Primal-Dual
Cops and Robbers

Bachelor Thesis of

Minh Tuan Ha

At the Department of Informatics
Institute of Theoretical Informatics

Reviewers:	PD Dr. Torsten Ueckerdt
	T.T.-Prof. Dr. Thomas Bläsius
Advisors:	PD Dr. Torsten Ueckerdt
	Paul Jungeblut

Time Period: 23rd May 2022 – 23rd September 2022

Statement of Authorship

Ich versichere wahrheitsgemäß, die Arbeit selbstständig verfasst, alle benutzten Hilfsmittel vollständig und genau angegeben und alles kenntlich gemacht zu haben, was aus Arbeiten anderer unverändert oder mit Abänderungen entnommen wurde sowie die Satzung des KIT zur Sicherung guter wissenschaftlicher Praxis in der jeweils gültigen Fassung beachtet zu haben.

Karlsruhe, September 23, 2022

Abstract

The game of *cops and robbers* is a pursuit and evasion game played on graphs in which both parties alternate in taking turns to move along the edges. While the cops try to catch the robber in a finite number of moves, the robber tries to evade the cops indefinitely. Contrary to the original game in which both parties move from vertex to vertex, we introduce a new variation for planar graphs with cops that advance on faces instead. In this variation, the *cop number*, denoted as $c^*(G)$, specifies how many cops are required to catch the robber, that is, by occupying all faces incident to the robber's position, even if the latter plays perfectly.

We first establish general lower and upper bounds on $c^*(G)$ for planar graphs G. From there on, we consider classes of graphs such as k-regular graphs and graphs with maximum degree $\Delta(G) \geq 5$. Here we proceed by using already proven results of the original game and applying them to our setting. Furthermore, we consider the effect of subdividing edges of graphs on the cop number $c^*(G)$. For this, we use K_4 as an example.

Deutsche Zusammenfassung

Bei dem Spiel *Cops and Robbers* handelt es sich um ein Verfolgungsspiel auf Graphen, in dem zwei Parteien sich abwechselnd auf den Kanten des Graphen fortbewegen. Das Ziel der Cops ist es, den Robber in endlich vielen Zügen zu fangen, während dieser versucht, den Cops unendlich lange zu entweichen. Im Gegensatz zu der urpsrünglichen Variante, in der sich beide Parteien von Knoten zu Knoten bewegen, untersuchen wir in dieser Arbeit eine Variante, in der sich die Cops auf den Facetten des Graphen bewegen. Hier bezeichnet $c^*(G)$ die sogenannte *Cop-Number*, die angibt, wie viele Cops benötigt werden, um einen perfekt spielenden Robber zu fangen. Dieser gilt als gefangen, wenn alle umliegenden Facetten von Cops besetzt werden.

Wir bestimmen zuerst allgemeine untere und obere Schranken für $c^*(G)$ auf beliebigen planaren Graphen G. Anschließend betrachten wir bestimmte Klassen von Graphen wie k-reguläre Graphen und Graphen mit Maximalgrad $\Delta(G) \geq 5$. Hierbei verwenden wir bereits bekannte Resultate aus der ursprünglichen Variante und passen diese an unser Szenario an. Des Weiteren untersuchen wir den Effekt, den Unterteilungen der Kanten auf die Cop-Number $c^*(G)$ haben. Dabei betrachten wir K_4 als spezifisches Beispiel.

Contents

1. Introduction

In mathematics and computer science, certain researchers are interested in the study of games. These games are used to model situations with multiple adversaries having conflicting interests. One particular category of those games are pursuit and evasion games. Researchers studying these are motivated by the fact that pursuit and evasion games model several real-world scenarios, such as cyber attacks [DN03]. In various cases, the existence of an intruder on a given network is considered. A set of pursuers is then tasked with capturing or eliminating the intruder. Modeling this scenario in a turn-based framework on a structure such as a graph results in many useful insights.

In 1983 Nowakowski & Winkler [NW83] and in 1978 Quillot [Qui78] independently introduced a pursuit and evasion game known as "Cops and Robbers" played on a graph G. In this game, the player controlling the cops chooses starting positions on vertices of G. After that, the player in control of the robber chooses a starting vertex. The cops and robber then move in alternate turns, with the cops moving on odd turns and the robber moving on even turns. A round of the game consists of the cops' turn and the robber's subsequent turn. During every turn, each cop or the robber either moves along an edge of G to a neighboring vertex or remains on their current vertex. Both players play with complete information, meaning that they each know each other's position at any time. The cop's objective is to catch the robber by moving onto the robber's position.

Let us consider the game played on the cycle graph shown in Figure 1.1 with one cop and a robber in play. The first player chooses one of the eight vertices to place the cop on, followed by the second player choosing the starting vertex of the robber. As the graph is symmetrical, let us assume that the cop starts on the bottom right vertex. Then it stands to reason that the robber would start on the opposite vertex, the one at the top left (Figure 1.2), to be located as far away from the cop as possible. Now the game commences with the cop's turn who can choose to move either left or right to a neighboring vertex or stand still. The cop cannot move directly onto the robber's position as there is no edge connecting the two vertices and it would therefore be an illegal move. If the cop decides to stand still, the robber may also stay put. If the cop moves in clockwise or counterclockwise rotation, the robber can choose to do the same. By doing this, the robber manages to keep the distance of 3 vertices between them after every round and therefore has a winning strategy.

Nevertheless, if the robber plays sub-optimally, for example by staying put on every turn, then one cop can be sufficient to capture the robber. We assume, however, that both parties play perfectly and consider under these circumstances, how many cops are required to catch the robber even if the latter plays optimally. In this case, that would be two cops that move in opposing directions (Figure 1.3). Hence, we have a winning strategy for the cops if two cops are in play.

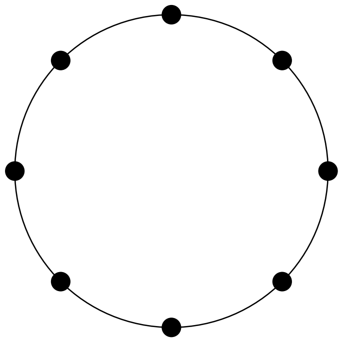

Figure 1.1: Cycle with 8 vertices.

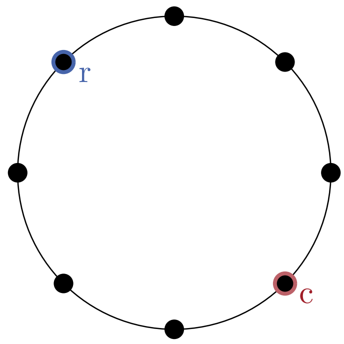

Figure 1.2: Winning strategy for robber with one cop in play.

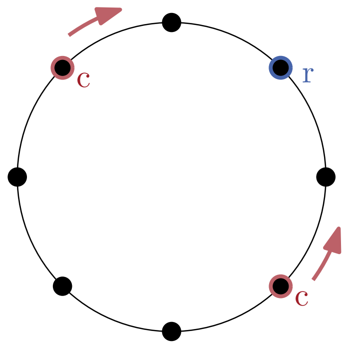

Figure 1.3: Winning strategy for cops with two cops in play.

Given an arbitrary graph G the following question arises: Do the cops have a winning strategy or can the robber evade the cops indefinitely? If only one cop is in play, we call G a *cop-win* graph if there is a winning strategy for the cop. Otherwise, we call G a *robber-win* graph if the robber has a strategy to never get caught. The cycle in (Figure 1.1) is one such example. When we consider multiple cops chasing the robber, the question becomes: How many cops suffice to have a winning strategy for the cops? This number is then called the *cop number* of G and denoted as $c(G)$. In our example, it would be $c(G) = 2$. This notion was first introduced in the paper by Aigner and Fromme in 1984 [AF84].

Without surprise, determining the cop number of any arbitrarily given graph turns out to be difficult. To be precise, it is considered EXPTIME-complete to determine whether $c(G) = k$ with G and k as inputs [Kin13]. For that reason, it is interesting to see whether one is able to determine or at least bound the cop number for specific sets or classes of graphs with a certain structure by smartly exploiting characteristics of that structure. As an example, showing that any tree is a cop-win graph is very easy: Let the cop start on a root of the tree and move down the distinctive path to the robber. As that is a winning strategy for the cop, trees have cop number 1.

In general, cop-win graphs have a very nice characterization due to the following observation about the robber's position right before his last move: If the robber is unable to prevent the cop from capturing him regardless of what the robber does, it means that the robber's vertex and all neighboring vertices are adjacent to the cop's position.

A vertex v is called a *corner* if there exists another vertex w such that w is adjacent to v and all of v's neighbors. A graph G is *dismantlable* if it can be reduced to K_1, the graph with only one vertex, by removing corners successively. Nowakowski & Winkler [NW83] have shown that cop-win graphs are the same class of graphs as dismantlable graphs. This characterization of dismantlable graphs allows cop-win graphs to be recognized in polynomial time. Trees belong to this class, as leaves in trees are corners and removing a leaf from a tree results in another tree. Other examples of dismantlable graphs are finite chordal graphs and graphs with a universal vertex, such as wheel graphs and complete graphs.

An obvious upper bound for the cop number would be the number of vertices n in G. Placing a cop on every vertex unquestionably leads to the robber's capture. However, the most well-known, yet unproven upper bound is known as *Meyniel's Conjecture*:

Conjecture 1.1 (Meyniel's Conjecture [Fra87]). *For any connected graph G with n vertices, we have that* $c(G) = \mathcal{O}(\sqrt{n})$.

Although this conjecture has first appeared in 1987 [Fra87] and made its appearance in many books [BN11, Bon16], it still remains unproven. What has been shown, however, is that the cop number is at most $\mathcal{O}\left(\frac{n \log \log n}{\log n}\right)$ [Fra87]. Later, this bound got first improved to $\mathcal{O}\left(\frac{n}{\log n}\right)$ [Chi08] and later on to $\mathcal{O}\left(\frac{n}{2^{(1-o(1))\sqrt{\log n}}}\right)$ [LP12]. It has also been proven that there exists an infinite family of graphs with $c(G) > \sqrt{n/8}$ and even $c(G) > \sqrt{n/2} - n^{0.2625}$ for n sufficiently large [Pra10].

In contrast to that, for planar graphs we know the answer with certainty. The result was published by Aigner and Fromme [AF84], stating that the cop number for any planar graph is at most 3. We reproduce said result in Section 4.2. As there exist planar graphs with cop number 3, such as the dodecahedral graph [AF84], this is a tight bound. Results of similar nature can be found for other classes of graphs [CM12].

1.1 Variants of the Game

The research area has been largely expanded around the idea of modifying the way in which the cops or the robber move and analyzing how these changes affect the strategies and outcome of the game, and with that, the cop number of graphs as well. This approach has been used to get a better insight into Meyniel's Conjecture.

A well-known variant was introduced by Seymour and Thomas [ST93] called "helicopter cops and robbers". In their version the robber is allowed to move at great speed to any other vertex along a path of the graph and cops can fly via helicopters to arbitrary vertices with some delay, giving the robber a chance to escape before a helicopter lands. In this version, two cops are necessary to catch the robber on a tree. For cycles the cop number becomes three. In fact, Seymour and Thomas came to the conclusion that in the helicopter variant $tw(G) + 1$ cops are necessary to catch the robber. As the treewidth $tw(G)$ of (planar) graphs is unbounded, the helicopter cop number is unbounded as well. One such example is the $n \times n$ grid graph G_n with treewidth $tw(G_n) = n$.

Another variation specifies how many cops must move each turn, how many must remain in the same position, and how many can do either [OO14]. In this variation, the variant they named "one-active-cop game", the one that only allows one cop to move each turn, has received the most attention. Later on, this variation was introduced by many others with different names, such as "lazy cops and robbers" [BBKP15, BBKP16, STW16] and "one-cop-moves game" [GY17]. Here, it has been shown that there exist planar graphs in which three lazy cops are not enough to catch the robber [GY17], contrary to the result of the original game on planar graphs. The graph constructed in [GY17] has over 300 000 vertices and shows that at least four lazy cops are necessary. An upper bound for the lazy cop variant remains to be found, apart from the simple upper bound we get from the planar separators, which we introduce in Section 3.2.

Apart from that, a variation also known as "active cops and robbers" that forces players to move every turn was introduced [AF84, NN98, BW00, GKS18]. Furthermore, there are variants that demand the cops to occupy all surrounding vertices to catch the robber [BCC+20, Sch22], add a decoy into the game [Isl14, Des16], or consider a fast robber that can move faster than the cops [FGK+10]. Other variants of the game that add either advantages or restrictions to the players include forcing a player to move randomly in a graph [KW13], forcing them to move along geodesic paths [FHMP16], allowing the cops to capture the robber at a distance [BCP10], giving the cops limited visibility [IKK06, CCD+17], letting the robber's position be known to the cops only through alarms and photo radars [CN00, CC06], allowing the robber to be invisible [DDTY13], and many others.

1.2 Primal-Dual Cops and Robbers

In this thesis, we consider a new modified version of the game for planar embedded graphs. Instead of moving on the vertices like the robber, we consider cops that move on the faces. Similar to the robber that can either stay still or move to a neighboring vertex, cops can either stay still on their current face or move to a neighboring face, that is a face that shares an edge with the current face. The robber is caught if all incident faces to the robber's position are occupied by cops. With this capture rule, we conclude a straightforward lower bound on the cop number in Section 3.1.

Definition 1.2. *We denote the cop number in our primal-dual variant as $c^*(G)$ for a planar embedded graph G.*

We continue with an introductory example.

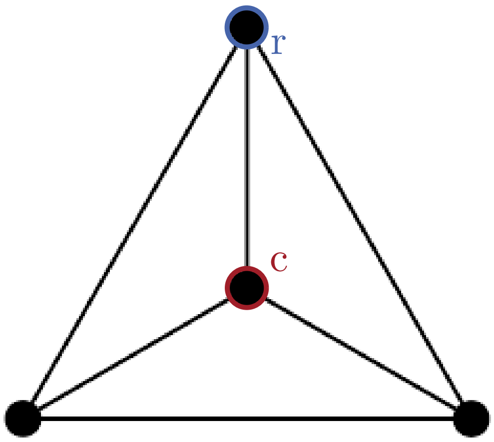

Figure 1.4: Starting position.

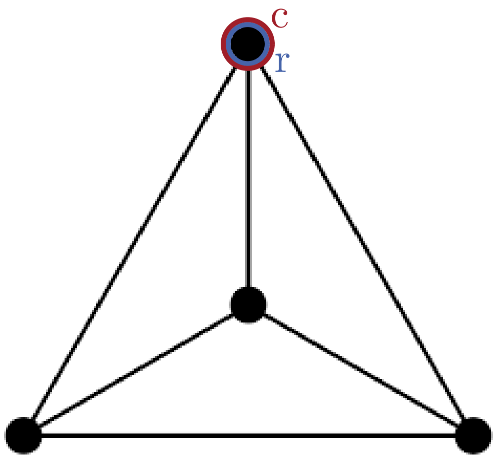

Figure 1.5: Robber is caught.

Let us assume that we play on K_4, the complete graph with four vertices (Figure 1.4). In the original game, the player controlling the cop may choose any vertex to start on. After that, the second player chooses a vertex to place the robber on. Let the cop be placed in the inner vertex (Figure 1.4). Then, no matter which vertex the robber starts on, the cop either wins immediately (if the robber chooses the same vertex) or wins in one move by moving onto the robber's position (Figure 1.5). This is a winning strategy for the cop and since we only needed one cop, the cop number of K_4 is $c(K_4) = 1$.

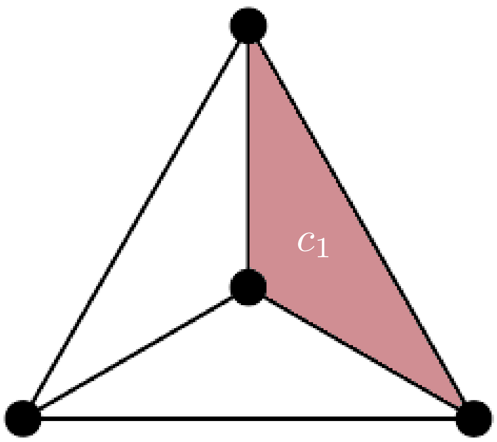

Figure 1.6: Cop on face.

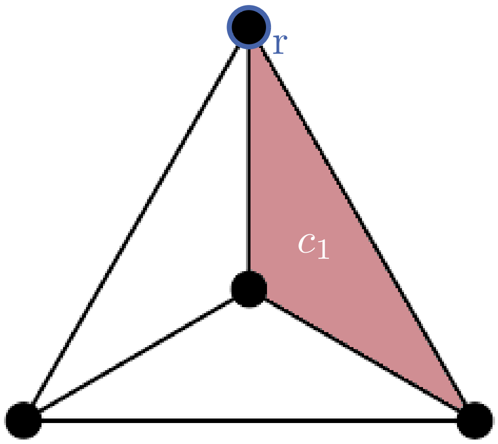

Figure 1.7: Robber cannot be caught by a single cop.

Now we consider the variation in which the cops are placed and played on the faces of K_4 instead of the vertices (Figure 1.6). Here, we quickly realize that one cop isn't sufficient to capture the robber. Due to the capture rule requiring all incident faces to the robber's position to be occupied by cops, a single cop cannot capture the robber by himself in this graph. As every vertex in K_4 has degree 3, no matter which vertex the robber is on, there is no winning strategy for the cop (Figure 1.7).

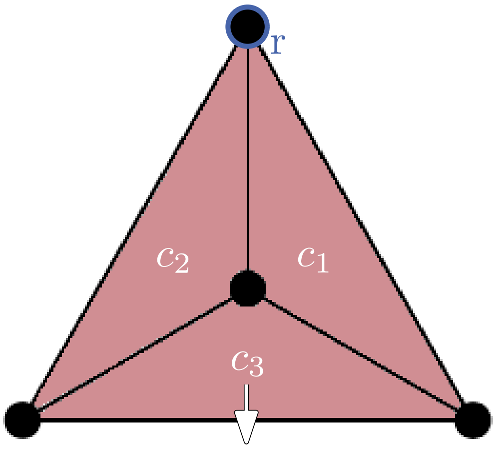

Figure 1.8: Three cops occupying the inner faces.

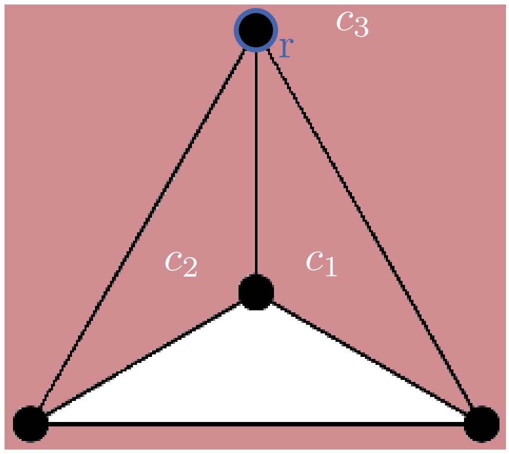

Figure 1.9: Robber is caught after one turn.

It turns out that we need exactly three cops. Let the cops begin on the inner three faces. If the robber chooses to start on the inner vertex, he is immediately caught. As K_4 is symmetrical, we can assume that the robber starts on the vertex at the top (Figure 1.8). Then in the next turn, cop c_3 can move to the outer face and with that, all faces incident to the robber are occupied (Figure 1.9). As this is a winning strategy, we have $c^*(K_4) = 3$.

1.3 Outline of this Thesis

In the next Chapter 2 we continue with definitions for technical terms and a formal introduction of the primal-dual variant of the game.

Chapter 3 begins with rough bounds that are rather obvious, some of which we have already hinted at. We first establish a lower bound based on the capture rule and later an upper bound with the help of planar separators.

The following Chapter 4 attempts to narrow these bounds down and provides insights for the cop number $c^*(G)$ depending on the highest vertex degree $\Delta(G)$. The main result of this chapter shows that the cop number $c^*(G)$ of planar graphs G with $\Delta(G) \geq 5$ is unbounded, but bounded for graphs with $\Delta(G) < 5$.

In Chapter 5 we consider subdividing edges of a graph and its effects on the cop number $c^*(G)$. We specifically use K_4 as an example.

Chapter 6 introduces other variants of this primal-dual variant and suggests further ideas for more research.

2. Preliminaries

In this chapter we establish some assumptions and notations in this thesis. We also introduce some of the basic concepts used throughout the thesis before we delve further into the game.

2.1 Assumptions and Notations

Unless specified otherwise, we consider undirected connected finite planar graphs $G = (V, E)$ in this thesis. V is the set of *vertices* or *nodes* of G and $E \subseteq V \times V$ is the set of *edges*, using the notation $E = \{(u, v) \mid u, v \in V\}$ for edges. As we consider undirected graphs, (u, v) refers to the same edge as (v, u). We may use $V(G)$ and $E(G)$ to denote the vertex or edge set of a graph G.

For a given graph $G = (V, E)$ and vertex $v \in V$, we define $G - v$ as the resulting graph by removing v from V and all edges incident to v. To be precise, $V(G - v) = V(G)/\{v\}$ and $E(G - v) = \{(x, y) \in E(G) \mid x \neq v, y \neq v\}$.

2.2 Planar Graph

An undirected graph $G = (V, E)$ is *planar* if there exists an *embedding* of the graph in the plane such that the edges intersect each other only at the endpoints. With a drawing, the plane is then divided into *inner faces* and one *outer face*. Any face can be the outer face with a different drawing of the same graph embedding.

2.3 Dual Graph

Given a connected, planar embedded graph $G = (V, E)$, we construct its dual graph $G^* = (V^*, E^*)$ in the following way:

- For each face f in G we have a vertex v_f in G^*.
- For each edge e in E we have an edge e^* in E^* that connects the corresponding vertices of the two faces incident to e.

The graph G is also referred to as the primal graph in this case. With this construction, G^* again is the embedding of a planar graph and we have $G^{**} = G$. Note that given a (primal) robber graph G, the cops in our variant of the game are then moving on the vertices of the dual graph G^*.

2.4 Euler's Formula

Euler's Formula gives us a correlation between the number of vertices of a graph G and the number of vertices of its dual graph G^*.

Theorem 2.1 (Euler's Formula)**.** *For a connected, planar embedded graph G with n vertices, m edges and f faces we have $n - m + f = 2$.*

Proof. We rewrite the equation to $m - (f-1) \stackrel{!}{=} n - 1$ and prove it by induction over the amount of inner faces $f - 1$.

Basis: Let $f - 1 = 0$.
Then G is a tree (connected and circle-free) and we have $m = n - 1$ and therefore $m - (f - 1) = n - 1$.

Induction: Let $f - 1 \geq 1$ (at least one inner face).
Let e be an edge between an inner face and the outer face. Consider $G' = G - e$ with n', m' and f'. Then we have $n' = n$, $m' = m - 1$ and $f' = f - 1$. As G' is still connected, by using the induction step on G' we get $m - (f - 1) = m' - (f' - 1) = n' - 1 = n - 1$. □

2.5 Formal Definition of Primal-Dual Cops and Robbers

To clear any possible remaining unclarities regarding our variant of the game, we give a mathematical definition of "primal-dual cops and robbers" for a planar embedded graph G.

The game is played on a planar embedded graph G by two parties, the cops and the robber. A round consists of the cops' turn followed by the robber's turn.

Let the position of the k cops in round i be denoted as a k-tupel

$$\text{pos}_c^i = (f_1, \ldots, f_k) \in V(G^*)^k$$

and the position of the robber as a single vertex $\text{pos}_r^i \in V(G)$.

Then we define a configuration on the i'th round as

$$\text{conf}^i = (\text{pos}_c^i, \text{pos}_r^i) \in V(G^*)^k \times V(G) =: \text{Conf}.$$

A cop strategy consists of a starting position pos_c^0 and a function $\lambda : \text{Conf} \to V(G^*)^k$ with

$$\lambda(\text{pos}_c^i, \text{pos}_r^i) = \lambda((f_1, f_2, \ldots, f_k), v) = (f_1', f_2', \ldots, f_k') = \text{pos}_c^{i+1}.$$

A cop turn with a mapping λ is considered a legal move when

$$(f_i, f_i') \in E(G^*) \cup \{(f, f) \mid f \in V(G^*)\} \; \forall i \in \{1, \ldots, k\}.$$

A robber's sequence of positions is similarly considered legal if

$$(\text{pos}_c^i, \text{pos}_c^{i+1}) \in E(G) \cup \{(v, v) \mid v \in V(G)\} \; \forall i \in \mathbb{N}_0.$$

With this, $(\text{pos}_c^0, \lambda)$ is considered a *winning strategy* for the cops, if for any starting position and moving sequence $(\text{pos}_r^0, \text{pos}_r^1, \ldots)$ of the robber, the latter is caught after a finite number of turns.

The robber is considered caught when we reach a configuration conf^i such that the robber is surrounded by cops after the cops' turn, that is when all incident faces $f_1, f_2, \ldots, f_{\deg(v)}$ of v are in pos_c^i. Note that $f_i = f_j$ with $i \neq j$ is possible.

The *cop number* $c^*(G)$ is then defined as the smallest k such that there exists a winning strategy for the cops. This cop number is well defined as occupying all faces guarantees the robber's capture and therefore $c^*(G) \leq 2 - |V(G)| + |E(G)|$ (see Theorem 2.1).

3. Simple Bounds

In this chapter we start with obvious bounds for the cop number $c^*(G)$ for any planar embedded graph G. In later chapters we try to narrow these bounds down.

3.1 Lower Bound

As already mentioned in Section 1.2, we get a lower bound due to the surrounding rule, as we require all faces incident to the robber's vertex to be occupied. This suggests that at least $\Delta(G)$ cops are necessary, as the robber may just sit on the vertex with the highest degree and never get caught. However, some edges might hold the same face on both sides. Such an edge is either an edge incident to a vertex with degree one or, if one were to remove this edge from the graph, the graph would no longer be connected. This means that only as many cops are required as the amount of unique faces the robber's vertex is incident to. If all vertices have at least one such edge, it may result in $c^*(G) < \Delta(G)$.

Definition 3.1. *Let G be a planar embedded graph. Then we define $\nabla(G)$ as the highest amount of unique faces incident to the same vertex.*

Another way to describe $\nabla(G)$ is the maximum amount of incident vertices a face in the dual graph G^* has.

Theorem 3.2. *For a planar embedded graph G we have $c^*(G) \geq \nabla(G)$.*

Note that we always have $\nabla(G) \leq \Delta(G)$. In the upcoming chapters we presume $\nabla(G) = \Delta(G)$ as this leads to the worst case for the cops in regards to $c^*(G)$ and $c^*(G) < \Delta(G)$ only occurs when every vertex in G has at least one edge that has the same face on both sides, such as in trees.

3.2 Upper Bound

Similar to the upper bound of $c(G) \leq n$ for an arbitrary graph G with $|V(G)| = n$ vertices in the original cops and robbers game, here in the primal-dual variant we can also derive an upper bound by placing a cop on every face of a planar embedded graph G.

Theorem 3.3. *For a planar embedded graph G we have $c^*(G) \leq 2 - |V(G)| + |E(G)|$.*

Proof. The equation $f = 2 - |V(G)| + |E(G)|$ is a direct conclusion of Theorem 2.1. By occupying all faces of the graph with cops, every vertex has all its incident faces occupied by a cop. □

With the planar separator theorem we lay the foundation for the circle separator that leads to a different upper bound for the cop number.

Theorem 3.4 (Planar Separator Theorem [LT79])**.** *Every planar graph G with $|V(G)| \geq 5$ has a $\frac{2}{3}$-balanced separator of size $\mathcal{O}(\sqrt{n})$. This separator can be found in $\mathcal{O}(n)$.*

A separator S is a subset of $V(G)$ and partitions the graph into two subsets A and B such that there are no edges (a, b) in $E(G)$ with $a \in A$ and $b \in B$. $\frac{2}{3}$-balanced indicates that both A and B have at most $\frac{2}{3}n$ vertices. By placing cops on all faces incident to the vertices in S, which requires at most $\Delta(G) \cdot \mathcal{O}(\sqrt{n})$ cops, we prevent the robber from ever changing his position from inside A to B or vice-versa.

Theorem 3.5 (Cycle Separator [MTTV97])**.** *For $\alpha \geq \frac{3}{4}$ and a triangular, planar graph G there exists an α-balanced cycle separator with $\mathcal{O}(\sqrt{n})$ vertices. This cycle can be found in $\mathcal{O}(n)$.*

As we can add edges to any planar graph G to turn it into a triangular planar graph G', we can find a cycle separator C of size $\mathcal{O}(\sqrt{n})$ in G'. As G and G' share the same vertices, by placing cops in G on all faces incident to the vertices in C we prevent the robber from ever leaving or entering the inner area of the cycle.

Now using the cycle separator we show that:

Theorem 3.6. *For a planar embedded graph G we have $c^*(G) \leq \mathcal{O}(\sqrt{n})$.*

Proof. We are going to use three cycles c_i to repeatedly shrink the robber's area down until there is only one vertex left for the robber. A cycle is controlled by assigning $\Delta(G)$ cops to each vertex in the cycle and occupying all incident faces. We use Theorem 3.5 to determine the first cop cycle to control. This partitions the graph into two subgraphs A and B. Without loss of generality, let the robber be contained in A. Then, applying Theorem 3.5 on A yields us a second cycle for the cops to control. This again partitions the area into two subgraphs A_1 and A_2. Without loss of generality, let the robber be contained in A_1. Using the theorem on $A1$ gives us a third cycle. Now, the robber is contained by two cycles. By freeing the cops of the other cycle and forming a new cycle by applying the theorem on the robber's subgraph, we further narrow the space down in which the robber can move, until there is only one vertex left and the robber is caught. As cycle separators are balanced, we guarentee that with each newly formed cycle the robber's area becomes smaller and as each cycle contains at most $\mathcal{O}(\sqrt{n})$ vertices and subsequently formed cycles contain less vertices, we require $3 \cdot \Delta(G) \cdot \mathcal{O}(\sqrt{n}) = \mathcal{O}(\sqrt{n})$ cops in total. □

3.3 Simple Results

Now we consider simple classes of graphs in regards to their cop number $c^*(G)$.

Trees T only have one face and therefore cop number $c^*(T) = 1$. Here, we have the special case of $c^*(T) = \nabla(T) \leq \Delta(T)$ (Theorem 3.2).

Cycles C have cop number $c^*(C) = 2$ as there are two faces in total that every vertex is also incident to. This conforms to Theorem 3.2 and Theorem 3.3.

With this, we have covered 2-regular (connected) graphs. In the next chapter we study regular graphs of higher degree.

4. k-Regular Graphs

In this chapter we consider planar k-regular graphs. A graph G is k-regular when all vertices have degree k. By taking advantage of this property we find cop strategies and determine bounds for the primal-dual cop number $c^*(G)$.

4.1 3-Regular Graphs

We begin with 3-regular graphs. Here, we reveal that cops are able to cover a distance to a target vertex faster than a robber and thus can chase a robber down.

Theorem 4.1. *Let G be a planar embedded, 3-regular graph. Then $c^*(G) = 3 = \Delta(G)$.*

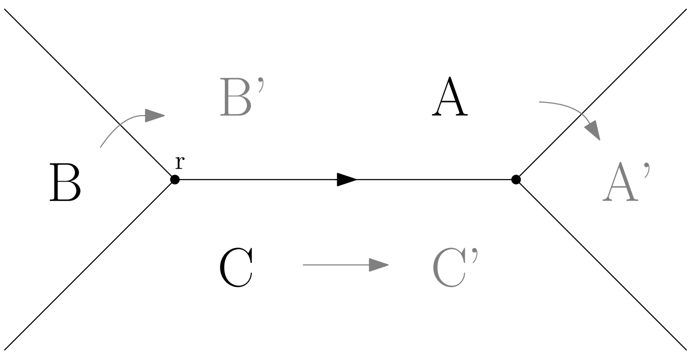

Figure 4.1: One cop comes closer to the robber with each move.

Proof. Let r be the vertex occupied by the robber and A, B, C be the incident faces. Let $d_i(f)$ denote the distance between cop_i and face f, namely the minimal amount of steps cop_i needs to take to reach face f. Let d be the total number of steps the three cops need to take in order to occupy A, B and C. Then let cop_1 go to A, cop_2 to B, cop_3 to C and we get

$$d = d_1(A) + d_2(B) + d_3(C).$$

After the robber moves to an adjacent vertex with incident faces $A', B' = A$ and $C' = C$ (Figure 4.1) we have

$$d_1(A') \leq d_1(A) + 1 \text{ and } d_2(B') \leq d_2(B) + 1$$

and therefore

$$d_{after} := d_1(A') + d_2(B') + d_3(C') \leq d_1(A) + 1 + d_2(B) + 1 + d_3(C) = d + 2.$$

As each cop can now take a step we get

$$d_{new} = d_{after} - 3 \leq d + 2 - 3 = d - 1.$$

Hence, the new total number of steps d_{new} the cops need to take decreases each turn by at least 1. If one cop has already reached a face incident to the robber's vertex, he can assume the role of the cop on A and move along while letting one of the other two cops move towards face C. Therefore, the total distance until capture becomes 0 after a finite number of moves and the game is over. □

Note that this proof does not use the properties of a plane and can therefore be applied to graphs embedded on surfaces of arbitrary genus such as Möbius strips and Klein bottles as well.

4.2 4-Regular Graphs

Now we consider 4-regular graphs where cops are not able to cover a distance to a target vertex faster than a robber anymore. In this section, we refer to our cops as *face-cops* and to cops that can move on vertices like in the original cops and robbers version as *vertex-cops*.

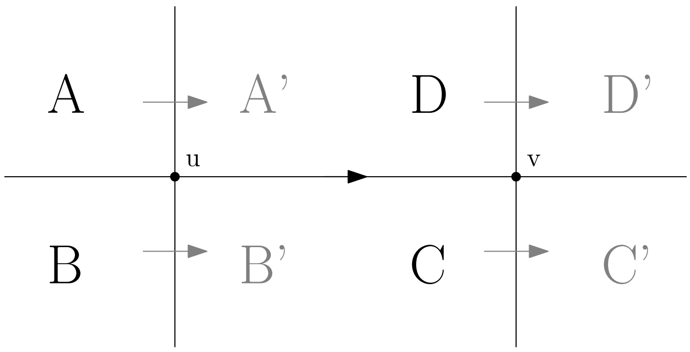

Figure 4.2: Four face-cops simulating a vertex-cop that can move on vertices.

Lemma 4.2. *Let G be a planar embedded, 4-regular graph. Then four face-cops can simulate one vertex-cop. Therefore, $c^*(G) \leq 4c(G)$.*

Proof. Let four face-cops occupy the four faces A, B, C, D surrounding a vertex u as shown in Figure 4.2. Then the robber would be caught if he is on u, just as he would be when a vertex-cop would occupy the same vertex. If the vertex-cop would want to move from u to an adjacent vertex v, the four face-cops can move to the surrounding faces A', B', C', D' of v with $A' = D$ and $B' = C$. □

A special case of planar 4-regular graphs are square-grid graphs. The vertices of these graphs correspond to the points in the plane with integer coordinates and two vertices are connected by an edge whenever the corresponding points are 1 distance apart from each other. We show that two vertex-cops suffice to catch the robber and therefore we have:

Theorem 4.3. *Let G be a planar embedded square-grid graph. Then $c^*(G) \leq 8$.*

Proof. By showing that two vertex-cops have a winning strategy in the original game, we conclude with Lemma 4.2 that eight face-cops suffice to catch the robber in the primal-dual variant of cops and robbers. As the vertices in G correspond to points with integer coordinates, let (x, y) refer to the vertex that corresponds to the point with such coordinates. For G with $n \times n$ vertices, let $(1, 1)$ refer to the vertex in the bottom left and (n, n) refer to the vertex at the top right. A winning strategy for the cops begins by placing both cops on $(1, 1)$. The strategy is to let one cop, cop_1, keep the same x-coordinate and the other cop, cop_2, keep the same y-coordinate as the robber while reducing the distance. Let (r_x, r_y) be the position of the robber and (cop_{1_x}, cop_{1_y}) or (cop_{2_x}, cop_{2_y}) be the position of cop_1 and cop_2 respectively. Cop_1 moves right until $cop_{1_x} = r_x$ and then continues to maintain that while cop_2 moves up and does the same with $cop_{2_y} = r_y$. As the board is finite, meaning that r_x, r_y are in $[1, n]$, the robber cannot prevent this. Now the cops act in the following way depending on the direction the robber moves:

- Left: Let cop_1 also move left and let cop_2 move right. Then the distance of cop_1 to the robber remains the same while cop_2 reduces the distance to the robber by 2 steps.
- Down: Let cop_1 move up and let cop_2 also move down. Then cop_1 reduces the distance by 2 while cop_2 maintains the distance before the robber moved.
- Right or up: Let both cops move in the same direction. Then both cops keep the same distance to the robber they had before the robber moved.

As the graph is finite, the robber can only go right and up a finite amount of times until he has to go left or down again. Therefore, the distance between the cops and the robber decreases steadily until the robber is eventually caught. □

Now we consider 4-regular graphs in general. Here, the paper from Aigner and Fromme [AF84] provides the result that three vertex cops suffice to catch the robber on any planar graph. For the sake of completeness the proof for $c(G) \leq 3$ is described here:

We begin by showing that a single vertex-cop can guard a geodesic path by himself. This is necessary in order to show that 3 vertex-cops and therefore 12 face-cops have a winning strategy on any planar graph G.

Lemma 4.4 ([AF84])**.** *Let G be a planar, 4-regular graph and P a shortest path between two different vertices u and v on G. Then four cops can, after a finite number of moves, prevent the robber R from entering P. That means, R will be immediately caught if he enters P.*

Proof. Let $d(x, y)$ denote the length of a shortest path between x and y. Let r be the vertex occupied by the robber R and c the vertex controlled by the four cops as in Figure 4.2. Suppose that for all $z \in V(P)$ it holds that

$$d(r, z) \geq d(c, z). \tag{4.1}$$

Then the cops can preserve this condition, meaning that the robber will not be able to enter P without being caught:

After the robber moves from vertex r to an adjacent vertex s we have for each vertex z on P

$$d(s, z) \geq d(r, z) - 1 \overset{4.1}{\geq} d(c, z) - 1.$$

Therefore, the cops can preserve the condition from equation (4.1) by moving in the right direction if necessary. For any z_0 in $V(P)$ with

$$d(s, z_0) = d(c, z_0) - 1 \tag{4.2}$$

the cops need to control the vertex one step closer to z_0 in comparison to c. So four cops can control a path as long as there are no two different vertices as in (4.2) on opposite sides of c on path P.

We show that there exist no such two vertices by contradiction. Suppose that there exist x, y in $V(P)$ with x and y being on opposite sides of c and

$$\begin{aligned} d(s,x) = d(c,x) - 1 \text{ and } d(s,y) \leq d(c,y) \text{ or} \\ d(s,y) = d(c,y) - 1 \text{ and } d(s,x) \leq d(c,x). \end{aligned} \tag{4.3}$$

This leads to the contradiction

$$d(x,y) \overset{(a)}{\leq} d(s,x) + d(s,y) \overset{(4.3)}{\leq} d(c,x) + d(c,y) - 1 \overset{(b)}{=} d(x,y) - 1 \tag{4.4}$$

due to the (a) triangle inequality and the (b) minimality of P. Therefore, such $x, y \in V(P)$ cannot exist.

It remains to show that the cops can reach the situation described in (4.1). As shown in (4.4), $d(r,z) < d(c,z)$ can only be valid for $z \in V(P)$ on a single side of c. By moving towards that side the cops force the situation after a finite number of moves. □

Now we show that 3 vertex-cops and therefore 12 face-cops suffice to catch the robber.

Theorem 4.5 ([AF84]). *Let G be a planar, 4-regular graph. Then c*(G) ≤ 12.*

Proof. We divide the game rounds into stages. In each stage i we assign an area R_i to the robber that he can still enter without being caught. At each subsequent stage, this area becomes smaller until the robber is eventually caught. In other words, $R_{i+1} \subsetneq R_i$ after a finite number of moves. We start with twelve cops controlling a vertex c as in Lemma 4.2 and let r_0 be the vertex occupied by the robber. Then R_0 is $G - c$ and at any stage of the game we have one of the two following situations:

a) Four cops control vertex u, R is on r and R_i is the section of $G - u$ containing r. This is also our starting situation.

b) $P1$ and $P2$ are two shortest paths between vertices u and v with $|P1|, |P2| \geq 1$ and disjoint except for u and v. With a planar embedding of G we get an interior region I and exterior region E formed by the two paths. Without loss of generality let r be in E. P1 is a shortest path in $P1 \cup P2 \cup E$ and P2 a shortest path in $P1 \cup P2 \cup E$ that is disjoint from $P1$. Four cops control P1, four cops control P2. Then $R_i = E$.

If we are in case (a) and u only has one neighbor v in R_i then the four cops on u move to control v. If $r = v$ then the robber is caught. Otherwise, R_{i+1} is contained in $R_i - v$ and we therefore reach case (a) again with $R_{i+1} \subsetneq R_i$. If u now has two neighbors s and t in R_i and P_{st} is a shortest path between s and t in R_i, let four of the remaining eight cops control P_{st} after a finite number of moves. Then we reach case (b) with $P1 = (s, u, t)$, $P2 = P_{st}$ (or $P1 = P_{st}$ and $P2 = (s, u, t)$ if $(s, t) \in E(G)$) and $R_{i+1} \subset R_i - V(P_{st}) \subsetneq R_i$.

In case (b) if there is no other path from u to v in R_i then R_i consists of disjoint components $A, B, C, \ldots$ attached to the vertices of $P1 \cup P2$. Without loss of generality, let r be contained in A attached to vertex a. Then move the remaining four free cops to the connection point a and we end up in case (a) with $u = a, R_i = A$ (Figure 4.3).

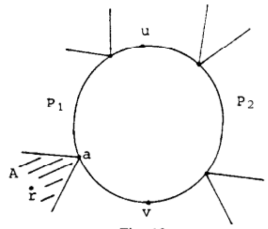

Fig. 10.

Figure 4.3: Case (b) with no other paths from u to v. From [AF84].

If there are other paths between u and v in $R_i \cup P1 \cup P2$ let Z be a shortest such path. Let $P(x, y)$ denote the subpath from x to y on P. Let w be the first vertex on Z after u which is also on $P1 \cup P2$. If $w \in V(P_1)$, then the path $P3 = Z(u, w) \cup P1(w, v)$ is also a shortest path due to P being a shortest path. $P3$ is disjoint from $P2$ and depending on how $P3$ partitions R_i we have one of the two cases in Figure 4.4.

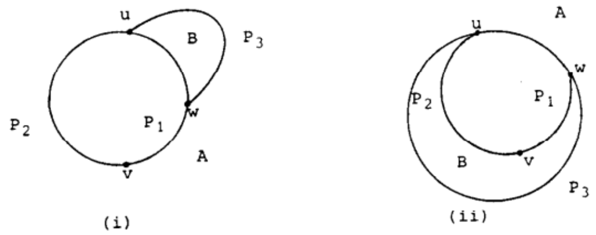

Figure 4.4: $P3$ partitions R_i with $w \in V(P_1)$. From [AF84].

If we are in case (i) and r is in A, then the last four free cops move to control $P3$. With $P2$ and $P3$ we arrive in situation (b) with $R_{i+1} \subsetneq R_i$. (There is at least one vertex on $P3(u, w)$ which is in R_i but not in R_{i+1}.) If r is in B, then the cops control $P1(u, w)$ and $P3(u, w)$ and we also arrive at case (b) with $R_{i+1} \subsetneq R_i$. Case (ii) is handled by a similar argument.

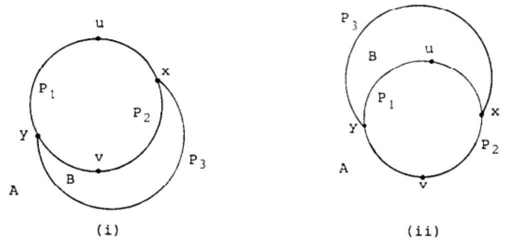

Figure 4.5: $P3$ partitions R_i with $w \in V(P2)$ and Z intersects $P1$. From [AF84].

If $w \in V(P2)$ and Z does not intersect $P1$ (except in u, v), then $P3 = Z(u, w) \cup P2(w, u)$ is another shortest path and we can repeat the argument above. Otherwise, let y be the first intersection of Z with $P1$, and x be the preceding intersection of Z with $P2$. By the minimality of $P1$ and $P2$, $P3 = P2(u, x) \cup Z(x, y) \cup P1(y, v)$ is another shortest path (Figure 4.5). Then we have one of the two situations in Figure 4.5 depending on whether $r \in A$ or $r \in B$, and the argument is as before. □

4.3 Low-Degree Graphs

Expanding on the results of Theorem 4.1 and Theorem 4.5 we show that the strategy used for 3-regular and 4-regular graphs can be applied to graphs G with $\Delta(G) \leq 3$ and $\Delta(G) \leq 4$ as well.

Corollary 4.6. *Let G be a planar embedded graph with $\Delta(G) \leq 3$. Then $c^*(G) = 3$.*

Corollary 4.7. *Let G be a planar embedded graph with $\Delta(G) = 4$. Then $c^*(G) \leq 12$.*

Proof. The transitions when moving between vertices of different degrees are described in the following.

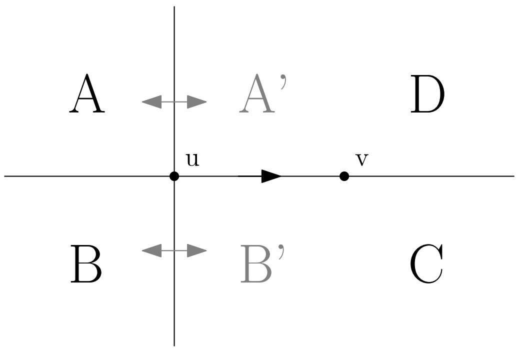

Figure 4.6: $\deg(u) = 4$ and $\deg(v) = 2$

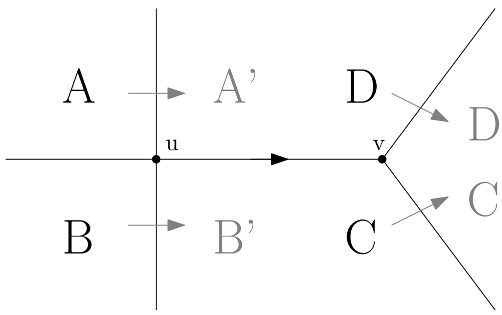

Figure 4.7: $\deg(u) = 4$ and $\deg(v) = 3$

We look at how four face-cops move in order to simulate a vertex-cop moving from a vertex u to an adjacent vertex v. For any vertex w with $\deg(w) = 2$ we uphold the invariant of having two cops on each incident face. We consider different scenarios depending on $\deg(u)$ and $\deg(v)$.

- $\deg(u) = 1$ or $\deg(v) = 1$
  Let $\deg(u) = 1$. Then the four cops are on the only face F incident to u. All faces incident to v are either F or adjacent to F and can be occupied by one of the cops in one move. As $\deg(v) \leq 4$ we have up to three unique faces that can all be covered by the four cops. Correspondingly, in the case of $\deg(v) = 1$ all cops can move to the only incident face.
- $\deg(u) = 4$ and $\deg(v) = 2$ (Figure 4.6)
  We start with four face-cops with each on their own face, of which two (C, D) are also the incident faces to v. Let the remaining two cops move to C and D and we end up with two cops on each face incident to v and thus, upholding the invariant.
- $\deg(u) = 2$ and $\deg(v) = 4$
  Assuming that we uphold the invariant, let one cop from each face move to the faces incident to v but not u.
- $\deg(u) = 4$ and $\deg(v) = 3$ (Figure 4.7)
  We start with four face-cops with each on their own face, of which two (C, D) are also the incident faces to v. Let the cops on A and B move to D and C respectively while the other two cops move to the third face incident to v.
- $\deg(u) = 3$ and $\deg(v) = 4$ (Figure 4.8)
  Each of the three faces has at least one face-cop. Regardless of which face has two cops, let one cop from face C and D each move to the faces not incident to u. Let the cop on A move to D and the last cop move to C.

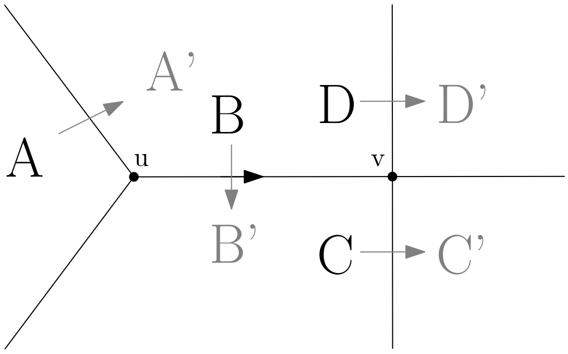

Figure 4.8: deg(u) = 3 and deg(v) = 4

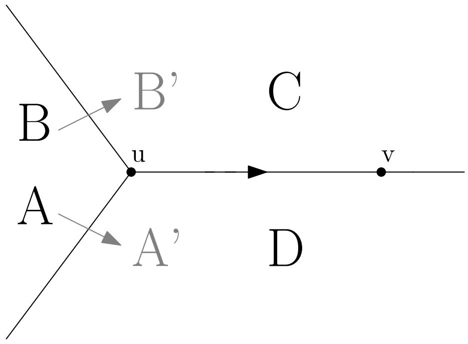

Figure 4.9: deg(u) = 3 and deg(v) = 2

- deg(u) = 3 and deg(v) = 2 (Figure 4.9)
  If there are two cops on a face incident to v, move the cop not incident to v to the other face incident to v. If there are two cops not incident to v, move one of them to each face incident to v. With this, we uphold the invariant.
- deg(u) = 2 and deg(v) = 3
  As we start with two cops in each incident face to u, move one cop to the face incident to v but not incident to u. We end up with at least one cop in each incident face of v.

So four face-cops can simulate a vertex-cop on graphs G with $\Delta(G) \leq 4$ as well. □

4.4 High-Degree Graphs

In this section we consider planar embedded graphs with $\Delta(G) \geq 5$ in general. Our approach until now does not work here anymore, as cops cannot keep pace with the robber when moving from a vertex with degree 5 to a neighboring vertex (Figure 4.10). When moving five cops surrounding a vertex with degree 5 to a neighboring vertex, while four cops manage to move to the faces incident to the new vertex, one cop does not manage to do so in one step. Therefore, face cops cannot simulate vertex cops anymore in graphs with $\Delta(G) \geq 5$.

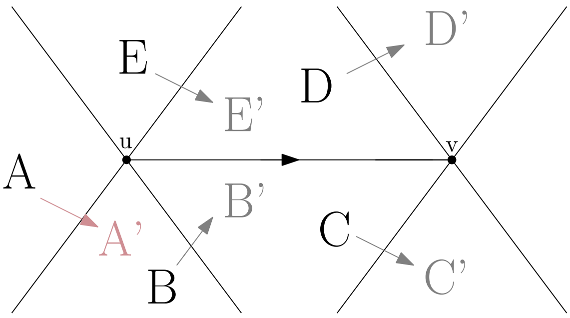

Figure 4.10: One of the five cops cannot reach a face incident to v with one step.

Instead, we use the main result in the paper of Nisse and Suchan [NS08] that proves the lower bound of the vertex-cop number in $n \times n$ square-grid graphs to be in $\Omega(\sqrt{\log n})$ when the robber has a higher speed than the cops, meaning that the robber can go a further distance than a cop in the same amount of time.

Theorem 4.8 (Nisse and Suchan [NS08])**.** *Let G_n be a square-grid graph with $n \times n$ vertices. Let p be the velocity of the vertex-cops and q be the velocity of the robber. Then for any $\frac{q}{p} > 1$ we have $c(G_n) > \Omega(\sqrt{\log n})$.*

In order to apply the theorem in our primal-dual cops and robber setting we need to artificially reduce the cops' speed.

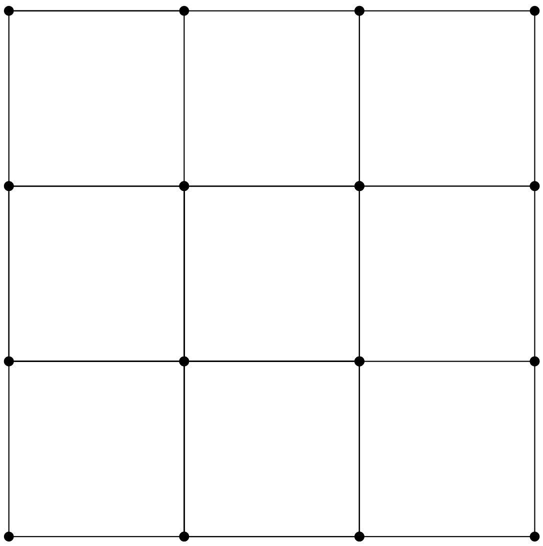

Figure 4.11: G_n

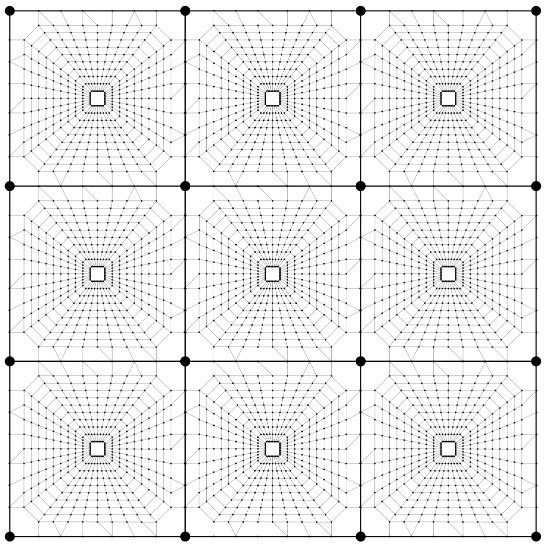

Figure 4.12: $G_{n,h}$

Let G_n be a square-grid, meaning that G_n consists of $n \times n$ vertices which we call *cities*. Each city is connected to its closest neighbors (in regard to the Euclidean metric), thus forming $(n-1)^2$ inner faces that we call *squares*. This is the graph that both robber and vertex-cops move on in the [NS08] setting (Figure 4.11). For easier intuition, we call the edges *highways*.

Let $G_{n,h}$ (with $h \geq 4$ even) be the graph that results from G_n after subdividing each highway h times. These new vertices on the highways will be called *checkpoints*. Each checkpoint will have degree 5, alternating between checkpoints that have two additional edges on one side of the highway and checkpoints that have two edges on the other side of the highway (Figure 4.12). These edges will be connected to *villages*, vertices that fill up the squares formed by G_n. Note that the total amount of vertices in $G_{n,h}$ now depends on n and h, and the way the villages are placed.

A highway is now a path $(c_0, c_1, \ldots, c_h, c_{h+1})$ with $\deg(c_0) = \deg(c_{h+1}) = 4$ (cities) and $\deg(c_i) = 5$ for $i \in \{1, \ldots, h\}$ (checkpoints). Let $N(c_i)$ be the neighborhood of c_i in counter-clockwise order. Then for any $i \in \{1, \ldots, h-1\}$ we have $N(c_i) = (c_{i-1}, v_1, v_2, c_{i+1}, v_3)$ and $N(c_{i+1}) = (c_i, v_4, c_{1+2}, v_5, v_6)$ or vice-versa, with v_i being villages. That means that checkpoints alternate between having two edges on one side of the highway and one edge on the other side and we therefore have $\frac{h}{2} \cdot 2 + \frac{h}{2} \cdot 1 = \frac{3}{2}\,h$ edges on each side of the highway.

As each side of a highway has $\frac{3}{2}\,h$ edges incident to checkpoints, each square has $4 \cdot \frac{3}{2}\,h = 6\,h$ edges incident to checkpoints.

In each square we construct villages connected to each other while ensuring two properties:

1. $\Delta(G_{n,h}) \leq 5$ and
2. A shortest path between two cities or checkpoints consists of highway edges only.

We construct the villages with nested *rings*. Each ring consists of $6h$ villages forming a cycle. A village is connected to the corresponding village on the next outer ring and the corresponding village on the next inner ring. By construction each village has degree 4 except for the villages in the inner most ring that have degree 3 (see Figure 4.13). With this Property 1 is fulfilled. By adding sufficient rings we can ensure Property 2. We show that $\frac{3}{2}h + 1$ rings are enough.

Lemma 4.9. *A shortest path between two checkpoints belonging to the same square for the robber only consists of highway edges.*

Proof. First we show that using the outer most ring instead of the highway takes more steps. Then we show that going into an inner ring is not faster than staying on a ring. Therefore, the robber should never leave the highway.

Let c_a and c_b be two checkpoints on highways belonging to the same square. As each checkpoint is connected to at least one village on the outer most ring, let v_a and v_b be such villages connected to c_a and c_b respectively (if there are two villages connected to the same checkpoint, choose the village with the closer distance to the other checkpoint). Let $P_h = (c_a, c_1, \ldots, c_\alpha, c_b)$ be a shortest path from c_a to c_b using edges on highways only and $P_v = (v_a, v_1, \ldots, v_\beta, v_b)$ be a shortest path from v_a to v_b through villages on the outer most ring only. Without loss of generality, let both paths go clockwise or both go counterclockwise. We show that $|P_v| \geq |P_h| - 1$.

In the case that P_h contains no cities, each checkpoint in $\{c_1, \ldots, c_\alpha\}$ is connected to *at least* one village in $\{v_1, \ldots, v_\beta\}$ that no other checkpoint is connected to and each village is connected to *at most* one checkpoint that no other village is connected to. Therefore, we have $|\{c_1, \ldots, c_\alpha\}| \leq |\{v_1, \ldots, v_\beta\}|$.

In the case that P_h contains one city we have $|\{c_1, \ldots, c_\alpha\}| \leq |\{v_1, \ldots, v_\beta\}| - 1$.

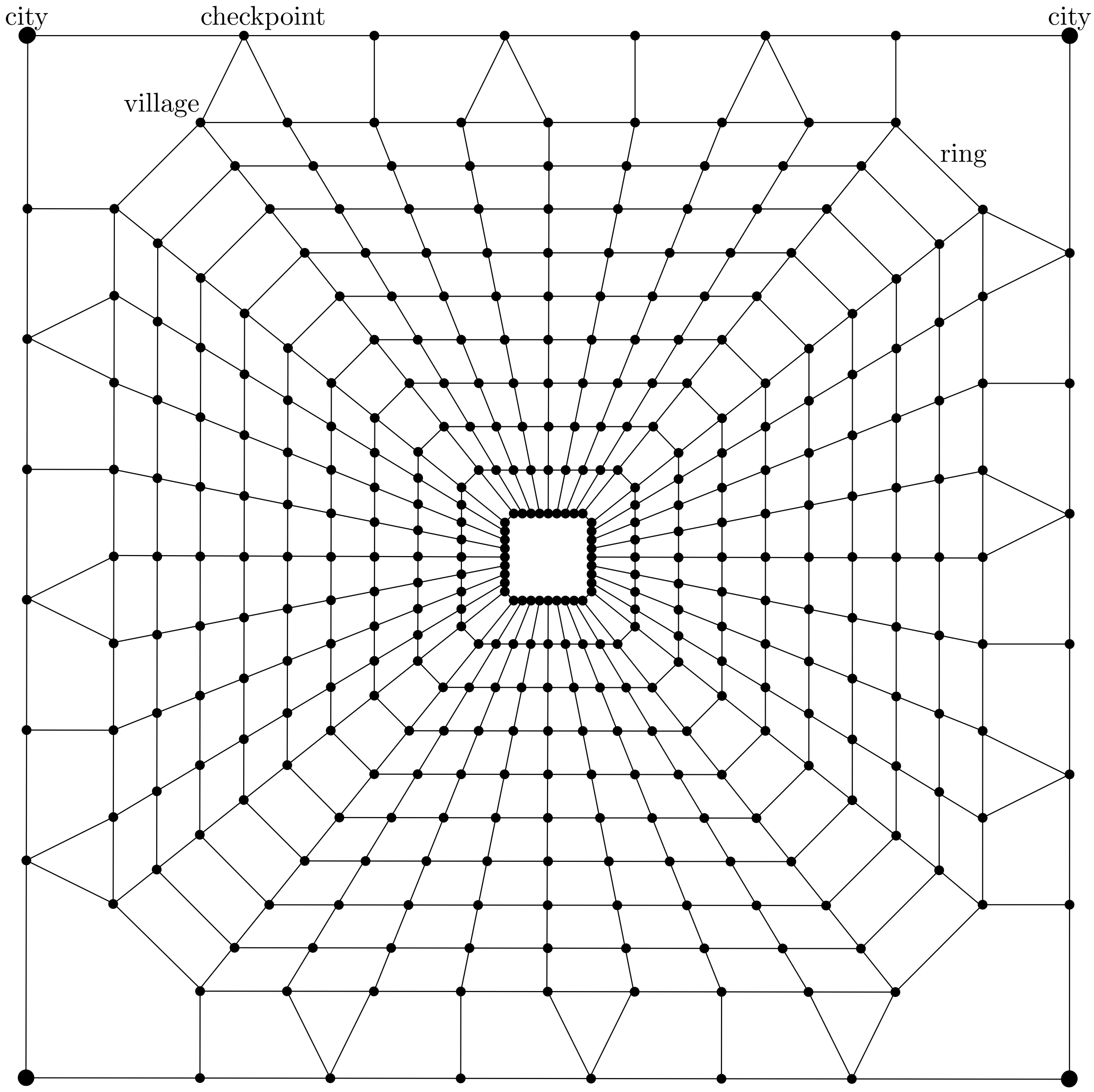

Figure 4.13: Construction of villages in a square; here $h = 6$ with 10 visible rings.

In the case that P_h contains two cities, P_h contains a full highway with h checkpoints. These checkpoints are connected to $\frac{3}{2}h$ villages that are in P_v. Therefore, $|P_v| \geq |P_h| - 2 + (\frac{3}{2} - 1)h \geq |P_h|$ for $h \geq 4$.

As P_h cannot contain more than two cities, we have shown $|P_v| \geq |P_h| - 1$. Since entering and exiting a village ring both takes one step, we have shown $|P_h| < |(c_a, v_a) \cup P_v \cup (v_b, c_b)|$.

Now we show that going through the next inner ring requires more steps than staying on the current ring of villages. Let v_a and v_b be two villages on the same ring r and v_a' and v_b' be villages on the next inner ring r' connected to v_a and v_b respectively. Let P_r be a shortest path from v_a to v_b on the ring r and $P_{r'}$ be a shortest path from v_a' to v_b'.

Assume that $P_{r'}$ stays on the ring r' and without loss of generality, let both paths go clockwise or both go counterclockwise. As each village between v_a and v_b corresponds to exactly one village between v_a' and v_b', we have $|P_{r'}| = |P_r|$ and therefore $|P_r| < |(v_a, v_a') \cup P_{r'} \cup (v_b', v_b)|$.

Now assume that $P_{r'}$ enters the next inner ring r'' at some point. Let v_a'' and v_b'' be villages on r'' connected to v_a' and v_b' respectively. By applying the above argument on $P_{r'} = (v_a', \ldots, v_b')$ and $P_{r''} = (v_a'', \ldots, v_b'')$, we inductively conclude that $P_{r'}$ must stay on r'.

A path $P = (c_0, \ldots, c_1, \ldots, c_2, \ldots, c_3)$ that switches between checkpoints and villages cannot be a shortest path either since for every subpath between two subsequent checkpoints c_i and c_{i+1}, there exists a shorter subpath using highway edges only as we have just shown. □

Lemma 4.10. *A shortest path between two cities belonging to the same square for the robber only consists of highway edges.*

Proof. As each city is connected to checkpoints only and no villages, this lemma is a conclusion of Lemma 4.9 by applying it to the checkpoints. □

Lemma 4.11. *A shortest path between two cities for the robber only consists of highway edges.*

Proof. The proof is by contradiction. Assume that a shortest path P between two cities contains a subpath going through villages. Since villages in a square are isolated from villages in another square, there must have been checkpoint c_1 before the subpath and a checkpoint c_2 after the subpath in P. According to Lemma 4.9 a shortest path from c_1 to c_2 uses highway edges only, a contradiction. □

Now we show the same for the cops.

Lemma 4.12. *A shortest path between two faces incident to cities or checkpoints belonging to the same square for the face-cops only consists of faces incident to highways.*

Proof. Note that every layer of faces between two village rings (and every layer between a highway and an outer most village ring) consists of the same amount of faces, namely $6h$ faces. Let $P = (f_a, f_1, \ldots, f_b)$ be a shortest path between two faces incident to cities or checkpoints belonging to the same square. Let layer$_{min}$ be the most inner layer that a face in P belongs to. We come to a contradiction by assuming that layer$_{min}$ is not the layer between the highway and outer most ring:

Let layer$_{min+1}$ be the next outer layer of layer$_{min}$ and let f_i be the last and f_j the first face belonging to layer$_{min+1}$ before and after the first face belonging to layer$_{min}$ respectively on

$$P = (f_a, \ldots, \underbrace{\ldots, f_i}_{\text{layer}_{min+1}}, \underbrace{\ldots}_{\text{layer}_{min}}, \underbrace{f_j, \ldots}_{\text{layer}_{min+1}}, \ldots, f_b).$$

Then either layer$_{min}$ is the inner most layer that contains only one face or the subpath $P_{sub} = (f_i, f_{i+1} \ldots, f_{j-1}, f_j)$ is not a shortest path between f_i and f_j.

If layer$_{min}$ is the inner most layer then $P = (f_a, \overbrace{\ldots}^{\geq 3/2\ h \text{ faces}}, f_{inner}, \overbrace{\ldots}^{\geq 3/2\ h \text{ faces}}, f_b)$ has length

$$|P| = 2 \cdot (\text{amount of village rings} - 1) + 3 \geq 2 \cdot \frac{3}{2}h + 3 = 3h + 3.$$

As each layer consists of $6h$ faces, the length of a shortest path staying on the layer of f_a and f_b can at most be

$$2 + \text{max amount of faces inbetween} = 2 + 3h - 1 = 3h + 1 < 3h + 3,$$

a contradiction to P being a shortest path.

If layer$_{min}$ is not the inner most layer then the amount of faces between f_i and f_j on layer$_{min+1}$ is the same as the amount of faces between f_{i+1} and f_{j-1} on layer$_{min}$. Therefore, a path that stays on layer_{min+1} and that is shorter than P_{sub} exists, a contradiction to P being a shortest path between f_a and f_b. By induction, we conclude that layer_{min} must be the layer between the highway and the outer most ring of villages. □

Lemma 4.13. *A shortest path between two faces incident to cities or checkpoints for the face-cops only consists of faces incident to highways.*

Proof. Let $P = (f_a, \ldots, f_b)$ be a shortest path between two faces incident to cities or checkpoints not belonging to the same square. Otherwise apply Lemma 4.12. Let f_i be the last and f_j be the first face incident to a highway after the first subpath of villages in $P = (f_a, \underbrace{\ldots, f_i}_{\substack{\text{incident to}\\ \text{highways}}}, \underbrace{\ldots}_{\substack{\text{between}\\ \text{village rings}}}, f_j, \ldots, f_b)$. Applying Lemma 4.12 to f_i and f_j yields a shorter subpath between f_i and f_j and by induction P cannot contain any villages. □

With these lemmas, we ensure that the second property is fulfilled, namely that a shortest path between two cities or checkpoints consists of highway edges only.

Theorem 4.14. *The number of face-cops c*(G) for graphs G with $\Delta(G) \geq 5$ can be in $\Omega(\sqrt{\log n})$.*

Proof. Since $G_{n,h}$ with robber and face-cops is equivalent to G_n with robber and vertex-cops that fulfill $\frac{v_{robber}}{v_{cops}} = \frac{3}{2} > 1$, Theorem 4.8 can be applied.

The amount of vertices in $G_{n,h}$ is the sum of

- cities: n^2
- checkpoints per highway × highways: $h \cdot 2n(n-1)$
- villages per square × squares: $(n-1)^2 \cdot (\frac{3}{2}h + 1)6h$.

This adds up to

$$\begin{aligned} & n^2 + h \cdot 2n(n-1) + (n-1)^2 \cdot (\frac{3}{2}h + 1)6h \\ = \ & 9n^2h^2 + 8n^2h + n^2 - 18nh^2 - 14nh + 9h^2 + 6h. \end{aligned}$$

With $h = 4$ we get $177n^2 - 344n + 168$ and the term is dominated by n^2 resulting in $\Omega(\sqrt{\log n})$ as a lower bound just like for G_n with n^2 vertices. □

5. Cop Graph K_4

In this chapter the cop graph is being analyzed. As opposed to the previous chapter, in which the face-cops were moving on the faces and the robber on the vertices, here the cops move on the vertices and the robber on the faces. We look at cop-graph K_4 specifically.

5.1 Subdivisions of K_4

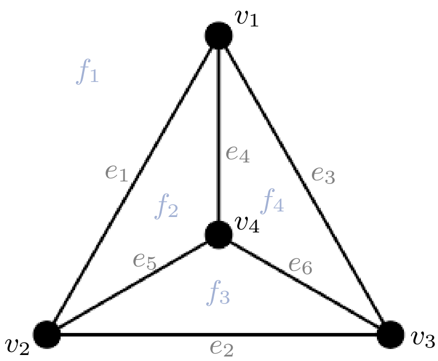

Figure 5.1: Cop-Graph K_4 labeled.

Lemma 5.1. *We have $c^*(K_4{}^*) = 3 = \Delta(K_4{}^*)$.*

Proof. Let the three cops occupy any three different vertices. Regardless of which face the robber occupies, either all three incident vertices are already occupied or two of them are and the cop not incident to the robber can move to the unoccupied vertex in the next move. With that, the robber is caught. □

As the dual graph of K_4 is K_4 itself, this is the minimal required amount of cops necessary to catch a robber on a (robber-)graph G^* with $\Delta(G^*) = 3$. It naturally raises the question: How many subdivisions on cop-graph K_4 are necessary until $c^*(G^*) > \Delta(G^*)$ for the dual graph G^*?

Lemma 5.2. *Let G be a subdivision of K_4 and U be the number of subdivisions. Then $\Delta(G^*) \geq \left\lceil \frac{U}{2} \right\rceil + 3$.*

Proof. The proof is by contradiction. Suppose that

$$\Delta(G^*) < \frac{U}{2} + 3. \tag{5.1}$$

Let $K_4 = (\{v_1, v_2, v_3, v_4\}, \{e_1, e_2, \ldots, e_6\})$ be the original graph before subdivision. Let u_i denote the number of vertices on edge e_i and U_i denote the number of vertices incident to face f_i that resulted from subdivision, i.e. $U_1 = u_1 + u_2 + u_3$. Since every face in K_4 has three incident vertices before subdivision, we have

$$U_i < \Delta(G^*) - 3 \overset{(5.1)}{<} \frac{U}{2}$$

for every $i \in \{1, 2, 3, 4\}$ and thus

$$U_1 + U_2 + U_3 + U_4 < 4 \cdot \frac{U}{2} = 2U.$$

By adding the number of all vertices incident to the faces that resulted from subdivision via the individual u_i we get

$$\begin{aligned} U_1 + U_2 + U_3 + U_4 &= (u_1 + u_2 + u_3) + (u_1 + u_4 + u_5) + (u_2 + u_5 + u_6) + (u_3 + u_4 + u_6) \\ &= \sum_{i=1}^{6} 2u_i = 2\sum_{i=1}^{6} u_i = 2U. \end{aligned}$$

With that we get $2U = \sum_{i=1}^{4} U_i < 2U$, a contradiction. □

Lemma 5.3. *Let G be a subdivision of K_4 and U be the number of subdivisions. If $\Delta(G^*) \leq U$ and $U \geq 6$ then $c^*(G^*) \geq U$.*

Proof. If there are less than U cops in the game, we show that the robber R can move to a face where he will not be caught in the next turn. Without loss of generality, let R be on face f_1 and let x_1 denote the number of cops incident to f_1. Since there are only three entry points to the vertices adjacent to f_1 (vertices that are connected to vertices not incident to this face), namely v_1, v_2 and v_3, we need to have $x_1 \geq u_1 + u_2 + u_3 = U_1$ in order to catch the robber in the next turn. The reasoning is that in the next turn no cop other than the x_1 cops already incident to f_1 can occupy the vertices that resulted from subdividing e_1, e_2 and e_3. If x_1 is large enough, the robber escapes to another face. He chooses the face f_i with the smallest x_i. Since there are less than U cops in the game, there exists a face f_i with $x_i < U_i$. As the robber can keep this situation up indefinitely, the cops may never catch the robber. □

Theorem 5.4. *Let G be a subdivision of K_4 and U be the number of subdivisions. If $\Delta(G^*) \leq U$ and $U \geq 6$ then $c^*(G^*) = U$.*

Proof. Assume that $u_i > 0$ for $i \in \{1, 2, 3, 4, 5, 6\}$. Then we place all U cops on the vertices in $V(G) \setminus \{v_1, v_2, v_3, v_4\}$. Let the robber start on face f. Let $v_a, v_b, v_c \in \{v_1, v_2, v_3, v_4\}$ be the incident vertices to f that originate from K_4. As each of those three vertices are connected to an edge that is not incident to f, let the next cop on those edges move to v_a, v_b and v_c respectively. This is possible since $u_i > 0$ for all edges. With this, the robber is caught.

Now assume that $u_i = 0$ for at least one edge. Let v_a and v_b be the endpoints of such an edge. Then place the cops like before on all vertices in $V(G) \setminus \{v_1, v_2, v_3, v_4\}$ with the exception of vertices adjacent to v_a and v_b. These cops go on v_a and v_b respectively and with this, the robber will be caught in the next turn regardless of his starting position. With Lemma 5.3 we conclude $c^*(G^*) = U$, but if $U < 6$ then we have $U < \Delta(G^*) = c^*(G^*)$. □

This results in the following table (Theorem 5.4):

U	minimal $\Delta(G^*)$	c*(G*)	U	minimal $\Delta(G^*)$	c*(G*)
1	4	4	11	9	11
2	4	4	12	9	12
3	5	5	13	10	13
4	5	5	14	10	14
5	6	6	15	11	15
6	6	6	16	11	16
7	7	7	17	12	17
8	7	8	18	12	18
9	8	9	19	13	19
10	8	10	20	13	20

Table 5.1: $c^*(G^*)$ depending on the amount of subdivisions U. G is a subdivision of K_4.

6. Variants

This chapter introduces some further variants on the primal-dual cops and robbers game. We consider a variant that simplifies the capture rule and a variant that restricts the movement of the robber. Hence, both variants benefit the cops.

6.1 Tag Variant

In this variant we change the capture rule from requiring all faces incident to the robber's vertex to be occupied by cops to only one of the faces having to be occupied. Similar to the original game, now one cop is able to catch the robber by himself. Let $c_t^*(G)$ denote the cop number for this tag variant. Obviously, we have $c_t^*(G) \leq c^*(G)$.

This change has an effect on the cop number of outerplanar graphs G. While we always have $c_t^*(G) = 1$ for outerplanar graphs (by occupying the outer face), the cop number $c^*(G)$ can be arbitrarily high, for example for outerplanar graphs constructed in the following way: Begin with a cycle of three vertices. At any step, for each edge incident to the outer face, add a new vertex that is connected to the endpoints of the edge, thus forming a new face (Figure 6.1). Repeat this step an arbitrary amount of times. As we know the lower bound to be $c^*(G) \geq \Delta(G)$ for this graph (see Section 3.1), the cop number $c^*(G)$ is unbounded.

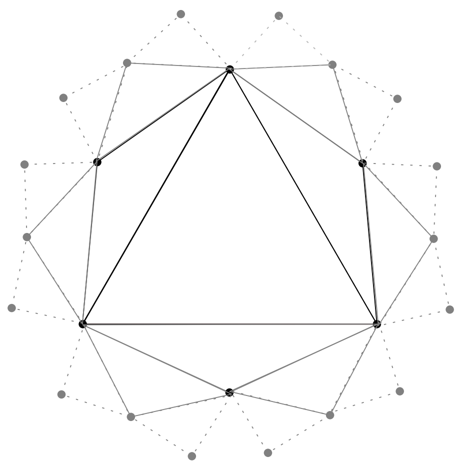

Figure 6.1: Construction of an outerplanar graph.

For 3-regular graphs the tag cop number $c_t^*(G)$ also becomes smaller than $c^*(G)$ and actually assumes the smallest number possible. This applies to graphs with $\Delta(G) \leq 3$ as well:

Theorem 6.1. *Let G be a planar embedded graph with $\Delta(G) \leq 3$. Then $c_t^*(G) = 1$.*

Proof. We show that with each step the cop reduces the total distance to the incident faces of the robber. Assume that the robber moves from vertex u to a neighboring vertex v. Let f_1, f_2, f_3 be the incident faces of u and f_2, f_3, f_4 be the incident faces of v. Note that f_1, f_2, f_3 and f_4 do not have to be distinct faces. Let $dist(c, f)$ denote the distance of the cop to face f, that is the number of steps the cop needs to reach the face. Let f_1 be the face with the smallest distance to the cop. Before the robber moves, for

$$dist(c, f_1) = d$$

we have

$$\begin{aligned} dist(c, f_2) &\leq d + 1, \\ dist(c, f_3) &\leq d + 1, \\ dist(c, f_4) &\leq d + 2. \end{aligned}$$

Therefore, as the total distance we have

$$total_{before} = dist(c, f_1) + dist(c, f_2) + dist(c, f_3) \leq 3d + 2.$$

Assume that $dist(c, f_2) = d + 1 = dist(c, f_3)$ and with that $total_{before} = 3d + 2$. After the robber moves, the cop reduces the distance to f_1 by 1. Then, we have

$$dist_{new}(c, f_1) = d - 1$$

and also

$$\begin{aligned} dist_{new}(c, f_2) &= dist(c, f_2) - 1 = d, \\ dist_{new}(c, f_3) &= dist(c, f_3) - 1 = d, \\ dist_{new}(c, f_4) &\leq min(dist_{new}(c, f_2), dist_{new}(c, f_3)) + 1 \leq d + 1. \end{aligned}$$

With that, the new total distance to the three faces incident to the robber becomes

$$total_{after} = dist_{new}(c, f_2) + dist_{new}(c, f_3) + dist_{new}(c, f_4) \leq 3d + 1 < 3d + 2 = total_{before}.$$

For $dist(c, f_2) = d$ or $dist(c, f_3) = d$ we have $dist_{new}(c, f_4) \leq d$ and subsequently $total_{after} \leq total_{before} - 2$. □

For 4-regular graphs the tag cop number is identical to the cop number in the original game, as a face-cop moves just as quickly as a vertex-cop.

Theorem 6.2. *Let G be a planar embedded graph with $\Delta(G) = 4$. Then $c_t^*(G) = 3 = c(G)$.*

For 5-regular graphs we have already shown in Section 4.4 that $c_t^*(G) \geq \Omega(\sqrt{\log n})$, as the cops in the proof never reached a face incident to the robber's vertex.

Theorem 6.3. *Let G be a planar embedded graph with $\Delta(G) \geq 5$. Then $c_t^*(G) \geq \Omega(\sqrt{\log n})$.*

For the subdivision of the cop graph K_4 from Chapter 5 two cops suffice, as one cop might not be able to cover three faces by himself and the second cop then covers the last face that the robber might be on, thus capturing the robber. One cop might not be sufficient as moving along the original edges of K_4 can give the robber enough time to escape to another face due to the subdivided the edges. This does not apply if one of the edges is not subdivided.

Theorem 6.4. *Let G be a subdivision of K_4. Then $c_t^*(G^*) \leq 2$.*

6.2 Restrictive Variant

In this variant, we add the rule that the robber cannot use an edge that has cops occupying both faces incident to the edge. If an edge has the same face on both sides, one cop occupying the face already blocks the edge. We denote the restrictive cop number as $c_r^*(G)$. Note that $c_r^*(G) \leq c^*(G)$.

For the square-grid graph from Section 4.2 we can implement the same idea with four restricting face-cops. Two face-cops fulfill the role of cop_1, the cop that upholds $cop_{1_x} = r_x$, by occupying the bottom two faces of the vertex that cop_1 controls and thus preventing the robber from moving down when $(cop_{1_x}, cop_{1_y}) = (r_x, r_y)$. The other two face-cops analogously occupy the two incident faces to the left of the vertex that cop_2 is supposed to control and thus preventing the robber from moving left when $(cop_{2_x}, cop_{2,y}) = (r_x, r_y)$. Once robber stands still the four face-cops can surround him. This happens at vertex (n, n) at the latest. We conclude:

Theorem 6.5. *Let G be a planar embedded square-grid graph. Then $c_r^*(G) \leq 4$.*

For the other results this variant does not seem to change the cop number. A new lower bound could be found for 4-regular graphs if two cops suffice to guard a geodesic path by blocking the robber from entering or exiting the path. This however, remains to be proven. A proof could look similar to the proof of Aigner and Fromme for $c(G) = 3$ on planar graphs [AF84].

7. Conclusion

In this thesis we studied the primal-dual variant of the cops and robbers game on planar graphs. We started by showing general lower and upper bounds on the cop number $c^*(G)$ of this variant in Chapter 3.

Following that, we determined some tight bounds and upper bounds for graphs G with $\Delta(G) \leq 4$ by analyzing the properties of those graphs in regards to their effect on the cops' movement and drawing analogies between the original and the primal-dual variant.

Aside from that, it remains to be proven what a tight bound for 4-regular graphs is. We were unable to prove an accurate lower bound as we were not able to show how less than four cops can guard a geodesic path.

For graphs G with $\Delta(G) \geq 5$ we constructed a graph that simulates face-cops as vertex-cops moving at a slower speed than the robber. Thus, we have shown by reduction that the cop number $c^*(G)$ is at least $\Omega(\sqrt{\log n})$.

We conclude that for $\Delta(G) \leq 4$ the cop number is bounded, but unbounded for $\Delta(G) \geq 5$. To be precise, for

- $\Delta(G) \leq 3$ we have $c^*(G) \leq \Delta(G)$ (Section 3.3, 4.1),
- $\Delta(G) = 4$ we have $c^*(G) \leq 12$ (Section 4.2) and
- $\Delta(G) \geq 5$ we have $c^*(G) \geq \Omega(\sqrt{\log n})$ (Section 4.4).

After changing perspectives and looking at graphs from the cop's perspective in Chapter 5, we determined that subdividing edges of K_4 results in a linear increase of the cop number on the resulting dual graph.

Although we have taken a brief look at further variants in Chapter 6, additional ideas may be considered and studied. Examples are: How does it affect the cop number $c^*(G)$ if all parties are forced to move every turn? What if we restrict the movement of the robber even more by already blocking edges that have a single cop on one of the two incident faces?

Bibliography

[AF84] M. Aigner and M. Fromme. A game of cops and robbers. *Discrete Applied Mathematics*, 8(1):1–12, 1984.

[BBKP15] Deepak Bal, Anthony Bonato, William Kinnersley, and Pawel Pralat. Lazy Cops and Robbers on Hypercubes. *Combinatorics, Probability and Computing*, 24:829–837, 01 2015.

[BBKP16] Deepak Bal, Anthony Bonato, William B. Kinnersley, and Paweł Prałat. Lazy Cops and Robbers played on random graphs and graphs on surfaces. *J. Comb.*, 7(4):627–642, 2016.

[BCC+20] Andrea C. Burgess, Rosalind A. Cameron, Nancy E. Clarke, Peter Danziger, Stephen Finbow, Caleb W. Jones, and David A. Pike. Cops that surround a robber. *Discrete Applied Mathematics*, 285:552–566, 2020.

[BCP10] Anthony Bonato, Ehsan Chiniforooshan, and Paweł Prałat. Cops and Robbers from a distance. *Theoretical Computer Science*, 411(43):3834–3844, 2010.

[BN11] Anthony Bonato and Richard Nowakowski. *The Game of Cops and Robbers on Graphs*. 09 2011.

[Bon16] Anthony Bonato. *Conjectures on Cops and Robbers*, pages 31–42. Springer International Publishing, Cham, 2016.

[BW00] Graham Brightwell and Peter Winkler. Gibbs Measures and Dismantlable Graphs. *Journal of Combinatorial Theory, Series B*, 78:141–166, 01 2000.

[CC06] Nancy Clarke and Emma Connon. Cops, Robber, and Alarms. *Ars Comb.*, 81, 10 2006.

[CCD+17] N. E. Clarke, D. Cox, C. Duffy, D. Dyer, S. Fitzpatrick, and M. E. Messinger. Limited Visibility Cops and Robbers, 2017.

[Chi08] Ehsan Chiniforooshan. A better bound for the cop number of general graphs. *Journal of Graph Theory*, 58:45–48, 05 2008.

[CM12] Nancy E. Clarke and Gary MacGillivray. Characterizations of k-copwin graphs. *Discrete Mathematics*, 312(8):1421–1425, 2012.

[CN00] Nancy Clarke and Richard Nowakowski. Cops, Robber, and Photo Radar. *Ars Comb.*, 56, 07 2000.

[DDTY13] Dariusz Dereniowski, Danny Dyer, Ryan M. Tifenbach, and Boting Yang. Zero-Visibility Cops and Robber Game on a Graph. In Michael Fellows, Xuehou Tan, and Binhai Zhu, editors, *Frontiers in Algorithmics and Algorithmic Aspects in Information and Management*, pages 175–186, Berlin, Heidelberg, 2013. Springer Berlin Heidelberg.

[Des16] Davita DesRoches. The Decoy game of cops and robber. Bachelor's thesis, Acadia University, 2016.

[DN03] N. Deo and Zoran Nikoloski. The game of cops and robbers on graphs: a model for quarantining cyber attacks. *Congressus Numerantium*, 01 2003.

[FGK+10] Fedor V. Fomin, Petr A. Golovach, Jan Kratochvíl, Nicolas Nisse, and Karol Suchan. Pursuing a fast robber on a graph. *Theoretical Computer Science*, 411(7):1167–1181, 2010.

[FHMP16] S.L. Fitzpatrick, J. Howell, M.E. Messinger, and D.A. Pike. A deterministic version of the game of zombies and survivors on graphs. *Discrete Applied Mathematics*, 213:1–12, 2016.

[Fra87] Peter Frankl. Cops and robbers in graphs with large girth and Cayley graphs. *Discrete Applied Mathematics*, 17(3):301–305, 1987.

[GKS18] Ilya Gromovikov, William B. Kinnersley, and Ben Seamone. Fully Active Cops and Robbers, 2018.

[GY17] Ziyuan Gao and Boting Yang. The Cop Number of the One-Cop-Moves Game on Planar Graphs. In Xiaofeng Gao, Hongwei Du, and Meng Han, editors, *Combinatorial Optimization and Applications*, pages 199–213, Cham, 2017. Springer International Publishing.

[IKK06] Volkan Isler, Sampath Kannan, and Sanjeev Khanna. Randomized Pursuit-Evasion with Local Visibility. *SIAM Journal on Discrete Mathematics*, 20(1):26–41, 2006.

[Isl14] Mohammad Moinul Islam. The Cop, Robber and Decoy Game. Master's thesis, Acadia University, 2014.

[Kin13] William B. Kinnersley. Cops and Robbers is EXPTIME-complete, 2013.

[KW13] Natasha Komarov and Peter Winkler. Capturing the Drunk Robber on a Graph. *Electronic Journal of Combinatorics*, 21, 05 2013.

[LP12] Linyuan Lu and Xing Peng. On Meyniel's conjecture of the cop number. *Journal of Graph Theory*, 71:192–205, 10 2012.

[LT79] Richard J. Lipton and Robert Endre Tarjan. A Separator Theorem for Planar Graphs. *SIAM Journal on Applied Mathematics*, 36(2):177–189, 1979.

[MTTV97] Gary L. Miller, Shang-Hua Teng, William Thurston, and Stephen A. Vavasis. Separators for Sphere-Packings and Nearest Neighbor Graphs. *J. ACM*, 44(1):1–29, jan 1997.

[NN98] S. Neufeld and R. Nowakowski. A game of cops and robbers played on products of graphs. *Discrete Mathematics*, 186(1):253–268, 1998.

[NS08] Nicolas Nisse and Karol Suchan. Fast Robber in Planar Graphs. In Hajo Broersma, Thomas Erlebach, Tom Friedetzky, and Daniel Paulusma, editors, *Graph-Theoretic Concepts in Computer Science*, volume 5344 of *Lecture Notes in Computer Science*, pages 312–323. Springer Berlin Heidelberg, Berlin, Heidelberg, 2008.

[NW83] Richard J. Nowakowski and Peter Winkler. Vertex-to-vertex pursuit in a graph. *Discret. Math.*, 43:235–239, 1983.

[OO14] David Offner and Kerry Ojakian. Variations of cops and robber on the hypercube. *The Australasian Journal of Combinatorics [electronic only]*, 59, 01 2014.

[Pra10] Pawel Pralat. When does a random graph have constant cop number? *Australasian Journal of Combinatorics*, 01 2010.

[Qui78] Alain Quilliot. *Jeux et pointes fixes sur les graphes.* PhD thesis, Ph. D. Dissertation, Université de Paris VI, 1978.

[Sch22] Samuel Schneider. Surrounding Cops and Robbers. Bachelor's thesis, Karlsruhe Institute of Technology, 2022.

[ST93] P.D. Seymour and R. Thomas. Graph Searching and a Min-Max Theorem for Tree-Width. *Journal of Combinatorial Theory, Series B*, 58(1):22–33, 1993.

[STW16] Brendan W. Sullivan, Nikolas Townsend, and Mikayla Werzanski. The 3x3 rooks graph is the unique smallest graph with lazy cop number 3, 2016.