Karlsruhe Institute of Technology

Title of the Thesis
Drawing hypergraphs as metro maps

Bachelor's Thesis of

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26. March 2019 – 25. July 2019

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I declare that I have developed and written the enclosed thesis completely by myself, and have not used sources or means without declaration in the text.

Karlsruhe, 25.07.2019

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(Fabian Walter Frank)

Abstract

A *metro map drawing* of a hypergraph is a visualization of a hypergraph in which each hyperedge of the hypergraph is drawn as a metro line and each vertex is represented by a station. A vertex v has a *vertex crossing* if at least two metro lines intersect at v.

Given a hypergraph $H = (V, E_H)$ and the ordering of every hyperedge, we show that it is NP-complete to decide whether there exists a metro map drawing with at most k vertex crossings, even if the embedding is fixed. Moreover we show some connections between support graphs and metro map drawings.

Zusammenfassung

Eine *Metromap-Darstellung* eines Hypergraphen ist eine Visualisierung eines Hypergraphen in der jede Hyperkante des Hypergraphen als eine Metrolinie und jeder Knoten als eine Metrostation dargestellt wird. Eine *Knotenkreuzung* ist ein Knoten, in welchem sich mindestens zwei Metrolinien kreuzen.

Für einen gegebenen Hypergraphen $H = (V, E_H)$ und eine gegebene Ordnung jeder Hyperkante zeigen wir, dass es NP-vollständig zu entscheiden, ob eine Metromap-Darstellung des Hypergraphen mit höchstens k Kreuzungsknoten existiert, auch wenn die Einbettung bereits vorgegeben ist. Des Weiteren zeigen wir Zusammenhänge zwischen Metromap-Darstellungen und Supports von Hypergraphen.

Acknowledgments

The topic of this bachelor thesis has its foundation in a Dagstuhl Workshop about Scalable Set Visualizations in 2017. The following people formed a working group with the title Set Visualization Using the Metro Map Metaphor: Robert Baker, Professor Nan Cao, Yifan Hu, Professor Michael Kaufmann, Professor Stephen Kobourov, Tamara Mchedlidze, Sergey Pupyrev, and Professor Alexander Wolff. Based on this idea I started my research. Therefore I am deeply grateful to this working group for their work.

To work on this thesis I spend some time with Professor Stephen Kobourov at the University of Arizona, who not only gave me the topic but also great advice that helped me to get a deeper understanding. My deepest appreciation goes to him for his great support. For this trip I got a scholarship from the Begabtenstiftung Informatik. I thank the Begabtenstiftung Informatik and especially Professor Dorothea Wagner and Professor Stephen Kobourov for giving me this opportunity.

I really appreciate the feedback offered by my two advisors Tamara Mchedlidze and Torsten Ueckerdt that also helped proofreading. I would also like to thank my fellow students Max Ruff and Jonas Flint for their insights and comments.

Contents

1 Introduction

London with its more than eight million inhabitants has one of the most famous metro systems in the world. It is the biggest metro in Europe with eleven different metro lines and 270 stations. Nevertheless it is possible to navigate through the underground of London and even tourists are able to understand quickly which lines they have to take by looking at the metro map. This map encodes the stations with their names, the metro lines and some geographical information at the same time. In spite of this overload of information, people are still able to read and receive the information they need.

Because of this benefits metro maps have been looked at intensively and people have tried to use metro map drawings to visualize other data. For example, Foo [Foo] turns personal memories into a metro map, Nesbitt [Nes04] and Stott et al. [Sto+05] use those metro map drawings to visualize relationships between PhD theses and items of a business plan, Sandvad et al. [San+01] for building Web-based guided tour systems, and Seskovec [Ses] uses it for visualizing historical events. One of the most popular applications is the visualization of the movies and movie genres (Figure 1.2) by the creators of the website Vodkaster. We will take this example to explain how we visualize some abstract data in this drawing style. The data we want to visualize is a set of movies and a set of genres such that each genre can contain many movies and each movie can also be part of more than one genre. One example for this is the movie "Wall-E" which belongs to the genres Animation and Romance. We transform this set data into a metro map by representing each movie as a station and each genre as a metro line that traverses exactly the movies that belong to this genre. In contrast to the actual metro map the set data does neither contain an order of the movies of that genre nor geographical positions for the movies. Therefore there is more freedom in generating a metro map drawing with such data set.

In our work we will mostly work with hypergraphs which are a generalization of graphs. A hypergraph $H = (V, E_H)$ can contain edges that are not just sets of two vertices but nonempty elements of the power set of the vertices. In other words the hyperedges $h \in E_H$ are sets of vertices $v \in V$ and therefore a way to encode set data. Going back to our movies and genres example, we can encode this data as a hypergraph by each movie being a vertex and each genre being a hyperedge containing exactly those movies that belong to the genre. Comparing this to the metro map drawing the vertices of the hypergraph are stations and the hyperedges are metro lines. The metro map drawing style actually visualize many properties of the hypergraph. For instance the number of necessary changes of metro lines we need from station A to station B is nothing else than the distance of the vertices A and B in the hypergraph H. The number of different metro lines at a station A is the degree of the vertex A. In order to make the metro map drawing less complicated it is useful to reduce the number of crossings that occur in the drawing. We will distinguish here between two different kind of crossings. The first kind of crossings takes place at the vertices of the metro map drawing, i.e. two metro lines cross

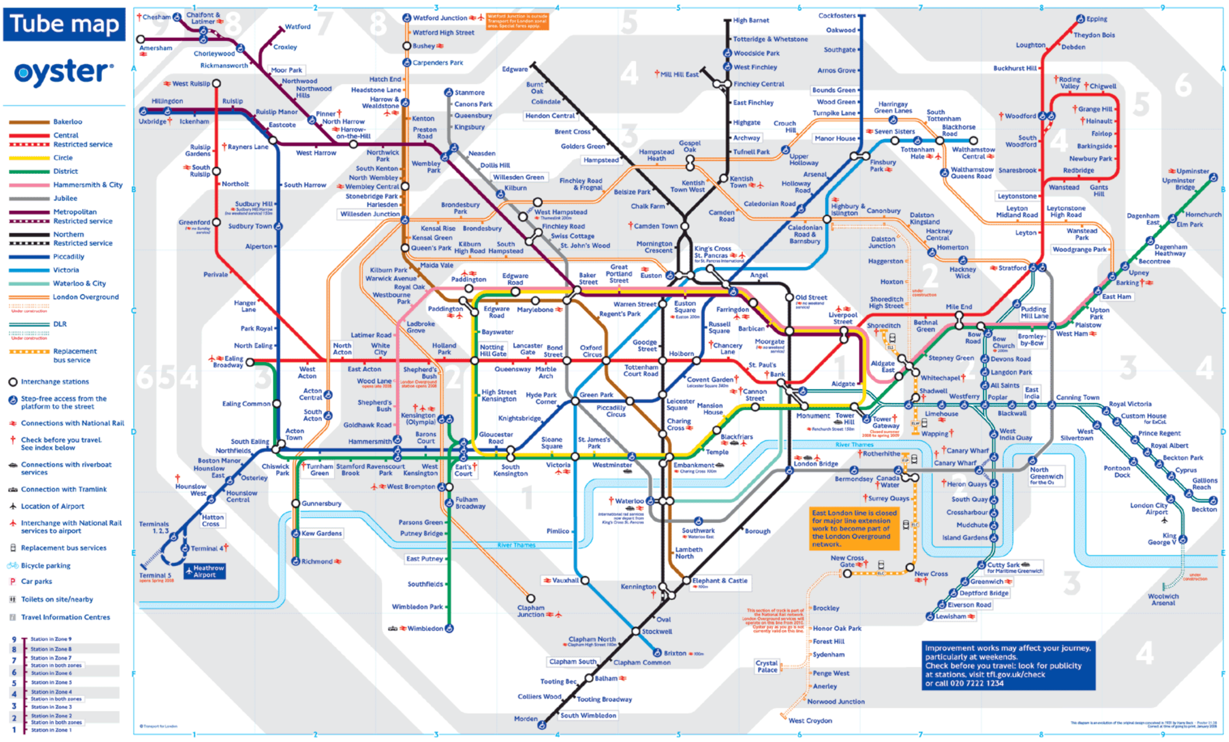


Figure 1.1: Map of the London underground system [15M]

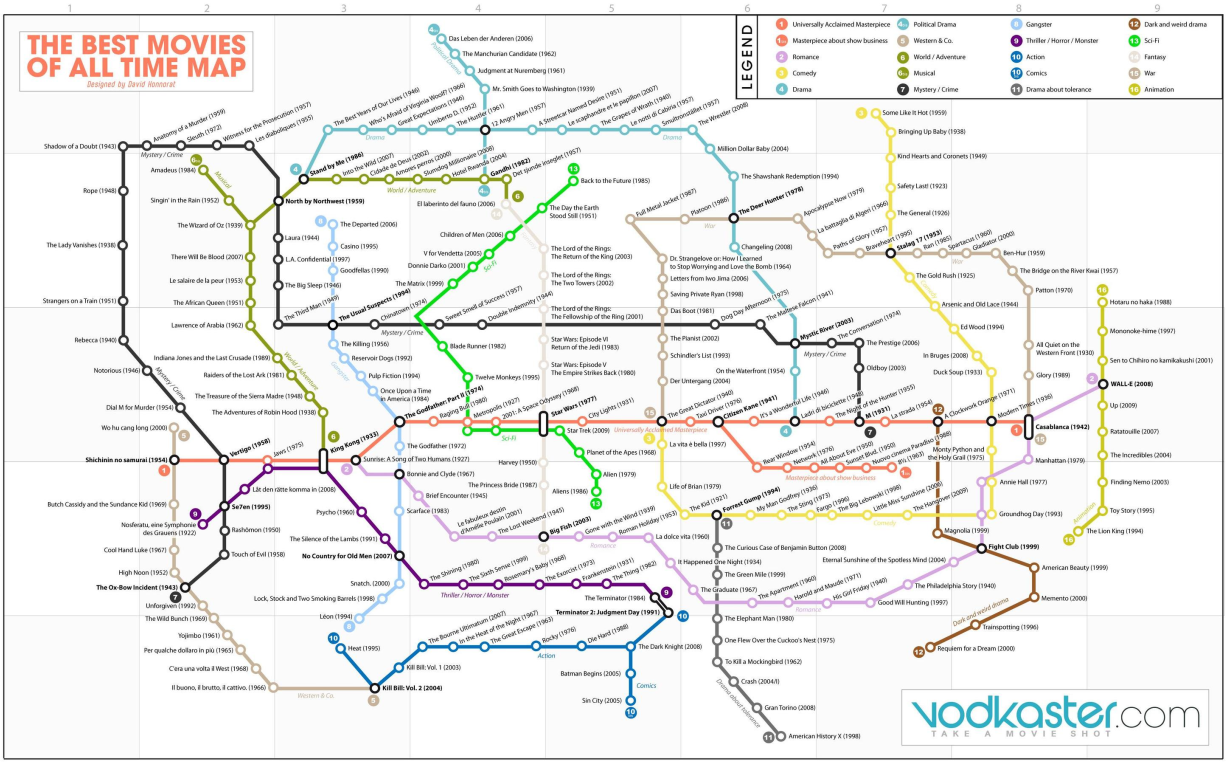


Figure 1.2: Metro map drawing of the data set movies and movie genres [Hon]

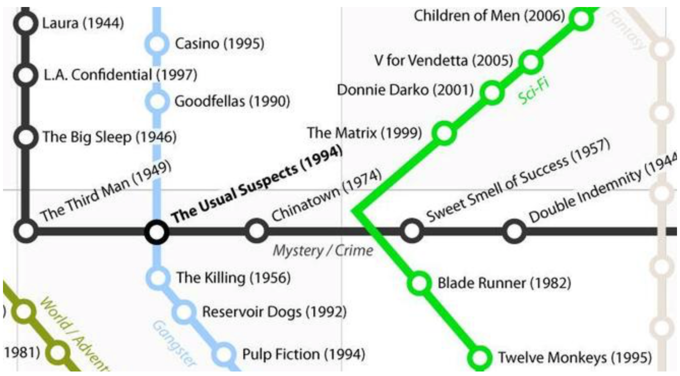


Figure 1.3: A vertex crossing between the blue and the black metro line at the station The Usual Suspect and an edge crossing between the green and the black metro line [Hon]

at a common vertex. Such a crossing can be seen in Figure 1.3 at the vertex The Usual Suspect. We will call those kind of crossings *vertex crossings*. The second kind of crossing is a crossing between two metro lines that occurs between vertices. One example for this kind of crossing is the crossing between the green and the black metro line in Figure 1.3. We will call those crossings *edge crossings*.

So far we discussed the connection between a hypergraph and its corresponding metro map drawing but a metro map drawing also induces a graph G with the same set of vertices in the following way. If there is a metro line such that the vertices u and v are directly connected without any vertices in between than we add the edge $\{u, v\}$ to G. In this transformation we lose the information which metro line actually uses a certain edge but nevertheless we know that each metro line is a path in G. So for each hyperedge $h \in E_H$ there is a path in G that contains exactly the vertices of h. We call graphs that have this attribute *path-based supports* of H. We can weaken the above condition of a hyperedge being represented by a path in G by not insisting on a path but only a connected subgraph. If each hyperedge h of H is a connected subgraph of G we call G a *support graph* of H.

1.1 Related Work

Since hypergraphs are a way to encode set data, drawing hypergraphs is a form of visualizing sets. But in comparison to the visualization of graphs there is less consense on how to draw a hypergraph. Therefore there a many different approaches. Mäkinen [Mäk90] describes the subset standard which is very similar to a Venn diagram. Every hyperedge is drawn as a closed curve that contains in the interior exactly the same vertices as the hyperedge. Johnson and Pollak [JP87] introduce so called vertex-based Venn diagrams. A vertex-based Venn diagram representation of a hypergraph H is a planar embedding of a graph G such that each vertex is represented by a face and for each hyperedge h of H the union of the corresponding faces of the vertices in h comprises a region whose interior is connected. They also show that this condition is equivalent to finding a planar support graph (Definition 2.6) for H which they show is NP-complete. Buchin et al. [Buc+09] give a polynomial-time algorithm to find a support that is a tree with bounded degrees on each vertex if such a support exists.

This work will give some theoretical results on path-based support graphs (Definition 2.7) and their relation to normal support graphs. Brandes et al. [Bra+10] give some results on the existence of path-based tree supports. In comparison we will look at path-based supports in general. Moreover we want to use those path-based support graphs to create drawings of hypergraphs in the metro map style under crossing minimization criteria.

Assume in the following that a planar path-based support is already given as well as an ordering for each hyperedge. Now we want to draw the metro lines on the support with as few crossings as possible and insist that all the crossings between metro lines that share at least one edge occur on the shared edges. Observe that at each vertex we are free to decide the ordering of the metro lines. This problem has been studied quite intensively and many variants of it are already NP-hard on relatively simple graphs. Fink and Pupyrev [FP13] proved that minimizing the number of crossings in such a setting (MLCM) is NP-hard even on caterpillars, Argryriou et al. [Arg+10] showed that minimizing the number of crossings is NP-hard for paths if insisted that each metro line terminates at a peripheral position (MLCM-SE). If further insisted on fixed peripheral positions (MLCM-fixedSE) the problem becomes polynomial (see [AGM08]).

1.2 Outline

In the second chapter we give some important defininitions for the following chapters. In Chapter 3 we give some results on path-based supports and compare them to general support graphs. Furthermore we look at the condensation (Definition 2.10) of hypergraphs and show that there are hypergraphs that have a planar path-based support but the condensation does not have one and vice versa.

In Chapter 4 we focus on the largest number k such that every hypergraph on k hyperedges can be drawn in a specific style without edge crossings.

In Chapter 5 we introduce a new problem where we want to find a metro map drawing that minimizes the number of vertex crossings. We show the hardness of this problem for given and variable embedding and give a polynomial-time algorithm for trees with given embedding. Furthermore we look at the MLCM problem for variable embedding and show its NP-hardness.

In Chapter 6 we give an Integer Linear Program (ILP) for minimizing the number of edges in a path-based support for a given hypergraph.

In Chapter 7 we summarize our results.

2 Preliminaries

First we need to give some necessary definitions. As mentioned above our goal is to visualize hypergraphs which are a generalized version of graphs in the sense that each hyperedge is a set of potentially more than just two vertices. We denote the power set of V by $\mathcal{P}(V)$.

Definition 2.1 (Hypergraph)**.** *A hypergraph $H = (V, E_H)$ is a pair of a set of vertices V and a set of hyperedges E_H such that $E_H \subseteq \{h \mid h \in \mathcal{P}(V) \setminus \{\emptyset\}\}$.*

In general, hyperedges are just a set of vertices and do not contain any information about the order. In this case we denote a hyperedge $h = \{1, 2, ..., k\}$ with curly brackets. A hypergraph with additional information about the ordering of the vertices within each hyperedge is called an *ordered* hypergraph. In this case we denote a hyperedge with ordering information $h = (a, b, ..., k)$ with ordinary brackets.

Our main goal is drawing hypergraphs as metro maps as defined below:

Definition 2.2 (Drawing of a graph G)**.** *A drawing of a graph $G = (V, E)$ is a representation of G in which each vertex $v \in V$ is depicted by a point in the plane and each edge $\{u, v\} \in E$ by an open continous curve between u and v with $u, v \in V$.*

Definition 2.3 (Metro map drawing)**.** *Let $H = (V, E_H)$ be a hypergraph. A* metro map drawing *of H is a graphical representation where each node in V is depicted by a point in the plane and each hyperedge $h \in E_H$ by an open continuous curve that passes through the points corresponding to the vertices in h.*

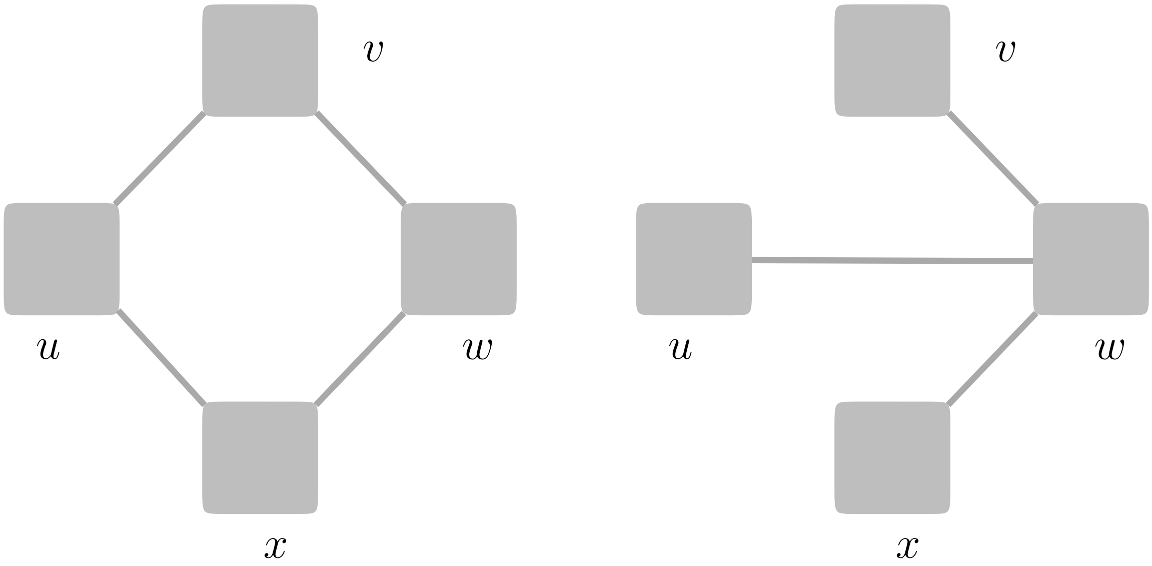
In order to present one way to create such drawings we need some further definitions on hypergraphs. At first we define the dual of a hypergraph:

Definition 2.4 (Dual of a hypergraph)**.** *The dual $H^* = (V^*, E_H^*)$ of a hypergraph $H = (V, E_H)$ is given by $V^* = E_H$ and $E_H^* = \bigcup_{v \in V}\{\{e \mid v \in e, e \in E_H\}\}$*

In our final representation as a metro map each hyperedge will have an ordering. This ordering corresponds to a path that contains exactly the vertices of the hyperedge. In our metro map drawing each of those paths will become a metro line. To specify this further we need a few definitions about support graphs.

Definition 2.5 (Induced Subgraph)**.** *The induced subgraph $G[S] = (S, E_S)$ of a graph $G = (V, E)$ and a subset $S \subseteq V$ is the graph that contains exactly the edges of G which have both endpoints in S. Therefore $E_S = \{\{u, v\} \in E \mid u, v \in S\}$.*

Definition 2.6 (Support graph)**.** *A support graph $G = (V, E)$ of a hypergraph $H = (V, E_H)$ is a graph such that for all $h \in E_H : G[h]$ is connected.*



(a) A minimal but not a minimum path-based support graph G of a hypergraph $H = (V, E_H)$ with $E_H = \{\{u, w, v\}, \{u, w, x\}\}$

(b) A minimum and therefore also a minimal path-based support graph G of a hypergraph $H = (V, E_H)$ with $E_H = \{\{u, w, v\}, \{u, w, x\}\}$

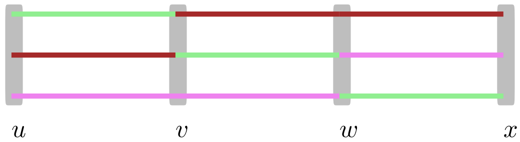
Figure 2.1: An example for a hypergraph H where a graph G is a minimal path-based support graph of H but not a minimum path-based support graph of H

Definition 2.7 (Path-based support graph)**.** *A path-based support graph $G = (V, E)$ of a hypergraph $H = (V, E_H)$ is a graph such that for all $h \in E_H$: The graph $G[h]$ contains a Hamiltonian path.*

It is clear that every path-based support graph of a hypergraph H is also a support graph of H but not the other way around. Furthermore every hypergraph $H = (V, E_H)$ does have a path-based support graph since the complete graph on $|V|$ vertices is a valid path-based support graph of H. In many cases we want to minimize the number of edges of this path-based support graph. We call a path-based support graph G of H *minimal* if any graph that we get by removing an edge in G is not a path-based support graph of H. Furthermore we call G a *minimum* path-based support graph of H if there is no other path-based support graph G' that does have less edges than G. It is clear that if a path-based support graph G of a hypergraph H is minimum it is also minimal but not the other way around as seen in Figure 2.1. In some of our problems we consider the case that we want to find a path-based support graph of an ordered hypergraph H. In this case we know how the paths for each hyperedge look like. For those instances a path-based support graph of H is minimal if for every edge $e = \{u, v\} \in E$ there exists an ordered hyperedge h in E_H with consecutive vertices u and v. In the case of ordered hypergraphs the definitions for minimum and minimal become equivalent and we will call those path-based supports graphs of H *minimum*.

We will use a path-based support graph G of H for our metro map drawings in the following way. A metro line of the hyperedge h of H is a path p in G that contains exactly the vertices of h. So basically a path-based support is the railwaysystem on which the

(a) A crossing free instance with four vertices



(b) Same instance as Figure 2.2a but with vertex crossings at v, w

Figure 2.2: Vertex crossings

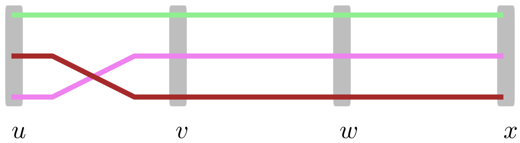


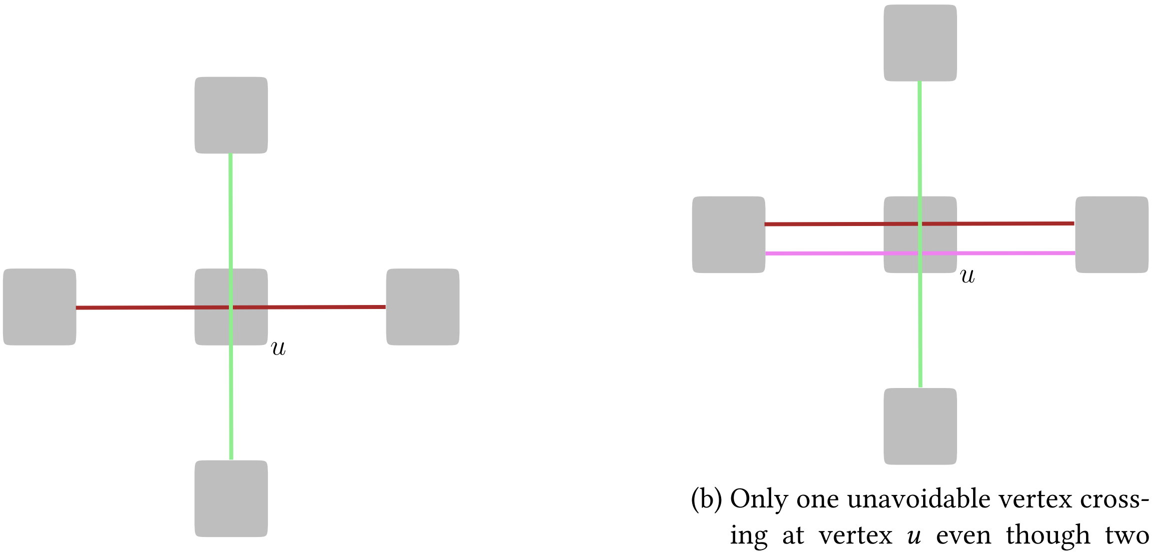
Figure 2.3: A line crossing between the pink and the brown metro line at the edge $\{u, v\}$

metro lines run. In the following we present a pipeline that results in a metro line drawing of a hypergraph H:

1. find a path-based support graph G of H with a path p_h in G for each hyperedge h of H.
2. find a drawing of G.
3. order the metro lines on the edges of G.

Each of those steps can have different optimization criteria. In step 1 we may want to find a minimal path-based support graph G of H or a planar path-based support graph. In step 2 we can optimize the number of crossings or find a straight-line drawing. In step 3 we may want to avoid crossings (Figure 2.2, Figure 2.3).

In the following we want to look at the crossings that can occur in a metro map drawing. In step 2 we choose a drawing for the path-based support graph. This drawing may contain a crossing between two edges e and e'. A metro line p_1 that uses the edge e and a metro line p_2 that uses the edge e' cross at this point. We call such a crossing an *edge crossing*. Observe that those kind of crossings can already be computed after step 2 of our metro map drawing procedure since the ordering of the hyperedges and the drawing of the support graph is enough to detect such crossings. But our metro map drawing can have a different type of crossings as well. Two metro lines can also cross at a vertex (Figure 2.4a). We denote this kind of crossing as a *vertex crossing*. For vertex crossing we do not distinguish between the number of crossings that take place at this vertex. So no matter how many lines intersect at a vertex it still only has one vertex crossing (Figure 2.4b). As seen in Figure 2.2b there are vertex crossings that can be avoided. But there are also vertex crossings that are unavoidable if we do not change the embedding as seen in Figure 2.4a. Even though some of the crossings can already be computed after step 2 (for example the crossing in Figure 2.4a), the way we order the metro lines on the edges of our support graph may change the number of vertex crossings (Figure 2.2). There is another kind of crossing which occurs if two metro lines p_1 and p_2 cross on a common



(a) An unavoidable vertex crossing at vertex u between the green and the brown metro line

(b) Only one unavoidable vertex crossing at vertex u even though two crossings occur in u (the brown and the green metroline, the pink and the green metroline)

Figure 2.4: an unavoidable vertex crossing

edge $e = \{u, v\}$ (Figure 2.3). We will call those crossings *line crossing*. For the chapters 3 and 4 we insist that no line crossings occur in our metro map drawings.

If the path-based support graph of our metro map is planar we call the metro map drawing planar, if the metro map drawing does not contain edge or vertex crossings we call it *crossing free*. A simple observation is that hyperedges of size two cannot enforce a vertex crossing since they do not traverse any vertices.

Another property of metro map drawings we want to discuss is monotonicity.

Definition 2.8 (x-monotone metro map drawing of a hypergraph)**.** *An x-monotone metro map drawing of a hypergraph $H = (V, E_H)$ is a metro map drawing of a hypergraph H such that $\forall h \in E_H$: the Hamiltonian path $(v_1, ..., v_k)$ crosses every line that is perpendicular to the x-axis at most once.*

A simple observation about hyperedges of size two is that since we do not care about the direction in which we traverse the path a hyperedge of size two will never violate the monotonicty condition.

In the following we want to give an example that shows that if we have a Hamiltonian graph G there does not have to be a straight line drawing D of G such that there is an x-monotone Hamiltonian path of G in D.

Example 2.9. Take the graph $G = (V, E)$ with $V = \{1, 2, 3, 4, 5, s, t\}$ and $E = \{\{u, v\} \mid u \in \{1, 2, 3, 4, 5\}, v \in \{2, 4, 5\}$ with $u \neq v\} \cup \{\{3, t\}\} \cup \{\{1, s\}\}$. This graph has a straight line planar drawing as seen in Figure 2.5 and is Hamiltonian because it contains the path $(s, 1, 5, 4, 2, 3, t)$. But if we insist that this graph is x-monotone for any Hamiltonian path we cannot draw the graph with straight lines: Since both vertices 1 and 3 are connected to the vertices 2,5,4 and the vertices 2,5,4 form a triangle, we know that either 1 or 3 has to

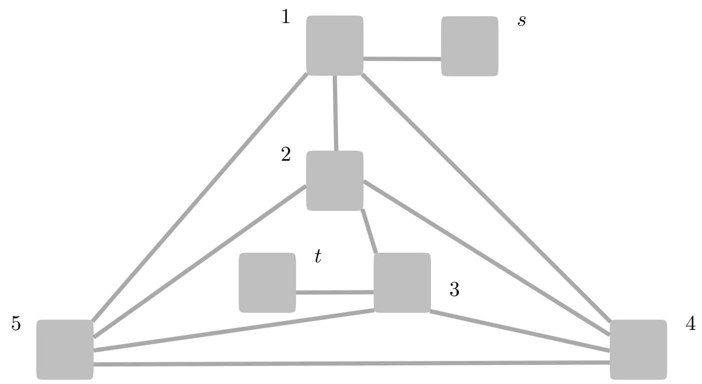


Figure 2.5: A planar straight line drawing

be inside the triangle. So without loss of generality assume 3 is inside the triangle. Since t has degree one and is therefore at the end of the path and connected to the vertex 3, t has to be inside the triangle as well. We further insisted on straight lines therefore t is inside the convex hull of 2,4,5 and so there cannot be any x-monotone Hamiltonian path in a planar straight line drawing of G.

Next we consider something called the *condensation* of a hypergraph. We define an equivalence relation $\sim$ on the vertices V of a hypergraph $H = (V, E_H)$ as follows:
$x \sim y \iff \forall h \in E_H : (x \in h \iff y \in h)$.

Definition 2.10 (Condensation of a hypergraph)**.** *The condensation $H_C = (V_C, E_{H_C})$ of a hypergraph $H = (V, E_H)$ is defined by removing all but one vertex $v_i \in V$ for each class i of the equivalence relation $x \sim y$. Formally, $V_C = \bigcup_i \{v_i\}$ and $E_{H_C} = \{h \cap V_C \mid h \in E_H\}$*

The condensation of a hypergraph is unique but different hypergraphs can have the same condensation. In order to create a hypergraph that equals its own condensation we can use the following trick:

Given a hypergraph $H = (V, E_H)$. We create a new hypergraph $H_{con} = (V, E_H \cup \bigcup_{v \in V} \{\{v\}\})$. Since for every vertex v in V there exists one hyperedge that only contains v, H_{con} is equal to its own condensation. We can use this trick to show that if a crossing problem for drawing hypergraphs as metro maps is NP-hard that it is also hard for the condensation of hypergraphs since adding hyperedges of size one does not change the problem. Even if we insist on hyperedges of size at least two we can add a unique vertex v_{clone} for each vertex $v \in V$ and add the hyperedge $\{v, v_{clone}\}$ to E_H. Those extra edges will not create any extra edge crossings.

In the following we want to show the relation between a metro map drawing of a hypergraph H and path-based support graphs of H.

Proposition 2.11. *For a hypergraph $H = (V, E_H)$: H has a planar metro map drawing if and only if H has a planar path-based support graph $G = (V, E)$.*

Proof. So assume that H has a planar metro map drawing D_H. Now we look at every pair of vertices u, v: if there is a metro line connecting u, v directly we add the edge $\{u, v\}$ to E. This creates a planar graph G because for every edge $e \in E$ there is a corresponding curve

in D_H and D_H is planar and by definition this is a path-based support graph (because if a metro line uses the edge $\{u, v\}$ we added it to E).
For the other direction assume $G = (V, E)$ is a planar path-based support graph of $H = (V, E_H)$. For every hyperedge $h \in E_H$ there exists a Hamiltonian path in the induced subgraph $G[h]$. Let $P_H = \bigcup_{h \in E_H} \{P_h\}$ be a valid set of Hamiltonian paths for each $h \in E_H$. Now split every edge e of E in e_k parallel parts where $e_k = |\{P_h \in P_H \mid e \in P_h\}|$. If no P_h contains e remove e. Now if $e = \{u, v\} \in E$ is in P_h add one of the created edges between the vertices u and v to the metro map drawing of the hyperedge h. □

The same argument can be used to show that the equivalency upholds also if we add the x-monotonicity condition.

3 Results on path-based supports

In this section we want to show some results about path-based support graphs of hypergraphs in comparison to other support graphs of hypergraphs. We further want to give some results on the correlation between finding a planar path-based support graph for the condensation of a hypergraph H and the original hypergraph H.

3.1 Hardness

Johnson and Pollak showed it is NP-complete to decide whether a hypergraph has a planar path-based support [JP87]. But even if a planar path-based support is given it remains NP-hard to verify it.

Proposition 3.1. *It is NP-complete to decide if a planar graph $G = (V, E)$ is a planar path-based support of the hypergraph $H = (V, E_H)$.*

Proof. First we observe that the problem is in NP since if the order is given for each hyperedge the solution can be verified in polynomial time.
Next we show that the problem is NP-hard. Hamiltonian path remains NP-complete even for planar graphs. Now we reduce planar Hamiltonian path to our problem: Take a graph $G = (V, E)$ and transform it to the hypergraph $H = (V, \{V\})$.

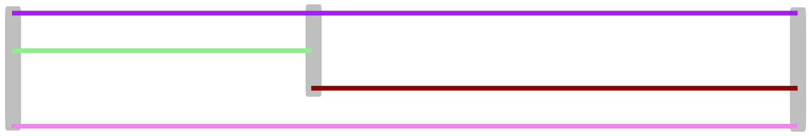
Let P be the Hamiltonian path in G. Since G has a Hamiltonian path the subgraph $G[V]$ has a Hamiltonian path. Therefore G is a path-based support graph of H.

If G is a planar path-based support of H each hyperedge h in H has a Hamiltonian path in $G[h]$. Since H contains a hyperedge with all the vertices V, G contains a Hamiltonian path. □

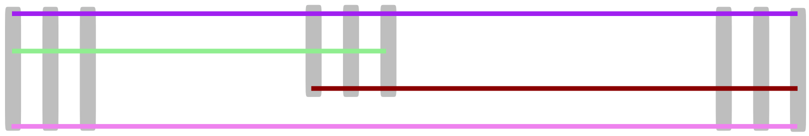
This NP-reduction shows that the Hamiltonian path problem is a specific version of the problem if a given graph G is a support graph for a hypergraph H. In other words the question if a graph of a certain graph class is a support graph for a given hypergraph H is NP-hard at least for all graph classes for which Hamiltonian path is NP-hard.

Proposition 3.2. *It is NP-complete to decide if a planar graph $G = (V, E)$ is an x-monotone path-based support of the hypergraph $H = (V, E_H)$.*

Proof. We can use the same transformation as in the NP-reduction for Proposition 3.1.



(a) A planar x-montone metro map drawing of the condensation of H



(b) A planar x-monotone metro map drawing of the original hypergraph H

Figure 3.1: Transformation of a planar x-monotone metro map drawing of the condensation of a hypergraph H to a planar x-monotone metro map drawing of H

Let P be the Hamiltonian path in G. This path induces an ordering of the vertices in G. After moving the vertices to positions that respect the ordering this results in an x-monotone path-based support graph.

If G is a path-based support of H each hyperedge in H has a Hamiltonian path in G. Since H contains a hyperedge with all the vertices in V, G contains a Hamiltonian path.

□

3.2 General results on path-based supports

Next we will look at the relationship between a hypergraph and its condensation. First we observe that in the x-monotone case an x-monotone planar drawing of the condensation can be transformed into an x-monotone planar metro map drawing of the originial hypergraph.

Proposition 3.3. *If the condensation $H_C = (V_C, E_{H_C})$ of a hypergraph $H = (V, E_H)$ has a planar x-monotone metro map drawing with k vertex crossings then H has a planar x-monotone metro map drawing with k vertex crossings.*

Proof. Use the metro map drawing of H_C in the following way. For every vertex $v \in V$ there exists a vertex $v_c \in V_C$ such that for every hyperedge $h \in E_H$ $v \in h$ if and only if $v_c \in h$. So take the x-position pos of vertex v_c in the drawing of H. There exists an $\epsilon > 0$ such that every hyperedge that contains v_c does not contain a vertex with x-position in the intervall $(pos, pos + \epsilon)$ in the drawing of H_C. Therefore we can place each v into this intervall of the drawing. If a hyperedge terminates at vertex v_c we extend it to the last vertex we inserted in the intervall. This method will neither create vertex nor edge crossings. An example of this method can be seen in Figure 3.1.

□

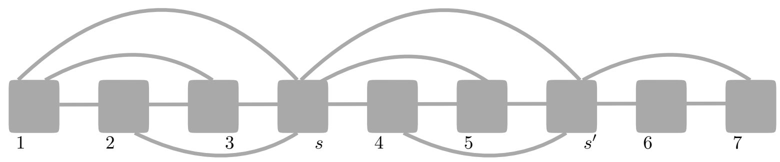


Figure 3.2: A possible planar x-monotone metro map drawing of H

Now we will show that the opposite statement is not true.

Proposition 3.4. *There is a hypergraph $H = (V, E_H)$ that has a planar x-monotone metro map drawing but its condensation $H_C = (V_C, E_{H_C})$ does not have a planar x-monotone drawing.*

Proof. Take the hypergraph $H = (V, E_H)$ with $V = \{1, 2, 3, 4, 5, 6, 7, s, s'\}$ and the hyperedges $E_H = \{\{u, v, s, s'\} \mid u, v \in V \backslash \{s, s'\}$ and $u \neq v\}$. There exists a planar x-montone metro map drawing as seen in Figure 3.2.
The condensation $H_C = (V_C, E_{H_C})$ of this hypergraph H has the vertex set $V_C = \{1, 2, 3, 4, 5, 6, 7, s\}$ and the hyperedges $E_{H_C} = \{\{u, v, s\} \mid u, v \in V_C \backslash \{s\}\}$. Now assume there is a planar x-montone metro map drawing of H_C. If s has the lowest or the highest x-value of the vertices of V_C in the metro map drawing the vertices $\{1, 2, 3, 4, 5, 6, 7\}$ are next to each other and will form a K_7. Because an x-monotone path containing the vertices of the hyperedge $\{u, v, s\}$ contains the edge $\{u, v\}$ due to the x-monotonicity. So from now on assume that s is neither the vertex with the lowest nor the vertex with the highest x-value of V_C in the drawing. Without loss of generality there are four vertices of H_C that have a higher x-value than s in the drawing (otherwise there are four vertices of H_C that have a lower x-value than s and we can use the same argument). We will call them V_{high}. Furthermore since s is not the vertex with the lowest x-value of V_C there is a vertex w of H_C that has a lower x-value than s in the drawing. Because an x-monotone path containing the vertices of the hyperedges $\{s, r, u\}$ contains the edge $\{r, u\}$ for every $r, u \in V_{high}$ and because an x-monotone path containing the vertices of the hyperedges $\{w, s, r\}$ contains the edge $\{s, r\}$ for every $r \in V_{high}$ the vertices $V_{high} \cup \{s\}$ will create a K_5. Contradiction. □

In the following we will use the notation $[A, N] := [A, N] \cap \mathbb{N}$ with $A, N \in \mathbb{N}$. We want to look at the relationship between the condensation of a hypergraph H and H in the non-monotone case. Therefore we need one more result which will compare path-based support graphs of hypergraphs to general support graphs of hypergraphs. It will show that the class of path-based support graphs is way more restrictive than the class of support graphs.

Proposition 3.5. *There is a graph that has a tree support but only path-based supports with $O(|V|^2)$ many edges.*

Proof. Take the star S_N with the center vertex s and the *peripheral* vertices $1, 2, 3, 4, 5..., N$. At first we observe that as long as s is in every hyperedge the star graph is a valid support for this hypergraph.

Take the hypergraph $H = (V, E_H)$ with $V = \{x \mid x \in [1, N]\} \cup \{s\}$ and $E_H = \{\{a, b, c, s\} \mid a, b, c \in V \backslash \{s\}$ and $a < b < c\} \cup \{\{u, s\} \mid u \in V \backslash \{s\}\}$ This graph contains the hyperedges $\{x, s\}$ with $x \in [N]$. Those hyperedges ensure that each path-based support graph contains those as well. So we already have N edges in our path-based support graph.

Now look at the hyperedges of the type $\{s, p, k, r\}$ with $p \in [1, N-2]$, $k \in [p+1, N-1]$ and $r \in [k+1, N]$. We observe that due to the fact that every hyperedge has to be connected by a path only two of those edges can be incident to the vertex s, which means that there has to be at least one edge connecting two peripheral vertices (Figure 3.3).

Case 1: the edge $\{p, k\}$ exists. Repeat the same argument for $\{s, p, k+1, r+1\}$. (adding one edge each step)

Case 2: the edge $\{p, k\}$ does not exist. Therefore for every r either p or k have to be connected to r (either the edge $\{p, r\}$ or the edge $\{k, r\}$ exists) (adding $N - k$ edges).

Now fix p. Either we never get to case 2 and therefore add $N - p - 1$ edges or we get to case 2 at the vertex q and also add $N - p - 1$ edges (case 1: $N - p - (N - q) - 1$, case 2: $(N - q)$). All the edges we had to add for a fixed p are incident to either p or q, meaning that if we remove those two vertices all the edges we have added in our support graph cannot be used for paths in the remaining graph. So if we increase p by 1 and remove the vertex q we can repeat the same argument with $N_i := N_{i-1} - 2$. In total we have to add at least $\lfloor(\frac{N-1}{2})\rfloor \cdot \lfloor(\frac{N}{2})\rfloor + N$ edges for a graph with $N + 1$ vertices.

□

Since planar graphs have at most $3 \cdot |V| - 6$ edges we can use this construction with $N \geq 9$ to show the following corollary:

Corollary 3.6. *There is a graph that has a planar tree support but not a planar path-based support.*

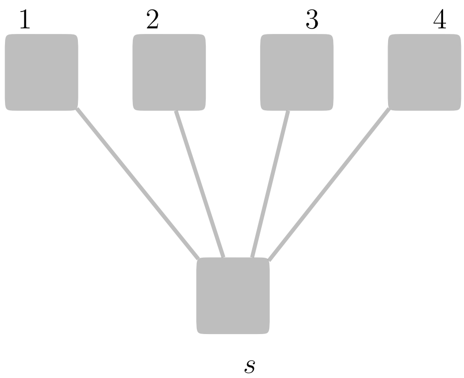
We can use the last proposition to show that if we do not insist on x-monotonicity then there is a hypergraph that does not have a planar path-based support but its condensation does and vice versa.

Proposition 3.7. *There is a hypergraph $H = (V, E_H)$ that has a planar path-based support but its condensation $H_C = (V_C, E_{H_C})$ does not have a planar path-based support.*

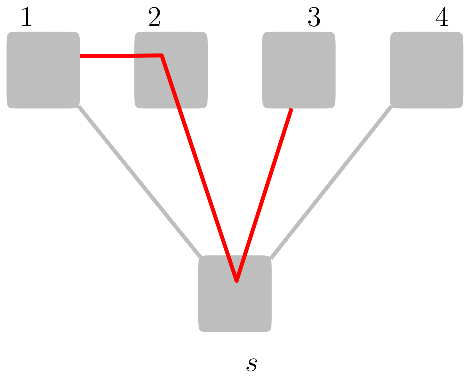
Proof. Take the hypergraph $H = (V, E_H)$ with $V = \{1, 2, 3, 4, 5, 6, 7, 8, 9, s, s'\}$ and $E_H = \{\{u, v, s, s'\} \mid u, v \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $u \neq v\} \cup \{\{u, s\} \mid u \in V \backslash \{s\}\}$. This graph has a planar path-based support as seen in Figure 3.4. Furthermore we have seen in Proposition 3.5 that its condensation cannot have a planar path-based support. □

Proposition 3.8. *There is a hypergraph $H = (V, E_H)$ that does not have a planar path-based support but its condensation $H_C = (V_C, E_{H_C})$ does have a planar path-based support.*

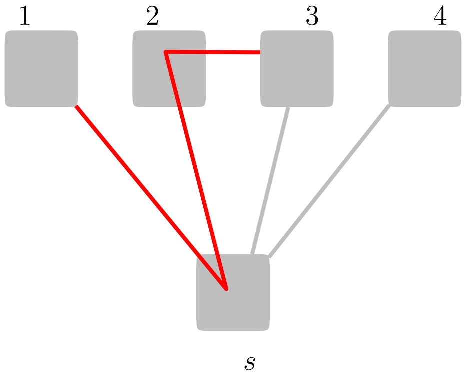
Proof. Take the hypergraph $H = (V, E_H)$ with $V = \{s, s'\} \cup [1, 15]$ and $E_H = \{\{u, v, s, s'\} \mid u, v \in V \setminus \{s, s'\}\} \cup \{\{i, i+1\} \mid i \in [1, 14]\} \cup \{\{1, 15\}\} \cup \{\{s, s'\}\}$. Its condensation $H_C = (V_C, E_{H_C})$ is $V_C = V \setminus \{s'\}$ and $E_H = \{\{u, v, s\} \mid u, v \in V \setminus \{s\}\} \cup \{\{i, i+1\} \mid i \in$



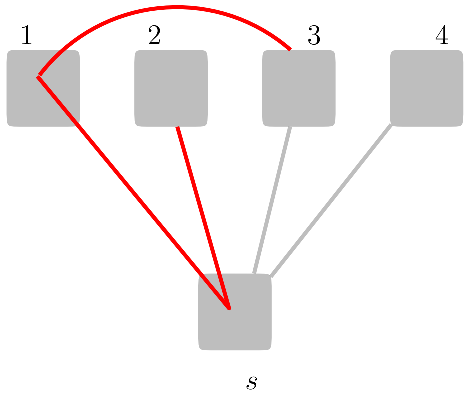
(a) a star



(b) A possible path containing the edge between the vertices 1 and 2



(c) A possible path using the edge between the vertices 2 and 3



(d) A possible path using the edge between the vertices 1 and 3

Figure 3.3: Different possibilites to draw the hyperedge $\{1, 2, 3, s\}$

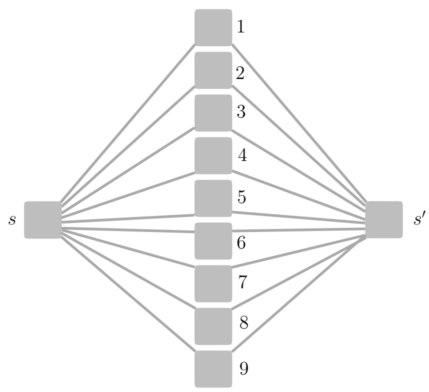
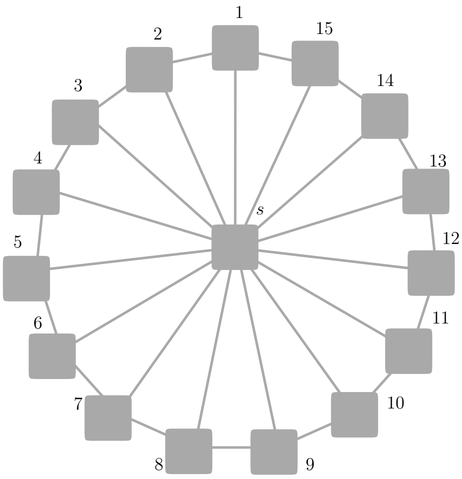
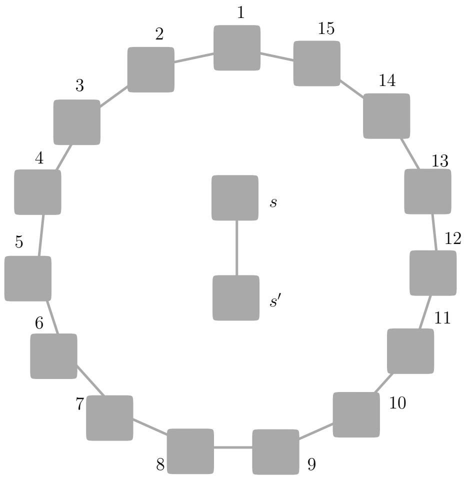


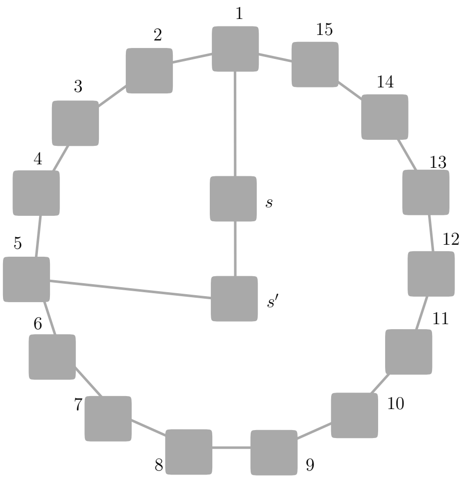
Figure 3.4: A valid drawing of hypergraph H



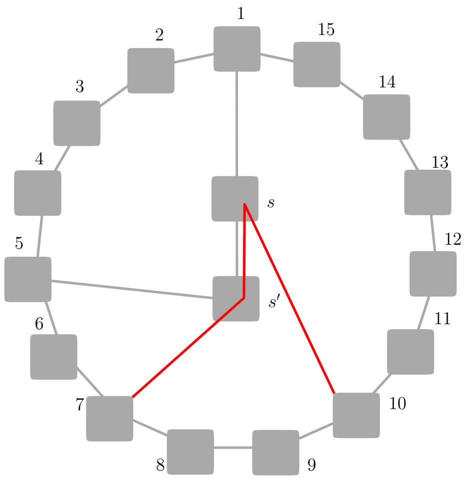
(a) A valid drawing of the condensation H_C



(b) The cycle C with the vertices s and s' inside C



(c) Creating a cycle that contains only vertices in $V_1 \cup \{s, s'\}$



(d) Creating a cycle that contains only vertices in $V_2 \cup \{s, s'\}$

Figure 3.5: Creating a path-based support graph for the hypergraph H

$[1, 14]\} \cup \{\{1, 15\}\} \cup \{\{s\}\}$. H_C does have a planar path-based support as seen in Figure 3.5a. Now we show that H does not have a planar path-based support by contradiction.

Assume H has a planar path-based support $G = (V, E)$. We know that the vertices $\{1, 2, 3..., 15\}$ form a cycle C. The vertices s and s' have to be connected by an edge. Therefore they can be either inside or outside of C. Without loss of generality assume they are inside (Figure 3.5b). Now pick the vertices $V_1 = \{1, 2, 3, 4, 5\}$. They cannot be connected all with each other because then they would form a K_5. So with the hyperedges $H_{sub} = \{\{u, v, s, s'\} \mid u, v \in V_1\}$ we know, that at least one of those hyperedges $\{u, v, s, s'\}$ with $u, v \in H_{sub}$ has to be connected by a path p that does not contain the vertices u and v in consecutive positions. We observe that p is either of the form (u, s, v, s') or of the form (u, s, s', v). In both cases we create a cycle that contains only vertices in $V_F = V_1 \cup \{s, s'\}$. We further observe that this cycle guarantees that the edge $\{s, s'\}$ touches a face in the drawing of G that only contains vertices of V_F (Figure 3.5c). Now pick the vertices $V_2 = \{6, 7, 8, 9, 10\}$ and $V_3 = \{11, 12, 13, 14, 15\}$. With the same argument the edge $\{s, s'\}$ touches a face in the drawing of G only containing vertices of V_2 and $\{s, s'\}$ touches a face only containg vertices of V_3 and $\{s, s'\}$. Therefore $\{s, s'\}$ touches three different faces in the drawing of G. Contradiction.

□

4 Existence of planar metro map drawings

In many cases it makes sense to assume that the number of hyperedges is way smaller than the number of vertices. In the following we give some results if the number of hyperedges is limited. Verroust et al. [VV04] show that every hypergraph with less than nine hyperedges has a planar support graph. In the following we will show that every hypergraph with less than six hyperedges has an x-monotone path-based support graph.

First we look at crossing free metro map drawings of hypergraphs. In this case every hypergraph with four or less hyperedges can be drawn.

Proposition 4.1. *Every hypergraph with at most hyperedges has a crossing free x-monotone metro map drawing.*

Proof. Every vertex can be contained in any subset of hyperedges. So if we do not consider the vertices that are only in one or none hyperedges and therefore easy to place we remain with eleven different kind of vertices. As we have seen in Theorem 3.3, it is sufficient to look at the condensation of the hypergraph H. So we only have to find a valid drawing such that all of those eleven vertices can be added without creating crossings. Figure 4.1 shows such a drawing. The hyperedges are represented by the four horizontal edges and the vertices by the eleven vertical ones. □

In the following we will introduce grid intersection graphs (GIG).

Definition 4.2 (Intersection graph)**.** *Let S be a finite family of sets. The* intersection graph *of S is a graph $G = (V, E)$ whose vertices corresponds to the sets, with $\{v_i, v_j\} \in E$ if and only if s_i and s_j intersect.*

Definition 4.3 (Grid intersection graph)**.** *Let I_1 and I_2 be finite families of horizontal and vertical intervals in the plane, such that no two horizontal or vertical lines intersect. The intersection graph of I_1 and I_2 is called grid intersection graph.*

We observe that a crossing free x-monotone metro map drawing of a hypergraph $H = (G, E_H)$ can be transformed into a GIG representation of a graph $G = (V \cup E_H, E)$ with $E = \{\{u, e\} \mid u \in V, e \in E_H \text{ and } u \in E_H\}$ and vice versa. Grid intersection graphs have been the interest of many different papers. One result about grid intersection graphs we can use is that if the created graph G is planar than we can find a crossing free x-monotone metro map drawing of H (see Hartman et al. [HNZ91]).

Next we want to show that four is a tight upper bound meaning that there exists a hypergraph with five hyperedges such that it cannot be drawn crossing free.

Proposition 4.4. *The dual hypergraph of K_5 does not have a crossing free metro map drawing.*

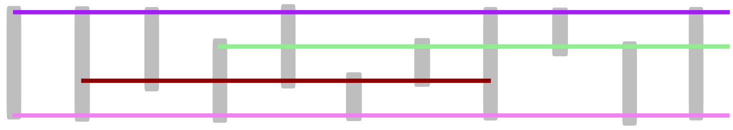


Figure 4.1: Crossing free x-monotone metro map drawing of any hypergraph with four hyperedges

Proof. Take the hypergraph $H = (V, E_H) = K_5^*$ with the vertices $V = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $E_H = \{\{1, 2, 3, 4\}, \{1, 5, 6, 7\}, \{2, 5, 8, 9\}, \{3, 6, 8, 10\}, \{4, 7, 9, 10\}\}$. The claim is that this graph cannot be drawn as a metro map without having at least one edge or vertex crossing. For the proof we will show that if we could find a crossing free metro map drawing of H we could also find a planar drawing of K_5.

So assume there is a crossing free drawing of H. First we observe that every vertex has degree two. Now we want to look at the paths for each hyperedge. Take the metro line $p_h = (a, b, c, d)$ with $a, b, c, d \in V$ of any hyperedge $h \in E_H$ and subdivide each edge in p_h such that $p_h^2 = (a, h_1, b, h_2, c, h_3, d)$. Since we do not have any crossings and each vertex has a degree of at most two in the hypergraph we can add the edges $E_{sub} = \{\{h_1, h_2\}, \{h_2, h_3\}\}$ without creating crossings. Next we will contract the edges in E_{sub}. This will not create any crossings. By repeating the same procedure for every hyperedge we create a graph K' which can be generated out of K_5 by subdividing each edge in K_5. Since we assumed that there existed a planar drawing and we did not create any crossings the transformation would give us a planar drawing for K_5. Contradiction. □

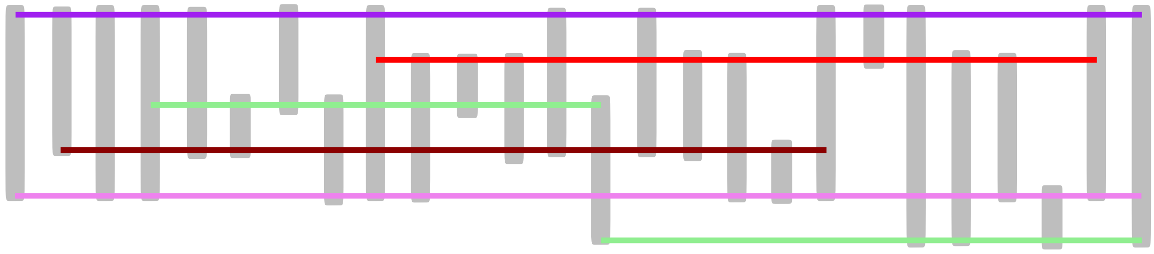
Corollary 4.5. *There is a hypergraph with five hyperedges that cannot be drawn as an x-monotone metro map without any crossings.*

So in the following we will allow vertex crossings but still insist that the drawing does not contain any edge crossings. A simple observation is that not every hypergraph with nine hyperedges has such a drawing. Take the graph $K_{3,3}$. We know that there is no planar drawing of this graph and it has exactly nine hyperedges. Furthermore every graph is also a hypergraph and hyperedges of size two cannot cause vertex crossings. Therefore we found a hypergraph with nine hyperedges that cannot be drawn without edge crossings.

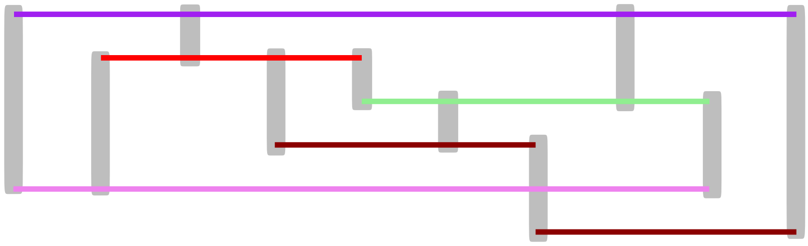
Next we want to show that we can draw each hypergraph that has at most five hyperedges as a planar x-monotone metro map. In the crossing free case it was enough to show that we find a crossing free x-monotone metro map drawing for a hypergraph containing all different vertex classes of the condensation relation. This is not the case if we allow vertex crossings because if we remove a vertex v that contains a vertex crossing from a planar x-monotone metro map drawing the remaining drawing will have an edge crossing. Therefore we have to show that if we remove a vertex with a vertex crossing there is a planar x-monotone metro map drawing for the remaining hypergraph.

Proposition 4.6. *Every hypergraph with five hyperedges can be drawn as a planar x-monotone metro map.*

Proof. For this proof we have to distinguish between two cases:



(a) Case 1: There exists a vertex of degree three



(b) Case 2: There exists at least one vertex of degree two

Figure 4.2: Planar drawings of hypergraphs with less than six hyperedges

Case 1: In the graph exists a vertex that has degree at least three. Figure 4.2a shows a way to draw every possible vertex with having exactly one vertex crossing. This crossing seen in the middle (between the green, brown, pink metro lines) is a crossing between three different hyperedges. So if one of those exists in the hypergraph we can use this vertex to get a drawing as in the picture.

Case 2: So from now on assume there does not exist a vertex of degree greater than two. Furthermore we assume that there exists one vertex of degree two since otherwise the drawing is trivial. Figure 4.2b shows a valid drawing with a vertex crossing between the pink and the brown hyperedge. □

Next we want to show that insisting on x-monotonicity may change if we find a planar path-based support graph for a hypergraph.

Proposition 4.7. *There is a hypergraph that has a planar metro map drawing but does not have a planar x-monotone metro map drawing.*

Proof. Take the following hypergraph $H = (V, E_H)$ with $V = \{1,2,3,4,5\}$ and $E_H = \{\binom{[1,5]}{3}\}$. We can think about this graph as K_5 and for every edge its compliment is taken. So at first we show that there is a planar metro map drawing of this hypergraph. The graph $K_5 - \{1,3\}$ (see Figure 4.3) is planar. Furthermore it is also a path-based support for H. The only edge that is missing is the edge $\{1,3\}$. But since each hyperedge has size three every hyperedges that contains the vertices 1 and 3 also contains another vertex. We can use the third vertex u to find a path $(1,u,3)$ that is in $K_5 - \{1,3\}$. Therefore there exists a planar metro map drawing for H with Proposition 2.11.

In the following we will show that there is no planar x-monotone metro map drawing of H. We prove this by contradiction. Every hyperedge is a subset of size three and for every possible subset of size three there is exactly one hyperedge. So without loss

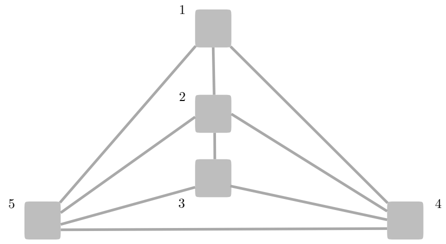
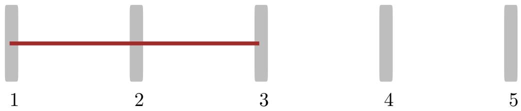
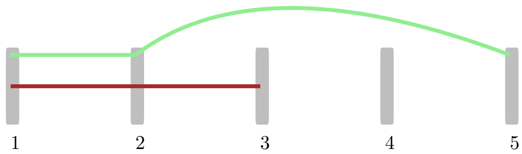


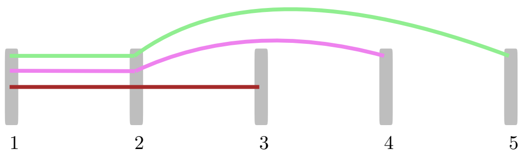
Figure 4.3: A planar path-based support graph for H



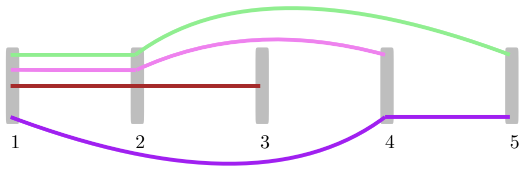
(a) The first hyperedge is inserted



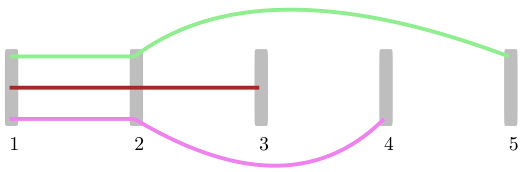
(b) The second hyperedge is inserted without loss of generality above the first



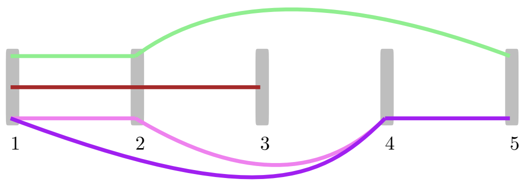
(c) The third hyperedge is inserted between the first two hyperedges



(d) The fourth hyperedge is inserted below the first three hyperedge



(e) The third hyperedge is inserted below the first two hyperedges



(f) The fourth hyperedge is inserted below the first three hyperedges

Figure 4.4: Showing that H does not have a planar x-monotone metro map drawing

of generality we assume a correct x-monotone metro map drawing D of the vertices V has the ordering $(1, 2, 3, 4, 5)$ of the vertices. Assume the hyperedge $\{1, 2, 3\}$ is placed corresponding to D (Figure 4.4a). Furthermore assume without loss of generality that in D the hyperedge $\{1, 2, 5\}$ is drawn above the hyperedge $\{1, 2, 3\}$ (Figure 4.4b). Next we look at the hyperedge $\{1, 2, 4\}$. There are two cases:

Case 1: The hyperedge $\{1, 2, 4\}$ is placed above the hyperedge $\{1, 2, 3\}$ (Figure 4.4c).

Next we look at the hyperedge $\{1, 4, 5\}$. Because of the other hyperedges it has to be below the hyperedge $\{1, 2, 3\}$. But now there is no way to add the hyperedge $\{1, 3, 5\}$ without creating an edge crossing. (Figure 4.4d)

Case 2: The hyperedge $\{1, 2, 4\}$ is placed below the hyperedge $\{1, 2, 3\}$ (Figure 4.4e). Next we look at the hyperedge $\{1, 4, 5\}$. Because of the other edges it has to be below the hyperedge $\{1, 2, 3\}$. But again there is no way to add the hyperedge $\{1, 3, 5\}$ without creating an edge crossing. (Figure 4.4f). $\square$

Another interesting problem occurs if we add one more restriction to the x-monotone metro map drawings. Given a hypergraph $H = (V, E_H)$ we also insist that in an x-monotone metro map drawing D of the hypergraph H which has vertex s with the lowest x-value $x(s)$ of the vertices V and the vertex t with the highest x-value $x(t)$ of the vertices V in D that for every metro line $p \in E_H$ and every y in the intervall $[x(s), x(t)]$ there exists a point with x-value y that lies on p. We call those metro map drawings *x-consecutive-monotone*. So for every metro line $p = (v_1, ..., v_n)$ that only contains vertices with x-values in $(x(s), x(t))$ we extend p by adding an x-montone curve from v_1 to a point with x-value $x(s)$ and by adding an x-montone curve from v_n to a point with x-value $x(t)$. We call an x-consecutive-monotone metro map drawing planar if it does not contain any crossings of metro lines that are not at vertices of G and crossing free if it is planar and does not contain vertex crossings.

Observation 4.8. *The hypergraph $K_3 = (\{1,2,3\}, \{\{1,2\}, \{1.3\}, \{2,3\}\})$ does not have a crossing free x-consecutive-monotone metro map drawing.*

Proof. Assume K_3 has a crossing free x-consecutive-monotone drawing. Since a crossing free x-consecutive-monotone drawing does not contain any crossings and the drawing is x-consecutive monotone the hyperedges of K_3 have the same ordering at each x- position. Without loss of generality assume that $(\{1,2\}, \{1,3\}, \{2,3\})$ is that ordering. Therefore the drawing cannot contain the vertex 2. Contradiction. □

In the following we want to show that if we allow vertex crossings every hypergraph with at most four hyperedges can be drawn in this style without edge crossings.

Proposition 4.9. *Every hypergraph with at most four hyperedges has a planar x-consecutive-monotone metro map drawing.*

Proof. With the same argument as for planar x-monotone metro map drawings it is sufficient to look at the condensation. We assume that there is at least one vertex of degree two because otherwise the drawing is trivial. To make notation easier we will denote the hyperedges with $E_H = \{1, 2, 3, 4\}$ and the vertices v as subsets $s \in E_H$ such that for every $h \in E_H$: $h \in s$ if and only if $v \in h$.

Case 1: There exists at least one vertex of degree three and another vertex of degree at least two.

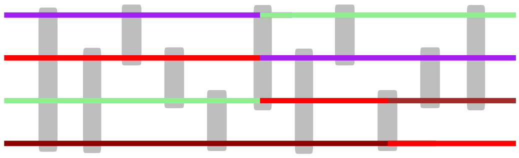
Under this condition we can find a valid drawing of the hypergraph (Figure 4.5a).

If we only have one vertex of degree greater than one then the drawing is trivial. So from now on we assume that there are only vertices of degree at most two. Now we distinguish between how many vertices of degree two exist.

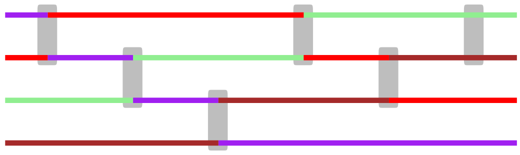
Case 2: There are at least five vertices of degree two.

Figure 4.5b is an example how to draw such a hypergraph if it contains at least five vertices of degree two.

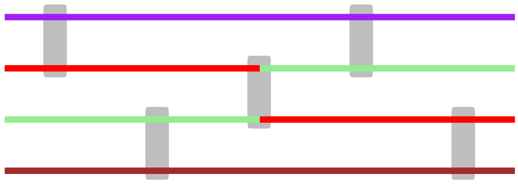
Case 3: There are at most four vertices of degree two. The drawing in Figure 4.5c is a valid drawing even for five hyperedges that contains a vertex crossing at a vertex $v = \{a, b\}$ with $a, b \in E_H$. The vertex of degree two that is missing in this drawing is the vertex $u = \{c, d\}$ with $\{c, d\} \cap \{a, b\} = \emptyset$. So assume that there is no vertex $v \in V$ such that the vertex $u = E_H \setminus v$ is not in V and $|V| \leq 4$. If V contains only two vertices the drawing



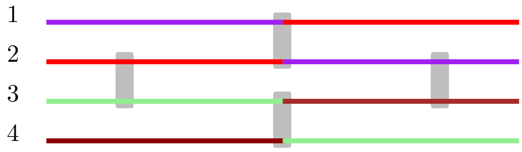
(a) There exists one vertex of degree at least three and one vertex of degree at least two



(b) There exist at least five vertices of degree two



(c) At least one vertex v of degree two does not exist and there exists a vertex crossing at a vertex u that is not contained in any hyperedge that contains v



(d) There are four vertices in V with the condition that for every vertex $v \in V$ there is a vertex $u \in V$ such that v is not contained in any hyperedge that contains u

Figure 4.5: Drawings of hypergraphs with four hyperedges

is trivial. So without loss of generality assume that $V = \{\{1, 2\}, \{3, 4\}, \{1, 4\}, \{2, 3\}\}$. Figure 4.5d is a valid drawing of this hypergraph.

□

Proposition 4.10. *C_5 does not have an x-consecutive-monotone metro map drawing without edge crossings.*

Proof. Assume for the sake of contradiction that there is an x-consecutive-monotone metro map drawing of C_5 which does not contain any edge crossings and let v be the vertex with the lowest x-value in this drawing and let $ord : E_H \rightarrow \{1, 2, 3, 4, 5\}$ be the order of the hyperedges at the x position $x(v)$ of vertex v. To make notation easier denote each hyperedge h by $ord(h)$. First we observe that for each hyperedge h there are two other hyperedges that each have one common vertex with h and the only way to change position is to have a common vertex since we do not allow edge crossings.

Case 1: The hyperedges 1 and 5 have a common vertex.

At position $x(v)$ there are three hyperedges $\{2, 3, 4\}$ separating the hyperedges 1 and 5. But since 1 and 5 have a common vertex with each other each of them has only a common vertex with one hyperedge of the hyperedges $\{2, 3, 4\}$. Therefore one of those hyperedges will separate 1 and 5 and they cannot have a common vertex. Contradiction.

Case 2: Two hyperedges h_1, h_2 with $|ord(h_1) - ord(h_2)| = 3$ have a common vertex.

Without loss of generality let h_1 be 1 and h_2 be 4. Since there are two hyperedges $\{2, 3\}$ separating 1 and 4 we know that either 1 or 4 has a common vertex with 2 and the other hyperedge has a common vertex with 3. Since each hyperedge has a common vertex with exactly two other hyperedges we further know that 5 has a common vertex with 2 and a common vertex with 3. But since 4 is separating 5 from both hyperedges 2 and 3 and 5 does not have a common vertex with 4 both hyperedges 2 and 3 need to have a common vertex with 4. Contradiction.

Case 3: No hyperedges h_1, h_2 with $|ord(h_1) - ord(h_2)| > 2$ have a common vertex.

This means that the hyperedge 1 has a common vertex with 2 and one with 3 and the hyperedge 5 has a common vertex with 4 and a common vertex with 3. Therefore 2 and 4 have a common vertex as well. Since 2 and 4 are separated by the hyperedge 3 either 2 and 3 or 3 and 4 have to change position. Since neither 2 nor 4 has a common vertex with 3 we get a contradiction. □

5 Minimizing crossings

In the following we want to minimize different kinds of crossings in our metro map drawing. In order to make the different optimization criteria more clear we need some further notation.

The input of the problems we study in the following is a pair (G, Π), where $G = (V, E)$ is a planar path-based support of a hypergraph $H = (V, E_H)$ and Π is a set of paths in G such that for every hyperedge $h \in E_H$ there exists exactly one path in Π, that is a Hamiltonian path in $G[h]$. We also consider the cases where additionally a planar embedding $\mathcal{G}$ of G is provided as a part of the input; in this case we adopt the notation $(G, \mathcal{G}, \Pi)$. Let $u \in V$ and $\{u_1, \ldots, u_k\} \subset V$ be the clockwise ordered neighbors of u as provided by $\mathcal{G}$. Let $P \subseteq \Pi$ be the paths containing the edge $\{u, u_i\}$. A *line-ordering at vertex u on the edge $\{u, u_i\}$*, $\text{lord}_u(u_i)$, is an ordering $p_1, \ldots, p_h$ of P.

A *metro map embedding* of $(G, \mathcal{G}, \Pi)$ is a set $\{\text{lord}_u(v), \text{lord}_v(u) \colon (u, v) \in E\}$. In case the embedding of G is not a part of the input, a *metro map embedding* of (G, Π) is an embedding $\mathcal{G}$ of G and a metro map embedding of $(G, \mathcal{G}, \Pi)$. Line orderings at v and at w on $\{v, w\} \in E$ imply a line crossing along $\{v, w\}$ if both contain the same pair of paths in the same order. Thus, if there is no line crossing along $\{v, w\}$ the line-ordering at v on $\{v, w\}$ is reverse of the line-ordering at w on $\{v, w\}$. Observe that a metro map embedding induces a cyclic order of the metro lines at a vertex v because the embedding $\mathcal{G}$ gives an ordering of the adjacent vertices of v which we will denote as ord_v and the metro map embedding gives an ordering of the metro lines at each edge of G. Therefore we can define the *line-ordering at vertex v $lord_v$* with v being a vertex v of G and $u_1, \ldots, u_k$ being the adjacent vertices of v in G ordered according to ord_v as the concatenation $\text{lord}_v(u_1) \oplus \cdots \oplus \text{lord}_v(u_k)$. Observe that a line-ordering at vertex v implies a vertex crossing at v if it contains two elements i and j two times in an alternating order, $i - j - i - j$.

For a path $p = (a, \ldots, b)$ we say that p *terminates* in its endpoints a and b.

Consider a metro map embedding of (G, Π) and an edge $\{u, v\}$ of G, and let $p_1, \ldots, p_h$ be the paths that contain the edge $\{u, v\}$ ordered according to $\text{lord}_v(u)$. Assume that for some i p_i, $1 \leq i \leq h$ terminates in v. We say that p_i is *lower peripheral* at v if $\forall j$, $1 \leq j \leq i$, p_j terminates in v. Similarly, we say p_i is *upper peripheral* at v if $\forall j$ with $i \leq j \leq h$: p_j terminates in v. We say that a path is *peripheral* if it is upper or lower peripheral at both of its end-vertices. We say that a path p is *constrained peripheral* if for both end-vertices s, t it is already given whether p is upper or lower peripheral at s and t.

Furthermore we call a vertex $v \in V$ with the property that every path $p \in \Pi$ that contains v terminates in v a terminal vertex. A simple observation about those terminal vertices is that they do not have a vertex crossing.

We study the following problems. Given a triple $(G, \mathcal{G}, \Pi)$ construct a metro map embedding with minimum number of vertex crossings or of line crossings. We denote these problems by V-MLCM, L-MLCM, respectively. In case the embedding is not provided we

Problem	Given		Find				
	Embedding	graph	Peripheral	Vcros	Lcros	Res.	Ref.
L-MLCM	✓	caterpillar	✗	✗	✓	NP-hard	[FP13]
L-MLCM-E	✗	double star	✗	✗	✓	NP-hard	Corollary 5.16
P-L-MLCM	✓	path	✓	✗	✓	NP-hard	[Arg+10]
CP-L-MLCM	✓	planar graph	✓	✗	✓	$O(\|\Pi\|^2 \cdot \|V\|)$	[AGM08]
CP-L-MLCM-E	✗	tree	✓	✗	✓	NP-hard	Theorem 5.14
V-MLCM	✓	planar bipartite graph	✗	✓	✗	NP-hard	Corollary 5.9
V-MLCM	✓	tree	✗	✓	✗	$O(\|\Pi\|^2) \cdot \|V\|)$	Theorem 5.10
V-MLCM-E	✗	planar bipartite graph	✗	✓	✗	NP-hard	Corollary 5.12

Table 5.1: This table gives an overview over different crossing minimization problems for metro map drawings. We distinguish between the cases where the embedding of the path-based support graph G is already given and the cases where it is part of the problem to find an embedding for G. Furthermore the graph class that is given says that the problem is considered for this class of graphs. We distinguish between the way we want to draw the metro map. Vcros means that we want to find a metro map drawing with a minimum amount of vertex crossings but do not allow any line crossings, whereas Lcros means that we want to minimize the number of line crossings but avoid vertex crossings. For the Line crossing problem we further distinguish between the case that each line has to terminate in a peripheral position (peripheral) and the case where there is no restriction on where the lines can terminate at a vertex. Res. stands for result and gives the computational complexity of this problem for this graph class and the information in the column Ref. gives the reference where this claim is proven.

denote the problems by V/L-MLCM-E. In case the paths of Π are required to be (constrained) peripheral we add (C)P to the problem notation. We observe that in these problem formulations only one type of crossing (either line or vertex) is allowed.

For our problems we will distinguish between *unavoidable crossings* and *avoidable crossings.*

Definition 5.1 (Common subpath). *A common subpath of two paths p_1 and p_2 in a graph G is a path $p_s = (v_1, \dots, v_n)$ with $n \geq 1$ such that $p_1 = (\dots, v_1, \dots, v_n, \dots), p_2 = (\dots, v_1, \dots, v_n, \dots)$ and adding any vertex of G to p_s violates this condition.*

Definition 5.2 (Unavoidable and avoidable crossing). *A crossing between two paths p_1 and p_2 on a common subpath p_s in a graph G with embedding $\mathcal{G}$ is unavoidable, if the paths p_1 and p_2 cross in every metro map embedding on p_s. Otherwise we call it avoidable.*

In the following we make an observation when two paths have an unavoidable vertex crossing. First we observe that the paths p_1 and p_2 cannot have a vertex crossing if there common subpath is empty. Furthermore we observe that p_1 and p_2 may have more than one common subpath and that we can consider each common subpath independently for finding unavoidable crossings.

Observation 5.3 (Unavoidable crossing). *Given a triple* $(G, \mathcal{G}, \Pi)$ *with* $G = (V, E)$ *being a planar path-based support graph of an ordered hypergraph* $H = (V, E_H)$*,* $\mathcal{G}$ *a planar embedding of* G *and* Π *being the corresponding paths for the ordered hyperedges of* H*. We want to formalize when two paths* p_1 *and* p_2 *have an unavoidable vertex crossing on a common subpath. If one path* p_1 *terminates on the common subpath we can order the paths on the common subpath according to the ordering of the other end of the subpath. Therefore they will not create an unavoidable crossing. So assume now that neither* p_1 *nor* p_2 *terminates on the common subpath. If the common subpath contains only one vertex* v $p_1 = (..., s_1, v, t_1, ...)$ *and* $p_2 = (..., s_2, v, t_2, ...)$ *have an unavoidable crossing on* (v) *if and only if* ord_v *contains the alternating order* $a - b - c - d$ *with* $a, c \in \{s_1, t_1\}$*,* $a \neq c$ *and* $b, d \in \{s_2, t_2\}$*,* $b \neq d$*. So assume now that the common subpath contains at least two vertices. Denote the common subpath as* $p_s = (u_1, ..., u_r)$*. Furthermore denote* p_1 *as* $(..., s_1, u_1, ..., u_r, t_1, ...)$ *and* p_2 *as* $(..., s_2, u_1, ..., u_r, t_2, ...)$*. The paths have an unavoidable crossing on* p_s *if and only if the order of the vertices* $\{s_1, s_2, x\}$ *in* ord_{u_1} *is the same as in* ord_{u_r} *with* $x = u_2$ *in* ord_{u_1} *and* $x = u_{r-1}$ *in* ord_{u_r} *(Figure 5.1).*

Next we want to show that it is sufficient to consider the unavoidable crossings to compute the minimum number of vertex crossings in a metro map embedding for $(G, \mathcal{G}, \Pi)$. Remember that we defined that there is one vertex crossing at a vertex v if there are at least two metro lines that cross in v and therefore we do not count the number of crossings between metro lines but the number of vertices where metro lines cross.

Definition 5.4 (V-graph). *Given a graph* $G = (V, E)$ *and a subset* $S \subseteq V$ *of vertices. Let* $G_{com} = (V_{com}, E_{com})$ *be a component of* $G - S$ *and* $V_v = V_{com} \cup \{s \in S \mid \exists v \in V_{com} : \{s, v\} \in E\}$*. Then we define* $G[V_v]$ *to be a V-graph of* S *in* G*.*

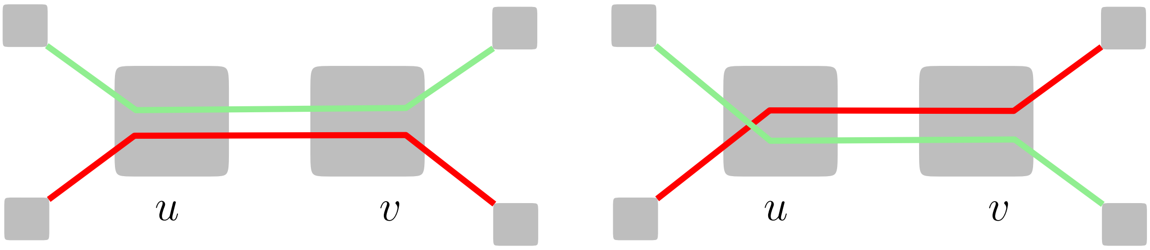
Definition 5.5 (Hitting Set Problem). *Given a hypergraph* $H = (V, E_H)$ *and an integer* k*. Is there a set* $S \subseteq V$ *with* $|S| \leq k$ *such that for all hyperedges* $h \in E_h$*, it holds* $S \cap h \neq \emptyset$*.*

Lemma 5.6. *Given a triple* $(G, \mathcal{G}, \Pi)$ *with* $G = (V, E)$ *being a planar path-based support graph of an ordered hypergraph* $H = (V, E_H)$*,* $\mathcal{G}$ *a planar embedding of* G*,* Π *being the corresponding paths for the ordered hyperedges of* H *and a set of sets* L *that contains every subpath* p_s *such that there exists* $p_1, p_2 \in \Pi$ *which have an unavoidable crossing on their common subpath* p_s*.* $(G, \mathcal{G}, \Pi)$ *has a metro map embedding with* k *vertex crossings if and only if* $H = (V, L)$ *has a hitting set of size* k*.*

Proof. For the proof we need the following result of Fink et al. [FP13]: If there are no unavoidable crossings than there is a solution without any (vertex) crossing.

First we observe that if we have a hitting set S of a hypergraph $H = (V_L, L)$ with $V_L = \bigcup_{l \in L} l$, than for every pair of paths $p_1, p_2 \in \Pi$ that has an unavoidable crossing on their common subpath p_s there exists a vertex $v \in p_s \cap S$. Furthermore we observe that two paths $p_1, p_2 \in \Pi$ do not have to cross more than once on a common subpath p_s.

Let v be a vertex crossing. Since we do not count the number of crossing at a vertex v but only if there is at least one crossing between two metro lines at v this means that



(a) Two paths not causing any crossings on their common subpath (u, v)

(b) Two paths with an unavoidable crossing on their common subpath (u, v)

Figure 5.1: Unavoidable and avoidable crossings of two paths

we can choose every line-ordering at vertex v $lord_v(u)$ with $\{u, v\} \in E$. Therefore for a path $p_a = (a_1, \dots, a_k, v, b_1, \dots, b_l)$ we can look at the subpaths $p_b = (a_1, \dots, a_k, v)$ and $p_b(v, b_1, \dots, b_l)$ independently.

So for our solution we will allow vertex crossings only at the vertices of S. Now we look at any V-graph $G_v = \{V_v, E_v\}$ of S in G. So take two paths $p_1, p_2 \in \Pi$ that have an unavoidable crossing on a common subpath $p_s = (a_1, \dots, a_k, v, b_1, \dots, b_k)$ with a nonempty intersection with G_v and $v \in S \cap V_v$. We know that such a vertex v exists because of the following argument. If p_s is completely in G_v we know because of S being a hitting set that there is such a vertex v. If p_s is not completely in G_v than there is a vertex $x \in V_v$ such that one of its neighbors on p_s is not in V_v. After the construction of G_v $x \in S \cap V_v$. With the observations earlier we can divide p_s in two from each other independent subpaths $p_a = (a_1, \dots, a_k, v, b_1, \dots, b_l)$ and $p_b = (a_1, \dots, a_k, v)$ and look at them independently. Since we can choose the line-ordering at vertex v we know that $p_1, p_2 \in \Pi$ do not have an unavoidable crossing on each of those subpaths. We can repeat the same argument for any pair of paths $p_1, p_2 \in \Pi$ and can conclude with the result of Fink et al. that there is a metro map drawing of G_v that has only vertex crossings at vertices in S.

Since the union of the V-graphs and S is G and the intersection of any two V-graphs is in $G[S]$ we can combine the drawing of each V-graph to get a metro map drawing for the triple $(G, \mathcal{G}, \Pi)$ with k vertex crossings.

So the remaining part is to show that if $(G, \mathcal{G}, \Pi)$ has a metro map drawing with k vertex crossings then L has a hitting set of size k. The way we choose L ensures that for every set $l \in L$ we have to pick at least one vertex since otherwise there are two paths p_1, p_2 that have an unavoidable crossing but there is no vertex on which they intersect. Therefore our metro map drawing has at least as many vertex crossings as a minimal hitting set of L. □

Theorem 5.7. *Given a triple $(G, \mathcal{G}, \Pi)$, with $G = (V, E)$ being a planar path-based support graph of an ordered hypergraph $H = (V, E_H)$, $\mathcal{G}$ a planar embedding of G and Π being the corresponding paths for the ordered hyperedges of H and an integer k. It is NP-complete to decide if there is a metro map drawing for $(G, \mathcal{G}, \Pi)$ with at most k vertex crossings (V-MLCM).*

Proof. First we show that the problem is in NP. A metro map drawing of $(G = (V, E), \mathcal{G}, \Pi)$ without line crossings is the set $\{\text{lord}(e) : e \in E\}$. Going through every vertex $v \in V$ we

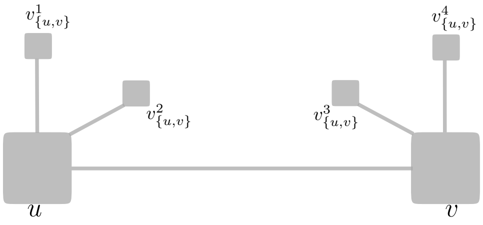


Figure 5.2: Representation of an edge

need to check whether it is a crossing vertex. This can be done by checking whether the line ordering at v contains two elements i, j two times in an alternating order $i - j - i - j$. This, can clearly be done in polynomial time.

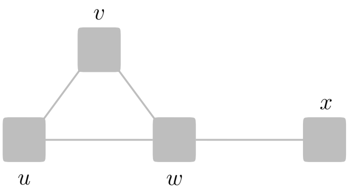
In the following we reduce Planar Vertex Cover to our problem. A *vertex cover* of $G' = (V', E')$ is defined as a subset of vertices $V_{\text{cov}} \subseteq V'$ such that $\forall \{u, v\} \in E' : u \in V_{\text{cov}} \vee v \in V_{\text{cov}}$. Given a planar graph G' Planar Vertex Cover asks whether G' has a vertex cover of size k. We set $V = V' \cup \{v_e^1, v_e^2, v_e^3, v_e^4 \mid e \in E'\}$ and $E = E' \cup \{(v_e^1, u), (v_e^2, u), (v_e^3, v), (v_e^4, v) \mid e = \{u, v\} \in E'\}$ – thus G is G' where for each edge we add four edges connected to newly introduced vertices; refer to Figure 5.2. We set $\Pi = \{p_e^1 = (v_e^1, u, v, v_e^3), p_e^2 = (v_e^2, u, v, v_e^4) \mid \{u, v\} \in E'\}$. The embedding $\mathcal{G}$ of G coincides with the embedding of G' and new edges are inserted so that for each edge $e = \{u, v\}$, vertices v, v_e^1, v_e^2 are consecutive in the ordering ord_u; the same holds for u, v_e^3, v_e^4 in the ordering ord_v. The embedding $\mathcal{G}$ guarantees that there is an unavoidable vertex crossing between the paths p_e^1 and p_e^2 for every $e = \{u, v\}$ in G. This unavoidable crossing occurs on the common subpath of p_e^1 and p_e^2 and therefore they cross either in u or in v.

Assume that G' has a vertex cover V_{cov} of size k. For each edge $e = \{u, v\}$ in G we define the line ordering as follows. If $u \in V_{\text{cov}}$ then $lord_u(v) = (p_e^1, p_e^2)$ and $lord_u(v) = (p_e^2, p_e^1)$. Therefore the paths $\{p_e^1, p_e^2\}$ enforce a vertex crossing at vertex u but no vertex crossing at vertex v. If $u \notin V_{\text{cov}}$ then $v \in V_{\text{cov}}$ and we set $lord_u(v) = (p_e^2, p_e^1)$ and $lord_u(v) = (p_e^1, p_e^2)$. Therefore the paths $\{p_e^1, p_e^2\}$ enforce a vertex crossing at v but they do not cross in vertex u. Thus $(G, \mathcal{G}, \Pi)$ has a metro map drawing with at most k vertex crossings.

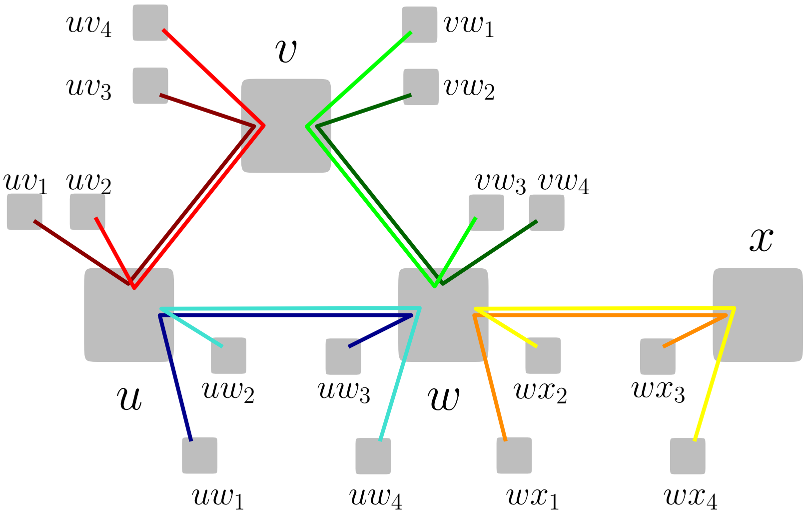
In the reverse direction, assume that $(G, \mathcal{G}, \Pi)$ has a metro map drawing with k vertex crossings. For each edge $e = \{u, v\} \in E' \cap E$, the corresponding paths p_e^1 and p_e^2 have an unavoidable crossing on the common subpath $\{u, v\}$ and therefore cross either at u or at v. Thus the vertices with vertex crossings of the metro map drawing are a vertex cover of G.

□

Example 5.8 (V-MLCM NP-reduction)**.** In the following we give an example for this NP reduction. Taken the graph $G = (V, E)$ with $V = \{u, v, w, x\}$ and $E = \{\{u, v\}, \{u, w\}, \{v, w\}, \{w, x\}\}$ (Figure 5.3a). Now we want to find a minimum vertex cover. We can transform this problem into the MLCM-V problem with the hypergraph $H = (V', E_H)$ with $V' = \{u, v, w, x, uv_1, uv_2, uv_3, uv_4, uw_1, uw_2, uw_3, uw_4, vw_1, vw_2, vw_3, vw_4, wx_1, wx_2, wx_3, wx_4\}$ and $E_H = \{(uv_1, u, v, uv_3), (uv_2, u, v, uv_4), (uw_1, u, w, uw_3), (uw_2, u, w, uw_4), (vw_1, v, w, vw_3), (vw_2, v, w, vw_4), (wx_1, w, x, wx_3), (wx_2, w, x, wx_4)\}$ and the drawing of



(a) A planar graph with four vertices



(b) The transformation of Figure 5.3a

Figure 5.3: Example for the NP-reduction of the V-MLCM problem from PLANAR VERTEX COVER

the planar path-based support G of H as seen in Figure 5.3b. In the example we choose the vertices $V_{cov} = \{u, w\}$ both for vertex cover and for the vertices with vertex crossings. This reduction uses a pair of hyperedges that have an unavoidable crossing on a subpath (u, v) to enforce a vertex crossing either in u or in v and thereby create a vertex cover. To make this relationship more obvious each of those conditions is represented by a different type of colouring. The condition for the edge $\{u, v\}$ is enforced by the two red coloured hyperedges, the condition for the edge $\{u, w\}$ by the two blue coloured, the condition for the edge $\{v, w\}$ by the two green coloured and the condition for the edge $\{w, x\}$ by the yellow and the orange coloured hyperedge.

Corollary 5.9. *Given a triple $(G, \mathcal{G}, \Pi)$, with $G = (V, E)$ being a planar bipartite path-based support graph of an ordered hypergraph $H = (V, E_H)$, $\mathcal{G}$ a planar embedding of G and Π being the corresponding paths for the ordered hyperedges of H and an integer k. It is NP-complete to decide if there is a metro map embedding for $(G, \mathcal{G}, \Pi)$ with at most k vertex crossings (V-MLCM).*

Proof. Since the problem is in NP for general planar graphs G' it is also in NP for planar bipartite graphs. Therefore we only need to show that the problem is NP-hard.

We know that the problem is NP-hard on general planar path-based support graphs $G' = (V', E')$. So given a triple $(G', \mathcal{G}', \Pi')$, with $G' = (V', E')$ being a planar path-based support graph of an ordered hypergraph $H' = (V', E'_H)$, $\mathcal{G}'$ a planar embedding of G' and Π' being the corresponding paths for the ordered hyperedges of H' and an integer k. The idea is to subdivide each edge $e' \in E'$ and subdivide e' also in the paths $p' \in \Pi'$ and hyperedges $h' \in E_H$ that contain e', name the new vertex v_e and define $V_{sub} = \bigcup_{e' \in E'} \{v_e\}$. Furthermore the embedding $\mathcal{G}$ of G coincides with the embedding $\mathcal{G}'$ of G'. In this way we create a planar bipartite graph $G = (V \cup V_{sub}, E)$. First we observe that we can transform every metro map embedding M' for $(G', \mathcal{G}', \Pi')$ with k crossings into a metro map drawing for $(G, \mathcal{G}, \Pi)$ with k crossings by using the same line-ordering for every vertex $v \in V'$ and using the line-ordering $lord_{v_{\{u,v\}}}(u)$ for the vertex $v_{\{u,v\}} \in V_{sub}$ on the edge $\{u, v_{\{u,v\}}\}$, which is the reversed line ordering of $lord_u(v_{\{u,v\}})$. Since M' is a valid metro map drawing for $(G', \mathcal{G}', \Pi')$ we know that there are no line crossings and therefore the vertices in V_{sub} will not contain a vertex crossing.

Next we want to show that if we have a metro map embedding M for $(G, \mathcal{G}, \Pi)$ with k crossings there is also a metro map embedding for $(G', \mathcal{G}', \Pi')$ with k crossings. Take any metro map embedding of $(G, \mathcal{G}, \Pi)$ witk k crossings. First we observe that if all the crossings occur on the vertices V we directly have a metro map embedding of $(G, \mathcal{G}, \Pi)$ witk k crossings. So assume M contains a vertex crossing at the vertex $v_{\{u,v\}} \in V_{sub}$. Since every path $p \in \Pi$ that contains $v_{\{u,v\}}$ also contains v we can shift every crossing of M that occurs in $v_{\{u,v\}}$ to vertex v. Therefore $v_{\{u,v\}}$ does not have a vertex crossing anymore but v will. In total we did not increase the number of vertex crossings so if we do the same procedure for every vertex $v \in V_{sub}$ we get a metro map embedding for $(G', \mathcal{G}', \Pi')$ with at most k vertex crossings.

□

Data: hyperedges: set of ordered hyperedges Π of a hypergraph H, graph: path-based support graph of H with given embedding
Result: cond: set of all subpaths that contain an unavoidable crossing

```
cond = [] ;
for p1 in hyperedges do
    for p2 in hyperedges, index(p1) < index(p2) do
        conSub = commonSubpaths(p1,p2) ;
        for ps in conSub do
            if unavoidableCrossing(ps, p1, p2, graph) then
                cond.add(ps) ;
        end
    end
end
```

Algorithm 1: Computing ILP

Given a triple $(G, \mathcal{G}, \Pi)$ with $G = (V, E)$ being a planar path-based support graph of an ordered hypergraph $H = (V, E_H)$, $\mathcal{G}$ a planar embedding of G and Π being the corresponding paths for the ordered hyperedges of H. We want to compute the set of sets L that contains every subpath p_s such that there exists $p_1, p_2 \in \Pi$ that have an unavoidable crossing on their common subpath p_s. Algorithm 1 computes L by selecting any two paths $p_1, p_2 \in \Pi$ taking any common subpath p_s of p_1 and p_2 and computing if the paths p_1 and p_2 have an unavoidable crossing on p_s. For the running-time of the algorithm we refer to Theorem 1 of the paper An Improved Algorithm for the Metro-line Crossing Minimization Problem by Noellenburg [Nöl09]. Noellenburg proofs that the algorithm runs in $O(|\Pi|^2 \cdot |V|)$.

Now we can use the algorithm to show that V-MLCM becomes polynomial if the path-based support graph is a tree.

Theorem 5.10. *Given a triple $(T, \mathcal{T}, \Pi)$, with $T = (V, E)$ being a tree and a path-based support graph of an ordered hypergraph $H = (V, E_H)$, $\mathcal{T}$ a planar embedding of T and Π being the corresponding paths for the ordered hyperedges of H. There is an algorithm that decides if there is a metro map drawing for $(T, \mathcal{T}, \Pi)$ with at most k vertex crossings (V-MLCM) in $O(|\Pi|^2 \cdot |V|)$ time.*

Proof. With Lemma 5.6 we know the following: Given a triple $(G, \mathcal{G}, \Pi)$ with $G = (V, E)$ being a planar path-based support graph of an ordered hypergraph $H = (V, E_H)$, $\mathcal{G}$ a planar embedding of G, Π being the corresponding paths for the ordered hyperedges of H and a set of sets L that contains every subpath p_s such that there exists $p_1, p_2 \in \Pi$ that have an unavoidable crossing on their common subpath p_s. $(G, \mathcal{G}, \Pi)$ has a metro map drawing with k vertex crossings if and only if L has a hitting set of size k. Therefore it is sufficient to show that we can compute a hitting set of L efficiently.

In order to compute a hitting set of L we use the fact that T is a tree. At first we pick any vertex r, denote it as root, do a breadth-first search starting at r and define for $v \in V$ the $level(v)$ as the distance from v to r.

Next we do a bottom up approach and add vertices to our solution V_{cros} as follows:

Step 1 Take for each list l in L the vertex v_l with the minimum level in l and call the union of those vertices $V_{min} = \{v_l \mid l \in L\}$.

Step 2 Select a vertex $v_{low} \in V_{min}$ such that $level(v_{low}) \geq level(v_l)$ for all $v_l \in V_{min}$.

Step 3 Add v_{low} to our solution V_{cros}, remove every list l that contains v_{low} and the corresponding v_l from V_{min} and go back to Step 2 until L is empty.

In the following we want to bound the running-time of the algorithm. With Algorithm 1 we can compute L in $O(|\Pi|^2 \cdot |V|)$ time. Furthermore doing a breadth-frist search on a tree is in $O(|V|)$ time. Next we look at the steps from our bottom up approach. Step 1 can be done in $O(|V| \cdot |\Pi|)$ time since we have to find a minimal element in $|\Pi|$ lists that contain at most $|V|$ elements. Since finding a maximal element in a list can be computed in linear time Step 2 is in $O(|\Pi|)$ time. In Step 3 we can go over every list $l \in L$ to check if it contains v_{low} which can be done in $O(|\Pi| \cdot |V|)$ time.

Now we bound the number of times we repeat Step 2 and 3. Since we selected v_{low} in Step 2 there is at least one set in L remaining that contains v_{low}. Therefore we decrease the number of elements in L by at least one and we can bound the number of iterations by $O(|\Pi|)$. In total the algorithm runs in $O(|\Pi|^2 \cdot |V|)$ time.

We observe that V_{cros} is a hitting set of L since we only removed sets from L that are hit by a vertex in V_{cros}. So it remains to show that V_{cros} is an optimal solution.

First we look at any path p in T and consider the level of the vertices in the path. We observe that since there is a unique path from each vertex v to the root r we can write p as the concatenation of two paths p_1 and p_2 such that the level of the vertices in p_1 is decreasing and the level of the vertices in p_2 is increasing.

Whenever the algorithm adds a vertex $v \in V$ to V_{cros}, there is a set $l \in L$ such that $v = v_l$ and l is not hit yet. We will denote this set as $l(v)$.

Now we claim that the set $S = \{l(v) \mid v \in V_{cros}\}$ of these sets is pairwise disjoint, forcing any hitting set to have cardinality at least $|S| = |V_{cros}|$.

Assume that there are two different sets $l(a), l(b)$ with $a, b \in V_{cros}$ such that $l(a) \cap l(b) \neq \emptyset$. Therefore there exists a vertex $x \in l(a) \cap l(b)$. Further assume without loss of generality that $level(a) \leq level(b)$. Since T is a tree we know that there exists a unique path p_b from x to b and a unique path p_a from x to a. We observe that since b is the vertex with minimal $level$ in $l(b)$ and a is the vertex with minimal $level$ in $l(a)$ that both paths are subpaths from the unique path from x to the root r and together with $level(a) \leq level(b)$ we know that $b \in l(a)$. Since $a \neq b$ the algorithm chooses b first and therefore removes $l(a)$ from L. So the algorithm will not select the vertex a. Contradiction.

□

In the following we want to give an ILP (Integer Linear Programming) for the V-MLCM problem. As seen in Lemma 5.6 it is sufficient to compute the set of subpaths L on which unavoidable crossings occur and then to compute a minimum hitting set of L. So we will give an ILP for the hitting set problem and use Algorithm 1 to compute L.

ILP

Given a hypergraph $H = (V, E_H)$. For each hyperedge $h \in E_H$ we want to select at least one vertex $v \in h$. Define the variables $x_v \in \{0, 1\}$ for all $v \in V$. The variable x_v being 1 corresponds to selecting v. Therefore we want to minimize the number of variables x_v that have value 1. We therefore define the following optimization function:

$$min \sum_{v \in V} x_v \tag{5.1}$$

Further we want to ensure that for each hyperedge we select at least one vertex:
$\forall h \in E_H$:

$$\sum_{v \in h} \geq 1 \tag{5.2}$$

We can interpret the hitting set problem also as the vertex cover problem for hypergraphs. To solve this problem there are different approximation algorithms [Hal02]. If the size of the sets is bounded by k there is a polynomial-time multiplicative k-approximation. In the general case vertex cover for hypergraphs can only be approximated logarithmically.

So far an embedding has already been given for our path-based support graph. In the following we will look at the same problem with the additional freedom of choosing the embedding as well. We will call this problem V-MLCM-E.

Proposition 5.11. *Given a tuple (G, Π), with $G = (V, E)$ being a planar path-based support graph of an ordered hypergraph $H = (V, E_H)$, Π being the corresponding paths for the*

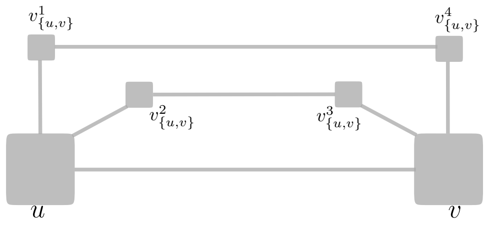


Figure 5.4: Representation of an edge

ordered hyperedges of H and an integer k. It is NP-complete to decide if there exists a metro map embedding for (G, Π) *with at most k vertex crossings (V-MLCM-E).*

Proof. With the same argument as in Theorem 5.7 we can see that the problem is in NP. Next we want to show that it is NP-hard.

We use a similar reduction as in Theorem 5.7. There the vertex positions were given which allowed us to encode a vertex cover instance into the metro map drawing. In this more general case we do not know which embedding is optimal. Therefore the idea is to reduce the number of possible embeddings and ensure that the embedding of Theorem 5.7 is optimal. We will add some edges to enforce a certain vertex placement. In the transformation of Theorem 5.7 we added four vertices $V_e = \{v_e^1, v_e^2, v_e^3, v_e^4\}$ for each edge $e = \{u, v\}$ and the two hyperedges $E_{H_e} = \{(v_e^1, u, v, v_e^3), (v_e^2, u, v, v_e^4)\}$. We will do the same transformation but we also add the edges $E_{emb} = \{\{v_e^1, v_e^4\}, \{v_e^2, v_e^3\}, \{v_e^3, v_e^4\}, \{v_e^1, v_e^2\}\}$ to E and the edges E_{emb} as hyperedges to E_H. The vertices v_e^2 and v_e^3 are connected and together with u and v they form a cycle of size four. Without loss of generality let v_e^1 be outside of this cycle in an embedding of G. Since v_e^1 and v_e^4 are connected v_e^4 has to be outside of this cycle as well. We observe the following: The vertices $\{a, b, c\}$ have the same ordering in ord_u and ord_v with $a = v_e^1$, $b = v_e^2$ and $c = v$ in ord_u and $a = v_e^3$, $b = v_e^4$ and $c = u$ in ord_v (see Figure 5.4). Therefore we know with Observation 5.3 that the two paths E_{H_e} have an unavoidable crossing on their common subpath (u, v) and we know that our solution has at least as many vertex crossings as the size of a vertex cover. Moreover by using a metro map embedding as seen in Theorem 5.7 we know there is one metro map embedding with exactly the same amount of crossings.

□

With the same argument as in Corollary 5.9 we can see that the problem remains NP-hard for bipartite path-based support graphs of a hypergraph H.

Corollary 5.12. *Given a tuple* (G, Π)*, with* $G = (V, E)$ *being a planar bipartite path-based support graph of an ordered hypergraph* $H = (V, E_H)$*,* Π *being the corresponding paths for the ordered hyperedges of H and an integer k. It is NP-complete to decide if there exists a metro map embedding for* (G, Π) *with at most k vertex crossings (V-MLCM-E).*

So far we have only considered vertex crossings. In the following we want to look at line crossings. Noellenburg [Nöl09] showed that the so called L-MLCM-T1 problem can be reduced to the L-MLCM-FixedSE problem in linear time. The problem is defined as follows:

Definition 5.13 (L-MLCM-T1). *Given a triple $(G, \mathcal{G}, \Pi)$, with $G = (V, E)$ being a path-based support graph of an ordered hypergraph $H = (V, E_H)$ with hyperedges E_H such that each hyperedge terminates in a vertex v of degree one, $\mathcal{G}$ an embedding of G, Π being the corresponding paths for the ordered hyperedges of H and an integer k. Decide whether $(G, \mathcal{G}, \Pi)$ admits a metro map embedding with at most k line crossings.*

Now we want to show that this problem becomes NP-hard even on trees if we do not fix the embedding.

Theorem 5.14. *Given a tuple (G, Π), with $G = (V, E)$ being a tree and a path-based support graph of an ordered hypergraph $H = (V, E_H)$ with hyperedges E_H such that each hyperedge terminates in a vertex v of degree one, Π being the corresponding paths for the ordered hyperedges of H and an integer k. It is NP-complete to decide if there exists a metro map embedding for (G, Π) with at most k line crossings (L-MLCM-T1-E).*

Proof. The number of line crossings can be computed in polynomial time if the embedding and the ordering at every edge of the path-based support graph is given. Therefore the problem is in NP. Now it remains to show that the problem is NP-hard. To show this we reduce the problem BIPARTITE CROSSING NUMBER ([GJ83]) to our problem. The problem is defined as follows: Given a bipartitioned multigraph $G = (L \cup R, E)$ and an integer k. Can G be embedded in the unit square such that all vertices in L are on the left boundary, all vertices in R are on the right boundary, all edges are within the square and there are at most k crossings. We observe that such embeddings only differ from each other in the order they embedded the vertices on the left boundary and in the order they embedded the vertices on the right boundary where the ordering orders the vertices according to their y-coordinate in the embedding. Therefore it is sufficient to represent a solution of the BIPARTITE CROSSING NUMBER problem as an order of the vertices in L which we will denote as ord_{left} and an order of the vertices in R which we will denote as ord_{right}.

In the following let $(\{u, v\}, n)$ be the n-th multiedge between u and v.

Transformation: Given the graph $G' = (L \cup R, E')$ with the integer k. The idea for the path-based support graph $G = (V, E)$ we create is to have one vertex s that is connected to the vertices of L and one vertex t that is connected to the vertices in R and to s. Since we insist that each terminal vertex has degree one we further have to add two vertices for each edge $e \in E'$. Formally, we define $G = (V, E)$ and the set of paths Π as follows: $V = L \cup R \cup \{s, t\} \cup \bigcup_{(\{u,v\},n) \in E} (v_{(\{u,v\},n),l} \cup v_{(u,v,n),r})$, $\Pi = \{(v_{(\{u,v\},n),l}, u, s, t, v, v_{(\{u,v\},n),r}) \mid \{u, v\} \in E'\}$ and E are the edges induced by the paths in Π. This construction ensures that every path $p \in \Pi$ has terminal vertices of degree one. Therefore each path can be identified by its first vertex.

Intuitively the edge $\{s, t\}$ is the inside of the unit square of the BIPARTITE CROSSING NUMBER problem. Every crossing will occur on this edge. A simple observation is that an optimal solution does not have crossings between two paths p_1,p_2 if they share the second or the second last vertex. Since the first and last vertex of each path is individual the first/last vertices can be placed according to the relative positioning of the last/first vertices such that p_1, p_2 do not intersect. Another observation is that two edges $e_1 =$

$\{l_1, r_1\}, e_2 = \{l_2, r_2\} \in G'$ with $l_1, l_2 \in L$ and $r_1, r_2 \in R$ cross in an embedding of the BIPARTITE CROSSING NUMBER problem if the order of l_1 and l_2 in ord_{left} is the reverse order of r_1 and r_2 in ord_{right}.

Assume that ord_{left}, ord_{right} is a solution for the BIPARTITE CROSSING PROBLEM with k crossings. Add the vertex t at the end to the ordering ord_{left} and order the vertices in $L \subseteq V$ around s according to ord_{left}. Add the vertex s at the end of the ordering ord_{right} and order the vertices in $R \subseteq V$ around t according to the reversed order of ord_{right}. This will create k crossings on the edge $\{s, t\}$. Moreover we know with the observation above that there are no more crossings on other edges.

In the reverse direction assume that $(G, \mathcal{G}, \Pi)$ has a metro map drawing with k crossings. With the observation above every crossing that does take place on a edge different to $\{s, t\}$ is unneccesary. Therefore we assume that no such crossing exists and every crossing happens on $\{s, t\}$. The line-ordering at vertex s (starting form t) and the reversed line-ordering at vertex t in reversed ordering (starting from s) induce orderings ord_{left} and ord_{right} with k crossings. □

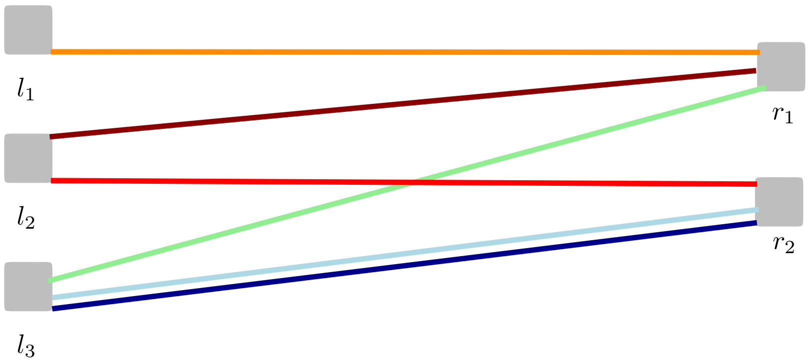
To understand the NP-reduction we give an example for the reduction.

Example 5.15 (L-MLCM-T1-E NP-reduction)**.** Given a bipartite multigraph $G_{bip} = (L \cup R, E_{bip})$ with $L = \{l1, l2, l3\}$, $R = (r1, r2)$,

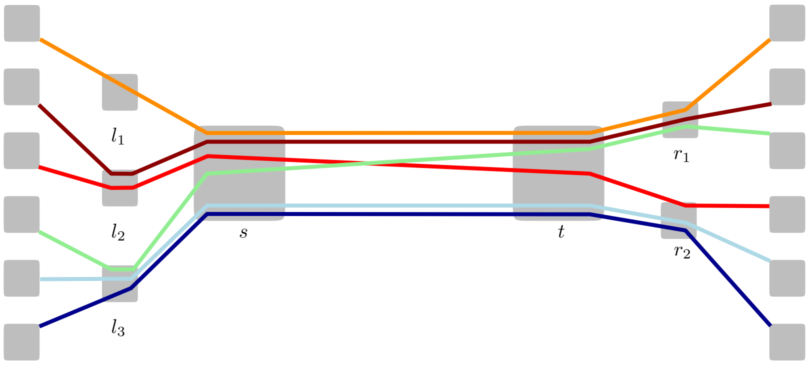
$E_{bip} = \{\{l1, r1\}, \{l2, r1\}, \{l2, r2\}, \{l3, r1\}, \{l3, r2\}, \{l3, r2\}\}$ and $k = 1$. A drawing of G_{bip} is seen in (Figure 5.5a). This drawing corresponds to a solution of the created L-MLCM-T1-E problem (Figure 5.5b) and vice versa.

Corollary 5.16. *Given a tuple (G, Π), with $G = (V, E)$ being a double star and a planar path-based support graph of an ordered hypergraph $H = (V, E_H)$ with hyperedges E_H, Π being the corresponding paths for the ordered hyperedges of H and an integer k. It is NP-complete to decide if there exists a metro map embedding for (G, Π) with at most k line crossings (L-MLCM-E).*

Proof. In the proof for Theorem 5.14 we can remove the terminal vertices for each path. The remaining graph will not be a T1-graph in general, since the new terminal vertices can have a higher degree. Nevertheless this is still a valid MLCM instance. With this exception the NP-reduction is equivalent to the proof of Theorem 5.14. □



(a) A solution for the bipartite crossing problem with one crossing



(b) A corresponding metro map drawing of the hypergraph H.

Figure 5.5: Example for the NP-reduction of L-MLCM-T1-E from BIPARTITE CROSSING NUMBER

6 Minimum path-based supports

In order to visualize a hypergraph as a metro map we require a path-based support graph at its core. The goal of this chapter is to minimize the number of edges of the path-based support graph. Brandes et al. [Bra+10] show by a reduction from Hamiltonian path that minimizing the number of edges in path-based support graphs is NP-hard even if the hypergraph is closed under intersection. We observe that since it is NP-hard to check if a given graph $G = (V, E)$ is a path-based support for a hypergraph H (Theorem 3.1), we should not just give a path-based support but also store the vertex-ordering for each metro line in order to use G for our metro map drawing.

In the following we give an ILP for the problem.

ILP

Given a hypergraph $H = (V, E_H)$ we want to find a minimum path-based support graph G. First we define an arbitrary ordering on the vertices $ord : V \to \{1, ...|V|\}$. We define the variables $x_{\{u,v\}} \in \{0, 1\}$ with $u, v \in V$ and $u \neq v$ such that $x_{\{u,v\}} = 1$ is equivalent to the edge $\{u, v\}$ being in the path-based support graph. Formally, $G = (V, E)$ with $E = \{\{u, v\} \mid x_{\{u,v\}} = 1\}$. We need for every $h \in E_H$ a Hamiltonian path p_h in $G[h]$. Therefore we define the variables $y^h_{\{u,v\}} \in \{0, 1\}$ with $u, v \in V$, $u \neq v$ and $h \in E_H$, where $y^h_{\{u,v\}} = 1$ corresponds to $\{u, v\}$ being an edge in the path p_h. To guarantee that the graph $G = (h, E_h)$ with $E_h = \{\{u, v\} \mid y^h_{\{u,v\}} = 1\}\}$ is a path we need some further variabes $z^{t,h}_{(v,u)} \in \{0, 1\}$ with $h \in E_H$, $u, v, t \in h$ and $u \neq v$. In order to guarantee that $G_h = (h, E_h)$ is a path we ensure that no vertex has degree greater than two, that G_h is connected and that $|E_h| = |h| - 1$.

So in the following we want to bound the degree of every vertex in G_h by two.
$\forall h \in E_H$, $\forall v \in V$:

$$\sum_{u \in V \setminus \{v\}} y^h_{\{u,v\}} \leq \begin{cases} 2, \text{if } v \in h \\ 0, \text{ otherwise} \end{cases} \tag{6.1}$$

We want to guarantee that G is connected. Therefore we transform G into a directed graph G' by replacing each edge $\{u, v\} \in E$ with two directed edges (u, v) and (v, u). We can guarantee that $G[h]$ is connected by finding a path from a fixed vertex $s_h \in h$ to any vertex $t \in h$ that only contains vertices in h. We choose for every hyperedge h one vertex s_h.

We observe that on a directed path from s_h to t the indegree of a vertex x minus the outdegree of x equals 1 if $x = t$, -1 if $x = s_h$ and 0 otherwise. We can further think of this condition as the existance of a flow of size one from s_h to t. Therefore there has to be a path from s_h to t.

$\forall h \in E_H, \forall v, t \in h$ with $t \neq s_h$:

$$\sum_{u \in V \setminus \{v\}} z^{t,h}_{(u,v)} - \sum_{u \in V \setminus \{v\}} z^{t,h}_{(v,u)} = \begin{cases} 1, \text{ if } v = t \\ -1, \text{ if } v = s_h \\ 0, \text{ otherwise} \end{cases} \tag{6.2}$$

Furthermore we want to ensure that we only use edges that are in G_h:
$\forall h \in E_h, \forall u, v, t \in h$ with $t \neq s_h$ and $u \neq v$:

$$z^{t,h}_{(u,v)} + z^{t,h}_{(v,u)} - y^h_{\{u,v\}} \leq 0 \tag{6.3}$$

Now we know that for every $h \in E_H$, G_h is connected and every vertex of G_h has degree at most two. To avoid creating a cycle we add the following condition:
$\forall h$:

$$\sum_{u,v \in V, ord(u) < ord(v)} y^h_{\{u,v\}} = |h| - 1 \tag{6.4}$$

In the following we want to ensure that every path p_h is in G:
$\forall u, v \in V$ with $u \neq v$, $\forall h \in E_H$:

$$y^h_{\{u,v\}} - x_{\{u,v\}} \leq 0 \tag{6.5}$$

With those conditions we guarantee that G is a path-based support graph of H. So the only thing missing is the optimization function for the path-based support graph:

$$min \sum_{u,v \in V, ord(u) < ord(v)} x_{\{u,v\}} \tag{6.6}$$

Correctness of ILP

Condition (6.2) ensures that for every hyperedge the graph $G[h]$ is connected since $\forall h \in E_H$ there is a path between s_h and any other vertex $t \in h$ and every of those edges is also picked for the path-based support graph, as ensured by condition (6.3) and (6.5). In combination with condition (6.1) and (6.4) the edges picked for this hyperedge have to be a path. Therefore the ILP will return a valid solution with $x_{\{u,v\}}$ being the edges of a support graph and $y^h_{\{u,v\}}$ being the edges of the Hamiltonian path in $G[h]$.
Assume $G = (V, E)$ is a valid path-based support graph of $H = (V, E_H)$. Therefore each hyperedge $h \in E_H$ is connected by a path p_h. Choose for every hyperedge $h \in E_H$ a vertex s_h. We set the variables $x_{\{u,v\}}$ to 1 if $\{u, v\} \in E$, $y^h_{\{u,v\}}$ to 1 if and only if $\{u, v\}$ is an edge in p_h and $z^{t,h}_{(u,v)}$ to 1 if and only if the edge $\{u, v\}$ is on the path p_h between the vertices s_h and t and u has a smaller distance to s_h on the path than v. Since each p_h is a Hamiltonian path

in $G[h]$ this setting of the variables will fulfill the conditions (6.2), (6.4) and (6.1). Since each p_h is a path in G condition (6.5) is fulfilled, because if we picked some variable $y^h_{\{u,v\}}$ we also picked the corresponding variable $x_{\{u,v\}}$. Condition (6.3) is also fulfilled because we did not set both $z^{t,h}_{(u,v)}$ and $z^{t,h}_{(v,u)}$ to one and if we picked one we know that the edge $\{u, v\}$ belongs to the path p_h and therefore we also picked the variable $y^h_{\{u,v\}}$.

7 Conclusion

In this thesis we looked at metro map drawings as a way to visualize hypergraphs. With the goal of finding a nice metro map drawing of a hypergraph we considered minimization problems for vertex crossings and line crossings of metro map drawings.

In Chapter 3 we proved that given a planar graph G and a hypergraph H it is NP-complete to decide if G is a planar path-based support of H. Furthermore we showed that there is a hypergraph H that has a planar path-based support but its condensation H_C does not and vice versa. However we did not consider if there is any property of hypergraphs H which ensures that both H and H_C do have a planar path-based support.

In Chapter 4 we proved that every hypergraph which has at most five hyperedges does have a planar x-monotone metro map drawing and that there is one hypergraph with nine hyperedges that does not have a planar metro map drawing. We left the following question open:

Question 7.1. *Is there a graph with k hyperedges that does not have a planar (x-monotone) metro map drawing. (with $k = 6, 7, 8$).*

Furthermore we proved that every hypergraph with at most four hyperedges has a planar x-consecutive monotone drawing and that the bound is tight because C_5 cannot be drawn planar x-consecutive monotone.

In Chapter 5 we proved that the V-MLCM and the V-MLCM-E problem are both NP-complete for general planar path-based support graphs G but there exists a polynomial-time algorithm for V-MLCM if G is a tree. An obvious question that occurs is if the same is true for the V-MLCM-E problem.

Question 7.2. *Is there a polynomial-time algorithm for the V-MLCM-E problem if the path-based support graph G is a tree.*

We gave a polynomial-time algorithm to transform the V-MLCM problem into a hitting set instance. Therefore we know that the problem can be approximated logarithmically in polynomial-time. It remains open to show if the problem has a constant factor approximation.

Question 7.3. *Is there a constant factor approximation for the V-MLCM problem.*

A different idea is to bound the number of hyperedges of E_H by a constant or by restricting the type of desirable support and ask the question if the V-MLCM problem remains NP-complete.

An interesting problem occurs if we bound the number of crossings between metro lines at every vertex. In such case we do not count the number of crossings on the whole graph but only insist that on each vertex there are at most a certain number of crossings between metro lines. This approach will create metro map drawings that do not have to complex crossing structures at vertices but will contain more vertices that have crossings. We did not look into this problem further but it is another optimization criteria for metro map drawings that seems desirable.

Another problem we considered is the L-MLCM-E problem which we proved to be NP-complete even if the path-based support graph is a double star.

Question 7.4. *Does the L-MLCM-E remain NP-complete if the path-based support graph is a star?*

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