# Generalized Sphere of Influence Graphs

Bachelor's Thesis of

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I declare that I have developed and written the enclosed thesis completely by myself. The following tools were used to create this thesis:

- *GeoGebra* and *ipe*, for the creation of figures and pictograms.
- *Overleaf*, for the creation of the thesis and for the correction of spelling, punctuation, and grammar.
- *WolframAlpha*, for calculations.

I have not used any other than the aids mentioned above. I have marked all parts of the thesis that I have included from referenced literature, either in their original wording or paraphrasing their contents. I have followed the by-laws to implement scientific integrity at KIT.

**Karlsruhe, 04.09.2026**

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(Florin Lucut)

## Abstract

For a set of points $P \subseteq \mathbb{R}^2$ we define the following conflict graph: to each point $p \in P$ we assign a circle $C_p$ centered on $p$ with the radius equal to the euclidean distance $\text{dist}(p, q)$ from $p$ to its nearest neighbor $q$. Two points $p$ and $q$ are adjacent if and only if their circles $C_p$ and $C_q$ intersect. The resulting graph is known as a *Sphere of Influence Graph* (SIG). In this thesis we show multiple properties of SIGs and we introduce *general* SIGs, which allow the circle $C_p$ of $p$ to have a radius smaller than $\text{dist}(p, q)$. We show that more graphs can be depicted this way. Furthermore, we introduce $\alpha$-SIGs, which interpolate between general SIGs ($\alpha = 0$) and disk contact graphs ($\alpha = 1$). We show that $\alpha$-SIGs are planar graphs for $\alpha > \sqrt{2} - 1$ and that $(\sqrt{2} - 1)$-SIGs are 1-planar graphs. Additionally, we examine the edge density of $\alpha$-SIGs in relation to $\alpha$. Lastly, we answer the question: which stars are $\alpha$-SIGs and for which value of $\alpha$?

## Deutsche Zusammenfassung

Für eine Menge von Punkten $P \in \mathbb{R}^2$ definieren wir den folgenden Konfliktgraph: jedem Punkt $p \in P$ weisen wir einen Kreis $C_p$ mit Mittelpunkt $p$ und Radius gleich dem euklidischen Abstand $\text{dist}(p, q)$ von $p$ zu seinem nächsten Nachbar $q$ zu. Zwei Punkte $s$ und $t$ sind benachbart genau dann wenn sich ihre Kreise $C_s$ und $C_t$ kreuzen. Der resultierende Graph wird *Sphere of Influence Graph* (SIG) benannt. In dieser Bachelorarbeit zeigen wir mehrere Eigenschaften von SIGs und wir führen *general* SIGs ein. Diese erlauben, dass der Kreis $C_p$ des Punktes $p$ einen kleineren Radius als $\text{dist}(p, q)$ hat. Wir zeigen, dass auf dieser Weise mehr Graphen dargestellt werden können. Desweiteren, führen wir $\alpha$-SIGs ein, die zwischen general SIGs ($\alpha = 0$) und Kreiskontaktgraphen (engl. disk contact graphs) ($\alpha = 1$) interpolieren. Wir zeigen, dass $\alpha$-SIGs planare Graphen sind falls $\alpha > \sqrt{2} - 1$ und, dass $(\sqrt{2} - 1)$-SIGs 1-planar sind. Zusätzlich, untersuchen wir die Kantendichte von $\alpha$-SIGs in Abhängigkeit von $\alpha$. Als Letztes, beantworten wir die Frage: welche Sterne sind $\alpha$-SIGs und für welche Werte von $\alpha$?

## Contents

# 1 Introduction

*Sphere of Influence Graphs* (SIGs) were introduced by Toussaint in the 1980's in relation to pattern recognition problems, and for studying the underlying structures of dot patterns [15, 16, 17]. Figure 1.1 gives an introductory example. The concept of SIGs can be understood as a form of playing connect the dots, which works in the following way: to each point $p$ we assign a the largest circle $C_p$ with center $p$ which contains no other point. This circle is the sphere of influence of $p$. The SIG is the conflict graph of the spheres of influence, that is, two points are adjacent if their corresponding spheres intersect. For a set of points $P \subseteq \mathbb{R}^2$ we therefore write $\mathrm{SIG}(P)$ to denote the sphere of influence graph that results from $P$. A formal definition is given in Section 1.1.

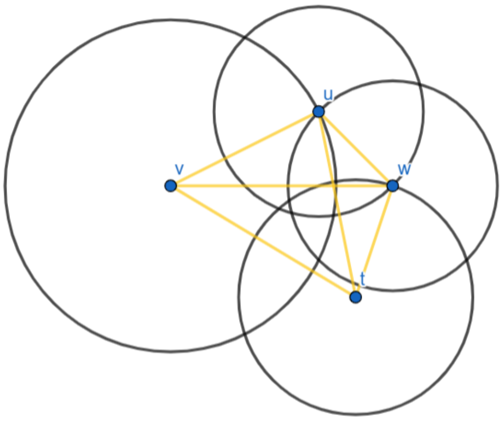



**Figure 1.1** An example of an SIG-drawing. The points are blue, the spheres of influence are black, and the edges are yellow.

We call a graph $G$ an SIG if it is isomorphic to the SIG of some set of points $V \subseteq \mathbb{R}^2$. Note that because of this connection we sometimes talk about vertices and sometimes about points, hence the name $V$ for vertices or alternatively the name $P$ for points.

Although some research has already been done on this field, there are still many unanswered questions. The most notable is the conjecture by Avis and Horton [2] which states that such a graph on $n$ nodes may have at most $9n$ edges.

▶ **Conjecture 1.1** (Avis and Horton [2]). *An* SIG *on $n$ nodes may have at most $9n$ edges.*

The first upper bound to the number of edges that an SIG might have was achieved in 1948 by Reifenberg [11] and independently in 1951 by Bateman and Erdős [3] — long before SIGs were introduced in the 1980's. They showed that in an $\mathrm{SIG}(P)$ the point $p \in P$ whose sphere of influence $C_p$ has the smallest radius will have at most 18 neighbors. This leads directly to an upper bound of $18n$ edges for an SIG on $n$ nodes.

The next big step forward was done in the year 1998 by Soss as part of their Master's thesis under the supervision of Toussaint [14]. They realize that we can imagine that each edge contributes a weight of 1 to the total count. If we redistribute this weight somehow among the vertices, and then show that no vertex ends up with more than 15 weight, then we have proved that no SIG on $n$ vertices has more than $15n$ edges. This method is also known as *discharging*, see [4] for an overview.

The proof that currently is the closest to proving the conjecture stems from the year 2020 and shows that a SIG on $n$ vertices may have at most $14.5n$ edges [6]. The authors arrive at this bound by improving upon the discharging method used by Soss.

Many more properties have been shown over the years.

The underlying idea for this thesis was to get a better answer to the Density Question:

▶ **Question 1.2** (Density Question). *What is the smallest constant $c > 0$ such that every* SIG *on $n$ vertices has at most $c \cdot n$ many edges?*

In particular, the hope was that we could use new advancements in the field of beyond-planarity (see [7] for example) that stem from graph classes that are defined by topology or combinatorics and transfer them the geometrically defined class of SIGs.

Because this endeavor did not prove fruitful, we introduce multiple modifications of SIGs. Chapter 3 concerns itself with general SIGs. The novel quality here is that we do not require the sphere of influence $C_p$ of a point $p$ to be as large as possible. We restrict ourselves to demanding that $C_p$ does not contain any other points besides $p$. To distinguish between the two settings we call the original SIGs well-formed, and the current general. This allows us to depict more graphs. A nice property of general SIGs is that they are hereditary, that is: every induced subgraph of an SIG is also an SIG.

We try to take this idea further in Chapter 4. Instead of using circles, we assign each point $p$ an annulus $a_p = (C^p_{\text{in}}, C^p_{\text{out}})$, which is composed of two circles $C^p_{\text{in}}, C^p_{\text{out}}$ centered on $p$. The parameter $\alpha$ states how thick the annuli are, i.e. $\alpha$ is the ratio between the radii of $C^p_{\text{in}}$ and $C^p_{\text{out}}$. The inside of an inner circle $C^p_{\text{in}}$ must remain empty, besides the point $p$. We call this the *empty-centers-condition*. The $\alpha$-SIG is the conflict graph of the spheres of influence, that is, two points are adjacent if and only if their corresponding annuli intersect.

For $\alpha = 0$ an inner circle $C_{\text{in}}$ must have radius zero and thus this is equivalent to the normal general SIG setting. For $\alpha = 1$ this formulation is equivalent to disk contact graphs: each vertex is assigned a disk, the disks are only allowed to touch (no proper overlaps), and in the resulting graph two vertices are touching if and only if their disks touch. The nice property here is that $\alpha$-SIGs interpolate between the general SIG setting ($\alpha = 0$) and disk contact graphs ($\alpha = 1$).

We examine $\alpha$-SIGs and find that $\alpha$-SIGs for $\alpha > \sqrt{2}-1$ are planar graphs. For $\alpha = \sqrt{2}-1$ we find that $\alpha$-SIGs are 1-planar graphs. A graph is $k$-planar if it admits a drawing where each edge is crossed by $k$ other edges or fewer. Table 1.1 sums up this relationship.

| $\alpha$ | $\alpha < \sqrt{2} - 1$ | $\alpha = \sqrt{2} - 1 = 0.4142\ldots$ | $\alpha > \sqrt{2} - 1$ |
|---|---|---|---|
| $\alpha$-SIG | for every $k \in \mathbb{N}_0$ not $k$-planar | 1-planar | planar |

**Table 1.1** The relationship between $\alpha$ and the $k$-planarity of $\alpha$-SIG$(P)$ for any set of points $P$.

We conduct a proof similar to Bateman and Erdős [3] and prove that in an $\alpha$-SIG the point $p$ whose outer circle $C^p_{\text{out}}$ has the smallest radius will have at most a constant number of neighbors. The tight values of $c(\alpha)$ are depicted in Table 1.2.

Similar to general and well-formed SIGs we distinguish between three settings for $\alpha$-SIGs. We call $\alpha$-SIG:

- *maximal*, if for each vertex $v$ its annulus $a_v$ cannot be enlarged without violating the empty-centers-condition;

| $i$ | $\alpha_i$ |
|---|---|
| 6 | 1 |
| 7 | $0.7355\ldots$ |
| 8 | $0.5307\ldots$ |
| 9 | $0.3680\ldots$ |
| 10 | $\sqrt{5}-2=0.2360\ldots$ |
| 11 | $0.1863\ldots$ |
| 12 | $\frac{2}{\sqrt{3}}-1=0.1547\ldots$ |
| 13 | $0.1074\ldots$ |
| 14 | $0.0798\ldots$ |
| 15 | $0.0677\ldots$ |
| 16 | $0.0503\ldots$ |
| 18 | $2\sqrt{2-\sqrt{3}}-1=0.0352\ldots$ |

| $\alpha$ | $c(\alpha)$ |
|---|---|
| $\alpha_7 < \alpha \leq \alpha_6$ | 6 |
| $\alpha_8 < \alpha \leq \alpha_7$ | 7 |
| $\alpha_9 < \alpha \leq \alpha_8$ | 8 |
| $\alpha_{10} < \alpha \leq \alpha_9$ | 9 |
| $\alpha_{11} < \alpha \leq \alpha_{10}$ | 10 |
| $\alpha_{12} < \alpha \leq \alpha_{11}$ | 11 |
| $\alpha_{13} < \alpha \leq \alpha_{12}$ | 12 |
| $\alpha_{14} < \alpha \leq \alpha_{13}$ | 13 |
| $\alpha_{15} < \alpha \leq \alpha_{14}$ | 14 |
| $\alpha_{16} < \alpha \leq \alpha_{15}$ | 15 |
| $\alpha_{18} < \alpha \leq \alpha_{16}$ | 16 |
| $0 \leq \alpha \leq \alpha_{18}$ | 18 |

**Table 1.2** The value of $c(\alpha)$ depending on $\alpha$.

- *well-formed*, if the size of the annuli result from growing them simultaneously and at the same rate from the vertices; or
- *general*.

We examine which stars $K_{1,n}$ fit into the different settings and for which values of $\alpha$. Table 1.3 depicts the results.

| | $\alpha = 0$ | $\alpha \leq \frac{2}{\sqrt{3}}-1$ | $\alpha \leq \sqrt{2}-1$ | $\alpha \leq 0.7013\ldots$ | $\alpha \leq 1$ |
|---|---|---|---|---|---|
| general | all stars are possible | | | | |
| maximal | $K_{1,3}$ | all stars are possible | | | |
| well-formed | $K_{1,3}$ | $K_{1,3}$ | $K_{1,4}$ | $K_{1,5}$ | $K_{1,6}$ |

**Table 1.3** Largest star $K_{1,n}$ that is **not** possible in the given $\alpha$-SIG setting.

## 1.1 Preliminaries

*Sphere of Influence Graphs* (SIGs) were introduced by Toussaint in the 1980's in relation to pattern recognition problems, and for studying the underlying structures of dot patterns [15, 16, 17]. In the introduction we mention that SIGs can be understood as a form of playing connect the dots. Given a set of points $V \subseteq \mathbb{R}^2$ in the plane, we let a circle $C_p$ grow out from each point $p \in V$ until the circle reaches another point. This circle is the sphere of influence of the corresponding point $p$. We now draw connections between two points $p,\ q \in V$ whenever their spheres of influence $C_p$ and $C_q$ intersect. The most straight-forward way is to draw the connections as straight lines.

The resulting drawing, composed of points, spheres of influence, and straight-line edges we call a *SIG-drawing*.

What we have essentially drawn is a graph $G = (V, E)$, that is composed of a set of vertices (or nodes) $V$, that are represented by the points in the plane, and a set $E \subseteq \{uv \mid u, v \in V, u \neq v\}$ of edges between vertices, that are represented by the lines between the points. Two points $u, v \in V$ are adjacent if and only if their spheres of influence $C_u$ and $C_v$ intersect.

We say that two spheres of influence $C_u, C_v$ *touch* or *have an improper intersection* if their intersection lies on the boundary of $C_u$ and $C_v$. Similarly, two spheres of influence $C_u, C_v$ *have a proper intersection* if they have more than a single point in common. Figure 1.2 provides examples for both of these cases. Note that the points $A$ and $C_1$ have an improper intersection, while $A$ has proper intersections with both $B_3$ and $C_{10}$. Also note that the point in which the spheres of influence of $A$ and $C_1$ touch is also a vertex, namely $B_1$.

We consider two different variations of SIGs. Depending on if we do or do not count improper intersections we may get different graphs. For a set of points $V \subset \mathbb{R}^2$, we define two graphs:

1. In the *Open Sphere of Influence Graph* (OSIG) of $V$ two vertices $u, v \in V$ are connected if and only if their spheres of influence $c_u$ and $c_v$ have a proper intersection.
2. In the *Closed Sphere of Influence Graph* (CSIG) of $V$ two vertices $u, v \in V$ are connected if their spheres of influence $c_u$ and $c_v$ have a proper or an improper intersection.

For a set of points $V$ we denote the CSIG of $V$ as CSIG$(V)$ and the OSIG as OSIG$(V)$. Note that for the set of points $V$ illustrated in Figure 1.1 we have CSIG$(V)$ = OSIG$(V)$, since every intersection is proper.

It should be noted that OSIG$(V)$ is a subgraph of CSIG$(V)$ for any set of points $V \subseteq \mathbb{R}^2$. This stems from the fact that some edges are present in the CISG but are omitted in the OSIG. Note that in the literature it is not always clear which setting the authors refer to. This is also half of the reason why in the introduction we have skipped this distinction. The other reason is that many properties apply to both CSIG and OSIG.

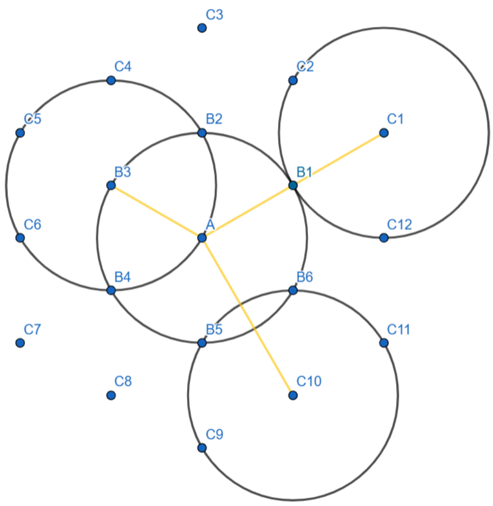



**Figure 1.2** Example of a CSIG-drawing. Only the spheres of influence of $A, B_3, C_1$ and $C_{10}$ and the edges $AB_3, AC_1, AC_{10}$ are depicted. In the OSIG case the edge $AC_1$ would not be a valid edge.

More formally, an CSIG is defined as follows.

▶ **Definition 1.3** (CSIG$(V)$). *Let $V \subseteq \mathbb{R}^2$ be a (finite) subset of points in the plane. For each point $v \in V$ we define* $\operatorname{rad}(V, v) := \min \{\|v - u\|_2 \mid u \in V,\ u \neq v\}$ *as the radius of the sphere of influence of $v$ (where $\|\cdot\|_2$ refers to the euclidean norm).*

*The corresponding CSIG is defined as* $\mathrm{CSIG}(V) = (V, E(V))$*, where $uv \in E(V)$ if and only if $\|u - v\|_2 \leq \operatorname{rad}(V, u) + \operatorname{rad}(V, v)$ for all $u, v \in V$ with $u \neq v$.*

This definition describes the CSIG setting. For a definition of OSIG$(V)$ one must replace $\leq$ by $<$ in the above definition. We will also write $\operatorname{rad}(v)$ for a point $v \in V$, when the corresponding point set $V$ can be inferred from the context. Note that we used the notation $E(V)$ to denote the set of edges of $\mathrm{SIG}(V)$, since these are directly defined by $V$. We will also occasionally write $E(G)$ for some graph $G$ to denote its set of edges.

The CSIG-drawing and OSIG-drawing are defined just as described at the start of this chapter. The difference between the two is that some edges present in the CSIG-drawing might be missing from the OSIG-drawing of the same set of points. All intersections in Figure 1.1 are proper and thus in this case the CSIG-drawing and the OSIG-drawing are the same. We call both $\mathrm{CSIG}(V)$ and $\mathrm{OSIG}(V)$ sphere of influence graphs, because of their similarity, even though they describe distinct graphs.

Observe that, by the above definition, SIGs do not contain loops or parallel edges. Both $\mathrm{CSIG}(V)$ and $\mathrm{OSIG}(V)$ are simple graphs.

Given a graph $G = (V, E)$, we say that $G$ is a CSIG (OSIG) if there exists a bijection $f : V \to P$ between the set of vertices $V$ and a set of points in the plane $P \subseteq \mathbb{R}^2$, such that $G$ and $\mathrm{CSIG}(P)$ ($\mathrm{OSIG}(P)$) are isomorphic.

## 1.2 Related Work

We have given a brief overview of the existing work on the field of sphere of influence graphs. In this section we want to offer more insights into the work and the proofs.

### 1.2.1 Density

Since $\mathrm{OSIG}(V) \subseteq \mathrm{CSIG}(V)$ for any set of points $V$, and we are interested in the maximal number of edges, it suffices to concern ourselves with the CSIG setting in this section. For brevity we will write SIG instead of CSIG.

▶ **Question 1.4** (Density Question). *Let $G = (V, E)$ be an SIG. What is the smallest constant $c > 0$ such that $|E| \leq c \cdot |V|$ holds for all possible choices of $G$?*

We will refer to this as the density question.

As stated in the introduction, it has been conjectured that $c = 9$ is the answer to the density question [2].

▶ **Conjecture 1.1** (Avis and Horton [2]). *An* SIG *on $n$ nodes may have at most $9n$ edges.*

We also know that $c \geq 9$ must hold. Take a closer look at the set of points $P$ depicted in Figure 1.2. Note that the vertex $A$ has 18 neighbors in $\mathrm{CSIG}(P)$. The points in $P$ form a section of the hexagonal lattice. For every $\varepsilon > 0$ we can find an a section $P_\varepsilon$ of the hexagonal lattice that contains $n$ nodes such that $\mathrm{CSIG}(P_\varepsilon)$ has $(9 - \varepsilon)n$ edges.

The first upper bound to the density question was achieved in 1948 by Reifenberg [12] and independently in 1951 by Bateman and Erdős [3] – long before SIGs were introduced in the 1980's. This curiosity stems from the fact that the authors were researching sets $\Gamma$ of circles in the plane with the property that the centers of none of the circles were contained in the interior of another. They showed that a circle $C \in \Gamma$ whose radius is the smallest among all circles in $\Gamma$, can be intersected by at most 18 other circles.

Translated into the language of SIGs their result states that the vertex $v$ with the smallest radius $\mathrm{rad}(v)$ may have at most 18 neighbors.

▶ **Lemma 1.5** (modified from Bateman and Erdős [3]). *The vertex $v \in V$ with the smallest radius $\mathrm{rad}(V, v)$, among all vertices in $V$, may have at most 18 neighbors in $\mathrm{SIG}(V)$.*

This simple result leads us to a first upper bound of the density.

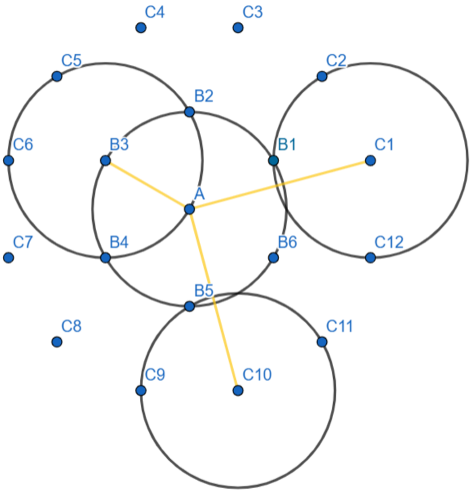



**Figure 1.3** A configuration of points in the plane $V$ such that the point $A$ has 18 neighbors in the resulting graph $\mathrm{OSIG}(V)$. Only the spheres of influence of $A, B_3, C_1$ and $C_{10}$ and the edges $AB_3, AC_1, AC_{10}$ are depicted. The remaining edges and spheres of influence have been omitted for clarity. This configuration is taken from [3].

▶ **Theorem 1.6.** *If $G$ is a CSIG with $m$ edges on $n$ vertices, then $m \leq 18 \cdot n$.*

**Proof.** We prove the above theorem by induction on the number of vertices of $G$.

For $n = 1$, the only possible SIG on one vertex does not have any edges, thus the statement holds.

Now assume that for every SIG with $m$ edges on $n \geq 1$ vertices, $m \leq 18 \cdot n$ holds, and consider a graph $G = \mathrm{SIG}(V)$ for some point set $V \in \mathbb{R}^2$ with $|V| = n + 1$ and $\widehat{m} = |E(V)|$. We will show that $\widehat{m} \leq 18 \cdot (n + 1)$.

Lemma 1.5 tells us that there exists a vertex $v \in V$ with at most 18 neighbors. Note that $G - v$ is a subgraph of the graph $G' = \mathrm{SIG}(V - v)$. Indeed, removing one vertex may only increase the radius of the remaining spheres of influence, resulting in more edges. In particular $G'$ is an SIG on $n$ vertices. If $G'$ has $m$ edges, we have $\widehat{m} \leq 18+m \leq 18+18 \cdot n = 18 \cdot (n+1)$. ◀

To show that this bound is tight, Bateman and Erdős [3] deliver the following arrangement of points: $V = V_\mathrm{a} \cup V_\mathrm{b} \cup V_\mathrm{c}$, where

- $V_\mathrm{a} = \{A = (0;\ 0)\}$
- $V_\mathrm{b} = \{B_i = (1;\ i \cdot 60^\circ + 30^\circ) \mid i = 0, 1, \ldots, 5\}$, and
- $V_\mathrm{c} = \{C_i = (1.9;\ i \cdot 30^\circ + 15^\circ) \mid i = 0, 1, \ldots, 11\}$ using polar coordinates (see Figure 1.3).

Here, every vertex $v \in V$ has $\mathrm{rad}(v) = 1$ and the vertex $A$ has an edge to all other vertices. Note that all intersections of the spheres of influence are proper and thus, in this example, $\mathrm{CSIG}(V) = \mathrm{OSIG}(V)$.

As part of his Master's thesis under the supervision of Toussaint, Soss has driven the constant in the density question down to 15, in the year 1998 [14].

▶ **Theorem 1.7** (Soss [14]). *If $G$ is an SIG with $m$ edges on $n$ vertices, then $m \leq 15 \cdot n$.*

To facilitate this proof, weights are assigned to the edges of the graph. The method used here is known as discharging (see [4] for an overview). Since we want to bound the total number of edges, we can imagine that each edge contributes a weight of 1 to the total count. If we redistribute this weight somehow among the vertices, and then show that no vertex ends up with more than 15 weight, then we have shown Theorem 1.7.

Soss distributed the weights as follows. Consider a set of points $V$ and $\mathrm{SIG}(V)$. Firstly, for each undirected edge $ab \in E(V)$ we create the two directed edges $(a, b)$ and $(b, a)$. Let the radii of the spheres of influence of $a$ and $b$ be $\mathrm{rad}(a)$ and $\mathrm{rad}(b)$ respectively. The edge $(a, b)$ is assigned a weight of:

- 1 if $\mathrm{rad}(a) \leq \frac{2}{3}\,\mathrm{rad}(b)$,
- $\frac{1}{2}$ if $\frac{2}{3}\,\mathrm{rad}(b) < \mathrm{rad}(a) < \frac{3}{2}\,\mathrm{rad}(b)$, or
- 0 otherwise.

We denote this weight function by $w((a, b))$ for all edges $ab \in E(\mathrm{WSIG}(V))$. The resulting directed graph is called the *Weighted Spheres of Influence Graph* (WSIG).

Soss goes on to prove that summing up the weights of the edges of the $\mathrm{WSIG}(V)$ for some point set $V$, gives the same value as counting the edges of $\mathrm{SIG}(V)$.

▶ **Lemma 1.8** (Soss [14]). *On any point set $V$ we have*

$$\sum_{(a,b)\in E(\mathrm{WSIG}(V))} w((a, b)) = |E(\mathrm{SIG}(V))|$$

With this in place, Soss proves Theorem 1.7 by showing that no vertex of a WSIG has more than 15 weight on its outgoing edges.

▶ **Theorem 1.9** (Soss [14]). *On any point set $V$ and any $v \in V$ we have:*

$$\sum_{(v,a)\in E(\mathrm{WSIG}(V))} w((v, a)) \leq 15$$

Theorem 1.7 follows directly from the above theorem.

Ismailescu, Kim, and Park [6] have improved upon the method devised by Soss in order to prove that the Density Question (1.4) can be answered with yes for a value of $c = 14.5$.

▶ **Theorem 1.10** (Ismailescu, Kim, and Park [6]). *If $G$ is an SIG with $m$ edges on $n$ vertices, then $m \leq 14.5 \cdot n$.*

They have modified the weight function in the following way: consider a constant $p \geq 1$ and let $\mathrm{rad}(a)$ and $\mathrm{rad}(b)$ be the radii of the spheres of influence of $a$ and $b$ respectively. Let $r = \frac{\mathrm{rad}(a)}{\mathrm{rad}(b)}$. The edge $(a, b)$ is assigned a weight $w_p((a, b))$ of:

- 1 if $r \leq \frac{1}{p}$,
- $\frac{1}{2}$ if $\frac{1}{p} < r < p$, and
- 0 if $p \leq r$.

It should be obvious that this construction is identical with the preceding one if we choose $p = \frac{3}{2}$. By choosing $p = 1.409$ and conducting a proof similar to Soss they show that every SIG $G$ on $n$ nodes has at most $14.5n$ edges.

As stated before, these results refer to the definition of SIG given further above: we only consider subsets of points $V \subseteq \mathbb{R}^2$ in the euclidean plane and the resulting SIG resulting from the euclidean norm $\|\cdot\|_2$. Generalizations – where a subset of points $V \subseteq \mathbb{R}^d$, for some natural number $d \geq 2$ is allowed and other norms are considered – have also been studied.

It has been proven that a SIG induced by the euclidean norm on $n$ vertices in $\mathbb{R}^d$ has at most $5^d \cdot n$ edges [10]. For $d = 2$ this gives a much weaker upper bound than the research mentioned above.

Additionally, Soss proved that a SIG $G$ on $n$ vertices induced by the infinite order Minkowski metric $\|\cdot\|_\infty$ in $\mathbb{R}^d$ has at most $(2^{2d-1} - 2^{d-1})n$ edges. They also give an asymptotic lower bound of $(2^{d+2} - 3d - 4) \cdot \frac{n}{9}$ on the maximum size of such a graph [14]. For $d = 2$ this gives a tight bound of $6n$.

### 1.2.2 Largest Clique

Another question we want to ask: which complete graphs $K_n$ are SIGs and which are not. It is rather easy to see that arbitrarily big $K_n$ cannot be SIGs because of the known bounds for the density of SIGs.

▶ **Question 1.11** (Largest Clique). *For a natural number $n$, let $K_n$ be the complete graph on $n$ vertices. What is the largest choice of $n$, such that $K_n$ is a SIG?*

We know that the complete graph $K_n$ on $n$ nodes has $\frac{1}{2}n(n-1)$ many edges. The contraposition of Theorem 1.10 tells us that a graph $G$ on $n$ nodes that has more than $14.5n$ edges cannot be a SIG. Note that $14.5 < \frac{1}{2}(n-1)$ if and only if $30 < n$. This tells us that for $n > 30$ the graph $K_n$ is not a SIG. This is a rather weak result, as is to be expected, since none of the specific properties of $K_n$ have been used.

Recall the result of Bateman and Erdős [3] from Lemma 1.5, which states that in an SIG($V$) for some set of points $V$ the vertex $v$ whose sphere of influence has the smallest radius may have at most 18 neighbors in SIG($V$). Because $1 + 18 = 19 < 20$ we follow that $K_{20}$ is not a CSIG and not an OSIG. We attribute this observation to Kézdy and Kubicki [8].

There have been other advancements made on answering Question 1.11. In 1997 Kézdy and Kubicki proved that $K_{12}$ is not a CSIG [8].

▶ **Theorem 1.12** (Kézdy and Kubicki [8]). *The complete graph on 12 nodes $K_{12}$ is not a CSIG.*

Additionally, approaching Question 1.11 from the other direction, Harary, Jacobson, Lipman, and McMorris have given a configuration showing that $K_8$ is a SIG, and they have conjectured that $K_9$ is smallest complete graph that is not a SIG. Figure 1.4 is taken from [5] and depicts an arrangement of points which shows that $K_8$ is an OSIG and a CSIG since in this case all intersections of spheres of influence are proper.

▶ **Conjecture 1.13** (Harary, Jacobson, Lipman, and McMorris [5]). *The graph $K_9$ is the smallest complete graph that is neither an* OSIG *nor a* CSIG.

### 1.2.3 Classification of SIGs

We have seen in the previous sections that for example $K_8$ is a SIG while we have also learned that some graphs for example dense graphs are not SIGs. The question: Which graphs are SIGs and which are not? seems particularly interesting. In other words: What is a classification or description of the class of OSIGs and of the class of CSIGs? Are these even different graph classes or can every CSIG also be depicted as a OSIG? This is a hard question, considering how relatively few advancements have been made on the topic of SIGs.

Recall that OSIGs only count proper intersections of spheres of influence, while in a CSIG two nodes $u, v$ are also adjacent if their spheres of influence $C_u, C_v$ are touching.

The authors of [5] show a property of SIGs that illustrates why it is hard to find such a characterization. They proved, that a subgraph of a SIG is not always a SIG. In particular, this means that there is no forbidden subgraph characterization for OSIGs or for CSIGs.

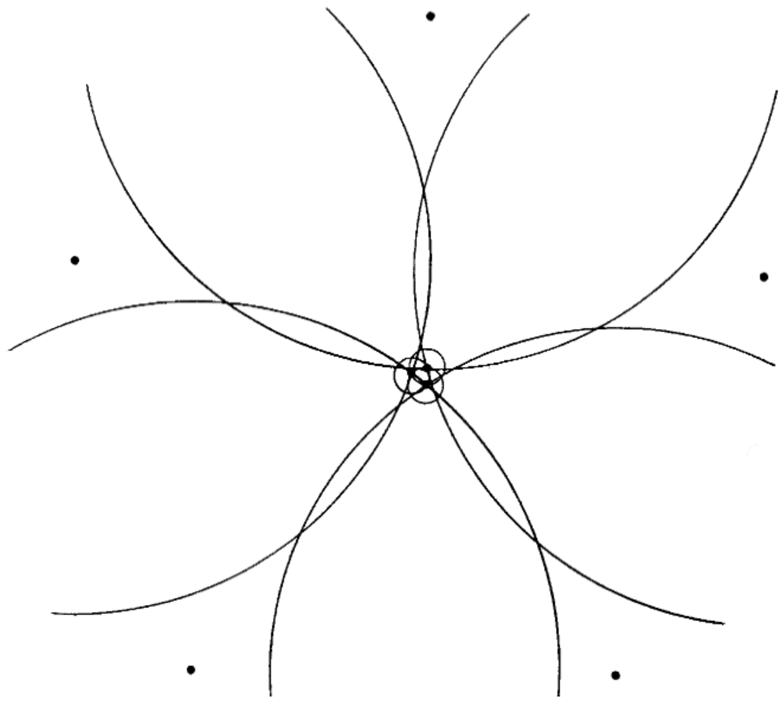


Figure 1.4 A SIG drawing of $K_8$ taken from [5].

▶ **Theorem 1.14** (Harary, Jacobson, Lipman, and McMorris [5]). *Not every induced subgraph of an* OSIG *is itself an* OSIG *and not every induced subgraph of a* CSIG *is itself a* CSIG*.*

To prove this theorem, the authors show that while the claw $K_{1,3}$ is not an OSIG, the graph $H$ that results by adding a node to one of the leaves of $K_{1,3}$ is an OSIG. Consider $S = \{(-1,\ 0),\ (0,\ 0), (\ 1, 0),\ (0,\ 1.9),\ (0,\ 2.9)\}$. It is easy to see that OSIG($S$) is isomorphic to $H$. See Figure 1.5 for an illustration.

Additionally, to prove this theorem for the CSIG setting the authors use another one of their results. A path $P_n$ on $n$ nodes is a CSIG if and only if $n = 1$ or $n$ is even. This means that, while $P_4$ is a CSIG, the path on three nodes $P_3$, which is an induced subgraph of $P_4$, is not a CSIG.

The authors also inspect multiple other types of graphs. We group some of their results together in the following lemma.

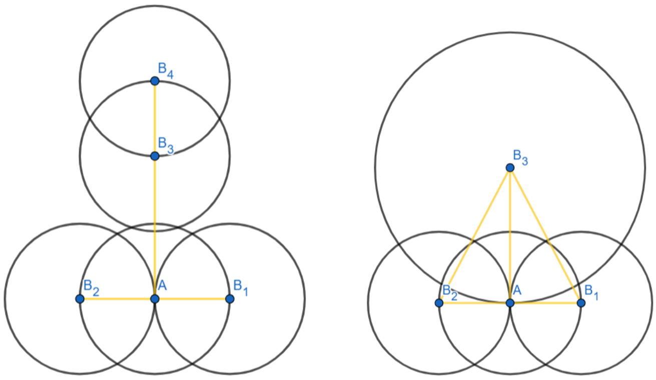
▶ **Lemma 1.15.** *Let $P_n$ denote the path on $n$ nodes and $C_n$ denote the cycle on $n$ nodes. The following statements are true.*

- *Every path $P_n$ is an* OSIG*.*
- *Every cycle $C_n$ is an* OSIG*.*
- *A path $P_n$ is a* CSIG *if and only if $n = 1$ or $n$ is even.*
- *A cycle $C_n$ is a* CSIG *if and only if $n = 3$ or $n$ is even.*

Note that the path $P_1$ on one node is the trivial graph $K_1$ and we consider it to be both an OSIG and a CSIG.

A direct implication of the above lemma is that OSIG and CSIG are in fact different graph classes. The path on three nodes $P_3$ is an example as it is an OSIG but not a CSIG.

To understand the next result, we must first know what a *factor* of a graph $G$ is. If




**(a)** The Graph $H$.

**(b)** $K_{1,3}$ is not an OSIG with the given points.

**Figure 1.5** The graph $H$ that results from adding a node to one of the leafs of $K_{1,3}$ is a OSIG (Figure 3.2a).

$G_1, \ldots, G_k$ are non-trivial graphs and $G$ is a graph, then a $\{G_1, \ldots, G_k\}$-factor of $G$ is a spanning subgraph of $G$ such that each component is isomorphic to one of the $G_i$'s. Let $P_n$ denote a path on $n$ vertices. A $\{P_2\}$-factor is usually referred to as a perfect matching.

In 1995 Jacobson, Lipman, and McMorris showed that a tree is a CSIG if and only if it has a perfect matching [11]. The authors also showed that a tree is an OSIG if and only if it has a $\{P_2, P_3\}$-factor.

▶ **Theorem 1.16** (Jacobson, Lipman, and McMorris [11]). *If $T$ is a tree, then:*

- *$T$ is a* CSIG *if and only if $T$ has a perfect matching.*
- *$T$ is an* OSIG *if and only if $T$ contains a $\{P_2, P_3\}$-factor.*

# 2 Properties of Sphere of Influence Graphs

Although the definition of SIGs may be very straightforward and intuitive, it seems there are good reasons for why there have been relatively few improvements on the questions posed in Section 1.2. In the following are some observations about properties of SIGs, that make working with them hard. All observations were made while trying to improve upon the known bounds for the Density Question (see Question 1.4). We will discuss how the shown properties influence the density of SIGs.

The first observation we want to make states that in the special case that for some set of points $V$ all spheres of influence have the same radius, the graph $\mathrm{CSIG}(V)$ has at most $9 \cdot |V|$ edges.

▶ **Observation 2.1.** Let $G$ be the sphere of influence graph $\mathrm{CSIG}(V)$ with $V \subseteq \mathbb{R}^2$ and $|V| = n$. If $\mathrm{rad}(V, u) = \mathrm{rad}(V, v)$ for all $u, v \in V$, then $|E(V)| < 9n$.

**Proof.** Because all radii are the same, each sphere of influence is now the smallest, and each vertex can only have at most 18 neighbors, according to Lemma 1.5. Therefore, $\deg(v) \leq 18$ for all $v \in V$. Using the handshake lemma, we have $2|E(V)| = \sum\limits_{v \in V} \deg(v) \leq 18|V|$. Dividing by two yields the desired bound. ◀

The related special case in which the radii have only two distinct values is already much more complicated. Up to scaling, we may assume that radii have the two possible values 1 and $r > 1$.

For simplicity, consider the case $r = 2$. We observe that a node $u$ with radius $\mathrm{rad}(u) = 1$ may have 9 neighbors with radius 2 and at the same time 6 neighbors with radius 1. Figure 2.1 provides an illustration.

The WSIG construction of Soss [14] gives us a weight of $9 \cdot 1 + 4 \cdot \frac{1}{2} = 11$ for $u$, which is already more than the desired value of 9. We can modify the weights. In particular, because radii have values 1 or $r = 2$, we only have three types of arcs:

1. an arc going from a node with radius 2 to a node with radius 1, we call this a $(2, 1)$-arc,
2. similarly a $(1, 2)$-arc, and
3. $(1, 1)$ and $(2, 2)$-arcs, which must be assigned weight $\frac{1}{2}$.

Note that a node with radius 2 can have 30 neighbors with radius 1, as can be seen in Figure 2.2. This tells us that a $(2, 1)$-arc must have weight $\leq \frac{9}{30} = 0.3$. Therefore, the corresponding $(1, 2)$-arc must have weight $\geq 0.7$. It seems plausible that we could prove a bound of $9n$ with weights 0.3 and 0.7 for $(2, 1)$ and $(1, 2)$-arcs respectively. One needs to prove that a node with radius 2 cannot have:

- 31 neighbors with radius 1
- 29 neighbors with radius 1 and 1 neighbor with radius 2
- 27 neighbors with radius 1 and 2 neighbors with radius 2
- 26 neighbors with radius 1 and 3 neighbors with radius 2

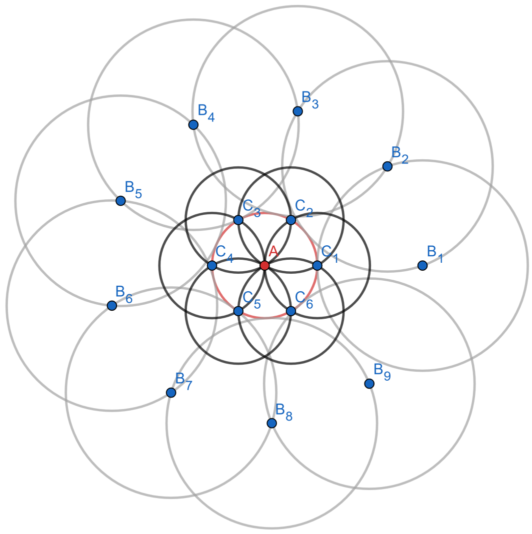



**Figure 2.1** A configuration of points where all spheres of influence have either radius 1 or radius 2. Note that the point $A$, which is colored in red, has nine neighbors with radius 2 and six neighbors with radius 1.

...

- 2 neighbors with radius 1 and 17 neighbors with radius 2

Note that $0.3 \cdot 30 = 9$ and thus $0.3 \cdot 31 > 9$. Also note that $0.3 \cdot 29 + 0.5 > 9$ while $0.3 \cdot 28 + 0.5 < 9$ and $0.3 \cdot 29 < 9$.

Something similar needs to be shown for a node with radius 1.

Note that this is only for the special case $r = 2$. For other values of $r$ the weights and numbers of neighbors are different. Indeed, for $r \to \infty$ the number of neighbors with radius 1 that a node with radius $r$ may have tends to $\infty$. This means that the weight of an $(r, 1)$-arc must tend to 0.

It seems possible that a weighing scheme that depends on $r$ could be devised, but we will not pursue this further.

The following observations deal with the CSIG-drawing $\Gamma$ that results from a given set of points $V$, as described in Section 1.1 of the introduction. Recall that an edge $uv \in E(V)$ is drawn as a straight line between $u$ and $v$. Therefore, there is an obvious correspondence between the edges in $E(V)$ and the set of lines in $\Gamma$.

We first observe that two edges $uv, st$ may intersect at an arbitrarily small angle. Note that if this were not the case, if whenever two edges $uv, st$ intersect their (smaller) angle had a value of at least $\beta > 0°$, we could sort all lines of $\Gamma$ into $\left\lceil \frac{180°}{\beta} \right\rceil$ according to their angle compared to the horizontal. Within a group edges cannot intersect, and thus the drawing of a single group is planar. It is known that a planar graph on $n$ nodes may have at most $3n - 6$ edges. Thus we would have a bound of $\left\lceil \frac{180°}{\beta} \right\rceil \cdot (3n - 6)$ for the density of a CSIG $G$ on $n$ nodes.

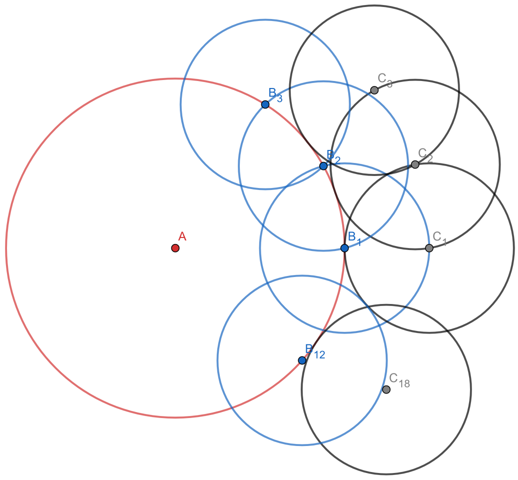



**Figure 2.2** A configuration of points where all spheres of influence have either radius 1 or radius 2. Note that the point $A$, which is colored in red, has radius 2 and has 30 neighbors with radius 1. The points are given by $B_i = (2; (i-1) \cdot 2 \cdot \arcsin\left(\frac{1}{4}\right))$ for $i \in \{1, \ldots, 12\}$ and $C_i = (3; (i-1) \cdot 2 \cdot \arcsin\left(\frac{1}{6}\right)$ for $i \in \{1, \ldots, 18\}$. Note that not all points are depicted.

▶ **Observation 2.2.** Let $V \subseteq \mathbb{R}^2$ be a set of points in the plane. Let $L$ be the set of lines corresponding to $E(V)$ in the SIG-drawing of SIG$(V)$. The (smaller) angle $\beta$ described by two intersecting lines $l_1$ and $l_2 \in L$ may be arbitrarily small.

**Proof.** Let $\beta$ be an angle smaller than 60° and greater than 0°. We will construct points $A, B, C, D$ in the plane such that the lines $AB, CD$ cross in a point $X$ at an angle of $\beta$, see Figure 2.3 for an illustration.

We will use a polar coordinate system, where the first coordinate represents the radius and the second coordinate is the angular coordinate. We will call the origin $O = (0;\ 0°)$. Consider the points $A = (1;\ 0°), B = (1;\ \beta), C = (1;\ 2\beta)$, and $D = (1;\ 3\beta)$.

Indeed, the SIG-drawing on the points $A, B, C, D$ contains both lines $AC$ and $BD$. By definition, the line $AC$ is present if and only if $\text{dist}(A, C) \leq \text{rad}(A) + \text{rad}(C)$.

Note that $\text{rad}(C) = \text{rad}(A) = \text{dist}(A, B) = \sqrt{1^2 + 1^2 - 2 \cdot 1 \cdot 1 \cdot \cos\beta} = \sqrt{2}\sqrt{1 - \cos\beta}$ and $\text{dist}(A, C) = \sqrt{1^2 + 1^2 - 2\cos(2\beta)} = \sqrt{2}\sqrt{1 - \cos(2\beta)}$. Thus $\text{dist}(A, C) \leq 2\,\text{rad}(A)$ if and only if $4\cos(\beta) - \cos(2\beta) \leq 3$. This holds for all values of $\beta$.

Take a look at the triangle $AOB$. By definition the angle at $O$ is $\beta$ and the two sides $OA, OB$ are of equal length. Thus, because the sum of angles in a triangle is 180°, the angle $\delta_1$ at $A$ must be $\frac{1}{2}(180° - \beta)$. Similarly, inspecting the triangle $AOC$ tells us that the angle $\delta_2$ at $A$ must be $\frac{1}{2}(180° - 2\beta)$.

The angle at $A$ (and at $C$) in the triangle $ABC$ is $\delta_1 - \delta_2 = \frac{\beta}{2}$. Thus, since the sum of the angles is 180°, the angle at $B$ is $180° - \beta$.

Let $X$ be the crossing point of the edges $AC$ and $BD$. Then the triangle $BXC$ is similar to the triangle $ABC$. This means that the angle at $X$ in the triangle $BXC$ is $180° - \beta$. The crossing angle between $AC$ and $BD$ is $180° - (180° - \beta) = \beta$. ◀

The next observation deals with the number of intersections an edge might have. Note that if there was a constant $k \in \mathbb{N}_0$ such that for all OSIGs $G$ with a SIG-drawing $\Gamma$ any

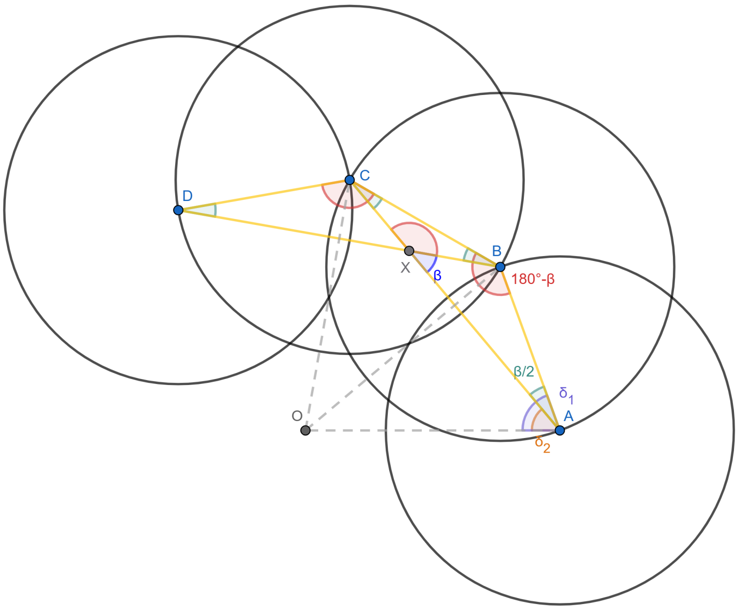



**Figure 2.3** CSIG-drawing of the given points $A, B, C, D$ (blue) with additional annotations. The edges $AC, BD$ cross at an angle of $\beta$. Here, $\beta = 40°$. Similar angles have the same color

edge in $\Gamma$ is crossed by at most $k$ other edges, it would follow that OSIGs are part of the well-studied class of $k$-planar graphs.

It is known that such a graph on $n$ nodes may have at most $3.81\sqrt{k}n$ edges [1]. If $k < \left(\frac{14.5}{3.81}\right)^2 \approx 14.48$ this would yield a better bound to the density of OSIGs (see Question 1.4).

▶ **Observation 2.3.** For every $n \in \mathbb{N}$, there exists a OSIG-drawing $\Gamma$ where some edge is crossed at least $n$ times.

**Proof.** For each $n \in \mathbb{N}$ we give a configuration of points where an edge is crossed $n$ times. Consider the points $A = (-1, 0)$, $B = (1, 0)$, $C = (0, 0.3)$ and the corresponding drawing of OSIG($\{A, B, C\}$), which is illustrated in Figure 2.4. Let $C_A, C_B, C_C$ be the respective spheres of influence. We observe that all spheres on influence $C_A, C_B, C_C$ intersect. In particular, the edge $AB$ is present.

We are interested in the portion of $C_C$ below the $x$-Axis, from the intersection point $S$ with $C_A$ to the intersection point $T$ with $C_B$.

We add new points $D_1, \ldots, D_n$ such that $C_A, C_B, C_C$ remain unchanged and the edge $AB$ is crossed by the edges $CD_1, \ldots, CD_n$. We achieve this by placing $D_1, \ldots, D_n$ equidistantly along the aforementioned portion of $C_C$ from $S$ to $T$. This means that $D_1, \ldots, D_n$ are all adjacent to $C$, and in particular the edges $CD_1, \ldots, CD_n$ must cross the edge $AB$. Note that this does not change the radii of $C_A, C_B, C_C$. ◀

The example given for Observation 2.3 might give the impression that SIGs are fan-crossing graphs, which is a well-studied graph class. Briefly speaking, a graph $G$ is a fan-crossing graph if it has a drawing $\Gamma$ in which for every edge $e$ of $G$ all edges crossing $e$ in $\Gamma$ have one node in common, that is they form a star in $G$ [7]. This need not be the case for OSIGs. Figure 1.3 depicts a configuration of points, showing that an edge might be crossed

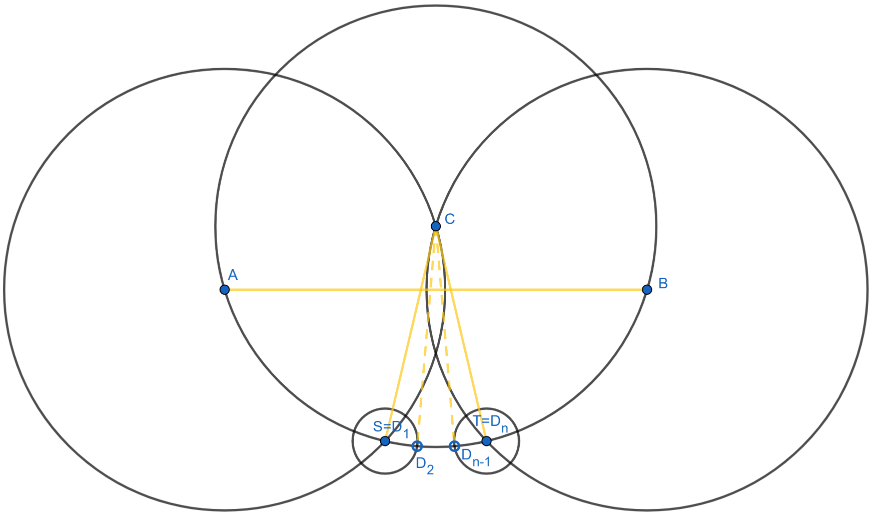



**Figure 2.4** An OSIG-drawing given by the points $A, B, C, D_1, \ldots, D_n$ where the edge $AB$ is crossed by the edges $CD_1, \ldots, CD_n$. The edges $AC, BC, AD_1, BD_n$ and other potential edges are omitted for simplicity.

by independent edges, although this is not depicted. Figure 2.5 highlights such an example in the same set of points.

The last observation we want to mention states that a point $p \in \mathbb{R}^2$ cannot be contained in more than six spheres of influence. We do not know how this may help with improving the upper bound on the density of SIG, but we wanted to include it anyways.

▶ **Observation 2.4.** If $\Gamma$ is a CSIG-drawing, then every point $p \in \mathbb{R}^2$, which need not be a vertex, is contained in at most six spheres of influence of $\Gamma$.

**Proof.** Without loss of generality, we may assume that the point $p$ is the origin $(0, 0) \in \mathbb{R}^2$ and is contained in seven spheres of influence (circles) $C_1, \ldots, C_7$ with the corresponding center points $p_1, \ldots, p_7$. We will use polar coordinates. We may assume that $p_1, \ldots, p_7$ are ordered around the origin with respect to their angular coordinates and that $p_1$ has angular coordinate $0°$.

Let $i \in \{1, \ldots, 7\}$ be the index for which the angle between the points $p_i$ and $p_{i+1}$, with the convention $p_8 = p_1$, is the smallest among all possible choices for $i$. We call this angle $\gamma$. Note that $\gamma$ is at most $\frac{360°}{7} \approx 51.4°$.

Let $a$ and $b$ be the radial coordinates of $p_i$ and $p_{i+1}$ respectively and let $c$ be the distance $\operatorname{dist}(p_i, p_{i+1})$ from $p_i$ to $p_{i+1}$. This creates a triangle with side lengths $a, b$, and $c$ and the angle $\gamma$ opposite the side with length $c$. The law of cosines states that $c^2 = a^2 + b^2 - 2ab\cos(\gamma)$. Because $\gamma < 60°$ we have $\cos(\gamma) > \frac{1}{2}$ this implies $2\cos(\gamma) > 1$. We may assume $a \leq b$. We have $a < 2b\cos(\gamma)$. This implies $a^2 < 2ab\cos(\gamma)$. Then $c^2 = a^2 + b^2 - 2ab\cos(\gamma) < a^2 + b^2 - a^2 = b^2$

This yields $\operatorname{rad}(p_i) \leq \operatorname{dist}(p_i, p_{i+1}) = c < b = \operatorname{dist}(p_i, p)$ Therefore, the radius of $p_i$ is smaller than the distance from $p_i$ to the origin. This means that the origin is not contained

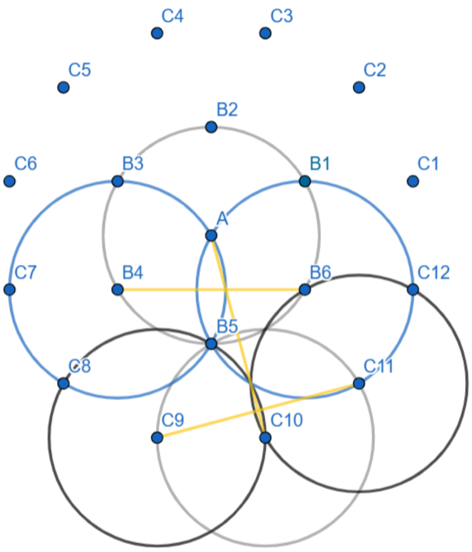



**Figure 2.5** The configuration of points shown in Figure 1.3 with other edges highlighted. Only the edges $AC_{10}, C_9C_{10}$, and $B_4B_6$ and the spheres of influence of the corresponding points are shown.

in $C_i$, a contradiction. ◀

# 3 General Sphere of Influence Graphs

The definition of SIGs given in Section 1.1 can be made more general: instead of just taking a set of points and deriving the spheres of influence and a graph from it, we can begin from a set of circles $\mathcal{C}$ such that the center of none of the circles is contained in another circle. Figure 3.1 gives an introductory example. Note the similarities and differences to Figure 1.1.

▶ **Definition 3.1** ($\mathrm{CSIG}(P, \mathcal{C})$). *Let $\mathcal{C}$ be a set of circles in the plane and $P \subseteq \mathbb{R}^2$ be the set of center points of all circles in $\mathcal{C}$, with the following property: For any circle $C \in \mathcal{C}$, its center point $p \in P$ is not contained in the interior of any other circle $\tilde{C} \in \mathcal{C}, C \neq \tilde{C}$. We call this the* SIG-condition*.*

*The corresponding closed sphere of influence graph is defined as $\mathrm{CSIG}(P, \mathcal{C})$ with vertices $P$ and edges $E$, where $pq \in E$ if and only if their corresponding circles $C_p$ and $C_q$ intersect or touch for all $p, q \in P, p \neq q$.*

*For a circle $C \in \mathcal{C}$ with corresponding center point $p \in P$ we call $C$ the circle of $p$. And we define $\mathrm{rad}(\mathcal{C}, p)$ as the radius of $C$. This we call the radius of $p$.*

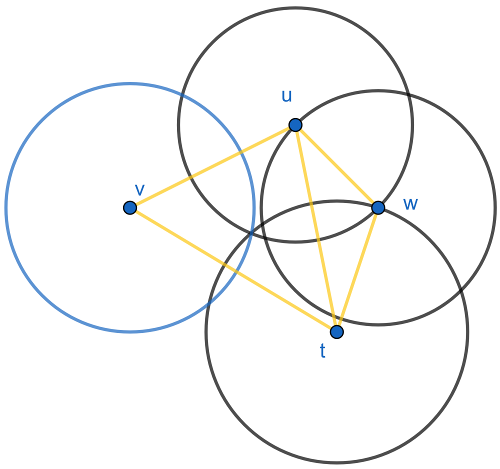



**Figure 3.1** An example of a general SIG-drawing. The coordinates of the points $u, v, w, t$ are the same as in Figure 1.1. Note that the sphere of influence $C_v$ of $v$ highlighted in blue, is smaller than it is in Figure 1.1.

Note that the above definition deals with the general CSIG setting. The general open sphere of influence graph $\mathrm{OSIG}(P, \mathcal{C})$ requests two circles (spheres of influence) $C_p, C_q$ to have a proper intersection for the vertices $p, q$ to be adjacent. Observe that in the example given in Figure 3.1 all intersections are proper, and therefore the resulting OSIG and CSIG are the same.

We choose the somewhat redundant notation, in order to retain more similarity with the conventional SIG case. To differentiate between the two cases, we call the a graph that fits the description in Section 1.1 a *well-formed* SIG and the current case a *general* SIG. SIG-drawings in the general setting are given analogously to the well-formed setting.

Notice that in the above definition, if we have a set of points $\mathcal{C}$ and a corresponding set of center points $P$ we have:

$$\mathrm{rad}(\mathcal{C}, p) \leq \min \{\|p - q\|_2 \mid q \in P,\ q \neq p\} = \mathrm{rad}(P, p)$$

for each point $p \in P$. Recall that $\mathrm{rad}(P, p)$ is the radius of $p$ in the case of well-formed SIG. This tells us that for any set of points $P$ and a corresponding set of circles $\mathcal{C}$, whose set of center points is $P$, that does not violate the SIG-condition, we have $\mathrm{CSIG}(P, \mathcal{C}) \subseteq \mathrm{CSIG}(P)$. Recall that $\mathrm{CSIG}(P)$ denotes the well-formed SIG that results from the set of points $P$. The same holds true for the OSIG setting.

We can say that a circle $C \in \mathcal{C}$ is maximal if its radius cannot be increased without violating the SIG-condition. If this holds for all $C \in \mathcal{C}$, then we call the set of circles $\mathcal{C}$ maximal. Note that for a given set of points $P$ the maximal set of circles $\mathcal{C}_{\max}(P)$ with center points $P$ is unique and is implicitly given by:

$$\mathrm{rad}(\mathcal{C}_{\max}(P), p) = \min_{q \in P, p \neq q} \|p - q\|_2$$

for all $p \in P$. Note that $\mathrm{SIG}(P) = \mathrm{SIG}(P, \mathcal{C}_{\max}(P))$.

Because of this, the maximum edge density of these two graph classes coincide (see Question 1.4). Likewise, if we ask ourselves what is the largest clique that is a general CSIG, this coincides with the largest clique that is a well-formed CSIG (see Question 1.11).

Assume that the complete graph $K_n$ on $n \in \mathbb{N}$ nodes is isomorphic to the general CSIG $G = \mathrm{CSIG}(P, \mathcal{C})$ for some set of circles $\mathcal{C}$ with center points $P$. This means that $G$ contains every edge $pq$ for all $p, q \in P$ with $p \neq q$. Note that the graph $G' = \mathrm{CSIG}(P, \mathcal{C}_{\max}(P))$, which is a well-formed CSIG, cannot contain fewer edges than $G$, because all spheres of influence are larger. Therefore, $G' = G$ and thus $K_n$ is a well-formed CSIG.

This proves: if $K_n$ is a general CSIG, then $K_n$ is also a well-formed CSIG. The other direction is trivial. Every well-formed CSIG is also a general CSIG. This result is formalized in the following lemma.

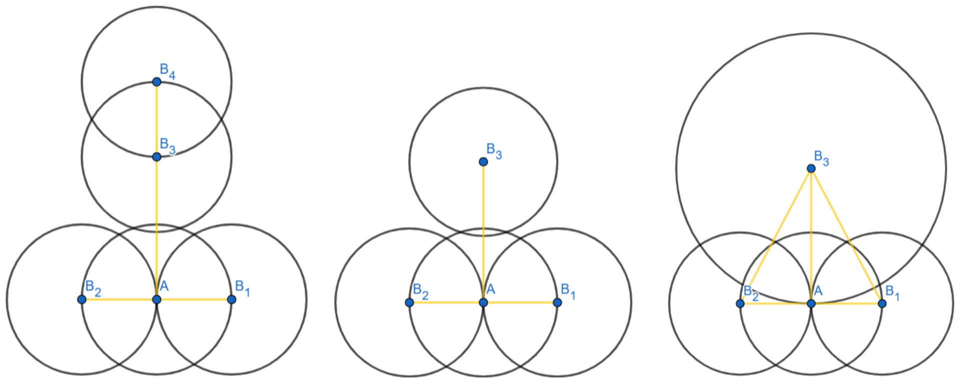
▶ **Lemma 3.2.** *The complete graph $K_n$ on $n$ nodes is a general CSIG if and only if it is a well-formed CSIG.*

The advantage of general SIGs over the well-formed setting becomes apparent when we ask the question: which graphs are general SIGs (both CSIG and OSIG), and which are not? Recall that the class of well-formed SIGs is not hereditary (some induced subgraphs of SIGs are not SIGs themselves, see Theorem 1.14). This means that there cannot be a forbidden subgraph description of well-formed SIGs. We do not have this obstacle when studying general SIGs. Figure 3.2 gives an intuition.

Consider the graph $G = \mathrm{CSIG}(P, \mathcal{C})$ for a set of circles $\mathcal{C}$ and the corresponding set of center points $P$. For any point $p \in P$ with corresponding circle $C \in \mathcal{C}$ the graph $\mathrm{CSIG}(P - p, \mathcal{C} - C)$ is identical to $G - p$, which results from $G$ by removing the vertex $p$ and all of its incident edges. Note that because we are in the general case the remaining circles in $\mathcal{C}$, and thus the rest of the graph, remain unchanged by the removal of $p$. In the well-formed case the radii of the remaining nodes might increase, thus adding new edges. This result is formalized in the following lemma, whose proof we have just conducted. Note that the case of OSIGs functions in the same way.

▶ **Lemma 3.3.** *Every induced subgraph of a general CSIG is itself* a general *CSIG and every induced subgraph of a general OSIG is itself* a general *OSIG.*

Recall that this lemma does not hold for the well-formed setting. We have seen that $K_{1,3}$ is not a well-formed OSIG, but the graph $H$ that results from adding a node to one of the leafs of $K_{1,3}$ is [5]. The above lemma tells us that $K_{1,3}$ is a general SOIG. Indeed Figure 3.2 illustrates this phenomenon.




**(a)** The Graph $H$.

**(b)** $K_{1,3}$ is a general OSIG.

**(c)** $K_{1,3}$ is not a well-formed OSIG with the given points.

**Figure 3.2** The graph $H$ that results from adding a node to one of the leafs of $K_{1,3}$ is a well-formed OSIG (Figure 3.2a). If we remove this additional node and its respective sphere of influence, we observe that $K_{1,3}$ is a general OSIG (Figure 3.2b). In the well-formed setting the sphere of influence of the point $B_3$ needs to grow. We get additional edges, that are not present in $K_{1,3}$ (Figure 3.2c). The points are given by $A = (0, 0)$, $B_1 = (1, 0)$, $B_2 = (-1, 0)$, $B_3 = (0, 1.9)$, and $B_4 = (0, 2.9)$.

Note that $K_{1,3}$ is also a general CSIG. Take a look at the set of points $P_b$ and circles $\mathcal{C}_b$ depicted in Figure 3.2b. The graph $\mathrm{CSIG}(P_b, \mathcal{C}_b)$ contains the edge $B_1B_2$ because the spheres of influence of $B_1$ and $B_2$ touch in the point $A$. We can remove this edge by setting $\mathrm{rad}(B_1) = 0.9$.

We observe that general SIGs are truly more expressive than well-formed SIGs. In fact every planar graph $G$ is also as a general CSIG. This stems from the fact that the class of planar graphs coincides with the class of disk contact graphs (also called disk touching graphs or coin graphs) [9, 13][1]. Disk contact graphs are the conflict graphs given by a set $\mathcal{D}$ of disks in the plane that are not allowed to overlap, i.e. no two disks $d_1, d_2 \in \mathcal{D}$ share more than one point.

Because a set $\mathcal{C}$ of circles that do not have proper intersections with each other does not violate the SIG-condition, every disk contact graph is also a general CSIG. In particular this means that all paths, cycles, stars, and trees are general CSIGs.

[1] [13] contains a discussion about how the proof (from which it follows that every planar graph is also a coin graph) given in [9] was forgotten for a long time.

# 4 $\alpha$-SIG

Since working with SIGs directly has proved to be quite difficult, we introduce a generalization of SIG, namely $\alpha$-SIG.

When computing the SIG-drawing induced by a set of points $P \subseteq \mathbb{R}^2$, we observe that a point $p \in P$ limits the size of the spheres of influence of the other points in $P$. But the point $p$ does not scale up, no matter how large its sphere of influence might be.

## 4.1 Introduction

The general idea behind *alpha-Sphere of Influence Graphs* ($\alpha$-SIG) is quite simple. Instead of circles, we work with annuli. In general, an annulus is the region between two circles that have the same center. We introduce an additional parameter $\alpha \in [0, 1]$ that defines how thick the resulting annuli are. The importance of this parameter will become apparent later.

For $\alpha$-SIGs we essentially study sets $\mathcal{A}$ of annuli in the plane with thickness $\alpha$ with the property that for each annulus $a = (C_{\text{in}}, C_{\text{out}}) \in \mathcal{A}$ there are no proper intersections between the inner circle $C_{\text{in}}$ of $a$ and the outer circle $\tilde{C}_{\text{out}}$ of another annulus $\tilde{a} = (\tilde{C}_{\text{in}}, \tilde{C}_{\text{out}}) \in \mathcal{A}$. Figure 4.1 gives a first example of an $\alpha$-SIG. A formal definition is given further down.

▶ **Definition 4.1** (Annulus). *An annulus $a = (C_{\text{in}},\ C_{\text{out}})$ with thickness $\alpha \in [0, 1]$ is composed of two circles $C_{\text{in}}, C_{\text{out}} \subseteq \mathbb{R}^2$ in the plane with the following two properties:*
1. *$C_{\text{in}}$ and $C_{\text{out}}$ have the same center.*
2. *If $C_{\text{out}}$ has radius $r > 0$ then the radius of $C_{\text{in}}$ is $\alpha \cdot r$.*

*We call the center point of $C_{\text{in}}$ and $C_{\text{out}}$ the center of the annulus $a$.*

We only consider annuli $a = (C_{\text{in}}, C_{\text{out}})$ where both circles $C_{\text{in}}, C_{\text{out}}$ have radii truly greater than zero.

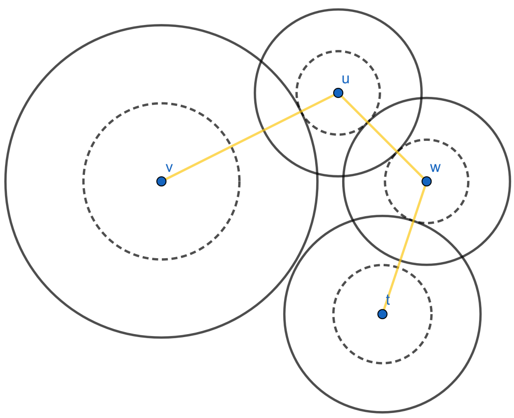



**Figure 4.1** A first example of an $\alpha$-SIG drawing with $\alpha = 0.5$. Note that annuli are depicted in black, their center points are blue and edges are yellow. Inner circles are depicted with dotted lines. The points $u, v, w, t$ have the same coordinates as in Figures 1.1 and 3.1.

Let $\mathcal{A}$ be a set of annuli with thickness $\alpha \in [0, 1]$ and $P \subseteq \mathbb{R}^2$ be the set of center points of all annuli in $\mathcal{A}$, with the following property: For each annulus $a = (C_{\text{in}}, C_{\text{out}}) \in \mathcal{A}$ the inner circle $C_{\text{in}}$ of $a$ does not have any proper intersections with any other inner circle $\tilde{C}_{\text{in}}$ or outer circle $\tilde{C}_{\text{out}}$ of an annulus $\tilde{a} = (\tilde{C}_{\text{in}}, \tilde{C}_{\text{out}}) \in \mathcal{A}$ with $a \neq \tilde{a}$. We call this the *empty-centers-condition*. Note the similarity to the SIG-condition in Chapter 3.

For a point $p \in P$ that corresponds to an annulus $a = (C^p_{\text{in}}, C^p_{\text{out}}) \in \mathcal{A}$ we call $C^p_{\text{in}}$ the inner circle of $p$ and $C^p_{\text{out}}$ the outer circle of $p$. The *radius* $\text{rad}(\mathcal{A}, p)$ of the point $p \in P$ corresponds to the radius of $C^p_{\text{out}}$. For brevity, we will also write $\text{rad}(p)$ when $\mathcal{A}$ can be inferred from the context.

In such a case, we define the graph $\alpha\text{-SIG}(P, \mathcal{A})$ with vertex set $P$ in the following manner: Two vertices $p, q \in P$, $p \neq q$ with corresponding annuli $a_p = (C^p_{\text{in}}, C^p_{\text{out}}) \in \mathcal{A}$ and $a_q = (C^q_{\text{in}}, C^q_{\text{out}}) \in \mathcal{A}$, are adjacent if and only if the outer circles $C^p_{\text{out}}$ and $C^q_{\text{out}}$ intersect. We call $\alpha\text{-SIG}(P, \mathcal{A})$ the *alpha-Spheres of Influence Graph* on points $P$ and annuli $\mathcal{A}$.

Note that this definition can be linked back to general SIG in the following way: the graph $\alpha\text{-SIG}(P, \mathcal{A})$ is the same graph as $\text{SIG}(P, \mathcal{C}_{\text{out}})$, where $\mathcal{C}_{\text{out}}$ is the set of outer circles of the points in $P$. This formulation highlights how the parameter $\alpha$ tells us what portion of the spheres of influence must remain empty.

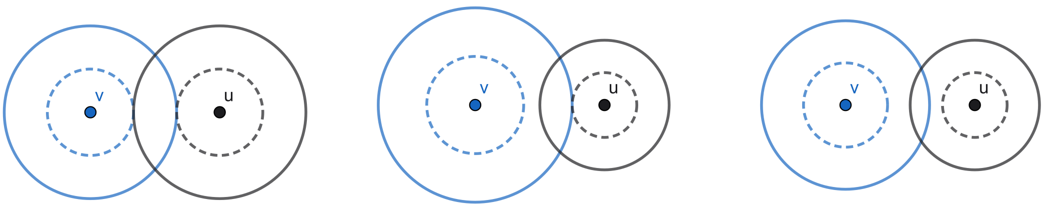
Recall that in the conventional well-formed SIG case we could imagine that we get a set of circles $\mathcal{C}_{\max}(P)$ from a set of points $P \subseteq \mathbb{R}^2$ in the following way: from each point $p \in P$ we let a circle $C_p$ grow until another point $q \in P$ lies on the border of $C_p$. We want to transfer this view point to $\alpha$-SIGs, together with some generalizations. Figure 4.2 depicts the different types of sets of annuli that we distinguish.

Let $\mathcal{A}$ be a set of annuli, and $P \subseteq \mathbb{R}^2$ be the set of center points of all annuli in $\mathcal{A}$.

Then $\mathcal{A}_{\text{grow}}(P)$ is the set of annuli that results from growing annuli from each point $p \in P$ simultaneously and at the same rate, until no annulus can be enlarged without violating the empty-centers-condition. (When an annulus cannot grow anymore it stops at its current size, but the other annuli continue to grow until they are stopped too.) Note that $\mathcal{A}_{\text{grow}}(P)$ is unique. An example can be observed in Figure 4.1.

We call $\mathcal{A}$ and the graph $\alpha\text{-SIG}(P, \mathcal{A})$:

1. *well-formed* if $\mathcal{A}$ is equal to $\mathcal{A}_{\text{grow}}(P)$, and
2. *maximal* if no annulus $a \in \mathcal{A}$ can be enlarged without violating the empty-centers-condition.
3. *general* if $\mathcal{A}$ fulfills the empty-centers-condition.




**(a)** A well-formed set of annuli. Here, $\text{rad}(v) = \frac{4}{3}$ and $\text{rad}(u) = \frac{4}{3}$.

**(b)** A maximal set of annuli. Here, $\text{rad}(v) = 1.5$ and $\text{rad}(u) = 1$.

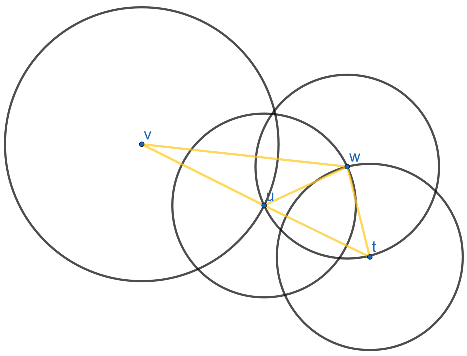
**(c)** A general set of annuli. Here, $\text{rad}(v) = 1.3$ and $\text{rad}(u) = 1$.

**Figure 4.2** The different types of annuli sets for the same two vertices $v = (0, 0)$ and $u = (2, 0)$. Here we set $\alpha = 0.5$. The node $v$ and its annulus are colored blue, while the node $u$ and its annulus is colored black. Inner circles are depicted with dotted lines.

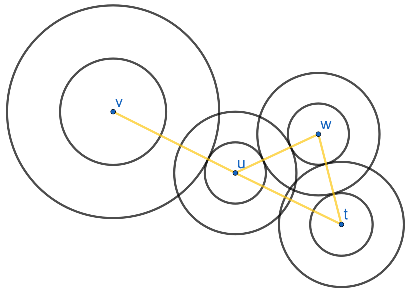
Note that if $\mathcal{A}$ is well-formed, then $\mathcal{A}$ is also maximal. The reverse direction need not be true. Note that both Figures 4.2a and 4.2b depict maximal sets of annuli, but Figure 4.2b is not well-formed.

We mostly concern ourselves with the general setting, since it is easiest to reason about and also the most expressive. Section 4.4 goes into further detail about the differences between the cases, by examining which stars $K_{1,n}$ are $\alpha$-SIGs and for which values of $\alpha$.

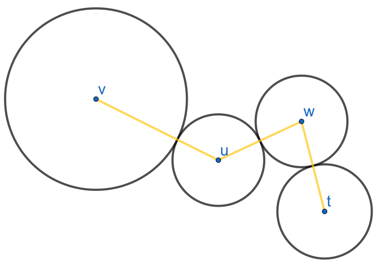
The beauty of general $\alpha$-SIGs comes from the fact that they provide an elegant interpolation between general SIGs (when we set $\alpha = 0$) and disks contact graphs (when we set $\alpha = 1$). (See also Chapter 3.) Figure 4.3 illustrates this property. It depicts the same vertices and the resulting well-formed $\alpha$-SIG drawing for different values of $\alpha$. We observe that different values of $\alpha$ may yield different graphs.




**(a)** $\alpha = 0$




**(b)** $\alpha = 0.5$




**(c)** $\alpha = 1$

**Figure 4.3** The same set of points $\{t, u, v, w\}$ yields different well-formed $\alpha$-SIGs, which we call $G_\alpha$, for different values of $\alpha \in \{0,\ 0.5,\ 1\}$. The edges of $G_\alpha$ are drawn in yellow. Note that $G_0$ is a well-formed CSIG and $G_1$ is a disk contact graph. It is important to see that $G_1$ is a proper subgraph of $G_{0.5}$, and $G_{0.5}$ is a proper subgraph of $G_0$.

Another nice property of $\alpha$-SIGs is that we can skip the distinction between CSIGs and OSIGs, if we are not as stringent with the value of $\alpha$. Assume the vertices $u, v \in P$ are adjacent in $\alpha$-SIG$(P, \mathcal{A})$, such that their outer circles $C^u_{\text{out}}$ and $C^v_{\text{out}}$ only have one point $x$ in common (improper intersection).

If $\alpha > 0$ and $|P| < \infty$ then we can find an $\varepsilon > 0$ and a modified set of annuli $\mathcal{A}'$ such that $(\alpha - \varepsilon)$-SIG$(P, \mathcal{A}') = \alpha$-SIG$(P, \mathcal{A})$. This is illustrated in Figure 4.4.

For some sufficiently small $\varepsilon$ we obtain $\mathcal{A}'$ from $\mathcal{A}$ by the following modification:

- For all points $p \in P$ with $p \neq u$ we set $\mathrm{rad}(\mathcal{A}', p) = \mathrm{rad}(\mathcal{A}, p)$. This means that all outer circles retain their size, while the inner circles shrink by a small amount.

- We enlarge the outer circle $C^u_{\text{out}}$ of $u$ the right amount such that the inner circle $C^u_{\text{in}}$ retains its radius. This is equivalent to $\alpha \cdot \text{rad}(\mathcal{A}, u) = (\alpha - \varepsilon) \cdot \text{rad}(\mathcal{A}', u)$. That means we set $\text{rad}(\mathcal{A}', u) = \frac{\alpha}{\alpha-\varepsilon}\,\text{rad}(\mathcal{A}, u)$.

It is important to choose $\varepsilon$ sufficiently small, such that no new edges are created. Again, we point to Figure 4.4: if we set $\varepsilon$ to a greater value, the outer circles of $u$ and $p$ might intersect, thus adding the edge $up$ to $(\alpha - \varepsilon)$-SIG$(P, \mathcal{A}')$. We needed $P$ to be finite for this part. Otherwise, it may be that for every $\varepsilon > 0$ there exists a point $p_\varepsilon \in P$ such that the edge $up_\varepsilon$ is present in $(\alpha - \varepsilon)$-SIG$(P, \mathcal{A}')$ but not in $\alpha$-SIG$(P, \mathcal{A})$.

Note that these modifications maintain the empty-centers-condition.

This way, we can allow for improper intersections.

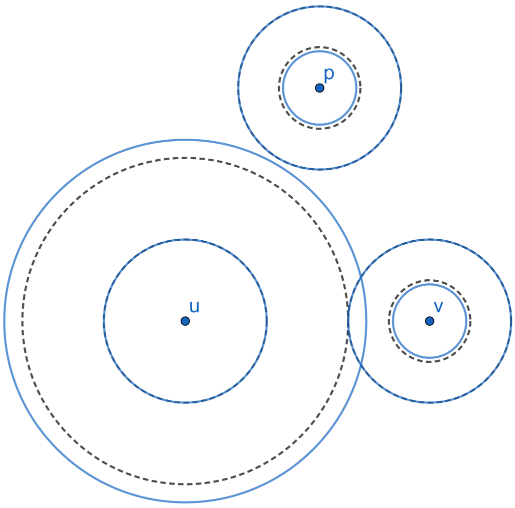



**Figure 4.4** A set of annuli $\mathcal{A}$ (depicted with black dotted lines) where the outer circles $C^u_{\text{out}}$ and $C^v_{\text{out}}$ of $u$ and $v$ respectively have an improper intersection. The set of annuli $\mathcal{A}'$ (depicted with continuous blue lines) shows that we can transform the given annuli so that the intersection becomes proper, without creating new edges.

Now that we have introduced $\alpha$-SIGs, let us take a look at some of their properties. In particular: what can we say about the distance of two neighboring vertices $u, v \in P$. Recall that in the case of SIG two vertices $u, v \in P$ are adjacent in CSIG$(P)$ if and only if the (euclidean) distance $\text{dist}(u, v)$ between $u$ and $v$ fulfills: $\text{dist}(u, v) \leq \text{rad}(u) + \text{rad}(v)$.

How does this translate to $\alpha$-SIGs? How close or far apart can two vertices $u$ and $v$ be, if we only know their radii $\text{rad}(u), \text{rad}(v)$, that is the radii of the respective outer circles? When $u$ and $v$ are as close to each other as possible, the outer circle of $v$ touches the inner circle of $u$ or vice versa, see Figures 4.2a and 4.2b.

▶ **Observation 4.2.** Consider an $\alpha$-SIG$(P, \mathcal{A})$ for some $P \subseteq \mathbb{R}^2$ and a set of annuli $\mathcal{A}$. Let $\text{dist}(u, v)$ denote the euclidean distance between two vertices $u, v \in P$. For two points $u, v \in P$ with $u \neq v$ and $\text{rad}(v) \geq \text{rad}(u)$ we have:

**1.** $\text{rad}(v) + \alpha \cdot \text{rad}(u) \leq \text{dist}(u, v)$ and $\text{rad}(u) + \alpha \cdot \text{rad}(v) \leq \text{dist}(u, v)$.
In particular, we have:

$$\begin{aligned} \text{dist}(v, u) &\geq \max\{\text{rad}(v) + \alpha \cdot \text{rad}(u),\ \text{rad}(u) + \alpha \cdot \text{rad}(v)\} \\ &= \max\{\text{rad}(u), \text{rad}(v)\} + \alpha \cdot \min\{\text{rad}(u), \text{rad}(v)\} \end{aligned}$$

**2.** The vertices $u$ and $v$ are adjacent if and only if $\mathrm{dist}(u,v) \leq \mathrm{rad}(v) + \mathrm{rad}(u)$.

**Proof.** Let $u, v \in P$ be two points with $u \neq v$. When $u$ and $v$ are as close to each other as possible, without violating the empty-centers-condition, the outer circle of $v$ touches the inner circle of $u$ or vice versa. See Figure 4.2b for an illustration. From this it follows that $\mathrm{rad}(v) + \alpha \cdot \mathrm{rad}(u) \leq \mathrm{dist}(u,v)$ and $\mathrm{rad}(u) + \alpha \cdot \mathrm{rad}(v) \leq \mathrm{dist}(u,v)$ must hold.

Consider the following. Rearranging $\mathrm{rad}(v) + \alpha\,\mathrm{rad}(u) \geq \mathrm{rad}(u) + \alpha\,\mathrm{rad}(v)$ yields $(1-\alpha)\,\mathrm{rad}(v) \geq (1-\alpha)\,\mathrm{rad}(u)$. Note that for $\alpha = 1$ we have an equality. Thus, this inequality holds if and only if $\mathrm{rad}(v) \geq \mathrm{rad}(u)$.

Together this gives us the first statement.

To see that the second statement holds, consider its contraposition. If the distance $\mathrm{dist}(u,v)$ between two points $u, v \in P$ is greater than $\mathrm{rad}(u) + \mathrm{rad}(v)$, then their outer circles cannot intersect. ◀

## 4.2 Minimum Value of $\alpha$ for Crossings

In the above section we have introduced general $\alpha$-sphere of influence graphs. We have seen that the graph class described by $\alpha = 1$ is equivalent to disk contact graphs, while $\alpha = 0$ gives us the general SIG case. Disk contact graphs are planar [9, 13]. This means that every 1-SIG-drawing is crossing free, while CSIG-drawings (which are equivalent to 0-SIG-drawings) may contain many crossings, see Observation 2.3. A crossing is given by two distinct edges $uv$ and $st$ of an $\alpha$-SIG-drawing $\Gamma$, that have exactly one point $x \notin \{u, v, s, t\}$ in common: $uv \cap st = \{x\}$.

The obvious next question is: *how small does $\alpha$ need to be in order for crossings to become possible?* The following lemma answers this question.

▶ **Lemma 4.3.** *The following statements hold:*

**1.** *If $\alpha \leq \sqrt{2} - 1$ then there exists an $\alpha$-SIG drawing $\Gamma$ that contains a crossing.*

**2.** *If $\alpha > \sqrt{2} - 1$ then there exists **no** $\alpha$-SIG drawing $\Gamma$ that contains crossings.*

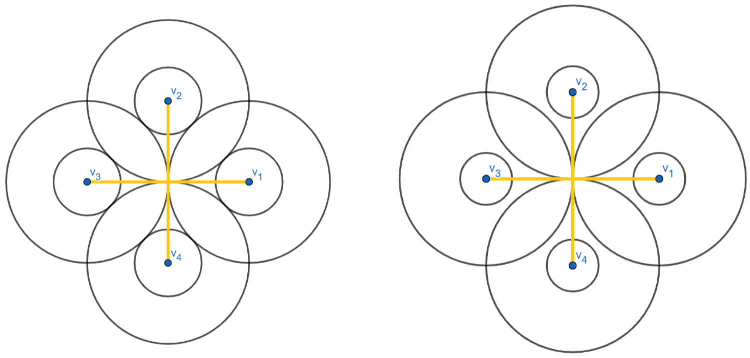
**3.** *If $\alpha = \sqrt{2} - 1$ then every crossing of an $\alpha$-SIG drawing $\Gamma$ is given by two edges $uv$ and $st$, which intersect at a right angle such that the radii of $u, v, s,$ and $t$ have the same value (see Figure 4.5a for an illustration).*

**Proof.** We will show the first statement by giving a set of points that create a crossing in the corresponding $\alpha$-SIG drawing $\Gamma$, for $\alpha \leq \sqrt{2} - 1$.

Consider the points $v_1 = (1,0), v_2 = (0,1), v_3 = (-1,0), v_4 = (0,-1)$, together with $\mathrm{rad}(v_i) = 1$ for all $i \in \{1,2,3,4\}$. See Figure 4.5 for an illustration. We see that the distance from each point to its nearest neighbor is $\sqrt{2} = 1 + (\sqrt{2} - 1) \geq 1 + \alpha \cdot 1$. Thus, Observation 4.2 telly us that the empty-centers-condition is not violated, and the origin $p_0 = (0,0)$ is contained in all outer circles. Therefore, the edges $v_2v_4$ and $v_1v_3$ both exist and cross in the point $p_0$.

Now, let us take a look at the second statement. We will prove this by contraposition. Assume that we have two edges $uv$ and $st$ that cross in $\Gamma$. We will now show that $\alpha \leq \sqrt{2} - 1$.

We can assume that the line $uv$ is horizontal and that $s$ is above and $t$ is below $uv$. Let us call the point of intersection $x$. Figure 4.6 provides an illustration. Note that each point on the line $st$ is contained in either the outer circle $C^s_{\mathrm{out}}$ of $s$ or the outer circle $C^t_{\mathrm{out}}$ of $t$. We can assume that $x$ is contained in $C^s_{\mathrm{out}}$. Consider the orthogonal projection $p$ of $s$ onto the line $uv$. We can assume that $p$ is contained in the outer circle $C^v_{\mathrm{out}}$ of $v$. Note that $p \in C^s_{\mathrm{out}}$.




**(a)** $\alpha = \sqrt{2} - 1$ **(b)** $\alpha = 0.3 < \sqrt{2} - 1 \approx 0.41$

**Figure 4.5** The resulting $\alpha$-SIG drawing for the points $v_1 = (1, 0), v_2 = (0, 1), v_3 = (-1, 0), v_4 = (0, -1)$ with $\mathrm{rad}(v_i) = 1$ for all $i \in \{1, 2, 3, 4\}$ for different values of $\alpha$. Edges other than $v_2v_4$ and $v_1v_3$ are omitted. We observe that for $\alpha = \sqrt{2} - 1$ the set of annuli is well-formed.

Let $d = \mathrm{dist}(s, p) > 0$ be the euclidean distance from $s$ to $p$. Up to scaling, we can assume that the distance from $v$ to $p$ is 1.

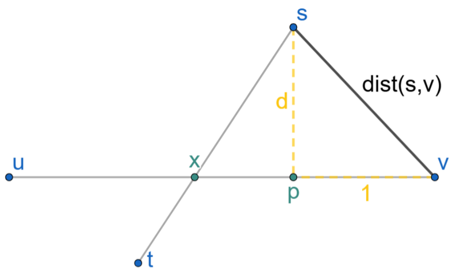



**Figure 4.6** If $\Gamma$ is an $\alpha$-SIG-drawing that contains two edges $uv$ and $st$ that cross, then we can assume that $uv$ and $st$ are positioned as depicted here.

We now have a right-angled triangle, created by the points $v, p, s$, whose hypotenuse has length $\mathrm{dist}(s, v)$ and the other sides have lengths $d$ and 1 respectively. See Figure 4.6 for an illustration. This gives us $\mathrm{dist}(s, v) = \sqrt{d^2 + 1}$.

Using $\mathrm{rad}(v) \geq 1$ (because $p \in C^v_{\mathrm{out}}$) and $\mathrm{rad}(s) > d$ (because $p \in C^s_{\mathrm{out}}$) together with Observation 4.2 gives us:

**1.** $\mathrm{dist}(s, v) \geq \mathrm{rad}(v) + \alpha \cdot \mathrm{rad}(s) \geq 1 + \alpha d$
**2.** $\mathrm{dist}(s, v) \geq \mathrm{rad}(s) + \alpha \cdot \mathrm{rad}(v) \geq d + \alpha$

Both of these inequalities must hold. This yields two upper bounds for $\alpha$. Rearranging $1+\alpha d \leq \sqrt{1 + d^2}$ yields $\alpha \leq \frac{\sqrt{1+d^2}-1}{d}$ and rearranging $d+\alpha \leq \sqrt{1 + d^2}$ yields $\alpha \leq \sqrt{1 + d^2} - d$.

Note that $\frac{\sqrt{1+d^2}-1}{d}$ is a monotone increasing function and $\sqrt{1 + d^2} - d$ is a monotone decreasing function. Additionally, for $d = 1$ we have $\frac{\sqrt{1+d^2}-1}{d} = \sqrt{1 + d^2} - d = \sqrt{2} - 1$.

If we additionally know that $\frac{\sqrt{1+d^2}-1}{d} \leq \sqrt{1 + d^2} - d$ if and only if $d \leq 1$, this yields the maximal upper bound of $\alpha \leq \sqrt{2} - 1$ which is achieved by setting $d = 1$.

Let us now tackle $\frac{\sqrt{1+d^2}-1}{d} \leq \sqrt{1 + d^2} - d$.

Rearranging yields $d^2 - 1 = (d + 1)(d - 1) \leq (d - 1)\sqrt{1 + d^2}$. We have already made the observation that the inequality holds for $d = 1$. For $d > 1$ we can simplify to $d+1 \leq \sqrt{1 + d^2}$.

Squaring both sides and simplifying yields $2d \leq 0$, a contradiction to $d > 0$.

Let us now consider $0 < d < 1$. Now we have $d - 1 < 0$ and therefore the same simplifying steps yield $2d \geq 0$. This holds for all $0 < d < 1$.

Let us now tackle the last statement. It remains to show that if $\alpha = \sqrt{2} - 1$, then $uv$ and $st$ cross at a right angle and $\mathrm{rad}(u) = \mathrm{rad}(v) = \mathrm{rad}(s) = \mathrm{rad}(t)$. Knowing that $\alpha = \sqrt{2} - 1$, we follows that $d = 1$, using the inequalities above. From this $\mathrm{dist}(s, v) = \sqrt{2}$ follows. We also have $\mathrm{rad}(s), \mathrm{rad}(v) \geq 1$, because $p \in C^s_{out}$ and $p \in C^v_{out}$. Therefore, $\sqrt{2} = \mathrm{dist}(s, v) \geq \mathrm{rad}(s) + \alpha\,\mathrm{rad}(v) \geq 1 + \left(\sqrt{2} - 1\right) \cdot 1 = \sqrt{2}$. We obtain $\mathrm{rad}(s) = \mathrm{rad}(v)$.

From this $x = p$ follows, since we assumed that $x \in C^s_{out}$. This tells us that $uv$ and $st$ form a right angle crossing.

Because the points $v, x, s$ form a right triangle, the triangle formed by $u, x, s$ must also have a right angle at $x$. Taking a closer look at the triangle $uxs$, we observe that we have the same setting as we had for the triangle formed by $v, p, s :$ the distance $\mathrm{dist}(s, x)$ can be assumed to be 1, up to scaling, and the distance $\mathrm{dist}(u, x)$ is unknown. Using the very same arguments, we can deduce that $\mathrm{rad}(u) = \mathrm{rad}(s)$.

Similarly, we can use the same arguments for the triangle formed by the points $t, x, v$. This yields $\mathrm{rad}(t) = \mathrm{rad}(v)$, proving the third statement.

◀

► **Corollary 4.4.** *If $G$ is an $\alpha$-SIG on $n \geq 3$ vertices for some $\alpha > \sqrt{2} - 1$, then $G$ has at most $3n - 6$ edges.*

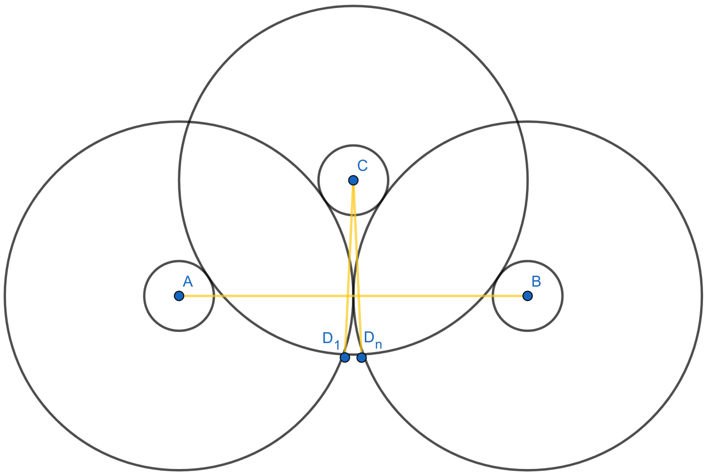
**Proof.** Consider the $\alpha$-SIG-drawing $\Gamma$ of $G$, with $\alpha > \sqrt{2} - 1$. Lemma 4.3 tells us that, crossings cannot appear in $\Gamma$. Therefore $\Gamma$ is a planar drawing. This means that $G$ is also planar and in particular we know that $G$ can have at most $3n - 6$ edges. ◀

► **Observation 4.5.** For every $n \in \mathbb{N}$ and $\alpha < \sqrt{2} - 1$ there exists an $\alpha$-SIG drawing $\Gamma$ where some edge is crossed at least $n$ times.

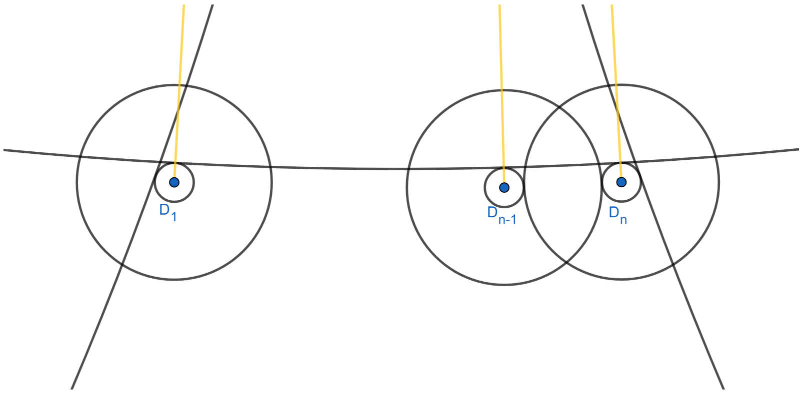
**Proof.** We will sketch out a configuration of points for each $n \in \mathbb{N}$ and for every $\alpha < \sqrt{2} - 1$ where an edge is crossed n times. Consider the points $A = (-1, 0)$, $B = (1, 0)$, and $C = \left(0, \sqrt{(1 + \alpha)^2 - 1}\right)$ with $\mathrm{rad}(A) = \mathrm{rad}(B) = \mathrm{rad}(C) = 1$ and the drawing $\Gamma$ depicted in Figure 4.7. Note that the distance $\mathrm{dist}(A, C) = \mathrm{dist}(B, C) = 1 + \alpha$ and thus the radii are valid.

Observe that a portion of the outer circle $C^C_{out}$ of $C$ below the line $AB$ is not contained in either of the outer circles of $A$ or $B$.

On this portion of $C^C_{out}$ we place the points $D_1, \ldots, D_n$ as sketched out in Figure 4.7. Note that the edge $AB$ is crossed by the edges $CD_1, \ldots, CD_n$. ◀




**(a)** The $\alpha$-SIG drawing containing the points $A, B, C, D_1, \ldots, D_n$. The edge $AB$ is crossed by the edges $CD_1, \ldots, CD_n$. Other edges are not depicted.




**(b)** A close-up of the above figure showing the points $D_1, \ldots, D_n$ and their annuli.

**Figure 4.7** An $\alpha$-SIG drawing where the edge $AB$ is crossed by the edges $CD_1, \ldots, CD_n$. Not all Note that we chose $\alpha = 0.2$ for clarity. This configuration works for all values $\alpha < \sqrt{2} - 1$.

## 4.3 Density in Relation to $\alpha$

In this section we want to investigate the connection between $\alpha$ and the density of $\alpha$-SIGs further. In the previous section, we have seen that for $\alpha > \sqrt{2} - 1$ the resulting graph $\alpha\text{-SIG}(P, \mathcal{A})$ is planar, independent of $P$ or $\mathcal{A}$. In order to reach a first result for smaller values of $\alpha$ we will modify the proofs of Bateman and Erdős [3] and Soss [14].

Consider an $\alpha$-SIG-drawing and its set of points $P \subseteq \mathbb{R}^2$. If $|P| < \infty$, there exists a point $p_0 \in P$ whose radius $\text{rad}(p_0)$ is the smallest. We will show that the number of neighbors of $p_0$ can be bounded, depending on $\alpha$.

▶ **Theorem 4.6.** *Let $G = \alpha\text{-SIG}(P, \mathcal{A})$ for a set of annuli $\mathcal{A}$ with center points $P \subseteq \mathbb{R}^2$, and $p_0 \in P$ be a point in $P$ whose radius $\text{rad}(\mathcal{A}, p_0)$ is the smallest among all points of $P$.*

*For every $\alpha$, there exists a constant $c(\alpha)$ such that $p_0$ has at most $c(\alpha)$ neighbors in $G$. The tight values for $c(\alpha)$ are given in Table 4.2.*

| $c(\alpha)$ | $\alpha$ | possible configuration |
|---|---|---|
| 6 | $\alpha \leq 1$ | 6 $\triangledown$ |
| 7 | $\alpha \leq 0.7355\ldots$ | 7 $\triangledown$ |
| 8 | $\alpha \leq 0.5307\ldots$ | 8 $\triangledown$ |
| 9 | $\alpha \leq 0.3680\ldots$ | 9 $\triangledown$ |
| 10 | $\alpha \leq \sqrt{5} - 2 = 0.2360\ldots$ | 10 $\triangledown$ or 10 $\triangle$ |
| 11 | $\alpha \leq 0.1863\ldots$ | 1 $\triangledown$ and 10 $\triangle$ |
| 12 | $\alpha \leq \frac{2}{\sqrt{3}} - 1 = 0.1547\ldots$ | 12 $\triangle$ or 12 $\triangle_{30}$ |
| 13 | $\alpha \leq 0.1074\ldots$ | 3 $\triangledown$, 2 $\triangle_{30}$, and 8 $\triangle$ |
| 14 | $\alpha \leq 0.0798\ldots$ | 2 $\triangledown$, 4 $\triangle_{30}$, and 8 $\triangle$ |
| 15 | $\alpha \leq 0.0677\ldots$ | 5 $\triangledown$ and 10 $\triangle$ |
| 16 | $\alpha \leq 0.0503\ldots$ | 4 $\triangledown$, 2 $\triangle_{30}$, and 10 $\triangle$ |
| 18 | $\alpha \leq 2\sqrt{2 - \sqrt{3}} - 1 = 0.0352\ldots$ | 6 $\triangledown$ and 12 $\triangle$ |

**Table 4.1** The number of neighbors $c(\alpha)$ the vertex with the smallest radius of a general $\alpha$-SIG may have depending on $\alpha$, and the corresponding configuration of neighbors.

Up to scaling, we can assume that $p_0$ has radius $\text{rad}(p_0) = 1$ and is positioned at the origin $p_0 = (0, 0) \in \mathbb{R}^2$. Observation 4.2 tells us that for two distinct points $p, q \in P$ we have $\text{dist}(p, q) \geq \text{rad}(p) + \alpha\, \text{rad}(q) \geq 1 + \alpha$ because $\text{rad}(p), \text{rad}(q) \geq \text{rad}(p_0) = 1$ . This means that all points in $P$ have mutual distance at least $1 + \alpha$.

We first show that it suffices to concern ourselves with the case that $\text{rad}(q) = 1$ for all points $q \in P$.

▶ **Lemma 4.7.** *Let $\alpha \in [0, 1]$ be a constant, and $\mathcal{A}$ a set of annuli with thickness $\alpha$ and centers $P \subseteq \mathbb{R}^2$ such that:*

**1.** *all points in $P$ have mutual distance at least $1 + \alpha$*

**2.** *$p_0 = (0,\ 0) \in P$ has the smallest radius $\text{rad}(\mathcal{A}, p_0) = 1$*

**3.** *all points $q \in P$ are adjacent to $p_0$ in $\alpha$-SIG$(P, \mathcal{A})$*

*Then there exists another set of points $S$ with $|P| = |S|$ and a set of annuli $\mathcal{A}_S$ that meet the above requirements, and additionally $\|q\|_2 \leq 2$ holds for all points $q \in S$.*

**Proof.** We use polar coordinates to describe points. Consider the following function $f$.

$$f((x;\ \phi)) = \begin{cases} (2;\ \phi) & \text{if } x > 2 \\ (x;\ \phi) & \text{otherwise} \end{cases} \qquad \text{for all } (x;\ \phi) \in P$$

Intuitively $f$ projects a point $(x;\ \phi)$ with radius $\text{rad}(\mathcal{A}, (x;\ \phi)) > 1$, close enough to $p_0$ such that $\text{rad}(\mathcal{A}_S, (x;\ \phi)) = 1$ would suffice for the two to be adjacent. We define $S = \{f(p) \mid p \in P\}$ and $\mathcal{A}_S$ by $\text{rad}(\mathcal{A}_S, q) = 1$ for all $q \in S$. Obviously $\|q\|_2 \leq 2$ holds for all $q \in S$, and because $f(p_0) = p_0$ we have $p_0 \in S$.

We need to show that all points in $S$ have mutual distance of at least $1 + \alpha$ and that all points $q \in S$ are adjacent to $p_0$ in $\alpha$-SIG$(S, \mathcal{A}_S)$.

First consider two points $p_1 = (x; \phi_x),\ p_2 = (y; \phi_y) \in P$. If $x, y \leq 2$ then $f(p_1) = p_1$ and $f(p_2) = p_2$ and thus $\text{dist}(f(p_1), f(p_2)) \geq 1 + \alpha$.

Let us now consider the case that $x > 2$ and $y \leq 2$. Again $f(p_2) = p_2$. We observe that $\text{rad}(\mathcal{A}, p_1) \geq x - 1$, because $p_1$ and $p_0$ are adjacent in $\alpha$-SIG$(P, \mathcal{A})$. Because $p_0$ has the smallest radius $\text{rad}(\mathcal{A}, p_0) = 1$ we have $\text{rad}(\mathcal{A}, p_2) \geq 1$ and we follow

$$\text{dist}(p_1, p_2) \geq \text{rad}(\mathcal{A}, p_1) + \alpha \cdot \text{rad}(\mathcal{A}, p_2) \geq x - 1 + \alpha$$

This means there is a ball $B$ with radius $x-1+\alpha$ and center point $p_1$ that contains no other points from $P$, in particular not $p_2$.

Now, let us take a look at $f(p_1)$. Note that $\text{dist}(f(p_1), p_1)$ is exactly $x-2$. Thus there is a ball $B'$ centered at $f(p_1)$ with radius $1+\alpha$ completely contained within $B$, because $(x-1+\alpha)-(x-2)=1+\alpha$. Therefore $p_2$ cannot be contained in $B'$ and the distance between $f(p_1)$ and $f(p_2)$ is at least $1+\alpha$.

The last case we need to consider is $x, y \geq 2$. Let $\psi = |\phi_x - \phi_y|$ be the angular difference between $p_1$ and $p_2$. A triangle is formed by the three points $p_0, p_1, p_2$. This triangle has side lengths $x, y$, and $\text{dist}(p_1, p_2)$. The angle at $p_0$ is $\psi$. Note that $\text{rad}(\mathcal{A}, p_1) \geq x-1$ and $\text{rad}(\mathcal{A}, p_2) \geq y-1$, because both $p_1$ and $p_2$ are adjacent to $p_0$ in $\alpha$-SIG$(P, \mathcal{A})$. We follow that $\text{dist}(p_1, p_2) \geq (y-1)+\alpha(x-1)$. Utilizing the law of cosines yields

$$\cos(\psi) = \frac{x^2+y^2-\text{dist}(p_1,p_2)^2}{2xy} \leq \frac{x^2+y^2-((y-1)+\alpha(x-1))^2}{2xy} \tag{4.1}$$

Another triangle is formed by the three points $f(p_0), f(p_1), f(p_2)$, where the two sides $f(p_0)f(p_1)$ and $f(p_0)f(p_2)$ have length 2 and the angle at $f(p_0)$ is $\psi$. We need to show that the side $f(p_1)f(p_2)$ has length $\text{dist}(f(p_1), f(p_2)) \geq 1+\alpha$. The law of cosines tells us that this is the case if and only if $\sqrt{8-8\cos(\psi)} \geq 1+\alpha$.

Rearranging yields $\cos(\psi) \leq 1-\frac{1}{8}(1+\alpha)^2$. Note that if we show this we have finished our proof. From this we can follow that for any two points $p_1, p_2 \in P$ the distance $\text{dist}(f(p_1), f(p_2))$ is at least $1+\alpha$. Observation 4.2 tells us first of all that this is sufficient for $\text{rad}(\mathcal{A}_S, q)=1$ to be valid for all $q \in S$ and second of all that all $q \in S$ are adjacent to $p_0$ in $\alpha$-SIG$(S, \mathcal{A}_S)$.

We will show $\cos(\psi) \leq 1-\frac{1}{8}(1+\alpha)^2$ by utilizing Equation (4.1) and proving the following:

$$\frac{x^2+y^2-((y-1)+\alpha(x-1))^2}{2xy} \leq 1-\frac{1}{8}(1+\alpha)^2$$

Rearranging yields the equivalent equation

$$(x-y)^2 - \frac{xy}{4}(1+\alpha)^2 \leq ((y-1)+\alpha(x-1))^2 \tag{4.2}$$

Additionally, because both $x$ and $y$ are greater than 2 we have

$$(x-y)^2 - \frac{xy}{4} \leq (x-2)^2 - 1 = x^2-4x+4-1 \leq x^2-2x+1 = (x-1)^2 \tag{4.3}$$

Without loss of generality we can assume $x \leq y$. This yields

$$(1+\alpha)^2(x-1)^2 = ((x-1)+\alpha(x-1))^2 \leq ((y-1)+\alpha(x-1))^2 \tag{4.4}$$

Using $0 \leq \alpha$ and Equations (4.3) and (4.4) results in

$$\begin{aligned}(x-y)^2 - \frac{xy}{4}(1+\alpha)^2 &\leq (1+\alpha)^2\left((x-y)^2 - \frac{xy}{4}\right) \\ &\underset{4.3}{\leq} (1+\alpha)^2(x-1)^2 \underset{4.4}{\leq} ((y-1)+\alpha(x-1))^2\end{aligned}$$

This proves Equation (4.2). ◀

The following lemma leads us to an upper bound on the number of neighbors that $p_0$, the point with the smallest radius, might have. If $\phi(\alpha) > 0$ is the minimum angle between two consecutive neighbors (ordered by angle around $p_0$) then $p_0$ can have at most $\left\lfloor \frac{360°}{\phi(\alpha)} \right\rfloor$ neighbors. Note that we can calculate $\phi(\alpha)$, given that we can focus on the case that $\text{rad}(q)=1$ for all $q \in P$.

▶ **Lemma 4.8.** *Let $p_0 = (0,\ 0)$, $p$, and $q$ be points in the plane $\mathbb{R}^2$ with $1+\alpha \leq \|p\|_2, \|q\|_2 \leq 2$ and $\operatorname{dist}(p,q) \geq 1+\alpha$ for a fixed value $\alpha \in [0,1]$.*

*Then the angle at $p_0$ in the triangle formed by the three points $p, p_0, q$ is of value at least*

$$\phi(\alpha) = \min\left\{\arccos\left(1 - \frac{1}{8}(1+\alpha)^2\right),\ \arccos\left(\frac{1}{1+\alpha}\right)\right\}$$

**Proof.** We can assume that the euclidean distance between $p$ and $q$ is exactly $1+\alpha$. Indeed, if the distance were greater we could minimize the angle by moving the points $p, q$ closer together. If both $p$ and $q$ have radial coordinate smaller than 2, then we could minimize the angle by moving both points further away from $p_0$, while keeping the distance $\operatorname{dist}(p,q)$ the same.

Let $x$ be the radial coordinate of $q$. Using the law of cosines, we follow that the angle $\phi$ at $p_0$ in the triangle given by $p, p_0, q$ has the value:

$$\arccos\left(\frac{2^2 + x^2 - (1+\alpha)^2}{2\cdot 2\cdot x}\right) = \arccos\left(\frac{4 + x^2 - (1+\alpha)^2}{4x}\right)$$

Differentiating with respect to $x$ for $x \in [1+\alpha,\ 2]$ we see that this function cannot have a minimum in the interior of this interval, and thus the minimum is achieved by $x = 1+\alpha$ or by $x = 2$. This yields:

$$\frac{4 + x^2 - (1+\alpha)^2}{4x} \underset{x=1+\alpha}{=} \frac{4 + (1+\alpha)^2 - (1+\alpha)^2}{4(1+\alpha)} = \frac{4}{4(1+\alpha)} = \frac{1}{(1+\alpha)}$$
$$\dots \underset{x=2}{=} \frac{4 + 2^2 - (1+\alpha)^2}{42} = \frac{8 - (1+\alpha)^2}{8} = 1 - \frac{1}{8}(1+\alpha)^2$$

This proves the lemma. ◀

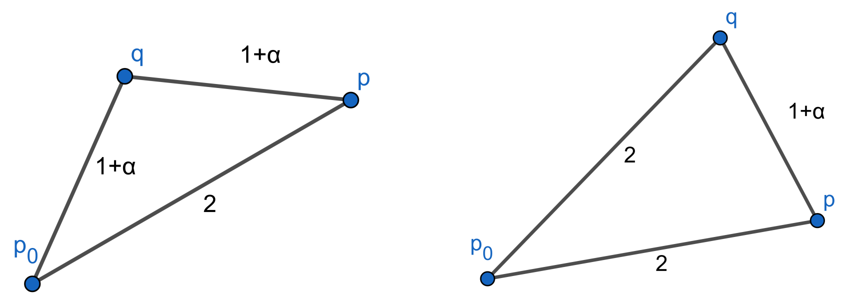
Taking a closer look at Lemma 4.8 and its proof, we see that we have two types of configurations that we are interested in, which stem from choosing either $x = 1+\alpha$ or $x = 2$. Recall that $x$ is precisely the distance from $p_0$ to $q$. This gives us the two triangles shown in Figure 4.8. These triangles are defined through their side lengths. In either case, the side $p_0p$ has length 2 and the side $pq$ has length $1+\alpha$. Choosing $x = 1+\alpha$ results in the *roof-shaped triangle*, and choosing $x = 2$ results in the *pizza-slice-shaped triangle*, as viewed from $p_0$. We will abuse notation and refer to the roof-shaped and pizza-slice-shaped triangle by the pictograms $\triangle$ and $\nabla$ respectively.

Note that for $\alpha = 1$ the two triangles coincide. This is because $1+\alpha = 2$ and thus we have an equilateral triangle. This tells us that the angle at $p_0$ is 60° and thus we can fit 6 triangles around $p_0$. This result tells us that for disk contact graphs the disk with the smallest radius will have at most 6 neighbors.

For $\alpha = 0$ the triangle $\triangle$ degenerates. Recall that for $\alpha = 0$ we find ourselves in the general SIG case and therefore we know that $p_0$ may have at most 18 neighbors (see Lemma 1.5).

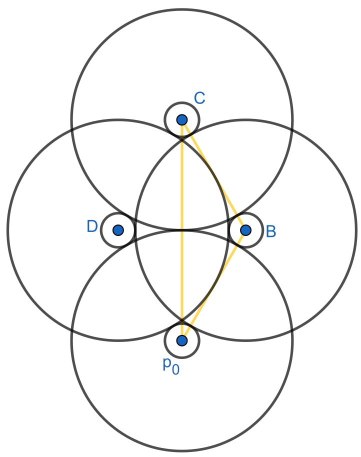
In order to prove how many neighbors $p_0$ may have, we just need to look at how many triangles of the types $\triangle$ and $\nabla$ we can place around $p_0$. Note that if the triangles cover a full 360° rotation around $p_0$, then the number of triangles is equal to the number of neighbors of $p_0$. For $\triangle$ we also need to account for parity: an uneven number of such triangles is impossible.

The following observations will help us.

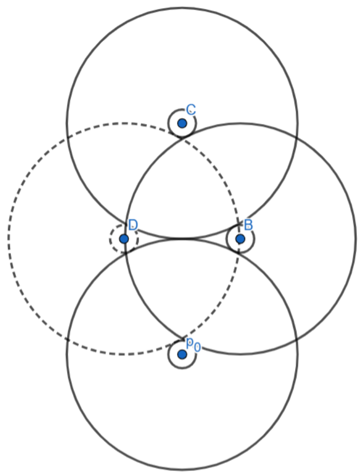



**(a)** Roof-shaped Triangle **(b)** Pizza-slice-shaped Triangle

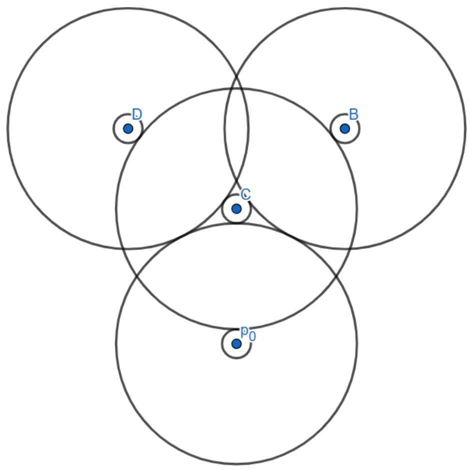
**Figure 4.8** The two configurations for the points $p$ and $q$ that minimize the angle at $p_0$. Note that we have set $\alpha = \sqrt{5} - 2$. For this value of $\alpha$ the angle at is the same in both triangles.




**(a)** Two placed with the long side adjacent. One is highlighted in yellow. We set $\alpha = \frac{2}{\sqrt{3}} - 1 \approx 0.1547$.




**(b)** Two placed with the long side adjacent with $\alpha = 0.12$. We observe that this is not a valid placement. Hence why the annulus of $D$ is drawn with dotted lines.




**(c)** Two placed with the short side adjacent with $\alpha = 0.12$. Note that this is a valid placement, opposed to Figure 4.9b.

**Figure 4.9** Different configurations of triangles ordered around a point $p_0$. Figures 4.9a and 4.9b illustrate that two can be placed with the long side adjacent if and only if $\alpha \geq \frac{2}{\sqrt{3}} - 1$. Figure 4.9c shows that placing two with the short sides adjacent is a valid placement for some values of $\alpha$ smaller than $\frac{2}{\sqrt{3}} - 1$.

▶ **Observation 4.9.** For $\alpha \geq \sqrt{5}-2$ $\triangledown$ triangles yield a smaller angle at $p_0$, while for $\alpha < \sqrt{5}-2$ triangles of the type $\triangle$ yield a smaller angle at $p_0$.

Formally:

$$\phi(\alpha) = \begin{cases} \arccos\left(1-\frac{1}{8}(1+\alpha)^2\right) & \text{for } \alpha \geq \sqrt{5}-2 \\ \arccos\left(\frac{1}{1+\alpha}\right) & \text{otherwise} \end{cases}$$

**Proof.** Recall that the angle at $p_0$ in $\triangledown$ is described by $\arccos\left(1-\frac{1}{8}(1+\alpha)^2\right)$ and the angle at $p_0$ in $\triangle$ is described by $\arccos\left(\frac{1}{1+\alpha}\right)$. We want to prove that:

$$\arccos\left(1-\frac{1}{8}(1+\alpha)^2\right) \leq \arccos\left(\frac{1}{1+\alpha}\right) \text{ if and only if } \alpha \geq \sqrt{5}-2$$

Simplifying and utilizing the fact that $\arccos(\cdot)$ is a monotone decreasing function, we arrive at:

$$\begin{aligned} 1-\frac{1}{8}(1+\alpha)^2 \geq \frac{1}{1+\alpha} &\iff (1+\alpha)-\frac{1}{8}(1+\alpha)^3 \geq 1 \\ &\iff 0 \geq (1+\alpha)^3 - 8\alpha = (\alpha-1)(\alpha^2+4\alpha-1) \end{aligned}$$

We have $\alpha - 1 \leq 0$ if and only if $\alpha \leq 1$ and $\alpha^2+4\alpha-1 > 0$ if and only if $\alpha \notin \left[-\sqrt{5}-2, \sqrt{5}-2\right]$. Because we limit $\alpha$ to $0 \leq \alpha \leq 1$ the above inequality is true if and only if $\alpha \geq \sqrt{5}-2$. ◀

▶ **Observation 4.10.** For $\alpha \geq \frac{2}{\sqrt{3}}-1$ two triangles of the type $\triangle$ placed such that the long sides are adjacent give a valid placement of points, meaning that the empty-centers-condition is not violated. All points have mutual distance at least $1+\alpha$.

For $\alpha < \frac{2}{\sqrt{3}}-1$ two triangles of the type $\triangle$ cannot be placed such that the long sides are adjacent.

For $\alpha \geq 2\sqrt{2-\sqrt{3}}-1 = 0.0352\ldots$ two triangles of the type $\triangle$ can be placed such that the short sides are adjacent. This is not possible for $\alpha < 2\sqrt{2-\sqrt{3}}-1$.

Figure 4.9 provides illustrations.

**Proof.** If we look at a $\triangle$ and orient it so that the long side is horizontal, what we need is that the height of this triangle is greater than $\frac{1}{2}(1+\alpha)$. That means, the angle at $p_0$ needs to be greater than 30°, because $\sin(30°) = \frac{1}{2}$. Using the fact that $\arccos(\cdot)$ is a monotone falling function we follow:

$$\arccos\left(\frac{1}{1+\alpha}\right) \geq 30° \quad \text{if and only if} \quad \frac{1}{1+\alpha} \leq \cos(30°) = \frac{\sqrt{3}}{2}$$

Rearranging yields $\alpha \geq \frac{2}{\sqrt{3}}-1$

Similarly, two $\triangle$ can be arranged such that the short sides coincide, if and only if two $\triangle$ angles are smaller than one $\triangledown$ angle. Figure 4.9c illustrates why this is. If the angle of a $\triangle$ is too small we run in to the problem that the points $B, D$ are too close together. Two points with radial coordinate 2 are closest together exactly when they describe a $\triangledown$. This leads to the equation:

$$2 \cdot \arccos\left(\frac{1}{1+\alpha}\right) \geq \arccos\left(1-\frac{1}{8}(1+\alpha)^2\right)$$

We calculate that this is the case if and only if $\alpha \geq 2\sqrt{2-\sqrt{3}}-1$. ◀

Note that the above proof gives us some insights. If we choose $\alpha = \frac{2}{\sqrt{3}} - 1$, we see that the angle at $p_0$ for a △ is exactly 30°, which means that we can place exactly 12 △ around $p_0$. Figure 4.10 illustrates this configuration. Observation 4.9 tells us that this is best possible, because $\frac{2}{\sqrt{3}} - 1 \approx 0.15 < \sqrt{5} - 2 \approx 0.23$.

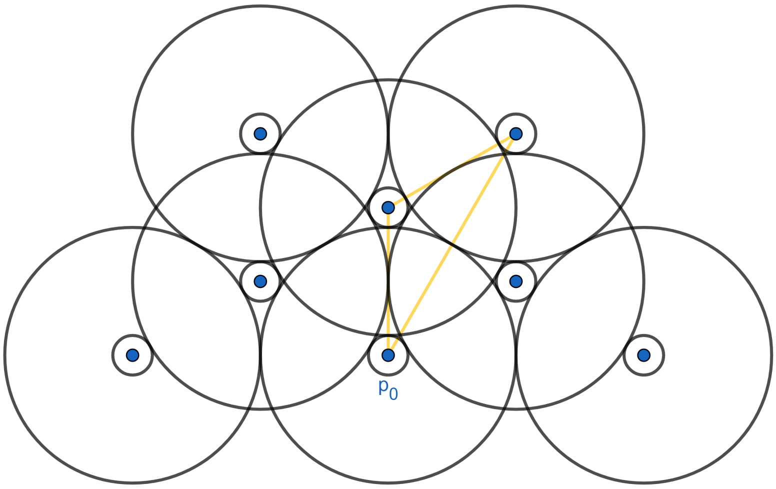



**Figure 4.10** Part of the configuration for $\alpha = \frac{2}{\sqrt{3}} - 1$ where $p_0$ has 12 neighbors. Here, $\alpha = \frac{2}{\sqrt{3}} - 1$. One of the △ is highlighted in yellow.

Note that the points in Figure 4.10 are a portion of the hexagonal lattice. This gives us a lower bound for the density question. It follows that for $\alpha = \frac{2}{\sqrt{3}} - 1$ the constant that answers the density question lies between 6 and 12.

Additionally, we observe that for $\alpha = 2\sqrt{2 - \sqrt{3}} - 1$ the angle in △ is 15° and the angle in ▽ is 30°. We observe that the we can place 2 △ then 1 ▽ then 2 △ and so on cyclically around $p_0$ and that this is a valid placement, which results in $p_0$ having 18 neighbors. Figure 4.11 illustrates a part of this configuration of points. Note the similarity to Figure 1.3.

Note that for $2\sqrt{2 - \sqrt{3}} - 1 \leq \alpha \leq \frac{2}{\sqrt{3}} - 1$ the angle 30° lies somewhere in between the angle given by △ and the angle given by ▽. Recall that here the angle of a △ is smaller than the angle of a ▽ if and only if $\alpha \leq \sqrt{5} - 2$. Both functions for the angles of △ and ▽ are strictly monotone falling, therefore 30° is always realizable in this interval. In particular we observe that the angle given by a ▽ is at least 30° for $\alpha \geq 2\sqrt{2 - \sqrt{3}} - 1$.

▶ **Observation 4.11.** If we exclude two △ placed with the short sides adjacent, the smallest angle occupied by two consecutive triangles around $p_0$ for $2\sqrt{2 - \sqrt{3}} - 1 \leq \alpha \leq \frac{2}{\sqrt{3}} - 1$ is $30° + \arccos\left(\frac{1}{1+\alpha}\right)$. This is achieved by a ▽ and a 30° triangle. Figure 4.12a provides an illustration.

**Proof.** For $\alpha = \frac{2}{\sqrt{3}} - 1$ two △ placed with the long sides adjacent (see Figure 4.9a) achieve the smallest angle. For $\alpha < \frac{2}{\sqrt{3}} - 1$ this is not a valid placement, but we can place a △ triangle and the remaining point as illustrated by Figure 4.12a.

We can assume that we have an equal sided triangle where each side has length $1 + \alpha$ formed by the three points that are not $p_0$. One of the points has radial coordinate 2. We call this point $C$, and the other two points $B$ and $D$. What is uncertain is how the triangle is oriented (rotated) with respect to $p_0$.

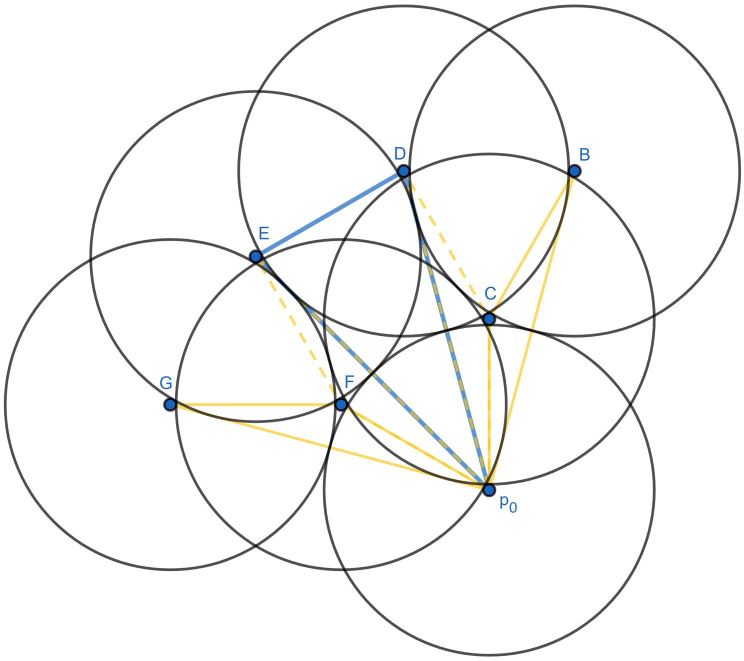



**Figure 4.11** A configuration of points around $p_0$ for $\alpha = 2\sqrt{2-\sqrt{3}}-1$. Note that the for this value of alpha the inner circle of an annulus coincides with the blue dot that depicts a point. Going clockwise around $p_0$ we have 2 then 1 and so on and so forth. Blue highlights a and yellow highlights a .

Consider Figure 4.12b. Let $\omega$ be the smaller of the two angles between the sides of the triangle and the line from $p_0$ to $C$. We can assume that $\omega$ is the angle at $C$ in the triangle $p_0, C, B$. Note that $\arccos\left(\frac{1}{1+\alpha}\right) \le \omega \le 30°$. We want to minimize the sum of the angles at $p_0$ in the triangles formed by the points $p_0, C, B$ and $p_0, C, D$. We will show that this minimum has the value $30° + \arccos\left(\frac{1}{1+\alpha}\right)$ and is achieved by $\omega = \arccos\left(\frac{1}{1+\alpha}\right)$.

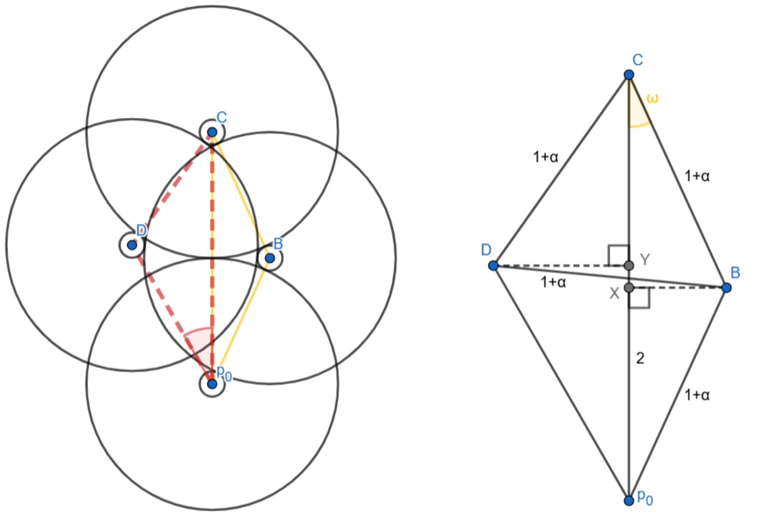
Using trigonometry we obtain that he angle at $p_0$ in the triangle given by $p_0, B, C$ is $\beta_1 = \arctan\left(\frac{(1+\alpha)\sin(\omega)}{2-(1+\alpha)\cos(\omega)}\right)$.

Similarly, the angle at $p_0$ in $p_0, C, D$ is $\beta_2 = \arctan\left(\frac{(1+\alpha)\sin(60°-\omega)}{2-(1+\alpha)\cos(60°-\omega)}\right)$. Our goal is to minimize $f(\omega) = \beta_1 + \beta_2$ for $\arccos\left(\frac{1}{1+\alpha}\right) \le \omega \le 30°$. We will omit the tedious calculus. We observe that $f$ is minimal for $\omega = \arccos\left(\frac{1}{1+\alpha}\right)$.

Note that $\beta_1 = 30°$ and $\beta_2 = \arccos\left(\frac{1}{1+\alpha}\right)$ for $\omega = \arccos\left(\frac{1}{1+\alpha}\right)$. In this case, the points $p_0, B, C$ form a . ◀

This new configuration defines a new type of triangle which we call the 30°-triangle. We will use the pictogram to refer to this triangle. It is important to note that this triangle is only valid for $\alpha \le \frac{2}{\sqrt{3}} - 1$ and for $\alpha = \frac{2}{\sqrt{3}} - 1$ it coincides with . This is another triangle where we need to take care of parity.

The above observation tells us that in fact for $\alpha < \frac{2}{\sqrt{3}} - 1$ a configuration of two then two and so on cyclically around $p_0$ is in fact minimal, i.e. the angle at $p_0$ is minimized




**(a)** A configuration of points depicting a triangle in yellow and a new type of triangle in red. Here, $\alpha = 0.1$. Note that the red angle at $p_0$ is exactly 30°.

**(b)** A configuration showing the distances between the points. The points $X, Y$ are the respective orthogonal projections of $B$ and $D$ onto the line $p_0C$.

**Figure 4.12** A possible configuration of points for $\alpha < \frac{2}{\sqrt{3}} - 1$.

by such a configuration. Figure 4.13 illustrates a portion of such a configuration.

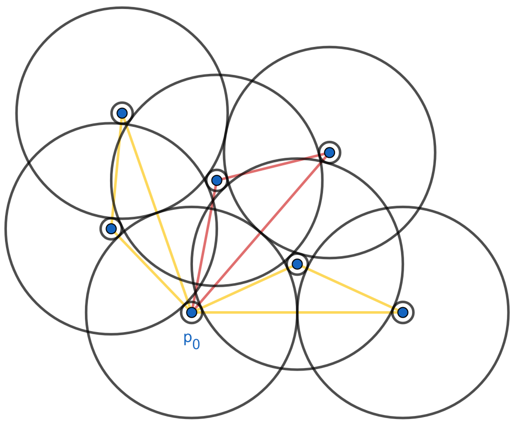



**Figure 4.13** A configuration of 3 and 2 around $p_0$. Here we have $\alpha = 0.1$. Every second triangle is highlighted: yellow for and red for .

Now we have everything we need to prove Theorem 4.6. For $\alpha \geq \frac{2}{\sqrt{3}} - 1$ we have and and do not run into problems. For $\alpha < \frac{2}{\sqrt{3}} - 1$ we have , , and and we need to be more careful.

▶ **Theorem 4.6.** *Let $G = \alpha\text{-SIG}(P, \mathcal{A})$ for a set of annuli $\mathcal{A}$ with center points $P \subseteq \mathbb{R}^2$, and $p_0 \in P$ be a point in $P$ whose radius $\operatorname{rad}(\mathcal{A}, p_0)$ is the smallest among all points of $P$.*

*For every $\alpha$, there exists a constant $c(\alpha)$ such that $p_0$ has at most $c(\alpha)$ neighbors in $G$. The tight values for $c(\alpha)$ are given in Table 4.2.*

| $c(\alpha)$ | $\alpha$ | possible configuration |
|---|---|---|
| 6 | $\alpha \le 1$ | 6 $\triangledown$ |
| 7 | $\alpha \le 0.7355\ldots$ | 7 $\triangledown$ |
| 8 | $\alpha \le 0.5307\ldots$ | 8 $\triangledown$ |
| 9 | $\alpha \le 0.3680\ldots$ | 9 $\triangledown$ |
| 10 | $\alpha \le \sqrt{5}-2 = 0.2360\ldots$ | 10 $\triangledown$ or 10 $\triangle$ |
| 11 | $\alpha \le 0.1863\ldots$ | 1 $\triangledown$ and 10 $\triangle$ |
| 12 | $\alpha \le \frac{2}{\sqrt{3}} - 1 = 0.1547\ldots$ | 12 $\triangle$ or 12 $\overset{30}{\triangle}$ |
| 13 | $\alpha \le 0.1074\ldots$ | 3 $\triangledown$, 2 $\overset{30}{\triangle}$, and 8 $\triangle$ |
| 14 | $\alpha \le 0.0798\ldots$ | 2 $\triangledown$, 4 $\overset{30}{\triangle}$, and 8 $\triangle$ |
| 15 | $\alpha \le 0.0677\ldots$ | 5 $\triangledown$ and 10 $\triangle$ |
| 16 | $\alpha \le 0.0503\ldots$ | 4 $\triangledown$, 2 $\overset{30}{\triangle}$, and 10 $\triangle$ |
| 18 | $\alpha \le 2\sqrt{2-\sqrt{3}} - 1 = 0.0352\ldots$ | 6 $\triangledown$ and 12 $\triangle$ |

**Table 4.2** The number of neighbors $c(\alpha)$ the vertex with the smallest radius of a general $\alpha$-SIG may have depending on $\alpha$, and the corresponding configuration of neighbors.

**Proof.** As in the rest of this section, we can assume that $p_0 = (0,0)$ and $\mathrm{rad}(p_0) = 1$. Recall that:

- the angle at $p_0$ in a $\triangle$ is given by $\arccos\left(\frac{1}{1+\alpha}\right)$, and
- the angle at $p_0$ in a $\triangledown$ is given by $\arccos\left(1 - \frac{1}{8}(1+\alpha)^2\right)$.

We have already established that $p_0$ may have six neighbors with $\alpha = 1$.

In order for $p_0$ to have seven neighbors we need 7 $\triangledown$ around $p_0$. Therefore $p_0$ may have seven neighbors if and only if:

$$7 \cdot \arccos\left(1 - \frac{1}{8}(1+\alpha)^2\right) \le 360°$$

Rearranging yields:

$$\alpha \le \sqrt{8 - 8\cos\left(\frac{360°}{7}\right)} - 1 = 0.7355\ldots$$

Note that this is consistent with Observation 4.9 because $0.73 > \sqrt{5} - 2$.

Similarly, eight neighbors are possible if and only if $8 \cdot \arccos\left(1 - \frac{1}{8}(1+\alpha)^2\right) \le 360°$. Rearranging for $\alpha$ yields $\alpha \le \sqrt{8 - 8\cos\left(\frac{360°}{8}\right)} - 1 = 2\sqrt{2-\sqrt{2}} - 1 = 0.5307\ldots$.

The same argument for nine neighbors yields $\alpha \le 0.3680\ldots$

Using the same method we follow that: $p_0$ may have ten neighbors if and only if $\alpha \le \sqrt{5}-2$. Note that for $\alpha = \sqrt{5} - 2$ $\triangledown = \triangle$ so both are possible. See Observation 4.9.

If we proceed in the same manner for the case of eleven neighbors we find that: eleven $\triangledown$ are possible if $\alpha \le \sqrt{8 - 8\cos\left(\frac{360°}{11}\right)} = 0.1269\ldots$. Observation 4.9 tells us that 11 neighbors are possible with a greater value of $\alpha$ if we use a configuration that uses $\triangle$ instead of $\triangledown$ whenever possible. Note that eleven $\triangle$ are not possible because of parity but 1 $\triangledown$ and 10 $\triangle$ are. We obtain: $p_0$ may have eleven neighbors if and only if $1 \cdot \arccos\left(1 - \frac{1}{8}(1+\alpha)^2\right) + 10 \cdot \arccos\left(\frac{1}{1+\alpha}\right) \le 360°$. This is not easy to rearrange for $\alpha$, but we calculate $\alpha \le 0.1863\ldots$, which is a greater value than we previously had.

The case of twelve neighbors is easier, as 12 $\triangle$ present a valid arrangement. We obtain: $p_0$ may have 12 neighbors if and only if $10 \cdot \arccos\left(\frac{1}{1+\alpha}\right) \le 360°$. Solving for $\alpha$ and using $\cos(30°) = \frac{\sqrt{3}}{2}$, we obtain $\alpha \le \frac{2}{\sqrt{3}} - 1 = 0.1547\ldots$.

Observation 4.10 tells us that for smaller values of $\alpha$ we cannot only use $\triangle$. We also have to consider arrangements that make use of $\triangledown$ and $\triangle_{30}$.

We know that the angle given by a $\triangle$ is the smaller than 30° for $\alpha \leq \frac{2}{\sqrt{3}} - 1$ and that 30° is smaller that the angle given by $\triangledown$ for $\alpha \geq 2\sqrt{2-\sqrt{3}} - 1$. This tells us that we want the amount of $\triangle$ and $\triangle_{30}$ to be as big as possible while keeping our arrangements valid. We can place two $\triangle$ then either a $\triangledown$ or two $\triangle_{30}$ and then we can go back to $\triangle$. For parity reasons we need an even amount of $\triangle$ and an even amount of $\triangle_{30}$.

Let us now take a look at the possible configurations for 13 neighbors. Figure 4.14 depicts the configuration that allows for the greatest value of $\alpha$.

Note that a configuration with 13 neighbors is not realizable as $1\triangledown + 12\triangle$. Similarly 3 $\triangledown$ and 10 $\triangle$ are not possible since we would need to place three $\triangle$ consecutively, which is not a valid placement (for $\alpha \leq \frac{2}{\sqrt{3}} - 1$). The configuration $5\triangledown + 8\triangle$ is possible if we arrange them cyclically in the order 2 $\triangle$, 1 $\triangledown$, 2 $\triangle$, 1 $\triangledown$, 2 $\triangle$, 1 $\triangledown$, 2 $\triangle$ and 2 $\triangledown$. This yields $\alpha \leq 0.1032\ldots$. We can do better than this with $3\triangledown + 2\triangle_{30} + 8\triangle$, because for $\alpha > 2\sqrt{2-\sqrt{3}} - 1$ a $\triangle_{30}$ yields a smaller angle than $\triangledown$. This arrangement yields $\alpha \leq 0.1074\ldots$, which is more than the best. Another possible configuration is $1\triangledown + 6\triangle_{30} + 6\triangle$, but this yields a smaller value of $\alpha \leq 0.1005\ldots$.

Therefore the best result (as in the largest value for $\alpha$) for 13 neighbors is achieved by $3\triangledown+2\triangle_{30}+8\triangle$, which yields $\alpha \leq 0.1074\ldots$. Can we do better? No, because $1\triangledown+4\triangle_{30}+8\triangle$ is not valid, since $1 + \frac{4}{2} = 3$, which is fewer than 4, and therefore we would need to place three $\triangle$ consecutively.

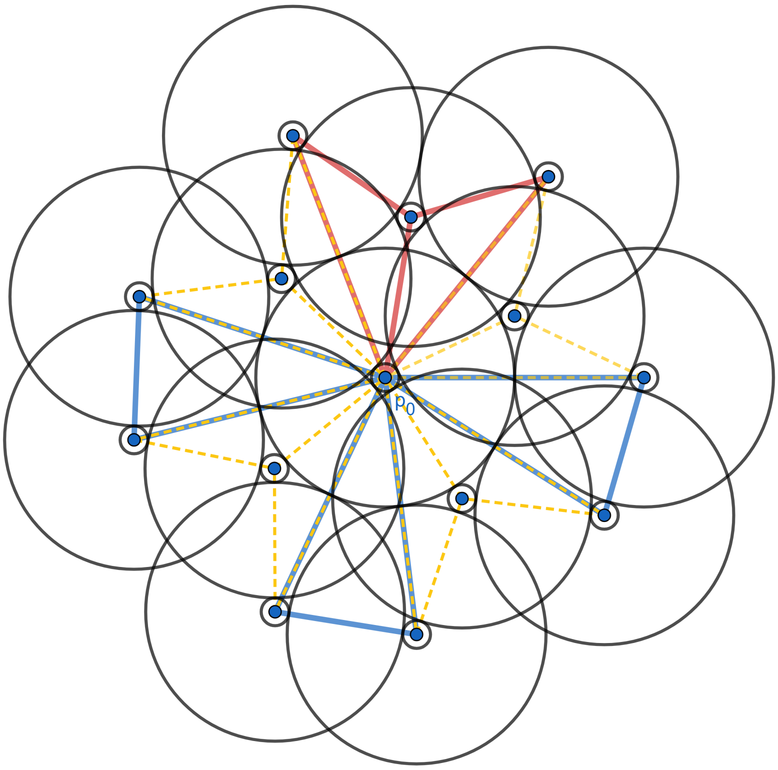



**Figure 4.14** An arrangement of points described as 3 $\triangle$, 2 $\triangle_{30}$, and 8 $\triangle$, in which the point $p_0$ has 13 neighbors. Here $\alpha = 0.1074\ldots$. Note that $\triangledown$ are highlighted with blue, $\triangle_{30}$ are highlighted with red, and $\triangle$ are highlighted with yellow dashed lines.

We will discuss the remaining configurations of $14, \ldots, 18$ neighbors more briefly. For the case that $p_0$ has 14 neighbors we obtain:

- $2\triangledown + 4\triangle_{30} + 8\triangle$ yields $\alpha \leq 0.0798\ldots$
- $0\triangledown + 8\triangle_{30} + 6\triangle$ yields $\alpha \leq 0.0641\ldots$, which is a smaller value for $\alpha$.

Note that 10 $\triangle$ are not a valid arrangement.

For the case of 15 neighbors we have:

- $5\triangledown + 0\triangle_{30} + 10\triangle$ yields $\alpha \leq 0.0677\ldots$
- note that $1\triangledown + 6\triangle_{30} + 8\triangle$ yields $\alpha \leq 0.0555\ldots$, which is a smaller value.

The case of 16 neighbors yields:

- $4\triangledown + 2\triangle_{30} + 10\triangle$ yields $\alpha \leq 0.0503\ldots$
- $0\triangledown + 8\triangle_{30} + 8\triangle$ yields $\alpha \leq \sqrt{6} - \sqrt{2} - 1 = 0.0352\ldots$ (a smaller value)

We will skip to the case of 18 neighbors. We do this because it turns out that there is not separate value of $\alpha$ for exactly 17 neighbors. For 18 neighbors we obtain:

- $6\triangledown + 0\triangle_{30} + 12\triangle$ yields $\alpha \leq 2\sqrt{2-\sqrt{3}} - 1 = 0.0352\ldots$.
- $8\triangledown + 0\triangle_{30} + 10\triangle$ yields a smaller value for $\alpha$. (For $\alpha < 2\sqrt{2-\sqrt{3}} - 1$ we find that $\triangle_{30}$ yields a greater angle than $\triangledown$, therefore $\triangle_{30}$ should not even be used. The configuration of 8 $\triangledown$ and 10 $\triangle$ might not even be valid, but this does not matter, since it requires a smaller value for $\alpha$ anyways.)

Let us call this specific value $\alpha_{18} = 2\sqrt{2-\sqrt{3}} - 1$.

For the case of 17 neighbors we first observe that a configuration with 12 $\triangle$ cannot be valid. For 10, 8, and 6 $\triangle$ we obtain:

- $3\triangledown + 4\triangle_{30} + 10\triangle$ yields $\alpha \leq \alpha_{18}$
- $1\triangledown + 8\triangle_{30} + 8\triangle$ yields $\alpha \leq \alpha_{18}$
- $1\triangledown + 10\triangle_{30} + 6\triangle$ yields $\alpha \leq \alpha_{18}$

It turns out that there is no separate value for 17 neighbors. For $p_0$ to be able to have 17 neighbors we need to set $\alpha$ so small that 18 neighbors are also possible. ◀

Note that Table 4.2 depicts the number of neighbors that $p_0$ might have in the general setting. Some of the configurations described are not valid in the maximal or general setting. Consider, for example, the case that $p_0$ may have 7 neighbors. The given configuration here is 7 $\triangledown$. Note that all neighbors of $p_0$ have radial coordinate 2 and radius 1. This means that the none of the inner circles of $p_0$'s neighbors bounds the outer circle of $p_0$.

For the well-formed and the maximal setting it is necessary that $p_0$ has at least one neighbors with radial coordinate $1 + \alpha$. This means that a given configuration needs to contain at least 2 $\triangle$ in order to be valid. Therefore the best we can achieve is $5\triangledown + 2\triangle$ which results in $\alpha \leq 0.7026\ldots$.

## 4.4 Characterization

We have seen that there are slight differences between well-formed, maximal, and general $\alpha$-SIGs. In short:

- The general setting allows for any set of annuli $\mathcal{A}$, as long as it does not violate the empty-centers-condition.
- A set of annuli $\mathcal{A}$ is maximal if none of the annuli can be enlarged without violating the empty-centers-condition.

- In the well-formed setting a set of annuli $\mathcal{A}_{\text{grow}}(P)$ is given by a set of points $P$ in the following manner: from each point $p \in P$ we grow an annulus $a_p$ simultaneously and at the same rate until no annulus can be enlarged without violating the empty-centers-condition. (When an annulus cannot grow anymore it stops at its current size, but the other annuli continue to grow until they are stopped too.)

We have also seen that for $\alpha > \sqrt{2} - 1$ all possible $\alpha$-SIG are planar graphs, and for $\alpha = \sqrt{2} - 1$ all possible $\alpha$-SIG are 1-planar graphs.

Furthermore, while examining the relations between $\alpha$ and the density of an $\alpha$-SIG, we have observed that there are slight differences between the cases, regarding what configurations are possible.

To further illustrate the differences between the different settings we analyze where stars $K_{1,n}$ fit in the different cases. In other words: Is the star $K_{1,n}$ a general, maximal or well-formed $\alpha$-SIG, and if so for what values of $\alpha$? Our results are summed up in Table 4.3.

| | $\alpha = 0$ | $\alpha \leq \frac{2}{\sqrt{3}} - 1$ | $\alpha \leq \sqrt{2} - 1$ | $\alpha \leq 0.7013\ldots$ | $\alpha > 0.7013\ldots$ |
|---|---|---|---|---|---|
| general | all stars are possible | | | | |
| maximal | $K_{1,3}$ | all stars are possible | | | |
| well-formed | $K_{1,3}$ | $K_{1,3}$ | $K_{1,4}$ | $K_{1,5}$ | $K_{1,6}$ |

**Table 4.3** Largest star $K_{1,n}$ that is **not** possible in the given $\alpha$-SIG setting.

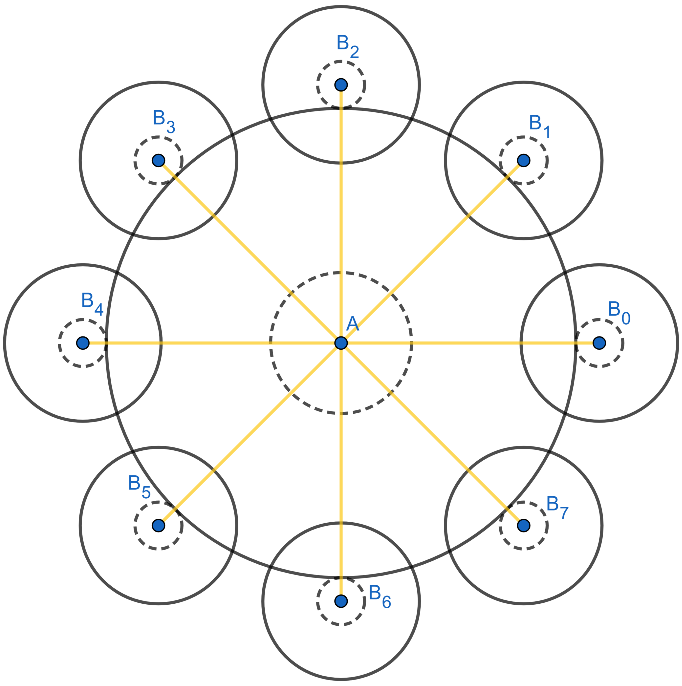



**Figure 4.15** A general $\alpha$-SIG-drawing depicting the star $K_{1,8}$. Vertices are blue, edges are yellow, and annuli are depicted in black. The concrete value of $\alpha$ is 0.3. Note that inner circles are depicted with dotted lines.

We first make the observation that we can depict $K_{1,n}$ as a general $\alpha$-SIG independently of $\alpha$. Consider the following configuration of points, that is also illustrated in Figure 4.15 for

$n = 8$. Once again, we use polar coordinates. We place the point $A$ at the origin and give it the radius $\mathrm{rad}(A) = 1$ and place $n$ points $B_0, \ldots, B_{n-1}$ with radial coordinate $1 + \varepsilon$ at the same distance from each other for an $\varepsilon$ and radii $\mathrm{rad}(B_i)$ small enough so that their outer circles do not intersect. In Figure 4.15 we have chosen $\varepsilon = 0.1$ and $\mathrm{rad}(B_i) = \frac{1}{3}$.

Note that for $\alpha > 0$. We can position the points $B_i$ so that the resulting set of annuli is maximal, as Figure 4.15 shows. Recall that for $\alpha = 0$ the maximal and well-formed setting coincide (see Chapter 3 for a more detailed description).

Recall that a result by Harary, Jacobson, Lipman, and McMorris [5] was that $K_{1,3}$ is not an OSIG or a CSIG. See Section 1.2.3 for a more detailed description. Recall that this is equivalent to saying: $K_{1,3}$ is not a well-formed $\alpha$-SIG for $\alpha = 0$.

Since we want to depict $K_{1,n}$ we stick to the following naming scheme for the remainder of this section: $A$ is the point that has connections to the remaining points $B_i$ for $i \in \{0, \ldots, n-1\}$.

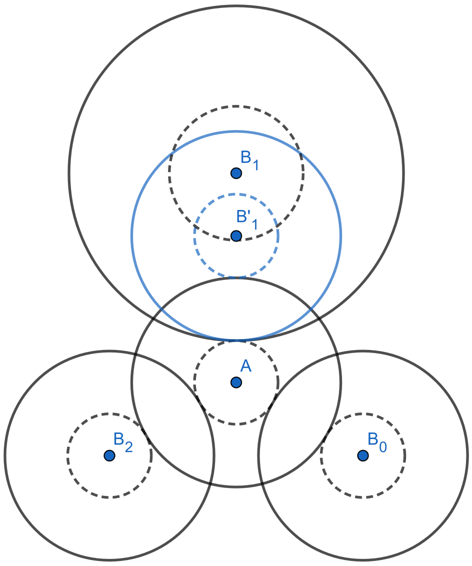


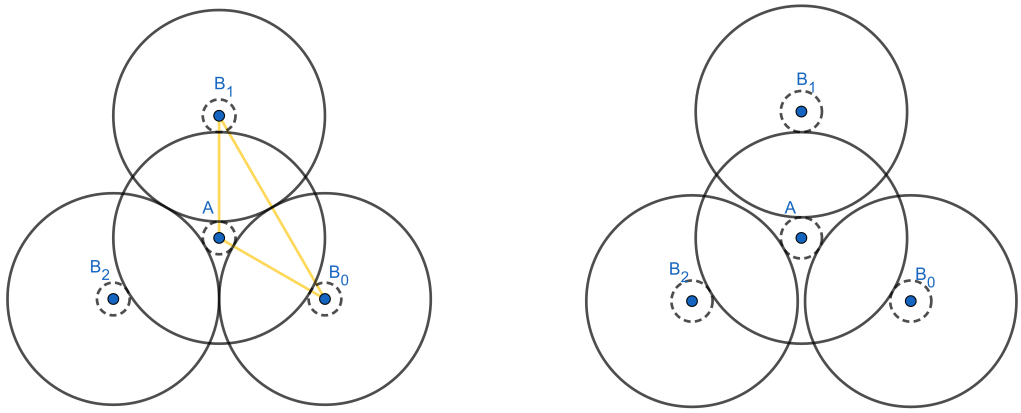

**Figure 4.16** The points $A, B_0, B_1, B_2$ show that $K_{1,3}$ is a well formed 0.4-SIG. Note that $\mathrm{rad}(B_1) = 1.6$ and all other points have radius 1. We can replace $B_1$ by the point $B_1'$ with $\mathrm{rad}(B_1') = 1$ and the resulting graph is unchanged. The annulus of $B_1'$ is colored blue. Note that edges are omitted and inner circles are depicted with dotted lines.

Important to know for the well-formed setting is that for two points $p, q$ whose distance $\mathrm{dist}(p, q)$ is the smallest among all pairs of points, their radii must have the same size of $\mathrm{rad}(p) = \mathrm{rad}(q) = \frac{1}{1+\alpha}\,\mathrm{dist}(p, q)$. For example, $\alpha = 0$ yields $\mathrm{dist}(p, q)$ and $\alpha = 1$ yields $\frac{1}{2}\,\mathrm{dist}(p, q)$. Of course, in such a case $p$ and $q$ are adjacent, because $\mathrm{rad}(p) + \mathrm{rad}(q) \leq \mathrm{dist}(p, q)$ (see Observation 4.2). This tells us that there must exist at least one $i$ such that $A, B_i$ are the closest points among all pairs of points. From this we follow that $\mathrm{rad}(B_i) \geq \mathrm{rad}(A)$ for all $i$. Up to scaling we can assume that $\mathrm{rad}(A) = 1$.

Next observation: If we can depict $K_{1,n}$ for some fixed $n$ such that one of the $B_i$'s has $\mathrm{rad}(B_i) > 1$, then we can depict $K_{1,n}$ such that $\mathrm{rad}(B_i) = 1$ for all $i \in \{0, \ldots, n-1\}$. Figure 4.16 illustrates this. If some point $B$ has $\mathrm{rad}(B) > 1$ then we know that its outer circle must meet the inner circle of $A$. Because the resulting graph is a star, the outer circle of $B$

does not intersect any other outer circles (besides the outer circle $C_{\text{out}}^{A}$ of $A$). In particular, this means that if we remove $B$ and replace it with the point $B'$ that has the same angular coordinate as $B$, but with radial coordinate $1+\alpha$ and $\text{rad}(B') = 1$, the resulting graph is still $K_{1,n}$.

In particular Figure 4.16 shows that $K_{1,3}$ is a well formed 0.4-SIG. Let us investigate $K_{1,3}$ further.




**(a)** $\alpha = \frac{2}{\sqrt{3}} - 1$

**(b)** $\alpha = \frac{2}{\sqrt{3}} - 1 + \varepsilon$ for some $\varepsilon > 0$

**Figure 4.17** The star $K_{1,3}$ is a well-formed $\alpha$-SIG if and only if $\alpha > \frac{2}{\sqrt{3}} - 1$. Note in Figure 4.17a that the points $A, B_0, B_1$ form a $\triangle$, which is highlighted in yellow. Edges are omitted.

If we want to depict $K_{1,3}$, it makes intuitive sense to arrange the points $B_i$ such that the angle difference between two consecutive points is 120° (see Figure 4.17). It is not possible to depict $K_{1,3}$ if the greater angle given by a $\triangle$ is 120° or more, because then the angular difference between two consecutive $B_i$'s must be at least 120° if their outer spheres should not meet. In total we would need more than 360°. Note the $\triangle$ highlighted in Figure 4.17a. The greater angle in $\triangle$ is greater than 120° if and only if the small angle is smaller than 30°. This means that the outer circles of the points $B_i$ must meet, for $\alpha \leq \frac{2}{\sqrt{3}} - 1$. Thus we can depict $K_{1,3}$ if and only if $\alpha > \frac{2}{\sqrt{3}} - 1$.

Similarly, we deduce that $K_{1,4}$ is a well-formed $\alpha$-SIG if and only if the great angle given by $\triangle$ is smaller than 90°. This is equivalent to $\arccos\left(\frac{1}{1+\alpha}\right) > 45°$. Rearranging yields $\alpha > \sqrt{2} - 1$.

The same arguments yields: $K_{1,5}$ is a well-formed $\alpha$-SIG if and only if the great angle given by $\triangle$ is smaller than 72°. This is equivalent to $\arccos\left(\frac{1}{1+\alpha}\right) > 54°$. Rearranging yields $\alpha > \frac{1}{\cos(54°)} - 1 = 0.7013\ldots$.

We know from a previous result that for $\alpha = 1$ the point with smallest radius, which must be $A$, may have at most 6 neighbors. The configuration that achieves this is shown in Figure 4.18. The neighbors of $A$ have edges between them. The depicted graph is not $K_{1,6}$. This means that $K_{1,6}$ is not a well-formed $\alpha$-SIG for any value of $\alpha \in [0, 1]$.

In fact, if we use the same method used for the other stars we obtain: $K_{1,6}$ is a well-formed $\alpha$-SIG if and only if the great angle in a $\triangle$ is smaller than 60°. This is equivalent to $\arccos\left(\frac{1}{1+\alpha}\right) > 60°$ which yields $\alpha > 1$, a contradiction.

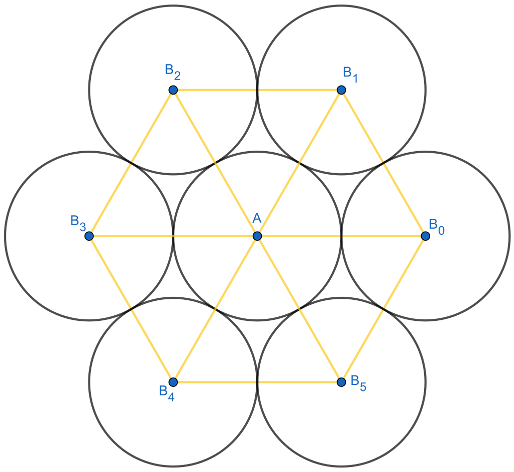



**Figure 4.18** The configuration of points for $\alpha = 1$ for which the point $A$ has six neighbors.

# 5 Conclusion

The hope for this thesis was to improve upon the currently best known answer of $c = 14.5$ to the Density Question.

▶ **Question 5.1** (Density Question)**.** *What is the smallest constant $c > 0$ such that every* SIG *on $n$ vertices has at most $c \cdot n$ many edges?*

For this purpose we have explored multiple properties of (well-formed) SIGs and we have introduced generalizations. One such generalization is posed by general CSIGs and OSIGs. We have observed that the general setting allows us to depict more graphs, but we have left the following problems unanswered.

▶ **Open Problem 5.2.** Is every well-formed CSIG also a well-formed OSIG?

We have seen that there are well-formed OSIGs that are not well-formed CSIGs, such as the path on three nodes $P_3$. The above question asks whether the class of CSIGs is a subclass of the class of OSIGs, or the two definitions actually describe completely different graph classes.

A similar question remains unopened for general SIGs.

▶ **Open Problem 5.3.** Do general CSIGs and general OSIGs describe different graph classes?

▶ **Open Problem 5.4.** Which graphs are general CSIGs (OSIGs), and which are not?

Because in the general setting both CSIGs and OSIGs are hereditary, it seems like the above question is easier to answer than the equivalent question for the well-formed setting.

Furthermore, we have introduced $\alpha$-SIGs, which interpolate between disk contact graphs ($\alpha = 1$) and SIGs ($\alpha = 0$). We have seen that $\alpha$-SIGs are planar graphs for $\alpha > \sqrt{2} - 1$ and 1-planar graphs for $\alpha = \sqrt{2} - 1$. We have also seen that in an $\alpha$-SIG the vertex whose sphere of influence has the smallest radius will have at most a constant number of neighbors, depending on $\alpha$. Additionally, we have introduced the distinction between well-formed, maximal, and general $\alpha$-SIGs, which, in fact, describe different graph classes.

We have left the following problem untouched, even though it merits some attention, in our opinion.

▶ **Open Problem 5.5.** Does there exist a value $\alpha' > 0$ such that every general CSIG is also a general $\alpha'$-SIG?

We finish with the problem that has started this research.

▶ **Open Problem 5.6.** Does every SIG on $n$ nodes have at most $9n$ edges?

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