P_n-free colorings of planar graphs

Bachelor Thesis of

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Statement of Authorship

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Karlsruhe, den 15. September 2020

Abstract

Axenovich et al. started the study of planar avoidable graphs. We say that a graph H is k-planar unavoidable if there exists a planar graph G such that every k-edge-coloring of G contains a monochromatic subgraph isomorphic to H, otherwise we call H k-planar avoidable. Similarly, we define the notion of k-outerplanar avoidable graphs. Axenovich et al. were in particular interested in 2-edge-colorings and showed that every path is 2-planar unavoidable. Building on their work, we continue the study of edge-colorings of planar graphs that do not contain a monochromatic path of a given length. Modifying a proof of Ding et al., we prove that paths of length at least 3 are 3-outerplanar avoidable and applying a result of Merker and Postle that there exists a natural number n such that paths of length at least n are 4-planar avoidable. Further, we study edge-colorings of the iterated triangulation, a specific family of planar graphs, that avoid a restricted class of long paths. Finally, we are interested in the complexity of the associated decision problem: Given a planar graph and a natural number k, is there a 3-edge-coloring that does not contain a monochromatic P_k? We attack the special case $k = 2$ by considering 3-regular, bridgeless, planar supergraphs.

Deutsche Zusammenfassung

Axenovich et al. haben sich als erste der Analyse von planar avoidable Graphen gewidmet. Ein Graph H heißt k-planar unavoidable, wenn ein planarer Graph G existiert, sodass jede k-Kantenfärbung H als monochromatischen Subgraphen enthält. Anderenfalls heißt H k-planar avoidable. Analog definiert man k-outerplanar avoidable Graphen. Axenovich et al. haben sich in erster Linie mit 2-Kantenfärbungen beschäftigt und konnten beweisen, dass alle Pfade 2-planar unavoidable sind. Wir führen diese Arbeit fort. Wir beweisen, ähnlich zu einem Resultat von Ding et al., dass Pfade der Länge 3 3-outerplanar avoidable sind und es eine natürliche Zahl n gibt, sodass Pfade der Länge n 4-planar avoidable sind. Insbesondere widmen wir uns Kantenfärbungen der iterated triangulation, einer Familie planarer Graphen, die keine langen monochromatischen Pfade einer Subklasse von Pfaden enthalten. Wir setzen uns des Weiteren mit dem folgenden Entscheidungsproblem auseinander: Gegeben ein planarer Graph G und eine natürliche Zahl k. Existiert eine 3-Kantenfärbung von G die keinen monochromatischen Pfad der Länge k enthält? Wir untersuchen den Spezialfall $k = 2$ indem wir 3-reguläre, brückenlose, planare Supergraphen betrachten.

Contents

1 Introduction

Ramsey theory deals with the appearance of substructures in large graphs. Typically, given a graph H we ask whether every large enough graph G or its complement $\overline{G}$ contains H as a subgraph. The smallest order n such that the claim above holds for every graph of order at least n, is called *Ramsey number* $R(H)$ of H. In 1990, Ramsey showed that Ramsey numbers exist for all complete graphs [10, p. 9.1.1]. As every graph is a subgraph of a complete graph, the same holds for all graphs. Equivalently, Ramsey's theorem states that for every graph H, there exists a natural number n such that every 2-edge-coloring of a complete graph on at least n vertices contains H as a monochromatic subgraph as we can consider one color class as a graph G and the other as its complement. Interestingly, Ramsey numbers even exist if we consider k-edge-colorings of complete graphs, where $k \geq 2$ [10, Theorem 9.1.3]. Lots of work has focused on determining bounds for Ramsey numbers [7].

Until now, we only considered Ramsey numbers of one single graph at a time. Generalizing this concept, we can define the Ramsey number $R(H_1, \ldots, H_k)$ as the smallest natural number n such that every k-edge-coloring of a complete graph on at least n vertices contains a copy of H_i in color i for some $i \in \{1, \ldots, k\}$. If all graphs H_i are isomorphic, we are in the same situation as above. Clearly, as there exists a complete graph containing all chosen graphs H_i as subgraphs, the Ramsey number $R(H_1, \ldots, H_k)$ exists for all graphs H_i.

As we wish to study Ramsey properties of planar graphs, we can consider two different Ramsey problems. The first is the study of edge-colorings of complete graphs where one color class is a planar graph. This problem was introduced by Steinberg and Tovey in 1993. They defined the planar Ramsey number $\mathrm{PR}(H_1, H_2)$ as the smallest natural number n, such that every edge-coloring in red and blue of a complete graph on at least n vertices where the red color class is a planar graph, contains a red copy of H_1 or a blue copy of H_2 [26]. This can be reformulated as follows. The number $n \coloneqq \mathrm{PR}(H_1, H_2)$ is the smallest number such that every planar graph G on n vertices contains H_1 or its complement $\overline{G}$ contains H_2 as a subgraph. Steinberg and Tovey calculated the values for all planar Ramsey numbers where H_1 and H_2 are complete graphs [26, Theorem 2]. Therefore, planar Ramsey numbers exist for all graphs H_1 and H_2 as every such graph is subgraph of a complete graph. Dudek and Ruciński computed the values of $\mathrm{PR}(G, H)$, where G and H are cycles or complete graphs [12].

We will concentrate on a different notion of planar Ramsey numbers. Instead of considering edge-colorings of complete graphs, we study edge-colorings of planar graphs. Then the Ramsey number of a graph H relative to all planar graphs is the smallest order of a planar graph such that every edge-coloring of G contains H as a monochromatic subgraph, if such a graph G exists. Note that this notion differs from the one given by

class	number of colors	result	minimum length	reference
outerplanar graphs	2	unavoidable		[3, Lemma 6]
outerplanar graphs	3	avoidable	3	Corollary 3.7
planar graphs	2	unavoidable		[3, Lemma 6]
planar graphs	4	avoidable	bounded	Theorem 6.4
planar graphs	5	avoidable	3	[16]

Table 1.1: A summary of results concerning avoidable and unavoidable paths of edge-colorings using a given number of colors. If paths of a certain length are avoidable, the minimum known length k for which P_k is avoidable, is specified.

Steinberg and Tovey as in the definition of planar Ramsey numbers, the graphs H_1 and H_2 are not necessarily planar, but in the definition above, if such a number exists, H is a subgraph of a planar graph, thus planar. Further, existence of planar Ramsey numbers and Ramsey numbers relative to all planar graphs have different implications. While the first one has implications for *every* planar graph, the latter only implies that there exists at least *one* specific, planar graph. Thus, the structural implications of these two notions differ widely. Informally, the second Ramsey problem studies unavoidable graphs in all edge-decompositions of certain planar graphs, the first unavoidable subgraphs of all planar graphs.

The study of the second Ramsey problem of planar graphs as given above has been initiated by Axenovich et al. [3]. As all monochromatic subgraphs of edge-colorings of planar graphs are necessarily planar, not all graphs appear as monochromatic subgraphs of large enough planar graphs. Thus, we are interested in the existence of Ramsey numbers. While Axenovich et al. focused on 2-edge-colorings of planar graphs, we will be concerned with 3-edge-colorings. They proved that arbitrarily long paths cannot be avoided in 2-edge-colorings of outerplanar graphs, i.e., for every given length k, there exists an outerplanar graph G such that every 2-edge-coloring of G contains P_k as a monochromatic subgraph [3, Lemma 6]. We wondered whether the same holds for 3-edge-colorings of planar graphs.

Chapter 2 contains preliminaries. In Chapter 3, we consider 3-edge-colorings of outerplanar graphs avoiding long paths. Chapter 4 gives an insight in P_k-free edge-colorings of a specific family of planar graphs, the iterated triangulation. In Chapter 5, we show that subdivisions of stars are 2-planar unavoidable. We prove that there exist paths that can be avoided in 4-edge-colorings of planar graphs in Chapter 6. Finally, we are interested in an associated decision problem concerning P_k-free colorings of planar graphs in Chapter 7. A summary of results concerning avoidable and unavoidable paths is given in Table 1.1.

2 Preliminaries

Roughly speaking, a planar graph is a graph that can be drawn in the plane without intersecting edges. We will call such a drawing a planar embedding. A formal definition is given by Diestel [10, p. 102]. Outerplanar graphs are planar graphs that admit a planar embedding where all vertices lie on the boundary of the outer face. We will be interested in edge-colorings of planar and outerplanar graphs avoiding long monochromatic paths. The path P_k denotes a path of length k, i.e., a path containing $k+1$ vertices and k edges.

Ramsey's theorem states that for every graph H and any number of d colors, every d-edge-coloring of a sufficiently large complete graph contains a monochromatic copy of H [10, Theorem 9.1.3]. Generalizing this concept, we get the following definition.

Definition 2.1 (d-planar avoidable, [3, p. 1])**.** Let G and H be graphs. We say that G *d-arrows* H if every d-edge-coloring of G contains a monochromatic copy of H. We write

$$G \to_d H$$

Let $\mathcal{F}$ be a class of graphs. We say that a graph H is *d-$\mathcal{F}$ unavoidable* if there is a graph $G \in \mathcal{F}$ such that G d-arrows H. Otherwise we call H *d-$\mathcal{F}$ avoidable.* If $\mathcal{F}$ is the class of planar graphs, we say that H is *d-planar unavoidable* or *d-planar avoidable* respectively. If $\mathcal{F}$ is the class of outerplanar graphs, we say that H is *d-outerplanar unavoidable* or *d-outerplanar avoidable.*

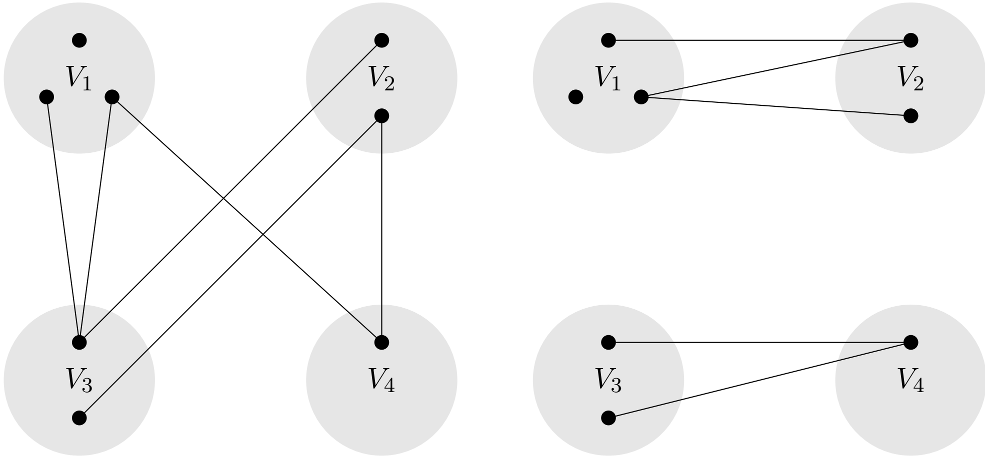
Therefore, if paths of lengths k are d-planar avoidable, every planar graph admits a d-edge-coloring such that no monochromatic component contains P_k as a subgraph, i.e., every d-edge-coloring is a P_k-free coloring. Often, we will write planar unavoidable instead of 2-planar unavoidable. Note that all d-planar unavoidable graphs for $d \geq 2$ are in particular 2-planar unavoidable.

Axenovich et al. started the study of planar avoidable graphs [3]. They noticed that all planar unavoidable graphs are bipartite, outerplanar graphs using a result of Gonçalves [15] and the Four-Color-Theorem [1].

Observation 2.2 ([3, p. 1])**.** *If H is a planar unavoidable graph, then H is bipartite and outerplanar.*

Proof. As H is planar unavoidable, there exists a planar graph G such that every 2-edge-coloring of G contains H as a monochromatic subgraph. If there is a 2-edge-coloring of G such that each color class is a bipartite graph, H is bipartite. Therefore, in order to show that H is bipartite, it suffices to prove that every planar graph can be edge-decomposed into two bipartite graphs.

Claim 2.2.1. *Every planar graph can be edge-decomposed into two bipartite graphs.*



(a) G' is a bipartite graph. (b) G'' is a bipartite graph.

Figure 2.1: Illustrations to the proof of Observation 2.2.

Proof of Claim. Let G be a planar graph. The Four-Color-Theorem [1] implies that there is a proper vertex-coloring c of G using four colors. Let $V_1, \dots, V_4$ denote the color classes of c. Every color class is an independent set, as c is a proper vertex-coloring. For all $1 \leq i < j \leq 4$ let E_{ij} denote the edges between V_i and V_j. As every color class is an independent set, the edges of G can be decomposed as follows:

$$E(G) = \bigcup_{1 \leq i < j \leq 4} E_{ij}$$

Let $G' := G[E_{13} \cup E_{14} \cup E_{23} \cup E_{24}]$ and $G'' := G[E_{12} \cup E_{34}]$. Then $G = G' \cup G''$. Note that G' is bipartite with parts $V_1 \cup V_2$ and $V_3 \cup V_4$, as by definition of G' there are no edges between the vertices of V_1 and V_2 and the same holds for the sets V_3 and V_4; see Figure 2.1a. Similarly, we see that G'' is bipartite; see Figure 2.1b. ⌟

Further, Gonçalves showed that every planar graph can be edge-decomposed into two outerplanar graphs [15]. Thus, H has to be an outerplanar graph as every subgraph of an outerplanar graph is outerplanar. □

For d-planar unavoidable graphs, where $d \geq 3$ even more is known. Later, we will see that all 3-planar unavoidable graphs are forests as Nash-Williams formula [23] implies that every planar graph can be edge-decomposed into three forests; see Proposition 6.2. All 4-planar unavoidable graphs are caterpillar forests as every planar graph can be edge-decomposed into four caterpillar forests, as has been shown by Gonçalves [14]. Hakimi, Mitchem, and Schmeichel proved that every planar graph can be edge-decomposed into five star forests [16]. Thus, every 5-planar unavoidable graph is a star forest.

3 P_k-free Colorings of Outerplanar Graphs

We are interested in edge-colorings of outerplanar graphs avoiding long monochromatic paths, i.e., we wish to find a positive integer $k \in \mathbb{N}$ such that all paths of length at least k are d-outerplanar avoidable for some number of colors d. Then, arbitrarily long paths can be avoided. Instead of considering all outerplanar graphs, we will first study one particular family of outerplanar graphs, called the universal outerplanar graphs $\mathrm{UOP}(n)$, where $n \in \mathbb{N}$. We will show that every outerplanar graph G is a subgraph of $\mathrm{UOP}(n)$ for sufficiently large n. Thus, if we can color the edges of $\mathrm{UOP}(n)$ avoiding long monochromatic paths for all $n \in \mathbb{N}$, then the same holds for every outerplanar graph. The following definition is due to Axenovich et al. [3].

Definition 3.1 (universal outerplanar graph)**.** A *universal outerplanar* graph $\mathrm{UOP}(n)$ is defined as follows: $\mathrm{UOP}(1)$ is a triangle. An edge on the outer face is called an outer edge. For $k > 1$, $\mathrm{UOP}(k)$ is an outerplanar graph that is a supergraph of $\mathrm{UOP}(k-1)$ obtained by introducing, for each outer edge xy, a new vertex v_{xy} and new edges $v_{xy}x$ and $v_{xy}y$. We say that x and y are the *parents* of v_{xy}. If v_{xy} is introduced in $\mathrm{UOP}(n)$, we say that v_{xy} is part of the *n-th generation*. We write $\mathrm{gen}(v_{xy}) = n$. An illustration to the construction of $\mathrm{UOP}(3)$ is given in Figure 3.1.

The following result is well-known.

Lemma 3.2. *Every outerplanar graph on n vertices is a subgraph of* $\mathrm{UOP}(n)$*.*

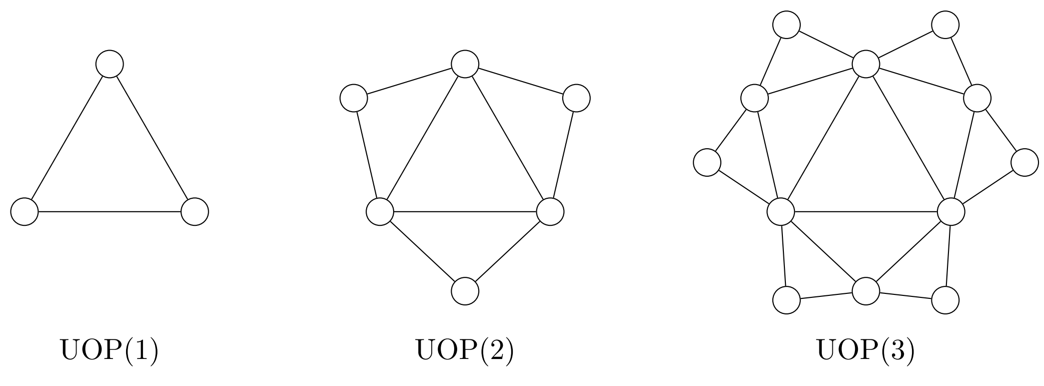
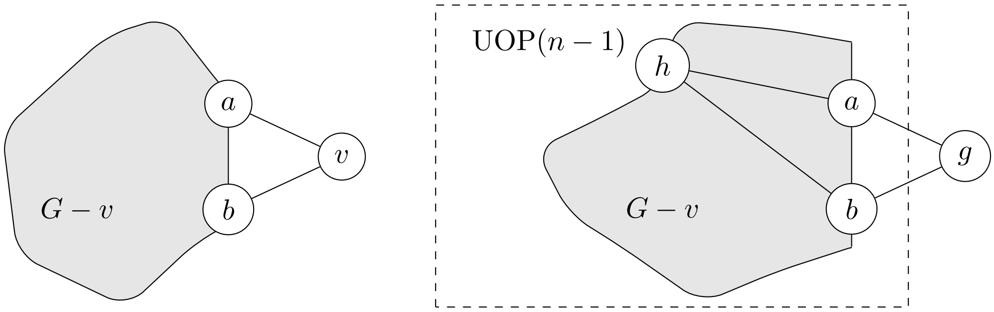


Figure 3.1: The iterative construction of the universal outerplanar graph UOP(3).



(a) $G - v$ is maximal outerplanar.

(b) The embedding of $G - v$ in UOP$(n-1)$ can be extended to an embedding of G in UOP(n).

Figure 3.2: Illustrations to the proof of Lemma 3.2.

Proof. We prove a stronger statement. Let G be an outerplanar graph on n vertices and consider a fixed outerplanar drawing of G. By abuse of notation, we also write G for its drawing. Without loss of generality, we may assume that G is maximal outerplanar.

Claim 3.2.1. *The outerplanar drawing G can be embedded in the drawing of* UOP(n) *as given in Definition 3.1.*

We will prove the claim by induction on the number of vertices. For $n \leq 3$, we have

$$G \subseteq K_3 = \mathrm{UOP}(1)$$

and therefore G is a subgraph of UOP(n). Let now $n > 3$ and assume the claim holds for $n-1$. As G is maximal outerplanar there is a vertex v of degree 2 in G. Let a and b be its neighbors. The drawing G is an inner triangulation as G is maximal outerplanar. Therefore $ab \in E(G)$; see Figure 3.2a. Note that $G - v$ is also an inner triangulation, thus maximal outerplanar. By induction, we have that $G - v \subseteq \mathrm{UOP}(n-1)$. Consider now UOP$(n)$. Then there are two vertices g, h such that g, a, b and h, a, b form triangles in UOP(n) as UOP(n) is an inner triangulation and ab is an inner edge in UOP(n); see Figure 3.2b.

Then $g \notin V(G - v)$ or $h \notin V(G - v)$, as ab is an outer edge of $G - v$. Without loss of generality $g \notin V(G - v)$. Then we can embed G in UOP(n) by mapping v to g. ☐

The case of edge-coloring outerplanar graphs using two colors is well-studied. Axenovich et al. proved that all paths are 2-outerplanar unavoidable.

Theorem 3.3 ([3, Lemma 6])**.** *For all $n \in \mathbb{N}$, we have* $\mathrm{UOP}(n^2) \to_2 P_n$.

We are now interested in edge-colorings of UOP(n) using three colors. Clearly, paths of length two cannot be avoided, as in any 3-edge-coloring of a graph that contains a vertex v of degree four, there are at least two edges of the same color incident to v. The strongest possible result would therefore be a coloring avoiding paths of length 3. Note that a 3-edge-coloring of a graph G corresponds to an edge-decomposition of G into three graphs. The following definition is due to Gonçalves [14].

Definition 3.4 (Star, Star arboricity)**.** A *star* is a tree in which all the edges are incident to the same vertex. The *star arboricity* $\mathrm{sa}(G)$ of a graph G is the smallest number of forests needed to cover all the edges of G such that each connected component of each forest is a star.

If $\mathrm{UOP}(n)$ admits an edge-decomposition into three star forests, paths of length at least 3 can be avoided. Our aim will be to construct such a decomposition.

In order to color $\mathrm{UOP}(n)$ such that we can avoid long paths, we have to understand the structure of $\mathrm{UOP}(n)$. We seek to iteratively construct a 3-edge-coloring from a vertex-coloring of $\mathrm{UOP}(n)$ by coloring new edges uv, where u is a vertex of the previous iteration with the color of the vertex u. Therefore, it will be helpful that almost all edges of $\mathrm{UOP}(n)$ are incident to a vertex in $\mathrm{UOP}(n-1)$.

Lemma 3.5. *Let $n \in \mathbb{N}$, $uv \in E(\mathrm{UOP}(n))$ and $\mathrm{gen}(u) \geq 2$. Then either u is parent of v or v is parent of u. In particular $\mathrm{gen}(u) \neq \mathrm{gen}(v)$.*

Proof. As $u \notin V(\mathrm{UOP}(1))$ there is a $k \in \mathbb{N}$ such that the edge uv has been introduced in $\mathrm{UOP}(k+1)$. Without loss of generality $u \in V(\mathrm{UOP}(k))$ and $v \notin V(\mathrm{UOP}(k))$. Then u is a parent of v by construction. □

By coloring the edges of $\mathrm{UOP}(n)$ for all $n \in \mathbb{N}$, we obtain a coloring of every outerplanar graph, as $\mathrm{UOP}(n)$ is universal by Lemma 3.2. Note that a 3-edge-coloring of $\mathrm{UOP}(n)$ where every color class is a star forest induces an edge-decomposition of $\mathrm{UOP}(n)$ into three star forests. Thus if we can decompose $\mathrm{UOP}(n)$ into three star forests, the same holds for every outerplanar graph.

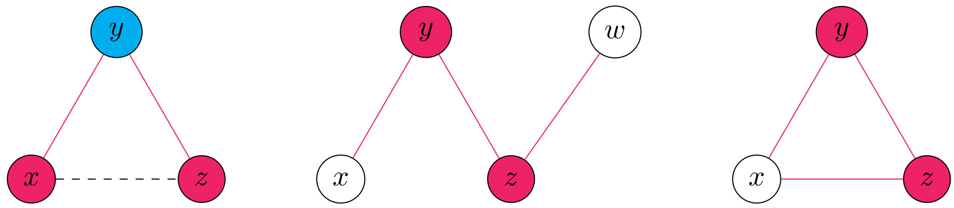
Theorem 3.6. *Every outerplanar graph can be edge-decomposed into at most three star forests.*

Proof. As we just noticed, a 3-edge-coloring of $\mathrm{UOP}(n)$ where every color class is a star forest, induces an edge-decomposition of $\mathrm{UOP}(n)$ into three star forests, hence, it will be sufficent to construct such a coloring.

Ding et al. showed a generalization of a similar statement. They proved that every so-called k-tree can be edge-decomposed into $k+1$ star forests [11, Theorem 4.1]. We proceed analogously to the proof given by Ding et al. Let $V(\mathrm{UOP}(1)) = \{v_1, v_2, v_3\}$. Consider the proper vertex-coloring c_1 where $c_1(v_i) = i$ for all $i \in \{1,2,3\}$. Let $c_i \colon V(\mathrm{UOP}(i)) \to \{1,2,3\}$ be the extension of the vertex-coloring c_{i-1} such that every vertex of the i-th generation has a different color than its parents. Then c_i is clearly a proper vertex-coloring since every newly introduced vertex is only adjacent to its parents.

Consider the edge-coloring $c_1' \colon E(\mathrm{UOP}(1)) \to \{1,2,3\}$ where $c_1(v_1v_2) = 1$, $c_1(v_2v_3) = 2$ and $c_1(v_3v_1) = 3$. Let $c_i' \colon E(\mathrm{UOP}(i)) \to \{1,2,3\}$ be the extension of the edge-coloring c_{i-1}' such that for any newly introduced edge uv with $\mathrm{gen}(v) = i$ we have $c_i(uv) = c_i(u)$. The edge-coloring c_i' is well-defined as for every newly introduced edge exactly one of its endpoints is part of the i-th generation.

Claim 3.6.1. *Let xyz be a monochromatic path in $\mathrm{UOP}(n)$, i.e., $c_n'(xy) = c_n'(yz)$. Then $c_n(y) = c_n'(xy)$.*



(a) If xyz is a monochromatic path in the edge-coloring of UOP(n) and the color of y is different to the path's color, then x and z are not connected.

(b) If there is a monochromatic path $xyzw$ of length 3 in the edge-coloring of UOP(n), then the vertices y and z have the same color as the edges of the path.

(c) If there is a monochromatic triangle xyz in the edge-coloring of UOP(n), then the vertices y and z have the same color as the edges of the triangle.

Figure 3.3: Illustration of the proof of Theorem 3.6. Dashed edges represent edges of the complement graph. White colored vertices represent a not specified color of a vertex.

Proof of Claim. Let $a := c'_n(xy)$.

Case 1. $\mathrm{gen}(y) \geq 2$. Suppose $c_n(y) \neq a$. By Lemma 3.5, x and y have a parent-child-relationship and by definition of c'_n we have $c'_n(x) = a$. Analogously $c'_n(z) = a$. In particular x and z are parents of y. But as c_n is a proper vertex-coloring x and z are not connected and therefore x and z cannot both be parents of y; see Figure 3.3a. This is a contradiction.

Case 2. $\mathrm{gen}(y) = 1$. If $\mathrm{gen}(x) \geq 2$ or $\mathrm{gen}(z) \geq 2$ then by definition of c'_n we have $c_n(y) = a$ as x or z are children of y. Thus, the claim holds. Otherwise $x, y, z \in V(\mathrm{UOP}(1))$. As all edges in $E(\mathrm{UOP}(1))$ are colored differently this is a contradiction to $c'_n(xy) = c'_n(yz)$. ⌟

Claim 3.6.2. *There is no monochromatic path of length* 3 *in* c'_n.

Proof of Claim. Suppose there is a monochromatic path $P = xyzw$ of length 3 in c'_n in color $a \in \{1, 2, 3\}$. Then by Claim 3.6.1 $c_n(y) = a$ and for the same reason $c_n(z) = a$; see Figure 3.3b. This is a contradiction as c_n is a proper coloring but y and z are adjacent. ⌟

Claim 3.6.3. *There is no monochromatic cycle in* c'_n.

Proof of Claim. By Claim 3.6.2 a monochromatic cycle C must have length 3. Suppose there is such a cycle $C = xyz$ in color a. Then by Claim 3.6.1 $c_n(y) = a$ and $c_n(z) = a$; see Figure 3.3c. This is a contradiction as c_n is a proper coloring but y and z are adjacent. ⌟

Therefore the color classes of c'_n induce an edge-decomposition of UOP(n) into three star forests. As any outerplanar graph G is a subgraph of UOP(n) by Lemma 3.2 for sufficiently large n, the claim also holds for all outerplanar graphs. □

Corollary 3.7. *P_k is 3-outerplanar avoidable for $k \geq 3$.*

Proof. Let G be an outerplanar graph. By Theorem 3.6 there is a coloring c of G such that each color class is a star forest. Then a longest monochromatic path in c has length at most two. □

4 P_k-free Colorings of the Iterated Triangulation

As long monochromatic paths are 2-outerplanar unavoidable, thus in particular 2-planar unavoidable, we are now interested in the case of edge-coloring planar graphs using three colors.

4.1 Star Arboricity of the Iterated Triangulation

One specific family of graphs has often proved to be useful. The following definition is due to Axenovich et al. [3].

Definition 4.1 (Iterated Triangulation)**.** An *iterated triangulation* is a plane graph $\mathrm{Tr}(k)$ defined as follows: $\mathrm{Tr}(0) = K_3$ is a triangle, $\mathrm{Tr}(i) \subseteq \mathrm{Tr}(i+1)$, $\mathrm{Tr}(i+1)$ is obtained from $\mathrm{Tr}(i)$ by inserting a vertex in each of the inner faces of $\mathrm{Tr}(i)$ and connecting this vertex with edges to all the vertices on the boundary of the respective face. An illustration to the construction of $\mathrm{Tr}(2)$ is given in Figure 4.1. If v is introduced in $\mathrm{Tr}(i)$ we say that v is part of the *i-th generation* and write $\mathrm{gen}(v) = i$. We see that there is exactly one vertex c of the first generation. We call c the *center*. Note that $\mathrm{Tr}(i)$ is a triangulation and each triangle of $\mathrm{Tr}(i)$ bounds a face of $\mathrm{Tr}(j)$ for some $j \leq i$.

The graph $\mathrm{Tr}(n)$ has proved to be a useful tool for showing that a graph is 2-planar-unavoidable. Indeed, for every graph G for which Axenovich et al. proved that G is 2-planar unavoidable, we have that G is already unavoidable in the family of iterated triangulations [3, proof of Theorem 1], i.e., for large enough $n \in \mathbb{N}$ we have

$$\mathrm{Tr}(n) \to_2 G.$$

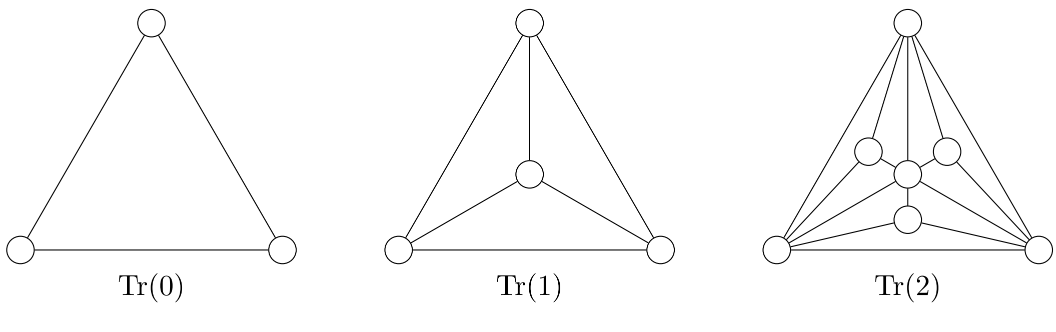
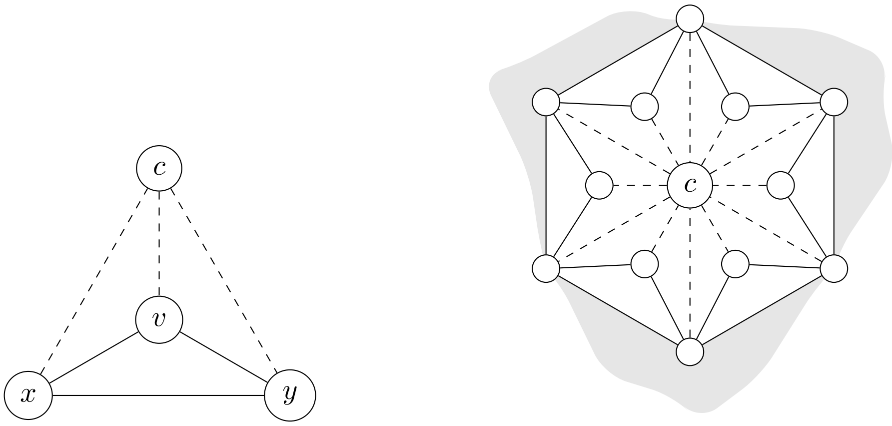


Figure 4.1: The iterative construction of the iterated triangulation Tr(2).



(a) Inserting a vertex v in a face having c on its boundary.

(b) The edges on the boundary of the face of $\mathrm{CTr}(n) - c$ containing c correspond to the edges $E(\mathrm{UOP}(n)) \setminus E(\mathrm{UOP}(n-1))$.

Figure 4.2: Illustrations to the proof of Proposition 4.2. Dashed edges represent edges incident to c, thus edges that do not belong to the embedding of $\mathrm{UOP}(n)$ in $\mathrm{CTr}(n)$.

One might get the impression that long paths cannot be avoided even when we restrict our graph $G := \mathrm{Tr}(n)$ to the induced graph $G[N(v) \cup \{v\}] \subseteq_{ind} \mathrm{Tr}(n)$ of the closed neighborhood of one vertex $v \in V(\mathrm{Tr}(n))$. But paths of length greater than four can be avoided in $G[N(v) \cup \{v\}]$ as $G[N(v) \cup \{v\}]$ is isomorphic to the universal outerplanar graph where a vertex v that is adjacent to all other vertices has been added.

Proposition 4.2. *Let $n \in \mathbb{N}_0$ be a non-negative integer. Consider the induced subgraph* $\mathrm{CTr}(n) := \mathrm{Tr}(n)[N(c) \cup \{c\}] \subseteq_{ind} \mathrm{Tr}(n)$ *of the neighborhood of the center c of* $\mathrm{Tr}(n)$*. Then for all $k \geq 5$*

$$\mathrm{CTr}(n) \not\twoheadrightarrow_3 P_k.$$

Proof. In order to prove that paths of length at least five are avoidable in 3-edge-colorings of $\mathrm{CTr}(n)$, we will extend an edge-coloring of $\mathrm{UOP}(n)$ avoiding long paths to an edge-coloring of $\mathrm{CTr}(n)$.

Claim 4.2.1. $\mathrm{CTr}(n) - c$ *is isomorphic to* $\mathrm{UOP}(n)$*.*

Proof of Claim. Note that we obtain $\mathrm{CTr}(n)$ from $\mathrm{CTr}(n-1)$ by inserting a vertex v in every face $f \in F(\mathrm{CTr}(n))$ having c on its boundary and joining v to all vertices on the boundary of f; see Figure 4.2a. For every edge xy bounding the face in $\mathrm{CTr}(n-1) - c$ containing c, a new vertex v has been inserted that is adjacent to x and y. We see inductively that the new edges that are not incident to c correspond to the edges $E(\mathrm{UOP}(n)) \setminus E(\mathrm{UOP}(n-1))$; see Figure 4.2b. ⌟

Thus $\mathrm{CTr}(n)$ can be obtained by inserting a vertex c in the outer face of $\mathrm{UOP}(n)$ and connecting c to all other vertices. We already proved in Theorem 3.6 that there is an

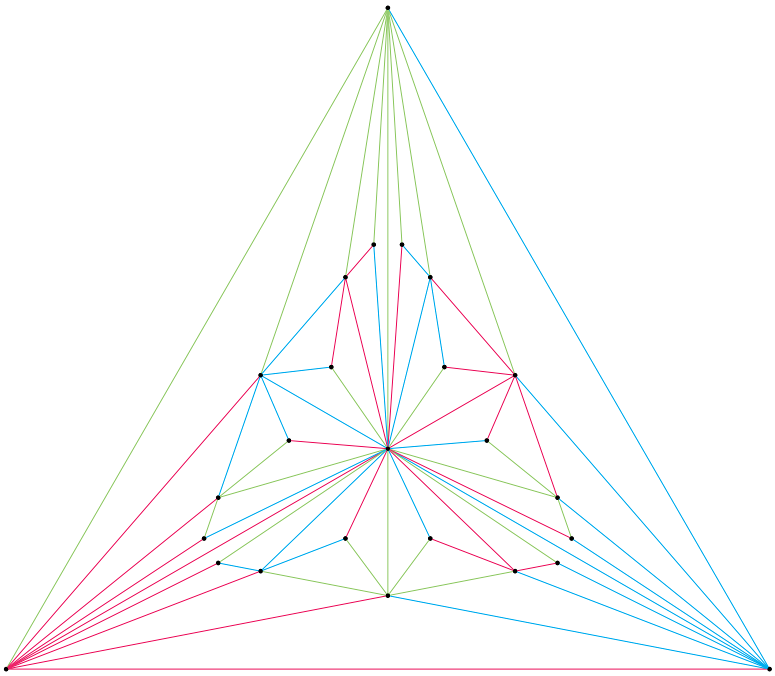


Figure 4.3: An edge-coloring of CTr(4) without a monochromatic P_5.

edge-coloring c'_n of UOP(n) in three colors such that every color class is a star forest. We extend the coloring c'_n used in the proof of Theorem 3.6 to a coloring h_n of CTr(n) as follows:

$$h_n \colon E(\mathrm{CTr}(n)) \to \{1,2,3\}$$
$$uv \mapsto \begin{cases} c_n(u), & v = c \\ c_n(v), & u = c \\ c'_n(uv), & \text{otherwise} \end{cases}$$

This means that we color the edges uc according to the color of u, i.e., we connect the centers of stars to c according to their color; see Figure 4.3. Note that every monochromatic path of length 3 has to pass through c and that no two centers of stars of the same color are connected in c'_n since the color classes of c'_n are star forests. Thus, the color classes of the edge-coloring h_n are trees rooted at c where all leaves have distance at most two from c. Therefore there is no monochromatic path of length greater than four in the coloring h_n. □

As the graph G' induced by the neighborhood of any vertex $v \in V(\mathrm{Tr}(n))$ is isomorphic to the universal outerplanar graph UOP(k) for some k, for large enough n, we can find long paths if the coloring uses only two colors for the edges of G'.

Observation 4.3. *Let C be an edge-coloring of $G \coloneqq \mathrm{Tr}(1+k+n^2)$. If there exists a vertex $v \in V(G)$ with $1 \leq \mathrm{gen}(v) \leq k$ such that C uses only two colors for the induced subgraph $G[N(v)]$, then C contains a monochromatic path of length n.*

Proof. $\mathrm{UOP}(n^2) \subseteq G[N(v)]$. As by Theorem 3.3 we have $\mathrm{UOP}(n^2) \rightarrow_2 P_n$, this shows the claim. □

We were able to decompose $\mathrm{UOP}(n)$ into three star forests (see Theorem 3.6). Using a similar approach, we can decompose $\mathrm{Tr}(n)$ into four star forests. Thus, if we use four colors, long paths are avoidable in $\mathrm{Tr}(n)$.

The result that we will prove is also an application of a result of Ding et al. They proved that every so-called k-tree can be edge-decomposed into $k+1$ star forests [11, Theorem 4.1]. As one can show that the iterated triangulation is a 3-tree, the claim holds. Our approach will be similar to the proof given by Ding et al.

Lemma 4.4. *The iterated triangulation $\mathrm{Tr}(n)$ can be edge-decomposed into at most four star forests for all $n \in \mathbb{N}_0$.*

Proof. We proceed analogously to the proof of Theorem 3.6; see Figure 4.4. Due to the definition of the iterated triangulation, we can inductively define a proper 4-vertex-coloring. This vertex-coloring enables us to define once again a 4-edge-coloring where every color class is a star forest. □

As $\mathrm{Tr}(n)$ can be edge-decomposed into four star forests, paths of length at least 3 are avoidable in $\mathrm{Tr}(n)$.

Corollary 4.5. *Let $\mathcal{T} \coloneqq \{\mathrm{Tr}(n) \mid n \in \mathbb{N}_0\}$. Paths of length at least 3 are 4-$\mathcal{T}$ avoidable, i.e., for all $n \in \mathbb{N}_0$ and all $k \geq 3$, we have*

$$\mathrm{Tr}(n) \nrightarrow_4 P_k.$$

Proof. This follows immediately from Lemma 4.4. □

As long paths are avoidable in 4-edge-colorings of $\mathrm{Tr}(n)$, we are now interested in edge-colorings using three different colors. We could hope that there is an edge-decomposition of $\mathrm{Tr}(n)$ into three star forests. Then long monochromatic paths in 3-edge-colorings of $\mathrm{Tr}(n)$ could be avoided. But this is not the case as the star arboricity of $\mathrm{Tr}(n)$ is four. We will prove this claim using a result of Gonçalves [14, Theorem 1].

Theorem 4.6. *For $n \geq 26$, we have*

$$\mathrm{sa}(\mathrm{Tr}(n)) = 4.$$

Proof. Our aim is to show that $\mathrm{Tr}(n)$ contains a subgraph of star arboricity four.

Claim 4.6.1. *There is a subgraph $T_3 \subseteq \mathrm{Tr}(n)$ for $n \geq 26$, such that*

$$\mathrm{sa}(T_3) = 4.$$

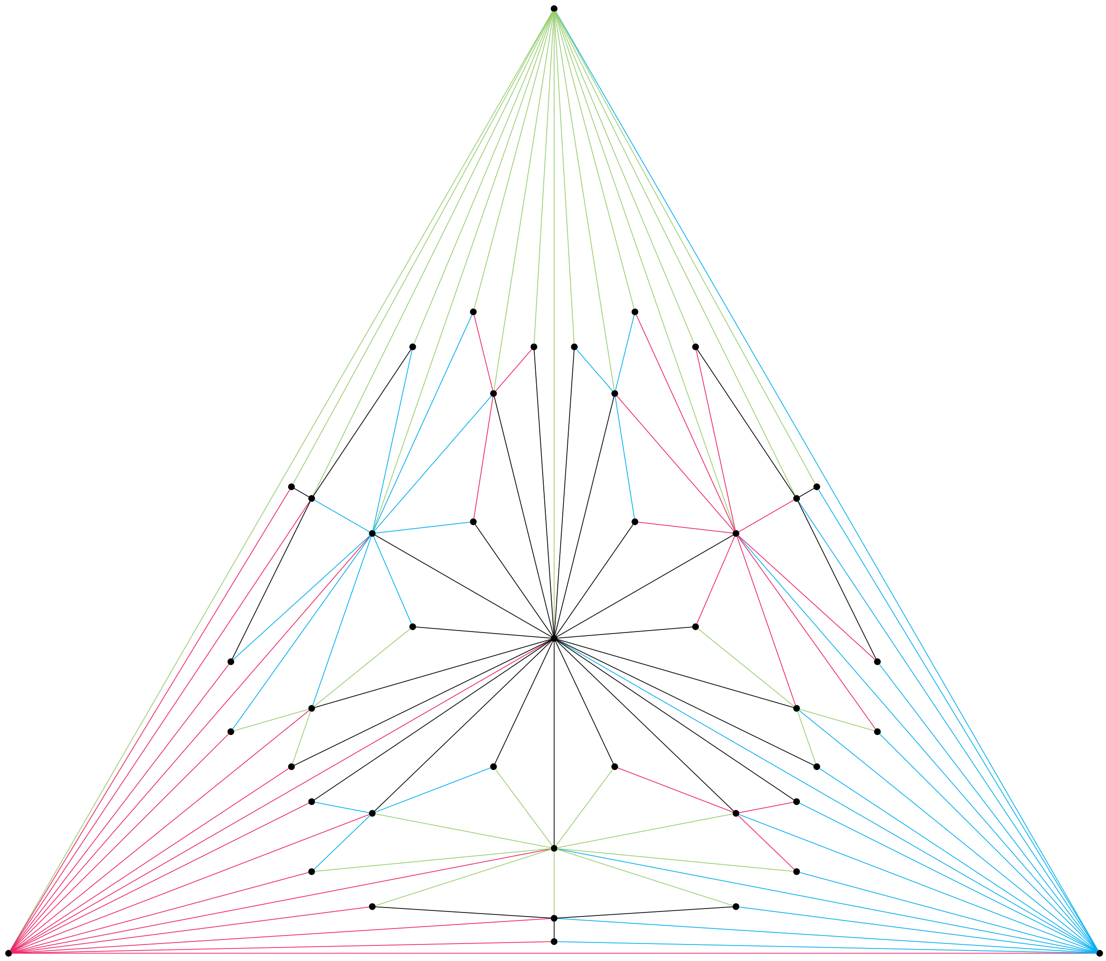


Figure 4.4: An edge-decomposition of Tr(4) into four star forests.

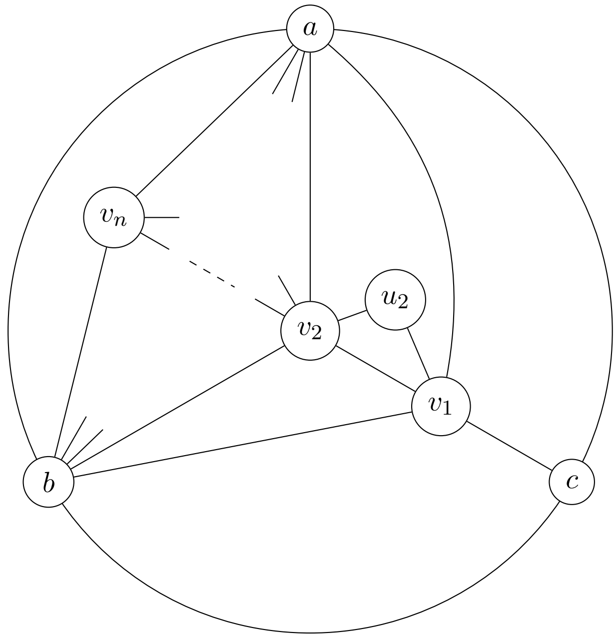


Figure 4.5: The embedding of T_3 in $\mathrm{Tr}(n)$ for $n \geq 26$.

Proof of Claim. Gonçalves constructed for all $k \in \mathbb{N}$ a graph T_k with star arboricity $k+1$ [14, Theorem 1]. We are interested in the case $k = 3$ as we will observe that T_3 is a subgraph of $\mathrm{Tr}(n)$ for sufficiently large n. Thus, we follow the construction of T_k given by Gonçalves for $k = 3$. The graph T_3 is constructed as follows. Let $\{a, b, c\} = V(\mathrm{Tr}(0))$. Let the vertex v_1 be the center of $\mathrm{Tr}(1)$. The induced subgraph $T_3[a, b, c, v_1] \subseteq_{ind} T_3$ of T_3 is a complete graph on four vertices. Let $n := 4k(k-1) + 2 = 4 \cdot 3 \cdot 2 + 2 = 26$. For $1 < i \leq n$ let v_i be the vertex with parents a, b, v_{i-1}. T_3 contains v_i and all three edges connecting v_i to its parent vertices. Therefore, the graph $T_3[a, b, c, v_1, \ldots, v_n] \subseteq \mathrm{Tr}(n)$ is a subgraph of $\mathrm{Tr}(n)$. For all $1 < i \leq n$ let u_i be a vertex of the $(n+1)$-th generation that is a child of v_{i-1} and v_i. T_3 contains u_i and the edges $u_i v_i$ and $u_i v_{i-1}$. Then $T_3 \subseteq \mathrm{Tr}(n+1)$; see Figure 4.5. ⌟

Note that an edge-decomposition of $\mathrm{Tr}(n)$ in i star forests induces such an edge-decomposition of T_3 as $T_3 \subseteq \mathrm{Tr}(n)$. Hence

$$4 = \mathrm{sa}(T_3) \leq \mathrm{sa}(\mathrm{Tr}(n))$$

holds. By Lemma 4.4, we have

$$\mathrm{sa}(\mathrm{Tr}(n)) \leq 4.$$

Thus $\mathrm{sa}(\mathrm{Tr}(n)) = 4$. □

4.2 Paths of Increasing Generation

In order to prove that long paths are 2-outerplanar unavoidable (see Theorem 3.3), Axenovich et al. extended monochromatic paths P in $\mathrm{UOP}(n)$ to monochromatic paths P' in $\mathrm{UOP}(n+1)$ such that the last vertex on P' is of the $(n+1)$-th generation. Thus, we wondered whether long paths in 3-edge-colorings of $\mathrm{Tr}(n)$ are unavoidable, even if we restrict our paths to paths P such that the vertices along P increase in generation. Inspired by the proof given by Axenovich et al. of Theorem 3.3, we introduce the notion of a ping.

Definition 4.7 (Ping)**.** Let $P = v_1 \dots v_k$ be a path in $\mathrm{Tr}(n)$. If for all $i \in \{1, \dots, k-1\}$ we have $\mathrm{gen}(v_i) < \mathrm{gen}(v_{i+1})$, we call P a *path of increasing generation* (ping). We write LP_k to denote a ping of length k.

Note that a ping is not a graph, but a fixed embedding of a subgraph in $\mathrm{Tr}(n)$. By abuse of notation, we will write $\mathrm{Tr}(n) \to_k LP_\ell$ to express that every k-edge-coloring of $\mathrm{Tr}(n)$ contains a monochromatic ping of length ℓ. We will use the notation $\mathrm{Tr}(n) \nrightarrow_k LP_\ell$ analogously.

However, there is a 3-edge-coloring of $\mathrm{Tr}(n)$ such that long monochromatic pings are avoided. Thus, using pings, we cannot prove that long monochromatic paths are 3-planar-unavoidable.

Theorem 4.8. *For all $n \in \mathbb{N}$ and $\ell \geq 3$:*

$$\mathrm{Tr}(n) \nrightarrow_3 LP_\ell$$

Proof. For a vertex $v \in V(\mathrm{Tr}(k))$ such that $\mathrm{gen}(v) = k$ and a color $i \in \{1, 2, 3\}$ let v_i denote the maximum length of a ping in color i ending in v in a given edge-coloring C_k of $\mathrm{Tr}(k)$ in three colors. Note that by definition of a ping, a ping in $\mathrm{Tr}(n)$ ending in a vertex $v \in V(\mathrm{Tr}(n))$ with $\mathrm{gen}(v) = k$ is a ping in $\mathrm{Tr}(k)$. Therefore, it suffices to show that for all $n \in \mathbb{N}$ there is an edge-coloring of $\mathrm{Tr}(n)$ in three colors such that for all $v \in V(\mathrm{Tr}(n))$ and for all $i \in \{1, 2, 3\}$ we have $v_i \leq 2$.

Claim 4.8.1. *There is a 3-edge-coloring C_n of $\mathrm{Tr}(n)$ such that for all $v \in V(\mathrm{Tr}(n))$ we have*

$$\sum_{i=1}^{3} v_i \leq 4 \qquad \textit{and} \qquad \max_{i \in \{1,2,3\}} v_i \leq 2.$$

We prove the claim by induction on n by extending an edge-coloring C_n of $\mathrm{Tr}(n)$ to an edge-coloring C_{n+1} of $\mathrm{Tr}(n+1)$. Let C_0 be a 3-edge-coloring of $\mathrm{Tr}(0)$. For $n = 0$ the length of a monochromatic ping is bounded by 0, as all vertices in $\mathrm{Tr}(0)$ are of generation 0. Assume now that for fixed $n \in \mathbb{N}$ the claim holds.

Let now $v \in V(\mathrm{Tr}(n+1))$ be a vertex of generation $n+1$. Then by definition of $\mathrm{Tr}(n+1)$, $N(v) = \{a, b, c\} \subseteq V(\mathrm{Tr}(n))$. Note that any ping P ending in v has to pass through $N(v)$. Thus $P - v$ is a ping in $\mathrm{Tr}(n)$ ending in $N(v)$.

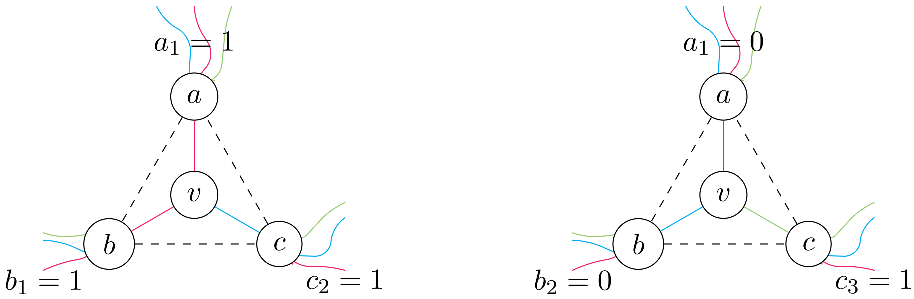


Figure 4.6: Illustrations of Case 1 (left) and Case 2 (right) in the proof of Theorem 4.8.

We call a vertex $u \in \{a, b, c\}$ *good*, if there are two distinct colors $i \neq j$ such that the length of a monochromatic ping in C_k ending in u in color i and j respectively is at most 1, i.e., $u_i, u_j \leq 1$. Otherwise we call u a *bad* vertex.

Case 1. There are at least two distinct good vertices. We may assume without loss of generality that a and b are good. Then by pigeonhole principle, there exists $i \in \{1, 2, 3\}$ such that $a_i, b_i \leq 1$. We may assume $i = 1$. We color the edges av and bv in color 1. By induction hypothesis, there exists $j \in \{1, 2, 3\}$ such that $c_j \leq 1$. We color the edge cv in color j. If $j = 1$, then $v_1 = \max(a_1, b_1, c_1) + 1 \leq 2$ by definition of a ping and $v_\ell = 0$ for $\ell \neq 1$. Thus $\sum_{m=1}^{3} v_m \leq 2 \leq 4$. Otherwise $j \neq 1$. We may assume $j = 2$. Then $v_1 = \max(a_1, b_1) + 1 \leq 2$, $v_2 = c_2 + 1 \leq 2$, $v_3 = 0$ and the claim holds for v; see Figure 4.6.

Case 2. There is at most one good vertex in $\{a, b, c\}$. Then there are at least two bad vertices. We may assume that a and b are bad. By induction hypothesis there exist $q, r, s \in \{1, 2, 3\}$ such that $a_q, b_r = 0$ and $c_s \leq 1$. Then we color av in q, bv in r and cv in s. If q, r and s are distinct, then $v_q = 1$, $v_r = 1$ and $v_s \leq 2$. Thus the claim holds. Otherwise we colored some of these edges in the same color, therefore the bounds still hold.

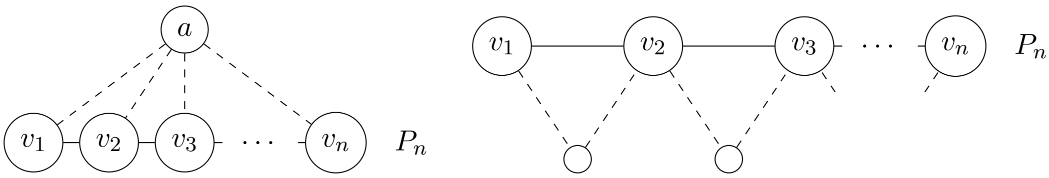
Note that C_{n+1} is welldefined as all edges in $E(\mathrm{Tr}(n+1)) \setminus E(\mathrm{Tr}(n))$ are incident to exactly one vertex of the $(n+1)$-th generation by definition of $\mathrm{Tr}(n+1)$. $\square$

Although, we were able to bound the length of monochromatic pings in a specific 3-edge-coloring of $\mathrm{Tr}(n)$, the same does not necessarily hold for the length of a longest monochromatic path.

Observation 4.9. *Let C be an edge-coloring of $\mathrm{Tr}(n)$ in at most three colors such that the length of a monochromatic ping in $\mathrm{Tr}(n)$ is at most $k \in \mathbb{N}$. Then there is a coloring C' of $\mathrm{Tr}(n+1)$ such that the length of a monochromatic ping in C' is at most $k+1$ but the length of a monochromatic path is at least $2(n-1)$ none the less.*

Proof. Let $a, b \in V(\mathrm{Tr}(n))$ be two distinct vertices of generation 0.

Claim 4.9.1. *There is a path $P_{n-1} = v_1 \dots v_n$ of length $n-1$ in $\mathrm{Tr}(n)$ such that for all $i \in \{1, \dots, n\}$ we have $\mathrm{gen}(v_i) = i$ and v_i is adjacent to a and b.*



(a) All edges v_iv_{i+1} of the path P_n belong to different faces.

(b) If all the edges that do not belong to P_n are colored the same, then the path represented by the dashed edges is a long monochromatic path.

Figure 4.7: Illustrations to the proof of Observation 4.9. The initial path P_n is represented by non-dashed edges. All edges that do not belong to P_n are dashed.

Proof of Claim. This can be seen by induction on the length of the path P_{n-1}. For $n = 1$, let v_1 be the center of the triangle. Then v_1 is clearly of generation 1 and is adjacent to a and b. Assume there is such a path P_{n-1} in $\mathrm{Tr}(n)$ such that every vertex v_i on P_{n-1} is adjacent to a and b. Then v_n is of generation n and therefore a, b and v_n form a face in $\mathrm{Tr}(n)$. Therefore, there is a vertex $v_{n+1} \in V(\mathrm{Tr}(n+1))$ of the $(n+1)$-th generation that is adjacent to a,b and v_n. Thus, we can extend P_{n-1} to a path P_n as stated. ⌟

As $\mathrm{Tr}(n) \subseteq \mathrm{Tr}(n+1)$, P_{n-1} is a path in $\mathrm{Tr}(n+1)$. Note that every edge v_iv_{i+1} belongs to a different face as $\mathrm{Tr}(n+1)$ is a plane graph and all v_i are adjacent to a; see Figure 4.7a. Therefore for every edge v_iv_{i+1} of P_{n-1} there is a vertex u_i in $\mathrm{Tr}(n+1)$ of the $(n+1)$-th generation that is adjacent to v_i and v_{i+1} by definition of $\mathrm{Tr}(n+1)$ and all the vertices u_i are distinct. Consider the coloring C' of $\mathrm{Tr}(n+1)$ that extends C where all new edges are colored in red. Then in particular all edges u_iv for $v \in \{v_i, v_{i+1}\}$ are red. Therefore, the path $v_1u_1v_2u_2 \dots u_{n-1}v_n$ is a monochromatic path of length $2(n-1)$; see Figure 4.7b. However by definition of a ping, the length of a monochromatic ping in C' is at most one more than the length of a monochromatic ping in C. □

Corollary 4.10. *Assume there exists $k \in \mathbb{N}$ and a sequence $(C_n)_{n\in\mathbb{N}_0}$ such that C_n is an edge-coloring of $\mathrm{Tr}(n)$ in at most three colors and the length of a monochromatic ping in C_n is at most k for all $n \in \mathbb{N}_0$. Then for all $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ and an edge-coloring C' in at most three colors of $\mathrm{Tr}(m)$ such that the length of a monochromatic ping in $\mathrm{Tr}(m)$ is bounded by $k+1$, but $\mathrm{Tr}(m)$ contains a monochromatic path of length n.*

Thus, bounding the length of a longest monochromatic ping in an edge-coloring of $\mathrm{Tr}(n)$ does not enable us to bound the length of a longest monochromatic path.

5 Subdivisions of Stars

We will be interested in graphs that we obtain by drawing new vertices on some or all edges of a star. These graphs are called subdivisions of stars; see Figure 5.1 for an illustration. A formal definition is given below. It is based on the definitions of subdivisions given by Diestel [10, p. 19] and Axenovich [2, p. 39].

Definition 5.1 (subdivision)**.** Let G be a graph and let $uv \in E(G)$ be an edge of G. A *single-edge-subdivision* of G is a graph $G' = (V(G) \cup \{c\}, E(G) \setminus \{uv\} \cup \{uc, cv\})$, where $c \notin V(G)$. Informally we obtained G' from G by inserting a new vertex on the edge uv. A *subdivision* of G is a graph that is obtained by a series of single-edge-subdivisions.

Our aim is to show that all subdivisions of stars are 2-planar unavoidable. We have already been interested in the iterated triangulation where we inserted iteratively new vertices in inner faces. Generalizing this approach, we obtain k-trees.

Definition 5.2 (k-tree, [11, p. 2])**.** Let $k \in \mathbb{N}$. A *k-clique* is a complete graph on k vertices. A *k-tree* is a graph defined inductively as follows: A k-clique is a k-tree. If G is a k-tree, and C is a k-clique of G, then a graph obtained from G by adding a new vertex and connecting it with edges to all vertices of C is a k-tree. Any subgraph of a k-tree is called a *partial k-tree.*

Due to this definition, $\mathrm{Tr}(n)$ is a 3-tree. We will be interested in a specific class of k-trees. The following definition is due to Ding et al. [11, Chapter 3].

Definition 5.3 ([11, p. 4])**.** Let $k \in \mathbb{N}$. Let $T(k, 0, r)$ be a k-clique. We call a k-clique that is introduced in $T(k, i, r)$ a k-clique of the *i-th generation.* The k-tree $T(k, i+1, r)$ is obtained from $T(k, i, r)$ by inserting for every k-clique C of the i-th generation r vertices $v_1, \dots, v_r$ and connecting each vertex v_i with edges to all vertices of C.

The construction of $T(k, i, r)$ reminds us of the construction of the universal outerplanar graph and the iterated triangulation. In order to get a better understanding of the

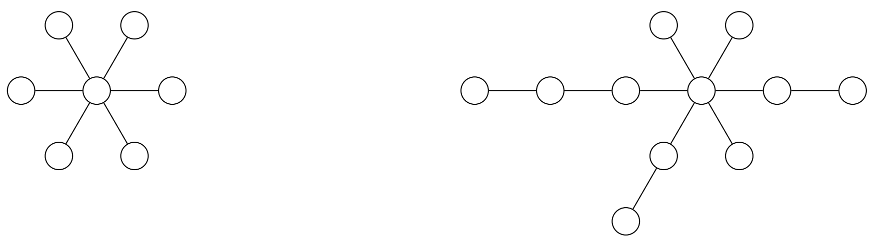


Figure 5.1: On the left, we see a star S and on the right a subdivision of the star S.

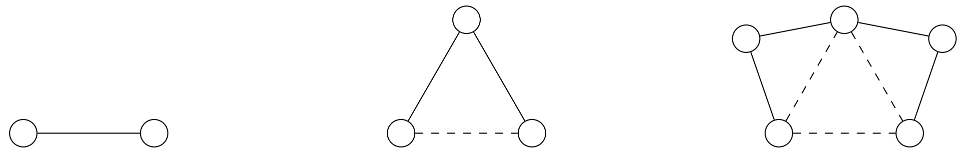


Figure 5.2: The iterative construction of $T(2, \ell, 1)$. The non dashed edges represent 2-cliques of the ℓ-th generation, the 2-cliques of smaller generation are represented as dashed edges.

definition above, we well relate the construction of $T(2, \ell, 1)$ to the definition of the univeral outerplanar graph. Having the definition of the universal outerplanar graph in mind, see Definition 3.1, wee can see inductively that $T(2, \ell, 1)$ is a subgraph of $\mathrm{UOP}(\ell)$ for all $\ell \in \mathbb{N}$. Note however that $T(2, \ell, 1)$ is in general not isomorphic to $\mathrm{UOP}(\ell)$.

Example 5.4. *The 2-tree $T(2, \ell, 1)$ is a subgraph of the universal outerplanar graph $\mathrm{UOP}(\ell)$ for all $\ell \in \mathbb{N}$.*

Proof. We will prove a stronger claim by induction on ℓ.

Claim 5.4.1. *For all $\ell \in \mathbb{N}$ the graph $T(2, \ell, 1)$ is isomorphic to a subgraph of $\mathrm{UOP}(\ell)$. Further, we can embed $T(2, \ell, 1)$ in $\mathrm{UOP}(\ell)$ such that the 2-cliques of the ℓ-th generation of $T(2, \ell, 1)$ are outer edges of $\mathrm{UOP}(\ell)$.*

Proof of Claim. Let $\ell = 0$. Then $T(2, 0, 1)$ is a single edge. Thus $T(2, 1, 1)$ is a triangle as we added a single new vertex and connected this vertex to the only 2-clique. Therefore, the 2-cliques of the first generation are outer edges of $\mathrm{UOP}(1)$ and $T(2, 1, 1) \cong \mathrm{UOP}(1)$. Assume the claim holds for some $\ell \in \mathbb{N}$. Then we obtain $\mathrm{UOP}(\ell + 1)$ from $\mathrm{UOP}(\ell)$ by adding a new vertex for every outer edge uv and connecting this vertex to u and v. As by induction $T(2, \ell, 1)$ can be embedded as a subgraph in $\mathrm{UOP}(\ell)$ such that all 2-cliques of the ℓ-th generation are outer edges, we see that the newly created vertices and edges of $T(2, \ell + 1, 1)$ correspond to newly added vertices and edges of $\mathrm{UOP}(\ell + 1)$. Hence, the claim holds. ⌟

Note that $\mathrm{UOP}(2) \ncong T(2, 2, 1)$ as one of the outer edges of $\mathrm{UOP}(1)$ is not a 2-clique of the first generation of $T(2, 1, 1)$; see Figure 5.2. □

Another class of k-trees that we will be interested in are the 1-trees $T(1, \ell, r)$. Indeed, the 1-trees $T(1, \ell, r)$ are trees were all non-leaves have exactly r children.

Example 5.5. *For all $\ell \in \mathbb{N}_0$ and $r \in \mathbb{N}$, the graph $T(1, \ell, r)$ is a tree that can be rooted such that every non-leaf has exactly r children and every leaf has distance ℓ from the root.*

Proof. Let $r \in \mathbb{N}$. For all $\ell \in \mathbb{N}_0$ let $T_\ell := T(1, \ell, r)$. By definition of T_0, there is a single vertex s of the 0-th generation. We show a stronger claim by induction on ℓ.

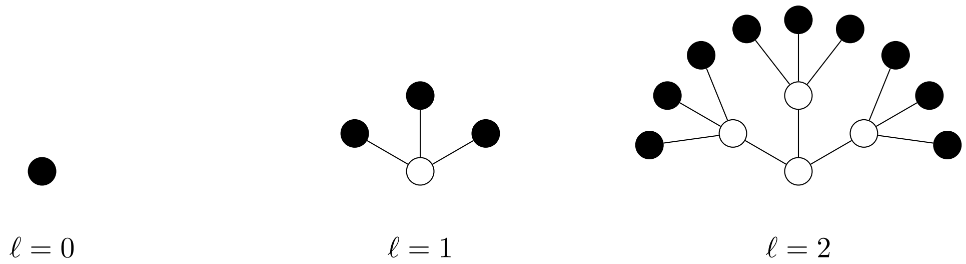


Figure 5.3: The iterative construction of $T(1,\ell,3)$. Black vertices represent 1-cliques of the ℓ-th generation.

Claim 5.5.1. *For all $\ell \in \mathbb{N}_0$ the graph T_ℓ is a tree with root s where all non leaves have r children and the leaves of T_ℓ are the vertices of the ℓ-th generation. Further for all leaves v of T_ℓ we have $\mathrm{dist}_{T_\ell}(s,v) = \ell$.*

If $\ell = 0$ then T_0 is a single vertex. In particular, $\mathrm{dist}_{T_0}(s,s) = 0$. Assume now the claim holds for some $\ell \in \mathbb{N}$. We get $T_{\ell+1}$ from T_ℓ by adding for every vertex v of the ℓ-th generation r new vertices and connecting these with edges to v; see Figure 5.3. Thus, as by induction all vertices of the ℓ-th generation were leaves in T_ℓ, they have exactly r children in $T_{\ell+1}$. The neighborhood of vertices of generation less than ℓ did not change and the newly added vertices are leaves in $T_{\ell+1}$. Hence, every non-leaf has r children. Since we added as many vertices as edges to the tree T_ℓ in such a way that $T(\ell+1)$ is connected, $T_{\ell+1}$ is a tree by the tree equivalence theorem [10, Corollary 1.5.3]. Further, the leaves of $T_{\ell+1}$ are the vertices of the $(\ell+1)$-th generation. Let v be a vertex of the $(\ell+1)$-th generation. Its parent p is a vertex of the ℓ-th generation. As $T_{\ell+1}$ is a tree, there is exactly one v-s path. This path has to pass through v, thus by induction we get

$$\mathrm{dist}_{T_{\ell+1}}(s,v) = \mathrm{dist}_{T_\ell}(s,p) + 1 = \ell + 1.$$

□

We wish to prove that $T(2,\ell,r)$ is a planar graph. In order to obtain this result, we consider the subgraph $T(2,1,r)$ of $T(2,\ell,r)$ that we will call a fishbone. The following definition is based on the definition of a fish given by Axenovich et al. [3].

Definition 5.6 (Fishbone)**.** A graph G is called a *fishbone* if $V(G) = \{x,y\} \cup S$, where $S \cap \{x,y\} = \varnothing$, x and y are each adjacent to each vertex in S, xy is an edge of G and S is an independent set, i.e., there are no edges between the vertices of S. We call S the set of *spine vertices*. A Fishbone on x and y is denoted by $\mathrm{Fb}_{x,y}$. Note that $\mathrm{Fb}_{x,y} = T(2,1,|S|)$ and $\mathrm{Fb}_{x,y}$ is a planar graph; see Figure 5.4.

The 2-tree $T(2,\ell,r)$ is obtained from $T(2,\ell-1,r)$ by replacing some edges with a fishbone with r spine vertices. This observation enables us to show inductively that $T(2,\ell,r)$ is a planar graph.

Observation 5.7. *The graph $T(2,\ell,r)$ is planar for all $\ell, r \in \mathbb{N}$.*

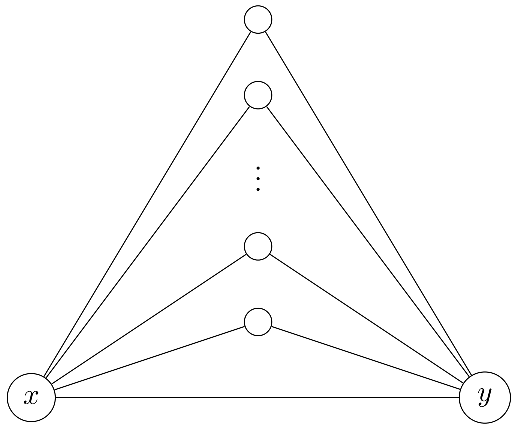


Figure 5.4: A fishbone $\mathrm{Fb}_{x,y}$ with k spine vertices.

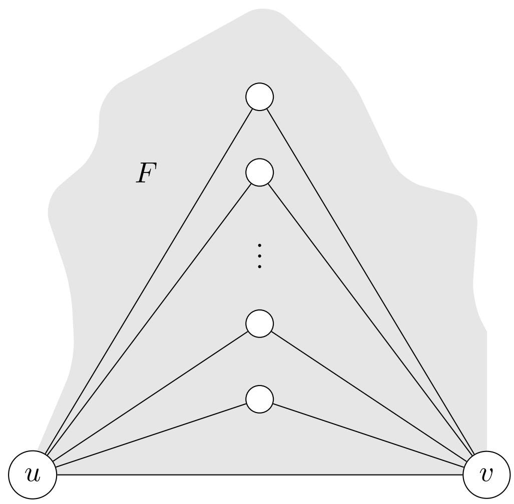


Figure 5.5: Replacing the edge uv with the fishbone $\mathrm{Fb}_{u,v}$ within the face F.

Proof. We will prove the claim by induction on ℓ. For $\ell = 0$ the graph $T(2, 0, r)$ is a single edge and therefore planar. Assume that there is a plane embedding of $T(2, \ell, r)$. We wish to show that $T(2, \ell + 1, r)$ is planar. The graph $T(2, \ell + 1, r)$ is obtained by replacing the edges uv of the ℓ-th generation in $T(2, \ell, r)$ with a fishbone $\mathrm{Fb}_{u,v}$ with r spine vertices. For such an edge uv let F be a face such that uv lies on its boundary. We can replace the edge uv with $\mathrm{Fb}_{u,v}$ by drawing all spine vertices in the face F; see Figure 5.5. Therefore, $T(2, \ell + 1, r)$ is clearly a planar graph. $\square$

Ding et al. showed that for all positive integers $\ell, r \in \mathbb{N}$, there exist constants $L, R \in \mathbb{N}$ such that every 2-edge-coloring of $T(2, L, R)$ contains a monochromatic subdivision of $T(1, \ell, r)$. Thus these trees are in a certain way planar unavoidable. Note that the monochromatic subdivisions of $T(1, \ell, r)$ in a 2-edge-coloring of $T(2, L, R)$ might however depend on the coloring. The following result follows immediately from a result of Ding et al. concerning 2-edge-colorings of certain 2-trees [11, Theorem 3.5].

Theorem 5.8. *For every tree T, there is planar graph G such that every coloring of G in two colors contains a monochromatic subdivision of T.*

Proof. Let $v \in V(T)$ be a vertex and consider the tree T as rooted at v. Let ℓ be the maximum distance of a vertex in $V(T)$ to the root v. Let r be the maximum number of children of a vertex in $V(T)$ considering the tree as rooted at v. The tree T is a subgraph of $T(1, \ell, r)$ as $T(1, \ell, r)$ is a tree that can be rooted such that every non-leaf has r children and every leaf has distance ℓ from the root as seen in Example 5.5. Ding et al. showed that there are positive integers $L, R \in \mathbb{N}$ such that every edge-coloring of $T(2, L, R)$ in blue and red contains a red $T(1, \ell, r)$ or a blue subdivision of $T(1, \ell, r)$ [11, Theorem 3.5]. The claim follows since $T(2, L, R)$ is planar by Observation 5.7. $\square$

As every subdivision of a subdivided star S contains S as a subgraph, we can prove that subdivisions of stars are 2-planar unavoidable. This generalizes a result of Axenovich et al. They showed that all generalized brooms are 2-planar unavoidable [3, Lemma 7].

Corollary 5.9. *All subdivisions of a star are 2-planar unavoidable.*

Proof. Let S be a subdivision of a star. Then every subdivision S' of S contains S as a subgraph. By Theorem 5.8 S is 2-planar unavoidable. $\square$

6 Bounded Diameter Arboricity

We were able to decompose every outerplanar graph into three star forests; see Theorem 3.6. Thus, paths of length 3 are 3-outerplanar avoidable. Similarly, we can consider decompositions of planar graphs into forests. However, such a decomposition might contain a long path as a subgraph. In order to bound the length of a longest path in an edge decomposition, we will use the notion of bounded diameter arboricity that was introduced by Merker and Postle [22].

6.1 Bounded Diameter Arboricity of Planar Graphs

We will now be interested in a better understanding of 3-planar unavoidable graphs. More precisely, we wish to identify necessary conditions for a graph H to be 3-planar unavoidable. Axenovich et al. discussed this question [3]. One can easily see that such a graph H has to be a forest. We will prove this claim in Proposition 6.2. In order to understand the details of the proof, we have to define arboricity that we already encountered in Definition 3.4 through the notion of star arboricity.

Definition 6.1 (Arboricity). The *arboricity* $\Upsilon(G)$ of a graph G is the smallest number of forests needed to cover all the edges of G.

Proposition 6.2 ([3, Theorem 3]). *Every 3-planar unavoidable graph is a forest.*

Proof. Note that an edge-decomposition of a graph G into three forests induces a 3-edge-coloring such that every monochromatic subgraph $H \subseteq G$ is a forest. Thus, it will be sufficient to show that all planar graphs have arboricity at most 3. This follows from Nash-Williams formula [23] that states the following.

Claim 6.2.1 ([10, Theorem 2.4.3]). *The edges of a graph G can be covered by at most k trees if and only if*

$$|E(G[U])| \leq k \cdot (|U| - 1) \tag{6.1}$$

for every non-empty set $U \subseteq V(G)$.

Set $k \coloneqq 3$ and let G be a planar graph. Let $U \subseteq V(G)$ be a non-empty set. If $|U| = 1$, then $G[U]$ contains no edges, thus Inequality 6.1 holds. If $|U| = 2$, the induced graph $G[U]$ contains at most one edge, therefore the Inequality 6.1 holds once again. Otherwise $G[U]$ is a planar graph on at least three vertices, as every subgraph of a planar graph is planar. Therefore, the number of edges in $G[U]$ is bounded by $3|U| - 6$. Hence, we get

$$|E(G[U])| \leq 3 \cdot |U| - 6 = 3 \cdot (|U| - 2) \leq 3 \cdot (|U| - 1).$$

As Inequaltiy 6.1 holds for every non-empty set $U \subseteq V(G)$, we have by Claim 6.2.1

$$\Upsilon(G) \leq 3.$$

Thus, every planar graph can be edge-decomposed into three forests. □

Using star arboricity, we showed that long monochromatic paths in 4-edge-colorings of $\mathrm{Tr}(n)$ can be avoided. Due to a result of Hakimi, Mitchem, and Schmeichel [16], a similar results holds for 5-edge-colorings of planar graphs, as every planar graph can be edge-decomposed into at most five star forests. As stars have diameter at most two, we can avoid paths of length at most 3 in star forest decompositions. Similarly, if we are able to decompose a graph into k forests of diameter at most d, paths of length at least $d+1$ are avoidable using k colors. We wish to avoid long paths in planar graphs using less than five colors, thus we might want to allow a larger diameter in our forest decomposition. The following definition generalizes this concept. It is due to Merker and Postle [22, Definition 1.1].

Definition 6.3 (bounded diameter arboricity)**.** The *diameter-d arboricity* $\Upsilon_d(G)$ of a graph G is the minimum number k such that the edges of G can be partitioned into k forests each of whose components have diameter at most d. The *bounded diameter arboricity* $\Upsilon_{bd}(\mathcal{G})$ of a class of graphs $\mathcal{G}$ is the minimum number k for which there exists a natural number d such that every $G \in \mathcal{G}$ has diameter-d arboricity at most k.

We are interested in the bounded diameter arboricity of planar graphs. If there exists some positive integer $k \in \mathbb{N}$ and a constant d such that every planar graph can be edge-decomposed into k forests of diameter at most d, long paths are k-planar avoidable. Merker and Postle introduced the notion of bounded diameter arboricity and studied its applications to planar graphs [22]. The following lemma is a simple application of one of their results concerning the bounded diameter arboricity of planar graphs [22, Theorem 3.5].

Theorem 6.4. *Let $g \in \mathbb{N}$ and let $\mathcal{P}_g$ be the class of planar graphs of girth at least g. There is a positive integer $d_g \in \mathbb{N}$ such that*

1. *if $g = 3$ and H is 4-$\mathcal{P}_g$ unavoidable*
2. *if $g = 4$ and H is 3-$\mathcal{P}_g$ unavoidable*
3. *if $g \geq 6$ and H is 2-$\mathcal{P}_g$ unavoidable*

then H is a forest of diameter at most d_g. In particular, all paths P_k for $k > d_g$ are 4-planar avoidable.

Proof. We will only prove the first claim, as all others can be verified in a similar way. Note that all planar graphs have girth at least 3, thus $\mathcal{P}_3$ is precisely the class of all planar graphs. Since $\Upsilon_{bd}(\mathcal{P}_3) = 4$, as has been shown by Merker and Postle [22, Theorem 3.5], there exists a positive integer $d := d_3 \in \mathbb{N}$ such that every planar graph can be edge-decomposed into four forests of diameter at most d. Therefore, if H is

4-planar unavoidable, then H has to be a subgraph of a forest of diameter at most d. As every such subgraph is a forest of diameter at most d, the claim follows. Note that any path P in a tree T of diameter at most d is in particular a shortest path by uniqueness of a-b paths in a tree. Thus

$$|E(P)| \leq \operatorname{diam}(T) \leq d.$$

Therefore, P_k is 4-planar avoidable for $k > d$ as all planar graphs can be edge-decomposed into four forests of diameter at most d. □

Merker and Postle calculated the bounded diameter arboricity not only for the class of all planar graphs, but also for several subclasses [22, Theorem 3.5]. We will only restate their proof for estimating the bounded diameter arboricity of the class of planar graphs [22, Theorem 3.5].

Theorem 6.5 ([22, Theorem 3.5])**.** *Let $\mathcal{P}$ denote the class of planar graphs. Then*

$$\Upsilon_{bd}(\mathcal{P}) = 4.$$

Let $\mathcal{A}_k$ denote the class of graphs with arboricity at most k. Merker and Postle showed that the bounded diameter arboricity of $\mathcal{A}_3$ is four [22, Theorem 1.3]. We already know that $\mathcal{P} \subseteq A_3$ as seen in the proof of Proposition 6.2. Thus $\Upsilon_{bd}(\mathcal{P}) \leq \Upsilon_{bd}(\mathcal{A}_3) = 4$. If we wish to show that the bounded diameter arboricity of the class of planar graphs is four, we have to find a family $\mathcal{G} \subseteq \mathcal{P}$ of graphs, such that for all $d \in \mathbb{N}$ there is a graph $G \in \mathcal{G}$ that cannot be edge-decomposed into three forests of diameter at most d.

Lemma 6.6 ([22, Lemma 3.4])**.** *Let $\mathcal{G}$ be a family of graphs with arboricity at most k and c a natural number. If there exists a sequence of graphs $G_1, G_2, \ldots$ in $\mathcal{G}$ such that $\operatorname{diam} G_i \geq i$ and $|E(G_i)| \geq k|V(G_i)| - c$ for all i, then $\Upsilon_{bd}(\mathcal{G}) \geq k+1$.*

Proof. Suppose $\Upsilon_{bd}(\mathcal{G}) \leq k$. Then there exists a positive integer $d \in \mathbb{N}$ such that for all $G \in \mathcal{G}$: $\Upsilon_d(G) \leq k$. Let $G := G_{cd+1}$. By assumption G can be edge-decomposed into k forests $F_1, \ldots, F_k$ such that each tree in F_j has diameter at most d for all j. Let $\mathcal{T}_j$ denote the set of components in F_j for all j. Note that every vertex $v \in V(F_j)$ belongs to exactly one component, as no two components share a vertex. Since every tree on n vertices has $n-1$ edges and all components of F_j are trees, we can bound the number of edges in every forest F_j for all j:

$$|E(F_j)| = \sum_{T \in T_j} |E(T)| = \sum_{T \in T_j} (|V(T)| - 1) = |V(F_j)| - |T_j| \leq |V(G)| - |T_j| \qquad (6.2)$$

Let $\mathcal{T} := \bigcup_{j=1}^{k} \mathcal{T}_j$. By definition, $\mathcal{T}$ is the set of all components of the edge-decomposition, thus of all trees of the decomposition. By Inequation 6.2, we get the following inequation that bounds the number of components in the edge-decomposition.

$$k|V(G)| - c \leq |E(G)| = \sum_{j=1}^{k} |E(F_j)| \leq \sum_{j=1}^{k} (|V(G)| - |\mathcal{T}_j|) = k|V(G)| - |\mathcal{T}|$$

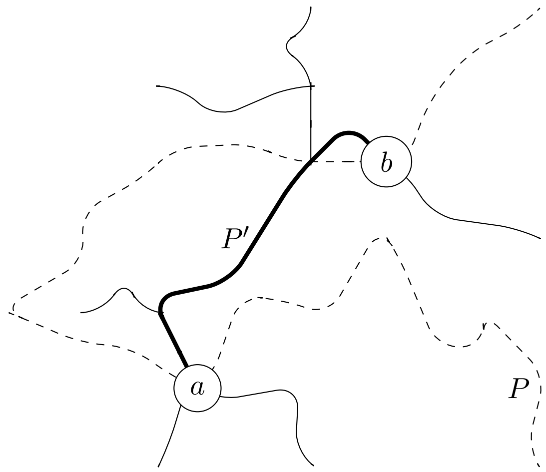


Figure 6.1: Illustration to the proof of Lemma 6.6. All non-dashed lines represent edges of the tree T. Dashed lines may contain edges of T. The path P is represented by dashed lines. The a-b-path P' that is represented by a thick line is a path in the tree T. The length of P' is smaller than the length of the a-b-path that is a subgraph of P. Therefore, P is not a shortest path, as we can easily construct a shorter path with the the same endvertices.

Thus $|\mathcal{T}| \leq c$. As the diameter of G is at least $cd + 1$, there is a shortest path $P = v_0 \dots v_{cd+1}$ in G. By pigeonhole principle there has to be a tree $T \in \mathcal{T}$ such that at least $d + 1$ edges of P belong to T. Let ℓ be the minimum index such that $a \coloneqq v_\ell \in V(T)$ and m be the maximum index such that $b \coloneqq v_m \in V(T)$. Note that $m \geq \ell + d + 1$, as $d + 1$ edges of P belong to T.

Suppose there is an a-b-path P' in T of length at most d. Note that the vertices $v_0, \dots, v_{\ell-1}$ and $v_{m+1}, \dots, v_{cd+1}$ are not vertices of T, in particular these vertices do not belong to the path P. Therefore, $v_0 \dots v_{\ell-1} P' v_{m+1}$ is a shorter path then P connecting v_0 and v_{cd+1}; see Figure 6.1. This is a contradiction, as P is a shortest path. Thus

$$\operatorname{diam}(T) \geq \operatorname{dist}_T(a, b) \geq d + 1.$$

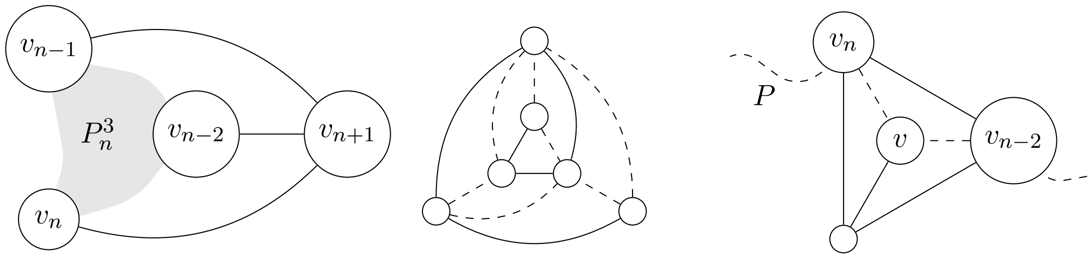
This is a contradiction to the choice of the edge-decomposition of G, since every tree in $\mathcal{T}$ has diameter at most d. Thus

$$\Upsilon_{bd}(\mathcal{G}) \geq k + 1.$$

Therefore the claim follows. □

Hence, in order to prove that the bounded diameter arboricity of the class of planar graphs is four, it suffices to see that there are planar triangulations of arbitrarily large diameter, as in this case we have a sequence $G_1, G_2, \dots$ of planar graphs such that $\operatorname{diam} G_i \geq i$ and $|E(G_i)| = 3|V(G_i)| - 6$, as all graphs G_i are planar triangulations.

Paths are planar graphs of large diameter. Therefore, if we are able to augment these graphs to planar triangulations while preserving large diameters, we can prove Theorem 6.5.



(a) Extension of a planar embedding of P_n^3 to a planar embedding of P_{n+1}^3.

(b) A planar embedding of P_6^3. All edges that do not belong to the underlying path are represented as dashed lines.

(c) A u-v-path P in P_{n+1}^3 between two vertices $u, v \in V(P_n^3)$ that goes through $v = v_{n+1}$ is not a shortest path.

Figure 6.2: Illustrations to the proof of Observation 6.8.

Definition 6.7 (Graph Power)**.** Let G be a graph and $k \in \mathbb{N}$ be a positive integer. The *k-th power* of G is the graph G^k where $V(G^k) = V(G)$ and $E(G^k) = \{ab \mid a, b \in V(G), a \neq b, \operatorname{dist}_G(a, b) \leq k\}$

Observation 6.8. *The third path power P_n^3 is a planar triangulation of diameter at least $n/3$ for all $n \geq 3$.*

Proof. We will first show that P_n^3 is a planar triangulation. Let $v_0 v_1 \dots v_n$ be the underlying path.

Claim 6.8.1. *There is a planar embedding of P_n^3 such that v_{n-2}, v_{n-1} and v_n are on the outer face for all $n \geq 3$. In particular, P_n^3 is a plane triangulation.*

Proof of Claim. We will prove the claim by induction on n. If $n = 3$ the claim holds since $P_n^3 = K_3$. Suppose the claim holds for some fixed $n \in \mathbb{N}$. By induction hypothesis, there is a planar embedding of P_n^3 such that v_{n-2}, v_{n-1} and v_n lie on the outer face. Then we can obtain P_{n+1}^3 from P_n^3 by adding a vertex v_{n+1} and connecting v_{n+1} to all vertices on the outer face of the planar embedding of P_n^3 such that v_{n-1}, v_n and v_{n+1} lie on the outer face; see Figure 6.2a. In particular all new faces are bounded by a triangle. Thus, the claim holds. See Figure 6.2b for an illustration of such an embedding. ⌟

Claim 6.8.2. *For all $n \in \mathbb{N}$ and all $1 \leq k \leq n$ we have:* $\operatorname{dist}_{P_n^3}(v_0, v_k) = \lceil \frac{k}{3} \rceil$.

Proof of Claim. We will prove the claim by induction on the number of vertices. If $1 \leq n \leq 2$, then $\operatorname{dist}(v_0, v_n) = 1$ as $P_n^3 = K_{n-1}$. Assume the claim holds for some fixed $n \in \mathbb{N}$. Then we obtain P_{n+1}^3 from P_n^3 by adding a vertex $v \coloneqq v_{n+1}$ and edges $\{v_i v \mid n-2 \leq i \leq n\}$. Note that $N(v)$ induces a triangle. Thus, the distances of vertices in P_n^3 to other vertices of P_n^3 do not change by adding v, as going through v is a detour; see Figure 6.2c. Hence for all $0 \leq k \leq n$, we have by induction hypothesis:

$$\operatorname{dist}_{P_{n+1}^3}(v_0, v_k) = \operatorname{dist}_{P_n^3}(v_0, v_k) = \left\lceil \frac{k}{3} \right\rceil$$

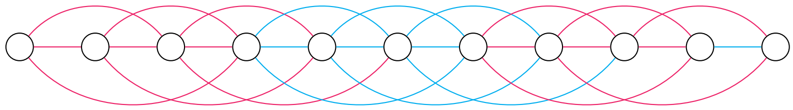


Figure 6.3: An edge-decomposition of P_{10}^3 into two graphs avoiding long paths.

Any shortest path connecting v_{n+1} to v_0 has to pass through $N(v_{n+1})$. Thus

$$\operatorname{dist}_{P_{n+1}^3}(v_0, v_{n+1}) = \min_{n-2 \leq i \leq n} \operatorname{dist}_{P_n^3}(v_0, v_i) + 1 = \left\lceil \frac{n-2}{3} \right\rceil + 1 = \left\lceil \frac{n+1}{3} \right\rceil$$

by induction hypothesis. ⌟

Therefore, as

$$\operatorname{diam}(P_n^3) \geq \operatorname{dist}_{P_n^3}(v_0, v_n) = \left\lceil \frac{n}{3} \right\rceil \geq \frac{n}{3}$$

the third path power P_n^3 is a planar triangulation of diameter at least $n/3$ for all $n \geq 3$. □

As third path powers are planar triangulations of arbitrarily large diameter, we can construct a sequence of graphs as needed in Lemma 6.6. This enables us to prove that the bounded diameter arboricity of planar graphs is four, as stated in Theorem 6.5. Thus, we cannot avoid long paths in decompositions of planar graphs into three forests.

Bounded diameter arboricity is however restricted to decompositions into forests. What happens if we allow decompositions into graphs containing cycles? We could hope that as we cannot avoid long paths in edge-decompositions of a family $\mathcal{G}$ of planar graphs into three forests, the same might hold if we consider edge-decompositions into three arbitrary graphs.

Proposition 6.9. *Let $\mathcal{G} := \{P_n^3 \mid n \in \mathbb{N}\}$ be the class of all third powers of paths. Then $\Upsilon_{bd}(\mathcal{G}) = 4$. However, for all $G \in \mathcal{G}$ and all $k \geq 6$, we have*

$$G \nrightarrow_2 P_k.$$

Proof. As $\mathcal{G}$ is a subclass of all planar graphs, we have $\Upsilon_{bd}(\mathcal{G}) \leq 4$, as the bounded diameter arboricity of planar graphs is four by Theorem 6.5. Let $H_i := P_{3i}^3$ for all $i \in \mathbb{N}$. By Observation 6.8 for all $i \in \mathbb{N}$,

$$\operatorname{diam} H_i \geq \left\lceil \frac{3i}{3} \right\rceil \geq i$$

and H_i is a planar triangulation, thus $|E(H_i)| = 3|V(H_i)| - 6$. Therefore by Lemma 6.6, we have

$$\Upsilon_{bd}(\mathcal{G}) = 4.$$

Claim 6.9.1. *For all $n \in \mathbb{N}$ there is an edge-decomposition of P_n^3 into two graphs G_1, G_2 such that $P_6 \nsubseteq G_i$ for all i.*

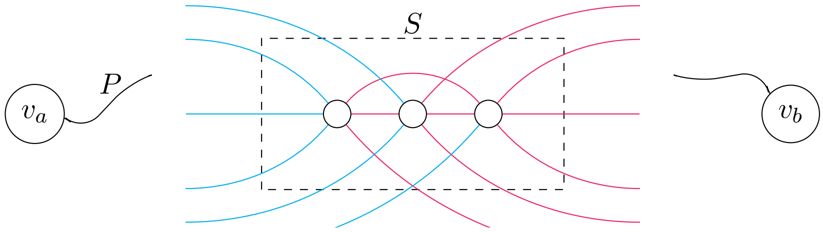


Figure 6.4: The edges entering S from the left represent edges connecting S to vertices of lower index, the edges entering S from the right represent edges connecting S to vertices of higher index. All edges connecting S to $P_n^3 - S$ are shown. Every v_a-v_b-path P in P_n^3 has to pass through S. Since c_n is a 2-edge-coloring that colors the edges entering S from the left in blue and the edges entering S from the right in red, P is not a monochromatic path.

Proof of Claim. Let $v_0 \dots v_n$ denote the underlying path of P_n^3. Consider the following edge-coloring c_n of P_n^3; see Figure 6.3.

$$c_n \colon E(H_n) \to \{1, 2\}$$
$$v_i v_j \quad (i < j) \mapsto \begin{cases} 1, & \text{if } i \equiv 0, 1, 2 \pmod 6 \\ 2, & \text{otherwise} \end{cases}$$

Let $G_1 := (V(P_n^3), \{e \in E(P_n^3) \mid c_n(e) = 1\})$ be the subgraph of P_n^3 in color 1 and $G_2 := (V(P_n^3), \{e \in E(P_n^3) \mid c_n(e) = 2\})$ the subgraph of P_n^3 in color 2.

Claim 6.9.2. *Every connected component of G_1 and G_2 respectively contains at most six vertices.*

Proof of Claim. Suppose there is a connnected component of G_1 containing at least seven vertices. Then there are two distinct vertices v_a, v_b such that $a \equiv b \pmod 6$ by pigeonhole principle. We may assume $a < b$.

Case 1. $a \equiv 3, 4, 5 \pmod 6$. Let $j \in \mathbb{N}$ be minimal such that $a \leq 6j$. Let P be a v_a-v_b-path in G_1. Then P is a path in P_n^3 as $G_1 \subseteq P_n^3$. Thus, P has to pass through $S := \{v_{6j}, v_{6j+1}, v_{6j+2}\}$ by definition of P_n^3. Note that $v_a, v_b \notin S$, as $a \equiv 3, 4, 5 \pmod 6$ and thus $b \equiv 3, 4, 5 \pmod 6$. However, all edges $v_k s$, where $k < 6j$ and $s \in S$ are colored in 2, and all edges $s v_{k'}$, where $s \in S$ and $k' \geq 6j$ are colored in 1; see Figure 6.4. Therefore, P is not a monochromatic path in P_n^3, hence not a path in G_1. This is a contradiction.

Case 2. $a \equiv 0, 1, 2 \pmod 6$. A similar argumentation as in the first case will work by letting j be minimal, such that $a \leq 6j + 3$ and considering $S' := \{v_{6j+3}, v_{6j+4}, v_{6j+5}\}$.

Analogously, every component of G_2 contains at most six vertices. ⌟

As every connected component in G_1 and G_2 respectively contains at most six vertices, a longest path in G_1 and G_2 respectively has length at most five. Therefore c_n is an edge-decomposition of P_n^3 avoiding paths of length at least six. ⌟

Thus, paths of length at least six are 2-$\mathcal{G}$ avoidable. $\square$

Although, we cannot avoid long paths in edge-decompositions of planar graphs into three forests, this does not imply that the same holds for decompositions of planar graphs into three graphs that might contain cycles.

6.2 Bounded Diameter Arboricity of the Iterated Triangulation

We already considered the family of iterated triangulations $\mathcal{T} \coloneqq \{\mathrm{Tr}(n) \mid n \in \mathbb{N}_0\}$. Our aim was to show that long paths in 3-edge-colorings cannot be avoided. Assume that long paths in 3-edge-colorings of graphs in $\mathcal{T}$ cannot be avoided. Then we have in particular

$$\Upsilon_{bd}(\mathcal{T}) \geq 4$$

as otherwise there is a positive integer $d \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ there is an edge-decomposition of $\mathrm{Tr}(n)$ into three forests of diameter at most d. In particular paths of length at least $d+1$ could be avoided. Thus, $\Upsilon_{bd}(\mathcal{T}) \geq 4$ is a necessary condition for unavoidability of long paths in 3-edge-colorings. If we are able to prove that the conditions of Lemma 6.6 hold for the family of iterated triangulations, then we have $\Upsilon_{bd}(\mathcal{T}) = 4$. Note that for all $n \in \mathbb{N}$, $\mathrm{Tr}(n)$ is a planar triangulation. Thus, it remains to show that the diameter of $\mathrm{Tr}(n)$ is arbitrarily large for large enough n.

Observation 6.10. *For all $n \in \mathbb{N}_0$ we have:* $\operatorname{diam} \mathrm{Tr}(n) \geq (n+2)/3$.

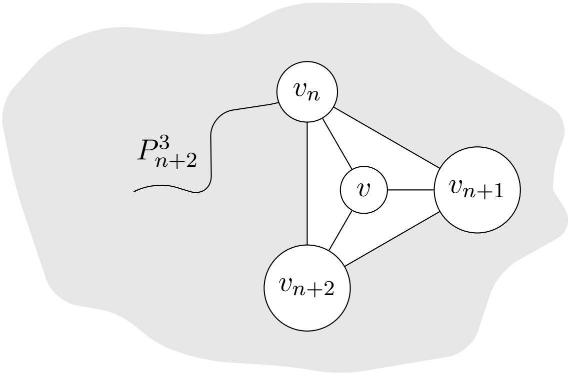
Proof. Let $n \in \mathbb{N}_0$. Consider P^3_{n+2}. Let $v_0 v_1 \dots v_{n+2}$ denote the underlying path.

Claim 6.10.1. *The third path power $P^3_{n+2} \subseteq_{ind} \mathrm{Tr}(n)$ is an induced subgraph of the iterated triangulation and there is an embedding of P^3_{n+2} such that v_n, v_{n+1}, v_{n+2} bound a face of $\mathrm{Tr}(n)$.*

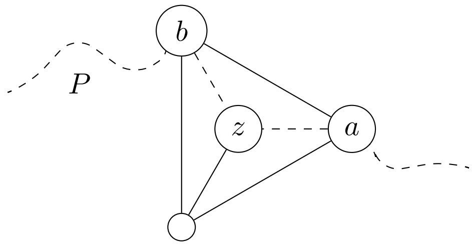
Proof of Claim. We prove the claim by induction on n. If $n = 0$, then $\mathrm{Tr}(n) = K_3 = P^3_2$. Assume the claim holds for some fixed $n \in \mathbb{N}_0$. By induction hypothesis, there exists a face f of $\mathrm{Tr}(n)$ bounded by v_n, v_{n+1}, v_{n+2}. In $\mathrm{Tr}(n+1)$, there is a vertex v in the face f connected to all vertices on f. Thus, we formed in particular a face f' bounded by the vertices $v_{n+1}, v_{n+2}, v \coloneqq v_{n+3}$; see Figure 6.5a. We obtain P^3_{n+3} from P^3_{n+2} by adding the vertex v_{n+3} and all edges in $\mathrm{Tr}(n+1)$ incident to v_{n+3}. By construction, we obtained P^3_{n+3} as an induced subgraph of $\mathrm{Tr}(n+1)$, since no new edge connects two vertices of generation less than $n+1$ and all edges incident to v_{n+3} belong to P^3_{n+3}. The face f' is the desired face of the embedding. $\lrcorner$

Claim 6.10.2. *Let $n \in \mathbb{N}_0$, $u, v \in V(\mathrm{Tr}(n))$, $k \coloneqq \max(\mathrm{gen}(u), \mathrm{gen}(v))$. Then any shortest u-v-path in $\mathrm{Tr}(n)$ is a path in $\mathrm{Tr}(k)$. In particular*

$$\mathrm{dist}_{\mathrm{Tr}(n)}(u,v) = \mathrm{dist}_{\mathrm{Tr}(k)}(u,v).$$



(a) Gray areas represent vertices and edges of $\mathrm{Tr}(n+1)$. The third path power P^3_{n+2} is a subgraph of $\mathrm{Tr}(n)$, thus a subgraph of $\mathrm{Tr}(n+1)$. The vertices v_n, v_{n+1}, v_{n+3} form a face f in $\mathrm{Tr}(n)$. By construction of $\mathrm{Tr}(n+1)$, a vertex v that is adjacent to all vertices on f is inserted. Hence, we obtain P^3_{n+3} as a subgraph of $\mathrm{Tr}(n+1)$.



(b) The path P is a path in $\mathrm{Tr}(n+1)$ between two vertices u, v of generation less than $n+1$ that passes through a vertex z of generation $n+1$. The edges of P are represented by dashed lines. The path P has to pass through at least two vertices a, b of the neighborhood of z. Thus P is not a shortest u-v-path.

Figure 6.5: Illustrations to proof of Observation 6.10.

Proof of Claim. We will prove the claim by induction on n. The claim clearly holds for $k = n$. Assume the claim holds for some $n \in \mathbb{N}_0$. Consider a shortest u-v path $P = v_1 \dots v_\ell$ in $\mathrm{Tr}(n+1)$. Suppose there exists $z \in V(\mathrm{Tr}(n+1)) \cap V(P)$ such that $\mathrm{gen}(z) = n+1$, then the neighborhood of z bounds a face of $\mathrm{Tr}(n)$; see Figure 6.5b. Thus, there are $a, b \in N_{\mathrm{Tr}(n+1)}(z) \subseteq V(\mathrm{Tr}(n))$ such that $P = v_1 \dots v_h azb v_{h+4} \dots v_\ell$. Then $P' = v_1 \dots v_h ab v_{h+4}$ is a shorter path as $ab \in E(\mathrm{Tr}(n)) \subseteq E(\mathrm{Tr}(n+1))$. This is a contradiction to the choice of P. Thus, P is a path in $\mathrm{Tr}(n)$. By induction, any shortest u-v-path in $\mathrm{Tr}(n)$ is a path in $\mathrm{Tr}(k)$. ⌟

Claim 6.10.3. *Let $n \in \mathbb{N}_0$. Consider the embedding of P^3_{n+2} in $\mathrm{Tr}(n)$ constructed in Claim 6.10.1. Let $u, v \in V(P^3_{n+2})$. Then*

$$\mathrm{dist}_{\mathrm{Tr}(n)}(u, v) = \mathrm{dist}_{P^3_{n+2}}(u, v).$$

Proof of Claim. We will show the claim by induction on n. If $n = 0$ the claim holds, as $P^3_{n+2} = \mathrm{Tr}(n) = K_3$. Assume the claim holds for some fixed $n \in \mathbb{N}_0$. Consider the embedding of $P^3_{(n+1)+2}$ in $\mathrm{Tr}(n+1)$. Let $k \coloneqq \max(\mathrm{gen}(u), \mathrm{gen}(v))$. Without loss of generality, we may assume $\mathrm{gen}(v) = k$.

Case 1. $k \le n$. Then by induction and Claim 6.10.2, we have

$$\mathrm{dist}_{P^3_{k+2}}(u, v) = \mathrm{dist}_{\mathrm{Tr}(k)}(u, v) = \mathrm{dist}_{\mathrm{Tr}(n)}(u, v).$$

As a shortest u-v-path in P^3_{k+2} is also a shortest u-v-path in P^3_{n+2} the claim follows.

Case 2. $k = n+1$. Note that $\mathrm{gen}(u) < k$ by definition of the embedding of P^3_{n+3}. Let P be a shortest u-v-path in $\mathrm{Tr}(n+1)$. The path P has to pass through some vertex w that is a neighbor of v in $\mathrm{Tr}(n+1)$. Note that every neighbor of v is a vertex of P^3_{n+2} by definition of the embedding of P^3_{n+3}, in particular $\mathrm{gen}(w) < \mathrm{gen}(v) = n+1$. Thus, $P - v$ is a shortest $u - w$ path in $\mathrm{Tr}(n+1)$. By induction and Claim 6.10.2

$$\mathrm{dist}_{\mathrm{Tr}(n+1)}(u,w) = \mathrm{dist}_{\mathrm{Tr}(n)}(u,w) = \mathrm{dist}_{P^3_{n+2}}(u,w)$$

follows. Hence, we may assume that $P - v$ is a path in P^3_{n+2}. Therefore P is a shortest u-v-path in P^3_{n+3}, in particular we have

$$\mathrm{dist}_{\mathrm{Tr}(n+1)}(u,v) = \mathrm{dist}_{P^3_{n+3}}(u,v)$$

as P is a shortest path. ⌟

Thus, as the diameter of P^3_{n+2} is at least $\lceil \frac{n+2}{3} \rceil$ by Observation 6.8, we have

$$\frac{n+2}{3} \leq \mathrm{diam}(P^3_{n+2}) \leq \mathrm{diam}(\mathrm{Tr}(n)).$$

Therefore, the claim follows. □

As the diameter of the iterated triangulation $\mathrm{Tr}(n)$ is arbitrarily large for large enough n, the family $\mathcal{T}$ of the iterated triangulation fulfills the conditions of Lemma 6.6. Thus, there exists no natural number d such that every iterated triangulation can be covered by three forests of diameter at most d.

Corollary 6.11. *The family of the iterated triangulation has bounded diameter arboricity 4, i.e.,*

$$\Upsilon_{bd}(\mathcal{T}) = 4.$$

Proof. By Observation 6.10, the diameter of $\mathrm{Tr}(n)$ is arbitrarily large for large enough n. As the iterated triangulation is a plane triangulation for all $n \in \mathbb{N}$, we have:

$$|E(\mathrm{Tr}(n))| = 3|V(\mathrm{Tr}(n)| - 6$$

Thus, the conditions of Lemma 6.6 hold for the family of the iterated triangulation. Hence, we have

$$\Upsilon_{bd}(\mathcal{T}) \geq 4$$

As the bounded diameter arboricity of the family of planar graphs $\mathcal{P}$ is four by Theorem 6.5, we have

$$\Upsilon_{bd}(\mathcal{T}) \leq \Upsilon_{bd}(\mathcal{P}) = 4$$

Thus, the claim follows. □

7 Complexity of 3PlanarAvoidP$_k$

We will discuss the complexity of the decision problem associated to P_k-free edge-colorings of planar graphs. We use standard terminology as used by Garey and Johnson [13].

Definition 7.1 (3PlanarAvoidP$_k$)**.** Consider the following decision problem that we will call 3PlanarAvoidP$_k$. Given a planar graph G, decide whether G admits an edge-decomposition into three graphs G_1, G_2, G_3 such that for all i we have $P_k \nsubseteq G_i$.

If long paths cannot be avoided, we might hope that 3PlanarAvoidP$_k$ is $\mathcal{NP}$-hard for large enough k. On the other hand, $\mathcal{NP}$-hardness of 3PlanarAvoidP$_k$ implies that long paths cannot be avoided.

Observation 7.2. *If for all $k \in \mathbb{N}$ there exists $k' \geq k$ such that 3PlanarAvoidP$_{k'}$ is $\mathcal{NP}$-hard, then for all $k \in \mathbb{N}$ P_k is 3-planar unavoidable.*

Proof. Let $d \in \mathbb{N}$ be a positive integer. By assumption, there exists a positive integer $d' \in \mathbb{N}$ such that $d' \geq d$ and 3PlanarAvoidP$_{d'}$ is $\mathcal{NP}$-hard. Therefore, there is a polynomial transformation P from Sat to 3PlanarAvoidP$_{d'}$. Consider an instance I of Sat that is not satisfiable. Then by definition of a polynomial transformation, $P(I)$ is a no-instance of 3PlanarAvoidP$_{d'}$. Note once again that an edge-decomposition of a graph into three graphs G_1, G_2, G_3 induces a 3-edge-coloring where every color class corresponds to one of the graphs G_i. As $P(I)$ is a no-instance of 3PlanarAvoidP$_{d'}$, $P(I)$ is a planar graph such that every 3-edge-coloring of $P(I)$ contains $P_{d'}$ as a monochromatic subgraph. Hence $P(I) \to_3 P_{d'}$. Since P_d is a subgraph of $P_{d'}$, the path P_d is 3-planar unavoidable. □

Broersma et al. investigated a similar decision problem concerning vertex decompositions instead of edge-decompositions of planar graphs. They were in particular interested in deciding for a fixed length $k \geq 1$ of a path and a given planar graph G if there is a 3-vertex-coloring, i.e., a decomposition of the vertices $V(G) = V_1 \cup V_2 \cup V_3$ such that $G[V_i]$ does not contain P_k as a subgraph for all $i \in \{1, 2, 3\}$. They proved that this problem is $\mathcal{NP}$-hard for all k [4, Theorem 5.3]. Note however that $G[V_1], G[V_2], G[V_3]$ is in general not an edge-decomposition of G as edges with endpoints in different color classes are not part of any of these graphs.

7.1 Complexity of 3PlanarAvoidP$_2$

Clearly, 3PlanarAvoidP$_1$ is in $\mathcal{P}$, as a path of length one cannot be avoided in an edge-decomposition of a graph G, if and only if G contains at least one edge. We will first consider 3PLANARAVOIDP$_2$. An edge-decomposition of a graph G into three graphs avoiding paths of length two, corresponds to a decomposition of G into three matchings; thus, a proper 3-edge-coloring of G. If the maximum degree $\Delta(G)$ of G is greater than 3, there is no proper 3-edge-coloring of G as in every 3-edge-coloring the same color will be assigned to at least two edges incident to a vertex of degree at least 4. By a result of Vizing [10, Proposition 5.3.2], we know that for all graph, we have

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1.$$

We wish therefore to decide for a planar graph G of maximum degree 3, if its chromatic index is 3 or 4.

Lemma 7.3. *Let G be a graph such that there exists $v \in V(G)$ with $1 \leq \deg(v) \leq 2$ and for all $u \in V$ with $u \neq v$ we have $\deg(u) = 3$. Then G is not 3-edge-colorable.*

Proof. Let $n := |V|$. Suppose there is a proper 3-edge-coloring c of G in colors $1, 2, 3$. Without loss of generality, there is an edge in color 1 incident to v and no edge in color 2 incident to v. For all $i \in \{1, 2, 3\}$ consider

$$X_i = \{(u, e) \mid u \in V, e \in E, e \text{ is incident to } u, c(e) = i\}.$$

Since c is a proper edge-coloring and for all $u \in V \setminus \{v\}$ we have $\deg(u) = 3$ there is exactly one edge incident to u colored in i for all $i \in \{1, 2, 3\}$. Once again, as c is a proper edge-coloring and one of the edges incident to v is colored in 1, there is exactly one edge incident to v colored in 1. Thus, $|X_1| = n$ and $|X_2| = n - 1$, as there is no edge in color 2 incident to v. However, counting from the perspective of edges, for all $i \in \{1, 2, 3\}$ $|X_i|$ has to be even, as every edge is incident to exactly two vertices. This is a contradiction as either n or $n - 1$ is odd. $\square$

Using the result above, we can prove that no 3-regular graph with a bridge[1] is 3-edge-colorable. Since Tait showed that every 3-regular, bridgeless, planar graph G is 3-edge-colorable if and only if G is 4-face-colorable [10, Exercise 6.23], we can characterize the class of planar, 3-regular, 3-edge-colorable graphs using the Four-Color-Theorem [1]. This characterization is well-known.

Corollary 7.4. *Let G be a planar, 3-regular graph. Then*

$$G \textit{ is } 3\textit{-edge-colorable} \iff G \textit{ is bridgeless.}$$

Proof. Let G be a planar 3-regular graph.

Case 1. G is bridgeless. Tait showed that a 3-regular, planar, bridgeless graph is 3-edge-colorable if and only if it is 4-face-colorable [10, Exercise 6.23]. The dual graph $G^\star$ of G is planar, as G is planar. Thus, $G^\star$ is four-colorable by the Four-Color-Theorem. Therefore G is 3-edge-colorable.

Case 2. G is not bridgeless. Let uv be a bridge in G. Let G' be the component of $G - uv$ containing v; see Figure 7.1. Suppose G is 3-edge-colorable. Then G' is 3-edge-colorable. Note that all vertices in G' except v have degree 3 and $\deg_{G'}(v) = 2$. This is a contradiction as G' is not 3-edge-colorable by Lemma 7.3. $\square$

[1] An edge e of a graph G is called *bridge* or *cut-edge*, if $G - e$ has more connected components than G.

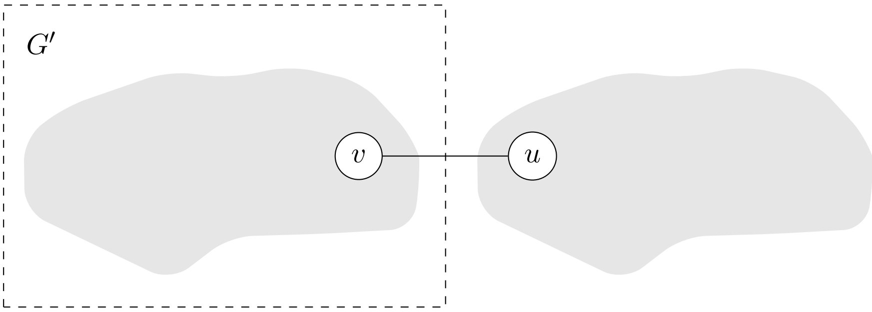


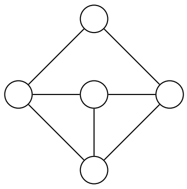
Figure 7.1: Illustration of the proof of Corollary 7.4. The edge uv is a bridge of the 3-regular graph G. The component G' of $G - uv$ containing v is represented by the gray area on the left side.

Note that we can check in linear time if a graph contains a bridge [25, Property 18.6]. Thus, we can decide in polynomial time if a given 3-regular, planar graph is 3-edge-colorable. Cole and Kowalik claim [6, p. 3] that due to this observation the edge-chromatic number of a planar graph with maximum degree 3 can be calculated in polynomial time without giving a proof. However, Holyer showed that for a 3-regular graph G that is not necessarily planar the problem of deciding if G is 3-edge-colorable is $\mathcal{NP}$-complete [19]. Vizing conjectured in 1965 that all planar graphs of maximum degree $\Delta \geq 6$ are Δ-edge-colorable giving a proof for planar graphs of maximum degree $\Delta \geq 8$ [27]. In 2001, Sanders and Zhao proved that all planar graphs of maximum degree $\Delta \geq 7$ are Δ-edge-colorable [24], the case $\Delta = 6$ is still open. We also know that there are planar graphs of maximum degree $\Delta = 3, 4$ and $\Delta = 5$ respectively that are not Δ-edge-colorable [27]. Nevertheless, the complexity of determining the chromatic index of a planar graph of maximum degree Δ, where $3 \leq \Delta \leq 5$ is still unknown [18, p. 124].

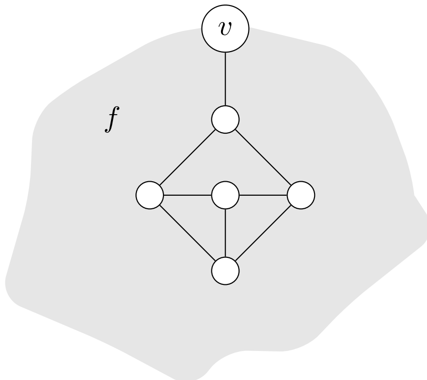
We will be interested in the complexity of deciding for a planar graph of maximum degree 3 whether it is 3-edge-colorable. Remember that the complexity of this problem corresponds to the complexity of 3PlanarAvoidP_2 as a proper 3-edge-coloring corresponds to an edge-decomposition into three graphs avoiding paths of length 2, and no graph of maximum degree at least 4 can be edge-decomposed into three graphs avoiding paths of length 2. In order to determine the complexity of the edge-coloring problem, one could aim to embed a planar graph G of maximum degree 3 into a planar, 3-regular, bridgeless graph H. If there exists such a graph H, then G is 3-edge-colorable as H is 3-edge-colorable by Lemma 7.4.

Definition 7.5 (3-regular-embeddable)**.** Let G be a planar, connected graph of maximum degree 3. Let $L(G) \coloneqq \{v \in V(G) \mid \deg(v) < 3\}$ denote the vertices in G of degree less than 3. A vertex $v \in L(G)$ is called *non-saturated*, a vertex $v \in V(G) \setminus L(G)$ is called *saturated*. We call G 3*-regular-embeddable* if there is a bridgeless 3-regular planar graph H, such that $G \subseteq H$.

Note that if we do not enforce the supergraph H to be bridgeless, every planar graph of maximum degree 3 would be 3-regular embeddable.



(a) A planar embedding of the graph A where all vertices but one have degree 3 and the only vertex of degree 2 lies on the outer face.



(b) Embedding the graph of Figure 7.2a into a face f adjacent to a non-saturated vertex v in a planar way. The face f is represented by the gray area.

Figure 7.2: Illustrations to Example 7.6.

Example 7.6. *Let G be a planar graph of maximum degree 3. Then there exists a planar, 3-regular graph H such that $G \subseteq H$.*

Proof. Let $L(G) \coloneqq \{v \in V(G) \mid \deg(v) < 3\}$ be the non-saturated vertices of G. Consider a planar embedding of G. Then we connect every non-saturated vertex $v \in L(G)$ to $3 - \deg(v)$ copies of the graph A represented in Figure 7.2a through edges connecting v and the only vertex of degree 2 in the copy of A. This can be done in a planar way by embedding the copies of A into a face f adjacent to v; see Figure 7.2b. The resulting graph H is planar by construction and 3-regular as we connected every non-saturated vertex v to $3 - \deg(v)$ new vertices and every new vertex has degree 3 in H. □

The 3-regular supergraph H of Example 7.6 is a 3-regular graph with bridges, therefore by Lemma 7.4 not 3-edge-colorable. As we are looking for a 3-edge-colorable, planar supergraph of a planar graph of maximum degree 3 we need our supergraph to be bridgeless.

By definition, a 3-regular-embeddable graph is a subgraph of a 3-regular, bridgeless graph, thus 3-edge-colorable as a subgraph of a 3-edge-colorable graph by Lemma 7.4. However, we will see that there exists a planar, 3-edge-colorable graph that is not 3-regular-embeddable; see Observation 7.10. In other words, being 3-regular-embeddable is sufficient, but not necessary for being 3-edge-colorable. Hence, we will not be able to reduce the problem of properly edge-coloring a planar graph of maximum degree 3 to the 3-regular case using 3-regular supergraphs of our initial graph.

Nevertheless, we will be interested in deciding whether a given planar graph is 3-regular embeddable. Hartmann, Rollin, and Rutter studied a similar augmentation problem [17]. They discussed in particular if we can augment a planar graph G of maximum degree 3 by adding edges such that the augmented graph is 3-regular and 2-connected, thus the augmented graph is 3-regular and bridgeless as the notions of 2-edge-connectivity

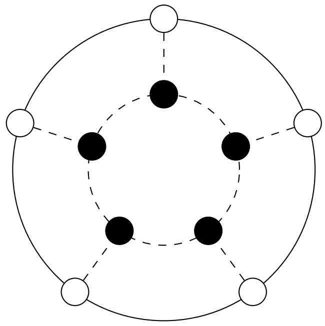


Figure 7.3: A bridgeless, 3-regular, planar supergraph H of C_5. All edges belonging to C_5 are represented as non-dashed lines, the edges in $E(H) \setminus E(C_5)$ are dashed. All vertices in $V(H) \setminus V(C_5)$ are represented as black vertices.

and 2-connectivity coincide in a graph of maximum degree 3. Reducing Planar3Sat to this augmentation problem, they showed that deciding whether there exists such an augmentation is $\mathcal{NP}$-hard [17, Theorem 3]. Clearly, if we can add edges to a given graph in a planar way such that the resulting graph is bridgeless and 3-regular, the initial graph is 3-regular-embeddable. However, not every 3-regular-embeddable graph can become 3-regular by only adding edges.

Example 7.7. *The cycle C_5 on five vertices is* 3*-regular-embeddable, but C_5 has no* 3*-regular supergraph H such that $V(H) = V(G)$.*

Proof. The graph C_5 is 3-regular-embeddable; see Figure 7.3.

Suppose there exists a 3-regular supergraph H of C_5 such that $V(H) = V(G)$. Consider the set $X := \{(v, e) \mid v \in V(C_5), e \in E(H) \setminus E(G), e \text{ is incident to } v\}$. Then counting from the perspective of vertices, $|X| = 5$ as every vertex of C_5 has exactly one incident edge in $E(H) \setminus E(G)$. However, counting from the perspective of edges, $|X|$ is even as every edge is incident to exactly two vertices. This is a contradiction. $\square$

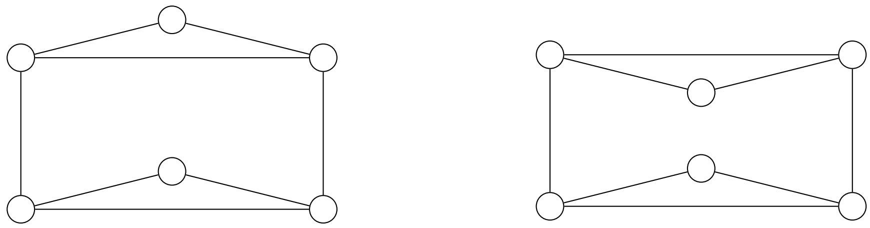
If a connected graph G is 3-regular-embeddable, then there is a planar embedding of G such that we can connect every non-saturated vertex of G via new vertices and edges to another non-saturated vertex on the same face, since G admits a 3-regular, bridgeless, planar supergraph.

Definition 7.8 (2-face-mappable)**.** Let G be a planar, connected graph of maximum degree 3. We say that G is 2*-face-mappable* if G has the following property:

There is a planar embedding of G and a map $\gamma\colon L(G) \to F(G)$, where $F(G)$ denotes the faces of the embedding, such that:

(P1) for every face in the image of γ at least two vertices of $L(G)$ have been mapped to f, i.e.,

$$\forall f \in \operatorname{Im}(\gamma)\colon \left|\gamma^{-1}(\{f\})\right| \geq 2.$$



(a) An embedding that is not fixed 2-face-mappable.

(b) A fixed 2-face-mappable embedding.

Figure 7.4: Not every planar embedding of a 2-face-mappable graph has to be fixed 2-face-mappable.

(P2) for all faces $f \in F(G)$ and every bridge uv on f, we have

$$\forall a \in \{u, v\} \colon \gamma^{-1}(\{f\}) \cap V(G_a) \neq \varnothing,$$

where G_a is the component of $G - uv$ containing a for $a \in \{u, v\}$.

We call such an embedding *fixed* 2*-face-mappable.*

Note that not every planar embedding of a 2-face-mappable graph has to be fixed 2-face-mappable; see Figure 7.4.

If a graph is 2-face-mappable, then there is an embedding such that we can map each vertex to a face and add new vertices and edges in a planar way without creating bridges such that the resulting graph is 3-regular. On the other hand, if a graph is 3-regular-embeddable, then there is such a mapping. This observation enables us to prove that the definitions of 3-regular-embeddable and 2-face-mappable coincide for connected, planar graphs.

Proposition 7.9. *Let G be a planar, connected graph of maximum degree 3. Then*

$$G \text{ is 2-face-mappable} \iff G \text{ is 3-regular-embeddable.}$$

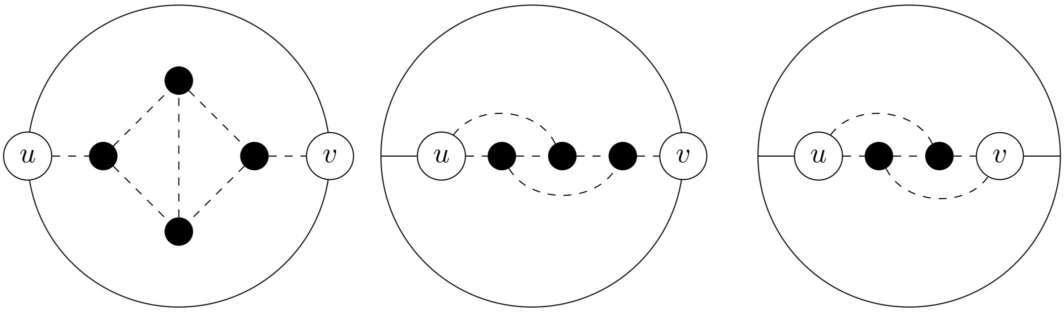
If G is 2-face-mappable, then we have in particular $\chi'(G) \leq 3$.

Proof. Let $L(G) \coloneqq \{v \in V(G) \mid \deg(v) < 3\}$ denote the non-saturated vertices in G.

Claim 7.9.1. *If G is 2-face-mappable, then G is 3-regular-embeddable.*

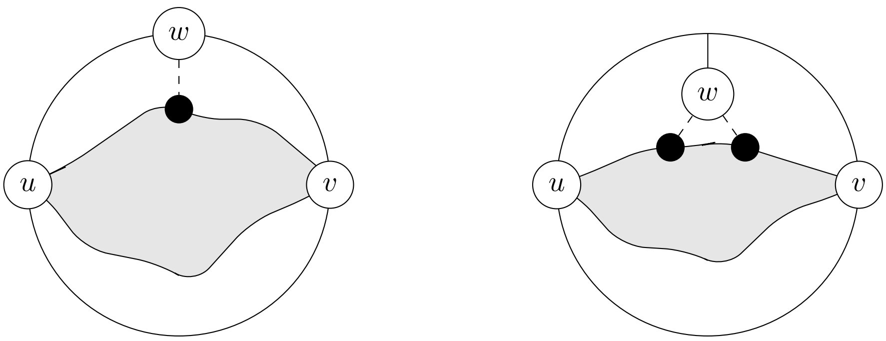
Proof of Claim. As G is 2-face-mappable, there is a planar embedding of G and a map $\gamma \colon L(G) \to F(G)$ as in Definition 7.8. Then we proceed as follows: For every face $f \in \mathrm{Im}(\gamma)$ there exist $u, v \in \gamma^{-1}(\{f\})$ such that $u \neq v$ by (P1). Without loss of generality, we may assume $\deg_G(u) \leq \deg_G(v)$.

There are three distinct cases depending on the degree of u and v in G. As u and v lie on the same face, we can add vertices and edges as shown in Figure 7.5. Black vertices represent new vertices, dashed edges represent new edges. Note that all new vertices have degree 3, and that we did not create multi-edges while preserving planarity. We did



(a) $\deg_G(u) = 2$, $\deg_G(v) = 2$ (b) $\deg_G(u) = 1$, $\deg_G(v) = 2$ (c) $\deg_G(u) = 1$, $\deg_G(v) = 1$

Figure 7.5: Illustration of the proof of Proposition 7.9. The non-saturated vertices u and v lie on the same face f. The boundary of f is represented by a circle. Black vertices represent new vertices, dashed edges represent new edges. All new edges and vertices are drawn within the former face f such that the resulting graph is planar and all represented vertices have degree 3.



(a) $\deg_G(w) = 2$. (b) $\deg_G(w) = 1$.

Figure 7.6: Illustration of the proof of Proposition 7.9. The graph $\widehat{G}'$ on the vertices u, v and the new created vertices is represented by the gray area within the former face f that is adjacent to the vertices u and v in G. The boundary of f is represented by circle. The vertex w is a non-saturated vertex adjacent to f in the initial graph G. We connect w to $\widehat{G}'$ by subdividing edges of $\widehat{G}'$. Black vertices represent vertices obtained by such an subdivision. The vertex w has degree 3 in the resulting graph.

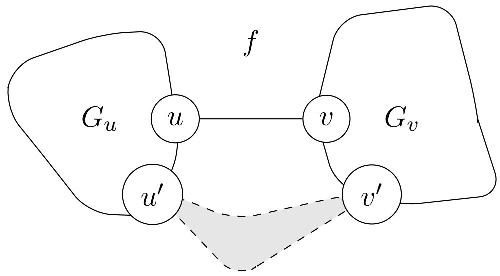


Figure 7.7: Illustration of the proof of Proposition 7.9. The edge uv is a bridge in the graph G that lies on the face f. The graph G_u is the component of $G - uv$ containing u, G_v the component containing v. The non-saturated vertices u' and v' are connected through new vertices and edges represented by gray areas and dotted lines. Therefore the former bridge uv is not a bridge in the augmented graph H.

not create bridges since the initial graph G is connected; thus there exists a u-v-path in G that is also a path of the augmentation. Another edge-disjoint u-v-path that only contains new edges can be easily be found in the augmented graph.

Let $\widehat{G}$ be the augmentation of G. Let $\widehat{G}'$ denote the subgraph of $\widehat{G}$ on the vertices u, v and the new vertices containing all new edges drawn in the former face f. Let $w \in \gamma^{-1}(\{f\}) \setminus \{u, v\}$. Then we can subdivide one of the edges of $\widehat{G}'$ and connect w with the newly created vertex; see Figure 7.6a. We have to do this twice if $\deg_G(w) = 1$; see Figure 7.6a. Note that once again, we did not create multi-edges and that all new vertices have degree 3. Similarly as before, we can show that we did not create bridges since G is connected.

Let H denote the graph that we obtain, if we proceed in such a way for every face of G. Note that H is well-defined, as every non-saturated vertex is mapped to exactly one face of G and the operations above do not effect any other face. Then H is a 3-regular, planar graph and $G \subseteq H$ by construction. Suppose H is not bridgeless. As by construction, we did not create any bridge, a bridge uv in H is a bridge in G. Then uv lies on the boundary of exactly one face $f \in F(G)$. Let G_u denote the component of $G - uv$ containing u, and G_v the component of $G - uv$ containing v. By (P2), there are vertices $u' \in V(G_u) \cap L(G)$ and $v' \in V(G_v) \cap L(G)$ such that u' and v' are mapped to f. By construction, there is a u'-v' path that only goes through edges in $E(H) \setminus E(G)$; thus, in particular does not contain uv, see Figure 7.7. Therefore $H - uv$ is still connected and uv is not a bridge, contradicting our assumption.

Therefore, H is a 3-regular, planar, bridgeless graph. ⌟

In particular, as $\chi'(H) = 3$ by Lemma 7.4, we have $\chi'(G) \leq 3$.

Claim 7.9.2. *If G is 3-regular-embeddable, then G is 2-face-mappable.*

Proof of Claim. Let G be 3-regular-embeddable. Then there exists a planar, 3-regular, bridgeless graph H such that $G \subseteq H$. We will call the edges $E(H) \setminus E(G)$ new edges and the vertices $V(H) \setminus V(G)$ new vertices. Consider a planar embedding of H. We will define a function $\gamma\colon L(G) \to F(G)$. Let $v \in L(G)$. All new edges that are incident to v lie within the same (former) face $f_v \in F(G)$, as if $\deg_G(v) = 2$ there is just one such edge and if $\deg_G(v) = 1$ the vertex v lies on the boundary of exactly one face of G. We set

$$\gamma(v) = f_v.$$

Suppose there exists $v \in L(G)$ such that there is no other vertex $u \in L(G)$ mapped to the face $\gamma(v)$.

Case 1. $\deg_G(v) = 2$. As no other vertex in $L(G)$ is mapped to f_v, but v has degree 3 in H, there is a new vertex v' adjacent to v; see Figure 7.8a. Then vv' is a bridge in H, as no other vertex on the boundary of f_v has an incident new edge within the former face f_v.

Case 2. $\deg_G(v) = 1$. Then v has exactly one neighbor u in G; see Figure 7.8b. Thus, uv is a bridge in H as no other vertex on the boundary of f_v has an incident new edge within the former face f_v.

In both cases, this is a contradiction, as H is bridgeless. Therefore the property (P1) required in Definition 7.8 holds.

Suppose the property (P2) does not hold for γ. Then there is a bridge $uv \in E(G)$ such that

$$\gamma^{-1}(f) \cap V(G_a) = \varnothing$$

for some $a \in \{u, v\}$, where G_a denotes the component of $G - uv$ containing a and f denotes the face such that uv lies on its boundary. Note that there is exactly one such face f. Without loss of generality, we may assume

$$\gamma^{-1}(f) \cap V(G_u) = \varnothing.$$

This means that no new edge incident to a vertex in G_u lies in the (former) face f. Hence, uv is still a bridge in H. This is a contradiction, as H is bridgeless.

Therefore γ is the function required in Definition 7.8 and G is 2-face-mappable. ⌟

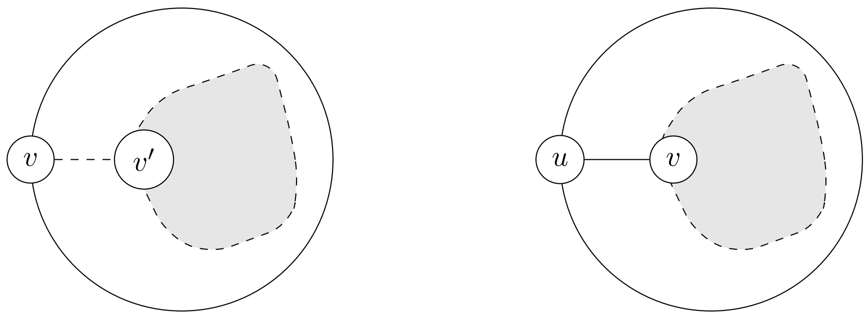
By Claim 7.9.1 and Claim 7.9.2, we see that the notions of 3-regular-embeddable and 2-face-mappable graphs coincide for connected, planar graphs of maximum degree 3. □

Using the characterization of 3-regular-embeddable graphs given in Proposition 7.9, we can now conclude that there exists a 3-edge-colorable graph of maximum degree 3 that is not 3-regular-embeddable. Thus, being 3-regular-embeddable is not necessary, but sufficient for being 3-edge-colorable.

Observation 7.10. *There exists a planar 3-edge-colorable graph G of maximum degree 3 that is not 3-regular-embeddable.*

Proof. Consider the graph G given in Figure 7.9a. As G is a bipartite graph, we have by a result of König [10, Proposition 5.3.1]

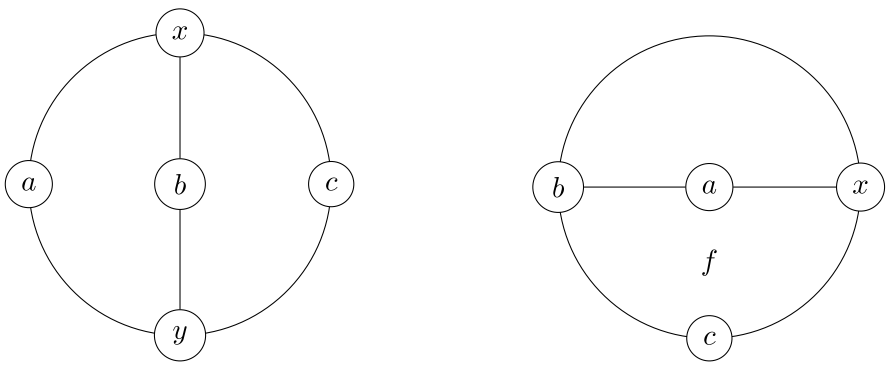
$$\chi'(G) = \Delta(G) = 3.$$



(a) $\deg_G(v) = 2$.

(b) $\deg_G(v) = 1$.

Figure 7.8: Illustration of the proof of Proposition 7.9. Dashed edges represent new edges, gray areas represent new vertices and edges. The non-saturated vertex v of G is adjacent to the former face f_v whose boundary is represented by a circle. No other non-saturated vertex is mapped to v. Therefore, the augmentation H contains a bridge.



(a) A 3-edge-colorable planar graph of maximum degree 3 that is not 3-regular-embeddable.

(b) An embedding of $G[a, b, c, x]$ where a, b and c lie on the same face f.

Figure 7.9: Illustrations to the proof of Observation 7.10.

Claim 7.10.1. *There is no embedding of G such that a,b and c lie on the same face.*

Proof of Claim. Note that G is 2-connected, since G admits an ear-decomposition [2, Theorem 25]: $G_0 = G[a, x, c, y]$, $G_1 = G$, where G_0 is a cycle on four vertices and G_1 can be obtained from G_0 by adding a G_0-path. Thus, in every planar embedding of G, every face is bounded by a cycle [10, Proposition 4.2.6].

Suppose there is a planar embedding of G, such that a,b and c lie on the same face f. As a and c are not connected and a,b,c form a path, x or y has to lie on f, since f is bounded by a cycle. Without loss of generality, we may assume that x lies on f. As b and x are connected, either a or c does not lie on the outer face. We may assume a does not lie on the outer face. Thus, we are in the situation of Figure 7.9b. There is no way to add y and the edges incident to y such that the embedding is still planar and a,b and c lie on the same face. This is a contradiction to the assumption and thus shows the claim. ⌟

Therefore, G is not 2-face-mappable, as the property (P1) cannot hold for any function $\gamma\colon L(G) \to F(G)$. By Proposition 7.9, a graph H is 3-regular-embedabble if and only if H is 2-face-mappable. Thus, G is not 3-regular-embeddable. □

Although not all 3-edge-colorable, planar graphs of maximum degree 3 are 3-regular-embeddable, we are still interested in deciding for a given graph if it is 3-regular-embeddable, i.e., 2-face-mappable by Proposition 7.9. For every 2-face-mappable graph G there exists a mapping from the non-saturated vertices of G to the faces of a fixed 2-face-mappable embedding. In some way, this mapping resembles a matching in a bipartite graph where one part contains all non-saturated vertices of G and the other all faces of the embedding. We can calculate the size of a maximum matching in a bipartite graph in polynomial time using flows as explained by Kleinberg and Tardos [20, p. 370]. By increasing the capacity of edges connecting the faces to the sink, we can modify this approach slightly so that multiple vertices can be mapped to the same face.

The situation of 2-face-mappable graphs is however somewhat different, as we wish that there is no face f such that exactly one vertex is mapped to f. Further, it is not clear how we can enforce that property (P2) holds. Thus, we will restrict our analysis to bridgeless, planar graphs of maximum degree 3.

We wish to decide for a bridgeless, planar graph G of maximum degree 3 if it is 2-face-mappable. In fact, if we consider the bipartite graph G' where one part consists of all non-saturated vertices, the other of the faces of G and we connect each vertex with its adjacent faces, we are looking for a subgraph of G' such that each non-saturated vertex in G has degree one in G' and no face has degree 1. Thus, we wish to decide whether G' admits a specific generalized factor as defined below. The following definitions are taken from Cornuéjols [8, p. 185].

Definition 7.11. Let G be a graph.

1. Let $b\colon V(G) \to \mathbb{N}_0$ be a mapping from the vertices of G to non-negative integers. The problem of deciding for a given graph G and such a mapping b whether there

exists a spanning subgraph $H \subseteq G$ such that

$$\forall v \in V(H) = V(G)\colon\ \deg_H(v) = b_v$$

is called the FactorProblem. We call the graph H a b-factor.

2. Let $B\colon V(G) \to \{S \subset \mathbb{N}_0\}$ be a mapping from all vertices v of G to a subset of $\{0,\dots,\deg_G(v)\}$. The problem of deciding for a given graph G and such a mapping B whether there exists a spanning subgraph $H \subseteq G$ such that

$$\forall v \in V(H) = V(G)\colon\ \deg_H(v) \in B(v)$$

is called the GeneralFactorProblem. We call the graph H a B-factor. Let $v \in V(G)$. We say that the set $B(v)$ has a *gap* of length $p \geq 1$ if there exists a non-negative integer $k \in B(v)$ such that $k+1, k+2, \dots, k+p \notin B(v)$ and $k+p+1 \in B(v)$.

While the FactorProblem can be solved in polynomial time [8], the GeneralFactorProblem is $\mathcal{NP}$-complete [8, p. 186]. Lovász who initiated the study of general factors, realized however that the gaps in the sets $B(v)$ of the definition above play an important role in determining the complexity of restrictions of the GeneralFactorProblem [21]. In 1988, Cornuéjols showed that if there are no gaps of length at least two, we can decide for a graph in polynomial time if it admits a general factor [8, Section 3]. This enables us to show that we can verify in polynomial time for a given planar embedding of a bridgeless, planar graph G of maximum degree 3 if it is fixed 2-face-mappable, i.e., if G is 3-regular-embeddable relative to the given embedding.

Theorem 7.12. *Let G be a planar, bridgeless graph of maximum degree 3. Then we can decide in polynomial time, whether a planar embedding of G is fixed 2-face-mappable.*

Proof. Consider the bipartite graph G' with parts

$$A \coloneqq L(G) = \{v \in V(G) \mid \deg(v) < 3\}$$

and $B \coloneqq F(G)$ the faces of the embedding. Let

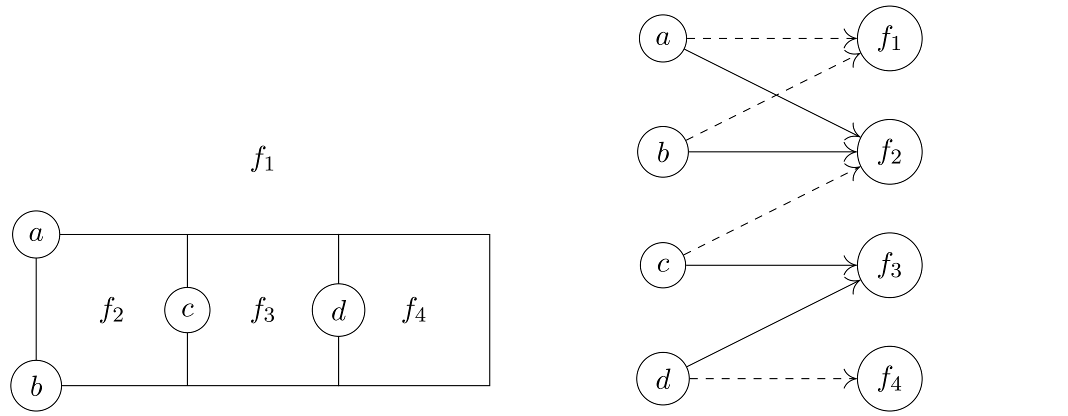
$$E(G') = \{vf \mid v \in L(G), f \in F(G), v \text{ lies on } f\}.$$

Let

$$\begin{aligned} B\colon V(G') &\to \{S \subseteq \mathbb{N}_0\} \\ u &\mapsto \begin{cases} \{1\}, & u \in A = L(G) \\ \{0,2,3,\dots,\deg_{G''}(u)\}, & u \in B = F(G). \end{cases} \end{aligned}$$

An example of the construction of G' is given in Figure 7.10.

Then (G', B) is an instance of the GeneralizedFactorProblem where no gap has length at least two and G' is bipartite. More precisely (G', B) is an instance



(a) The inital graph G. Only vertices of degree at most two are drawn explicitly.

(b) The graph G' and a B-factor H of G'. All represented edges belong to G'. The edges of H are the non-dashed edges.

Figure 7.10: An example of the polynomial transformation proposed in Theorem 7.12.

of the BIPARTITE1FACTORANTIFACTORPROBLEM, a restricted version of the GENERALIZEDFACTORPROBLEM where the initial graph H is bipartite with parts A and B, $B(a) = \{1\}$ for all $a \in A$ and $B(b) = \{0, 2, 3, \ldots, \deg_H(b)\}$. As Cornuéjols showed that the BIPARTITE1FACTORANTIFACTORPROBLEM polynomially reduces to the EDGEANDTRIANGLEPARTITIONINGPROBLEM [8, Theorem 1] and the latter can be solved in polynomial time as has been shown by Cornuéjols, Hartvigsen, and Pulleyblank [9], we can decide in polynomial time whether G'' admits B-factor.

Claim 7.12.1. *The graph G' admits a B-factor if and only if the embedding of G is fixed 2-face-mappable.*

Proof of Claim. Let G' admit a B-factor H. We wish to define a map, $\gamma\colon L(G) \to F(G)$. As for all vertices $v \in L(G) = A$ we have $\deg_H(v) = 1$, there is exactly one face $f_v \in F(G) = B$ such that $v f_v \in E(H)$. By definition of G', v lies on the face f_v. We set $\gamma(v) = f_v$ for all $v \in L(G)$. Thus γ is well-defined. As for all $f \in F(G) = B$, we have $\deg_H(f) \neq 1$, we see that

$$\forall f \in F(G)\colon \left|\gamma^{-1}(f)\right| \neq 1.$$

Thus, property (P1) holds. As G is bridgeless, we do not have to show property (P2). Therefore, G is fixed 2-face-mappable.

Conversely, if G is fixed 2-face-mappable and $\gamma\colon L(G) \to F(G)$ is the map given in Definition 7.8, we get a B-factor H of G' as follows. Let $V(H) = V(G')$ and $E(H) = \{v\gamma(v) \mid v \in L(G) = A\}$. Then similarly as before, we see that $E(H) \subseteq E(G')$ and that H is a B-factor. ⌟

As the construction of G' can be done in polynomial time and we can decide in polynomial time if G' admits a B-factor, we can decide in polynomial time if G is fixed 2-face-mappable.

□

7.2 Complexity of DemandFlow

As we already pointed out earlier, we can find a matching in a bipartite graph in polynomial time by calculating the flow of an auxiliary network. We will proceed similarly as in the proof of Theorem 7.12. Once again, we wish to find a map as required in the definition of 2-face-mappable graphs in order to decide for a planar, bridgeless graph whether it is fixed 2-face-mappable, but instead of B-factors, we will use flows. Imagine a water source connected to a target via pipes that are able to carry a certain amount of water. This situation can be modeled by a *network*. We will use the definition of a *flow* given by Diestel [10, p. 151]. A network $N = (G, s, t, c)$ is a directed graph G with a *capacity* $c\colon E(G) \to \mathbb{N}_0 \cup \{\infty\}$ and two vertices $s, t \in V(G)$ called the sink and the target, respectively. A flow is a map $\varphi\colon E(G) \to \mathbb{N}_0$ such that the amount of flow $\varphi(e)$ going through an edge $e \in E(G)$ cannot exceed the capacity $c(e)$, i.e.,

$$\forall e \in E(G)\colon \varphi(e) \leq c(e)$$

and further such that flow is preserved within the network excepting source and target, i.e.,

$$\forall v \in V(G) \setminus \{s, t\}\colon \sum_{uv \in E(G)} \varphi(uv) - \sum_{vu \in E(G)} \varphi(vu) = 0.$$

The *value* $|\varphi|$ of the flow φ is the sum of flow leaving the source, i.e,

$$|\varphi| = \sum_{sv \in E(G)} \varphi(sv).$$

Once again, we consider the bipartite graph G' where one part consists of all non-saturated vertices, the other of the faces of the chosen planar embedding. A vertex and a face are adjacent in G' if the vertex lies on the face. We wish to find a map from non-saturated vertices to adjacent faces in G' such that every vertex is mapped to exactly one face and no face is mapped to exactly one vertex. While the first condition can be modeled using flows, to express the second condition we will introduce a more general notion of demand flows.

Definition 7.13. Consider the following decision problem DemandFlow: Given a network $N \coloneqq (G, c, s, t)$, a positive integer $k \in \mathbb{N}$ and a function $d\colon E(G) \to \mathbb{N}_0$ that we will call demand. Is there an integral flow φ such that for every $e \in E(G)$ we have

$$\varphi(e) \geq d(e) \text{ or } \varphi(e) = 0$$

and the flow value is at least k? We call such a flow *feasible*. We are also interested in restricted variants of DemandFlow for fixed $\ell \in \mathbb{N}$ where $\mathrm{Im}(d) \subseteq \{0, \ell\}$. We call this decision problem DemandFlow$_\ell$.

Note that the problem on a network where we have a demand function d and ask whether there is a flow φ such that for every $e \in E(G)$, we have

$$d(e) \leq \varphi(e)$$

can be solved in polynomial time. This problem can be reduced to a flow problem without a demand function as has been discussed by Kleinberg and Tardos [20, pp. 382–384].

Proposition 7.14. *If* DemandFlow$_2 \in \mathcal{P}$*, then we can decide in polynomial time if a given planar embedding of a planar, connected, bridgeless graph G of maximum degree 3 is fixed 2-face-mappable.*

Proof. Consider the directed bipartite graph G' with parts

$$A \coloneqq L(G) = \{v \in V(G) \mid \deg(v) < 3\}$$

and $B \coloneqq F(G)$ the faces of the embedding. Let

$$E(G') = \{vf \mid v \in L(G), f \in F(G), v \text{ lies on } f\}.$$

Let G'' be the directed graph that we obtain from G' by adding vertices s and t and joining s to all vertices in A and t to all vertices in B with an edge, i.e.,

$$E(G'') = E(G') \cup \{sv \mid v \in L(v)\} \cup \{ft \mid f \in F(G)\}.$$

Let

$$\begin{aligned} d\colon E(G'') &\to \mathbb{N}_0 \\ uv &\mapsto \begin{cases} 0, & u \notin F(G) \\ 2, & \text{otherwise} \end{cases} \end{aligned}$$

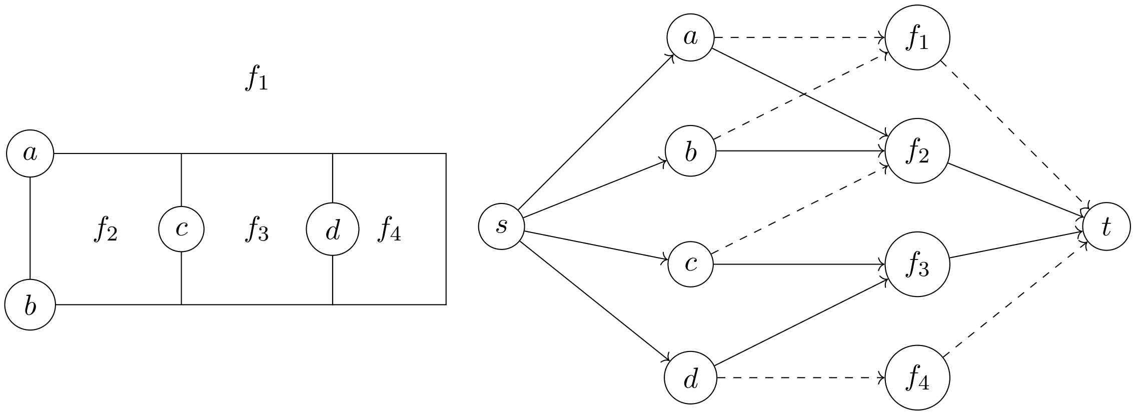
be the demand function and let

$$\begin{aligned} c\colon E(G'') &\to \mathbb{N}_0 \\ uv &\mapsto \begin{cases} \infty, & u \in F(G) \\ 1, & \text{otherwise} \end{cases} \end{aligned}$$

be the capacity function. Consider (G'', s, t, c) together with d and $k \coloneqq |L(v)|$, we call this instance I. I is an instance of DemandFlow$_2$. An example is given in Figure 7.11.

Claim 7.14.1. *The instance I is a yes-instance of* DemandFlow$_2$ *if and only if the embedding of G is fixed 2-face-mappable.*

Proof of Claim. If I is a yes-instance, then there is a feasible flow φ of value at least $|L(G)|$. We wish to define a function $\gamma\colon L(G) \to F(G)$. As s is only mapped to vertices of $L(G)$ and the capacity of each such edge is one, φ has value $|L(G)|$. Thus, for every vertex $v \in L(G)$, there is exactly one face $f \in F(G)$ such that $\varphi(vf) = 1$. We set



(a) The inital graph G. Only vertices of degree at most two are drawn explicitly.

(b) The graph G'' and a feasible flow. Dashed edges represent edges with 0-flow, all other edges have positive flow.

Figure 7.11: An example of the polynomial transformation proposed in Proposition 7.14.

$\gamma(v) = f$. As for every edge ft, where $f \in F(G)$, we have $\varphi(ft) = 0$ or $\varphi(ft) \geq 2$ we see that

$$\forall f \in \mathrm{Im}(\gamma) \colon \left|\gamma^{-1}(f)\right| \geq 2.$$

Thus, property (P1) holds. As G is bridgeless, we do not have to show property (P2).

Similarly, if G is fixed 2-face-mappable, the map $\gamma \colon L(G) \to F(G)$ given in Definition 7.8 defines a feasible flow. ⌟

As we can construct G'' in polynomial time and we assumed DemandFlow$_2$ to be in $\mathcal{P}$, we can decide in polynomial time if G is fixed 2-face-mappable. □

We already proved in Theorem 7.12 that we can decide in polynomial time for a given planar embeddding of a planar, bridgeless graph of maximum degree 3 if it is fixed 2-face-mappable. The proof of Proposition 7.14 is indeed very similar to the approach chosen in Theorem 7.12. Therefore, at first glance, it might seem likely that DemandFlow$_2$ is in $\mathcal{P}$. However, we will see in Theorem 7.16 that DemandFlow$_2$ is $\mathcal{NP}$-complete. As this is the case, there exists a polynomial transformation from every problem in $\mathcal{NP}$ to DemandFlow$_2$, thus it is not surprising that we were able to find a polynomial transformation from the fixed 2-face-mappable problem to DemandFlow$_2$.

In order to prove $\mathcal{NP}$-completeness of DemandFlow$_2$, we will prove that the decision problem DemandFlow$_3$ is $\mathcal{NP}$-complete. As DeamndFlow$_3$ is a restricted variant of DemandFlow, $\mathcal{NP}$-completeness of DemandFlow follows.

Lemma 7.15. *The decision problem DemandFlow$_3$ is $\mathcal{NP}$-complete.*

Proof. Büning and Lettmann showed that it is $\mathcal{NP}$-complete to determine the satisfiability of a given 3SAT-instance where every variable appears in at most three clauses [5, Theorem 3.1.4. 2]. We call this problem 3Bounded3Sat. We wish to show that the restricted variant Exactly3Bounded3Sat of 3Bounded3Sat where every variable appears in exactly three clauses is also $\mathcal{NP}$-complete.

Claim 7.15.1. *The decision problem Exactly3Bounded3Sat is $\mathcal{NP}$-complete.*

Proof of Claim. As we can verify for an instance of Exactly3Bounded3Sat and a given variable assignment whether it is satisfying in polynomial time, Exactly3Bounded3Sat is in $\mathcal{NP}$. Further, 3Bounded3Sat is reducible to Exactly3Bounded3Sat as for every variable x that appears only once we can add two clauses $\mathrm{x} \lor \mathrm{x} \lor \overline{\mathrm{x}}$ and for every variable x that appears twice, we can add a single clause $\mathrm{x} \lor \mathrm{x} \lor \overline{\mathrm{x}}$. As all new clauses are clearly always satisfiable and this construction can be done on polynomial time, the claim holds. ⌟

DemandFlow$_3 \in \mathcal{NP}$ as we can verify in polynomial time if a given flow is feasible.

Claim 7.15.2. *Exactly3Bounded3Sat $\propto$ DemandFlow$_3$.*

Let I be an instance of Exactly3Bounded3Sat. Let $X := \{x_1, \dots, x_n\}$ be the variables of I and let t_x be the number of clauses containing the variable $x \in X$. As I is an instance of Exactly3Bounded3Sat, we have $t_x = 3$ for all $x \in X$. For every variable $x \in X$ let x and $\overline{\mathrm{x}}$ be the literals of x. Let $L := \{\mathrm{x}, \overline{\mathrm{x}} \mid x \in X\}$ denote the set of all literals of I and C the set of all clauses. Consider the following directed graph G. Let

$$V(G) := \{s, t\} \cup X \cup L \cup C.$$

Let

$$\begin{aligned} E_{sX} &:= \{sx \mid x \in X\} \\ E_{XL} &:= \{x\mathrm{x}, x\overline{\mathrm{x}} \mid x \in X\} \\ E_{Ls} &:= \{\mathrm{x}s, \overline{\mathrm{x}}s \mid x \in X\} \\ E_{LC} &:= \{\ell c \mid \ell \in L, c \in C, \ell \in c\} \\ E_{Ct} &:= \{ct \mid c \in C\} \end{aligned}$$

and let $E(G) := E_{sX} \cup E_{XL} \cup E_{Ls} \cup E_{LC} \cup E_{Ct}$. We define the capacity function c as follows.

$$\begin{aligned} c\colon E(G) &\to \mathbb{N}_0 \\ e &\mapsto \begin{cases} t_x, & e = sx \in E_{sX} \\ t_x, & e = x\ell \in E_{XL} \\ t_x, & e = \mathrm{x}s \in E_{Ls} \text{ or } e = \overline{\mathrm{x}}s \in E_{Ls} \\ 1, & e \in E_{LC} \\ 1, & e \in E_{Ct} \end{cases} \end{aligned}$$

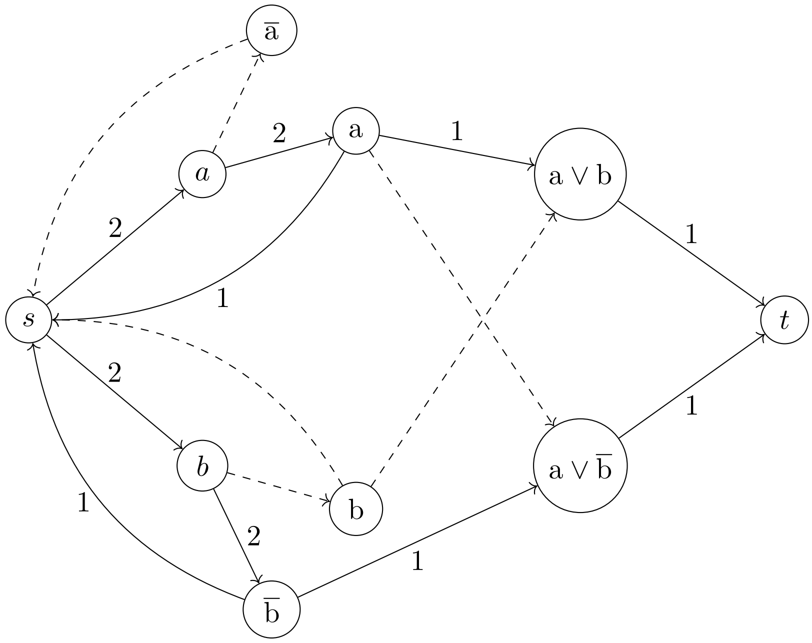


Figure 7.12: An example of the polynomial transformation proposed in Lemma 7.15 representing a feasible flow of the network (G, s, t, c) corresponding to the Sat-instance $(\mathrm{a} \vee \mathrm{b}) \wedge (\mathrm{a} \vee \overline{\mathrm{b}})$. Dashed edges represent edges with 0-flow. For simplicity, the transformation is illustrated for an instance of SAT instead of Exactly3Bounded3Sat.

Let d be the demand defined as follows.

$$d\colon E(G) \to \mathbb{N}_0$$
$$e \mapsto \begin{cases} t_x, & e = x\ell \in E_{XL} \\ 0, & \text{otherwise} \end{cases}$$

Let I' denote the instance of DemandFlow given by the network (G, s, t, c), the demand d and $k := |C|$. An example is given in Figure 7.12. Note that I' is an instance of DemandFlow$_3$ as $t_x = 3$ for all $x \in X$.

Claim 7.15.3. *I is satisfiable $\iff$ I' is a yes-instance of DemandFlow$_3$.*

Proof of Claim. Let I be satisfiable. We wish to define a feasible flow φ. There exists a variable assignment satisfying all clauses. For every clause c, we can pick a literal ℓ that is true. We set $\varphi(\ell c) = 1$ and $\varphi(ct) = 1$. Let t_ℓ be the number of times the literal ℓ has been chosen. Let x be the variable associated to ℓ. We set $\varphi(x\ell) = t_x$, $\varphi(\ell s) = t_x - t_\ell$ and $\varphi(sx) = t_x$. For all variables and all literals that have not been considered so far, we set the flow of all incident edges to 0. This defines a feasible flow of value $|C|$. Note that the edges E_{Ls} were necessary in order to preserve flow.

If I' is a yes-instance of DemandFlow$_3$, there is a feasible flow φ' of value at least $|C|$. As the edges E_{Ct} define an s-t-cut of the graph G, the flow φ' is of value exactly $|C|$. Thus, for every clause c there is exactly one literal ℓ such that $\varphi'(\ell c) = 1$ by flow preservation. Further, for every variable $x \in X$, at most one of the edges $x\mathrm{x}$ and $x\overline{\mathrm{x}}$ has positive flow by definition of the demand d. Therefore we can set x to true, if $x\mathrm{x}$ has positive flow and to false otherwise. This defines a satisfying assignment, as in every clause there is at least one literal that is true. ⌟

As Exactly3Bounded3Sat is $\mathcal{NP}$-complete and the given transformation can be calculated in polynomial time, DemandFlow$_3$ is $\mathcal{NP}$-complete. □

However, in order to verify for a fixed embedding if it is 2-face-mappable, we needed DemandFlow$_2$ in Proposition 7.14; thus, the demand for each edge was 0 or 2. The same construction as in Lemma 7.15 will not work in order to prove $\mathcal{NP}$-completeness for DemandFlow$_2$ as we could only consider SAT-instances where each variable appears in exactly two clauses. The problem Sat(2) of deciding for such SAT-instances if they are satisfiable is solvable in linear time as has been shown by Büning and Lettmann [5, Theorem 3.1.4 3.]. Though, we can simulate edges of demand 3 and capacity 3 by small graphs that only have demand 2 or 0 for edges as we will see in the following proof.

Theorem 7.16. *The decision problem DemandFlow$_2$ is $\mathcal{NP}$-complete.*

Proof. Clearly, DemandFlow is in $\mathcal{NP}$ as we can verify in polynomial time for a given flow if it is feasible. We already proved that DemandFlow$_3$ is $\mathcal{NP}$-complete. Using the reduction of Exactly3Bounded3Sat to DemandFlow$_3$ proposed in Lemma 7.15, we will show that DemandFLow$_2$ is $\mathcal{NP}$-complete. Let I be an instance of the decision problem Exactly3Bounded3Sat and let I' be the constructed instance of DemandFlow$_3$ as in Lemma 7.15. Let (G, s, t, c) be the network, d the demand and k the value of a feasible flow of I. Note that there are only two types of edges in G depending on their demand. Let $e \in E(G)$ be an edge.

Case 1. $d(e) = 0$. The demand of this edge is also a valid demand for an instance of DemandFlow$_2$.

Case 2. $d(e) = 3$. By construction of I', $e \in E_{XL}$, thus $c(e) = 3$ as every variable in I appears in exactly three clauses. We can replace the edge $e = gh$ by the graph H_e given in Figure 7.13. We will show that e behaves in the same way as H_e. Let $\widehat{G}$ be the graph where we replaced the edge e by H_e. Let $\widehat{c}$ denote the capacity and $\widehat{d}$ the demand of the network $(\widehat{G}, s, t, \widehat{c})$. For all edges of $E(G) \cap E(\widehat{G})$ the maps d and $\widehat{d}$, and c and $\widehat{c}$ respectively, agree.

Claim 7.16.1. *The network G admits a feasible flow if and only if $\widehat{G}$ admits a feasible flow.*

Proof of Claim. Let φ be a feasible flow of G. Then we can construct a feasible flow $\widehat{\varphi}$ of $\widehat{G}$ as follows. If $\varphi(e) = 0$, we can set $\widehat{\varphi}(\hat{e}) = 0$ for all edges $\hat{e} \in E(H_e)$, otherwise $\varphi(e) = 3$ as $c(e) = d(e) = 3$ and we can set $\widehat{\varphi}(\hat{e}) = \widehat{c}(\hat{e})$ for all $\hat{e} \in E(H_e)$. For all other edges, we set $\widehat{\varphi}$ to the value of φ. This defines a feasible flow as the outgoing flow of g in

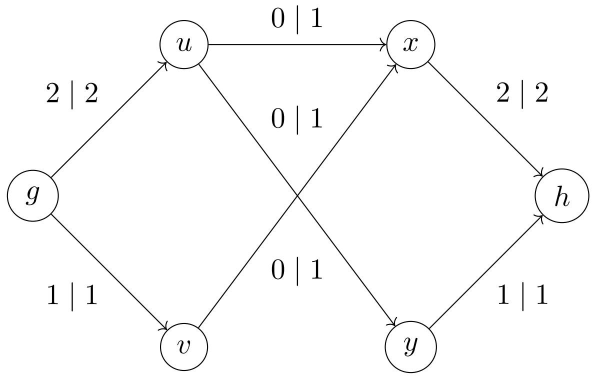


Figure 7.13: Every edge $e = gh$ in the instance I' of DemandFlow$_3$ where $c(e) = 3$ and $d(e) = 3$ can be replaced by the graph given above. All vertices except g and h do not belong to the graph G of the instance I'. An edge e' labeled $p \mid \ell$ denotes an edge of demand $d(e') = p$ and capacity $c(e') = \ell$.

H_e is equal to the flow going through $e = gh$ and the same holds for the incoming flow of h in H_e. Note that the flow values of φ and $\widehat{\varphi}$ are identical.

Let $\widehat{\varphi}$ be a feasible flow of $\widehat{G}$. If $\widehat{\varphi}(gu) > 0$, then $\widehat{\varphi}(gu) = 2$ as $\widehat{c}(gu) = \widehat{d}(gu) = 2$. By flow preservation, we have $\widehat{\varphi}(ux) = 1$ and $\widehat{\varphi}(uy) = 1$, thus by the same argument $\widehat{\varphi}(yh) = 1$ and $\widehat{\varphi}(xh) \geq 1$. As $\widehat{d}(xh) = 2$ and $\widehat{c}(xh) = 2$, we have $\widehat{\varphi}(xh) = 2$, thus by flow preservation $\widehat{\varphi}(vx) = 1$ and $\widehat{\varphi}(gv) = 1$. Similarly, we can show that $\widehat{\varphi}$ assigns the same values to the edges of H_e if $\widehat{\varphi}(gv) > 0$. Thus, if the outgoing flow of g in H_e is positive, it is equal to 3, otherwise it is zero. Therefore, we set the flow of $e = gh$ in G to the outgoing flow of g in H_e, i.e., $\varphi(e) = \widehat{\varphi}(gu) + \widehat{\varphi}(gv)$ and for all other edges $e' \in E(G) \setminus \{e\}$, we set $\varphi(e') = \widehat{\varphi}(e')$. This defines a feasible flow as seen above as the flow values of φ and $\widehat{\varphi}$ are identical. ⌟

We proceed in such a way for all edges of G. Let I'' be the obtained instance of DemandFlow, let G'' be the underlying graph and d'' its demand. Then we see inductively by Claim 7.16.1 that G admits a feasible flow if and only if G'' admits a feasible flow. By construction of the instance I'', we have $\mathrm{Im}(d'') \subseteq \{0, 2\}$. Thus I'' is an instance of DemandFlow$_2$. As G was constructed through a polynomial transformation from Exactly3Bounded3Sat to DemandFlow$_3$, we have a polynomial transformation from Exactly3Bounded3Sat to DemandFlow$_2$. Therefore DemandFlow$_2$ is $\mathcal{NP}$-complete as Exactly3Bounded3Sat is $\mathcal{NP}$-complete. □

8 Conclusions

Our aim was to show that all paths are not only 2-planar unavoidable as has been shown by Axenovich et al. [3, Lemma 6], but also 3-planar unavoidable. While this question remains open, we were able to show that there exist paths that are 4-planar avoidable. Axenovich et al. studied the iterated triangulation intensively. If all paths are 3-planar unavoidable, it seems likely that they are also 3-planar unavoidable in the class of iterated triangulations as the iterated triangulations cannot be edge-decomposed into three forests of bounded diameter. However, there exists a family of planar graphs that cannot be covered by three forests of bounded diameter, but admits an edge-decomposition into two graphs (which may contain cycles) of bounded diameter. Thus, in order to show that all paths are 3-$\mathcal{T}$ unavoidable, it is not sufficient, but necessary to know that the bounded diameter arboricity of the class of iterated triangulations $\mathcal{T}$ is 4.

Another possible approach of proving that all paths are 3-planar unavoidable is to show that the decision problem 3PlanarAvoidP$_k$ is $\mathcal{NP}$-complete for large enough k. We were interested in the special case $k = 2$. A graph G is a yes-instance of 3PlanarAvoidP$_2$ if and only if G admits a 3-edge-coloring that does not contain P_2 as a subgraph, i.e., if G is 3-edge-colorable. As all 3-regular, bridgeless, planar graphs are 3-edge-colorable by a result of Tait, we aimed to reduce 3PlanarAvoidP$_2$ to the problem of determining whether a graph G admits a 3-regular, bridgeless, planar supergraph. In the latter case, we call G 3-regular-embeddable. It turned out that being 3-regular-embeddable implies being 3-edge-colorable, but not vice-versa. Thus, we cannot reduce 3PlanarAvoidP$_2$ to being 3-regular-embeddable. Although we did not determine the complexity of the 3-regular-embeddable problem, we were able to show that we can decide in polynomial time for a given planar embedding of a bridgeless, connected graph G whether G admits a 3-regular, bridgeless, planar supergraph that respects the embedding of G.

The complexity of 3PlanarAvoidP$_k$ is still unknown for $k \geq 2$. The special case $k = 2$ is of greater interest as it is part of the study of the chromatic index of planar graphs. Vizing conjectured that all planar graphs of maximum degree $\Delta \geq 6$ are Δ-edge-colorable [27]. A proof for $\Delta \geq 7$ was given by Sanders and Zhao [24]. Although planar graphs of maximum degree $3 \leq \Delta \leq 5$ are known that are not Δ-edge-colorable [27], the complexity of the associated decision problems remain unknown [18, p. 124].

While all paths are 2-outerplanar unavoidable [3, Lemma 6], the same does not hold for edge-colorings of outerplanar graphs using three colors. As every outerplanar graph can be edge-decomposed into three star forests and all stars are 3-outerplanar unavoidable, the 3-outerplanar unavoidable graphs are precisely the star forests.

Bibliography

[1] Kenneth Appel and Wolfgang Haken. "Every planar map is four colorable". In: *Bulletin of the American Mathematical Society* 82.5 (1976), pp. 711–712.

[2] Maria Axenovich. *Graph Theory*. Lecture notes. 2020. URL: `https://www.math.kit.edu/iag6/lehre/graphtheory2019w/media/main.pdf`.

[3] Maria Axenovich, Ursula Schade, Carsten Thomassen, and Torsten Ueckerdt. "Planar Ramsey graphs". In: *Electronic Journal of Combinatorics* 26.4 (2019), Paper No. 4.9, 14. DOI: `10.37236/8366`. URL: `https://doi.org/10.37236/8366`.

[4] Hajo Broersma, Fedor V Fomin, Jan Kratochvil, and Gerhard J Woeginger. "Planar graph coloring avoiding monochromatic subgraphs: Trees and paths make it difficult". In: *Algorithmica* 44.4 (2006), pp. 343–361.

[5] Hans Kleine Büning and Theodor Lettmann. *Propositional logic: deduction and algorithms*. Vol. 48. Cambridge University Press, 1999.

[6] Richard Cole and Łukasz Kowalik. "New linear-time algorithms for edge-coloring planar graphs". In: *Algorithmica* 50.3 (2008), pp. 351–368.

[7] David Conlon, Jacob Fox, and Benny Sudakov. "Recent developments in graph Ramsey theory". In: *Surveys in combinatorics 2015*. Vol. 424. London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 2015, pp. 49–118.

[8] Gérard Cornuéjols. "General factors of graphs". In: *Journal of Combinatorial Theory, Series B* 45.2 (1988), pp. 185–198.

[9] Gerard Cornuéjols, David Hartvigsen, and W Pulleyblank. "Packing subgraphs in a graph". In: *Operations Research Letters* 1.4 (1982), pp. 139–143.

[10] Reinhard Diestel. *Graph theory*. Fifth. Vol. 173. Graduate Texts in Mathematics. Springer, Berlin, 2017, pp. xviii+428. ISBN: 978-3-662-53621-6. DOI: `10.1007/978-3-662-53622-3`. URL: `https://doi.org/10.1007/978-3-662-53622-3`.

[11] Guoli Ding, Bogdan Oporowski, Daniel P Sanders, and Dirk Vertigan. "Partitioning graphs of bounded tree-width". In: *Combinatorica* 18.1 (1998), pp. 1–12.

[12] Andrzej Dudek and Andrzej Ruciński. "Planar Ramsey numbers for small graphs". In: vol. 176. 36th Southeastern International Conference on Combinatorics, Graph Theory, and Computing. 2005, pp. 201–220.

[13] Michael R Garey and David S Johnson. *Computers and intractability*. Vol. 174. freeman San Francisco, 1979.

[14] Daniel Gonçalves. “Caterpillar arboricity of planar graphs”. In: *Discrete Mathematics* 307.16 (2007), pp. 2112–2121.

[15] Daniel Gonçalves. “Edge partition of planar graphs into two outerplanar graphs”. In: *Proceedings of the thirty-seventh annual ACM Symposium on Theory of Computing.* 2005, pp. 504–512.

[16] S Louis Hakimi, John Mitchem, and Edward Schmeichel. “Star arboricity of graphs”. In: *Discrete Mathematics* 149.1-3 (1996), pp. 93–98.

[17] Tanja Hartmann, Jonathan Rollin, and Ignaz Rutter. “Regular Augmentation of Planar Graphs”. In: *Algorithmica* 73.2 (2015), pp. 306–370.

[18] Frédéric Havet. *Combinatorial Optimization – Algorithms for telecommunications.* Lecture notes. 2014. URL: https://www-sop.inria.fr/members/Frederic.Havet/Cours/coloration.pdf.

[19] Ian Holyer. “The NP-completeness of edge-coloring”. In: *SIAM Journal on Computing* 10.4 (1981), pp. 718–720.

[20] Jon Kleinberg and Eva Tardos. *Algorithm design.* Pearson Education India, 2006.

[21] László Lovász. “The factorization of graphs. II”. In: *Acta Mathematica Academiae Scientiarum Hungarica* 23.1-2 (1972), pp. 223–246.

[22] Martin Merker and Luke Postle. “Bounded diameter arboricity”. In: *Journal of Graph Theory* 90.4 (2019), pp. 629–641.

[23] C St JA Nash-Williams. “Decomposition of finite graphs into forests”. In: *Journal of the London Mathematical Society* 1.1 (1964), pp. 12–12.

[24] Daniel P Sanders and Yue Zhao. “Planar graphs of maximum degree seven are class I”. In: *Journal of Combinatorial Theory, Series B* 83.2 (2001), pp. 201–212.

[25] Robert Sedgewick. *Algorithms in C, Part 5: Graph Algorithms.* Pearson Education, 2001.

[26] Richard Steinberg and Craig A. Tovey. “Planar Ramsey numbers”. In: *Journal of Combinatorial Theory. Series B* 59.2 (1993), pp. 288–296. ISSN: 0095-8956. DOI: 10.1006/jctb.1993.1070. URL: https://doi.org/10.1006/jctb.1993.1070.

[27] V. G. Vizing. “Critical graphs with given chromatic class”. In: *Akademiya Nauk SSSR. Sibirskoe Otdelenie. Institut Matematiki. Diskretnyĭ Analiz. Sbornik Trudov* 5 (1965), pp. 9–17.