number and introduce notations for book embeddings with fixed vertex ordering.

In Chapter 3 we give lower and upper bounds on the local page number of special graph classes. We start with a characterization of outerplanar graphs and continue with a lower and an upper bound for planar graphs. Even when the class of graphs is not restricted, we can give bounds on the number of embedded edges and pages of a k-local book embedding. Additionally, we show that the gap between global and local page number can be arbitrarily large. Finally, lower and upper bounds on the local page number of complete graphs and k-trees are given.

We consider the problem of finding a k-local book embedding for a given graph with fixed vertex ordering in Chapter 4. We show $\mathcal{NP}$-completeness for the case of k-local book embeddings with fixed $k \geq 3$.

In Chapter 5 we introduce the union page number and compare the local, union, and global versions of book embeddings.

2. Preliminaries

In the following chapter we introduce basic concepts and notations that are used throughout this thesis. In particular, we define global and local versions of covers and book embeddings, discuss k-trees and stellations, and introduce satisfiability problems.

If not stated otherwise, we assume graphs to be finite, simple (that is no loops nor multiple edges), and undirected. Consider a graph G. Let $V(G)$ denote the vertex set of G and let $E(G)$ denote the edge set of G. If there is an edge $\{v, w\} \in E(G)$, then we simply write $vw \in E(G)$ and say v and w are *adjacent*. The *degree* $\deg(v)$ of a vertex v is the number of edges that are incident to v. The *outdegree* and *indegree* of a vertex v in a directed graph is denoted by $\deg_{\text{out}}(v)$ and $\deg_{\text{in}}(v)$, respectively.

A *complete graph on n vertices*, denoted by K_n or n-clique, consists of n vertices which are pairwise adjacent. A complete graph on three vertices is also called a *triangle*. If the vertex set of a graph G can be partitioned into two sets A and B with $|A| = m$ and $|B| = n$, and we have $E(G) = \{ab\colon\ a \in A, b \in B\}$, then G is called *complete bipartite* and is denoted by $K_{m,n}$. A graph G' is a subgraph of G, denoted by $G' \subseteq G$, if $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$. A subgraph is induced by a subset X of $V(G)$ and denoted by $G[X]$ if $E(G[X]) = \{vw \in E(G)\colon\ v, w \in X\}$.

Next, we consider embeddings of graphs into the Euclidean plane $\mathbb{R}^2$. We embed the vertices of a graph by placing them at pairwise distinct positions. Edges are represented by Jordan curves. More precisely, an edge vw is represented by a continuous injective map $f\colon [0, 1] \to \mathbb{R}^2$ with $f(0) = v$ and $f(1) = w$, where the endpoints v and w are the only vertices lying inside the edge. If any two edges of an embedding of a graph do not cross, we call the embedding a *plane graph*. After removing all vertices and edges of a plane graph, the connected components of $\mathbb{R}^2$ are called *faces*. There is exactly one unbounded face, which we call *outer face*. All other faces are called *inner faces*. A vertex or an edge is incident to a face f if it is contained in the closure of f.

A *planar graph* is a graph G for which there exists a plane graph that is isomorphic to G. A graph G is called *outerplanar* if there is a plane graph representing G such that all vertices are incident to the outer face. A connected plane graph G is called an *inner triangulation* if all inner faces are triangles and the outer face is bounded by a cycle, or if $G \cong K_2$. Note that an inner triangulation on n vertices has exactly $2n - 3$ edges, where $n \geq 2$.

Notions that are used in this thesis but not introduced can be found in [Wes01].

2.1 Global and Local Covering Number

In this section we formalize the concept of global and local covering numbers, following the definitions of Knauer and Ueckerdt [KU16].

Let G and H be graphs. A map $\varphi\colon V(G) \to V(H)$ is called a *homomorphism* if for every edge $vw \in E(G)$ there is an edge $\varphi(v)\varphi(w) \in E(H)$. If for every edge $xy \in E(H)$ there is an edge $vw \in E(G)$ with $\varphi(v) = x$ and $\varphi(w) = y$, then we call φ *edge-surjective*.

For a cover, an *input graph* H and a set of graphs $\mathcal{G}$, called *covering class*, is given. The graph H is covered by a subset of graphs from $\mathcal{G}$ if every covering graph is a subgraph of H and every edge of H is contained in some covering graph. More formally, for an input graph H, a *$\mathcal{G}$-cover of size k* is an edge-surjective homomorphism $\varphi\colon G_0 \dot\cup \ldots \dot\cup G_{k-1} \to H$, where $G_i \in \mathcal{G}$ for $i \in \{0, \ldots, k-1\}$ and $\dot\cup$ denotes the vertex disjoint union. If φ restricted to G_i is injective for every $i \in \{0, \ldots, k-1\}$, then φ is called *injective*.

The *global covering number* $c_g^{\mathcal{G}}(H)$ is the minimum size of an injective $\mathcal{G}$-cover of H. The *local covering number* $c_l^{\mathcal{G}}(H)$ is the minimum k such that there is an injective $\mathcal{G}$-cover of H and each vertex of H is contained in at most k covering graphs. Here, the size of the $\mathcal{G}$-cover is not restricted. More precisely, we define

$$\begin{aligned} c_g^{\mathcal{G}}(H) &= \min\left\{\text{size of } \varphi\colon \varphi \text{ is an injective } \mathcal{G}\text{-cover of } H\right\}, \\ c_l^{\mathcal{G}}(H) &= \min\left\{\max_{v \in V(H)} \left|\varphi^{-1}(v)\right| \colon \varphi \text{ is an injective } \mathcal{G}\text{-cover of } H\right\}. \end{aligned}$$

If the covering class $\mathcal{G}$ is the set of forests, then $c_g^{\mathcal{G}}(G)$ is called *arboricity* and is denoted by $a(G)$. If $\mathcal{G}$ is restricted to the set of star forests, then $c_g^{\mathcal{G}}(G)$ is called *star arboricity*.

2.2 Global and Local Page Number

The concepts of *books*, *book embeddings*, and *page numbers* were introduced by Bernhart and Kainen [BK79]. Based on this, we propose a local version of book embeddings and define notations used in the next chapters.

Definition 2.1. *A* book with k pages *is a set of k half planes, called* pages*, in 3-dimensional space such that they have a common boundary, called* spine*.*

Definition 2.2. *A* book embedding with k pages *(or k-*page book embedding*) embeds a graph G in a book with k pages such that the vertices lie at pairwise distinct positions on the spine and every edge is embedded on exactly one page, where no two edges on the same page cross. Moreover, every page contains at least one edge.*

Definition 2.3. *The* (global) page number $p(G)$ *of a graph G (sometimes referred to as* book thickness *[BK79] or* stack number *[HR92]) is the minimum $k \in \mathbb{N}_0$ such that there exists a book embedding with k pages for G.*

Consider a book embedding Γ of a graph G. Let $\mathcal{P}(\Gamma)$ be the set of pages of Γ. For a page $P \in \mathcal{P}(\Gamma)$ and a vertex $v \in V(G)$ we denote $\deg_P(v)$ as the the number of edges that are incident to v and are embedded on page P. Let $V(P)$ denote the vertex set of P with $V(P) = \{v \in V(G)\colon \deg_P(v) > 0\}$ and $E(P)$ the edge set of P with $E(P) = \{e \in E(G)\colon e \text{ is embedded on page } P\}$. Note that isolated vertices are not contained in the vertex set of any page.

While $p(G)$ is the global page number for a graph G, $p_G(v)$ denotes the number of pages on which a vertex v has incident edges with respect to a graph G, that is $p_G(v) = |\{P \in \mathcal{P}(\Gamma)\colon \exists\, w \in V(P)\colon vw \in E(P)\}|$.

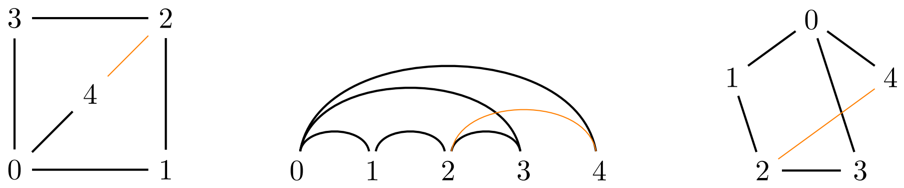


Figure 2.1: Planar, linear, and circular layout of a graph

Here, we propose a local version of *book embeddings* and *page numbers*, following the global concepts.

Definition 2.4. *A* k-local book embedding with t pages *of a graph G is a book embedding with t pages such that every vertex $v \in V(G)$ has incident edges on at most k pages, that is $p_G(v) \leq k$ for every vertex $v \in V(G)$.*

Definition 2.5. *The* local page number $p_l(G)$ *of a graph G is the minimum $k \in \mathbb{N}_0$ such that there exists a k-local book embedding for G.*

Note that a book embedding with k pages is k-local. Hence, we have $p_l(G) \leq p(G)$ for every graph G.

Next, we introduce definitions and notations for book embeddings of graphs with fixed vertex ordering. For a book embedding Γ of a graph G with an ordering $\prec$ on the vertex set $V(G)$, we assume that the vertices are embedded on the spine according to $\prec$. For vertices v and w, we say v is to the *left of* w and w is to the *right of* v if $v \prec w$. We define that $v \preccurlyeq w$ if $v \prec w$ or $v = w$. Moreover, we write $v \succ w$ if $w \prec v$, and $v \succcurlyeq w$ if $w \preccurlyeq v$. For vertex sets V and V', we write $V \prec V'$ or $V' \succ V$ if for all vertices $v \in V$ and $v' \in V'$ we have $v \prec v'$. For vertices v, w, and x with $v \prec w \prec x$, we say that w is between v and x. We say two vertices v and w lie on the spine consecutively if there is no vertex between v and w.

For a book embedding of a graph G with an ordering $\prec$ on $V(G)$ and a vertex set $X \subseteq V(G)$, let the *span* of X be $\operatorname{sp}(X) = \{v \in V(G) \colon \exists x, x' \in X \colon x \preccurlyeq v \preccurlyeq x'\}$. Intuitively, the span contains all vertices lying between the leftmost and the rightmost vertex of X.

In Chapter 4 we investigate the problem of finding a k-local book embedding for a given vertex ordering. For this, we define the following decision problem for any fixed integer $k \geq 1$.

Definition 2.6. k-LOCAL BOOK EMBEDDING WITH FIXED VERTEX ORDERING

Given a graph G with a linear ordering $\prec$, is there a k-local book embedding of G such that the vertices are ordered on the spine according to $\prec$?

In Definition 2.1 we define a spine to be a straight line. We call this embedding *linear layout* of a book embedding. However, we can restrict the spine to a line segment that contains all vertices embedded on the spine. Identifying the endpoints of this line segment results in a *circular layout*. Figure 2.1 shows a linear and the corresponding circular layout of a book embedding.

Consider a circle with chords. A graph is called *circle graph* if its vertex set can be identified with a set of chords and there is an edge between two vertices if and only if the corresponding chords intersect. Note that chords having a common endpoint do not intersect. A proper coloring of a circle graph can also be seen as a coloring of the chords such that no two chords of the same color intersect. Figure 2.2 shows a coloring of a circle

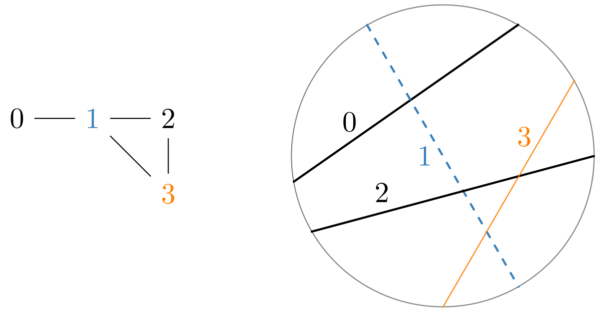


Figure 2.2: Coloring of a circle graph and the corresponding chords

graph and the corresponding chords. Hence, finding a book embedding for a graph with fixed vertex ordering can be considered as coloring problem of a circle graph, which is $\mathcal{NP}$-hard [GJMP80].

In the figures of this thesis we use linear and circular layouts equivalently. When illustrating a book embedding of a graph, the edge set of each page is indicated by a unique color.

2.3 k-Trees and Stellations

In the following section we define k-trees and stellations, following the definitions of Ganley and Heath [GH01], and Bernhart and Kainen [BK79], respectively. We present a stellation of a planar graph for which local and global page number differ. Stellations of a triangle can be embedded in three pages, which is proven afterwards.

Definition 2.7. *A* k-tree *with $k \geq 1$ is a complete graph on k vertices or a graph defined inductively as follows: If G is a k-tree and $K \subseteq G$ is a k-clique in G, then adding a new vertex which is incident exactly to the k vertices of K results in a k-tree. When constructing a k-tree according to the inductive definition above, we start with a k-clique which we call* central k-clique*. We remark that every k-clique of a k-tree can be chosen as central k-clique.*

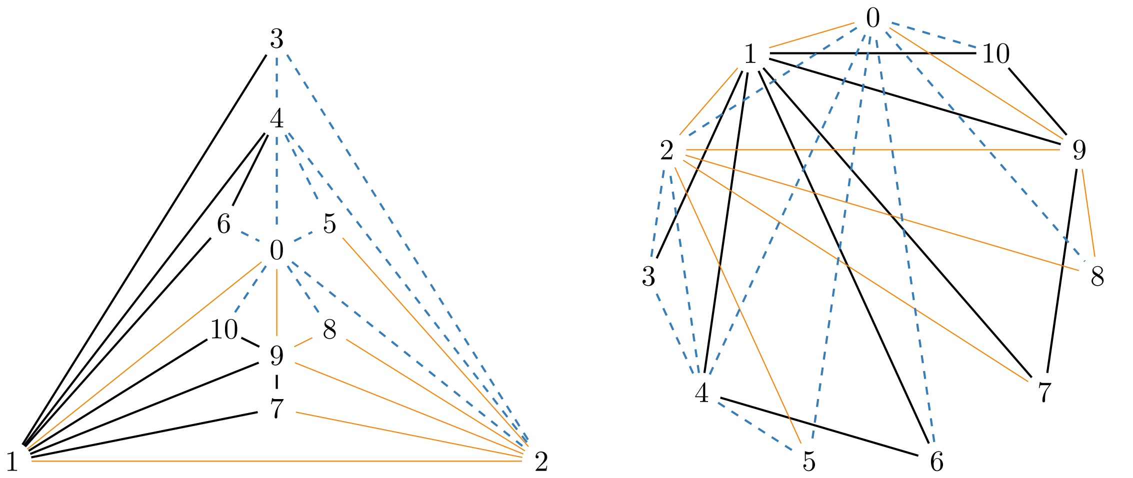
Definition 2.8. *A* stellation $\mathrm{ST}(G)$ *of a 2-connected plane graph G is the result of placing a new vertex in each face of G and adding edges to each vertex around the face. A stellation of a planar graph is also planar. For $n \geq 1$ we define $\mathrm{ST}^n(G)$ as $\mathrm{ST}(\mathrm{ST}^{n-1}(G))$, where $\mathrm{ST}^0(G) \cong G$.*

Note that repeated stellations of a triangle ($\mathrm{ST}^n(K_3)$ for $n \geq 0$) are stacked triangulations (planar 3-trees). Bernhart and Kainen [BK79] presented $\mathrm{ST}^2(K_3)$ as a planar graph that has global page number 3. See Figure 2.3a for a planar embedding of $\mathrm{ST}^2(K_3)$. However, there is a 2-local book embedding with three pages for this graph, as shown in Figure 2.3b. We present $\mathrm{ST}^9(K_3)$ as repeated stellation of a triangle with local page number 3 in Section 3.1.

Heath [Hea84] proved that $\mathrm{ST}^n(K_3)$ has global page number at most 3 not only for $n = 2$ but for every $n \geq 0$. We follow his proof presented in [Hea84].

Proposition 2.9 (Heath [Hea84])**.** *Every stellation of a triangle can be embedded in at most three pages, that is $p(\mathrm{ST}^n(K_3)) \leq 3$ for every $n \geq 0$.*

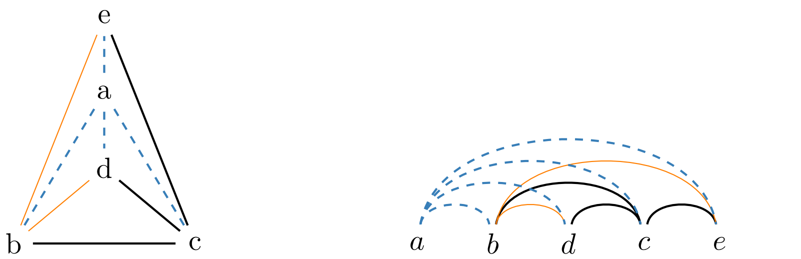
Proof. We prove the proposition inductively starting with $\mathrm{ST}(K_3)$. Figure 2.4a shows a planar embedding of $\mathrm{ST}(K_3)$, while Figure 2.4b shows a book embedding of $\mathrm{ST}(K_3)$ with three pages.



(a) Planar embedding

(b) 2-local book embedding with three pages

Figure 2.3: Embeddings of $\mathrm{ST}^2(K_3)$. The vertices 0, 1, and 2 are the vertices of the initial triangle.



(a) Planar embedding

(b) Book embedding satisfying Conditions (i) and (ii) for d and e.

Figure 2.4: Embeddings of $\mathrm{ST}(K_3)$

When placing a new vertex z in a triangular face with incident vertices u, v, and w, we maintain the following conditions:

(i) The vertex z and one of u, v, and w lie on the spine consecutively.

(ii) The edges uz, vz, and wz are embedded on three pairwise distinct pages.

Figure 2.4b shows a book embedding of $\mathrm{ST}(K_3)$ with three pages fulfilling the two conditions above. The vertices a, b, and c form the initial triangle. Vertex d is placed in the inner face, and e is placed in the outer face. Note that d and c lie on the spine consecutively, also c and e, which satisfies Condition (i). Condition (ii) is clearly met for the added vertices d and e.

Next, we consider a copy of K_4 that has a vertex of degree 3 and place new vertices in the inner faces. Note that the copies of K_4 having a vertex of degree 3 are exactly those which are created in the previous step. Let a, b, and c be the vertices of a triangle, and let d be a vertex placed in this triangle fulfilling Conditions (i) and (ii). We have three triangular face in which new vertices f, g, and h are placed according to Figure 2.5. Note that c satisfies Condition (i) for d, and f is the vertex that is not adjacent to c. The situation for g and h is symmetric.

The new vertices are embedded on the spine as follows (see Figure 2.6). Recall that d and c lie on the spine consecutively. Without loss of generality, d is embedded to the left of c.

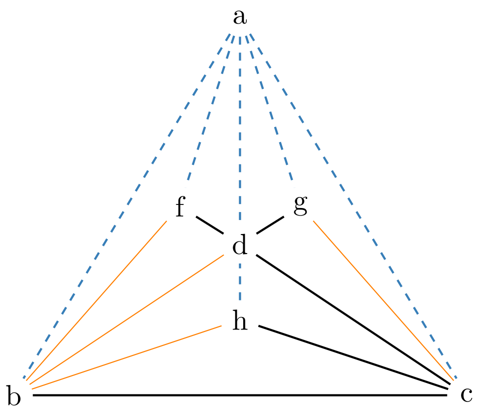


Figure 2.5: The vertices a, b, c, and d induce a copy of K_4, where d fulfills Conditions (i) and (ii) by inductive hypothesis. The vertices f, g, and h are added in the current induction step.

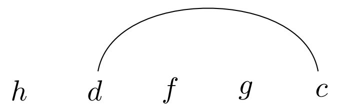


Figure 2.6: Recall that d and c lie on the spine consecutively, say $d \prec c$. The new vertices f, g, and h are embedded so that $h \prec d \prec f \prec g \prec c$ and there is no vertex between h and d.

The new vertices are embedded so that $h \prec d \prec f \prec g \prec c$ and there is no vertex between h and d. Note that c fulfills Condition (i) for g, and d for f and h.

Now, we embed the edges between a, b, c, and d and the new vertices f, g, and h. The embedding is shown in Figure 2.7. For this, let P_a, P_b, and P_c be three pages on which ad, bd, and cd are embedded, respectively. Due to Condition (ii), the three page are pairwise distinct. For $x \in \{a, b\}$ and $y \in \{f, g, h\}$, we embed xy on page P_x (if xy exists). Similarly, the edge hc is embedded on page P_c. However, we embed cg on page P_b. It remains to embed the edges that are incident to d and to one of f, g, and h. These edges are embedded such that Condition (ii) is met, that is each of f, g, and h has incident edges on three pairwise distinct pages.

Below, we prove that any two edges on the same page do not cross. First, consider an edge xy with $x \in \{a, b\}$ and $y \in \{f, g, h\}$. Recall that xy and xd are embedded on the same page P_x. Every edge crossing xy but not xd is incident to d or to a vertex which is embedded between d and y (see Figure 2.8). More precisely, every edge that crosses xy on

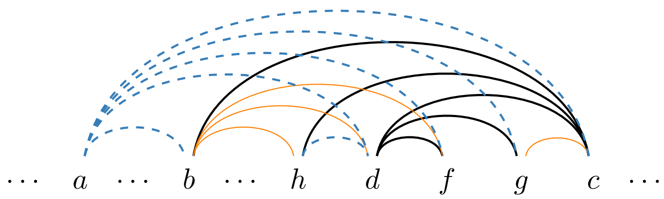


Figure 2.7: Vertex d satisfies Conditions (i) and (ii). The vertices f, g, and h are added in the current step so that Conditions (i) and (ii) are met.

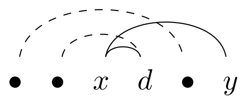


Figure 2.8: Every edge crossing xy but not xd is incident to d or a vertex between d and y.

page P_x is incident to d or f. By construction, however, all edges incident to d, f, or y that are embedded on P_x are incident to x, and thus do not cross.

Similarly, there is no edge but bd and ad crossing hc but not dc (compare Figure 2.7). However, both edges are not embedded on P_c. Thus, the edge hc does not cross any edges on P_c. The edges cg, dh, and df do not cross any other edges since the respective vertices are embedded consecutively. Recall that dg is embedded on page P_c. The edge dg is crossed only by af and bf, which are not embedded on P_c.

After embedding new vertices as described above, we show that Conditions (i) and (ii) still hold for all other copies of K_4 with a vertex of degree 3. Recall that f, g, and h are embedded next to d or between d and c, where d and c are consecutive by Condition (i) (see Figure 2.6). Note that each edge can fulfill Condition (i) for at most one vertex. Additionally, each vertex occurs at most once in the role of d. Hence, we can place one vertex in each face and embed all added edges in three pages. □

2.4 Satisfiability Problem

In this section we introduce the satisfiability problems SAT and 3SAT, as defined by Garey and Johnson [GJ79]. We use these problems in Section 4.1 for proving $\mathcal{NP}$-completeness of k-LOCAL BOOK EMBEDDING WITH FIXED VERTEX ORDERING.

Let U be a set of Boolean variables. For a variable $u \in U$ we say u is a *positive literal* and $\bar{u}$ is a *negative literal*. A map $t\colon U \to \{\text{true}, \text{false}\}$ is called a *truth assignment*. If $t(u) = \text{true}$ for a variable u, then we say the literal u is true and $\bar{u}$ is false. Similarly, the literal u is false and $\bar{u}$ is true if $t(u) = \text{false}$. A *clause* over U is a finite set of literals in U, joined together by the Boolean *or*. A clause C is satisfied by truth assignment t if there is a literal in C that is true. A set of clauses $\mathcal{C}$ is satisfiable if there is a truth assignment such that every clause in $\mathcal{C}$ is satisfied.

Definition 2.10. SAT

Given a set U of Boolean variables and a set $\mathcal{C}$ of clauses over U, is $\mathcal{C}$ satisfiable?

Definition 2.11. 3SAT

Given a set U of Boolean variables and a set $\mathcal{C}$ of clauses over U, where each clause $C \in \mathcal{C}$ contains exactly three literals, is $\mathcal{C}$ satisfiable?

Gary and Johnson proved $\mathcal{NP}$-completeness for both problems [GJ79].

3. Bounds on the Local Page Number

In the following chapter we give bounds on the number of edges and pages in k-local book embeddings and on the local page number of special graph classes. For instance, outerplanar graphs, planar graphs, complete graphs, and k-trees are considered. Additionally, we investigate the gap between global and local page number. The results of this chapter are summarized in Table 3.1.

3.1 Planar Graphs

Bernhart and Kainen [BK79] proved that graph a can be embedded in a 1-page book if and only if it is outerplanar. Similarly, we give a characterization of graphs with local page number 1.

Proposition 3.1. *For every graph G the following statements are equivalent:*

(i) G has local page number at most 1.

(ii) G has global page number at most 1.

(iii) G is outerplanar.

Proof. Let G be a graph with local page number at most 1. We prove that G can be embedded in a 1-page book. Consider a book embedding Γ of G. Any two adjacent edges lie on the same page. Thus, each connected component is embedded on a single page. We construct a book embedding Γ' by sorting the vertices such that all vertices belonging to the same connected component are embedded consecutively on the spine. By construction, two edges belonging to different connected components do not cross in Γ'. Since two connected components can be embedded independently in Γ', all connected components can be embedded on the same page.

Since the local page number is always less than or equal to the global page number, it follows that (i) and (ii) are equivalent. Additionally, Bernhart and Kainen [BK79] proved that a graph has global page number at most 1 if and only if it is outerplanar. □

In contrast to outerplanar graphs, there are planar graphs that cannot be embedded in a 1-local book embedding. Yannakakis [Yan89] proved that the global page number of planar graphs is at most 4. Although this already implies that for every planar graph there exists a 4-local book embedding, we present a simpler prove for the local page number.

	Local Page Number			
	Lower Bound		Upper Bound	
General Graphs	$\frac{\lvert E(G)\rvert}{2\lvert V(G)\rvert-3}$	(Cor. 3.8)	$p(G)$	
Outerplanar	1		1	(Prop. 3.1)
Planar	3	(Prop. 3.4)	4	(Prop. 3.2)
k-Tree	k	(Prop. 3.21)	$k+1$	(Prop. 3.22)
Stellation of K_3	3	(Prop. 3.4)	3	([Hea84], Prop. 2.9)
K_n with $n \geq 5$	$\lceil\frac{n-1}{4}\rceil$	(Prop. 3.11)	$\lceil\frac{n}{2}\rceil - 1$	(Prop. 3.12)
K_2, K_3	1		1	
K_4, K_5, K_6	2	(Obs. 3.13 and 3.14)	2	(Obs. 3.13 and 3.14)
K_7, K_8, K_9	3	(Obs. 3.13 and 3.15)	3	(Obs. 3.13 and 3.15)
K_{10}, K_{11}	4	(Cor. 3.19)	4	(Cor. 3.19)

Table 3.1: Lower and upper bounds on the local page number

Proposition 3.2. *For every planar graph there exists a 4-local book embedding.*

Proof. We start with a 3-orientation of a planar graph, which we use to find a 4-local book embedding. Chrobak and Eppstein [CE91] proved that for every planar graph there exists a 3-orientation, that is there is an orientation of the edges such that $\deg_{\text{out}}(v) \leq 3$ for every vertex v.

Let G be a planar graph. Based on a 3-orientation of G we construct a 4-local book embedding Γ with $|V(G)|$ pages. Let $\mathcal{P}(\Gamma)$ be the set of pages of Γ with $\mathcal{P}(\Gamma) = \{P_v \colon v \in V(G)\}$. We embed an edge vw that is oriented from v to w on page P_w.

Next, we prove that Γ is 4-local. Consider a vertex $v \in V(G)$. All edges that are oriented towards v are embedded on the same page P_v. Since there are at most three edges that are oriented from v to a neighbor of v, it follows that $p_G(v) \leq 4$. Hence, Γ is 4-local. $\square$

While for every planar graph there exists a 4-page book embedding [Yan89], it is not known whether there is a planar graph with global page number 4. However, Bernhart and Kainen [BK79] presented a planar graph with global page number 3. More precisely, they proved that $p(\mathrm{ST}^2(K_3)) = 3$. The following lemma prepares a proof for $p_l(\mathrm{ST}^9(K_3)) = 3$.

Lemma 3.3. *Let G be a complete graph on four vertices, and let Γ be a 2-local book embedding for G. Then there is an edge $vw \in E(G)$ such that $\mathcal{P}_v = \mathcal{P}_w$ and $|\mathcal{P}_v| = |\mathcal{P}_w| = 2$, where $\mathcal{P}_v$ and $\mathcal{P}_w$ are the sets of pages that contain edges incident to v or w, respectively.*

Proof. Let $\mathcal{P}(\Gamma)$ denote the set of pages of Γ. Since G is not outerplanar, Γ has at least two pages. In addition, Γ has less than four pages by Lemma 3.6 as $|E(G)| = 6 > 4 = 2{\cdot}2{\cdot}4 - 3{\cdot}4$.

Next, we consider the cases of Γ having two or three pages. If $|\mathcal{P}(\Gamma)| = 2$, then there are two vertices v and w having incident edges on both pages. Since G is a complete graph, the edge vw exists.

On the other hand, any three edges form a triangle in G. Thus, we have a triangle T with all three edges embedded on pairwise distinct pages if $|\mathcal{P}(\Gamma)| = 3$. Let v be the fourth vertex of G, that is $v \in V(G) \setminus V(T)$. We have $|\mathcal{P}_v| = 2$ since the vertices in $V(T)$ do not share a common page. Choose any two of the three pages containing T for v. Since the

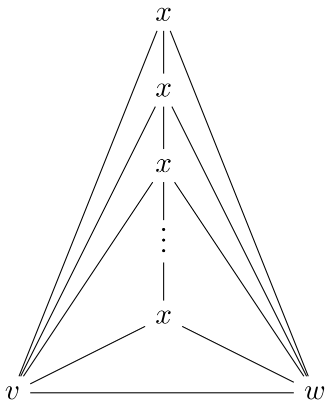


Figure 3.1: By Lemma 3.3 there exists an edge $vw \in V(G_1)$ with $\mathcal{P}_v = \mathcal{P}_w$. There is a set $X \subseteq V(G_8)$ of nine vertices (marked with x) incident to v and w and inducing a path in G_8.

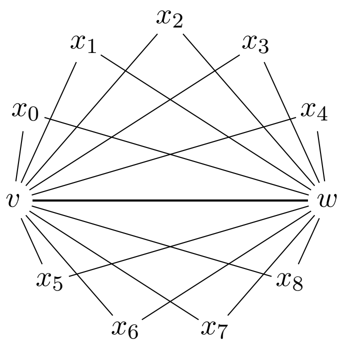


Figure 3.2: There are five vertices in X that are incident two v and w and lie in the same part of the circle.

three edges of T are embedded on pairwise distinct pages, there is a vertex $w \in V(T)$ with $\mathcal{P}_v = \mathcal{P}_w$, and v and w are adjacent, which proves the lemma. □

Proposition 3.4. *For every $n \geq 9$ we have $p_l(\mathrm{ST}^n(K_3)) = 3$.*

Proof. Since stellations of K_3 have global page number at most 3 (see Proposition 2.9), we have $p_l(\mathrm{ST}^n(K_3)) \leq p(\mathrm{ST}^n(K_3)) \leq 3$ for every $n \geq 0$. We shall prove that $p_l(\mathrm{ST}^9(K_3)) \geq 3$. For $n \geq 9$, it follows that $p_l(\mathrm{ST}^n(K_3)) \geq 3$ since $\mathrm{ST}^9(K_3)$ is a subgraph of $\mathrm{ST}^n(K_3)$.

Let $G_0, \dots, G_9$ denote subgraphs of $\mathrm{ST}^9(K_3)$ such that $K_3 = G_0 \subseteq \cdots \subseteq G_9$ and G_{i+1} is constructed by placing a vertex in each inner face of G_i and connecting it to all vertices around the face for $i \in \{0, \dots, 8\}$. Suppose there is a 2-local book embedding for $\mathrm{ST}^9(K_3)$.

First, consider G_1, which is a complete graph on four vertices. By Lemma 3.3, there is an edge $vw \in E(G_1)$ such that $\mathcal{P}_v = \mathcal{P}_w$ and $|\mathcal{P}_v| = |\mathcal{P}_w| = 2$, where $\mathcal{P}_v$ and $\mathcal{P}_w$ are the sets of pages that contain edges incident to v or w, respectively. We call these two pages P and P', that is $\mathcal{P}_v = \mathcal{P}_w = \{P, P'\}$. By construction, there is a set X of nine vertices $x_0, \dots, x_8 \in V(G_8)$ that are incident to v and w and induce a path in G_8 (see Figure 3.1).

Consider a circular book embedding of G_8 as shown in Figure 3.2. Observe that the circle is partitioned by edge vw so that at least five vertices of X lie in the same part. Without loss of generality, we say $x_0, \dots, x_4$ are embedded in the same part. Now, consider a linear

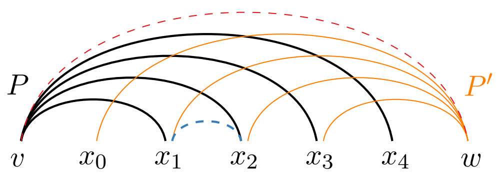
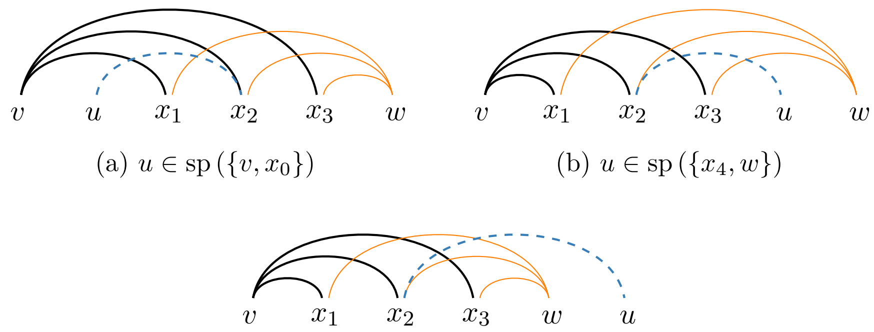


Figure 3.3: Dashed edges are embedded either on P or on P'. The edge x_1x_2 is contained in a triangle that is embedded on a single page.



(a) $u \in \mathrm{sp}(\{v, x_0\})$ (b) $u \in \mathrm{sp}(\{x_4, w\})$

(c) $u \notin \mathrm{sp}(\{v, w\})$

Figure 3.4: The edge x_2u is embedded either on P or on P'. If $u \notin \{x_1, x_3\}$, then x_2u crosses edges on both pages.

book embedding with $v \prec x_0 \prec x_1 \prec x_2 \prec x_3 \prec x_4 \prec w$ and no other vertex of X is embedded in $\mathrm{sp}(\{x_0, x_4\})$ (see Figure 3.3).

Recall that P and P' are the only pages containing edges incident to v or w. Without loss of generality, we have $vx_4 \in E(P)$. Note that the edges x_0w, x_1w, x_2w, and x_3w cross vx_4, and thus are embedded on page P'. Since $x_0w \in E(P')$, we have $vx_1, vx_2, vx_3 \in E(P)$.

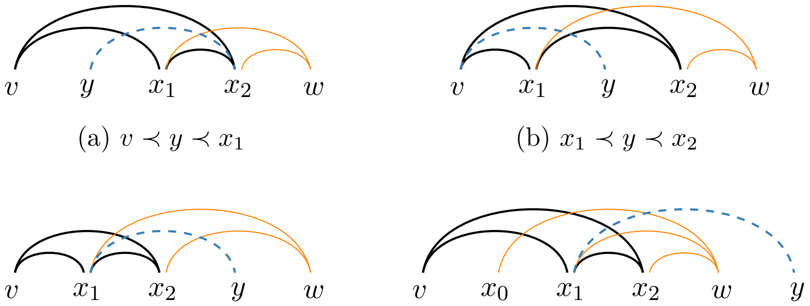
Now, we consider x_2 and its neighborhood. Recall that X induces a path in G_8. Hence, x_2 has a neighbor $u \in X$. We prove that $u \in \{x_1, x_3\}$, and thus at least one of the edges x_1x_2 and x_2x_3 exists. Suppose $u \notin \{x_1, x_3\}$. Recall that $x_2 \in V(P) \cap V(P')$, which implies that x_2u is embedded on P or P'. We distinguish the following three cases and observe that x_2u crosses edges on both pages (see Figure 3.4).

If $u \in \mathrm{sp}(\{v, x_0\})$, then x_2u crosses vx_1 and wx_1. Symmetrically, x_2u crosses vx_3 and wx_3 if $u \in \mathrm{sp}(\{x_4, w\})$. Finally, x_2u crosses vx_3 and wx_1 if $u \notin \mathrm{sp}(\{v, w\})$. Therefore, we have $u \in \{x_1, x_3\}$. Without loss of generality, we assume $u = x_1$ and thus $x_1x_2 \in E(G_8)$.

Note that x_1 and x_2 form two triangles with v and with w. Recall that $vx_1, vx_2 \in E(P)$ and $wx_1, wx_2 \in E(P')$. Hence, we have three edges embedded on a common page and forming a triangle T. Without loss of generality, we assume $x_1x_2 \in E(P)$ and hence $v \in V(T)$.

Next, consider the vertex $y \in V(G_9)$ that is incident to all vertices of T. Note that the vertices v, x_1, and x_2 have incident edges on P and P'. Hence, we have $vy, x_1y, x_2y \in E(P) \cup E(P')$. Recall that $vx_1, vx_2, x_1x_2 \in E(P)$ and $wx_1, wx_2 \in E(P')$. Figure 3.5 illustrates the following four cases.

If y is embedded between v and x_1, the edge yx_2 crosses vx_1 and wx_1. On the other hand, the edge vy crosses x_1x_2 and wx_1 if y is embedded between x_1 and x_2. If y is embedded between x_2 and w, then the edge x_1y crosses vx_2 and wx_2. Finally, x_1y crosses vx_2 and



(a) $v \prec y \prec x_1$ (b) $x_1 \prec y \prec x_2$

(c) $x_2 \prec y \prec w$ (d) $y \notin \mathrm{sp}(\{v, w\})$

Figure 3.5: In all four cases there is an edge incident to y that cannot be embedded in a 2-local book embedding.

wx_0 if $y \notin \mathrm{sp}(\{v, w\})$. In all four cases, there is an edge incident to y crossing edges on P and P', and thus cannot be embedded. Therefore, there is no 2-local book embedding for $\mathrm{ST}^9(K_3)$. $\square$

Note that stellations of a triangle are planar, which leads to the following corollary.

Corollary 3.5. *There are planar graphs with local page number 3.*

3.2 General Graphs

Given a graph, we can find a lower bound on its local page number even when the class of graphs is not restricted. We also give bounds on the number of embedded edges and the number of pages of a k-local book embedding. For special graph classes, however, better bounds are presented in the next sections.

Lemma 3.6. *Let $n \geq 1$. Every graph G on n vertices for which there exists a k-local book embedding with t pages has at most $2kn - 3t$ edges. Moreover, we have $|E(G)| = 2kn - 3t$ if and only if for all pages P the subgraph embedded on P is an inner triangulation and $p_G(v) = k$ for all vertices $v \in V(G)$.*

Proof. Let Γ be a k-local book embedding of a graph G with t pages, and let $\mathcal{P}(\Gamma)$ be the set of pages of Γ. We can find an upper bound on the number of edges:

$$|E(G)| = \sum_{P \in \mathcal{P}(\Gamma)} |E(P)| \overset{(*)}{\leq} \sum_{P \in \mathcal{P}(\Gamma)} (2\,|V(P)| - 3) = 2 \sum_{P \in \mathcal{P}(\Gamma)} |V(P)| - 3t \overset{(**)}{\leq} 2kn - 3t.$$

Inequality $(*)$ holds since every subgraph G' of G that is embedded on a single page is outerplanar and thus has at most $2\,|V(G')| - 3$ edges. We have an equality in $(*)$ if and only if G' is an inner triangulation.

The book embedding Γ is k-local, so every vertex is counted at most k times in the sum $\sum_{P \in \mathcal{P}(\Gamma)} |V(P)|$. Hence, inequality $(**)$ holds. If $p_G(v) = k$ for all vertices $v \in V(G)$, then $(**)$ is an equality. On the other hand, if we have $\sum_{P \in \mathcal{P}(\Gamma)} |V(P)| = kn$, then every vertex has incident edges on exactly k pages since Γ is k-local. $\square$

For further usage of Lemma 3.6 it is convenient to state results not depending on a the number of pages of a given book embedding.

Corollary 3.7. *Let $n \geq 1$. Every graph G on n vertices with local page number at most k has at most $(2n-3)k$ edges.*

Proof. Let Γ be a k-local book embedding of a graph G on n vertices with t pages. Lemma 3.6 bounds the number of edges of G to be at most $2kn - 3t$. However, every vertex can have incident edges on at most t pages, so $k \leq t$. It follows that

$$|E(G)| \leq 2kn - 3t \leq 2kn - 3k = (2n-3)k.$$

□

Corollary 3.8. *Every graph G on n vertices and m edges has local page number at least $m/(2n-3)$.*

Lemma 3.9. *If a graph G on n vertices has a k-local book embedding Γ, then Γ has at most $kn/2$ pages.*

Proof. Let t be the number of pages of Γ. Since empty pages are not allowed, the number of pages can be bounded by the number of edges with $t \leq |E(G)|$. With Lemma 3.6 it follows that $t \leq |E(G)| \leq 2kn - 3t$ and thus $t \leq kn/2$. □

Note that there exist graphs on n vertices and an integer k for which every k-local book embedding has less than $kn/2$ pages. For instance, a complete graph on six vertices can be embedded such that $p_{K_6}(v) \leq 2$ for every vertex v (see Observation 3.13). However, every 2-local book embedding of K_6 has exactly three pages. At least three pages are needed since the global page number equals 3. By Lemma 3.6, every 2-local book embedding of a 6-vertex graph with at least four pages has at most $2 \cdot 2 \cdot 6 - 3 \cdot 4 = 12$ edges, which is less than the number of edges of K_6. Hence, at most three pages can be used.

On the other hand, there exist graphs for which the bound of Lemma 3.9 is best possible. A k-regular graph G on n vertices can be embedded such that every edge lies on its own page. In such an embedding we have $p_G(v) = k$ for every vertex $v \in V(G)$, and the number of pages equals the number of edges, that is $t = |E(G)| = kn/2$.

Next, we consider graphs for which the global and local page number differ. We show that the gap can be arbitrarily large.

Proposition 3.10. *There exist n-vertex graphs with local page number at most k but global page number $\Omega\left(\sqrt{k} n^{1/2-1/k}\right)$.*

Proof. Malitz [Mal94] proved that there exist k-regular n-vertex graphs which require $\Omega(\sqrt{k} n^{1/2-1/k})$ pages. Embedding every edge on its own page results in a k-local book embedding. Hence, the local page number of a k-regular graph is at most k. □

3.3 Complete Graphs

In this section we find a lower and an upper bound on the local page number of complete graphs. Additionally, we specify the exact local page number for complete graphs on at most eleven vertices.

Proposition 3.11. *Let $n \geq 2$. The local page number of a complete graph on n vertices is greater than $\lceil (n-1)/4 \rceil$.*

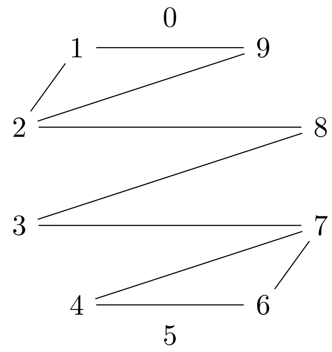


Figure 3.6: Page P_0 of K_{10}

Proof. By Corollary 3.8, the local page number of a graph on n vertices and m edges is at least $m/(2n-3)$. For the local page number of a complete graph it follows

$$p_l(K_n) \geq \frac{|E(K_n)|}{2\,|V(K_n)|} = \frac{\binom{n}{2}}{2n} = \frac{n(n-1)}{4n} = \frac{n-1}{4}.$$

Since the local page number is integer, it is at least $\lceil (n-1)/4 \rceil$. □

Proposition 3.12. *Let $n \geq 5$. The local page number of a complete graph on n vertices is at most $\lceil n/2 \rceil - 1$.*

Note that the global page number of a complete graph on n vertices equals $\lceil n/2 \rceil$ [BK79], so the local page number is strictly smaller.

Proof. First, let n be even. We construct a $(n/2-1)$-local book embedding Γ for K_n with t pages such that $t = n/2$. Let $\mathcal{P}(\Gamma)$ be the set of pages with $\mathcal{P}(\Gamma) = \{P_0, \ldots, P_{t-1}\}$. Let the vertices $0, \ldots, (n-1)$ of K_n lie on the spine in this ordering. For the construction, all vertices are taken modulo n.

Figure 3.6 shows how edges are embedded on page P_0. The construction is rotated for embedding on other pages.

For $i \in \{0, \ldots, t-1\}$ we define

$$\begin{aligned} V(P_i) &= V(K_n) \setminus \{i, i+t\} \\ E(P_i) &= \{\{i+1, i+2\}, \{i+t+1, i+t+2\}\} \\ &\quad \cup \{ab\colon\ a, b \in \{i, \ldots, n+i-1\},\ a+b \in \{2i+n, 2i+1+n\}, \\ &\qquad a, b \not\equiv i \mod t\}. \end{aligned}$$

Every vertex $v \in V(K_n)$ has edges on at most $t-1$ pages since $i, i+t \notin V(P_i)$ for $i \in \{0, \ldots, t-1\}$.

Every page P_i can be embedded without any intersecting edges: The edges $\{i+1, i+2\}$ and $\{i+t+1, i+t+2\}$ do not cross any other edges since their end points lie on the spine consecutively. Suppose there are two edges intersecting on page P_i, that is there exist edges $ac, bd \in E(P_i)$ such that $a < b < c < d$. By construction, $a + c \in \{2i+n, 2i+1+n\}$ and $b + d \in \{2i+n, 2i+1+n\}$. It follows that $b + d \geq a+1+c+1 \geq 2i+2+n > 2i+1+n$, which is a contradiction.

The constructed graph is isomorphic to K_n: Each page has $n-1$ edges. Hence, the total number of edges equals $t(n-1)$, which is $\binom{n}{2}$. It remains to show that no edge occurs twice, that is for $i, j \in \{0, \ldots, t-1\}, i \neq j\colon\ E(P_i) \cap E(P_j) = \emptyset$. Let $ab \in E(P_i) \cap E(P_j)$.

By construction, $a + b \in \{2i + n, 2i + 1 + n\} \cap \{2j + n, 2j + 1 + n\}$, so the intersection is nonempty. Since n is even, $2i + n$ and $2j + n$ are even, and $2i + 1 + n$ and $2j + 1 + n$ are odd. Hence, we have $2i + n = 2j + n$ or $2i + 1 + n = 2j + 1 + n$. In both cases it follows that $i = j$.

Now, let n be odd. By the first part, a complete graph on $n + 1$ vertices has local page number at most $\lceil (n + 1)/2 \rceil - 1$. Since K_n is a subgraph of K_{n+1}, we can bound the local page number with $p_l(K_n) \leq p_l(K_{n+1}) \leq \lceil (n + 1)/2 \rceil - 1 = \lceil n/2 \rceil - 1$. □

After proving bounds on complete graphs with arbitrarily many vertices, we continue with exact results for small complete graphs. Both the lower and the upper bound are not best possible. For instance, we prove $p_l(K_9) = 3$ in Observation 3.15, so we have $\lceil (9 - 1)/4 \rceil = 2 < p_l(K_9) < 4 = \lceil 9/2 \rceil - 1$.

Observation 3.13. *Every complete graph on n vertices with $5 \leq n \leq 8$ has local page number $\lceil n/2 \rceil - 1$.*

Proof. Proposition 3.12 shows that the local page number of a complete graph K_n is at most $\lceil n/2 \rceil - 1$. Thus, for K_5 and K_6 the local page number is at most 2. Since both graphs are not outerplanar, it is exactly 2 (see Proposition 3.1).

By Proposition 3.12, the local page number of complete graphs on seven or eight vertices is at most 3. Suppose there exists a 2-local book embedding Γ of K_7 with t pages. Since K_7 has global page number 4 [BK79], Γ uses at least four pages. With Lemma 3.6 it follows that at most $2 \cdot 2 \cdot 7 - 3 \cdot 4 = 16$ edges can be embedded, which is less than the number of edges of K_7. Hence, the local page number of K_7 is exactly 3. Since K_7 is a subgraph of K_8, it follows that K_8 has local page number 3. □

Observation 3.14. *The complete graph on four vertices has local page number 2.*

Proof. The complete graph K_4 is not outerplanar. By Proposition 3.1, its local page number is at least 2. Since K_4 is a subgraph of K_5 the local page number is exactly 2. □

Observation 3.15. *The complete graph on nine vertices has local page number 3.*

Proof. Figure 3.7 shows a 3-local book-embedding of K_9. Hence, the local page number is at most 3. However, the global page number of K_9 is 5 [BK79]. By Lemma 3.6, a 9-vertex graph for which there exists a 2-local book embedding with at least five pages has at most $2 \cdot 2 \cdot 9 - 3 \cdot 5 = 21$ edges, which is less than the number of edges of K_9. Thus, the local page number of K_9 is strictly greater than 2. □

The following lemma is helpful for proving further results on graphs containing K_9 as a subgraph in this section and in the next chapter.

Lemma 3.16. *Every 3-local book embedding of K_9 has exactly six pages.*

Note that the global page number of K_9 equals 5 [BK79]. Hence, there exists no book embedding of K_9 which is optimal for local and global page number.

Proof. Let Γ be a 3-local book embedding of K_9 with t pages. Let $\mathcal{P}(\Gamma)$ be the set of pages of Γ. By Lemma 3.6, we have $36 = |E(K_9)| \leq 2 \cdot 3 \cdot 9 - 3t = 54 - 3t$, which implies $t \leq 6$.

Suppose $t \leq 5$. Let $0, \ldots, 8$ denote the vertices of K_9. All vertices are taken modulo 9. Consider a partition $E_1, \ldots, E_4$ of $E(K_9)$ with $E_i = \{\{a, b\} : a + i \equiv b \mod 9\}$. The parts E_1, E_2, and E_4 form 9-cycles, while E_3 consists of three 3-cycles, as shown in Figure 3.8.

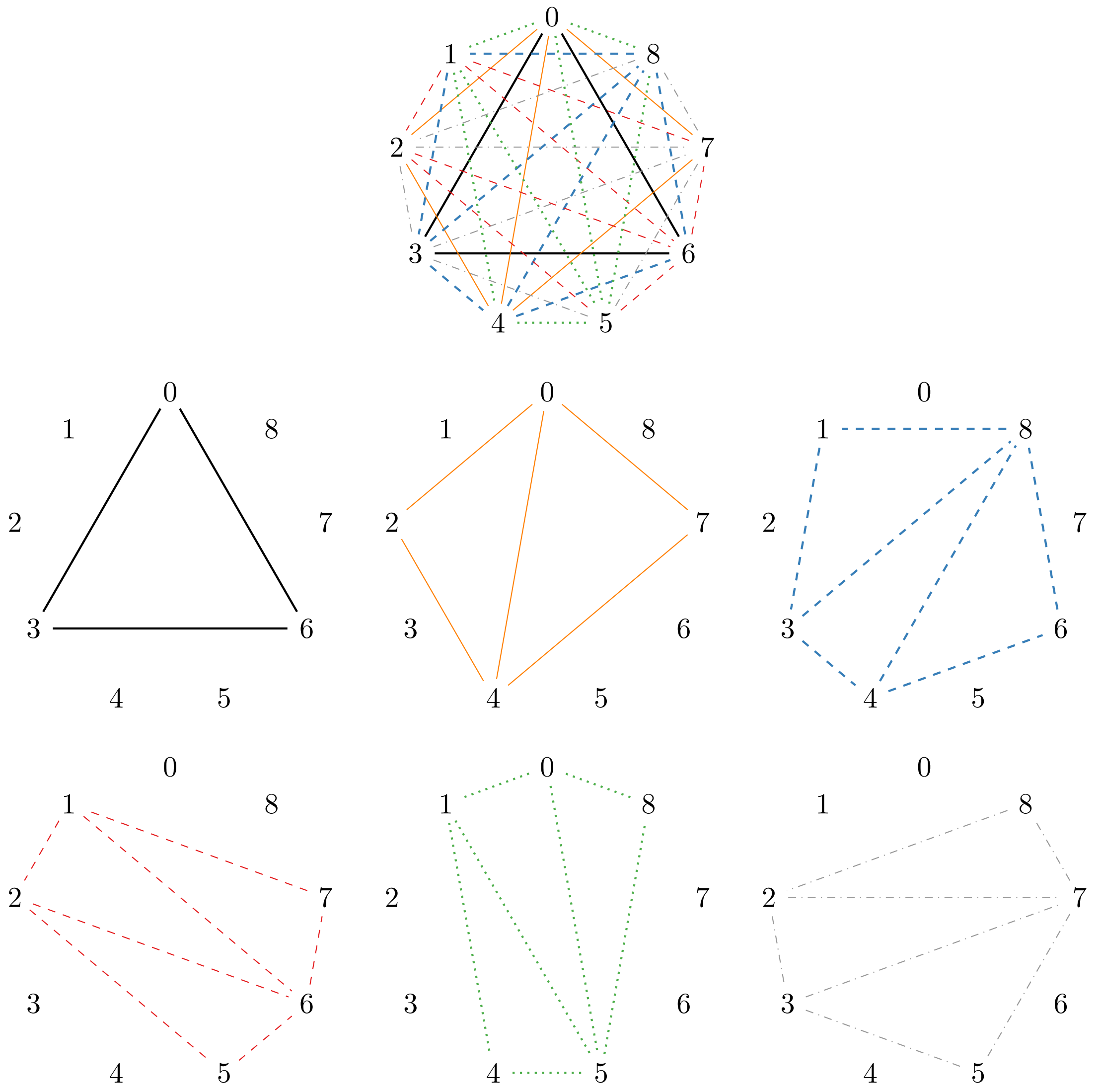


Figure 3.7: 3-local book embedding of K_9

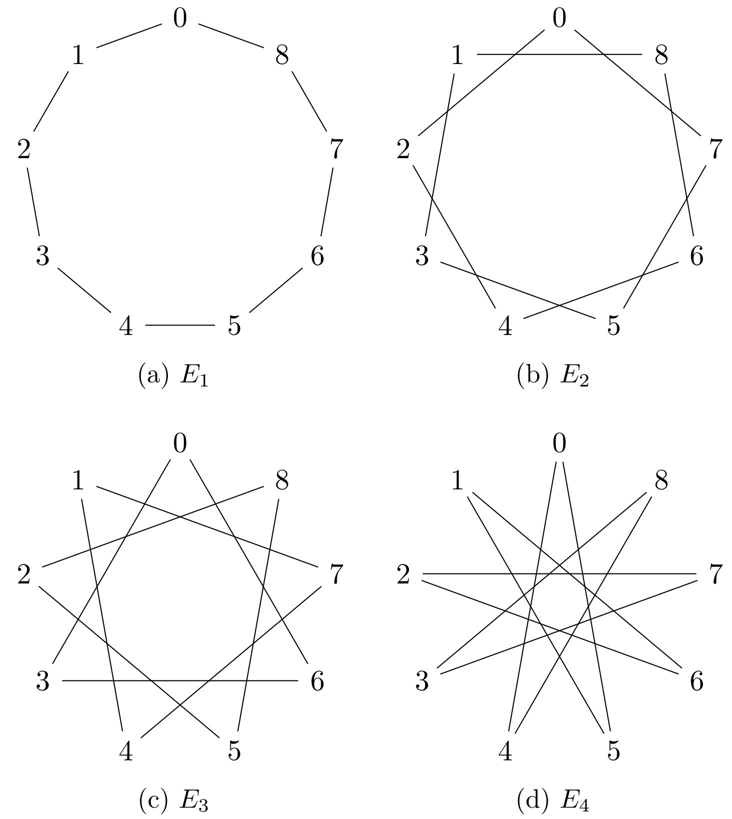


Figure 3.8: Edge sets $E_1, \dots, E_4$

To a graph embedded on a page P, edges can be added such that P embeds an inner triangulation. Without loss of generality, we assume that for every $P \in \mathcal{P}(\Gamma)$ the embedded subgraph is an inner triangulation. However, edges can be embedded on multiple pages. We denote the number of edges that are embedded twice in Γ as $d(\Gamma)$ with $d(\Gamma) = |E(\mathcal{P}(\Gamma))| - |E(K_9)|$, where $|E(\mathcal{P}(\Gamma))| = \sum_{P \in \mathcal{P}(\Gamma)} E(P)$. By Lemma 3.6, the number of edges in Γ is bounded by $|E(\mathcal{P}(\Gamma))| \leq 2 \cdot 3 \cdot 9 - 3 \cdot 5 = 39$. With $|E(K_9)| = 36$, this implies $d(\Gamma) \leq 3$. Below, we try to embed the edge sets $E_1, \dots, E_4$ such that the vertex sets of all pages induce inner triangulations and $d(\Gamma) \leq 3$. We show that such an embedding does not exist.

In E_4 any two non-adjacent edges cross. Hence, at most two edges from E_4 are embedded on the same page. Since $|E_4| = 9$, it follows that at least five pages are necessary for embedding E_4, and thus $t = 5$. Let $\mathcal{P}_4$ be the set of pages on which exactly two edges from E_4 are embedded. With $t = 5$, we have $|\mathcal{P}_4| \geq 4$.

Consider edges $e_0, e_1 \in E_3$ that lie on the same page P. If e_0 and e_1 are non-adjacent, then all other edges from E_3 cross one of e_0 and e_1, and thus are not embedded on the same page. Otherwise, let $e_0 = vw$ and let $e_1 = vx$. If $wx \in E(P)$, then no edge in E_4 is embedded on page P, which is a contradiction. Thus $wx \notin E(P)$. All edges $e \in E_3 \setminus \{vw, vx, wx\}$ cross e_0 or e_1, and thus are not embedded on P. Hence, on every page at most two edges from E_3 are embedded. Let $\mathcal{P}_3$ be the set of pages on which exactly two edges from E_3 are embedded. With $t = 5$, we have $|\mathcal{P}_3| \geq 4$.

Let $\mathcal{P}'$ be the set of pages on which two edges from E_3 and two edges from E_4 are embedded, that is $\mathcal{P}' = \mathcal{P}_3 \cap \mathcal{P}_4$. Since $\mathcal{P}_3$ and $\mathcal{P}_4$ each contain at least four pages and $t = 5$, we have $|\mathcal{P}'| \geq 3$. Consider a page $P \in \mathcal{P}'$, as shown in Figure 3.9. We count the number of edges in E_1 that are embedded on P. Let $e_0, e_1 \in E(P) \cap E_3$ and $e_2, e_3 \in E(P) \cap E_4$.

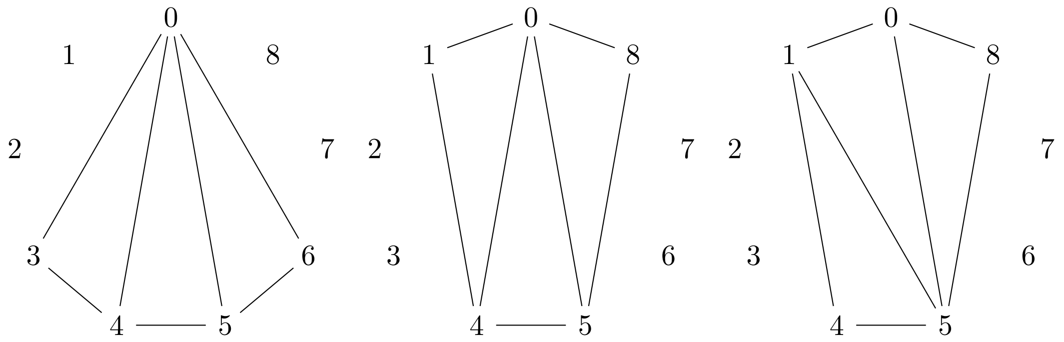


Figure 3.9: Possible inner triangulations on pages in $\mathcal{P}'$

First, assume that e_0 and e_1 are adjacent. Without loss of generality, $e_0 = \{0, 3\}$ and $e_1 = \{0, 6\}$. We have $e_2 = \{0, 4\}$ and $e_3 = \{0, 5\}$ since all other edges in E_4 cross e_0 or e_1. Hence, the edges $\{3, 4\}$, $\{4, 5\}$, and $\{5, 6\}$ are embedded on P.

Now, assume that e_0 and e_1 are non-adjacent. Without loss of generality, $e_0 = \{1, 4\}$ and $e_1 = \{5, 8\}$. Thus, e_2 or e_3 is incident to vertex 0, as shown in Figure 3.9. Since the subgraph embedded on P is an inner triangulation, the edges $\{0, 1\}$, $\{0, 8\}$, and $\{4, 5\}$ are embedded on P.

In both cases, every page $P \in \mathcal{P}'$ contains at least three edges of E_1. If $|\mathcal{P}'| \geq 4$, then $\sum_{P \in \mathcal{P}'} |E_1 \cap E(P)| \geq 12$. With $|E_1| = 9$, it follows that $d(\Gamma) \geq 3$. If $|\mathcal{P}'| = 3$, then $\sum_{P \in \mathcal{P}'} |E_1 \cap E(P)| = 9$. In this case, the subgraph embedded on $P_3 \in \mathcal{P}_3 \setminus \mathcal{P}'$ induces one edge in E_1, and the subgraph embedded on $P_4 \in \mathcal{P}_4 \setminus \mathcal{P}'$ induces another edge in E_1. Thus, we have

$$\sum_{P \in \mathcal{P}(\Gamma)} |E_1 \cap E(P)| = \sum_{P \in \mathcal{P}'} |E_1 \cap E(P)| + |E_1 \cap E(P_3)| + |E_1 \cap E(P_4)| \geq 9 + 1 + 1 = 11.$$

However, with $|E_1| = 9$ this implies $d(\Gamma) \geq 2$.

Since $d(\Gamma) \leq 3$, at most one additional edge in E_1 may be induced by the vertex set of any page. Consider the edge set E_2. Embedding an edge from E_2 on a page $P \in \mathcal{P}'$ induces an edge from E_1 on P since there are at most two consecutive vertices v and w on the spine with $v, w \notin V(P)$. Hence, at most one edge from E_2 can be embedded on pages in $\mathcal{P}'$ and at least eight edges from E_2 are embedded on pages in $\mathcal{P}(\Gamma) \setminus \mathcal{P}'$. Since $|\mathcal{P}(\Gamma) \setminus \mathcal{P}'| \leq 2$, there is one page $P' \in \mathcal{P}(\Gamma) \setminus \mathcal{P}'$ on which at least four edges $e_0, \ldots, e_3 \in E_2$ are embedded. The edges $e_0, \ldots, e_3$ may not cross, and thus form a path on four vertices, which forbids embedding any edge from E_4. This contradicts the fact that five pages are necessary for embedding all edges in E_4. Therefore, a 3-local book embedding with five pages of K_9 does not exist. □

Lemma 3.17. *The local page number of K_{10} is at least 4.*

Proof. Suppose there is a 3-local book embedding Γ with t pages of K_{10}. By Lemma 3.6, we have $45 = |E(K_{10})| \leq 2 \cdot 3 \cdot 10 - 3t = 60 - 3t$, which implies $t \leq 5$. Thus, there is a 3-local book embedding with five pages of K_9, which contradicts Lemma 3.16. □

Lemma 3.18. *The local page number of K_{11} is at most 4.*

Proof. We construct a 4-local book embedding Γ with eleven pages for K_{11}. Let $\mathcal{P}(\Gamma)$ be the set of pages with $\mathcal{P}(\Gamma) = \{P_0, \ldots, P_{10}\}$. Let $0, \ldots, 10$ denote the vertices of K_{11}. All vertices are taken modulo 11.

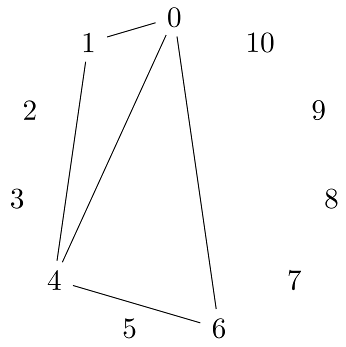


Figure 3.10: Page P_0 of K_{11}

Consider a partition $E_1, \ldots, E_5$ of $E(K_{11})$ with $E_i = \{\{a,b\} : a + i \equiv b \mod 11\}$. Each part contains exactly one cycle on eleven vertices. We construct P_0 such that exactly one edge of each part is embedded on P_0, as shown in Figure 3.10. The construction is rotated for embedding on other pages so that each edge is embedded exactly once.

For $i \in \{0, \ldots 10\}$ we define

$$\begin{aligned} V(P_i) &= \{i, i+1, i+4, i+6\} \text{ and} \\ E(P_i) &= \{\{i, i+1\}, \{i, i+4\}, \{i, i+6\}, \{i+1, i+4\}, \{i+4, i+6\}\}. \end{aligned}$$

Since the construction is rotated exactly $|V(K_{11})|$ times, there is an integer k such that for all vertices $v \in V(K_{11})$ we have $p_{K_{11}}(v) = k$. With $|V(P)| = 4$ for all pages $P \in \mathcal{P}(\Gamma)$, it follows that $k = \sum_{P \in \mathcal{P}(\Gamma)} |V(P)| / |V(K_{11})| = 4$. Hence, Γ is a 4-local book embedding of K_{11}. $\square$

Corollary 3.19. *The local page numbers of K_{10} and K_{11} equal 4.*

Proof. By Lemmas 3.17 and 3.18, we have $p_l(K_{10}) \geq 4$ and $p_l(K_{11}) \leq 4$. Since K_{10} is a subgraph of K_{11}, it follows that $4 \leq p_l(K_{10}) \leq p_l(K_{11}) \leq 4$. $\square$

3.4 k-Trees

In this section we give a lower and an upper bound on the local page number of k-trees.

Proposition 3.20. *For $k \in \{1, 2\}$ the local page number of a k-tree is at most k.*

Proof. Since 1-trees are trees, they are outerplanar and have local page number 1 by Proposition 3.1. Rengarajan and Veni Madhavan [RVM95] proved that every 2-tree can be embedded in two pages. Hence, the global page number, and therefore also the local page number, is at most 2. $\square$

Note that there exist k-trees with local page number 1 and 2, respectively. Trees with at least one edge are 1-trees and have local page number 1. In addition, there are 2-trees that are not outerplanar, and thus have local page number 2.

While the page number of k-trees is at most k for $k \in \{1, 2\}$, Vandenbussche, West, and Yu [VWY09] proved that there exist k-trees with global page number $k+1$ for every $k \geq 3$. We prove that there are k-trees that have local page number k. However, it remains open whether there is a k-tree with local page number $k+1$ for $k \geq 3$.

Proposition 3.21. *For every $k \geq 3$, there is a k-tree G which has a k-page and k-local book embedding. In particular, we have $p(G) = p_l(G) = k$.*

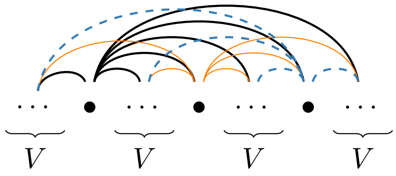


Figure 3.11: 3-tree with local page number 3. All edges incident to a vertex $v \in V(K)$ (marked with •) and to a vertex in V are embedded on page P_v.

Proof. Let $k \geq 3$. We construct a k-tree G that has local page number k. See Figure 3.11 for an illustration of the upcoming construction. Let K be a copy of K_k and let V be a vertex set of size $(2(k(k-1))^k)(k+1)+1$. We construct G by adding edges between each vertex of V and the k-clique K. For this let

$$\begin{aligned} V(G) &= V(K) \cup V \\ E(G) &= E(K) \cup \{uv\colon\ u \in V(K),\, v \in V\}\,. \end{aligned}$$

First, we show that there is a k-local book embedding Γ with k pages for G as shown in Figure 3.11. For this, we take one page P_v for each vertex $v \in V(K)$ and embed all edges incident to v on P_v. Let $\mathcal{P}(\Gamma) = \{P_v\colon\ v \in V(K)\}$ be the set of pages. By construction, every edge in $E(G)$ is incident to at least one vertex in $V(K)$. For edges $uv \in E(K)$, we choose any page of P_u and P_v. We observe that for every page P the subgraph embedded on P is a star, and thus any two edges on P do not cross. Additionally, we have $|\mathcal{P}(\Gamma)| = |V(K)| = k$, and thus Γ is k-local.

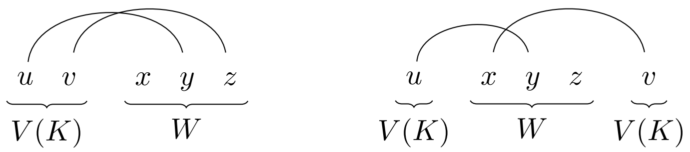
Now, we prove that there is no $(k-1)$-local book embedding for G. Suppose to the contrary that there is a $(k-1)$-local book embedding Γ for G with a page set $\mathcal{P}(\Gamma)$. Let $\prec$ be the linear ordering of the vertices on the spine. Recall that all edges are incident to at least one vertex in $V(K)$. Since Γ is $(k-1)$-local and K has exactly k vertices, we have $|\mathcal{P}(\Gamma)| \leq k(k-1)$.

We have $(2(k(k-1))^k)(k+1)+1$ vertices in V and k vertices in $V(K)$. By pigeon hole principle, there is a set $W \subseteq V$ of size $2(k(k-1))^k+1$ such that $v \notin \operatorname{sp}(W)$ for all vertices $v \in V(K)$. For $v \in V(K)$ and $w \in W$, let $P_{w,v}$ denote the page on which the edge vw is embedded. We say that two vertices w and x in W have the same *edge assignment* if $P_{w,v} = P_{x,v}$ for all vertices $v \in V(K)$. Since $|\mathcal{P}(\Gamma)| \leq k(k-1)$ and $\deg(w) = |V(K)| = k$ for $w \in W$, there are at most $(k(k-1))^k$ vertices in W that have pairwise distinct edge assignments. With $|W| = 2(k(k-1))^k+1$, it follows that there are three vertices x, y, and z in W having the same edge assignment. Without loss of generality, we assume that $x \prec y \prec z$.

Next, we prove that $p_G(y) = k$. With this, we conclude that Γ is not $(k-1)$-local. Consider two distinct vertices $u, v \in V(K)$. We prove that uy and vy are embedded on different pages, that is $P_{y,u} \neq P_{y,v}$. Without loss of generality, we have $u \prec v$ and $u \prec x$. Recall that $x \prec y \prec z$ and that $u, v \notin \operatorname{sp}(W)$. See Figure 3.12 for an illustration. We distinguish whether v is embedded to the left of W or to the right of W.

Recall that x, y, and z have the same edge assignment, that is $P_{x,q} = P_{y,q} = P_{z,q}$ for every vertex $q \in V(K)$. In the first case, we have $u \prec v \prec x$. As shown in Figure 3.12a, the edges uy and vz cross. Hence, we have $P_{y,u} \neq P_{z,v} = P_{y,v}$. In the second case, we have $u \prec x \prec y \prec z \prec v$ (see Figure 3.12b). Thus, the edges uy and xv cross. It follows that $P_{y,u} \neq P_{x,v} = P_{y,v}$.

We conclude that $P_{y,u} \neq P_{y,v}$ for any two distinct vertices $u, v \in V(K)$. With $|V(K)| = k$, it follows that $p_G(y) = k$, which contradicts the assumption that Γ is $(k-1)$-local. Therefore,



(a) Case 1: $v \prec x$
The edges uy and vz cross.

(b) Case 2: $z \prec v$
The edges uy and xv cross.

Figure 3.12: Constructed k-tree with a $(k-1)$-local book embedding. The vertices x, y, and z have the same edge assignment.

the local page number of G is k. Since the global page number cannot be smaller than the local page number, it follows that $p(G) = k$. $\square$

Ganley and Heath [GH01] proved that every k-tree can be embedded using $k+1$ pages. Their proof for the global page number can be simplified for proving an upper bound for the local page number.

Proposition 3.22. *For $k \geq 1$ every k-tree has local page number at most $k+1$.*

Proof. Let G be a k-tree on n vertices, where $n \geq k$. Fix any k-clique as central k-clique K and an ordering of $V(G) \setminus V(K)$ in which the vertices are added to the graph in order to construct the k-tree G. We denote the vertices of G with $0, \ldots, n-1$ so that $0, \ldots, k-1$ are the vertices of K and for $v, w \in V(G) \setminus V(K)$ the vertex v is added to the graph before w if $v < w$. We construct a $(k+1)$-local book embedding Γ for G with an arbitrary ordering of the vertices on the spine. For this, let $\mathcal{P}(\Gamma) = \{P_0, \ldots, P_{n-1}\}$ be a set of n pages. The edges are embedded so that $vw \in E(P_v)$ if and only if $v < w$. We observe that every edge embedded on page P_v is incident to v for $v \in V(G)$. Hence, for every page $P \in \mathcal{P}(\Gamma)$ the subgraph embedded on P is a star, and thus any two edges on P do not cross.

Next, we prove that Γ is $(k+1)$-local. Consider a vertex $v \in V(G)$. By construction of a k-tree, there are at most k vertices that are adjacent to v and are embedded before v. Thus, there are at most k pages that contain edges uv with $u < v$ and $u \in V(G)$. However, all edges vw with $v < w$ and $w \in V(G)$ are embedded on page P_v. Therefore, there are at most $k+1$ pages that contain edges incident to v. $\square$

4. $\mathcal{NP}$-Completeness for k-Local Book Embedding with Fixed Vertex Ordering

In the following chapter we show that k-LOCAL BOOK EMBEDDING WITH FIXED VERTEX ORDERING is $\mathcal{NP}$-complete for $k = 3$. After that, we extend this result by proving $\mathcal{NP}$-completeness for $k \geq 3$. Note that it can be tested in polynomial time whether a graph with a given vertex ordering can be embedded in a 1-local book embedding due to the stack structure of pages. Here, each connected component is considered independently. However, the case $k = 2$ remains open.

4.1 $\mathcal{NP}$-Completeness for $k = 3$

We start with a variation of 3SAT and reduce it to the case $k = 3$.

Definition 4.1. 2-3SAT

Let U be a set of Boolean variables. Let $\mathcal{C}$ be a set of clauses, where each clause consists of exactly three literals, joined together by the Boolean or. *In $\mathcal{C}$ each variable occurs exactly once. Let $\mathcal{C}'$ be a set of clauses, where each clause consists of exactly two literals. In $\mathcal{C}'$ each variable occurs exactly once as positive literal and exactly once as negative literal. Moreover, every clause in $\mathcal{C}'$ contains exactly one positive and one negative literal. Given an instance $(U, \mathcal{C}, \mathcal{C}')$ of* 2-3SAT*, is the corresponding SAT instance $(U, \mathcal{C} \cup \mathcal{C}')$ satisfiable?*

Proposition 4.2. 2-3SAT *is $\mathcal{NP}$-complete.*

Proof. Since 2-3SAT is a special case of the satisfiability problem, 2-3SAT is in $\mathcal{NP}$ [GJ79]. We reduce 3SAT to 2-3SAT. Let $(U_3, \mathcal{C}_3)$ be an instance of 3SAT. We construct an instance $(U, \mathcal{C}, \mathcal{C}')$ of 2-3SAT that is satisfiable if and only if $(U_3, \mathcal{C}_3)$ is satisfiable.

Let $u \in U_3$ be a variable which occurs n times in $\mathcal{C}_3$ with $n \geq 1$. Let $u_0, \ldots, u_{n-1} \in U$ be n new variables. All indices are taken modulo n. Let $\{(\bar{u}_i \vee u_{i+1})\colon\ i \in \{0, \ldots, n-1\}\} \in \mathcal{C}'$. For a clause $C_3 \in \mathcal{C}_3$ that contains u, we have a corresponding clause $C \in \mathcal{C}$. The new clause C is obtained from C_3 by replacing u by u_i and $\bar{u}$ by $\bar{u}_i$ for some $i \in \{0, \ldots, n-1\}$ such that each u_i occurs exactly once in $\mathcal{C}$ for $i \in \{0, \ldots, n-1\}$.

The instance $(U, \mathcal{C}, \mathcal{C}')$ can be constructed in polynomial time since $|U| = |U_3|$, $|\mathcal{C}| = |\mathcal{C}_3|$, and $|\mathcal{C}'| \leq |\mathcal{C}_3|$.

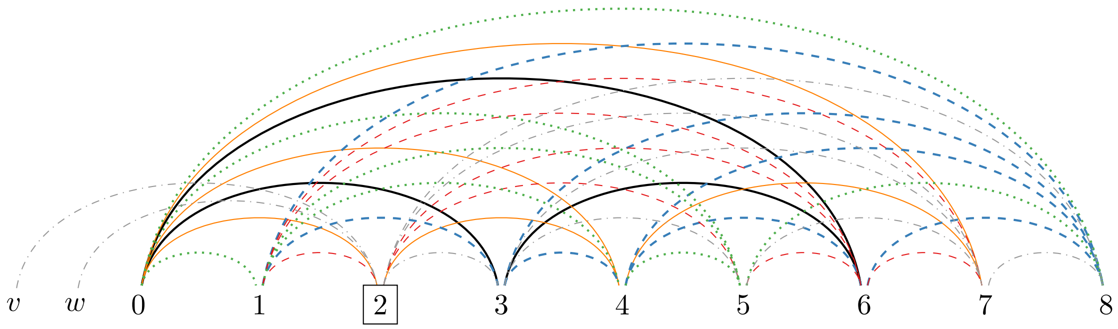


Figure 4.1: 3-local book embedding of K_9 – all edges leaving vertex 2 are embedded on the same page

First, assume that $(U_3, \mathcal{C}_3)$ is satisfiable. Let $t_3 \colon U_3 \to \{\text{true}, \text{false}\}$ be a truth assignment satisfying all clauses in $\mathcal{C}_3$. We construct a truth assignment $t \colon U \to \{\text{true}, \text{false}\}$ that satisfies all clauses in $\mathcal{C} \cup \mathcal{C}'$. For a variable $u \in U_3$ that occurs n times in $\mathcal{C}_3$ and $i \in \{0, \dots, n-1\}$, let $t(u_i) = t_3(u)$. All clauses in $\mathcal{C}'$ are satisfied since each clause contains one positive and one negative literal. All clauses in $\mathcal{C}$ are satisfied since each clause in $\mathcal{C}$ corresponds to a clause in $\mathcal{C}_3$.

Now, assume that $(U, \mathcal{C}, \mathcal{C}')$ is satisfiable. Let $t \colon U \to \{\text{true}, \text{false}\}$ be a truth assignment that satisfies all clauses in $\mathcal{C} \cup \mathcal{C}'$. Let $u \in U_3$ be a variable that occurs n times in $\mathcal{C}_3$. The clause $(\bar{u}_i \vee u_{i+1})$ is equivalent to the implication $t(u_i) \Rightarrow t(u_{i+1})$. Thus, the set of clauses $\{(\bar{u}_i \vee u_{i+1})\colon i \in \{0, \dots, n-1\}\}$ implies $t(u_0) \Leftrightarrow \dots \Leftrightarrow t(u_{n-1})$. We construct a truth assignment $t_3 \colon U_3 \to \{\text{true}, \text{false}\}$ that satisfies all clauses in $\mathcal{C}_3$. Let $t_3(u) = t(u_i)$ for some $i \in \{0, \dots, n-1\}$. Since each clause in $\mathcal{C}_3$ corresponds to a clause in $\mathcal{C}$, all clauses in $\mathcal{C}_3$ are satisfied. □

We use the following lemma to prove $\mathcal{NP}$-completeness of k-LOCAL BOOK EMBEDDING WITH FIXED VERTEX ORDERING for $k = 3$.

Lemma 4.3. *Let K be a copy of K_9 with $V(K) = \{0, \dots, 8\}$. Let G be a graph with $V(G) = V(K) \cup \{v, w\}$. Let Γ be a 3-local book embedding of G with a linear ordering $\prec$ of $V(G)$, where $0 \prec \dots \prec 8$ and $v, w \notin \operatorname{sp}(V(K))$. Then, $\{2, v\}$ and $\{2, w\}$ are embedded on the same page and such an embedding Γ exists.*

Proof. Let $\mathcal{P}(\Gamma)$ be the set of pages of Γ. By Lemma 3.16, the subgraph K is embedded on exactly six pages. Let v and w be vertices with $v, w \notin \operatorname{sp}(V(K_9))$. Figure 4.1 shows that it is possible to find a 3-local book embedding of K_9 such that the edges $\{2, v\}$ and $\{2, w\}$ can be embedded.

With $|E(K_9)| = 36 = 2 \cdot 3 \cdot |V(K_9)| - 3\,|\mathcal{P}(\Gamma)|$ and Lemma 3.6, it follows that for every page $P \in \mathcal{P}(\Gamma)$ the subgraph of K embedded on P is an inner triangulation and that $p_G(x) = 3$ for all vertices $x \in V(K_9)$. Recall that a single edge is an inner triangulation. Hence, there is exactly one page $P \in \mathcal{P}(\Gamma)$ with $0, 1 \in V(P)$ and a set $\mathcal{P}' \subseteq \mathcal{P}(\Gamma)$ of exactly five pages that contain all edges incident to 0 or 1.

Suppose $\{2, v\}$ and $\{2, w\}$ are embedded on different pages, say $\{2, v\} \in E(P_v)$ and $\{2, w\} \in E(P_w)$ with $P_v, P_w \in \mathcal{P}(\Gamma)$. Since $|\mathcal{P}'| = 5$ and $|\mathcal{P}(\Gamma)| = 6$, we have $P_v \in \mathcal{P}'$ or $P_w \in \mathcal{P}'$. Without loss of generality, we assume that $P_v \in \mathcal{P}'$. There is no vertex $x \in \{3, \dots, 8\}$ with $x \in V(P_v)$ as triangulating a graph with vertices in $\{0, 1\}$ and vertices

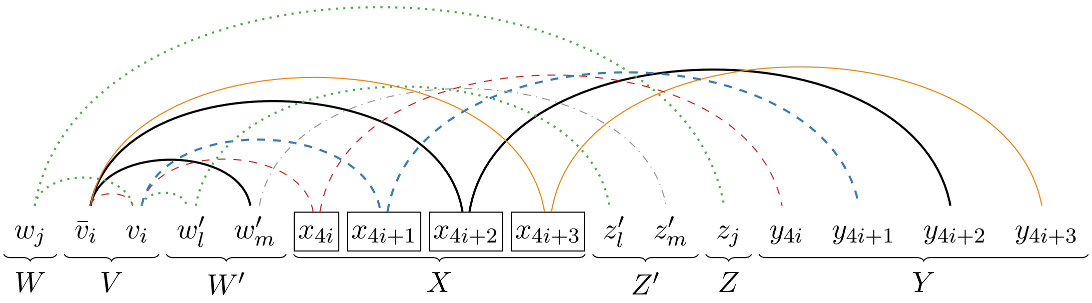


Figure 4.2: Constructed graph for variable $u_i \in U$ and clauses $C_j \in \mathcal{C}$ and $C'_l, C'_m \in \mathcal{C}'$ with $u_i \in C_j, u_i \in C'_l$, and $\bar{u}_i \in C'_m$. The shown vertices in X are part of the gadget presented in Lemma 4.3, which is indicated by boxes. All edges leaving such a vertex to vertices in V or Y are embedded on the same page.

in $\{3, \ldots, 8\}$ results in an edge crossing $\{2, v\}$. Thus, we have $V(P_v) \subseteq \{0, 1, 2\}$ and $E(P_v) \subseteq \{\{0,1\}, \{0,2\}, \{1,2\}\}$. Since edges $\{1, x\}$ with $x \in \{3, \ldots, 8\}$ are not embedded on P_w, no edge embedded on P_w crosses any edge embedded on P_v. Hence, we can construct a page P' with $V(P') = V(P_v) \cup V(P_w)$ and $E(P') = E(P_v) \cup E(P_w)$. Let Γ' be a book embedding with $\mathcal{P}(\Gamma') = \mathcal{P}(\Gamma) \setminus \{P_v, P_w\} \cup \{P'\}$. The constructed book embedding Γ' is 3-local and embeds K_9 in five pages, which contradicts Lemma 3.16. Therefore, all edges leaving vertex 2 to a vertex $v \notin \operatorname{sp}(V(K_9))$ are embedded on the same page. $\square$

The following theorem is the main result of this section.

Theorem 4.4. 3-LOCAL BOOK EMBEDDING WITH FIXED VERTEX ORDERING *is $\mathcal{NP}$-complete.*

Proof. Given a graph G, a linear ordering $\prec$, and a book embedding Γ we can check in polynomial time whether the vertices are embedded on the spine according to $\prec$, whether on every page the embedded subgraph is a plane graph, and whether $p_G(v) \leq 3$ for every vertex $v \in V(G)$. Hence, 3-LOCAL BOOK EMBEDDING WITH FIXED VERTEX ORDERING is in $\mathcal{NP}$.

We reduce 2-3SAT to 3-LOCAL BOOK EMBEDDING WITH FIXED VERTEX ORDERING. Given a 2-3SAT instance $(U, \mathcal{C}, \mathcal{C}')$ we construct a graph G with a linear ordering $\prec$ for which there is a 3-local book embedding if and only if $\mathcal{C} \cup \mathcal{C}'$ is satisfiable over U. Let $V(G) = V \cup X \cup Y \cup W \cup W' \cup Z \cup Z'$ and $E(G) = E_V \cup E_X \cup E_W \cup E_{W'} \cup E_Z \cup E_{Z'}$. The vertex sets and edge sets are constructed below and are illustrated in Figure 4.2.

E_V
V

Let $U = \{u_0, \ldots, u_{r-1}\}$. For $i \in \{0, \ldots, r-1\}$ and a variable $u_i \in U$ we have two vertices v_i and $\bar{v}_i$ and an edge $v_i\bar{v}_i$. We embed $\bar{v}_i$ to the right of v_i if there is a clause in $\mathcal{C}$ which contains $\bar{u}_i$ and to the left of v_i otherwise. Recall that the variable u_i is contained at most once in $\mathcal{C}$, so at most one of the literals u_i and $\bar{u}_i$ is contained in $\mathcal{C}$. For $i, j \in \{0, \ldots, r-1\}$ with $i < j$ we have $\{v_i, \bar{v}_i\} \prec \{v_j, \bar{v}_j\}$. Let

$$\begin{aligned} V &= \{v_i, \bar{v}_i \colon\ i \in \{0, \ldots, r-1\}\} \text{ and} \\ E_V &= \{v_i\bar{v}_i \colon\ i \in \{0, \ldots, r-1\}\}. \end{aligned}$$

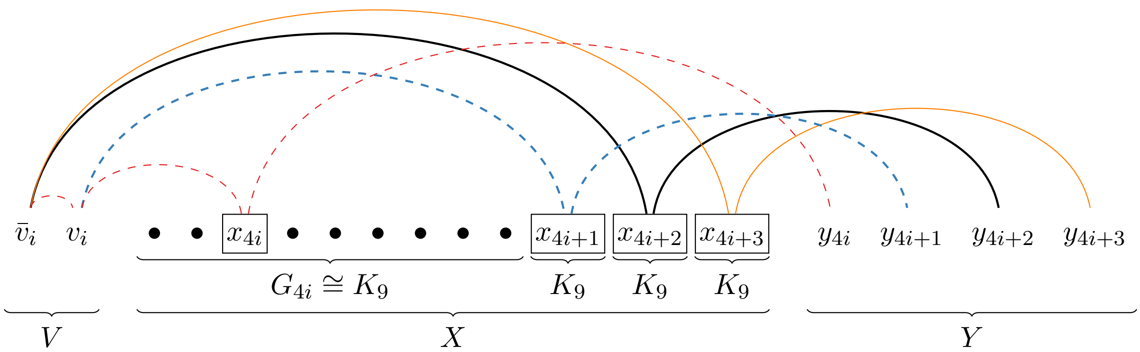


Figure 4.3: The vertex x_{4i} is the vertex of $V(G_{4i})$ which has exactly two vertices of $V(G_{4i})$ to the left. All vertices marked with boxes, like x_{4i}, are contained in a copy of K_9. Lemma 4.3 applied to such a vertex proves that all edges to V or Y are embedded on the same page.

E_X

X, Y

We take $4r$ disjoint copies $G_0, \ldots, G_{4r-1}$ of K_9 with $V \prec V(G_0) \prec \cdots \prec V(G_{4r-1})$. Let X and Y be vertex sets with

$$
\begin{aligned}
X &= \bigcup_{i=0}^{4r-1} V(G_i) \text{ and} \\
Y &= \{y_0, \ldots, y_{4r-1}\}.
\end{aligned}
$$

For $i \in \{0, \ldots, 4r-1\}$ let x_i denote the vertex in $V(G_i)$ which has exactly two vertices of $V(G_i)$ to the left. In the figures of this chapter, these vertices are marked with boxes. We embed Y such that $y_0 \prec \cdots \prec y_{4r-1}$ and $V(G_{4r-1}) \prec Y$. We connect the K_9-gadgets in X with vertices in V and Y as shown in Figure 4.3. For this let

$$
\begin{aligned}
E_X = \; & \{v_i x_{4i},\, v_i x_{4i+1},\, \bar{v}_i x_{4i+2},\, \bar{v}_i x_{4i+3} \colon\, i \in \{0, \ldots, r-1\}\} \\
& \cup \{x_i y_i \colon\, i \in \{0, \ldots, 4r-1\}\} \\
& \cup \bigcup_{i=0}^{4r-1} E(G_i).
\end{aligned}
$$

E_W

W

Let $\mathcal{C} = \{C_0, \ldots, C_{s-1}\}$ be the set of clauses with three variables each. Let W be a set of vertices with

$$
W = \{w_0, \ldots, w_{s-1}\},
$$

$w_0 \prec \cdots \prec w_{s-1}$, and $W \prec V$. For $i \in \{0, \ldots, r-1\}$ and $j \in \{0, \ldots, s-1\}$ we add edges $v_i w_j$ if $u_i \in C_j$ or $\bar{v}_i w_j$ if $\bar{u}_i \in C_j$. For this let

$$
\begin{aligned}
E_W = \; & \{v_i w_j \colon\, u_i \in C_j,\, i \in \{0, \ldots, r-1\},\, j \in \{0, \ldots, s-1\}\} \\
& \cup \{\bar{v}_i w_j \colon\, \bar{u}_i \in C_j,\, i \in \{0, \ldots, r-1\},\, j \in \{0, \ldots, s-1\}\}.
\end{aligned}
$$

$E_{W'}$

W'

Let $\mathcal{C}' = \{C'_0, \ldots, C'_{t-1}\}$ be the set of clauses with exactly two variables each. We take t disjoint copies $G'_0, \ldots, G'_{t-1}$ of K_9. For $l \in \{0, \ldots, t-1\}$ let $V(G_l) = \{a_{l,0}, \ldots, a_{l,8}\}$. Let W' be a set of vertices with

$$
W' = \bigcup_{l=0}^{t-1} V(G'_l) \cup \{w'_l, a_l \colon\, l \in \{0, \ldots, t-1\}\}.
$$

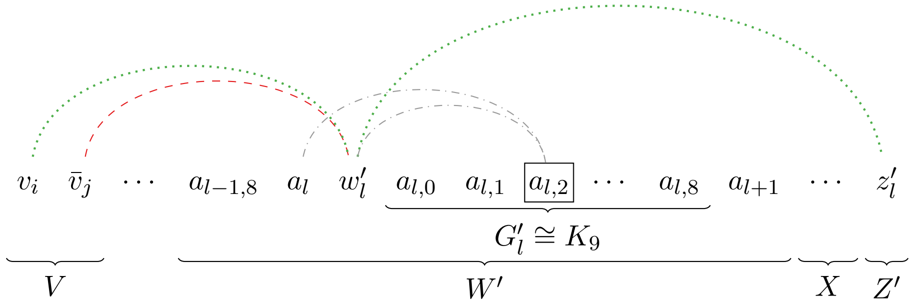


Figure 4.4: Constructed graph for clause $C_l \in \mathcal{C}'$ with $C_l = (u_i \vee \bar{u}_j)$. Here, we have $t(u_i) = \text{true}$ and the edges $v_i w'_l$ and $w'_l z'_l$ are embedded on the same page P'_l. By Lemma 4.3 all edges leaving vertex $a_{l,2}$ are embedded on the same page.

Figure 4.4 shows how W' is embedded into $\prec$. We embed W' on the spine such that $a_l \prec w'_l \prec a_{l,0} \prec \cdots \prec a_{l,8}$ for $l \in \{0, \ldots, t-1\}$, $a_{m,8} \prec a_{m+1}$ for $m \in \{0, \ldots, t-2\}$, and $V \prec W' \prec V(G_0)$. Again, we can apply Lemma 4.3 to $a_{l,2}$, which is marked by boxes in the figures of this chapter. For $i \in \{0, \ldots, r-1\}$ and $l \in \{0, \ldots, t-1\}$, we add edges $v_i w'_l$ if $u_i \in C'_l$, or $\bar{v}_i w'_l$ if $\bar{u}_i \in C'_l$. Additionally, we add edges $a_l a_{l,2}$ and $w'_l a_{l,2}$ for $l \in \{0, \ldots, t-1\}$. For this let

$$\begin{aligned} E_{W'} = \; & \{v_i w'_l \colon u_i \in C'_l,\, i \in \{0, \ldots, r-1\}, l \in \{0, \ldots, t-1\}\} \\ & \cup \{\bar{v}_i w'_l \colon \bar{u}_i \in C'_l,\, i \in \{0, \ldots, r-1\}, l \in \{0, \ldots, t-1\}\} \\ & \cup \{a_l a_{l,2},\, w'_l a_{l,2} \colon l \in \{0, \ldots, t-1\}\} \\ & \cup \bigcup_{l=0}^{t-1} E(G'_l). \end{aligned}$$

$E_Z, E_{Z'}$ Finally, we have two vertex sets Z and Z' with

Z, Z'
$$\begin{aligned} Z &= \{z_0, \ldots, z_{s-1}\} \text{ and} \\ Z' &= \{z'_0, \ldots, z'_{t-1}\}. \end{aligned}$$

Let $X \prec Z' \prec Z \prec Y$. Let $z_0 \prec \cdots \prec z_{s-1}$ and $z'_0 \prec \cdots \prec z'_{t-1}$. Let

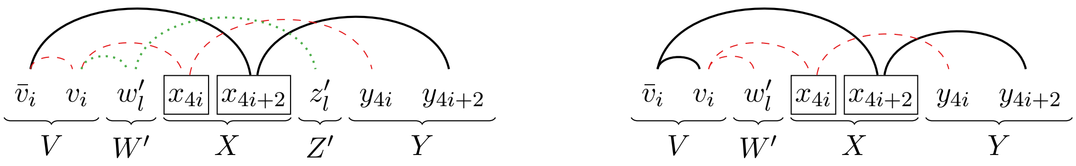
$$\begin{aligned} E_Z &= \{w_j z_j \colon j \in \{0, \ldots, s-1\}\} \text{ and} \\ E_{Z'} &= \{w'_l z'_l \colon l \in \{0, \ldots, t-1\}\}. \end{aligned}$$

The graph G and a vertex ordering $\prec$ can be constructed in polynomial time since $|V(G)| \in \mathcal{O}(|U| + |\mathcal{C}| + |\mathcal{C}'|)$.

Below, we prove that there is a 3-local book embedding for the constructed graph with respect to the ordering $\prec$ if and only if $(U, \mathcal{C}, \mathcal{C}')$ is satisfiable.

First, assume that $(U, \mathcal{C}, \mathcal{C}')$ is satisfiable. Let $t \colon U \to \{\text{true}, \text{false}\}$ be a truth assignment that satisfies all clauses in $\mathcal{C} \cup \mathcal{C}'$ over U. We find a 3-local book embedding for G by embedding all constructed edge sets. The embedding of some edge sets is illustrated in Figure 4.2.

E_X We start by taking $4r$ pairwise distinct pages $P_0, \ldots, P_{4r-1}$. For $i \in \{0, \ldots, r-1\}$ and $j \in \{0, \ldots, 3\}$, we embed the edges vx_{4i+j} with $v \in \{v_i, \bar{v}_i\}$ on P_{4i+j}. For $i \in \{0, \ldots, 4r-1\}$, we embed the edges $x_i y_i$ on page P_i as shown in Figure 4.3. By Lemma 4.3, all subgraphs G_i can be embedded in a 3-local book embedding since all edges leaving vertex x_i to vertices in V or Y are embedded on the same page P_i for $i \in \{0, \ldots, 4r-1\}$.



(a) We have $t(u_i) = \text{true}$. Hence, $\bar{v}_i v_i$ is embedded on page P_{4i} (which also contains $\bar{v}_i x_{4i}$). The edge $v_i w'_l$ is embedded on page P'_l (which also contains $w'_l z'_l$).

(b) We have $t(u_i) = \text{false}$. Hence, $\bar{v}_i v_i$ is embedded on page P_{4i+2} (which also contains $\bar{v}_i x_{4i+2}$). The edge $v_i w'_l$ is embedded on page P_{4i} (which also contains $v_i x_{4i}$).

Figure 4.5: Embedding of the edges $\bar{v}_i v_i$ and $v_i w'_l$ depending on the truth assignment of the variable corresponding to v_i

E_V For $i \in \{0, \dots, r-1\}$, we embed $v_i \bar{v}_i$ on page P_{4i} (which already contains an edge incident to v_i) if $t(u_i) = \text{true}$. Otherwise, we embed $v_i \bar{v}_i$ on page P_{4i+2} (which already contains an edge incident to $\bar{v}_i$). Both cases are shown in Figure 4.5. Hence, for every vertex $v \in V$ we have $p_{G[V \cup X]}(v) = 2$ if the corresponding literal is true, and $p_{G[V \cup X]}(v) = 3$ otherwise.

$E_{W'}, E_{Z'}$ Next, we embed all edges $w'_l z'_l \in E_{Z'}$ on new pages P'_l for $l \in \{0, \dots, t-1\}$. Consider two adjacent vertices v_i and w'_l. If $t(u_i) = \text{true}$, then $p_{G[V \cup X]}(v_i) = 2$ by construction and we can embed $v_i w'_l$ on page P'_l, as shown in Figure 4.5a. All edges embedded on page P'_l are incident to w'_l, and thus do not cross. On the other hand, if $t(u_i) = \text{false}$, then we embed $v_i w'_l$ on page P_{4i} which already contains the edges $v_i x_{4i}$ and $x_{4i} y_{4i}$, as shown in Figure 4.5b. Hence, the number of pages that contain edges incident to v_i does not increase. Edges on P_{4i} do not cross since $V \prec W' \prec X \prec Y$. Similarly, an edge $\bar{v}_i w'_l$ is embedded on page P'_l if $t(u_i) = \text{false}$ and on page P_{4i+2} otherwise.

Since clauses in $\mathcal{C}'$ consist of exactly one positive and exactly one negative literal, every vertex $w'_l \in W'$ is adjacent to two vertices $v_i, \bar{v}_j \in V$ for some $i, j \in \{0, \dots, r-1\}$ and to no other vertex in V. This corresponds to the clause $C'_l \in \mathcal{C}'$ with $C'_l = (u_i \vee \bar{u}_j)$. Recall that $w'_l z'_l$ is embedded on page P'_l. Since t is a satisfying truth assignment, we have $t(u_i) = \text{true}$ or $t(u_j) = \text{false}$. As discussed above, an edge $v w'_l$ is embedded on page P'_l if the literal corresponding to v is true. Thus, one of $v_i w'_l$ and $\bar{v}_j w'_l$ is embedded on P'_l, as shown in Figure 4.4. Hence, we have $p_{G[V \cup \{w'_l\} \cup Z']}(w'_l) = 2$, and thus we can embed $w'_l a_{l,2}$ on a new page $\hat{P}_l$. We also embed $a_l a_{l,2}$ on $\hat{P}_l$. Lemma 4.3 applied to the copy of K_9 containing $a_{l,2}$ implies that G'_l can be embedded in a 3-local book embedding.

E_W, E_Z Finally, consider $w_j \in W$ and three vertices v_{i_0}, v_{i_1}, and v_{i_2} in V that are incident to w_j. Let u_{i_0}, u_{i_1}, and u_{i_2} be the corresponding variables and $C_j \in \mathcal{C}$ the clause corresponding to w_j. Without loss of generality we have $C_j = (u_{i_0} \vee u_{i_1} \vee u_{i_2})$. Since all clauses in $\mathcal{C}$ are satisfied, at least one of u_{i_0}, u_{i_1} and u_{i_2} is true. Without loss of generality we assume that $t(u_{i_0}) = \text{true}$.

Let $w'_l \in W'$ be the vertex in W' for which the edge $v_{i_0} w'_l$ exists, that is $u_{i_0} \in C'_l$ (see Figure 4.6). Recall that an edge $v w'_l$ is embedded on page P'_l if the literal corresponding to v is true. Since u_{i_0} is true, we have $v_{i_0} w'_l \in E(P'_l)$. We embed $w_j z_j$ and $w_j v_{i_0}$ on page P'_l. Note that edges between vertices in W and Z and edges between vertices in W' and Z' do not cross. The edge $w_j v_{i_0}$ does not cross any other edge on page P'_l since $W \prec V \prec W'$.

Recall that P_{4i_1} and P_{4i_2} are the pages on which the edges $v_{i_1} x_{4i_1}$ and $v_{i_2} x_{4i_2}$ are embedded, respectively. We embed $w_j v_{i_1}$ on page P_{4i_1} and $w_j v_{i_2}$ on page P_{4i_2}.

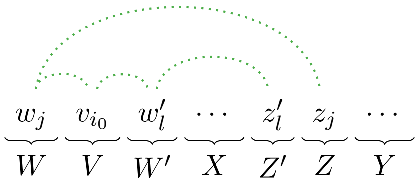


Figure 4.6: We have $t(u_{i_0}) = \text{true}$, $u_{i_0} \in C'_l$, and $u_{i_0} \in C_j$. The figure shows edges embedded on page P'_l.

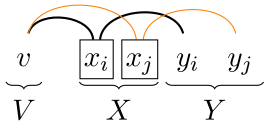


Figure 4.7: For $i \neq j$ the edges $x_i y_i$ and $x_j y_j$ cross. By Lemma 4.3 all edges leaving a vertex in X to vertices in V or Y are embedded on the same page. We observe that any two edges between V and X are embedded on different pages.

This increases the number of pages containing edges incident to w_j but leaves $p_G(v) = p_{G[V \cup X]}(v) \leq 3$ for $v \in \{v_{i_1}, v_{i_2}\}$. Hence, there are at most three pages containing edges incident to w_j. The embedded edges do not cross any other edges on their pages since $W \prec V \prec X$.

Now, assume there is a 3-local book embedding Γ with the page set $\mathcal{P}(\Gamma)$ for the constructed graph G. We find a truth assignment $t \colon U \to \{\text{true}, \text{false}\}$ that satisfies $\mathcal{C} \cup \mathcal{C}'$ over U. Let G' be the subgraph of G restricted to the vertex sets V and X, that is $G' = G[V \cup X]$. For $i \in \{0, \ldots, r-1\}$, a variable $u_i \in U$, and the corresponding vertex $v_i \in V$ let

$$t(u_i) = \begin{cases} \text{true}, & \text{if } p_{G'}(v_i) = 2, \\ \text{false}, & \text{if } p_{G'}(v_i) = 3. \end{cases}$$

For any $i, j \in \{0, \ldots, 4r-1\}$ with $i \neq j$ the edges $x_i y_i$ and $x_j y_j$ cross, and thus are embedded on different pages (see Figure 4.7). By construction, the subgraphs G_i are isomorphic to K_9 for $i \in \{0, \ldots, 4r-1\}$. Recall that x_i is the vertex in $V(G_i)$ that has neighbors in V and X. By Lemma 4.3, the edges $v x_i$ and $x_i y_i$ are embedded on the same page for $v \in V$ and $i \in \{0, \ldots, 4r-1\}$. We observe that any two edges leaving V to vertices in X are embedded on different pages. Since v_i is adjacent to x_{4i} and x_{4i+1}, we have $p_{G'}(v_i) \geq 2$ for $i \in \{0, \ldots, r-1\}$. We have $p_{G'}(v_i) \leq 3$ since Γ is 3-local. Hence, the truth assignment t is well-defined.

Next, we observe that the edges $v w'_l$ and $w'_l z'_l$ are embedded on different pages if the literal corresponding to $v \in V$ is false. Let u be the literal corresponding to v. Let $\bar{v} \in V$ be the vertex that is corresponding to $\bar{u}$. Let u be false and thus $p_{G'}(v) = 3$. Suppose

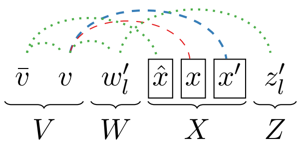


Figure 4.8: We have $t(u) = \text{false}$, $p_{G'}(v) = 3$, and $p_{G'}(\bar{v}) = 2$. If $v w'_l$ and $w'_l z'_l$ are embedded on the same page P, then $w'_l z'_l$ and $\bar{v}\hat{x}$ cross on page P for some $\hat{x} \in X$.

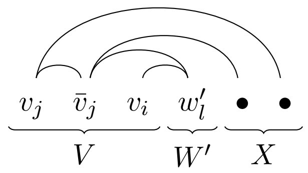


Figure 4.9: Consider the case that $v_i w_l'$ and $\bar{v}_j w_l'$ are embedded on the same page $\hat{P}$. One of v_j and $\bar{v}_j$ has an edge to X that is embedded on $\hat{P}$. However, $v_i \in V(\hat{P})$ but there is no edge between v_i and X embedded on $\hat{P}$.

there is a page P that contains vw_l' and $w_l'z_l'$ as shown in Figure 4.8. Recall that v is adjacent to two vertices x and x' in X such that vx and vx' are embedded on different pages. Since $w_l'z_l'$ crosses all edges between V and X, we have $vx, vx' \notin E(P)$. Note that $p_{G'}(v) = p_{G[V \cup X]}(v) = 2$ if $v\bar{v}$ is embedded on the same page as vx or vx'. With $p_{G'}(v) = 3$, it follows that $v\bar{v} \in E(P)$. However, we have $p_{G'}(\bar{v}) = 2$ and thus there is an edge from $\bar{v}$ to a vertex in X that is embedded on P. This is a contradiction since the page P contains $w_l'z_l'$ which crosses all edges between V and X.

Now, we prove that all clauses in $\mathcal{C}'$ are satisfied. Consider a clause $C_l' \in \mathcal{C}'$ with $C_l' = (u_i \vee \bar{u}_j)$ for some $i, j \in \{0, \ldots, r-1\}$ and $l \in \{0, \ldots, t-1\}$. Figure 4.4 shows vertex $w_l' \in W'$ and its neighborhood. By construction, we have $v_i w_l' \in E(G)$ and $\bar{v}_j w_l' \in E(G)$. The vertices $a_{l,0}, \ldots, a_{l,8}$ form a complete graph on nine vertices. By Lemma 4.3, the edges $a_l a_{l,2}$ and $w_l' a_{l,2}$ are embedded on the same page $P \in \mathcal{P}(\Gamma)$. All edges leaving w_l' to vertices in V or Z' cross $a_l a_{l,2}$, and thus are not embedded on P. In particular, the edge $w_l'z_l'$ is embedded on a page $P' \in \mathcal{P}(\Gamma)$ with $P' \neq P$. Together, we have $w_l' \in V(P)$ and $w_l' \in V(P')$. Since Γ is 3-local, we can use P, P', and at most one new page in order to embed the edges $v_i w_l'$ and $\bar{v}_j w_l'$.

Hence, one of $v_i w_l'$ and $\bar{v}_j w_l'$ is embedded on P', or $v_i w_l'$ and $\bar{v}_j w_l'$ are embedded on the same page $\hat{P} \in \mathcal{P}(\Gamma)$. Without loss of generality, we assume $p_{G'}(\bar{v}_j) = 3$, and $t(u_j) = \text{true}$. Consider the first case, that is $v_i w_l' \in E(P')$ or $\bar{v}_j w_l' \in E(P')$. As discussed above, the edges $\bar{v}_j w_l'$ and $w_l'z_l'$ are embedded on different pages since $\bar{u}_j$ is false (see Figure 4.4). With $w_l'z_l' \in E(P')$, it follows that $\bar{v}_j w_l' \notin E(P')$. Similarly, we have $v_i w_l' \notin E(P')$ if $t(u_i) = \text{false}$, which is a contradiction. Hence, we have $t(u_i) = \text{true}$ and C_l' is satisfied.

The second case, namely if $v_i w_l', \bar{v}_j w_l' \in E(\hat{P})$, is illustrated in Figure 4.9. Recall that $G' = G[V \cup X]$ and $p_{G'}(\bar{v}_j) = 3$. Since Γ is 3-local and $\bar{v}_j w_l' \notin E(G')$ but $\bar{v}_j \in E(\hat{P})$, there is an edge in G' that is incident to $\bar{v}_j$ and is embedded on $\hat{P}$. Thus, there is a vertex $x \in X$ such that $\bar{v}_j x \in E(\hat{P})$ or $v_j x \in E(\hat{P})$. Since any two edges leaving V to vertices in X are embedded on different pages, there is no vertex $x' \in X$ with $v_i x' \in E(\hat{P})$. With $v_i \in V(\hat{P})$, it follows that $p_{G'}(v_i) = 2$, and thus $t(u_i) = \text{true}$. Therefore, C_l' is satisfied.

Next, we prove that all clauses in $\mathcal{C}$ are satisfied. Consider a clause $C_j \in \mathcal{C}$ and the corresponding vertex $w_j \in W$ for $j \in \{0, \ldots, s-1\}$. Without loss of generality, we assume that $C_j = (u_0 \vee u_1 \vee u_2)$. Thus, the vertex w_j is adjacent to v_0, v_1, and v_2. Since Γ is 3-local, at least two of $w_j v_0$, $w_j v_1$, $w_j v_2$, and $w_j z_j$ are embedded on the same page.

Suppose C_j is not satisfied, that is $t(u_0) = t(u_1) = t(u_2) = \text{false}$, and $p_{G'}(v_0) = p_{G'}(v_1) = p_{G'}(v_2) = 3$. See Figure 4.10 for an illustration. By construction, for $i \in \{0, 1, 2\}$ the vertex v_i is to the right of $\bar{v}_i$ since $v_i \in C_l$. Recall that w_j is embedded to the left of V. Hence $w_j v_i$ crosses every edge from $\bar{v}_i$ to a vertex in W'. Since $p_{G'}(\bar{v}_i) = 2$, $\bar{v}_i v_i$ is embedded on a page that contains edges between $\bar{v}_i$ and X. With $p_{G'}(v_i) = 3$, it follows that $w_j v_i$ is embedded on a page which contains edges between $\{\bar{v}_i, v_i\}$ and X. Since there are no two

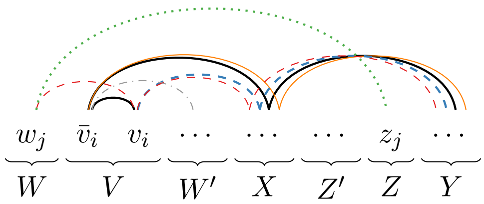


Figure 4.10: We have $t(u_i) = $ false and $p_{G'}(\bar{v}_i) = 2$. The edge $w_j v_i$ is embedded on a page that contains an edge between v_i and X.

edges between V and X that are embedded on the same page, any two of $w_j v_0$, $w_j v_1$, and $w_j v_2$ are embedded on different pages.

Recall that all edges leaving a vertex in X to vertices in V or Z are embedded on the same page. Hence, every page that contains an edge between V and X also contains an edge between X and Y. However, $w_j z_j$ crosses every edge between X and Y, and thus cannot be embedded on the same page as one of $w_j v_0$, $w_j v_1$, and $w_j v_2$. Hence, we have $p_G(w_j) = 4$, which is a contradiction since Γ is 3-local. Therefore, C_j is satisfied. $\square$

4.2 $\mathcal{NP}$-Completeness for $k \geq 3$

The following construction is used to reduce k-LOCAL BOOK EMBEDDING WITH FIXED VERTEX ORDERING for $k = 3$ to the case $k > 3$. See Figure 4.11 for an illustration of the upcoming construction.

Construction 4.5. *We construct a graph $G(k)$ depending on an integer $k \geq 2$ and a linear ordering $\prec$ of $V(G(k))$. Let $m = k - 2$ and let $n = (k-1)^2$. Let K be a copy of $K_{k-1,n+2}$ with parts $V = \{v_0, \ldots, v_m\}$ and $W = \{w_0, \ldots, w_n, z\}$. Let Y be a set of $n+1$ vertices with $Y = \{y_0, \ldots, y_n\}$. Let $G(k)$ be a graph with*

$$\begin{aligned} V(G(k)) &= Y \cup \{x\} \cup V(K) \text{ and} \\ E(G(k)) &= E(K) \cup \{y_i w_i,\ x w_i \colon\ i \in \{0, \ldots, n\}\}. \end{aligned}$$

Let $\prec$ be a linear ordering of $V(G(k))$ with $Y \prec \{x\} \prec V \prec W$. Let $y_n \prec \cdots \prec y_0$, $v_0 \prec \cdots \prec v_m$, and $w_0 \prec \cdots \prec w_n \prec z$.

Lemma 4.6. *Let $k \geq 2$ and let $G(k)$ be a graph with a linear ordering $\prec$ of $V(G(k))$ created according to Construction 4.5. Let $m = k-2$ and let $n = (k-1)^2$. We denote the vertices and subsets of $V(G(k))$ according to Construction 4.5. Then, there is a k-local book embedding of $G(k)$ into $\prec$ with $p_{G(k)}(x) = 1$. In addition, for every k-local book embedding of $G(k)$ into $\prec$, there exists some $i \in \{0, \ldots, n\}$ such that the edges $y_i w_i$ and $x w_i$ are embedded on the same page.*

Proof. First, we find a k-local book embedding Γ that embeds $G(k)$ into $\prec$. For this let $\mathcal{P}(\Gamma) = \{P_0, \ldots, P_{k(k-1)}\}$ be the set of pages. Figure 4.11 shows the constructed graph and some page sets for $k = 4$. We construct Γ so that $v_j \in V(P_r)$ if and only if $j \equiv r \mod |V|$, where $j \in \{0, \ldots, m\}$ and $r \in \{0, \ldots, k(k-1)-1\}$. For $i \in \{0, \ldots, n\}$ and $j \in \{0, \ldots, m\}$, we embed the edge $v_j w_i$ on page P_r with $r = \min\{s \in \mathbb{N}_0 \colon\ s \geq i,\ s \equiv j \mod |V|\}$. For $j \in \{0, \ldots, m\}$, we embed the edges $v_j z$ on page P_r with $r = n + j$. Note that $v w_n$ and vz are embedded on the same page for every $v \in V$. It remains to embed the edges between Y and W and between x and W. For $i \in \{0, \ldots, n-1\}$ we embed $y_i w_i$ on page P_i. All edges incident to x are embedded on $P_{k(k-1)}$. Additionally, we embed $y_n w_n$ on $P_{k(k-1)}$.

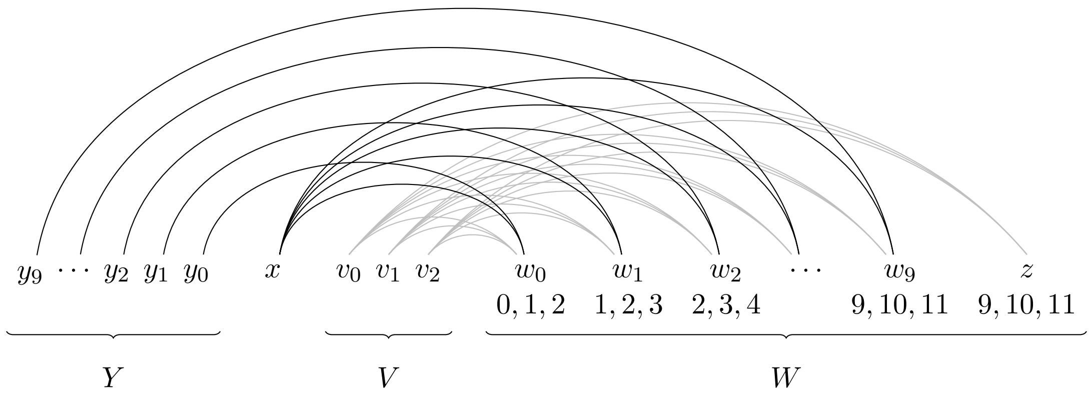


Figure 4.11: Construction 4.5 for $k = 4$ and $n = 9$. In a book embedding as constructed in Lemma 4.6, we have $\mathcal{P}_0 = \{P_0, P_3, P_6, P_9\}$, $\mathcal{P}_1 = \{P_1, P_4, P_7, P_{10}\}$, and $\mathcal{P}_2 = \{P_2, P_5, P_8, P_{11}\}$. The indices below a vertex $w \in W$ indicate the pages that contain edges incident to w, for example $w_0 \in V(P_0) \cap V(P_1) \cap V(P_2)$. Additionally, we have $w \in V(P_{12})$ for all vertices $w \in W$.

Next, we show that any two edges embedded on the same page do not cross. Recall that K is a copy of $K_{k-1,n+2}$ with parts V and W, as described in Construction 4.5. Consider Γ restricted to K. By construction, there is at most one vertex of V in the vertex set of each page. Hence, for every page $P \in \mathcal{P}(\Gamma)$ the subgraph of K embedded on P is a star, and thus any two edges do not cross.

Consider the edges between Y and W. Recall that $y_r w_r$ is embedded on P_r for $r \in \{0, \ldots, n-1\}$. However, for a vertex $w_i \in W$ with $i \in \{0, \ldots, n\}$ and $w_i \in V(P_r)$, we have $r = \min\{s \in \mathbb{N}_0 \colon s \geq i,\ s \equiv j \mod |V|\}$ for some $j \in \{0, \ldots, m\}$, and thus $r \geq i$. Edges between w_n and Y and between z and V are embedded on pages $P_s \in \mathcal{P}(\Gamma)$ with $s > r$. Hence, for all vertices $w \in W$ that are embedded to the right of w_r, we have $w \notin V(P_r)$. Finally, recall that all edges incident to x and the edge $y_n w_n$ are embedded on page $P_{k(k-1)}$. These edges do not cross since the edges incident to x form a star and all edges crossing $y_n w_n$ are incident to z but $z \notin V\left(P_{k(k-1)}\right)$.

Now, we prove that for every k-local book embedding Γ of $G(k)$ there exists some $i \in \{0, \ldots, n\}$ such that the edges $y_i w_i$ and $x w_i$ are embedded on the same page. Let $\mathcal{P}(\Gamma)$ be the set of pages of Γ. For $j \in \{0, \ldots, m\}$, let $\mathcal{P}_j$ be the set of pages that contain edges incident to v_j, that is $\mathcal{P}_j = \{P \in \mathcal{P}(\Gamma) \colon v_j \in V(P)\}$. Since Γ is k-local, we have $|\mathcal{P}_j| \leq k$ for $j \in \{0, \ldots, m\}$. For a vertex $w \in W$, a *free page* of w with respect to v_j is a page $P \in \mathcal{P}_j$ such that $v_j w$ can be embedded on P without crossing any other edges on P for some $j \in \{0, \ldots, m\}$. We denote the number of free pages of w with respect to v_j by $f_j(w)$. Let $f(w) = \sum_{j=0}^{m} f_j(w)$ be the sum of free pages of a vertex $w \in W$ over all v_j. Since $|V| = k-1$ and $|\mathcal{P}_j| \leq k$ for $j \in \{0, \ldots, m\}$, we have $f(w) \leq k(k-1)$ for all vertices $w \in W$. When embedding $G(k)$, we need to take care that $f(w) \geq k-1$ for every vertex $w \in W$, otherwise there are edges in $E(K)$ that cannot be embedded.

Below, we denote z by w_{n+1}. Recall that w_i and w_{i+1} are embedded consecutively on the spine and $v \prec w_i \prec w_{i+1}$ for $v \in V$ as shown in Figure 4.12. Thus, every edge crossing $v w_i$ also crosses $v w_{i+1}$. However, there might be edges that cross $v w_{i+1}$ but do not cross $v w_i$. If a page P is a free page of w_{i+1} with respect to $v \in V$ for $i \in \{0, \ldots, n\}$, then we observe that P is also a free page of w_i with respect to v. It follows that $f(w_i) \geq f(w_{i+1})$ for $i \in \{0, \ldots, n\}$.

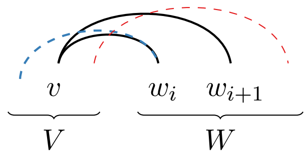


Figure 4.12: Recall that no vertex is embedded between w_i and w_{i+1}. Every edge crossing vw_i also crosses vw_{i+1} but there may exist edges crossing vw_{i+1} but not vw_i.

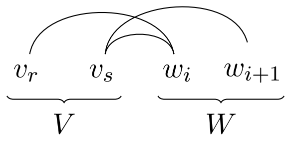


Figure 4.13: $v_r w_i, v_s w_i \in E(P)$ but $v_s w_{i+1} \notin E(P)$. Thus P is free for w_i but not free for w_{i+1} with respect to v_s.

Suppose that for all $i \in \{0, \dots, n\}$ the edges $y_i w_i$ and $x w_i$ are embedded on different pages. For a vertex $w_i \in W$ with $i \in \{0, \dots, n\}$ we consider two cases: There are two edges incident to w_i and to a vertex in V that are embedded on the same page, or the edges between V and w_i are embedded on pairwise distinct pages. We prove that $f(w_i) > f(w_{i+1})$ in both cases. Since a free page of w_{i+1} is also free for w_i, it suffices to find a page that is free for w_i but not free for w_{i+1} with respect to some $v \in V$. In the first case, let v_r and v_s be the two vertices in V for which $v_r w_i$ and $v_s w_i$ are embedded on the same page P, where $r, s \in \{0, \dots, m\}$ and $r < s$ (see Figure 4.13). Thus, we have $P \in \mathcal{P}_r \cap \mathcal{P}_s$. Since $v_r w_i$ and $v_s w_{i+1}$ cross, P is not a free page of w_{i+1} with respect to v_s. It follows that $f(w_i) > f(w_{i+1})$.

In the second case, we have $k-1$ edges between V and w_i which are embedded on pairwise distinct pages. See Figure 4.14 for an illustration. Recall that Γ is k-local and that the edges $y_i w_i$ and $x w_i$ are embedded on different pages. Hence, there is a page $P \in \mathcal{P}(\Gamma)$ on which $v_j w_i$ and one of $y_i w_i$ and $x w_i$ is embedded for some $j \in \{0, \dots, m\}$. Note that $P \in \mathcal{P}_j$ since $v_j \in V(P)$. However, all edges between w_{i+1} and a vertex in V cross the edges $y_i w_i$ and $x w_i$. Thus, P is not a free page of w_{i+1} with respect to v_j, so $f(w_i) > f(w_{i+1})$.

Recall that $f(w_0) \leq k(k-1)$ and that $f(w) \geq k-1$ for every $w \in W$. With $f(w_i) > f(w_{i+1})$ for $i \in \{0, \dots, n\}$ it follows that

$$f(z) = f(w_{n+1}) \leq f(w_0) - (n+1) \leq k(k-1) - ((k-1)^2 + 1) = k - 2 < k - 1,$$

which is a contradiction. Therefore, there is an index $i \in \{0, \dots, n\}$ such that the edges $y_i w_i$ and $x w_i$ are embedded on the same page. $\square$

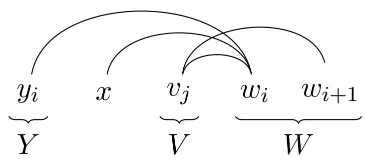


Figure 4.14: One of $y_i w_i$ and $x w_i$ is embedded on the same page as $v_j w_i$. All edges between w_{i+1} and a vertex in V cross the edges $y_i w_i$ and $x w_i$.

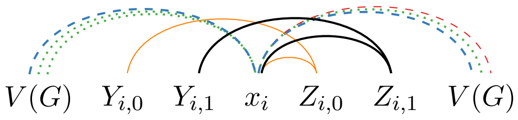


Figure 4.15: Reduction for $k = 5$. By Lemma 4.6 for every $l \in \{1, 2\}$ there is vertex $w \in Z_{i,l}$ and a vertex $y \in Y_{i,l}$ such that yw and wx_i are embedded on the same page. All edges from $x_i \in V(G)$ to another vertex in $V(G)$ cross yw, and thus are embedded on at most three pages.

Next, we use the presented construction and Lemma 4.6 to prove the following theorem. This extends Theorem 4.4 by proving $\mathcal{NP}$-completeness for k-LOCAL BOOK EMBEDDING WITH FIXED VERTEX ORDERING and $k \geq 3$.

Theorem 4.7. *k-LOCAL BOOK EMBEDDING WITH FIXED VERTEX ORDERING is $\mathcal{NP}$-complete for $k \geq 3$.*

Proof. k-LOCAL BOOK EMBEDDING WITH FIXED VERTEX ORDERING is $\mathcal{NP}$-complete for $k = 3$ by Theorem 4.4. Given a graph G, a linear ordering $\prec$, and a book embedding Γ we can check in polynomial time whether the vertices are embedded on the spine according to $\prec$, whether on every page the embedded subgraph is a plane graph, and whether $p_G(v) \leq k$ for every vertex $v \in V(G)$. Hence, k-LOCAL BOOK EMBEDDING WITH FIXED VERTEX ORDERING is in $\mathcal{NP}$ for every $k \geq 3$.

In order to prove $\mathcal{NP}$-completeness, we reduce the problem for $k = 3$ to the case $k > 3$ using Construction 4.5. Given a graph G and a linear ordering $\prec_G$ of $V(G)$, we construct a graph G' and a linear ordering $\prec_{G'}$ of $V(G')$ such that there is a 3-local book embedding of G into $\prec_G$ if and only if there is a k-local book embedding of G' into $\prec_{G'}$. We construct G' such that $G \subseteq G'$, and embed vertices $v, w \in V(G)$ on the spine such that $v \prec_{G'} w$ if and only if $v \prec_G w$. Below, we denote both orderings by $\prec$.

Let $k > 3$ and let $n = |V(G)|$. Let $V(G) = \{x_0, \ldots, x_{n-1}\}$. Without loss of generality we have $x_0 \prec \cdots \prec x_{n-1}$. We use Construction 4.5 to reduce the number of pages that can contain edges in $E(G)$ for each vertex in $V(G)$ as shown in Figure 4.15. For this we take $n(k-3)$ copies $G_{i,l}$ of the graph in Construction 4.5, where $i \in \{0, \ldots, n-1\}$ and $l \in \{0, \ldots, k-4\}$. For $G_{i,l}$ we denote the vertex x in Construction 4.5 by $x_{i,l}$. For $i \neq j$ and $l, m \in \{0, \ldots, k-4\}$, the graphs $G_{i,l}$ and $G_{j,m}$ are disjoint. For $i \in \{0, \ldots, n-1\}$ and $l, m \in \{0, \ldots, k-4\}$ let $x_{i,l} = x_{i,m} = x_i$ and let $V(G_{i,l}) \cap V(G_{i,m}) = \{x_i\}$. Recall that $x_i \in V(G)$. For constructing G' let

$$\begin{aligned} V(G') &= \bigcup_{i=0}^{n-1} \bigcup_{l=0}^{k-4} V(G_{i,l}) \\ E(G') &= \bigcup_{i=0}^{n-1} \bigcup_{l=0}^{k-4} E(G_{i,l}) \cup E(G). \end{aligned}$$

For $G_{i,l}$ we denote the vertex set Y in Construction 4.5 by $Y_{i,l}$ and the vertex set $V(K)$ by $Z_{i,l}$. Recall that $x_0 \prec \cdots \prec x_{n-1}$. For $i \in \{0, \ldots, n-1\}$ and $l \in \{0, \ldots, k-4\}$ we embed $V(G_{i,l})$ as described in Construction 4.5. We embed G' on the spine such that $V(G_{i,l}) \prec V(G_{j,m})$ if $i < j$ or if $i = j$ and $l < m$, where $i, j \in \{0, \ldots, n-1\}$ and $l, m \in \{0, \ldots, k-4\}$. For $i \in \{0, \ldots, n-1\}$ let $Y_{i,0} \prec \cdots \prec Y_{i,k-4} \prec \{x_i\} \prec Z_{i,0} \prec \cdots \prec Z_{i,k-4}$.

The graph G' can be constructed in polynomial time since $|V(G')| \in \mathcal{O}(nk^3)$.

Next, we prove that there is a 3-local book embedding of G if and only if there is a k-local book embedding of G'. First, assume there is a 3-local book embedding Γ of G according to $\prec$. By Lemma 4.6, there is a k-local book embedding $\Gamma_{i,l}$ of $G_{i,l}$ such that $p_{G_{i,l}}(x_i) = 1$ for $i \in \{0, \ldots, n-1\}$ and $l \in \{0, \ldots, k-4\}$. Let $\mathcal{P}(\Gamma)$ be the set of pages of Γ and let $\mathcal{P}(\Gamma_{i,l})$ be the set of pages of $\Gamma_{i,l}$, where all page sets are pairwise disjoint. We construct a k-local book embedding Γ' with page set $\mathcal{P}(\Gamma') = \bigcup_{i=0}^{n-1} \bigcup_{l=0}^{k-4} \mathcal{P}(\Gamma_{i,l}) \cup \mathcal{P}(\Gamma)$ by embedding each edge on its page according to Γ or $\Gamma_{i,l}$, respectively. In Γ' no two edges on the same page cross since this is true for Γ and $\Gamma_{i,l}$. For all vertices $v \in V(G') \setminus \{x_i \colon i \in \{0, \ldots, n-1\}\}$ we have $p_{G'}(v) \leq k$ since Γ and $\Gamma_{i,l}$ are k-local. For $i \in \{0, \ldots, n-1\}$ we have $p_G(x_i) \leq 3$ and $p_{G_{i,l}}(x_i) = 1$ for each $l \in \{0, \ldots, k-4\}$. It follows that $p_{G'}(x_i) \leq k$. Hence, Γ' is k-local.

Now, assume there is a k-local book embedding Γ of G'. Consider a vertex $x_i \in V(G)$ for $i \in \{0, \ldots, n-1\}$. We prove that $p_G(x_i) \leq 3$. By Lemma 4.6, for every $l \in \{0, \ldots, k-4\}$ there is vertex $w \in Z_{i,l}$ and a vertex $y \in Y_{i,l}$ such that yw and wx_i are embedded on the same page. We denote this page by P_l. Note that all edges vx_i with $v \notin \operatorname{sp}(V(G_{i,l}))$ cross yw, and thus are not embedded on P_l (see Figure 4.15). Recall that $Y_{i,0} \prec \cdots \prec Y_{i,k-4} \prec \{x_i\} \prec Z_{i,0} \prec \cdots \prec Z_{i,k-4}$. Thus, we have $P_l \neq P_m$ if $l \neq m$ for $l, m \in \{0, \ldots, k-4\}$. Hence, there are $k-3$ pairwise distinct pages $P_0, \ldots, P_{k-4}$ with $x_i \in V(P_l)$ but $v \notin V(P_l)$ for all vertices $v \in V(G) \setminus \{x_i\}$ and $l \in \{0, \ldots, k-4\}$. Since Γ is k-local, there are at most three pages embedding edges between vertices in $V(G)$. Therefore, Γ restricted to G is 3-local. $\square$

5. Union Page Number

After considering the global and local page number, we introduce another version of book embeddings. We investigate the relations between the three versions of page numbers in the following chapter.

Definition 5.1. *A k-*union book embedding *embeds a graph G in a book with k pages such that the vertices lie on the spine and every edge is embedded on exactly one page, where no two edges belonging to the same connected component of a page cross.*

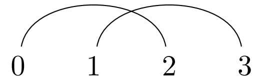
Figure 5.1 shows two example situations of edges embedded on a page, one is permitted and one is forbidden in a k-union book embedding. With this, we define the union page number.

Definition 5.2. *The* union page number $p_u(G)$ *of a graph G is the minimum $k \in \mathbb{N}_0$ such that there exists a k-union book embedding for G.*

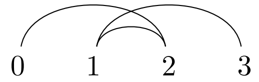
First, we observe that the union page number can be considered to be between local and global page number. Additionally, we present graphs for which the local page number is strictly smaller than the union page number or the union page number is strictly smaller that the global page number.

Proposition 5.3. *For every graph G we have $p_l(G) \leq p_u(G) \leq p(G)$.*

Proof. Let G be a graph. Since every book embedding with k pages is also a k-union book embedding, we have $p_u(G) \leq p(G)$. Next, we prove that $p_l(G) \leq p_u(G)$. Let Γ be a k-union book embedding of G. We construct a k-local book embedding Γ'. For every page of Γ, we embed each connected component on its own page. By definition of a k-union book embedding, no two edges that belong to the same connected component of a page



(a) permitted – the crossing edges belong to different connected components



(b) forbidden – the crossing edges belong to the same connected component

Figure 5.1: Permitted and forbidden situations in a 1-union book embedding

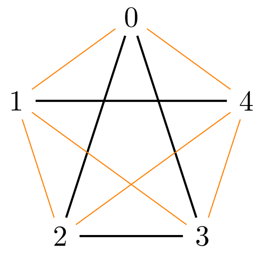
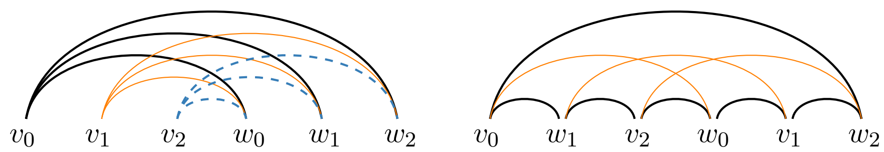


Figure 5.2: If one page has two connected components, then the other contains a subgraph that is not outerplanar.



(a) 3-page book embedding

(b) 2-union book embedding

Figure 5.3: Embeddings of $K_{3,3}$ with partite sets $V = \{v_0, v_1, v_2\}$ and $W = \{w_0, w_1, w_2\}$

cross. Thus, the constructed embedding Γ' is a book embedding. Since every vertex is contained in at most one connected component of each page of Γ, it follows that Γ' is k-local. Hence, we have $p_l(G) \leq p_u(G)$. □

There are graphs for which local and global page number coincide, for instance outerplanar graphs (see Proposition 3.1) or graphs with global page number 2. On the other hand, we present graphs G with $p_l(G) < p_u(G)$ or $p_u(G) < p(G)$ in the following observations.

Observation 5.4. *There is a graph G with $p_l(G) < p_u(G)$.*

Proof. In Observation 3.13 we show that the local page number of a complete graph on five vertices equals 2. However, we shall prove that $p_u(K_5) = 3$. Since the global page number of K_5 is 3 [BK79], we have $p_u(K_5) \leq p(G) = 3$. Suppose there is a 2-union book embedding Γ for K_5. Since there is no 2-page book embedding for K_5, there is a page P in Γ containing two connected components.

Let G be the subgraph of K_5 embedded on page P. Note that G is a subgraph of the disjoint union of an edge and a triangle, as illustrated in Figure 5.2. Hence, the subgraph G' that is embedded on the second page contains $K_{2,3}$, and thus is connected but not outerplanar. It follows that G' cannot be embedded on a single page, which is a contradiction. Therefore, we have $p_u(K_5) = 3$, and thus $p_l(K_5) = 2 < 3 = p_u(K_5)$. □

Observation 5.5. *There is a graph G with $p_u(G) < p(G)$.*

Proof. Since $K_{3,3}$ is not planar, we have $p(K_{3,3}) > 2$ and $p_u(K_{3,3}) > 1$. Nevertheless, Figure 5.3 shows a 3-page and a 2-union book embedding of $K_{3,3}$. Therefore, we have $p_u(K_{3,3}) = 2 < 3 = p(K_{3,3})$. □

After presenting graphs for which the three versions of page numbers differ, we analyze how large the gaps can be. We find that the gap between union and global page number can be arbitrarily large, whereas the ratio between union and local page number is bounded by a constant.

Proposition 5.6. *There are n-vertex graphs with union page number $k+1$ but global page number $\Omega\left(\sqrt{k}n^{1/2-1/k}\right)$.*

Proof. Malitz [Mal94] proved that there exist k-regular n-vertex graphs which require $\Omega(\sqrt{k}n^{1/2-1/k})$ pages. Let G be a k-regular graph on n vertices. We construct a $(k+1)$-union book embedding Γ with the page set $\mathcal{P}(\Gamma) = \{P_0, \ldots, P_k\}$ for G. Vizing [Viz64] proved that every graph with maximum degree Δ has chromatic index at most $\Delta + 1$. Hence, G has chromatic index at most $k+1$. Let $c\colon E(G) \to \{0, \ldots, k\}$ be a proper edge coloring of G. We assign the edges to pages such that $e \in E(P_i)$ if and only if $c(e) = i$ for $i \in \{0, \ldots, k\}$. For every page, each connected component is a single edge since c is a proper edge coloring. Therefore, the page set $\mathcal{P}(\Gamma)$ together with an arbitrary vertex ordering forms a $(k+1)$-union book embedding. □

Proposition 5.7. *For every graph G we have $\frac{p_u(G)}{p_l(G)} \leq 4$.*

Proof. Let G be a graph on n vertices with local page number k and $n \geq 1$. In Corollary 3.7 we show that G has at most $(2n-3)k$ edges. Next, we construct a $4k$-union book embedding by covering G with trees. Recall that the arboricity $a(G)$ of G denotes the covering number $c_g^{\mathcal{G}}$, where $\mathcal{G}$ is the set of forests (see Section 2.1).

Nash-Williams [NW64] proved $a(G) = \max\{|E(H)|/(|V(H)|-1)\colon H \subseteq G, |V(H)| > 1\}$. Note that every subgraph of G has local page number at most k, and thus Corollary 3.7 can be applied. With this, it follows that

$$\begin{aligned}
a(G) &= \max\left\{\tfrac{|E(H)|}{|V(H)|-1}\colon\ H \subseteq G, |V(H)| > 1\right\} \\
&\leq \max\left\{\tfrac{(2|V(H)|-3)k}{|V(H)|-1}\colon\ H \subseteq G, |V(H)| > 1\right\} \\
&\leq \max\left\{\tfrac{2k(|V(H)|-1)}{|V(H)|-1}\colon\ H \subseteq G, |V(H)| > 1\right\} \\
&= 2k.
\end{aligned}$$

Since every forest can be covered by two star forests [AMR92], there is a decomposition of G into $4k$ star forests. We create a $4k$-union book embedding by placing each star forest on its own page and choosing an arbitrary vertex ordering. For every page, each connected component is a star, and thus is crossing-free. Hence, it follows that

$$\frac{p_u(G)}{p_l(G)} \leq \frac{4k}{k} = 4.$$

□

6. Conclusions

Based on the concept of global book embeddings and the local covering number, we introduced k-local book embeddings. In Chapter 3 we gave bounds on the local page number of some graphs, namely outerplanar graphs, planar graphs, complete graphs, and k-trees. However, there are gaps between the lower and upper bounds, which leads to the question of the exact local page numbers. Similar problems can be formulated for the union page number.

For instance, for every planar graph there exists a 4-local book embedding, and there is a planar graph with local page number 3.

Question 6.1. *Is there a planar graph G with $p_l(G) \geq 4$?*

Question 6.2. *Is there a planar graph G with $p_u(G) \geq 4$?*

In contrast, there is a planar graph with global page number 3 [BK79]. While Yannakakis [Yan89] proved that every planar graph can be embedded in a 4-page book, it is open whether or not there exists a planar graph with global page number 4.

For complete graphs we gave a lower bound and an upper bound, which differ by a factor 2.

Question 6.3. *What is the local page number of a complete graph on n vertices?*

For every $k \geq 3$, we proved that there is a k-tree with local page number k, but the upper bound on the local page number of k-trees is $k+1$. Vandenbussche, West, and Yu [VWY09] proved that there exist k-trees with global page number $k+1$ for every $k \geq 3$. For the local page number, however, the following question remains open.

Question 6.4. *Is there a k-tree G with $p_l(G) = k + 1$ for $k \geq 3$?*

In Chapter 4 we proved that k-LOCAL BOOK EMBEDDING WITH FIXED VERTEX ORDERING is $\mathcal{NP}$-complete for any fixed $k \geq 3$. However, whether a graph can be embedded in a 1-local book embedding can be tested in polynomial time. This leads to the case $k = 2$, which remains open.

Question 6.5. *Is* 2-LOCAL BOOK EMBEDDING WITH FIXED VERTEX ORDERING *$\mathcal{NP}$-complete?*

We have not considered the complexity of finding a k-local book embedding if there is no vertex ordering given. Since a graph has global page number at most 2 if and only if it is a subgraph of a planar Hamiltonian graph [BK79], it is $\mathcal{NP}$-complete to test whether a graph has global page number at most 2 [Wig82]. The same question can be asked for the local and union page number.

Question 6.6. *Given a graph G and an integer k, is it $\mathcal{NP}$-complete to decide whether or not we have $p_l(G) \leq k$?*

Question 6.7. *Given a graph G and an integer k, is it $\mathcal{NP}$-complete to decide whether or not we have $p_u(G) \leq k$?*

In Chapter 5 we investigated the gaps between local, union, and global page numbers. While the ratio between union and local page number is bounded by a constant, whether the difference is bounded.

Question 6.8. *Is there a constant c such that $p_u(G) - p_l(G) < c$ for every graph G?*

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