Bachelor Thesis

Bounding the maximum Face Length of Girth-Planar Maximal Graphs

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Abstract

The girth of a graph is the size of the smallest cycle contained in the graph.

A planar graph G is called girth-planar maximal with respect to some integer g, if $\text{girth}(G) \geq g$ and the addition of any non edge $e \in \binom{V(G)}{2} \setminus E(G)$ results in a non-planar graph or reduces the girth of G to a value less than g.

Some early results on the face length of girth-planar maximal graphs were given by Axenovich, Ueckerdt and Weiner in 2016 [1] for $g = 6$. Fernándes, Sieger and Tait computet bounds for the probability of girth-planar maximal graphs to appear in random graphs [7]. However in their case, maximality was seen in an extremal setting.

This thesis presents results on the face lengths of girth-planar maximal graphs. In particular, we prove that the maximum face length is bounded by a function of g and give upper and lower bounds. Furthermore a tight characterization is shown for $g \leq 6$ and some special classes of planar graphs.

Contents

1 Introduction

1.1 Definitions and Notation

For more detailed definitions also see Diestel [6].

A *graph* is a pair $G = (V, E)$ of sets containing vertices and edges, where $E \subseteq \binom{V}{2}$. Here $\binom{X}{k}$ denotes the set of all k-element subsets of a given set X. If not clear from the context, the vertex and edge set shall be referred to as $V(G)$ and $E(G)$. For an edge $e = \{v, w\} \in E(G)$ also write $e = vw = wv$. Note that this thesis only covers simple undirected graphs. The order of a graph is the number of vertices, denoted by $|G| := |V|$. The number of edges is denoted by $\|G\| := |E|$.

For another graph G' we write $G' \subseteq G$ if $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G) \cap \binom{V(G')}{2}$. Two graphs G and H are called isomorphic if there exists a bijective map $\Phi : V(G) \to V(H)$ such that for all $v, w \in V(G)$ the following equivalence holds: $\Phi(v)\Phi(w) \in E(H) \iff vw \in E(G)$. The function Φ is called a graph isomorphism and we write $H \simeq G$. If an isomorphism exists between two graphs and no distinction is necessary, both graphs will be treated as if they were equal without applying the isomorphism. Especially if $H' \subseteq H$ and $G' \subseteq G$ s.t. $G' \simeq H'$ the notation $H' \subseteq G$ shall also be valid.

A *walk* of length n, $W = v_1e_1v_2e_2 \cdots v_ne_nv_{n+1}$ is a sequence of vertices and edges of G such that $e_i = v_iv_{i+1}$ for $i = 1, \ldots, n$. A walk might use vertices and edges more than once. A walk is called a *closed walk* if $v_1 = v_{n+1}$. If not stated otherwise, a graph is always connected, meaning that for each pair of vertices, there exists a walk that has both vertices as endpoints.

This thesis heavily uses paths, a special class of graphs. A path P is a graph such that for $n := |P|$, $V(P) = \{v_1, \ldots, v_n\}$ and $E(P) = \{v_iv_{i+1} \mid 1 \leq i < n\}$, v_1, v_n are called the endpoints. Paths can also be defined as walks with no repeating edges or vertices. The *length* of a path is the number of edges.

A cycle C is a graph on a vertex set $V(C) = \{v_0, \ldots, v_{n-1}\}$ and edge set $E(C) = \{v_iv_{i+1} \mid 0 \leq i < n \mod n\}$. Note that for any path $P = uPv$ s.t. $\|P\| > 1$ one can create a cycle by adding a single edge $e = uv$. Similar to paths, the length of a cycle is the number of edges, $\|C\|$. For a graph G the *girth*, denoted by $\text{girth}(G)$, is the length of the smallest cycle contained in G. Graphs that contain no cycles, i.e. the trees and forests, have $\text{girth}(G) = \infty$.

A common visualization for graphs is drawing a graph in the two-dimensional plane where vertices are represented as points and edges as curves. A *plane graph* M is a drawing of a graph in the plane, such that no two edges cross. Similar to graph isomorphisms, a plane graph can be mapped to a general graph. General graphs that have a plane drawing are called *planar graphs*, the plane graph is sometimes referred to as *embedding*. It is not necessary to characterize planar graphs by finding a plane drawing, as proven by Kuratowski [13] and Wagner [18].

For a plane drawing, the union of all edges splits the two-domensional plane into connected regions, called faces. When identified with a planar graph, the frontier of

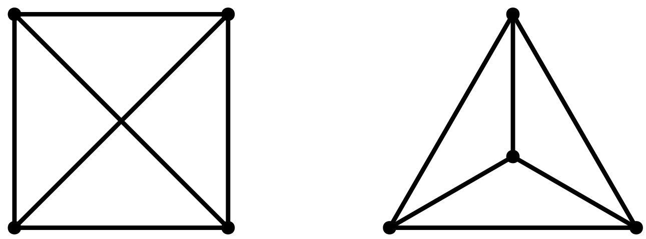

Figure 1: A drawing of the graph K_4 (left). A plane drawing of K_4 (right).

each face can be described by a closed walk along the edges that comprise the frontier. The minimum length of such a walk is called the *face length*. For 2-connected planar graphs, each face is bounded by a cylce. Thus the walk corresponds to a cycle C and the face length is $\|C\|$. We define $\mathrm{f}_{\max}(G)$ as the maximum face length over all embeddings of a planar graph.

A planar graph G is called maximal planar if no non-edge $e \in \binom{V(G)}{2} \setminus E(G)$ can be added such that $G + e$ is still a planar graph. This definition allows for any planar graph to have a maximal planar supergraph on the same vertex set. In other words, for all planar graphs H there exists a maximal planar graph H' with $V(H) = V(H')$ such that $H \subseteq H'$. The process of finding such supergraphs is called triangulating a planar graph and is used extensively in algorithms dealing with planar graphs. Some algorithms are the computation of a separator for the Planar Separator Theorem [15] or a mixed max-cut algorithm by Shih, Wu and Kuo [16]. Such triangulations may be computed in linear time and space. For an exemplary algorithm see [3].

There are two well known equivalent characterizations of being maximal planar. Firstly, a graph is maximal planar if and only if each face is bounded by a triangle. Hence the name triangulation. This fact leads to the second equivalent characterization, which is that a graph is maximal planar if and only if there are exactly $3n - 6$ edges, where $n = |G|$. The last part is a consequence of Euler's Formula.

For this thesis we will be considering girth-planar maximal (gpm) graphs, which is a natural extension of being maximal planar.

Definition 1.1:
A planar graph G is called *girth-planar maximal* with respect to some $g \in \mathbb{N}, g \geq 3$ (gpm-g) if for any non-edge e, $G + e$ is either not planar or has girth strictly less than g.

Examples and facts about girth-planar maximal will be given in Section 3 and Subsection 5.3

1.2 Main Results

The main results of this thesis are listed here and will be proven in Section 4. A very general result that holds for all classes of planar graphs and all girths $g \in \mathbb{N}, g \geq 3$ is a sufficient condition for the property of being girth-planar maximal.

Proposition 4.3:
Let G be a 2-connected planar graph with $\mathrm{girth}(G) \geq g$. If $\mathrm{f}_{\max}(G) < 2(g-1)$ then G is girth-planar maximal.

Using this result yields an exact characterization for girth-planar maximal graphs of girth $g \in \{3, 4, 5, 6\}$. Note that the case of $g = 3$ is exactly the case of maximal planar graphs.

Lemma 4.6:
Let G be a planar graph with $\mathrm{girth}(G) \geq g$ for $g \in \{4, 5\}$. Then G is gpm-g if and only if $\mathrm{f}_{\max}(G) < 2(g-1)$.

Lemma 4.7:
Let G be a 2-connected gpm-g graph for $g = 6$. Then $\mathrm{f}_{\max}(G) < 2(g-1)$.

In the case of $g \geq 7$ the sufficient condition proposed in Proposition 4.3 is no longer necessary to obtain girth-planar maximal graphs. As shown in the following theorem, a larger maximal face length can be achieved.

Theorem 4.9:
There exists a gpm-g G graph having $\mathrm{f}_{\max}(G) = 3g - 12$.

The most important result of this thesis concerns an upper bound for the face length of girth-planar maximal graphs as a function of g.

Theorem 4.13:
Let G be a 2-connected gpm-g graph. Then $\mathrm{f}_{\max}(G) < R(2g; g-3)$ where $R(2g; g-3)$ is the multicolor Ramsey Number for K_{2g} and $g-3$ colors.

Additionally, several structural observations about girth-planar maximal graphs have been made. Some interesting results are a minimum length for ears on cycles bounding the faces of some embedding and forbidden substructures. The results can be found in Subsections 4.1 and 4.3 respectively.

2 Preliminaries

In this section more definitions and notation are presented that are used throughout the thesis. Some less common notation that was introduced in the introduction is reiterated here for an easier lookup.

Definition 2.1 (Some Notation)**:**
This definition introduces some general mathematical notation.

(i) For $n \in \mathbb{N}$ define $[n] \coloneqq \{1, \dots, n\}$

(ii) For some set X and $0 \leq k \leq |X|$ define $\binom{X}{k} \coloneqq \{S \subseteq X \mid |S| = k\}$

(iii) For $x \in \mathbb{R}$ define $\lfloor x \rfloor \coloneqq \max(-\infty, x] \cap \mathbb{Z}$ and $\lceil x \rceil \coloneqq \min[x, \infty) \cap \mathbb{Z}$ as the floor and ceiling function.

Definition 2.2 (Graph Notation)**:**
Let G, G_1, G_2 be graphs.

(i) For some subset $V' \subseteq V$ or $E' \subseteq E$ define the *induced subgraph* $G[V'] \coloneqq (V', E(G) \cap \binom{V'}{2})$ and $G[E'] \coloneqq (\bigcup_{e \in E'} e, E')$.

(ii) The union $G = G_1 \cup G_2$ is defined as the graph $G = (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$. The intersection is defined similarly.

Remark: If G_2 is only one edge e or a single vertex, also write $G = G_1 \cup e = G_1 + e$.

Definition 2.3 (Graph Parameters)**:**
Let G be a graph and $u, v \in V(G)$. Define the following parameters:

(i) $N_G(u) \coloneqq \{x \in V(G) \mid ux \in E(G)\}$ is the neighborhood of u

(ii) $\deg_G(u) \coloneqq |N_G(u)|$ is the degree of u

(iii) $|G| \coloneqq |V(G)|$ and $\|G\| \coloneqq |E(G)|$

(iv) $\mathrm{girth}(G) \coloneqq \min\{|C| \mid C \subseteq G, C \text{ is a cycle}\}$ is the girth of G

(v) $\mathrm{dist}_G(u, v) \coloneqq \min\{\|P\| \mid P = uPv \subseteq G \text{ is a path}\}$ is the distance of u and v.

(vi) $\mathrm{diam}(G) \coloneqq \max\{\mathrm{dist}_G(x, y) \mid x, y \in V(G)\}$ is the diameter of G

(vii) If G is planar with some plane embedding M and $f \in F(M)$, then $\ell(f) \coloneqq \min\{\|W\| \mid W \text{ is a closed walk spanning } \partial f\}$

(viii) $\mathrm{f}_{\min}(G) \coloneqq \min\{\ell(f) \mid f \in F(M) \text{ for all plane embeddings } M \text{ of G}\}$ and $\mathrm{f}_{\max}(G) \coloneqq \max\{\ell(f) \mid f \in F(M) \text{ for all plane embeddings } M \text{ of G}\}$

If it is clear from the context, the subscript G will be dropped.

Definition 2.4 (More on plane graphs)**:**
Let $M = (V, E)$ be a plane graph. The set of all faces is denoted as $F(M)$.
For $f \in F(M)$ define ∂f as the subgraph of M containing all edges and vertices that border the face f.

Definition 2.5 ((k-)ears)**:**
Let G be a graph and $C \subseteq G$ a cycle. A path $P = uPv \subseteq G$ with endpoints u, v is called an ear of C if $V(P) \cap V(C) = \{u, v\}$.
If $\|P\| = k$ then P is called a k-ear.
The *width* of the ear is $\text{dist}_C(u, v)$.

Definition 2.6 (Maximal Planar)**:**
A planar graph G is called maximal planar, if and only if $G' = G + e$ is not planar for every $e \in \binom{V(G)}{2} \backslash E(G)$.

Definition 2.7 (Girth-Planar Maximal)**:**
Let $g \in \mathbb{N}$, $g \geq 3$. A planar graph G having $\text{girth}(G) \geq g$ is called girth-planar maximal with respect to g (gpm-g) if and only if

$$\forall e \in \binom{V(G)}{2} \backslash E(G) \ : \ \text{girth}(G + e) < g \text{ or } G + e \text{ is not planar}$$

Remark: A gpm graph G with $\text{girth}(G) = 3$ is maximal planar. Thus the notion girth-planar maximal is a generalization of maximal planarity.

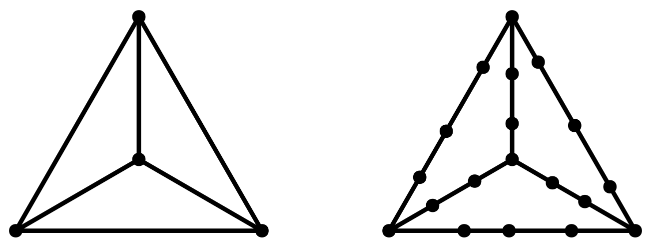

Figure 2: The graph K_4 (left). A subdivision of K_4 (right).

Definition 2.8 (Subdivision)**:**
A graph G is obtained from a graph H by *subdividing* an edge $e = xy$ if G is obtained by deleting the edge e from H and adding a path P of length at least 2 with endpoints x, y but inner vertices not in $V(G)$.

A graph G is called a *subdivision* of some graph H, if G can be obtained from H by a sequence of edge-subdivisions.

2.1 Path Arithmetic

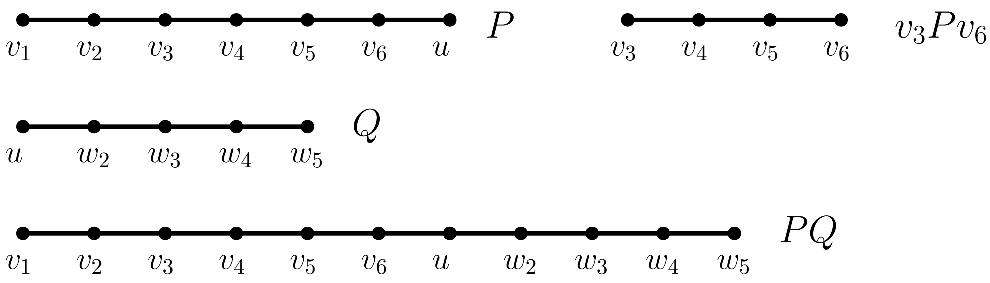

Figure 3: Visual examples for path arithmetic. Trimming of paths (top) and concatenation with $u = v_7 = w_1$ (bottom).

As paths and cycles are used extensively throughout this thesis, some notation will be introduced to describe common operations on paths like trimming and concatenation.

Let $P = (\{v_1, \ldots, v_n\}, E)$ be a path and $v_i, v_j \in V(P)$ be two vertices with $i < j$. Then v_iPv_j is the subpath on the vertices $v_i, \ldots, v_j$. In particular $P = v_1Pv_n$. If two paths P and Q share some endpoints, for example $P = Pv, Q = vQ$, then the paths may be concatenated, denoted by $PQ = PvvQ$ where $V(PQ) = V(P) \cup V(Q)$ and $E(PQ) = E(P) \cup E(Q)$. By using this notation, an edge $e = vw$ may also be interpreted as a path of length one. Visual examples are given in Figure 3. For easier concatenation, paths may also be written in reverse by swapping the endpoints used for path trimming. For example v_jPv_i as a graph is the subpath v_iPv_j but is seen as $v_jv_{j-1}v_{j-2} \cdots v_{i+1}v_i$.

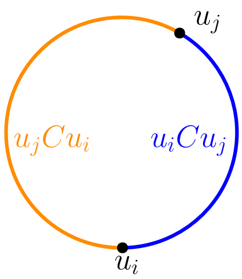

Figure 4: Visual examples for cycle arithmetic. In particular the difference between u_iCu_j and u_jCu_i.

Similar operations can be defined for cycles. Let $C = u_0 \cdots u_m u_0$ be a cycle. All indices are seen modulo $m+1$. Then for $i < j$ the graph u_iCu_j denotes the path $u_i \cdots u_j$. Contrary to paths, swapping the endpoints does not denote the same subgraph. Rather u_jCu_i is the part of the cycle complementary to u_iCu_j. In other words $(u_iCu_j)(u_jCu_i) = C$.

Using this notation, paths can also be extended to cycles by simply attaching the left endpoint to the right side. If $P = uPv$ is a path then $Pu = uPvu$ is the cycle on the same vertex and edge set as P but with the additional edge $uv \notin E(P)$.

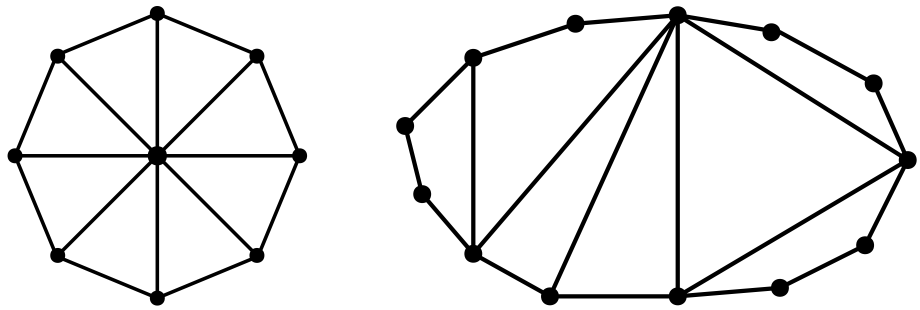

Figure 5: A wheel on 7 vertices (left). An outerplanar graph (right)

2.2 Common Graph Classes

The following section will introduce the reader to some common graph types and classes that are used in this thesis.

Definition 2.9 (Complete Graph)**:**
The complete graph is given by $K_n = ([n], \binom{[n]}{2})$

Definition 2.10 (Wheel)**:**
A graph W is called a wheel if W consists of a cycle C and one additional universal vertex, called the center vertex.
$W = (\{0\} \cup [n], \{\{i, i+1\} \mid 0 \leq i < n \mod n\} \cup \{\{i, n\} \mid 0 \leq i < n\})$
Subdivisions of this graph are also called wheels and the subdivisions of the edges containing the center vertex are the spokes.

Definition 2.11 (Star and Spider)**:**
A graph S is called a star if there exists a unique vertex $u \in V(S)$ such that every edge $e \in E(S)$ contains u. A spider is a subdivision of a star. A subdivision of one edge containing u is called a leg.

Definition 2.12 (Outerplanar)**:**
A planar graph G is called outerplanar, if there exists some embedding M and $f \in F(M)$ sucht that $V(\partial f) = V(G)$.

2.3 Known Theorems and Tools

In this thesis, some well known theorems and tools will be used. The most important ones will be mentioned here, mostly without proof. If not stated otherwise, proofs for the respective theorems can be found in Diestel [6].

Theorem 2.13 (Euler's Formula)**:**
Let G be a planar graph and M some plane embedding. Define the numbers $n \coloneqq |G|, m \coloneqq \|G\|, f \coloneqq |F(M)|$. Then the following equality holds:

$$n - m + f = 2 \tag{2.1}$$

Remark: As n and m do not depend on the embedding, the number of faces of a planar graph is consistent across all drawings.

Theorem 2.14 (Ramsey's Theorem)**:**
For graphs P_i, $i \in [k]$ define

$$R(P_1, \ldots, P_k) \coloneqq \min\{n \in N \mid \text{Any coloring of the edges of } K_n \text{ into } k \text{ colors contains a monochromatic copy of at least one } P_i\}$$

Then $R(P_1, \ldots, P_k) < \infty$

Remark: For $P_1 = K_{m_1}, \ldots, P_k = K_{m_k}$ also write $R(P_1, \ldots, P_k) = R(m_1, \ldots, m_k)$ and if $m_1 = \ldots = m_k = m$ write $R(m_1, \ldots, m_k) = R(m; k)$

Lemma 2.15:
Let u, v be some vertices and P, Q distinct paths with $P = uPv, Q = uQv$. Then $G \coloneqq P \cup Q$ contains a cycle.

Proof. Assume that G does not contain a cycle. Then G must be a tree. For any pair of vertices in a tree, there exists exactly one path connecting those vertices. Now consider the pair u, v. By construction of G there are two distinct u, v-paths in G, P and Q. This contradicts the uniqueness of a u, v-path. Thus G is not a tree and must contain a cycle. ■

3 Observations about Girth-Planar Maximal Graphs

Whilst maximal-planar graphs are well studied, the notion of maximal planar graphs of girth larger than 3 has not been explored in great detail. To familiarize the reader with the concept of girth-planar maximal graphs, some facts and observations, as well as some examples will be stated.

3.1 Extension of Maximal Planar

As mentioned in the introduction, the notion of girth-planar maximal graphs is a natural extension of maximal planar graphs. Indeed, each graph of girth at least g has a girth-planar maximal supergraph. Recall that there are three easy characterizations of maximal planar graphs. Edge maximality with respect to edge addition, the number of edges and the structure of the faces.

The first characterization has been extended to being the definition of girth-planar maximal graphs. The second statement about the edge count can also be formulated as an application of Euler's Formula.

Lemma 3.1:
Let G be a connected planar graph, $g \in \mathbb{N}$ and $g \leq \operatorname{girth}(G) < \infty$. Then

$$\|G\| \leq \frac{g}{g-2}(|G| - 2)$$

Proof. Define $n := |G|, m := \|G\|, l := m - n + 2$. Now fix some plane embedding M of G. By Euler's Formula we have $l = |F(M)|$.

$$D := \{(e, f) \in E(G) \times F(M) \mid e \in E(\partial f)\}$$

As $\operatorname{girth}(G) < \infty$ we know that G is not a tree. Furthermore G is connected and thus the frontier ∂f contains a cycle for each $f \in F(M)$. Hence it must contain at least g edges. This is because in a connected plane graph that is no tree, each face must be adjacent to some other face. Furthermore, an edge can only be contained in the frontier of at most two faces.

Using the facts mentioned above, one can derive some inequalities using the set D:

$$|D| = \sum_{f \in F(M)} |\{e \in E(G) \mid e \in E(\partial f)\}| \geq \sum_{f \in F(M)} g = l \cdot g \tag{3.1}$$

$$|D| = \sum_{e \in E(G)} |\{f \in F(M) \mid e \in E(\partial f)\}| \leq \sum_{e \in E(G)} 2 = 2m \tag{3.2}$$

Finally, one can rearrange Euler's Formula and conclude the proof:

$$n - m + l = 2 \overset{\cdot g}{\iff} gm = gn + gl - 2g \overset{3.2}{\leq} gn + 2m - 2g$$
$$\iff (g-2)m \leq g(n-2) \iff m \leq \frac{g}{g-2}(n-2)$$

■

Since every girth-planar maximal graph still has girth at least g, the bound of Lemma 3.1 also applies. A simple calculation verifies that indeed, for the smallest possible girth $g = 3$, the bound is exactly the edge count for maximal planar graphs. However for girth at least five the fraction is not always a natural number and for girth at least four examples of girth-planar maximal graphs can be constructed, that do not even meet the bound in the floor function, see Example 3.2.

It is therefore apparent, that this extremal view on girth-planar maximal graphs cannot be used as a characterization and only describes a subclass of gpm graphs. Indeed, if a graph has equality on the bound for the edge count, adding another edge immediately implies that the resulting graph cannot have girth at least g. Thus by definition the original graph was girth-planar maximal.

For a closer look at this extremal setting, see Subsection 5.3.

What remains is a characterization of girth-planar maximal graphs based on the structure of each face. Similar to the fact that triangles bound the faces of maximal planar graphs, one might find a good condition on the length of the faces of gpm graphs. Results concerning this approach to girth-planar maximality can be found in Section 4.

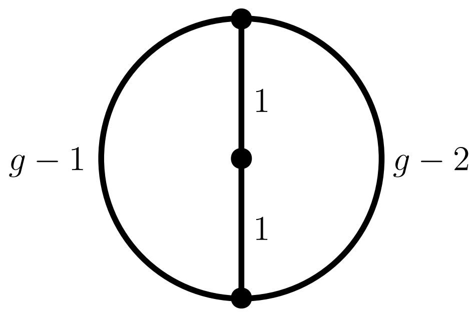

Figure 6: Girth-planar maximal graph G of girth g and maximum face length $\mathrm{f}_{\max}(G) = 2g - 3$. G also does not satisfy $\|G\| = \left\lfloor \frac{g}{g-2}(|G| - 2) \right\rfloor$.

Example 3.2: A minimal counterexample for a gpm graph that does not meet the bound of Lemma 3.1 is the graph G, that is the union of a cycle of length $2g - 3$ and a 2-path (a path on 2 edges) as seen in Figure 6. Using Proposition 4.3 it is immediate that G is indeed a gpm graph. Counting the number of edges in G gives

$$\|G\| = 2g - 3 + 2 = 2g - 1$$

but since

$$|G| = 2g - 3 + 1 = 2g - 4 + 2 = 2(g - 2) + 2$$

the upper bound is

$$\frac{g}{g-2}(|G| - 2) = \frac{g}{g-2}(2(g - 2) + 2 - 2) = 2g$$

3.2 Connectedness

In graph theory, many problems become easier, when there are some restrictions to graph parameters other than the girth. One being the connectedness. Recall that a graph is k-connected, if there exists a k-subset of its vertex set, such that removing those vertices leaves at least two disconnected components, but the graph is still connected after removing any $k-1$ element subset.

Connectednesss also plays a role when dealing with planar graphs. If a planar graph has conectivity at least 2, then the frontier of each face is a cycle. As another consequence of Euler's Formula and the bound on the edge count for planar graphs, one can derive that every planar graph must contain a vertex v such that $\deg(v) \leq 5$. This implies that all planar graphs of order at least 7 are at most 5-connected. Furthermore it has been shown, that 3-connected planar graphs are among the class of graphs that have a unique embedding [8].

The connectivity of girth-planar maximal graphs is even more restricted than for common planar graphs as is shown in the following lemma.

Lemma 3.3:
Let $g \in N, g \geq 6$ and G a gpm-g graph. Then the connectivity of G is at most 2.

Proof. Assume that the connectivity of G is at least 2, as being 1-connected concludes the proof. We will prove a stronger statement and show that there exists a vertex of degree at most 2. If such a vertex exists, one may delete its two neighbors and recieve disconnected components. Due to Lemma 3.1 we have that

$$\|G\| \leq \frac{g}{g-2}(n-2)$$

where $n := |G|$ Using the Handshake Lemma we get that for any graph $2\|G\| = \sum_{v \in V(G)} \deg(v)$. Thus the average degree of a vertex can be written as $\frac{2\|G\|}{|G|}$ Using the bound provided by Lemma 3.1 we get

$$\frac{2\|G\|}{n} \leq \frac{2g}{g-2}\frac{n-2}{n} \overset{g \geq 6}{\leq} \frac{12}{4}\frac{n-2}{n} = 3\frac{n-2}{n} < 3$$

Thus the average degree is less than 3 implying the existence of a vertex of degree at most 2. ■

The thesis mostly restricts itself to 2-connected girth-planar maximal graphs, especially in section 4. This is mostly due to the fact that 2-connected planar graphs only have faces bounded by cycles, but also because there are 1-connected girth-planar maximal graphs that can achieve arbitrarily large maximum face lengths, as can be seen in Example 3.4.

Note that the graph presented there is also 1-edge-connected. It is still an open question whether 2-edge-connected but 1-vertex-connected girth-planar maximal graphs have a bounded maximum face length.

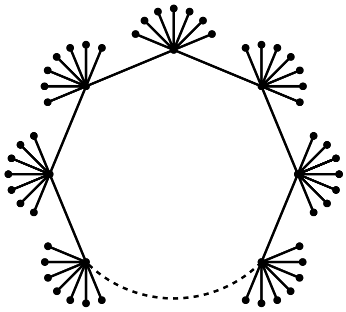

Figure 7: A 1-connected girth-planar maximal graph with arbitrary large f_{max}.

Example 3.4: Let G be a graph constructed from a cycle C of size g. For each $v \in V(C)$ add k vertices $u_i^{(v)}$ and edges $vu_i^{(v)}$, $i = 1, \ldots, k$. The graph also seen in Figure 7 is girth planar maximal, as its diameter is $\lfloor g/2 \rfloor + 2$. However a shortest closed walk around the outer face has length $g + 2gk = g(1 + 2k)$. This is because a closed walk containing all vertices of the outer face must also contain all edges of the entire graph.

3.3 More Examples of Girth-Planar Maximal Graphs

Example 3.5: Let C be a cycle of length $g \leq \|C\| \leq 2g-3$. Then C is gpm-g as can again be seen from Proposition 4.3.

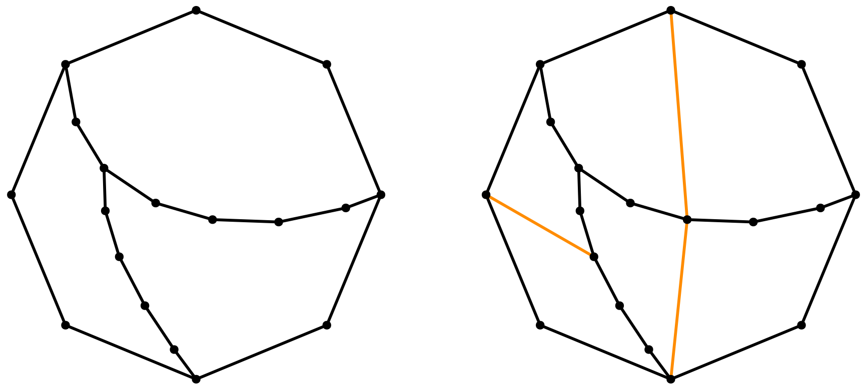

Figure 8: A graph of girth 8 (left). A gpm supergraph on the same vertexset for girth 6 (right). New edges are colored orange.

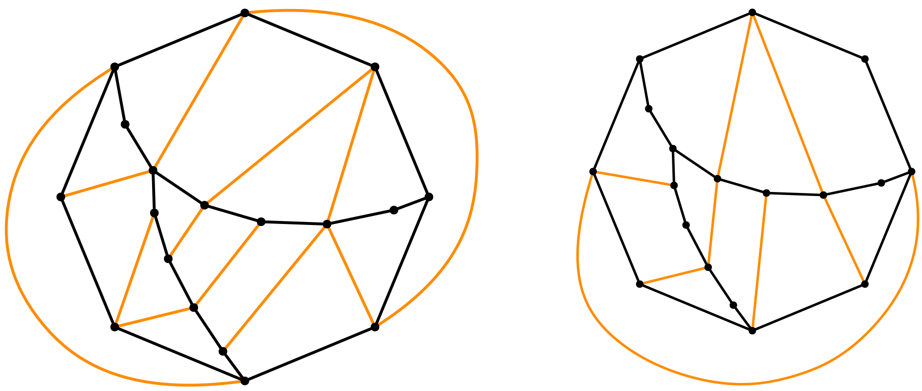

Figure 9: A gpm supergraph of girth 4 where each face has length 4 (left). A gpm supergraph of girth 4 with less edges (right). New edges are colored orange.

Example 3.6: The definition of girth-planar maximality allows for every planar graph of girth g to have a gpm supergraph with respect to g. Figure 8 shows a drawing of a planar graph with girth 8. Note that for this graph, every drawing is equivalent w.r.t. the relative positioning of the faces. The graph is already gpm for $g = 8$. For $g = 7$ only one edge has to be inserted into the largest face in order to create a gpm supergraph. A drawing for a gpm supergraph for $g = 6$ is depicted on the right in Figure 8.

Note that similar to creating plane triangulations, it is possible to create gpm supergraphs with certain additional properties. Figure 9 depicts two variants for a gpm supergraph for $g = 4$. The left drawing aims to create as many faces of length 4 as possible. Indeed, every face in the drawing has length 4. The second drawing adds as few edges as possible.

4 Bounds on the Maximum Face Length

4.1 Structure of Faces in Girth-Planar Maximal Graphs

As maximally planar graphs can be characterized by examining the face length in an arbitrary embedding, one can ask whether such a characterization also exists for girth-planar maximal graphs. Proposition 4.3 gives a sufficient condition for this property based on the face length and indeed, for girth-planar maximal graphs of girth 3, in other words the maximal planar graphs, this characterization is tight. Thus any planar graph with an embedding that contains a face of length at least $2(g-1) = 2(3-1) = 4$ is not maximal planar. Lemma 4.6 and 4.7 extend the characterization for girth-planar maximality using the face length to a girth of at most 6.

The following two lemmas will aid in proving a sufficient condition for a graph to be girth-planar maximal and will also appear during consecutive proofs.

Lemma 4.1:
Let G be a graph, $C \subseteq G$ a cycle of length k and $u, v \in V(C)$. Then $\operatorname{dist}(u, v) \leq \lfloor k/2 \rfloor$.

Proof. Assume not. Then $\operatorname{dist}(u, v) \geq \lfloor k/2 \rfloor + 1$. As u and v lie on a cycle there must be two distinct paths $P = uCv, Q = vCu$ connecting u to v that form said cycle. P and Q are edge disjoint and thus $\|P\| + \|Q\| = \|C\|$. As per the condition on the distance between u and v we get $\|P\|, \|Q\| \geq \lfloor k/2 \rfloor + 1$ and finally

$$\|C\| = \|P\| + \|Q\| \geq 2\lfloor k/2 \rfloor + 2 \geq k - 1 + 2 > k$$

a contradiction. ■

Remark: Note that if $k = 2g - 3$ for some $g \in \mathbb{N}$ then $\operatorname{dist}(u, v) \leq \lfloor (2g-3)/2 \rfloor = g - 2$

Lemma 4.2:
Let G be a graph, $u, v \in V(G)$ such that $uv \notin E(G)$ and $0 < \operatorname{dist}(u, v) = k < \infty$. Then $G + e$ contains a cycle of length $k + 1$

Proof. As $\operatorname{dist}(u, v) < \infty$ there must be a path P connecting u to v in G. Then it is easy to see that $uPvvu = uPve$ must be a cycle of length $\|P\| + 1 = k + 1$. Since all edges used in the cycle are either already contained in $P \subseteq G$ or the edge is e, the resulting cycle must lie in $G + e$. ■

Proposition 4.3 (Sufficient condition for gpm)**:**
Let G be a 2-connected planar graph with $\operatorname{girth}(G) \geq g$. *If* $\mathrm{f}_{\max}(G) < 2(g-1)$ *then G is girth-planar maximal.*

Proof. Recall that a planar graph is gpm iff no edge can be added without breaking planarity or reducing the girth. If an edge has been added that would break planarity, the edge must have endpoints that do not lie on the same face. For the sake of contradiction assume that G is not gpm and that there exists some non-edge $e \in \binom{V(G)}{2} \setminus E(G)$ such that $G + e$ is planar and $\operatorname{girth}(G + e) \geq g$.

Then the edge must have endpoints that lie on the same face. As G is 2-connected each face is bound by a cycle. Let C be the cycle bounding the face containing the endpoints of e. Let u, v denote the endpoints of e. As $\|C\| \leq 2g - 3$ Lemma 4.1 guarantees $\operatorname{dist}(u, v) \leq g - 2$. By Lemma 4.2 $G + e$ must have girth strictly less than g. Since e was an arbitrary non-edge that does not break planarity the statement is proven. ■

To understand the faces of girth-planar maximal graphs, one must understand the shortest paths between two vertices on the same face. Lemma 4.2 states that every pair of vertices on a face in a 2-connected gpm graph must be connected by a shortest path of length at most $g - 2$. For large faces those paths cannot be entirely contained in the frontier of the face. Therefore the cycle bounding the face must have ears.

The next lemmas examine the presence of 1- and 2-ears in girth-planar maximal graphs.

Lemma 4.4:
Let $g \in \mathbb{N}$ and G be a 2-connected gpm-g graph. Then, each face is bounded by a chordless cycle.

Proof. Let $f \in F$ be a face of some plane embedding. Using the connectivity of G it is immediately clear that ∂f must be a cycle. Let $C = \partial f = u_0 \cdots u_m u_0$, all indices modulo $m + 1$. If $\|C\| < 2(g - 1)$ a chord implies the existence of a cycle of length less than g, as the endpoints would have a distance of at most $g - 2$, a contradiction. Thus we may assume $\|C\| \geq 2(g - 1)$. As a chord is an edge, by planarity not other ear may cross the chord. W.l.o.g the endpoints of the chord are u_0 and u_k with $k \geq g - 1$ (a smaller k results in a cycle of length less than g).
Choose $v = u_{-\lfloor g/2 \rfloor}, w = u_{\lceil g/2 \rceil - 1}$. As there cannot be any other ear crossing the chord, a shortest vw-path P must use either u_0 or u_k. The vertices always split the path into two sections. The minimum length of each possible section is displayed in Figure 10.

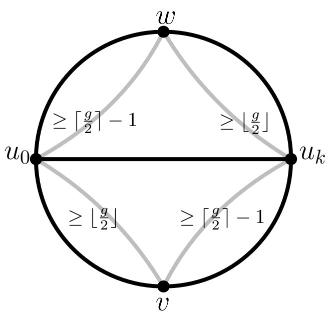

Figure 10: C with chord and minimum lengths between v, w and u_i

***Case* 1:** Assume $u_0 \in V(P)$. By choice of v, $\operatorname{dist}(v, u_0) \geq \left\lfloor \frac{g}{2} \right\rfloor$, as there is a path along C of length $\left\lfloor \frac{g}{2} \right\rfloor$ that, together with a shorter path, would break the girth-condition, since their union would contain a cycle with a length bounded by the path lengths. The same

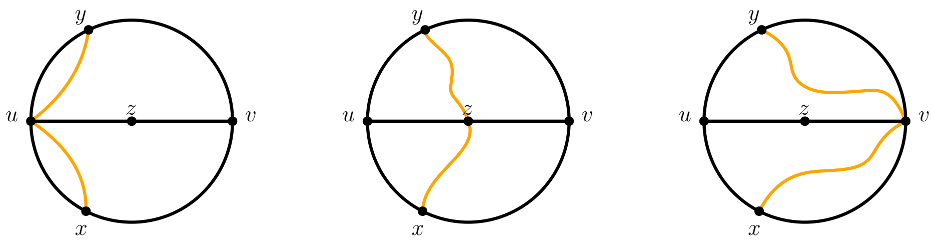

Figure 11: The three base cases for paths crossing with a 2-ear. P using u (left). P uses v (center). P uses v (right).

argument yields $\text{dist}(w, u_0) \geq \lceil \frac{g}{2} \rceil - 1$, as $g = \lceil \frac{g}{2} \rceil - 1 + \lfloor \frac{g}{2} \rfloor + 1$ and $\lfloor \frac{g}{2} \rfloor + 1 \geq \lceil \frac{g}{2} \rceil$. Thus we get $\|P\| = \|vPu_0\| + \|u_0Pw\| \geq \lfloor \frac{g}{2} \rfloor + \lceil \frac{g}{2} \rceil - 1 = g - 1$. Together with the edge vw, Pvw then forms a cycle of length at least g and since P is the shortest vw-path, the addition of vw will not break the girth condition, a contradiction.

***Case* 2:** Assume $u_k \in V(P)$. Note that vPu_k, together with $vCu_{-1}u_0$ and the chord u_0u_k contains a cylce of length at least $\lfloor \frac{g}{2} \rfloor + 1 + \|vPu_k\| \geq g$, because of the girth condition. This implies $\|vPu_k\| \geq \lceil \frac{g}{2} \rceil - 1$. Similarly, $\|u_kPw\| \geq \lfloor \frac{g}{2} \rfloor$. As seen in the first case, this is a contradiction. ∎

Lemma 4.5:
Let G be a 2-connected girth-planar graph with $\text{f}_{\max}(G) \geq 2(g-1)$ and a face f admitting said bound. Then f is bounded by a 2-ear free cycle.

Proof. Let C be the cycle bounding f. Assume there exists a 2-ear E on vertices u, z, v such that $z \notin E(G)$. Then u and v split C into two paths, call them the sides of E. As $\|C\| \geq 2(g-1)$ one can choose two vertices x, y on C having $\text{dist}_C(x, y) = g - 1$ such that these vertices lie on different sides of E. Now choose the vertices in such a way, that $\text{dist}_C(u, x) = \lfloor g/2 \rfloor$ and $\text{dist}_C(u, y) = \lceil g/2 \rceil - 1$ exactly as in Lemma 4.4. As $\text{dist}_C(x, y) \geq g - 1$ there must be a x, y-path P of length $\|P\| \leq g - 2$. By planarity $V(P) \cap \{u, z, v\} \neq \emptyset$. Thus there are three cases (see Figure 11).

***Case* 1:** $u \in V(P)$. This case is equivalent to case 1 from the proof of Lemma 4.4. Thus it can never happen.

***Case* 2:** $z \in V(P)$. Assume that $\|zP\| \leq \lfloor g/2 \rfloor - 1$. Then

$$g \leq \|yCz\| + \|uEz\| + \|zPy\| \leq \left\lceil \frac{g}{2} \right\rceil - 1 + 1 + \left\lfloor \frac{g}{2} \right\rfloor - 1 < g. \tag{4.1}$$

Thus $\|zP\| \geq \lfloor g/2 \rfloor$ and consequently $\|Pz\| \leq \lceil g/2 \rceil - 2$. Plug this in the girth condition and recieve

$$g \leq \|xCu\| + \|uEz\| + \|xPz\| \leq \left\lfloor \frac{g}{2} \right\rfloor + 1 + \left\lceil \frac{g}{2} \right\rceil - 2 < g, \tag{4.2}$$

another contradiction.

***Case* 3:** $v \in V(P)$. As the ear has length 2, the proof of Lemma 4.4 does not hold anymore. There may exist a path of length at most $g - 2$ connecting x to y. However

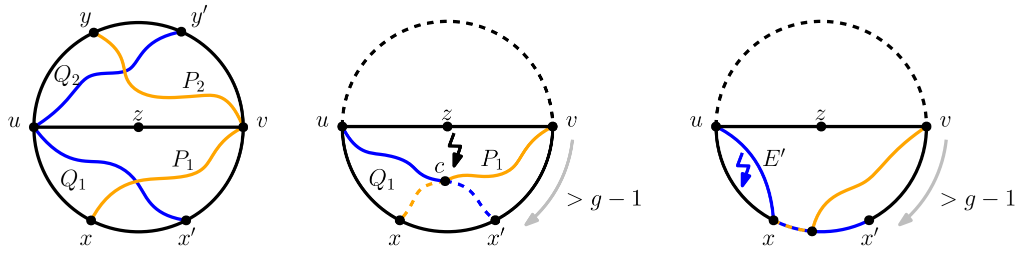

Figure 12: Depiction of the symmetric path Q using u while crossing the 2-ear (left). P_1 and Q_1 intersect in a vertex not on C (middle). P_1 and P_2 intersect only on C resulting in an ear with short width (right). If Q_1 would not have such an ear then P_1 must have one, otherwise there will be a crossing like the middle figure.

v still splits P into two parts, $P_1 = Pv$ and $P_2 = vP$. Now assume that there exits $P' \in \{P_1, P_2\}$ with $\|P'\| \leq \lfloor (g-2)/2 \rfloor - 1 < \lfloor (g-2)/2 \rfloor = \lfloor g/2 \rfloor - 1$. This yields the following inequalities:

$$g \leq \|yCu\| + \|E\| + \|P'\| \leq \left\lceil \frac{g}{2} \right\rceil - 1 + 2 + \left\lfloor \frac{g}{2} \right\rfloor - 2 = g - 1 \tag{4.3}$$

$$g \leq \|xCu\| + \|E\| + \|P'\| \leq \left\lfloor \frac{g}{2} \right\rfloor + 2 + \left\lfloor \frac{g}{2} \right\rfloor - 2 = 2\left\lfloor \frac{g}{2} \right\rfloor \leq g \tag{4.4}$$

Indeed, (4.3) forces $\|P_2\| \geq \lfloor g/2 \rfloor - 1$ because of the girth condition. Thus $\|P_1\| \leq \lceil g/2 \rceil - 1$. Note that (4.4) has $\|P'\| \geq \lfloor g/2 \rfloor - 2$ and $\|P'\| \geq \lfloor g/2 \rfloor - 1$ if g is odd. Combining the result gives $g - 3 \leq \|P\| \leq g - 2$.
Due to the girth condition, we know that $\|uCv\|, \|vCu\| \geq g - 2$. Furthermore one can construct a symmetrical path Q using vertices x', y' with $\text{dist}_C(x', y') = g - 1$ and $\text{dist}_C(v, x') = \lfloor g/2 \rfloor$, as well as $\text{dist}_C(v, y') = \lceil g/2 \rceil - 1$. Now consider $\|uCv\|$ (the length of the section where x and x' are placed). We will examine the cases $\max\{\|uCv\|, \|vCu\|\} > g - 1$ and $\|uCv\| = \|vCu\| = g - 1$.

***Case* 3.1:** Assume $\max\{\|uCv\|, \|vCu\|\} > g - 1$ and that w.l.o.g $\|uCv\| \geq \|vCu\|$. Thus x is placed in the larger section. In this case, u, x, x', v appear as listed in order along C, as $\text{dist}_C(u, x) = \lfloor g/2 \rfloor \leq \lceil g/2 \rceil = g - \lfloor g/2 \rfloor \leq \text{dist}_C(x, v)$ and symmetrically for x'. As $\|P_1\| \leq \lceil g/2 \rceil - 1$ it is clear that $xCv \neq P_1$. By planarity P_1 and Q_1 must cross (Figure 12). Since $x = x'$ can only happen if $uCv = g - 1$ and g is odd, we get $x \neq x'$.
Let c be the first vertex in the intersection as seen from u (Figure 12, middle). Then $\max\{\|cP_1x\|, \|cQ_1x'\|\} > 0$, as $x \neq x'$. If $c \notin \{x, x'\}$ and one of the following: $c \notin V(C)$ or $c \in V(xCx')$ holds, it follows that $\|cP_1x\|, \|cQ_1x'\| > 1$ and we get a cycle of length at most

$$g \leq \|E\| + \|vP_1c\| + \|cQ_1u\| \leq 2 + 2(\left\lceil \frac{g}{2} \right\rceil - 2) < g. \tag{4.5}$$

If $c \in \{x, x'\}$ and the cycle formed by the intersecting paths still has length greater than g there must ne an ear E' contained in Q with endpoints in uCx (or an ear contained in

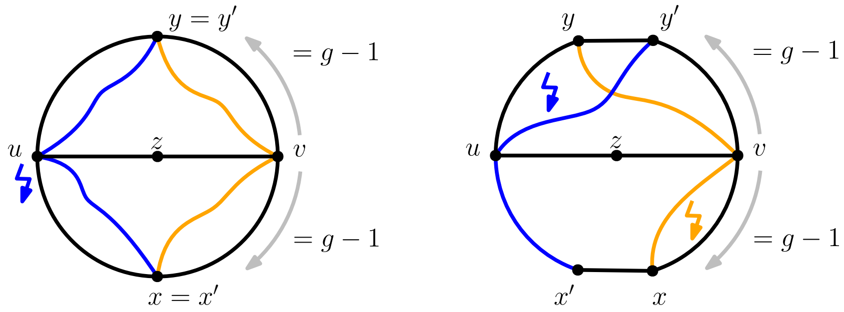

Figure 13: The case where $\|uCv\| = \|vCu\| = g-1$. The endpoints pairwise coincide, contradicting case 1 (left). There is either an ear in the lower part, or a short cycle in the top part (right).

P, but w.l.o.g it is sufficient to only consider Q because of symmetry). Thus, as Q was chosen to contain this ear, $c = x$. Therefore the ear must have length at most $\|Q_1\| - 1$, as there is still a path of length greater than 0 connecting x' tp x (see Figure 12, right). We then have

$$\|E'\| \leq \left\lceil \frac{g}{2} \right\rceil - 2 < \left\lfloor \frac{g}{2} \right\rfloor = \|uCx\|. \tag{4.6}$$

Consequently $E' \neq uCx$ and by Lemma 2.15 $E' \cup uCv$ contains a cycle of size at most

$$\|E'\| + \|uCx\| \leq \left\lceil \frac{g}{2} \right\rceil - 2 + \left\lfloor \frac{g}{2} \right\rfloor < g. \tag{4.7}$$

***Case* 3.2:** Now consider $\|uCv\| = \|vCu\| = g-1$. It follows that $\|C\| = 2(g-1)$ and for each $w \in V(C)$ there exists a unique $w' \in V(C)$ with $\text{dist}_C(w, w') = g-1$. Therefore u, x', x, v, y', y is the counter-clockwise ordering of these vertices along C. If g is odd, $x = x'$ and $y = y'$ as $\lfloor g/2 \rfloor = \lceil g/2 \rceil - 1 = (g-1)/2$. Now observe that Q is a x, y-path of length at most $g-2$ passing through u. This contradicts case 1 as such paths may never exist (Figure 13, left).
Thus only g even is left. Assume P_1 has an ear. Then the ear has length at most $\lceil g/2 \rceil - 1$ but so does xCv, resulting in a cycle of length less than g. Therefore $P_1 = xCv$ and $\|P_1\| = \lceil g/2 \rceil - 1$. The same holds for Q_1. Then $\|P_2\| \leq \lfloor g/2 \rfloor - 1$ implying $P_2 \neq vCy$ as $\|vCy\| = \lfloor g/2 \rfloor$ by the choice of y. This again results in a small cycle in $P_2 \cup vCy$ (Figure 13, right).

Overall the cases lead to a contradiction to the gpm condition of the underlying graph. Thus the graph could not have been girth-planar maximal, proving the statement. ■

Lemma 4.6:
Let G be a planar graph with $\text{girth}(G) \geq g$ *for $g \in \{4, 5\}$. Then G is gpm-g if and only if* $\text{f}_{\max}(G) < 2(g-1)$.

Proof. First consider the case $g = 4$. Assume that G is gpm but still has a face with length at least $2(g-1) = 6$. By Lemma 4.4 G cannot have a chord. Thus each shortest

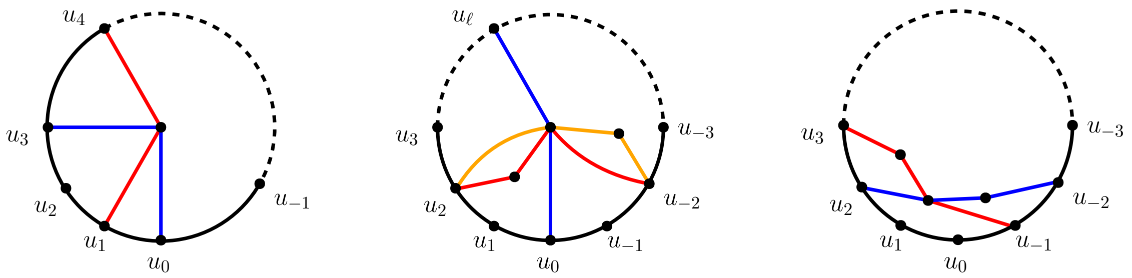

Figure 14: Two 2-ears with neighboring endpoints must create a 3-cycle (left). A two-ear crossed by a three ear must create one of the two 4-cycles (red, orange) (center). Two crossing 3-ears with neighboring endpoints must create a 4-cycle (right).

path connecting 2 vertices in C must have length at least 2. As G is gpm and the addition of any non-edge into the face creates a cycle of length $g-1=3$, all such shortest paths must have length at most 2.
Let $C = u_0 \dots u_m, m \geq 2g-3 = 5$ be the cycle bounding the large face. Then $\text{dist}_C(u_0, u_3) \geq g-1 = 3$. Therefore there must exist a path of length $g-2=2$ containing an ear that connects u_0 to u_3 in G. As the ear cannot be a chord, the entire path is an ear. The same holds for u_1, u_4. As G is a planar graph, both 2-ears must intersect on a vertex not on C. Otherwise, one path would have length at least 3. Let z be the vertex in the intersection. As both paths have length 2, the intersection is unique. Then u_0, u_1, z forms a triangle, see Figure 14 contradicting $\text{girth}(G) \geq 4$.

Now consider the case $g=5$. Let C be defined as above with $m \geq 7$. Assume there exists a 2-ear with endpoints u_0, u_ℓ and $\ell \in \{3, m-2\}$. Using the same idea as in Lemma 4.4 select vertices $u, v = u_2, u_{-2}$. Then $\text{dist}_C(u,v) \geq g-1 = 4$ and there must exist a path of length at most 3 connecting u to v. This short path cannot use u_0, u_ℓ. Indeed, as there are no 4-ears the used vertex would split the ear into two parts, one being just an edge and therefore there is a chord. Thus it must use the middle vertex of the 2-ear, z. This also implies that the short path contains an ear of length 2 or 3. By pigeonhole principle $uz \in E(G)$ or $vz \in E(G)$ as the ear intersecting z would otherwise have length 4, see Figure 14. This creates a 4-cycle, a contradiction.
Thus there cannot be any 2-ears on C and every short path that uses an ear must use a 3-ear. By the same argumentation as for $g=4$, connecting the vertices u_0, u_4 and u_1, u_5 respectively leads to the intersection of two 3-ears. This creates a cycle of length less than 4.

Thus it is clear by contradiction, that a girth-planar maximal graph of girth at least 4 or 5 must satisfy $\text{f}_{\max}(G) < 2(g-1)$. The second implication of the equivalence arises from Proposition 4.3. ■

Remark: Note that the case of $g=4$ also immediately follows from Lemma 4.5

Lemma 4.7:
Let G be a 2-connected gpm-g graph for $g=6$. Then $\text{f}_{\max}(G) < 2(g-1)$.

Proof. The proof follows from Lemma 5 in [1] as it was shown that a 2-connected planar graph that is maximal with respect edge addition for both planarity and retaining girth 6, can only have faces bounded by cycles of length at most 9. ■

As shown above, the sufficient condition of Proposition 4.3, $\mathrm{f}_{\max}(G) \leq 2g-3$ for a graph to be girth-planar maximal is indeed necessary if $g \in \{3,4,5,6\}$ and gives a tight characterization of girth-planar maximal graphs based on the face length.

Recall that a graph is girth-planar maximal if for any embedding no non-edge can be added without breaking the girth or planarity. Thus a graph is not girth planar maximal if some embedding can be constructed that allows for the insertion of a non edge into a face without breaking the girth. This immediately results in a characterization for girth-planar maximal outerplanar graphs.

Corollary 4.8:
Let G be a 2-connected outerplanar graph. Then G is gpm-g if and only if $|G| < 2(g-1)$

Proof. It is clear form Proposition 4.3 that if $|G| < 2(g-1)$, G must be gpm. Assume that $|G| \geq 2(g-1)$ and that G is gpm. Fix the outerplanar embedding. Then the outer face is bounded by all vertices. Using Lemma 4.4 the outer face must be bounded by a chordless cycle. As all vertices of G lie on this cycle, G must be a cycle. However a cycle of length at least $2(g-1)$ cannot be gpm, as there exist vertices that have distance at least $g-1$. Connecting both vertices with an edge thus cannot create a cycle of length less than g. A contradiction. ■

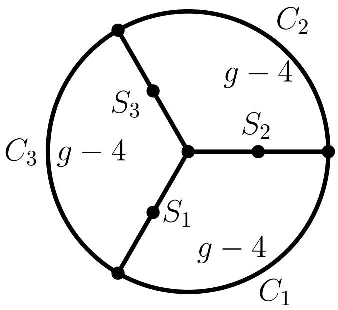

Figure 15: A gpm-g subdivision of a wheel as described in Thm 4.9.

4.2 Lower Bound on the Face Length

Theorem 4.9:
There exists a gpm-g G graph having $\mathrm{f}_{\max}(G) = 3g - 12$.

Proof. Define G as seen in Figure 15 as a wheel with three spokes of length 2 each. The spokes split the outer cycle into three parts of size $g-4$. The graph has exactly 4 faces. All inner faces containing the center vertex of the wheel have a face length of exactly g. By construction the outer face has a length of $3g-12$. Using Proposition 4.3 it is sufficient to show that every non edge inserted into the outer face creates a cycle of length less than g.

Denote the cycle bounding the outer face by C, the spokes by S_1, S_2, S_3 and the sections of the outer face as C_1, C_2, C_3. Let $u, v \in V(C)$ s.t. $\mathrm{dist}_C(u,v) \geq g-1$. Note that u and v can never be in the same section of C. W.l.o.g. $u \in V(C_1)$ and $v \in V(C_2)$. Observe that $C_1 \cup C_2 \cup S_1 \cup S_3$ forms a cycle in G containing both u and v. The length of the cycle is

$$\|C_1\| + \|C_2\| + \|S_1\| + \|S_3\| = 2(g-4) + 4 = 2g - 4 \leq 2g - 3.$$

By this inequality there must exist a path of length at most $g-2$ in G with endpoints u and v as shown in Lemma 4.1 and Lemma 4.2.

Therefore on cannot insert a non-edge into the outer face and not reduce the girth. It is clear that the graph has indeed girth g, as each cycle in G either contains two sections C_i, C_j or one section C_i and two spokes S_j, S_k. Thus G is a gpm-g graph. ■

Theorem 4.10:
For $g \geq 7$ there exists a gpm-g graph with no crossing ears and having $\mathrm{f}_{\max}(G) = \frac{5}{2}g + o(g)$.

Proof. Consider the following construction: Let G be a graph consisting of a cycle C split into 3 sections C_1, C_2, C_3 having

$$\|C_i\| = \left\lfloor \frac{2}{3}g \right\rfloor + \left\lfloor \frac{1}{6}g - \frac{3}{2} \right\rfloor.$$

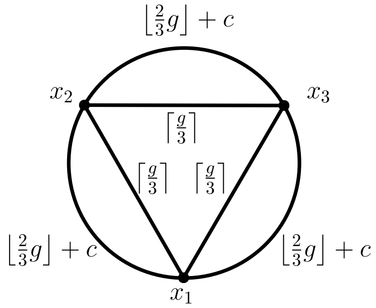

Figure 16: A gpm-g graph for $g \geq 7$ as described in Thm 4.10. $c = \left\lfloor \frac{1}{6}g - \frac{3}{2} \right\rfloor$
.

The endpoints of each segment is x_i. The endpoints are pairwise connected by disjoint paths P_1, P_2, P_3 of length $\left\lceil \frac{g}{3} \right\rceil$, such that $P_i = x_i P_i$ and $C_i = x_i C_i$, see Figure 16.

We shall now show that G is indeed gpm by proving that every two distinct vertices in G lie on a cycle of length at most $2g-3$. Then the gpm property follows from Lemma 4.1 and Lemma 4.2. Observe that for $\{i,j,k\} = \{1,2,3\}$ $P_i \cup C_j \cup C_k$ is a cycle in G. The length of any such cycle is

$$\|P_i\| + \|C_j\| + \|C_k\| = 2\left(\left\lfloor \frac{2}{3}g \right\rfloor + \left\lfloor \frac{1}{6}g - \frac{3}{2} \right\rfloor\right) + \left\lceil \frac{g}{3} \right\rceil = g + \left\lfloor \frac{2}{3}g \right\rfloor + 2\left\lfloor \frac{1}{6}g - \frac{3}{2} \right\rfloor$$
$$\leq g + \frac{2}{3}g + \frac{1}{3}g - 3 = 2g - 3.$$

Now consider any pair of vertices $u, v \in V(G)$. The vertices may either lie on a cycle segment or one of the center paths respectively. Since there are only two vertices, they lie on at most two cycle segments. Thus we can distinuish two cases:

***Case* 1:** W.l.o.g $u \in V(C_1)$. Using the symmetry of G and potential relabeling we get $v \in V(C_2)$ or $v \in V(P_3)$. Then u and v lie on the cylce $C_1 \cup C_2 \cup P_3$ and thus have a distance of at most $g-2$.

***Case* 2:** W.l.o.g $u \in V(P_1)$ and $v \in V(P_2)$. In this case the vertices lie on the cylce $P_1 \cup P_2 \cup P_3$. The cycle has length $3\left\lceil \frac{g}{3} \right\rceil$. Since by construction $\|P_i\| \leq \|C_j\|$ the distance is again at most $g-2$, as the upper bound from case 1 also is an upper bound here. Note that the inequalities only hold since $g \geq 7$.

For the length of the outer face we have $\|C\| = 3\|C_i\|$ and the following inequalities:

$$\|C\| = 3\left(\left\lfloor \frac{2}{3}g \right\rfloor + \left\lfloor \frac{1}{6}g - \frac{3}{2} \right\rfloor\right) \leq 3\left(\frac{2}{3}g + \frac{1}{6}g - \frac{3}{2}\right) = \frac{5}{2}g - \frac{9}{2},$$

but also

$$\|C\| \geq 3\left(\frac{2}{3}g - 1 + \frac{1}{6}g - \frac{3}{2} - 1\right) = \frac{5}{2}g - \frac{21}{2}.$$

This concludes the proof as $\|C\|$ has both an upper bound and a lower bound admitting $\frac{5}{2}g + o(g)$. ■

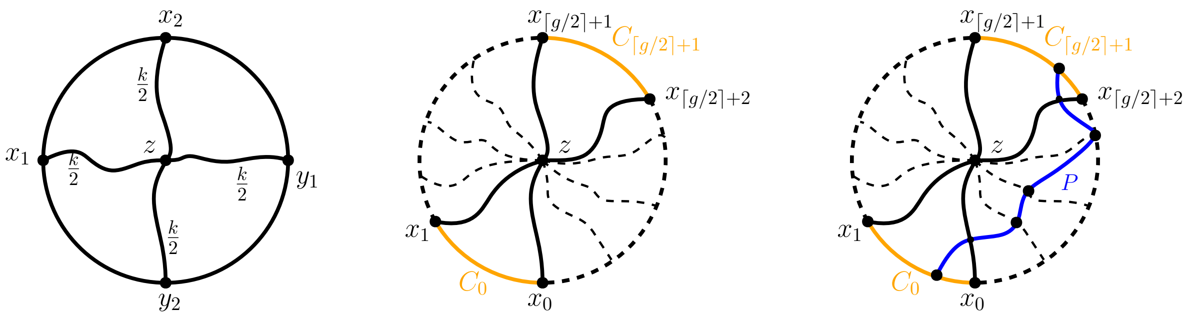

Figure 17: The paths connecting $x_i y_i$ for $i = 1, 2$ must cross (left). Each path has a subpath of length at most $k/2$ ending in z, implying all subpaths must have length exactly $k/2$. The wheel like structure formed by all crossing paths (center). A path of length $g-2$ not containing z must cross all other subpaths (right).

4.3 Upper Bound on the Face Length

The main goal of this thesis is to find a function of g bounding the maximum face length of gpm-g graphs. Indeed, using Ramsey Theory such a bound can be given, however it is quite large.

Let G be a 2-connected girth planar maximal graph. As seen in Lemma 4.15 all vertices sitting on the bounding cycle of the same face must have distance at most $g-2$ and at least 3. We now construct an auxiliary complete graph on the vertices of the bounding cycle. Create a coloring on the edges of the new graph as follows: For any edge uv define $c(uv) = \text{dist}_G(u, v) \in [g-2]$. Ramsey Theory then states that if there are enough vertices on the face, we must find a monochromatic K_{2g}-copy. Chosen correctly, this copy can then be used to find a short cycle in G, contradicting the girth-condition as there are other ears that must intersect with the shortest paths connecting the $2g$ vertices of the K_{2g}-copy.

The proof heavily makes use of the fact, that no girth-planar maximal graph can contain a large spider. The following key lemmas will be used in the proof.

Lemma 4.11:
Let G be a gpm-g graph and C a cycle bounding the outer face of a drawing of G. If there exist vertices y_1, y_2, y_3, y_4 in order around C such that $\text{dist}(y_i, y_j) = k$ for $i \neq j$ the shortest y_1, y_3- and y_2, y_4-paths (P_1 and P_2) intersect exactly in one vertex z and $\|x_j P_i z\| = k/2$.

Proof. For visualization also see Figure 17.
As all paths must lie on the same side of C, by planarity P_1 and P_2 must cross. Define u as the first vertex appearing in the intersection $V(P_1) \cap V(P_2)$ as seen traversing P_1 from y_1 and symmetrically w as the last vertex. Note that

$$\|y_1 P_1 u\| + \|u P_1 w\| + \|w P_1 y_3\| = \|P_1\| = k. \tag{4.8}$$

Thus we get $\min\{\|y_1 P_1 u\|, \|w P_1 y_3\|\} \leq \lfloor k/2 \rfloor \leq k/2$. Similarly, one can observe that $\min\{\|y_2 P_2 u\|, \|u P_2 y_4\|\}, \min\{\|y_2 P_2 w\|, \|w P_2 y_4\|\} \leq k/2$. Assume that $\|y_1 P_1 u\| < k/2$.

Then for $P = \text{argmin}\{\|uP_2y_2\|, \|uP_2y_4\|\}$

$$\|y_1P_1uP\| = \|y_1P_1u\| + \|P\| < \frac{k}{2} + \|P\| \leq \frac{k}{2} + \frac{k}{2} = k,$$

contradicting $\text{dist}(y_1, y_2) = \text{dist}(y_1, y_4) = k$. The same observation is made if $\|wP_1y_3\| < k/2$ by exchanging u with w in the inequalities. Thus $\|y_1P_1u\| \geq k/2$ and $\|wP_1y_3\| \geq k/2$. Accounting for the total length of P_1 (4.8), we get $\|y_1P_1u\| = \|wP_1y_3\| = k/2$, also implying that k must be even. Furthermore and again using (4.8) is is clear that $\|uPw\| = 0$ and thus $u = w$. Note that since $V(P_1) \cap V(P_2) \subseteq V(uPw)$ the claim is proven by renaming u and w to z. ■

Lemma 4.12:
Let $g \in \mathbb{N}$, G be a gpm-g graphand C a cycle bounding the outer face. Then G cannot contain a spider with $2\lceil g/2\rceil + 2$ legs of size at most $g/2 - 1$ that end in C.

Proof. Assume otherwise. Let S be the spider contained in G. The center point shall be denoted by z. A leg of the spider is a path P_i with endpoint x_i. All endpoints $x_0, \ldots, x_{2\lceil g/2\rceil+1}$ appear in order along C.

The endpoints split C into $2\lceil g/2\rceil + 2$ sections C_i where each section is a path with endpoints x_i and x_{i+1} (see Figure 17, center). All indices are modulo $2\lceil g/2\rceil + 2$.

Claim 1: For each section C_i there exists at least one vertex u and a vertex v such that $u, v \in V(C_i)$ and $\text{dist}_S(u, z) = \lfloor g/2\rfloor, \text{dist}_S(v, z) = \lceil g/2\rceil - 1$.

Proof of Claim 1: Indeed, consider the path $P := zP_ix_iC_ix_{i+1}$. Then $\|P\| \geq g/2$ as $\|P_{i+1}\| \leq g/2 - 1$ and $\text{girth}(G) \geq g$. Thus one can choose vertices u and v on P such that they have the desired distance to z along P. Furthermore both vertices must lie on C_i as they cannot lie on P_i, since P_i is too short. It remains to show that there does not exists a shorter u, z- and v, z-path.

Assume otherwise and let Q be the shorter path. Then $Q \neq Pu$ and by Lemma 2.15 $D = Pu \cup Q$ must contain a cycle. This cycle has length $\|D\| \leq \|Pu\| + \|Q\| < 2\|Pu\| = 2\lfloor g/2\rfloor \leq g$. This is however a contradiction to $\text{girth}(G) \geq g$. Similarly, one can obtain the same result for Pv since $2(\lceil g/2\rceil - 1) \leq 2(g/2 + 1 - 1) = g$.

Claim 2: Let P be an arbitrary path starting in C_0, ending in $C_{\lceil g/2\rceil+1}$ and having length at most $g - 2$. Then $z \in V(P)$.

Proof of Claim 2: Assume that $z \notin E(P)$. Then by planarity P still has to intersect with either $P_1, \ldots, P_{\lceil g/2\rceil+1}$ or $P_{\lceil g/2\rceil+2}, \ldots, P_0$ (see Figure 17, right). Since all legs only meet in z the path has to contain at least $\lceil g/2\rceil$ vertices that intersect with a leg. Note that for P_i and P_{i+1} both legs together with P form a cycle in G. As each leg has length at most $g/2 - 1$ the section of P that connects from P_i to P_{i+1} must contain at least 2 edges. Thus $\|P\| \geq 2(\lceil g/2\rceil + 1 - 1) \geq g$, a contradiction.

Now choose $u \in V(C_0)$ and $v \in V(C_{\lceil g/2\rceil+1})$ according to Claim 1. Then $\text{dist}_G(u, v) \leq g - 2$ by Lemma 4.2. Let P be a shortest u, v-path in G. By Claim 2 P must go through

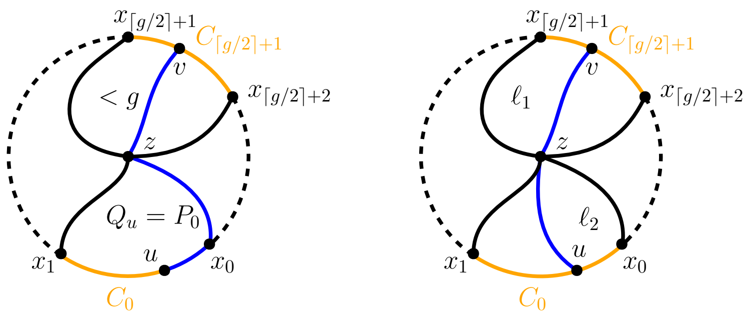

Figure 18: The case where $Q_u = P_0$. The cycle bounded by Q_v and $vCP_{\lceil g/2\rceil+1}$ must have length less than g (left). The case where $Q_u \neq P_0$ and $Q_v \neq P_{\lceil g/2\rceil+1}$. The cycles created by the paths have lengths ℓ_1, ℓ_2 with $\ell_1 + \ell_2 < 2g$ (right).

z. Furthermore P cannot be contained in S as can be derived from Claim 1, since $\text{dist}_S(u, v) = \text{dist}_S(u, z) + \text{dist}_S(z, v) = g - 1 > \|P\|$.

Define $Q_u, Q_v = uPz, zPv$. Then $Q_u \neq P_0$ or $Q_v \neq P_{\lceil g/2\rceil+1}$ as otherwise P would be contained in S.

***Case* 1:** $Q_u = P_0$. Then $Q_v \cup P_{\lceil g/2\rceil+1}$ must contain a cycle of length at least g. However the cycle must have a length of at most $\|Q_v\| + \|P_{\lceil g/2\rceil+1}\| \leq g - 2 - \lfloor g/2\rfloor + \lceil g/2\rceil - 1 = 2\lceil g/2\rceil - 3 < g$ a contradiction.

***Case* 2:** $Q_u \neq P_0$ and $Q_v \neq P_{\lceil g/2\rceil+1}$. Then $P \cup P_0 \cup P_{\lceil g/2\rceil+1} \cup C$ must contain at least two cycles, one contained in $Q_u \cup P_0 \cup uCx_0$ and the other in $Q_v \cup P_{\lceil g/2\rceil+1} \cup vCx_{\lceil g/2\rceil+1}$ both of length at least g. The entire structure only has at most $2g - 3$ edges by construction, due to the path lengths. Applying the pigeonhole principle to both cycles yields a contradiction as one cycle must have length less than g.

In conclusion all cases lead to a contradiction as is illustrated in Figure 18 (right) and therefore the graph G cannot contain the spider. ■

Theorem 4.13 (Large bound on the Face Length)**:**
Let G be a 2-connected gpm-g graph. Then $\text{f}_{\max}(G) < R(2g; g-3)$ where $R(2g; g-3)$ is the multicolor Ramsey Number for K_{2g} and $g - 3$ colors.

Proof. For the sake of contradiction assume that G has a drawing with a face of length at least $N := R(2g; g-3)$. Let C be the cycle bounding this face. Now choose $V(C)$. as the vertex set of the auxiliary graph. Then at least $R(2g; g-3)$ vertices have been chosen. Impose the following coloring on the edges: $c(uv) := \text{dist}_G(u, v)$. By the proof of Lemma 4.15 it is clear that c colors the auxiliary graph in at most $g - 3$ colors. By Ramsey Theory there exists a vertex set $X = \{x_0, \ldots, x_{2g-1}\}$ s.t. these vertices form a monochromatic copy of K_{2g} in the complete graph. All indices modulo $2g$. Denote its color by k.

It is clear that C is split into $2g$ sections $C_0, \ldots, C_{2g-1}$ where $C_i = x_iCx_{i+1}$. For $i, j \in [2g], i \neq j$ define $P(i, j)$ as the shortest x_i, x_j-path in G. As by the choice of the vertices of the complete auxiliary graph, all x_i must have distance exactly k. Thus each such path has to have an ear of length at most k if $|i - j| > 1$.

Let $y_i, y_j, y_k, y_l \in X$ be vertices on C appearing in the order of their indices. Then by Lemma 4.11 the paths $P(i, k), P(j, l)$ must intersect in a unique vertex z.

Now consider all such crossing paths. As each pair of distinct paths intersect exactly in the middle and the union of all paths only intersect in z as well. Define $S := \bigcup P(i, j)$ as the untion of all such paths that contain z. Indeed, S is a spider with legs $P_i = x_i P(i, j) z$ (independent of j). For P_i define x'_i as the first vertex in the intersection $V(P_i) \cap V(C)$ as seen from z. As the x'_i are ordered in the same way as the x_i one can also define the spider S' accordingly.

Note that each leg has length at most $k/2 \leq (g - 2)/2 = g/2 - 1$. Then the contradiction follows from Lemma 4.12.

■

Theorem 4.13 shows that the maximum face length of a girth-planar maximal graph is bounded by some function of g. However, the proposed bound is very large and does not come close to more tight bounds proven for several special graph classes. I thus conjecture, that the general bound for all girth-planar maximal graphs is still linear in g, as is consistend with all current observations.

Conjecture 4.14:
There exists a constant $c > 0$ such that for every gpm-G graph G, $\mathrm{f}_{\max}(G) \leq cg$ holds.

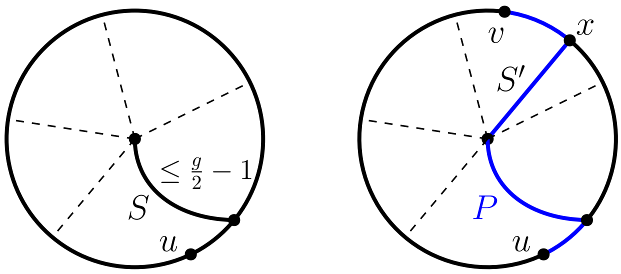

Figure 19: The short spoke S and the vertex u (left). A shortest u, v-path using S and S' (right).

4.4 Results for Special Classes of Graphs

Lemma 4.15:
Let G be a gpm-g graph with some embedding M and $f \in F(M)$ with $C = \partial f$ such that $\ell(f) \geq 2(g-1)$. Let $u, v \in V(C)$ s.t. $\mathrm{dist}_C(u, v) \geq g - 1$. Then every shortest u, v path must contain a k-ear for some k with $2 \leq k \leq \mathrm{dist}(u, v) \leq g - 2$.

Proof. As G is gpm, each non-edge that can be inserted into f must create a cycle of length less than g. Thus there must be a shortest u, v-path of length at most $g - 2$. This implies $\mathrm{dist}_G(u, v) \leq g - 2$ and since $\mathrm{dist}_C(u, v) \geq g - 1$ this shortest path is not a subgraph of C. Therefore the part of the path not intersecting with C is a collection of ears having a length of less than $g - 2$, as each ear is a subpath of the shortest path. The lower bound is given by Lemma 4.4. ■

The union of all such short ears then forms a subgraph of G whose structure can then be used to gain knowledge about whether the underlying graph is girth-planar maximal. The simplest such graph is the intersection in exactly one vertex. This implies that G is a wheel.

Proposition 4.16 (About Wheels)**:**
Let $g \in \mathbb{N}$, G a gpm-g wheel and let c be the center vertex of the wheel. Then $\mathrm{f}_{\max}(G) \leq 3g$

Proof. Note that a wheel has a unique embedding on the sphere with respect to the relative positioning of the faces. Let C be the cycle bounding the face that does not have c on its frontier.

Assume that $\|C\| > 3g$. Then there exist pairs of vertices with a distance at least $g - 1$ along C. Lemma 4.15 then implies that there must exist ears of length at most $g - 2$. Since each ear crosses c, at least one spoke of the wheel must have length at most $g/2 - 1$. Let S be any spoke of length at most $g/2 - 1$. Now choose a vertex u on $C \cup S$ such that $\mathrm{dist}_{C \cup S}(u, c) = \lfloor g/2 \rfloor$. Such a vertex always exists as G has sufficiently many vertices. Furthermore, u lies on C, as all vertices on S have distance at most $g/2 - 1 < \lfloor g/2 \rfloor$ from c, see Figure 19.

Claim: $\mathrm{dist}(u, c) = \lfloor g/2 \rfloor$.
Assume otherwise. Then there exists a shorter u, c-path P. This path cannot be contained in $uCSc$ which is the shortest u, c-path in $C \cup S$. Thus by Lemma 2.15 there

must exist a cycle contained in $P \cup uCSc$ with length bounded by

$$\|P \cup uCSc\| \leq \|P\| + \|uCSc\| < 2\left\lfloor \frac{g}{2} \right\rfloor \leq g,$$

a contradiction. Hence there cannot be any u, c-path shorter than $\mathrm{dist}_{C\cup S}(u, c)$, proving the claim.

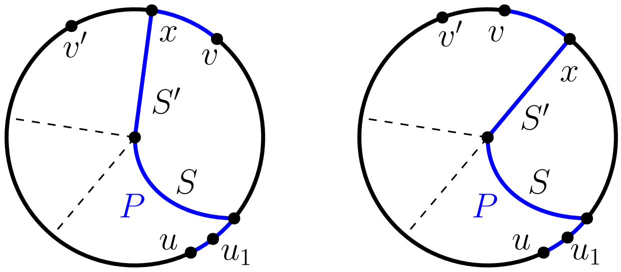

Figure 20: The vertex v' is always to the left of v and has distance at most g to v along C.

Let $C = u_0 \cdots u_m u_0$ for some $m \geq 3g$ and $u_0 = u$. We denote $v = u_{g-1}$. Observe that $\mathrm{dist}_C(u, v) = g - 1$ as there are only two possible u, v-paths on C, one of which has length $g-1$ and the other $m+1-(g-1) \geq 2g$. Again by Lemma 4.15 there must exist a u, v-path P in G that contains an ear E and has length at most $g-2$. By the properties of the wheel it contains c. In the wheel, every ear with respect to the outer cycle must consist of exactly two spokes. Thus there are two spokes, S and S' s.t. $E = S \cup S'$. W.l.o.g $S \subseteq uPc$. Since by the claim $\mathrm{dist}(u, c) = \lfloor g/2 \rfloor$ the other part of P containing S' must satisfy

$$\ell := \|S'\| \leq \|cPv\| = \|P\| - \|uPc\| \leq \left\lceil \frac{g}{2} \right\rceil - 2 < \frac{g}{2} - 1.$$

Let $x = u_i$ be the endpoint of S' on C. Now define $k := \lfloor g/2 \rfloor - \ell$ and $v' = u_{i+k}$, see Figure 20. Observe that similar to the choice of u, $\mathrm{dist}_{C\cup S'}(v', c) = k + \ell = \lfloor g/2 \rfloor$. Therefore the claim also holds and $\mathrm{dist}(v', c) = \lfloor g/2 \rfloor$.

***Case* 1:** $\mathrm{dist}_C(u, v') \geq g - 1$. Lemma 4.15 guarantees that a shortest u, v'-path P' must use an ear and contain c. This path can be partitioned into $uP'c$ and $cP'v'$. However, as both subpaths cannot have a shorter length than $\mathrm{dist}(u, c)$ and $\mathrm{dist}(v', c)$ respectively, it follows that

$$\|P'\| = \|uP'c\| + \|cP'v'\| \geq 2\left\lfloor \frac{g}{2} \right\rfloor \geq g - 1 > g - 2 \geq \mathrm{dist}(u, v') = \|P'\|,$$

a contradiction.

***Case* 2:** $\mathrm{dist}_C(u, v') < g - 1$. By the choice of v', the vertex must not lie in the section $uu_1 \cdots u_{i-1} x u_{i+1} \cdots v$ of C. Otherwise $v' \in \{u_{i+1}, \ldots v\}$ implying $\mathrm{dist}_C(v', x) \leq \mathrm{dist}_C(v, x)$ and leading to

$$\mathrm{dist}_{C\cup S'}(v', c) = \mathrm{dist}_C(v', x) + \ell \leq \mathrm{dist}_C(v, x) + \ell = \mathrm{dist}_{C\cup S'}(v, c) < \left\lfloor \frac{g}{2} \right\rfloor.$$

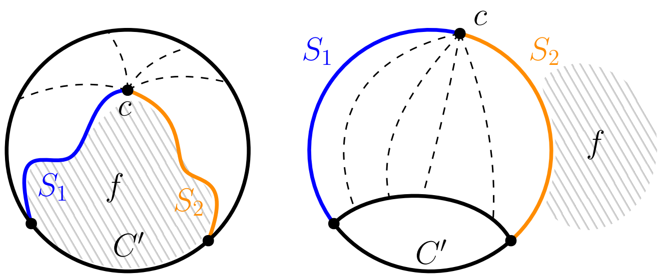

Figure 21: An inner face f of the wheel with spokes S_1 (blue) and S_2 (orange) (left). A different embedding of the wheel in the plane having f as the outer face (right).

Therefore $\|uCv'\| \geq \|uCv\| = g - 1$. Now consider $v'Cu$ as the only other option for a u, v'-path along C.

$$\|v'Cu\| = \|C\| - \|uCv'\| = \|C\| - (\|uCv\| + \|vCv'\|) > 3g - (g - 1 + \|vCv'\|). \quad (4.9)$$

Finally observe that $\|vCv'\| \leq 2\|xCv'\|$ as $\text{dist}_C(x, v) \leq \text{dist}_C(x, v')$. Thus we get the bound

$$\|vCv'\| \leq 2k = 2\left(\left\lfloor \frac{g}{2} \right\rfloor - \ell\right) \leq 2\left\lfloor \frac{g}{2} \right\rfloor \leq g.$$

Pluggin this bound into (4.9) yields

$$\|v'Cu\| > 3g - (g - 1 + g) \geq g.$$

Therefore $\text{dist}_C(u, v') = \min\{\|uCv'\|, \|v'Cu\|\} \geq g-1$, a contradiction to the assumption of case 2.

For the second part of the proof, assume that a face f that contains c, has length larger than $3g$. By the properties of the wheel this face must be bounded by three paths. Two spokes and a section of C. Denote the spokes by S_1 and S_2 and the section of C by C', see Figure 21.

Claim: $\|S_1\| + \|S_2\| \leq 2g - 1$ and $\|S_i\| + \|C'\| \leq 2g - 1$ for $i = 1, 2$.
Indeed, assume not and the length of two paths, p_1 and p_2, sum up to at least $2g$. Choose vertices u and v s.t. they lie exactly in the middle of each section. Then the distance to either one of the endpoints is at least $\lfloor p_1/2 \rfloor$ and $\lfloor p_2/2 \rfloor$ respectively. Note that ∂f only has three vertices of degree larger than 2.

Consider a shortest u, v-path P. The path can only be contained in the cycle bounding the face, see Figure 22. Indeed, otherwise P would have to use at least 2 additional edges for an ear. However, still half of the sections need to be used in order to reach a vertex that can be used as an endpoint of an ear. As the lengths of each half only vary by at most 1, the path cannot be shorter.

Thus a lower bound for the length of P is the floor of each half

$$\|P\| \geq \left\lfloor \frac{p_1}{2} \right\rfloor + \left\lfloor \frac{p_2}{2} \right\rfloor > \frac{p_1 + p_2}{2} - 2 \geq g - 2.$$

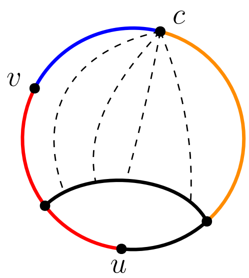

Figure 22: The only possibility for a shortest u, v-path (red) up to symmetry and choice of the sections containing u and v.

Since the inequality is strict there cannot be a u, v-path of length at most $g - 2$, a contradiction.

Therefore using the fact that the face has length at least $3g$, the remaining section has to have a length greater than g. Symmetrically this calculation can be done for all 3 sections:

$$\|S_1\| + \|S_2\| \leq 2g - 1 \implies \|C'\| \geq 3g - (\|S_1\| + \|S_2\|) \geq 3g - (2g - 1) = g + 1,$$
$$\|S_1\| + \|C'\| \leq 2g - 1 \implies \|S_2\| \geq 3g - (\|S_1\| + \|C'\|) \geq 3g - (2g - 1) = g + 1$$

It follows that $\|C'\| + \|S_2\| \leq 2g + 2$, contradicting the claim. Thus the graph could not have been girth planar maximal.

Altogether it was shown that each possible face in a wheel has to have length at most $3g$, if the wheel is to be gpm-g. ■

Remark: Note that by Theorem 4.9 the bound of $3g$ is tight up to an additive constant $c \leq 12$.

In the proof of the above proposition, one can observe that the condition of the graph being a subdivision of a wheel can be weakend. Indeed, it is only necessary that there exists a common vertex c that is used by all shortest paths connecting the vertices u and v' that are chosen according to the proof of Proposition 4.16.

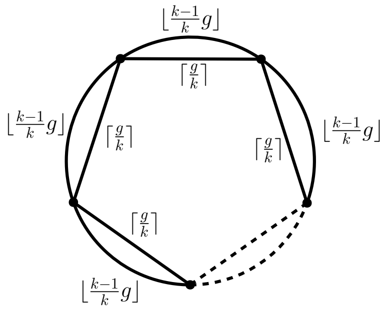

Figure 23: The graph $W_{g,k}$ as defined in Def. 4.17. Note that there are always exactly k inner paths depicted by the straight lines.

The constructions given in Subsection 4.2 can be generalized to allow for more sections along the cycle bounding the outer face. However if a fourth section is added to the construction, the graphs are no longer girth-planar maximal.

The idea for the construction of Theorem 4.10 stems from a more general graph of girth at least g that is defined in Def. 4.17.

Definition 4.17:
Let $k, g \in N$. Define the graph $W_{g,k}$ as follows:

(i) Let C be a cycle on the vertices $u_0, \ldots, u_{\lfloor \frac{k-1}{k} g \rfloor k-1}$

(ii) Let $P_i, i \in \{0, \ldots k-1\}$ be a path of length $\lceil \frac{g}{k} \rceil$ with endpoints $u_i, u_{i+\lfloor \frac{k-1}{k} g \rfloor} \in V(C)$ but otherwise disjoint from C and all other P_j, $j \neq i$

(iii) $W_{g,k} := C \cup P_0 \cup \ldots \cup P_{k-1}$

Remark: For a drawing of $W_{g,k}$, see Figure 23

Lemma 4.18:
The graph $W_{g,k}$ is only girth-planar maximal for $k = 3$

Proof. Consider the sections $C_0 = u_0 C u_1$ and $C_1 = u_{\lfloor k/2 \rfloor} C u_{\lfloor k/2 \rfloor + 1}$. For each section choose a vertex $x \in V(C_0)$ and $y \in V(C_2)$ such that the distance to the endpoints of each section is at least $\lfloor \frac{k-1}{2k} g \rfloor$. A shortest x,y-path then has to use at least $\lfloor k/2 \rfloor$ P_is as well as $\lfloor \frac{k-1}{2k} g \rfloor$ edges along each section C_i. This yields a lower bound of

$$2\left\lfloor \frac{k-1}{2k} g \right\rfloor + \left\lfloor \frac{k}{2} \right\rfloor \cdot \left\lceil \frac{g}{k} \right\rceil > \frac{k-1}{k} g - 2 + \frac{k}{2} \cdot \frac{g}{k} - \frac{g}{k} = \frac{k-2}{k} g + \frac{g}{2} - 2$$

Since $k \geq 4$ it follows that $2/k \leq 1/2$ and thus

$$\frac{k-2}{k} g + \frac{g}{2} - 2 \geq \frac{k-2}{k} g + \frac{2g}{k} - 2 = g - 2$$

concluding the proof as the inequality is scrict. ■

5 Related Work

The graph parameters used to describe girth-planar maximal graphs tend to be quite useful in many other aspects of graph theory. Especially girth and planarity on their own have great applications both in graph theory and in several algorithms.

5.1 Planar Graphs in Algorithms

Probably the most popular and useful aspect of planar graphs in algorithm theory is the Planar Separator Theorem [15]. A separator is a vertex set $S \subseteq V(G)$ such that $G - S$ is split into two disconnected subgraphs G_1, G_2. A recursive algorithm can then use the subgraphs and combine the respective results using the separator. It is crucial for the analysis of such algorithms that the separator has as few vertices as possible and that G_1 and G_2 are similar in size. The Planar Separator Theorem ensures $|S| \in \mathcal{O}(\sqrt{n})$, $|G_1|, |G_2| \leq \frac{2}{3}n$, where $n := |G|$. Its practicality follows from the fact that such a separator can be calculated in $\mathcal{O}(n)$ time.

One early application of the Planar Separator Theorem was the analysis of an algorithm for nested dissection [10]. This algorithm computes a Cholesky Decomposition of a sparse matrix such that the decomposition is still sparse by using the matrix as the adjacency matrix of a graph. Using separators the graph then can be split into smaller parts resulting in the preservation of many zero-entries. The version proposed by George Alan could be generalized for graph classes admitting a $f(n)$-separator theorem [14]. A $f(n)$-separator theorem is a generalized version of the Planar Separator Theorem where for $\frac{1}{2} \leq \alpha < 1, \beta < 1$ $|S| \leq \beta f(n)$ and $|G_1|, |G_2| \leq \alpha n$. Bounds on the sparsity of the decomposition and both time and space complexity were given using this theorem [12][11].

As girth-planar maximal graphs also allow for a separator of size $g - 1$ as seen in Lemma 4.15, the question can be raised if one can always find a separator that satisfies a $(g - 1)$-separator theorem. Indeed, for girth-planar maximal graphs with a maximum face length of at least $2g - 2$ one can always find a planar separator of size $g - 1$ that splits the graph into two components. However, it is not known whether girth-planar maximal graphs of hight maximum face length always have a balanced separator. Note that this is not possible for a girth of at most 6. Therefore this approach cannot improve on the Planar Separator Theorem for general planar graphs.

5.2 Coloring Planar Graphs

A topic that is studied quite extensively is graph colorings. It is known that planar graphs can be colored using 4 colors. It is however not clear which planar graphs have chromatic number 3. Indeed, it is $\mathcal{NP}$-complete to determine whether a planar graph is 3-colorable [9]. Thus there are still many open areas concerning the coloring of planar graphs.

Instead of checking for proper colorings of a graph, one might also look into improper colorings with some restrictions. An improper coloring is a coloring where vertices or edges of the same color might be adjacent. All vertices or edges of the same color induce a subgraph. For proper vertex coloring the subgraphs all are independent sets and in the case of edge colorings all subgraphs are matchings. For improper colorings one can examine how close the induces subgraphs get to independent sets or matchings. This is done by finding partitions of the graph into forests of small, easy to understand graphs, like trees.

Axenovich, Ueckerdt and Weiner showed that a planar graph of girth at least 6 can be split into two path-forests such that each path has length at most 14 [1]. The paths were constructed along the faces of a girth-planar maximal graph with a girth at least 6. It was in this paper that the upper bound for the maximum face length for girth-planar maximal graphs was first discussed, but only for the fixed girth of 6.

For general graphs, bounding the chromatic number by restricting the girth does not work as was shown by Erdős [6]. Planar graphs on the other hand have a trivial bound as discussed above. There are however some variants to colorings that are not as easy to find good bounds for. One example is total colorings. A proper total coloring of a graph is a coloring of the vertex and edge set such that no adjacent or incident elements have the same color. For any graph with highest degree Δ it is clear that the total chromatic number χ'' must be at least Δ. It was conjectured by Behzad [2] and Vizing [17] that for any graph $\chi'' \leq \Delta + 2$. The conjecture was verified for $\Delta \geq 9$ by Borodin [5].

As planar graphs have a lot of structure to work with, the total chromatic number can be calculated precisely for many planar graphs. As shown by Borodin, Kostochka and Woodall [4] restricting the girth can be helpful to determining the exact total chromatic number.

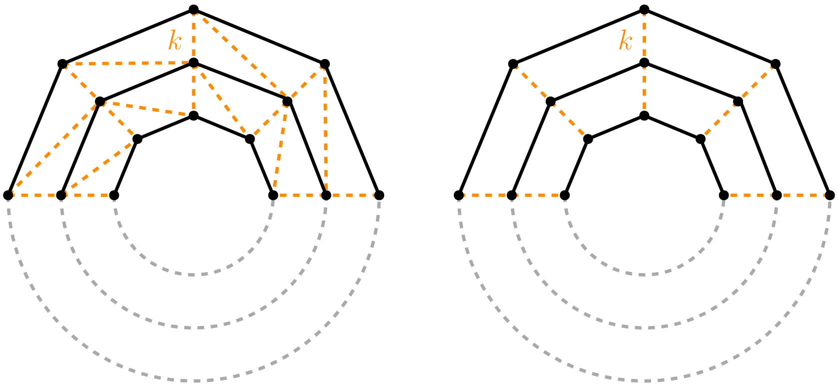

Figure 24: A construction of a gpm graph with a maximum number of edges and $g = 2k + 1$ (left). Construction for $g = 2k + 2$ (right). The concentric cycles all have length g.

5.3 Extremal Version of GPM

Girth-planar maximal graphs can also be considered in an extremal setting. Here the goal is to determine the maximum number of edges an arbitrary girth-planar maximal graph on n vertices can have. This number is given by Euler's Formula through

$$\|G\| = \left\lfloor \frac{g}{g-2}(n-2) \right\rfloor =: \mathrm{ex}(n, \text{gpm-}g), \;\; G \in \mathrm{EX}(n, \text{gpm-}g)$$

where $\mathrm{EX}(n, \text{gpm-}g)$ is the set of extremal gpm-g graphs, i.e. the gpm graphs that have exactly the maximum number of edges. As shown by Fernándes, Sieger and Tait [7] this bound is achieved by constructing plane graphs of girth g with each face also having length g, see Figure 24. For parity reasons, they only considered the cases where the above bound does not need the floor function to yield a natural number.

6 Conclusion

This thesis discussed bounds on the maximum face length for girth-planar maximal graphs. In particular lower and upper bounds as seen in Proposition 4.3 and Theorem 4.13. Additionally, several smaller results were stated considering tight bounds on the face length for girths 3 to 6 in Lemma 4.6 and 4.7 but also for outerplanar graphs (Corollary 4.8) and wheels (Proposition 4.16). Here the result was even stronger stating that girth-planar maximal outerplanar graphs are exactly the cycles of length g to $2g-3$.

Some questions regarding the faces of girth-planar maximal graphs still remain open. It was shown in Theorem 4.9 and 4.10 that a tight characterization for girth-planar maximality is not possible if no further restrictions are imposed on the graph. However, it might be possible to identify the graph classes where such a characterization still holds, similar to outerplanar graphs.

Furthermore, the very large upper bound could be improved drastically. As proposed in Conjecture 4.14 it might even be true that the bound is linear, as no graphs are known that admit a superlinear bound.

For a more general view of the potential graphs that can be considered for girth-planar maximality, it is also interesting to look at graphs embeddable on other surfaces than the plane, like the torus or the Klein Bottle.

As an application of girth-planar maximal graphs with a maximal face length greater than $2g-3$, a $(g-1)$-separator theorem could be formulated that, together with an algorithm for finding girth-planar maximal supergraphs could be useful in the field of algorithmic graph theory.

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Declaration of Authorship

I hereby declare that I am the sole author of this bachelor thesis and that I have not used any sources other than those listed in the bibliography and identified as references. I further declare that I complied with the *Satzung des Karlsruher Instituts für Technologie zur Sicherung guter wissenschaftlicher Praxis* in its valid version.

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