Karlsruher Institut für Technologie

Bachelor thesis

Asymmetric hypergraphs

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Abstract

An k-graph is a hypergraph where every edge has size k. If every automorphism on a k-graph G is the identity we say that G is asymmetric. Furthermore we call G minimal asymmetric, if G is asymmetric and every non-trivial induced subgraph on at least two vertices of G is not asymmetric. In 2016 Schweitzer and Schweitzer [13] confirmed a conjecture by Nešetřil [14] that there are only finitely many minimal asymmetric graphs.

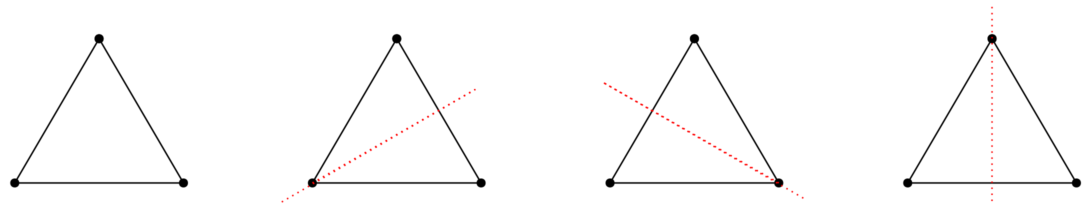
In this thesis we study minimal asymmetry of k-graphs. We extend a result by Jiang and Nešetřil [19] to show, that in contrast to the case $k = 2$, for every $k \geq 3$ there are infinitely many asymmetric k-graphs that are linear and have maximum degree 2. We generalize the concept of the degree of asymmetry introduced to graphs by Erdős and Rényi [12] to the setting of k-graphs. Then, we consider regular asymmetric k-graphs which have some interior symmetry as every vertex has the same degree. Finally, we analyze an algorithm proposed by Luks [5] which decides in exponential running time whether a given hypergraph is minimal asymmetric.

Contents

1 Introduction

Symmetries play an import role in our lives as we often associate them with beauty and perfection. From art like the *Vitruvian Man* by Leonardo da Vinci that depicts a man with idealistic body proportions which is at least superficially symmetric to architecture like the *Taj Mahal* where we can see symmetry and harmony to perfection, symmetric elements can be found in many aspects of our history.

One can also consider symmetries of mathematical objects. For instance consider an equilateral triangle T, then there are the 3 mirror axes, as show in Figure 1.



(a) Triangle T (b) The first mirror axis (c) The second mirror axis (d) The third mirror axis

Figure 1: Triangle T and its 3 mirror axes

Or even simpler, we can take an equidistant chain C of an odd number of indistinguishable objects and reflect all objects on the middle one, as shown in Figure 2.

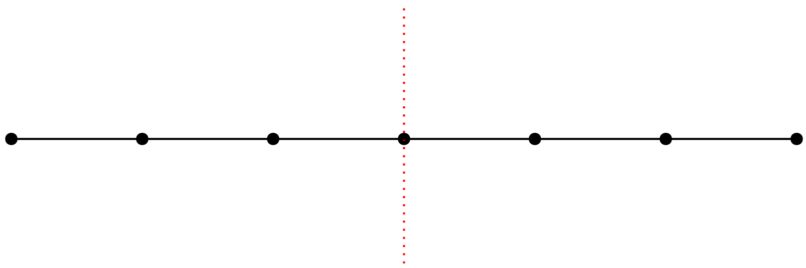
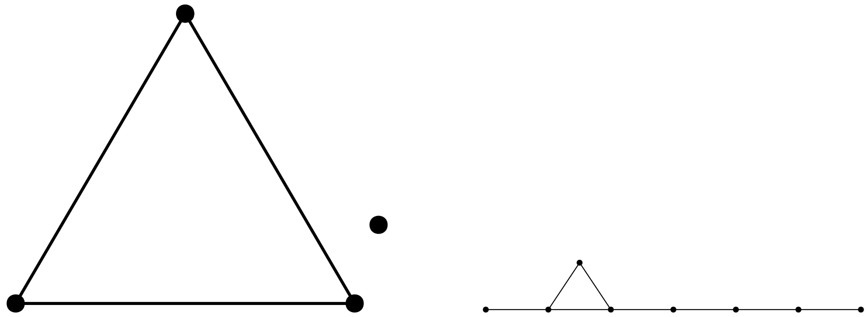


Figure 2: Chain C and its mirror axis

But as beautiful symmetry might seem as interesting are asymmetric things, things that are almost "perfect" and symmetric or things where we are hopeless to find symmetries at all even if we were to ignore some imperfections. We can destroy the symmetries of our triangle T by adding just another point that is not in any mirror-axis and we destroy the symmetry of our chain C by adding a simple triangle as depicted in Figure 3.



T with an additional point C with an extra triangle

Figure 3: Destruction of symmetry

Here we study asymmetry of discrete objects called k-graphs, a generalisation of graphs.

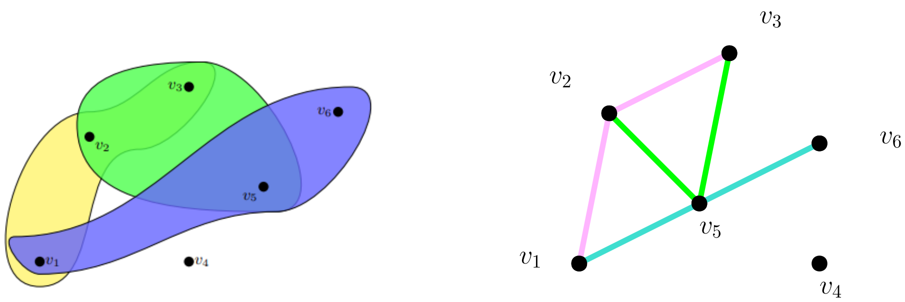
1.1 Basic Definitions

For any $n \in \mathbb{N}$ we denote the set $[n] := \{1, 2, 3, \ldots, n\}$. Let V be a finite set. We define $\binom{V}{k}$ as the set of all subsets of size k of V.

A *hypergraph* is a tuple $G := (V(G), E(G))$ of a vertex set $V(G)$ and an edge-set $E(G) \subseteq \bigcup_{k \in [|V(G)|]} \binom{V(G)}{k}$. We call $v \in V(G)$ a vertex and $e \in E(G)$ and edge.

For $k \in \mathbb{N}, k \geq 2$ we call a hypergraph $G := (V(G), E(G))$ a *k-graph* if every edge has size k, i.e. $|e| = k$ for every $e \in E(G)$. If every edge $e \in E(G)$ has size 2 we call G a *graph*.

Example: We give an 3-graph H with vertex set $V(H) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and edge set $E(H) = \{\{v_1, v_2, v_3\}, \{v_2, v_3, v_5\}\{v_1, v_5, v_6\}\}$ We can represent hypergraphs by enclosing vertices in one edge, as depicted in Figure 4a or by connecting them in a line of the same color, as depicted in Figure 4b. The latter will be helpful in depicting more complex examples later.



(a) H drawn *enclosing*.   (b) H drawn *connecting*.

Figure 4

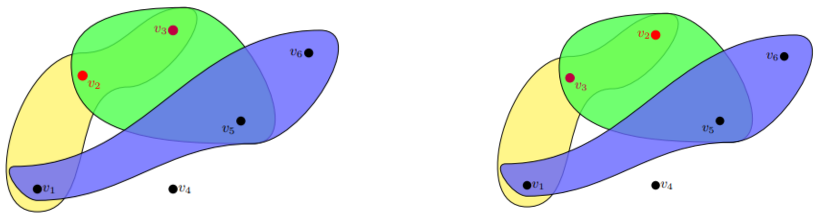
Let $G = (V(G), E(G))$ be a hypergraph. We call $G' = (V(G'), E(G'))$ a *subgraph* of G if $V(G') \subseteq V(G)$ and if for every edge $e \in E(G')$ also $e \subseteq V(G')$ and $e \in E(G)$. Furthermore G' is an *induced subgraph* if $E(G') = \{e : e \in E(G), e \subseteq V(G')$. We call G' a *proper* (induced) subgraph of G if $G' \neq G$.

Let $G = (V(G), E(G))$ be a hypergraph and $v \in V(G)$ a vertex. We call $deg(v) := |\{e \in E(G) : v \in e\}|$ the *degree* of v and $\max\limits_{v \in V(G)} deg(v)$ the *degree* of G. In our example graph H above we have $deg(v_1) = 2$, $deg(v_4) = 0$ and $deg(H) = 2$.

We call a hypergraph $G = (V(G), E(G))$ *linear* if any two distinct edges of G intersect in at most one vertex, i.e. if for every $e_1, e_2 \in E(G)$ with $e_1 \neq e_2$ it holds that $|e_1 \cap e_2| \leq 1$.

Let $G = (V(G), E(G))$ be a hypergraph. For any function $\Phi \colon V(G) \mapsto V(G)$ define for any set $S \subseteq V(G)$, $\Phi(S) = \{\Phi(v) : v \in S\}$. We say that a bijection $\Phi \colon V(G) \mapsto V(G)$ is an *automorphism* on G if for every edge $e \in E(G)$ also $\Phi(e) \in E(G)$.

Example: Automorphism Φ on H depicted in Figure 5 with $\Phi(v_2) = v_3, \Phi(v_3) = v_2$ and $\Phi(w) = w$ for all other $w \in V(H)$.



(a) H before automorphism Φ (b) H after automorphism Φ

Figure 5: The automorphism Φ on H with highlighted non-invariant vertices.

We call an automorphism Φ on a hypergraph $G = (V(G), E(G))$ an involution if Φ^2 is the identity, so if $(\Phi(\Phi(v)) = v$ for each vertex $v \in V(G)$.

Let $G = (V(G), E(G))$ be a hypergraph and let Φ be an automorphism of G. We call a set $S \subseteq V(G)$ (e.g. $S \in E(G)$) *invariant* under Φ if $\Phi(S) = S$ i.e. $\Phi(v) \in S$ for every vertex $v \in S$.

We call a hypergraph $G = (V(G), E(G))$ *asymmetric* if the only automorphism Φ on G is the identity so $\Phi(v) = v$ for every vertex $v \in V(G)$. If furthermore every induced subgraph G' of G is no longer asymmetric we call G *minimal asymmetric*.

Example: Let H_1 and H_2 be the graphs as depicted in Figure 6.

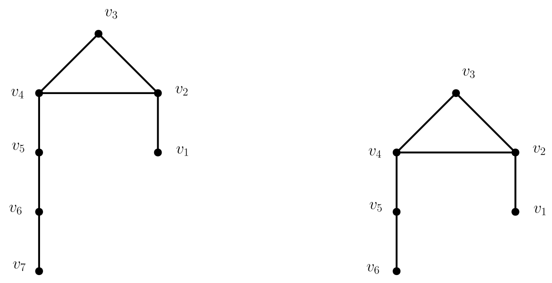


Figure 6: An asymmetric graph H_1 and a minimal asymmetric graph H_2.

We claim that H_1 is asymmetric but not minimal asymmetric and H_2 is minimal asymmetric. Let us first show that H_2 is asymmetric. So let Φ be any automorphism on H_2. We will later see in Lemma 1.1 that the sets of vertices of the same degree are invariant under any automorphism. Thus v_1 and v_6 are either both invariant under Φ or we have $\Phi(v_1) = v_6$ and $\Phi(v_6) = v_1$. The latter would also induce $\Phi(v_2) = v_5$ and $\Phi(v_5) = v_2$ as v_2 is the only neighbour of v_1 and v_5 the only neighbour of v_6. But this is already a contradiction as $deg(v_2) = 3 \neq 2 = deg(v_5)$. Thus Φ leaves v_1 and v_6 invariant. Therefore we also know that $\Phi(v_5) = v_5$ and $\Phi(v_2) = v_2$. Now v_4 is the only vertex that is a direct neighbour to both v_2 and v_5 and thus $\Phi(v_4) = v_4$. The only vertex left is v_3 and thus also $\Phi(v_3) = v_3$. So indeed Φ must be the identity and thus H_2 is asymmetric.
By a straightforward case by case analysis, which is not provided here, we can see that every proper non-trivial induced subgraph H_2' of H_2 has an automorphism.

One can show that H_1 is asymmetric by the same argument that we used to show that H_2 is asymmetric. It is also clear that H_2 is a proper induced subgraph of H_1 so indeed H_1 is asymmetric but not minimal asymmetric.

Let $G = (V(G), E(G))$ be a hypergraph. For every $v \in V(G)$ we define the *link* of v as $L_v := \{e \setminus \{v\} : e \in E(G), v \in e\}$.

In Section 4 we consider random k-graphs where each edge occurs with a probability $p \in (0,1)$ on potentially countable infinite many vertices. On a k-graph $G = (V(G), E(G))$ on countable infinite many vertices $V(G)$ we define the edge set $E(G)$ by $E(G) \subseteq [V(G)]^k$ where $[V(G)]^k$ is the set of all subset of $V(G)$ with size k.
Formally let V be a countable set and $k \in \mathbb{N}$. We now give a probability space $\mathcal{G}(V,p,k)$ for all k-graphs G with with vertex set $V(G) = V$. Let E be the set of all k-subsets of V, so $E := \{\{u_1, \dots, u_k\} : u_i \in V \text{ for all } i \in [k] \text{ and } u_i \neq u_j \text{ if } i \neq j\}$ and let $p \in (0,1)$. It should be noted that if V was countable, then E is countable too. Define for every $e \in E$ its own probability space by $\Omega_e := \{\mathbb{0}, \mathbb{1}\}, \mathbb{P}_e(\mathbb{1}) = p$ and $\mathbb{P}_e(\mathbb{0}) = 1 - p$. With this we define the product space $\Omega = \prod\limits_{e \in E} \Omega_e$, so $\mathcal{G}(V,p,k) = (\prod\limits_{e \in E} \Omega_e, \bigotimes\limits_{e \in E} \{\emptyset, \{\mathbb{0}\}, \{\mathbb{0}, \mathbb{1}\}, \{\mathbb{1}\}\}, \bigotimes\limits_{e \in E} \mathbb{P}_e)$. Strictly speaking the elements $w \in \mathcal{G}(V,p,k)$ are 0-1-vectors, but we can interpret them as the edges and non-edges in a k-graph G

with vertex set V. Note that if $|V| = n$ for some $n \in \mathbb{N}$ we denote our probability space with $\mathcal{G}(n, p, k)$.

Let $G = (V(G), E(G))$ be a k-graph. G can be made symmetric by removing or adding new edges to $E(G)$, e.g. by just removing all edges in $E(G)$. We call such an operation that makes G symmetric a *symmetrization* s and the set of all symmetrizations of G we call $S(G)$. For any $s \in S(G)$ let α_s be the number of edges that are removed and β_s be the number of edges are added to $E(G)$ by s. Then we define the *degree of asymmetry* $A(G) := \min\limits_{s \in S} \alpha_s + \beta_s$. Note that if G is symmetric then $A(G) = 0$.

For formal definitions of graph theoretical objects the reader is referred to Distel [4].

1.2 Historical Background

In 1939, Frucht [16] was one of the first to study the automorphism group of graphs. He showed that for every abstract group $\mathcal{G}$ there is a graph whose automorphism group is isomorphic to $\mathcal{G}$. 10 years later he gave a 3-regular asymmetric graph, the Frucht-Graph [17].

In 1963 Erdős and Rényi [12] introduced the concept of *degree of asymmetry* which measures how much one has to change a given hypergraph to make it symmetric. They also showed that not only almost every finite random graph is asymmetric but also that almost every graph has a high degree of asymmetry.

Brewer et.al. [1] considered it from the opposite direction and asked how much one has to change a symmetric graph to make it asymmetric.

Quintas [15] studied extremal values for the number of edges and vertices of an asymmetric graph. The minimum number of edges m an asymmetric graph with n vertices must contain was studied by Shelah [18].

In 1988, Nešetřil conjectured at an Oberwolfach seminar [14] that there are only a finite number of minimal asymmetric graphs. Nešetřil and Sabidussi [7, 8, 10], among other things, found the 18 minimal asymmetric graphs of Figure 7. The conjecture was confirmed to be true, in 2016, by Schweitzer and Schweitzer [13]. They showed that the minimal involution free graphs are exactly the ones that are also minimal asymmetric and that the 18 graphs in Figure 7 are, up to isomorphism, the only graphs that are minimal involution free and thus minimal asymmetric.

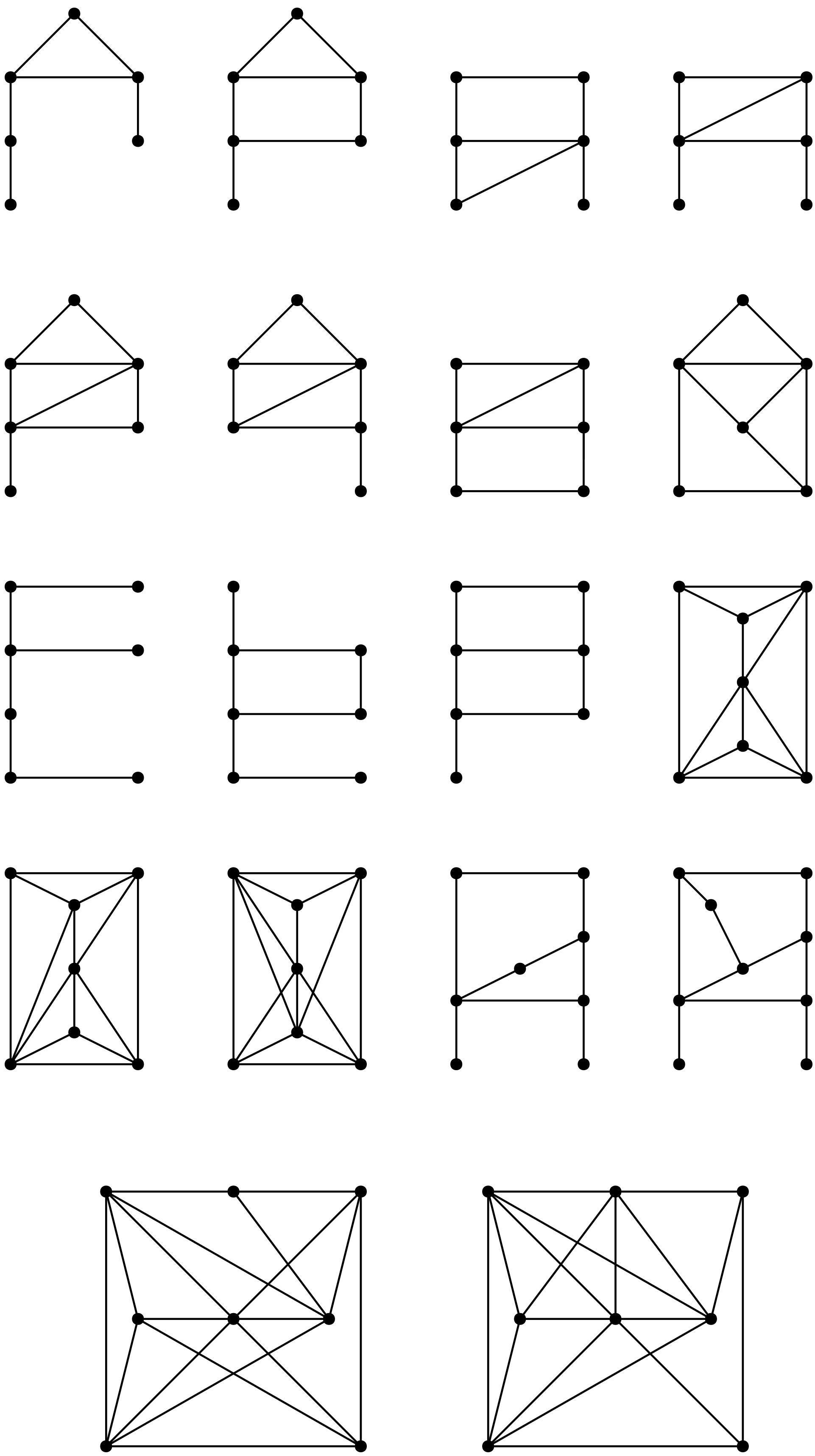


Figure 7: All 18 minimal asymmetric graphs

2021 Nešetřil and Jiang [19] refuted equivalent statements for hypergraphs and showed the following Theorem.

Theorem. *2.1(Jiang Nešetřil [19])*
For every $k \geq 3$ there are infinitely many minimal asymmetric k-graphs.

1.3 Results of this Thesis

After presenting some basic observations in Section 1.4, already mentioned by Erdős and Rényi [12], we consider the sparsity of asymmetric k-graphs.

In the Section 2 we give the main construction used by Jinag and Nešetřil [19]. We then introduce the concept of a path in a k-graph and modify their construction to show the following result.

Theorem. *2.2*
For every $k \geq 3$, $k \in \mathbb{N}$ there are infinitely many minimal asymmetric k-graphs that are linear and have maximum degree 2.

Then in Section 3 we consider regular asymmetric k-graphs. There we introduce the concept of *edge-asymmetry* and the *dual* of a hypergraph for showing the following statement.

Theorem. *3.1 There are infinitely many 2-regular asymmetric k-graphs for any $k \in \mathbb{N}$ with $k \geq 3$.*

In Section 4 we then consider the results of Erdős and Rényi [12] and generalise their results from graphs to k-graphs, so we show the following two Theorems.

Theorem. *4.4 Let $k \in \mathbb{N}$ with $k \geq 3$ and let $G = (V(G), E(G))$ be a random k-graph in $\mathcal{G}(V(G), 0.5, k)$ with an infinite countable vertex set $V(G)$. Then G is with a probability 1 symmetric.*

Theorem. *4.5 For a k-graph G with $|V(G)| = n$ the degree of asymmetry $A(G)$ can be bounded by $A(G) \leq \binom{n-1}{k-1}\frac{n-1}{2(n-k+1)}$.*

In the last Section we then survey an algorithm due to Luks, [5] that calculates all automorphism of a hypergraph: We modify it slightly to decide whether a given hypergraph G is minimal asymmetric, so we show the following Theorem.

Theorem. *5.17 Let $G = (V(G), E(G))$ be a hypergraph. For some constant c' it can be decided in $\mathcal{O}(c'^{|V(G)|})$ time whether G is minimal asymmetric.*

1.4 Basic Results

The following two Lemmata are general results about asymmetry on hypergraphs, already mentioned by Erdős and Rényi [12].

Let $G = (V(G), E(G))$ be a hypergraph and $v \in V(G)$. We call the set $N_E(v) := \{e \in E(G) : v \in e\}$ the *adjacency set* of v.

Lemma 1.1. Let $G = (V(G), E(G))$ be a hypergraph and $\Phi : V(G) \mapsto V(G)$ an automorphism. Then the degree $deg(v)$ of every vertex is invariant under Φ, so for every vertex $v \in V(G)$ it holds that $deg(v) = deg(\Phi(v))$.

Proof. Assume the statement is false, i.e. let there be a vertex $v \in V(G)$ with $deg(v) \neq deg(\Phi(v))$. Then either $|N_E(v)| < |N_E(\Phi(v))|$ or $|N_E(v)| > |N_E(\Phi(v)|$. Thus, there either is an edge $e \in N_E(v)$ with $e \notin N_E(\Phi(v))$ or there is an edge $e' \in N_E(\Phi(v))$ with $e' \notin N_E(v)$. This would be a contradiction to our assumption that Φ is a proper automorphism, as either $\Phi(e) \notin E(G)$ or $e^* \notin E(G)$ where e^* is the set with $\Phi(e^*) = e'$. Thus, indeed it follows $deg(v) = deg(\Phi(v))$ for every vertex $v \in V(G)$. $\square$

Lemma 1.2.

(i) If $G = (V(G), E(G))$ is an asymmetric hypergraph then its *complement graph* $H = (V(H), E(H))$ with $V(G) = V(H)$ and $E(H) = \bigcup_{k \in [|V(G)|]} \binom{V(G)}{k} \setminus E(G)$ is also asymmetric.

(ii) If $G = (V(G), E(G))$ is an asymmetric k-graph for some $k \in \mathbb{N}$ with $k \geq 2$ then its *k-complement* $H = (V(H), E(H))$ with $V(H) = V(G)$ and $E(H) = \binom{V(G)}{k} \setminus E(G)$ is also asymmetric.

Proof.
(i) Let G be a hypergraph and H be its complement. Assume that H is not asymmetric, so there is a non-trivial automorphism Φ on H. Let S be a arbitrary subset of $V(H)$. We know, due to the automorphism properties of Φ, that $S \in E(H)$ if and only if $\Phi(S) \in E(H)$.

Now because Φ is a bijection of $V(H) = V(G)$ we can also look at it as a function on G. Thus let S again be an arbitrary subset of $V(G)$. By definition of H, if $S \in E(G)$ then $S \notin E(H)$. Thus $\Phi(S) \notin E(H)$. But then $\Phi(S) \in E(G)$. Therefore Φ is also a automorphism on G. The statement follows by contraposition.

(ii) Let $G = (V(G), E(G))$ be an asymmetric k-graph. Note that the auxiliary hypergraph $A = (V(A), E(A))$ with $V(A) = V(G)$ and $E(A) = \bigcup_{j \in [|V(A)|], j \neq k} \binom{V(A)}{j} \cup E(G)$ is also asymmetric as any automorphism on A would also be an automorphism on G. Also note that the k-complement H of G is exactly the complement graph of A. Thus the statement of (ii) just follows from (i). $\square$

By similar argumentation we can show equivalent statements for minimal asymmetric hypergraphs.

The following lemma gives some intuitions about the behaviour of the degrees of asymmetric and minimal asymmetric hypergraphs. There we will already see a difference between asymmetry and minimal asymmetry as minimal asymmetric hypergraphs have stronger structural restrictions. We will also see in the proof that Lemma 1.2 is not a only useful tool but also a nice help in changing the perspective when thinking about asymmetry problems.

Proposition 1.3. *For every $n \in \mathbb{N}$ there is no minimal asymmetric k-graph G on n vertices with $deg(G) = \binom{n-1}{k-1}$. However there are asymmetric k-graphs on n vertices for some $n \in \mathbb{N}$ that achieve this bound.*

Proof. Assume there is a minimal asymmetric k-graph $G = (V(G), E(G))$ with maximum degree $\binom{n-1}{k-1}$, so assume there exists a vertex $v \in V(G)$ with $deg(v) = \binom{n-1}{k-1}$. Consider the k-graph $G' = (V(G'), E(G'))$ with $V(G') = V(G) \setminus \{v\}$ and $E(G') = \{e : e \in E(G), v \notin e\}$. If G' is asymmetric then G was not minimal asymmetric. If G' is symmetric then there is a non-trivial automorphism Φ on G'. Let Φ' be a extension of Φ to G with $\Phi'(v) = v$ and $\Phi'(u) = \Phi(u)$ for all $u \in V(G), u \neq v$. Let $e \in E(G)$ be an arbitrary edge. If $v \in e$ then $\Phi'(e) \in E$. Thus Φ' is also a non-trivial automorphism on G, a contradiction to the asymmetry of G.

Let $G = (V(G), E(G))$ be any asymmetric graph. We have already seen an example of an asymmetric 2-graph so the set of asymmetric k-graphs in non-empty. If there is a vertex $v \in V(G)$ with $deg(v) = 0$ we can take the edge-complement H of G and know that the degree of v in H is equal to $\binom{n-1}{k-1}$. By Lemma 1.2 H is also asymmetric. If there is no vertex in $V(G)$ with a degree of 0 we know that the extension $G' = (V(G'), E(G'))$ of G, with $V(G') = V(G) \cup x$ for a vertex x disjoint from $V(G)$ and $E(G') = E(G)$ is asymmetric. Then we again can take the complement of G' to show the statement. □

Next we show that in contrast to asymmetric hypergraphs, symmetric hypergraphs G can appear independently of the sparsity of G.

Proposition 1.4. *For any $n \in N$ with $n \geq 2$ and $k \leq n$ there is a symmetric k-graph with m edges for any $0 \leq m \leq \binom{n}{k}$.*

Proof. Let $[n]$ be the vertex set of our k-graph. For $n = 2$ the statement is trivial so let $n \geq 3$ and $u, v \in [n]$ with $u \neq v$. We define two sets of possible edges:

$$S_{nice} := \{e \subseteq [n] : |e| = k \text{ and } u, v \in e \text{ or } u, v \notin e\}$$

and

$$S_p := \{e \subseteq [n] : |e| = k \text{ and } u \in e, v \notin e \text{ or } u \notin e, v \in e\}$$

It is easy to check that $S_{nice} \cap S_p = \emptyset$ and that $S_{nice} \cup S_p$ contains all possible k-edges in the vertex set $[n]$. We say two edges $e_1, e_2 \in S_p$ are a pair (e_1, e_2) if $e_1 = e_2 \cup v \setminus u$. Then it is clear that $v \in e_1$ and $u \in e_2$. Now we can define a sequence of k-graphs $G_0, G_1, \dots, G_{\binom{n}{k}}$ with vertex set $[n]$: Let $G_0 = \{[n], \emptyset\}$. Next we define G_i for $i \in [\binom{n}{k}]$.

Case 1: $i \geq |S_p|$.
Let $E_{help} \subseteq S_{nice}$ be any subset with $|E_{help}| = i - |E_p|$. Then let $E(G_i) := S_p \cup E_{help}$

Case 2: $i < |S_p|$ and i is odd.
Let $E_{help} \subset S_p$ be a subset of $\frac{i-1}{2}$ distinct pairs in S_p and let $e \in E_{nice}$ be an arbitrary edge. Then define $E(G_i) := E_{help} \cup e$.

Case 3: $i < |S_p|$ and i is even.
Let $E_{help} \subset S_p$ be a set of $\frac{i}{2}$ distinct pairs in S_p. Then let $E(G_i) := E_{help}$.

In each case define the automorphism $\Phi \colon [n] \mapsto [n]$ as $\Phi(v) = u$, $\Phi(u) = v$ and $\Phi(w) = w$ for all other $w \in [n], w \notin \{u, v\}$. Then it is easy to see that in every case Φ defines a proper non-trivial automorphism of G_i because otherwise, if there is an edge $e \in E(G_i)$ with $u \in e$ and $v \notin e$ then the edge e' with $e' = e \cup v \setminus u$ is also in $E(G_i)$.

$\square$

2 Minimal Asymmetric k-graphs

The problem of finding infinitely many minimal asymmetric k-graphs for $k \geq 3$ was first considered by Jiang and Nešetřil [19].

Theorem 2.1. *(Jiang Nešetřil [19])*
For every $k \geq 3$ there are infinitely many minimal asymmetric k-graphs.

We modify their construction to show the following result.

Theorem 2.2. *For every $k \geq 3$, $k \in \mathbb{N}$ there are infinitely many minimal asymmetric k-graphs that are linear and have maximum degree 2.*

2.1 Construction by Jiang and Nešetřil

We first give the construction of Jiang and Nešetřil used in [19] that we call the JN-graph. Then we fix an inaccuracy in Lemma 8 of [19] and prove some other useful properties of the JN-graph. Afterwards we construct a family of k-graphs by extending the JN-graph $\mathcal{G}_{k,t}$ and show that every graph in this family is minimal asymmetric, linear and has max degree of 2.

Construction 2.3. JN-graph

For $k \geq 3$ and $t \geq 2$ we define the JN-graph $\mathcal{G}_{k,t} = (X_{k,t}, \mathcal{E}_{k,t})$ as the k-graph with vertex set

$X_{k,t} = \{v_i : i \in [tk]\} \cup \{u_i : i \in [tk]\} \cup \{v_i^j : i \in [tk], j \in [k-3]\}$ and edge set

$\mathcal{E}_{k,t} = \mathcal{E}_L \cup \mathcal{E}_{cyc}$.

We define the set of *L-edges* $\mathcal{E}_L$ and the set of *cyclic-edges* $\mathcal{E}_{cyc}$ by

$\mathcal{E}_L := \bigcup_{i \in [tk]} E_i$ with $E_i = \{v_i, u_i, v_i^1, v_i^2, \dots, v_i^{k-3} v_{i+1}\}$

and for the following pairs of indices

$P := \{(i,j) : j \in [k-3], i = j + sk; s \in \{0, 1, \dots, t-1\}\}$ let

$\mathcal{E}_{cyc} := \bigcup_{(i,j) \in P} E_{i,j}$ with $E_{i,j} = \{v_i^j, v_{i+1}^j, \dots, v_{i+k-1}^j\}$.

In Figure 8 we give an example of the JN-graph for $k = 6$ and $t = 3$.

Here and in the rest of this section let any index i with $i > tk$ or $i \leq 0$ be replaced by $i - tk$ and $i + tk$ respectively unless mentioned otherwise. With this we guarantee that all our vertices and edges are indeed a subset of our vertex set defined above. To be consistent with the original authors we start counting at 1 instead of 0.

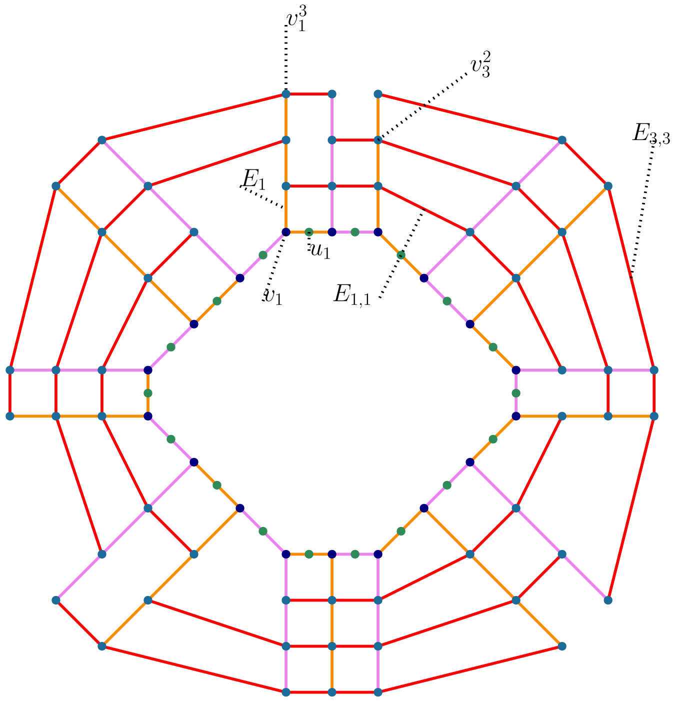


Figure 8: The JN-graph $\mathcal{G}_{6,3}$ with parameters $k = 6, t = 3$

Lemma 2.4. Let $k, t \in \mathbb{N}$ with $k = 3$ and $t \geq 2$ let $\mathcal{G}_{k,t}$ be the JN-graph. Let furthermore Φ be an automorphism on $\mathcal{G}_{k,t}$ then:

(i) $\Phi(u_j) \in \{u_i : i \in [tk]\}$ for every $j \in [tk]$.

(ii) $\Phi(E_j) \in \mathcal{E}_L$ for all $j \in [tk]$.

(iii) If $\Phi(E_1) = E_j$ for some $j \in [tk]$ then either $\Phi(v_1) = v_j, \Phi(v_2) = v_{j+1}$ or $\Phi(v_1) = v_{j+1}, \Phi(v_2) = v_j$.

(iv) If $\Phi(E_1) = E_j$ for some $j \in [tk]$ then either $\Phi(E_i) = E_{j+i-1}$ or $\Phi(E_j) = E_{i-j+1}$ for all $i \in [tk]$. We call the first kind of mapping a *cyclic shift* and the second kind a *reflection*.

Proof. Let $U := \{u_i : i \in [tk]\}$, then the only vertices of $\mathcal{G}_{k,t}$ with a degree of 1 are those in U. Therefore by Lemma 1.1 the set U is invariant under Φ and the first property holds.

Observe that all edges $E \in \mathcal{E}_L$ have a vertex $u \in E$ with $deg(u) = 1$ and all vertices of edges in $\mathcal{E}_{cyc}$ have a degree of 2. We know that $\Phi(E)$ is an edge of $\mathcal{G}_{k,t}$ and also that $deg(\Phi(u)) = 1$. Thus $\Phi(E) \in \mathcal{E}_L$, which shows the second property.

The only vertices with $v \in E_1$ and $v \in E$ for another edge $E \neq E_1, E \in \mathcal{E}_L$ are v_1 and v_2. Due to (ii) we know that $\Phi(E_1) = E_j$ for some $j \in [tk]$. Furthermore, v_j and v_{j+1} are the only vertices in E_j with $v_j \in E', v_{j+1} \in E''$ for other edges $E', E'' \neq E_j; E', E'' \in \mathcal{E}_L$.

Assume $\Phi(v_1) = w \notin \{v_j, v_{j+1}\}$ then $w \in E^*$ for some edge $E^* \in \mathcal{E}_{cyc}$. Every vertex in E^* has a degree of 2 but E_1, E_{tk} both have a vertex with a degree of 1, a contradiction to Lemma 1.1. Thus $\Phi(v_1) \in \{v_j, v_{j+1}\}$ and by the same argument also $v_2 \in \{v_j, v_{j+1}\}$ which shows (iii).

Consider now v_1 and v_2 and their possible mappings by Φ according to (ii) and (iii). So if $\Phi(v_1) = v_j$ and $\Phi(v_2) = v_{j+1}$ for some $j \in [tk]$ then we also know by (iii) that $\Phi(v_3) = v_{j+2}$ and inductively $\Phi(v_i) = v_{i+j-1}$ for all $i \in [tk]$. It directly follows that also $\Phi(E_i) = E_{i+j-1}$ for all $i \in [tk]$. Otherwise, $\Phi(v_1) = v_j$ and $\Phi(v_2) = v_{j-1}$ for some $j \in [tk]$. Then $\Phi(v_3) = v_{j-2}$ and inductively $\Phi(v_i) = v_{j-i+1}$ for all $i \in [tk]$. And again it follows that $\Phi(E_i) = E_{j-i+1}$ for all $i \in [tk]$, which shows (iv). □

Jiang and Nešetřil stated the following Lemma, here with give an independent proof.

Lemma 2.5. (Jiang Nešetřil [19])
Let $k, t \in \mathbb{N}$ with $k = 4$ or $k \geq 6$ and $t \geq 2$. Let $\mathcal{G}_{k,t}$ be the JN-graph. Let furthermore Φ be a homomorphism on $\mathcal{G}_{k,t}$. Then if E_1 is invariant under Φ, so $\Phi(w) \in E_1$ for every vertex $w \in E_1$, then Φ is the identity.

Proof. To show the statement we construct some sets of indexes for each E_ℓ, $\ell \in [tk]$, helping us to find possible mappings $\Phi'(E_1)$ for some automorphism Φ' complying with Lemma 2.4 $(iii), (iv)$. For any $\ell \in [tk]$ a vertex $x \in E_\ell \setminus \{v_\ell, u_\ell, v_{\ell+1}\}$ has $deg(x) = 2$ and by construction $x \in E$ for some edge $E \in \mathcal{E}_{cyc}$. Note that for any edge $E_{i,j} \in \mathcal{E}_{cyc}$, $E_{i,j} \cap E_\ell \neq \emptyset$ if $\ell \in \{i, \ldots, i+k-1\}$. So $E_{i,j}$ intersects an "interval" of edges. If we consider it from the opposite perspective we fix an ℓ and the set E_ℓ to determine all i such that there exists an $j \in [k-3]$ with $E_{i,j} \in \mathcal{E}_{cyc}$ and $E_{i,j} \cap E_\ell \neq \emptyset$. This leads to the following definition of the *forward set* $F(E_\ell)$. For any edge $E_{i,j} \in \mathcal{E}_{cyc}$ let

$$I(E_{i,j}) := \{E_\ell : \ell \in \{i, \ldots, i+k-1\}\}$$

and for any set $E_\ell \in \mathcal{E}_L$ let

$$F(E_l) := \{(\ell - i) \mod k : \exists j \text{ with } E_\ell \in I(E_{i,j})\}.$$

Then the forwards set $F(E_\ell)$ contains the "distance from the starting points" of the intersecting edges of E_ℓ. For example the forward set $F(E_1)$ of E_1 in $\mathcal{G}_{6,3}$ is given by $F(E_1) = \{0, 4, 5\}$, see Figure 9.

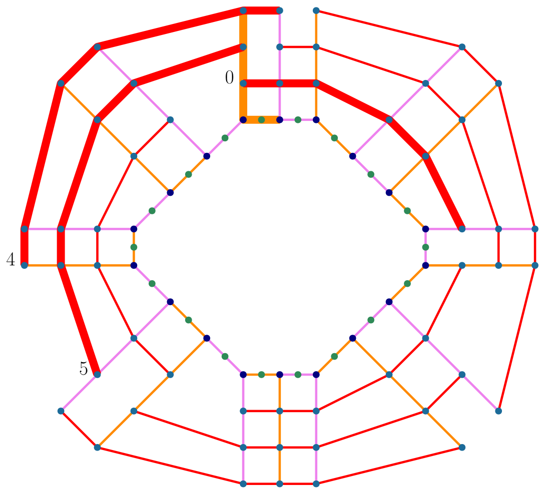


Figure 9: Construction of the forward set of E_1 in $G_{6,3}$

Similarly we define the *backwards set* $B(E_\ell)$ that contains the "distance to the ending points" of the intersecting edges of edge E_l. Let us first rename all edges in $\mathcal{E}_{cyc}$ so $E^*_{i,j} = E_{i+k-1,j}$ for all i with $i+k-1 \in [tk]$ and $(i,j) \in P$. Then define $\mathcal{E}^*_{cyc} := \{E^*_{i,j} : E_{i.j} \in \mathcal{E}_{cyc}\}$. Note that $\mathcal{E}_{cyc}$ and $\mathcal{E}^*_{cyc}$ contain the same elements we only renamed the indices to define the following for each edge $E^*_{i,j} \in \mathcal{E}^*$:

$$I(E^*_{i,j}) := \{E_\ell : \ell \in \{i-k+1, \dots, i\}\}$$

and

$$B(E_\ell) = \{(i-\ell) \mod k : \exists j \text{ with } E_\ell \in I(E^*_{i,j})\}.$$

In the following all indices in the backward and forward set are considered $\mod k$. Now let $E_\ell \in \mathcal{E}_L$ for some $\ell \in [tk]$. If we compare $F(E_\ell)$ and $F(E_{\ell+1})$ we see that for any $x \in F(E_\ell)$ with $x \neq k-1$ there is an $x' \in F(E_{\ell+1})$ with $x' = x+1$. This is due to the fact that $x = (\ell - i)$ for some j with $E_\ell \in I(E_{i,j})$. Because $x \neq k-1$ we know that $E_{\ell+1} \in I(E_{i,j})$ and thus $((l+1)-i) = x+1 \in F(E_{\ell+1})$.

If $x \in F(E_\ell)$ with $x = k-1$ then again there exist an j with $x = (l-i)$ and $E_\ell \in I(E_{i,j})$ but now $E_{\ell+1} \notin I(E_{i,j})$ because $\ell - i = k-1$. Then there must be an j' and an edge $E_{i+k,j'}$ with $E_{\ell+1} \in I(E_{i+k,j'})$ and thus $0 \in F(E_{\ell+1})$. So all in all we know that $F(E_{\ell+1}) = \{(x+1) : x \in F(\ell)\}$ and similarly it can be shown that $B(E_\ell) = \{(x-1) : x \in B(E_{\ell+1})\}$.

Next we want to show a different viewpoint of the possible edges in $\mathcal{E}_{cyc}$ meaning we will look at sets of size k that intersect with k sets in $\mathcal{E}_L$.

Claim: If Φ' is an automorphism on $\mathcal{G}_{k,t}$ with $\Phi'(E_1) = E_j$, then $F(E_1) \in \{F(E_j), B(E_j)\}$.

For proving the claim we consider "*strips*" of vertices. We noticed in the definition of the set $I(E_{i,j})$ that every $E_{i,j} \in \mathcal{E}_{cyc}$ intersects some E_ℓ. This leads to the following definitions. For any edge $E_\ell \in \mathcal{E}_L$ we define the upper part of E_ℓ as

$$U(E_\ell) := E_\ell \setminus \{v_\ell, u_\ell, v_{\ell+1}\}.$$

Let then

$$Q := \{(x_0, \dots, x_{k-1}) : x_0 \in U(E_\ell), \dots, x_{k-1} \in U(E_{\ell+k-1}) \text{ for } \ell \in [tk]\}$$

and

$$Strip(\alpha, \beta) := \{x = (x_0, \dots, x_\beta, \dots, x_{k-1}) : x_\beta \in E_\alpha \in \mathcal{E}_L, x \in Q\}$$

Note that, according to our observations above, Q contains all k-tuples which intersect exactly k consecutive edges in $\mathcal{E}_L$. Thus $Strip(\alpha, \beta)$ contains all $(k-3)^k$ different k-tuples $q \in Q$ that start in $E_{\alpha-\beta}$.

Let Φ' be any automorphism with $\Phi'(E_1) = E_i$ for an edge $E_i \in \mathcal{E}_L$. Assume now $F(E_1) \notin \{F(E_i), B(E_i)\}$. Because $|F(E)| = |F(E')| = |B(E')| = k$ for any two edges $E, E' \in \mathcal{E}_L$ there are $a, b \in F(E_1)$ with $a \notin F(E_i), b \notin B(E_i)$. Therefore $Strip(i, a) \cap E_{cyc} = \emptyset$ as well as $Strip(i, k-b) \cap E_{cyc} = \emptyset$. Thus we know that $E_{a,j}, E_{b,j'} \in \mathcal{E}_{cyc}$ for some $j, j' \in [k-3]$ by $a, b \in F(E_1)$. But we also know that either $\Phi(E_{a,j}) \notin \mathcal{E}_{cyc}$ or $\Phi(E_{b,j'}) \notin \mathcal{E}_{cyc}$. This is a contradiction to Lemma 2.4 (iv). Thus we can conclude that that $F(E_1) = F(E_i)$ or $F(E_1) = B(E_i)$, otherwise Φ would not be a proper automorphism.

Before we consider the two remaining cases $F(E_1) = F(E_i)$ or $F(E_1) = B(E_i)$ let us study some additional properties of the forward and backward sets. By definition, every forward set is a subset of $\{0, 1, \dots, k-1\}$ that contains $k-3$ distinct elements. Thus, the elements of any forward or backward set are also given by its 3 missing elements. So we define for any edge $E_\ell \in \mathcal{E}_L$ the missing elements of its forward and backward set:

$$M_F(E_\ell) = \{0, 1, \dots, k-1\} \setminus F(E_\ell)$$

and

$$M_B(E_\ell) = \{0, 1, \dots, k-1\} \setminus B(E_\ell).$$

First note that by definition $M_F(E_1) = \{1, 2, 3\}$ and thus $M_B(E_1) = \{k-2, k-3, k-4\}$. Then it is clear that for $k = 4$ and $k \geq 6$ the backward and forward set of E_1 differ so $M_F(E_1) \neq M_B(E_1)$.
The missing elements of two consecutive edges $E_{\ell'}, E_{\ell'+1}$ behave like the forward and backward set themselves in the sense that $M_F(E_{\ell+1}) = \{(x+1) : x \in M_F(E_\ell)\}$ and $M_B(E_{\ell+1}) = \{(x-1) : x \in M_B(E_\ell)\}$. Also note that by definition the sets $M_F(E_\ell)$ and $M_B(E_\ell)$ for any edge $E_\ell \in \mathcal{E}_L$ are always of the form $\{x, (x+1), (x+2)\}$ for some $x \in \{0, 1, \dots, k-1\}$. So if there now is an edge $E_\ell \in \mathcal{E}_L$ with $F(E_\ell) = F(E_{\ell'})$ for another edge $E_{\ell'}$ then also $M_F(E_\ell) = M_F(E_{\ell'})$. Thus $\ell' = (\ell + sk) \mod tk$ for some $s \in \{0, 1, \dots, t-1\}$ as ℓ and ℓ' must be a multiple of k edges apart as only then $M_F(E_\ell) = M_F(E_{\ell'})$.

Next we want to make a similar statement comparing edges E_ℓ and $E_{\ell'}$ with $F(E_\ell) = B(E_{\ell'})$. Let $E_\ell \in \mathcal{E}_L$ be an arbitrary edge and let $x \in M_F(E_\ell)$. Then for some j there is no edge $E_{\ell-x,j} \in \mathcal{E}_{cyc}$ with $E_{\ell-x,j} \cap E_\ell \neq \emptyset$. So by definition there is no edge $E_{\ell-x+k-1,j} \in \mathcal{E}^*_{cyc}$, where $\mathcal{E}^*_{cyc}$ is the set of relabeled cyclic edges from before, with $E_{\ell-x+k-1,j} \cap E_\ell \neq \emptyset$ and thus by definition $(\ell - x + k - 1 - \ell) = (x + k - 1) \notin B(E_\ell)$. Therefore $(x + k - 1) \in M_B(E_\ell)$. It follows that $M_B(E_\ell) = \{k - 1 - x : x \in M_F(E_\ell)\}$. By definition of the JN graph $M_F(E_1) = \{1, 2, 3\}$ and thus $M_F(E_\ell) = \{\ell, (\ell+1), (\ell+2)\}$ for every edge $E_\ell \in \mathcal{E}_L$. If now $M_F(E_\ell) = M_B(E_{\ell'})$ then

$$\{\ell, \qquad (\ell + 1),(\ell + 2)\} \tag{2.1}$$
$$=\{(k - 1 - (\ell' + 2)), \qquad (k - 1 - (\ell' + 1)),(k - 1 - \ell')\}. \tag{2.2}$$

Thus we see that $\ell \mod k = (k - 3 - \ell') \mod k$.

Finally we can come back to our two remaining cases $F(E_1) = F(E_\ell)$ and $F(E_1) = B(E_\ell)$. So let us assume Φ is an automorphism with $\Phi(E_1) = E_\ell$ for some edge $E_\ell \in \mathcal{E}_L$. Then Lemma 2.4 (*iii*) implies $\Phi(v_1) \in \{v_\ell, v_{\ell+1}\}$.

case 1: $F(E_1) = F(E_\ell)$:

Let us further assume that $\Phi(v_1) = v_\ell$. Then we know by Lemma 2.4 (*iii*) and (*iv*) that $\Phi(v_i) = v_{i+\ell}$ and $\Phi(u_i) = u_{i+\ell-i}$ for all $i \in [tk]$. This tells us also that $\Phi(E_{i,j}) = (E_{i+\ell-1,j})$ for all edges $E_{i,j} \in \mathcal{E}_{cyc}$, so $\Phi(v_i^j) = v_{i+\ell-1}^j$ for all remaining vertices $v_i^j \in \mathcal{X}_{k,t} \setminus V_0$ where $V_0 := \{v_i, u_i : i \in [tk]\}$. So indeed we have a valid automorphism Φ. If we assume $\Phi(v_1) = v_{\ell+1}$ then Lemma 2.4 (*iii*) and (*iv*) as well as our previous analysis tell us if $F(E_1) \neq B(E_\ell)$ that Φ is no proper automorphism since there would be an edge $E \in \mathcal{E}_{cyc}$ with $E \cap E_1 \neq \emptyset$ and $\Phi(E) \notin \mathcal{E}_{k,t}$.

case 2: $F(E_1) = B(E_\ell)$:

If $\Phi(v_1) = v_\ell$ then Lemma 2.4 (*iii*) and (*iv*) as well as our previous analysis tell us again if $F(E_1) \neq F(E_\ell)$ that Φ is no proper automorphism. If $F(E_1) = F(E_j)$ we refer to the previous case. Thus let us assume that $\Phi(v_1) = v_{\ell+1}$. Then we know by (*iii*) and (*iv*) that $v_i = v_{\ell-i+1}$ and $u_i = u_{\ell-i+1}$ for all $i \in [tk]$. We also know by analyzing the backward sets that $\Phi(E_{i,j}) = E_{\ell-i+1,k-2-j}$ for all $E_{i,j} \in \mathcal{E}_{cyc}$. Thus $\Phi(v_i^j) = v_{\ell-i+1}^{k-2-j}$ for all remaining vertices. And again Φ is a proper automorphism.

We have already have seen, because $k \neq 3$ and $k \neq 5$, that $F(E_1) \neq B(E_1)$ so the only automorphism that leaves E_1 invariant is given by case 1 and thus the identity.

□

It is easy to check that in the case $F(E_1) = B(E_j)$ the given automorphism is also an involution.

In their Lemma 8.2 Jiang and Nešetřil [19] claim, that *the only automorphism of $\mathcal{G}_{k,t}$ which leaves the set E_1 invariant (i.e. for each vertex $v \in E_1, \Phi(v) \in E_1$) is the identity.*

However their provided proof does not extend to the cases $k = 3$ and $k = 5$. Indeed for both cases we give the following explicit examples in Figure 10 and Figure 11 where it fails.

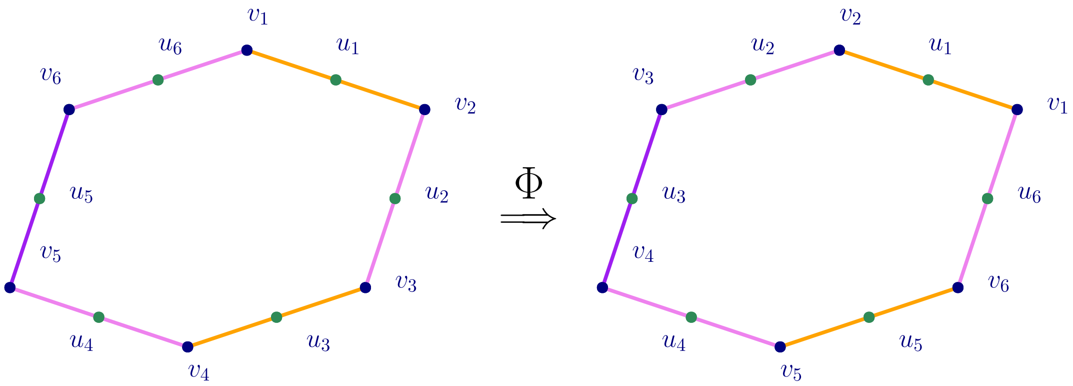


Figure 10: A non-trivial automorphism on $\mathcal{G}_{3,2}$ that leaves $E_1 = \{v_1, u_1, v_2\}$ invariant.

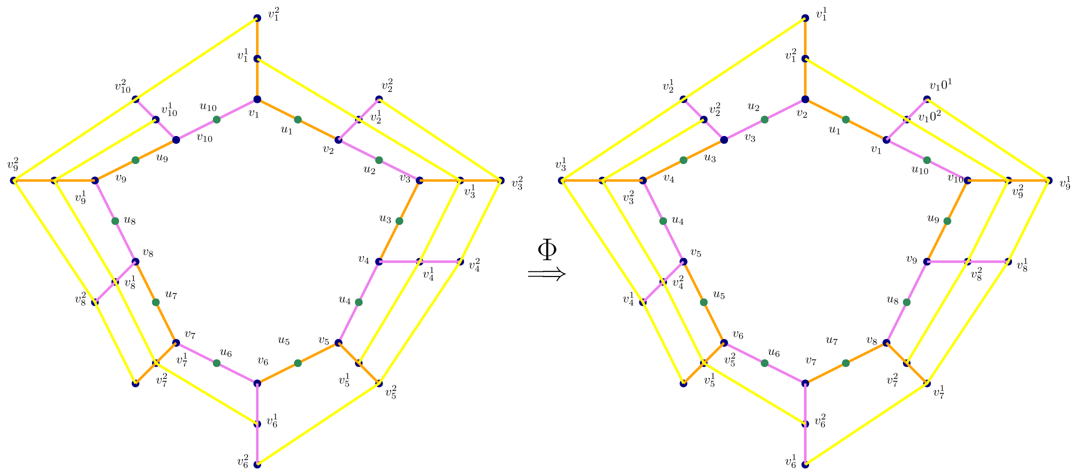


Figure 11: A non-trivial automorphism on $\mathcal{G}_{5,2}$ that leaves $E_1 = \{v_1, v_1^1, v_1^2, u_1, v_2\}$ invariant.

We can even state the following Lemma.

Lemma 2.6. Let $k \in \{3, 5\}$ and $t \geq 2$, $t \in \mathbb{N}$ and let $\mathcal{G}_{k,t}$ be the JN-graph. Then there are two distinct automorphisms that leave the set E_1 invariant, so in particular one automorphism that is *not* the identity.

Proof. First we consider $k = 5$. In this case, if we compare the backward and forward set of E_1 we see that $F(E_1) = \{0, 4\}$ but also that $B(E_1) = \{0, 4\}$. Thus case 1 of Lemma 2.5 gives us an automorphism that leaves E_1 invariant, i.e. the identity, but case 2 also gives an involution that leaves E_1 invariant. Similarly for $k = 3$, all backward and forward sets off all edges are equal as they all are empty. Thus again there is a non-trivial automorphism that leaves E_1 invariant. $\square$

Later this will be a problem because our main Construction 2.10 would produce symmetric 3 and 5-graphs. Thus we will have to consider those cases separately and give another construction that only allows one automorphism that leaves the set E_1 invariant, the identity.

Jiang and Nešetřil [19] mention the following statement without providing a proof. Here we give a proof. Recall that in the definition for minimal asymmetric k-graphs we required that on every non-trivial induced subgraph on at least two vertices of the considered graph should be a non-trivial automorphism. The next statement is even stronger than that because we show the property not only for all induced subgraphs but for all subgraphs of the JN-graph.

Lemma 2.7. (Jiang Nešetřil [19])
For $t, k \in \mathbb{N}$ with $t \geq 2, k \geq 3$ let $\mathcal{G}_{k,t}$ be the JN-graph and $\mathcal{G}'_{k,t}$ be an arbitrary subgraph of $\mathcal{G}_{k,t}$ with at least two vertices. Then $\mathcal{G}'_{k,t}$ is symmetric.

Proof. Let $\mathcal{G}'_{k,t} = (\mathcal{V}', \mathcal{E}')$ be a subgraph with at least two vertices of the JN-graph $\mathcal{G}_{k,t}$. If there are vertices $u, v \in \mathcal{V}'$ with $deg_{\mathcal{G}'_{k,t}}(u) = deg_{\mathcal{G}'_{k,t}}(v) = 0$ then $\mathcal{G}'_{k,t}$ is symmetric. So let us assume there is at most on vertex $v \in \mathcal{V}'$ with $deg_{\mathcal{G}'_{k,t}}(v) = 0$. Due to $\mathcal{G}'_{k,t}$ having at least two vertices we know that $|\mathcal{E}'| \geq 1$. Now if there are two vertices $u, v \in \mathcal{V}$ with $deg_{\mathcal{G}'_{k,t}}(u) = deg_{\mathcal{G}'_{k,t}}(v) = 1$ and there is an edge $E \in \mathcal{E}'$ with $u, v \in E$ then again there is a non-trivial autmorphism Φ with $\Phi(u) = v, \Phi(v) = u$ and $\Phi(w) = w$ for all other vertices $w \in \mathcal{V}'$. Thus, let us assume that for every edge $E \in \mathcal{E}'$ there is only one vertex $v \in E$ with $deg(v) = 1$.

As we noted earlier $\mathcal{G}'_{k,t}$ contains at least one edge E, so there are two cases either $E \in \mathcal{E}_{cyc}$ or $E \in \mathcal{E}_L$.

case 1: $E \in \mathcal{E}_L$.
Suppose that $E = E_i$ for an $i \in [tk]$. We know that $deg_{\mathcal{G}'_{k,t}}(u_i) = 1$, thus $deg_{\mathcal{G}'_{k,t}}(v_i) \neq 1 \neq deg_{\mathcal{G}'_{k,t}}(v_{i+1})$. This directly tells us that $E_{i-1}, E_{i+1} \in \mathcal{E}'$. Therefore, we can conclude inductively that $\mathcal{E}_L \subseteq \mathcal{E}'$. This also yields that $\mathcal{V}' = \mathcal{X}_{k,t}$ where $\mathcal{X}_{k,t}$ is the vertex set of $\mathcal{G}_{k,t}$. But then we also know that $\mathcal{E}_{cyc} \subseteq \mathcal{E}'$ because every vertex v_i^j for $i \in [tk], j \in [k-3]$ must have a degree of 2. But then $\mathcal{G}'_{k,t} = \mathcal{G}_{k,t}$.

case 2: $E \in \mathcal{E}_{cyc}$.
There must be at least one vertex $v \in E$ with $deg_{\mathcal{G}'_{k,t}}(v) = 2$. Thus, there is an edge $E' \in \mathcal{E}'$ with $E' \in \mathcal{E}_L$. Therefore, this case can just be reduced to the previous one.

We conclude that $\mathcal{G}'_{k,t}$ is in both cases just our original JN-graph $\mathcal{G}_{k,t}$ and we already saw in the proof of Lemma 2.5 that $\mathcal{G}_{k,t}$ is not asymmetric. □

Lemma 2.8. (Jiang Nešetřil [19])
For $t, k \in \mathbb{N}$ with $t \geq 2, k \geq 4$ let $\mathcal{G}_{k,t}$ be the JN-graph and let $\mathcal{G}'_{k,t}$ be a non-trivial subgraph of $\mathcal{G}_{k,t}$ with $E_1 \in E(\mathcal{G}')$. Then there is a non-trivial automorphism Φ on $\mathcal{G}'$ with $\Phi(u_1) = u_1$ that leaves E_1 invariant.

Proof. Let $\mathcal{G}_{k,t}$ be the JN-graph for $k, t \in \mathbb{N}$ with $k \geq 4$, $t \geq 2$ and let $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$ be a non-trivial subgraph of $\mathcal{G}_{k,t}$ with $E_1 \in \mathcal{E}'$. We now construct a non-trivial automorphism on $\mathcal{G}'$ that leaves E_1 and u_1 invariant. Let s be the maximal index such that $E_1, E_2, \ldots, E_s \in \mathcal{E}'$..

case $s = 1$:
Let s' be the index such that $E_{s+1}, \ldots, E_{s'-1} \notin \mathcal{E}'$ and $E_{s'} \in \mathcal{E}'$. Now if $s' = 1$ then $deg(v_1) = deg(v_2) = 1$. Thus define Φ with $\Phi(v_1) = v_2$, $\Phi(v_2) = v_1$ and $\Phi(w) = w$ for all other $w \in \mathcal{V}'$. Otherwise, so for $s' \neq 1$, we know that $deg(v_{s'}) = deg(u_{s'}) = 1$ thus define $\Phi(v_{s'}) = u_{s'}$, $\Phi(u'_s) = v_{s'}$ and $\Phi(w) = w$ for all other $w \in \mathcal{V}$.

case $s \in \{2, \ldots, tk - 1\}$:
Here it is clear that $deg(v_{s+1}) = deg(u_s) = 1$ hence define $\Phi(v_{s+1}) = u_s$, $\Phi(u_s) = v_{s+1}$ and $\Phi(w) = w$ for all other vertices $w \in \mathcal{V}'$.

case $s = tk$:
In this case we know that $\mathcal{V}' = \mathcal{X}_{k,t}$. We also know that G' is a proper subgraph of $\mathcal{G}_{k,t}$ and thus there is an edge $E_{i,j} \in \mathcal{E}_{cyc}$ with $E_{i,j} \notin E(\mathcal{G}')$. Now choose an $i' \neq 1$ such that $E_{i'} \cap E_{i,j} \neq \emptyset$. This is possible because all our edges have size $k \geq 4$. Then $deg(v^j_{i'}) = deg(u_{i'}) = 1$ and therefore define $\Phi(v^j_{i'}) = u_{i'}$, $\Phi(u_{i'}) = (v^j_{i'})$ as well as $\Phi(w) = w$ for all other vertices in $V(\mathcal{G}')$.

□

Before we continue we will sketch how Jiang and Nešetřil showed Theorem 2.1. They extended the JN-graph $\mathcal{G}_{k,t}$ to the following k-graph $\mathcal{G}'_{k,t} = (V(\mathcal{G}'_{k,t}), E(\mathcal{G}'_{k,t}))$ with $V(\mathcal{G}'_{k,t}) := \mathcal{X}_{k,t} \cup x_0$ for some vertex x_0 disjoint from $\mathcal{X}_{k,t}$ and $E(\mathcal{G}'_{k,t}) := \mathcal{E}_{k,t} \cup \{E_0\}$ for $E_0 = \{x_0, v_1, v_2, u_1, v^1_1 \ldots v^{k-3}_1\}$. We give in Figure 12 the construction with parameter $t = 3$, $k = 6$. They then proceeded to show that $\mathcal{G}'_{k,t}$ is minimal asymmetric by using Lemma 2.5. Note that the edge E_0 prevents *reflections*.

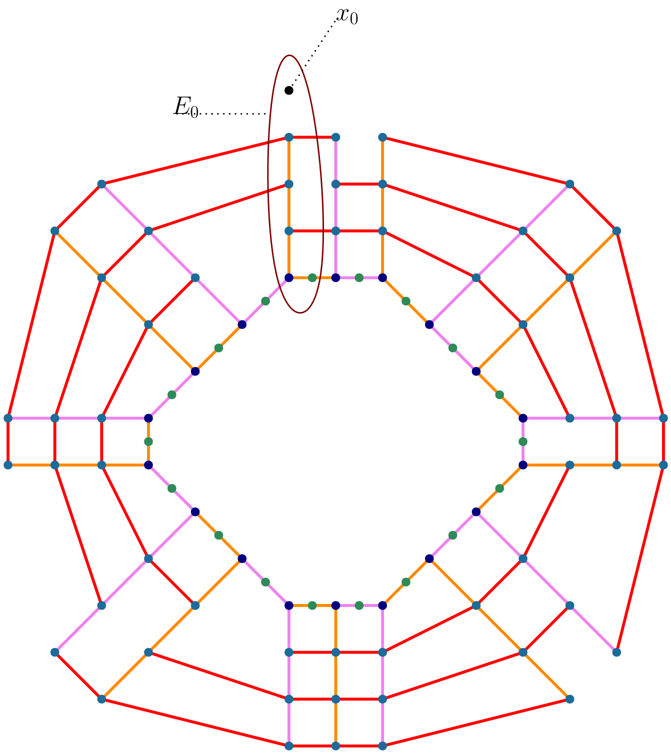


Figure 12: The k-graph $\mathcal{G}'_{6,3}$

2.2 Proof of the main Theorem 2.2

Lemma 2.9. (i) For $t, k \in \mathbb{N}$ with $t \geq 2, k \geq 3$ let $\mathcal{G}_{k,t}$ be the JN-graph and let $\mathcal{G}'$ be a non-trivial subgraph of $\mathcal{G}_{k,t}$ with $E_1, E_2 \in E(\mathcal{G}')$. Then there is an automorphism Φ on $\mathcal{G}'$ with $\Phi(u_1) = u_1, \Phi(u_2) = u_2$ that leaves E_1 and E_2 invariant.

(ii) Let $\mathcal{G}_{k,t}$ be the JN-graph for $k, t \in \mathbb{N}$ with $k = 3$, $t \geq 2$ and let $\mathcal{G}'$ be a non-trivial subgraph of $\mathcal{G}_{k,t}$ with $E_1, E_2, \ldots, E_r \in E(\mathcal{G}')$ and $r \in \{3, 4\}$. Then there is an automorphism Φ on $\mathcal{G}'$ with $\Phi(u_1) = u_1, \ldots \Phi(u_r) = u_r$ that leaves $E_1, \ldots, E_r$ invariant.

Proof. (i) Here we consider a non-trivial subgraph $\mathcal{G}'$ of $\mathcal{G}_{k,t}$ with $E_1, E_2 \in E(\mathcal{G}')$. Let s be, as above in Lemma 2.8, the maximal index such that $E_1, E_2, \ldots, E_s \in E(\mathcal{G}')$. The cases $s = 3, \ldots, tk - 1$ and $s = tk$ work similar as the second case of the proof of Lemma 2.8. So let $s = 2$ and s' be the index such that $E_3, \ldots, E_{s'-1} \notin E(\mathcal{G}')$ and $E_{s'} \in E(\mathcal{G}')$. The subcase $s' \neq 1$ then works as above. Let us now consider the case $s' = 1$. Note that because $k \geq 4$ we know $v_3^1, v_4^1 \in E_{1,1}$. Then if $E_{1,1} \in E(\mathcal{G}')$ we also know that $v_3^1, v_4^1 \in V(\mathcal{G}')$ with $deg(v_3^1) = deg(v_4^1) = 1$. Thus we define $\Phi(v_3^1) = v_4^1$, $\Phi(v_4^1) = v_3^1$ and $\Phi(w) = w$ for all other $w \in V(\mathcal{G}^*)$. Otherwise $E_{1,1} \notin E(\mathcal{G}')$, then define $\Phi(v_1^1) = v_1$, $\Phi(v_1) = v_1^1$ and $\Phi(w) = w$ for all other $w \in V(\mathcal{G}')$.

(ii) Let s again be the maximal index such that $E_1, E_2, \ldots, E_s \in E(\mathcal{G})'$. Note that

$s \neq 3t$ as then $\mathcal{G}' = \mathcal{G}_{3,t}$. Thus we are left with cases $s = r$ and $s \in \{r+1, \dots, 3t-1\}$ and those can be solved just as the second case of the proof of Lemma 2.8. $\square$

We now give the construction used for proving Theorem 2.2.

Construction 2.10. Let $k, t_1, t_2, \dots, t_{k-1} \in \mathbb{N}$ such that $k \geq 3$ and $2 \leq t_1 < t_2 < \dots < t_{k-1}$. For $\ell \in [k-1]$ let $\mathcal{G}^\ell := (\mathcal{X}^\ell, \mathcal{E}^\ell)$ be a k-graph isomorphic to $\mathcal{G}_{k,t_\ell}$ from Construction 2.3 such that $X^{\ell'}s$ are pairwise vertex-disjoint. We refer to the edges in $\mathcal{G}^\ell$ isomorphic to E_i and $E_{i,j}$ in $\mathcal{G}_{k,t_\ell}$ as E_i^ℓ and $E_{i,j}^\ell$ respectively. Let x_0 be an additional vertex disjoint from $\bigcup_{\ell \in [k-1]} \mathcal{X}^\ell$ and let $E_0 := \{x_0, u_1^1, u_1^2, \dots, u_1^{k-1}\}$. Then we define our final k-graph $\mathcal{G} = \mathcal{G}_{t_1,t_2,\dots,t_{k-1}} := (\mathcal{V}, \mathcal{E})$ with $\mathcal{V} := \bigcup\limits_{\ell \in [k-1]} \mathcal{X}^\ell \cup \{x_0\}$ and $\mathcal{E} := \bigcup\limits_{\ell \in [k-1]} \mathcal{E}^l \cup E_0$. We give a sketch of this construction in Figure 13.

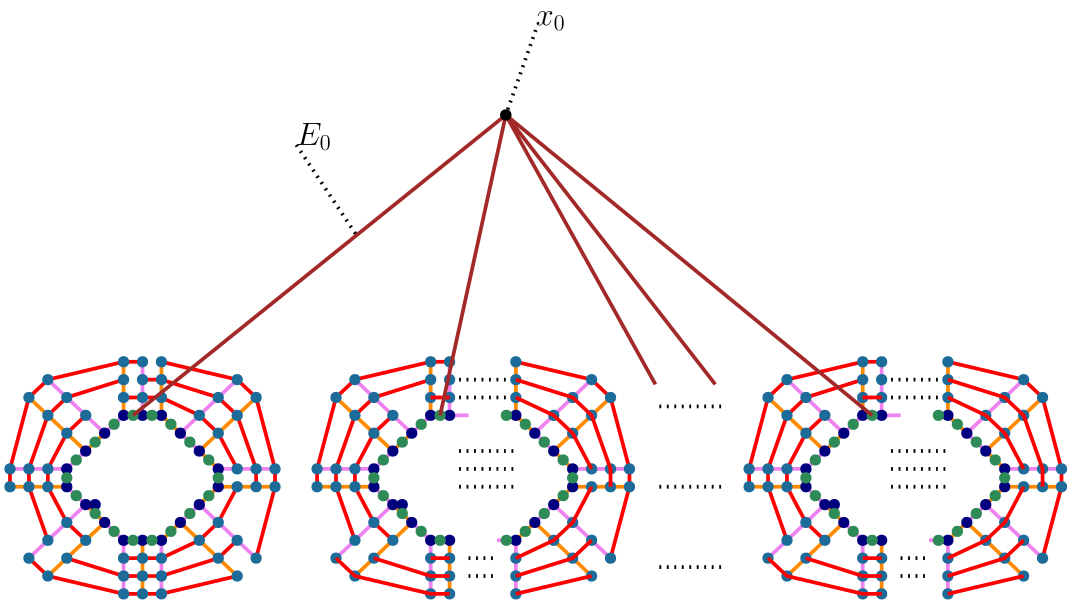


Figure 13: Sketch of the graph $G_{t_1,\dots,t_{k-1}}$

We want to use similar techniques as Jiang and Nešetřil in [19] to show that the k-graph $\mathcal{G}$ in Construction 2.10 is minimal asymmetric. For that we still have to solve some problems. We have seen that for the cases $k = 3$ and $k = 5$ Lemma 2.5 does not hold. Thus we have to do some extra construction for those. Secondly ,in the general cases, so $k = 4$ and $k \geq 6$, we want to have an easy argument why an automorphism on $\mathcal{G}$ leaves the underlying JN-graphs invariant. Therefore we introduce the following definition of a *path* in a hypergraph.

Let G be a hypergraph and $u, v \in V(G)$ with $u \neq v$. We call a set of edges $\mathcal{P}$ an $u-v$*-path* if $\mathcal{P} = \{e\}$ with $u, v \in e$ or $\mathcal{P} = \{e_1, \dots, e_i\}$ for an positive integer i such that there are edges $e_j, j \in [i]$ with $u \in e_1$, $v \in e_i$, $e_j \cap e_{j+1} \neq \emptyset$ for $j \in [i-1]$, $e_j \neq e_l$ for $j, l \in [i], j \neq l$. We say there is a path between a vertex v and an edge e if there is a $v-v'$-path for a vertex $v' \in e$. Two paths $\mathcal{P}_1, \mathcal{P}_2$ are *edge-disjoint* if $\mathcal{P}_1 \cap \mathcal{P}_2 = \emptyset$.

Let $G = (V(G), E(G))$ be a hypergraph and $u, v \in V(G)$. We say that u and v are *connected* if there is an $u - v$-Path in G. Let $E \in E(G)$ be an edge, we say u and E are *connected* if there is a vertex $v' \in E$ such that u and v' are connected.

Lemma 2.11. For $t, k \in \mathbb{N}, k \geq 3$ and $t \geq 2$ let $E \in \mathcal{E}_{k,t}$ be an edge of the JN-graph $\mathcal{G}_{k,t} = (\mathcal{X}_{k,t}, \mathcal{E}_{k,t})$. Then for any two vertices $u, v \in E$ with $u, v \notin \{u_1, u_2, ..., u_{tk}\}$ there are two edge disjoint $u - v$-paths.

Proof. Case 1: $E = E_i \in \mathcal{E}_L$ for some $i \in [tk]$:

We consider the k-graph $G'_{k,t} := (\mathcal{X}_{k,t}, \mathcal{E} \setminus \{E_i\})$ and show that u, v are still connected. If we look at u we see that either $u \in E_{i+1}$ or $u \in E_{i-1}$ or there is an edge $E^* \in \mathcal{E}_{cyc}$ with $u \in E^*$ and $E^* \cap (E_{i+1} \cup E_{i-1}) \neq \emptyset$. So without loss of generality we can say that u is connected to E_{i+1}. Otherwise it is connected to E_{i-1} and a similar argument can be applied. Because $v_{j+1} \in E_j$ and $v_{j+1} \in E_{j+1}$ for any $j \in [tk], j \neq i$ there is a path between u and E_{i-1}. With the same argument we can show that v is connected to E_{i-1} or to E_{i+1}. These paths can now be combined by considering their earliest intersecting Edge. Note that they always intersect in E_{i-1} or in E_{i+1} and thus there is a $u - v$ path in $G'_{k,t}$

Case 2: $E = E_{i,j} \in \mathcal{E}_{cyc}$ for some $j \in [k-3]$, $i = j + sk$; $s \in \{0, 1, \ldots, t-1\}$.

We now consider the k-graph $G'_{k,t} := (\mathcal{X}_{k,t}, \mathcal{E}_{k,t} \setminus \{E_{i,j}\})$. For u, v there are distinct vertices v_i and $v_{i'}$ and edges E', $E'' \in \mathcal{E}_L$ with $u, v_i \in E'$ and $v, v_{i'} \in E''$. Similarly to Case 1, v_i and $v_{i'}$ are still connected and thus there is a $u - v$-path in $G'_{k,t}$.
So in both cases there exists another path additionally to $\{E\}$, connecting the two vertices u, v.

□

Lemma 2.12. Let G be a k-graph and Φ be an automorphism of G. Then the number of edge-disjoint paths between any two vertices of G is invariant under Φ.

Proof. Take any two vertices $u, v \in V(G)$. If they are not connected in G they can't be connected in $\Phi(G)$. Assume $\Phi(u)$ and $\Phi(v)$ are connected, then there is a $\Phi(u) - \Phi(v)$ path $\mathcal{P} := \{E_1, E_2, \ldots, E_l\}$. Because there is no $u-v$-path in G we find an Edge $E \in \mathcal{P}$ such that there is no edge $E' \in E(G)$ with $\Phi(E') = E$. Let $\mathcal{P}(u, v) = \{\mathcal{P}_1, \ldots, \mathcal{P}_c$ a set of edge disjoint $u - v$ -paths in G. Now take any $P \in P(u, v)$ and see that $\Phi(P)$ is a $\Phi(u) - \Phi(v)$-path. Furthermore, any two paths $P_i, P_j \in P(u, v)$ are by definition edge-disjoint and thus $\Phi(P_i), \Phi(P_j)$ again must be edge-disjoint. There also can not be an extra path in $\Phi(G)$ which can be seen by reversing the preceding argument. □

Finally we can prove Theorem 2.2 by showing that our Construction 2.10 provides a minimal asymmetric k-graph which is linear and has a maximum degree of 2.

Proof. We first consider the general case.

Case 1: $k = 4$ or $k \geq 6$.

We will show that $\mathcal{G}_{t_1,t_2,\ldots,t_k-1}$ is minimal asymmetric for every $2 \leq t_1, \ldots, t_{k-1}$. We begin by showing that $\mathcal{G}_{t_1,t_2,\ldots,t_k-1}$ is asymmetric and afterwards that each induced subgraph is symmetric.

Let Φ be an automorphism on $\mathcal{G} := \mathcal{G}_{t_1,t_2,\ldots,t_k-1}$. First we show that E_0 is invariant under Φ, i.e. for every $v \in E_0$, $\Phi(v) \in E_0$. Assume that there is an edge $E \neq E_0$ with $\Phi(E_0) = E$. Note that E is an edge of some JN-graph $\mathcal{G}^l$ and also note that every pair of distinct vertices $u, v \in E_0$ with $u \neq v$ each $u-v$-Path contains E_0. Thus the maximal number of edge disjoint $u - v$ Paths us 1. By Lemma 2.11 and because $k \geq 3$ there are two vertices $u, v \in E_0$ such that there are two edge disjoint $\Phi(u) - \Phi(v)$-paths in $\mathcal{G}^l$ and thus in $\mathcal{G}_{t_1,\ldots,t_k}$. Therefore, by Lemma 2.12, we have a contradiction and E_0 is indeed invariant under Φ.

Because x_0 is the only vertex in E_0 with degree 1 we know that $\Phi(x_0) = x_0$. Assume that $\Phi(u_{1,i}) = u_{1,j}$ for some $i \neq j \in \mathbb{N}$. Because $E_{1,i}$ is the only edge, other than E_0, that contains $u_{1,i}$ we know that for every vertex $w \in E_{1,i}$ also $\Phi(w) \in \mathcal{G}_{k,i}$. If there now is an edge $E \in \mathcal{E}(\mathcal{G}_{k,i})$ with $\Phi(E) \notin \mathcal{E}(\mathcal{G}_{k,j})$ then again there is a vertex $w' \in E$ such that there are two edge-disjoint $w-w'$-paths for a vertex $w \in E_{1,i}$. But there are no longer two edge disjoint $\Phi(w)-\Phi(w')$-paths. Thus by Lemma 2.11 and Lemma 2.12 $\Phi(\mathcal{G}_{k,t_i}) = \mathcal{G}_{k,t_j}$. But this is a contradiction because $t_i \neq t_j$ and thus $|V(G_{k,t_i})| \neq |V(G_{k,t_j})|$. So $\Phi(u_{1,i}) = u_{1,i}$ for every $i \in [k-1]$.

Then we know that $E_{1,i}$ is also invariant under Φ for every $i \in [k-1]$. Therefore Lemma 2.5 yields that Φ restricted to $\mathcal{G}^{t_i}$ is the identity so Φ is the identity. We conclude the proof that $\mathcal{G}$ is asymmetric.

Now in order to show that $\mathcal{G}$ is even minimal asymmetric let $\mathcal{G}'$ be an arbitrary proper induced subgraph on at least two vertices of $\mathcal{G}$.

Case 1(a): $E_0 \notin E(\mathcal{G}')$.

Assume there are connected vertices in $\mathcal{G}'$. If not then all vertices have degree 0, thus there would be a non-trivial automorphism. Then these connected vertices are in a subgraph on at least two vertices of some $\mathcal{G}_{k,t_j}$. Therefore Lemma 2.7 yields a non-trivial automorphism $\mathcal{G}'$ restricted to $\mathcal{G}_{k,t_j}$. Note here that the statement in Lemma 2.7 is even stronger than needed because it covers all subgraphs not only the induced ones that we need. The given non-trivial automorphism Φ' on $\mathcal{G}_{k,t_j}$ can easily be extended to to the whole subgraph $\mathcal{G}'$ defining $\Phi'(w) = w$ for all other vertices $w \in V(\mathcal{G})$.

Case 1(b): $E_0 \in E(\mathcal{G}')$.

Then there is an $\ell \in [tk]$ such that the vertex set of $\mathcal{G}_{k,t_l}$ is not fully contained in our subgraph, so $V(\mathcal{G}_{k,t_l}) \nsubseteq V(\mathcal{G}')$. Then there is at least one vertex $w \in V(\mathcal{G}_{k,t_l})$ with $w \notin V(\mathcal{G}')$. Thus Lemma 2.8 yields an automorphism Φ' on $\mathcal{G}'$ restricted to $V(\mathcal{G}_{k,t_l})$ with $\Phi'(u_1^l) = u_1^l$ that can easily be extended to the rest of $\mathcal{G}'$ by defining $\Phi'(w) = w$ for all other vertices $w \in V(\mathcal{G}')$.

Thus $\mathcal{G}$ is indeed a minimal asymmetric k-graph as claimed. It is easy to check that $\mathcal{G}$ is linear and has maximum degree 2.

Case 2: $k = 5$.

This case is similar to the general case. We only have to modify the construction a little to prevent the reflection automorphism on the underlying JN-graphs. So let $\mathcal{G}_{5,t}$ and $\mathcal{G}_{5,t'}$ be two JN-graphs with $t, t' \in N$, $t, t \geq 2$ and $t \neq t'$. To indicate to which underlying k-graph an object belongs we denote it with an apostrophe for example u_1 if we talk about the vertex $u_1 \in V(\mathcal{G}_{5,t})$ and u_1' if we talk about $u_1 \in V(\mathcal{G}_{5,t'})$. Let x_0 be an extra vertex disjoint from $V(\mathcal{G}_{5,t}) \cup V(\mathcal{G}_{5,t'})$ and let $E_0 := \{x_0, u_1, u_2, u_1', u_2'\}$. Then define $\mathcal{F} = (V(\mathcal{F}), E(\mathcal{F}))$ with $V(\mathcal{F}) = V(\mathcal{G}_{5,t}) \cup V(\mathcal{G}_{5,t'}) \cup \{x_0\}$ and $E(\mathcal{F}) = E(\mathcal{G}_{5,t}) \cup E(\mathcal{G}_{5,t'}) \cup \{E_0\}$. A sketch of this construction can be seen in Figure 14.

Similarly to case 1 we see that E_0 must be invariant under any automorphism Φ on $\mathcal{F}$. Then it also follows that $\{u_1(t), u_2(t)\}$ and $\{u_1(t'), u_2(t')\}$ are invariant under Φ because $t \neq t'$. Thus there can not be a reflection automorphism on $\mathcal{F}$. The rest of the analysis is similar to the general case by considering Lemma 2.8.

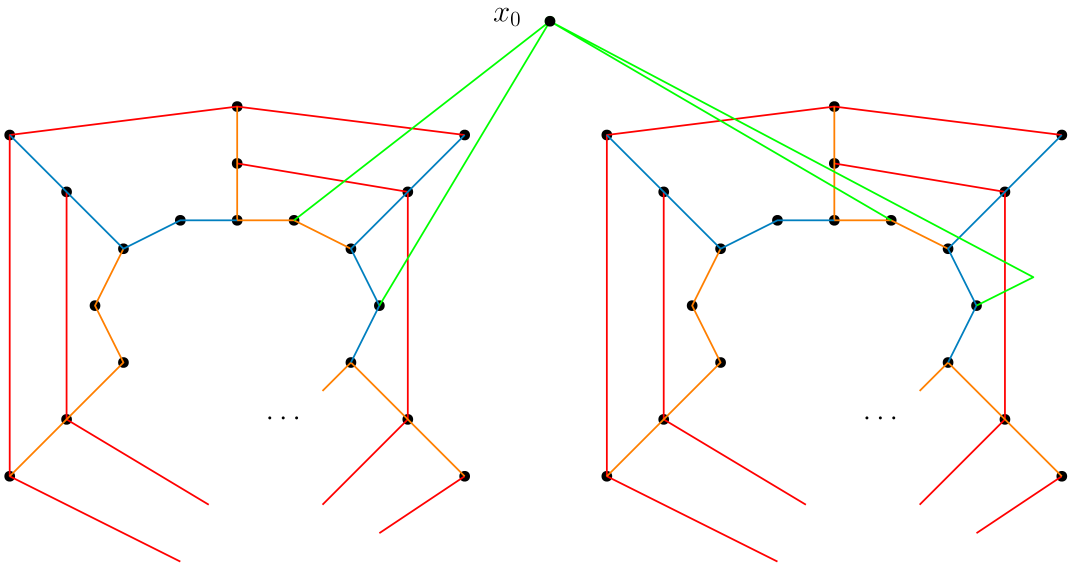


Figure 14: Sketch of the graph $\mathcal{F}$

case 3: $k = 3$.

Let $\mathcal{G}_{3,t}$ and $\mathcal{G}_{3,t'}$ be two disjoint JN-graphs with $t, t' \in \mathbb{N}$, $t, t' \geq 2$ and $t \neq t'$. As before in case 2 to indicate to which underlying k-graph an object belongs we denote it with an apostrophe. We then introduce new vertices x_0, y_0, z_0 disjoint from those of $\mathcal{G}_{3,t}, \mathcal{G}_{3,t'}$ and 3 new edges E_x, E_y, E_z with $E_x := \{x_0, u_1, u_2\}$, $E_y := \{y_0, u_1', u_2'\}$, $E_z := \{z_0, u_3, u_3'\}$. With this we define the 3-graph $\mathcal{H}_{t,t'} = (V(\mathcal{H}_{t,t'}), E(\mathcal{H}_{t,t'}))$ where $V(\mathcal{H}_{t,t'}) := V(\mathcal{G}_{3,t}) \cup V(\mathcal{G}_{3,t'}) \cup \{x_0, y_0, z_0\}$ and $E(\mathcal{H}_{t,t'}) := E(\mathcal{G}_{3,t}) \cup E(\mathcal{G}_{3,t'}) \cup \{E_x, E_y, E_z\}$. A sketch of this construction can be seen in Figure 14.

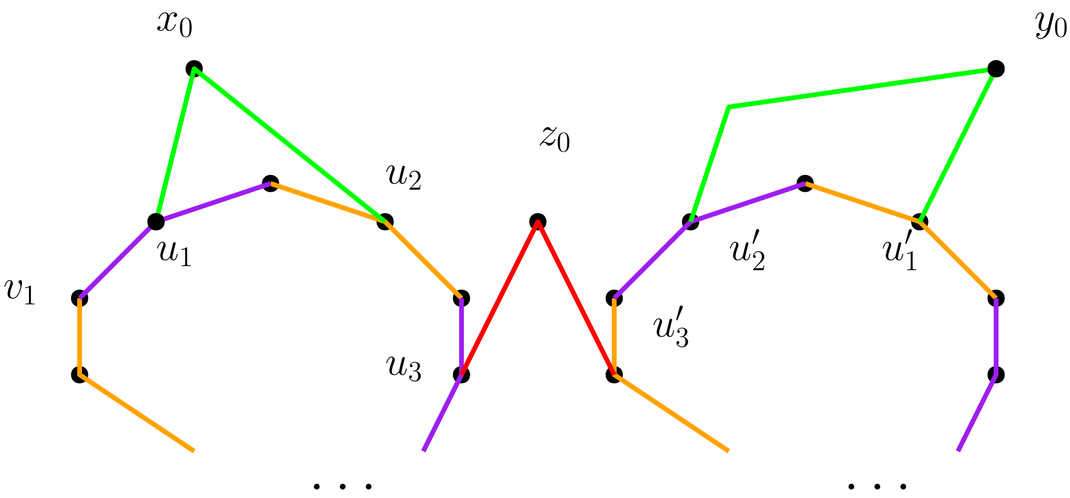


Figure 15: Sketch of the graph $\mathcal{H}_{t,t'}$

Now we show that $\mathcal{H}_{t,t'}$ is asymmetric. So let Φ be an arbitrary automorphism on $\mathcal{H}_{t,t'}$. Assume $\Phi(z_0) \neq z_0$, then there is an edge $E \neq E_z$ with $E \in E(\mathcal{H}_{t,t'})$ with $\Phi(E) = E_z$. By connectivity reason, namely Lemma 2.12 we are already at a contradiction. Thus $\Phi(z_0) = z_0$. Then it also follows that $V(\mathcal{G}_{3,t}) \cup \{x_0\}$ and $V(\mathcal{G}_{3,t}) \cup \{y_0\}$ are invariant under Φ. Therefore $\Phi(u_3) = u_3$ and $\Phi(u'_3) = u'_3$. Note that E_1, E'_1, E_2, E'_2 are the only other edges besides E_z in which every vertex has a degree of 2, thus $E_1 \cup E_2$ and $E'_1 \cup E'_2$ are invariant under Φ. It immediately follows that $\Phi(v_1) = v_1$, $\Phi(v_2) = v_2$ and $\Phi(v_3) = v_3$ as well as the equivalent statements for v'_1, v'_2, v'_3. Thus there can not be a reflection mapping and also no cyclic mapping according to Lemma 2.4 on either of the subgraphs induced by $V(G_{3,t})$ and $V(G_{3,t'})$. Therefore Φ must be identity.

Next we show that every non-trivial induced subgraph H^* of $\mathcal{H}_{t,t'}$ on at least two vertices has non-trivial automorphism. Assume $E_z \notin E(H^*)$ then we either need to find an automorphism on a subgraph of an JN-graph or we need to find an automorphism on some disjoint vertices. The former is given by Lemma 2.8 and the latter is trivial. So let $E_z \in E(H^*)$, then there are two cases.

Case 3(a): $E_x, E_y \notin E(H^*)$:
Assume that there is another vertex w, other then u_3, u'_3, with $w \in V(G_{3,t}) \cup V(G_{3,t'})$ and also $w \in V(H^*)$. If there is no such w then there is a non-trivial automorphism Φ' with $\Phi'(u_3) = u'_3$. Without loss of generality we say $w \in V(G_{3,t})$. Now if $E_3 \notin E(H^*)$ then there is a non-trivial automorphism Φ' with $\Phi'(u_3) = z_0$. Otherwise $E_3 \in E(H^*)$, but then there is a non-trivial automorphism that leaves u_3 invariant by Lemma 2.8.

Case 3(b): $E_x \in E(H^*)$ or $E_y \in E(H^*)$.

We can again assume that $E_3 \in E(H^*)$. Then Lemma 2.8 yields the required non-trivial automorphism.

Therefore $\mathcal{H}_{t,t'}$ is indeed a minimal asymmetric 3-graph. It is easy to check that it is linear and has a maximum degree of 2.

It should be mentioned that there are infinitely many possibilities to choose indices $t_1, \ldots, t_{k-1}$ that suffice to the conditions in Construction 2.10. There are also infinitely many ways to choose indices t, t' that suffice to the constructions presented in case 2 and case 3. Thus indeed we have shown that for every $k \in \mathbb{N}$ with ≥ 3 there are infinitely many linear asymmetric k-graphs with maximum degree 2.

$\square$

3 Regular Asymmetric k-graphs

In this section we want to study asymmetry and minimal asymmetry of regular hypergraphs. We often look at sets of vertices with distinct degrees if we want to prove the asymmetry of some k-graph G. So it might be reasonable to assume that if all vertices of G have the same degree then G is symmetric. But in 1969 Baron and Imrich [6] already showed that this is not the case for graphs.

Theorem 3.1. *There are infinitely many* $2-$*regular asymmetric* k*-graphs for any* $k \in \mathbb{N}$ *with* $k \geq 3$.

Here we construct some 2-regular k-graphs that are asymmetric. For this we introduce the concept of edge-asymmetry on graphs that will be very similar to our normal (vertex)-asymmetry and shall be very helpful in proving the following theorem.

Recall the definition of the *adjacency set*.

Definition 3.2. (*adjacency set*)

Let $G = (V(G), E(G))$ be a hypergraph and $v \in V(G)$. We call the set $N_E(v) := \{e \in E(G) : v \in e\}$ the *adjacency set* of v

Definition 3.3. (*edge-automorphism*)

Let $G = (V, E)$ be a graph. We call a bijection $\Theta : E \mapsto E$ an *edge-automorphism* if for every vertex $v \in V$ and set of adjacent edges $N_E(v)$ there is another vertex $v' \in V$ with $N_E(v') = \{\Theta(e) : e \in N_E(v)\}$.

Definition 3.4. (*edge-asymmetric*)

Let $G = (V, E)$ be a graph. We call G *edge-asymmetric* if the only edge automorphism of G is the identity.

We will later see that edge-asymmetry is equivalent to asymmetry on non-trivial graphs, even though there are trivial examples like Figure 16, where a k-graph G is symmetric but edge-symmetric.

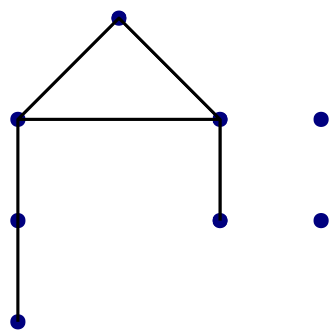


Figure 16: The graph Q, consisting of one of the 18 minimal asymmetric graphs with two disjoint vertices

Any edge automorphism on Q would induce a non-trivial automorphism on one of the 18 minimal asymmetric graphs. But there obviously is a non-trivial automorphism that just swaps the two disjoint vertices.

Definition 3.5. (*dual of a hypergraph*)
Let $G = (V(G), E(G))$ be a hypergraph. Then its dual graph is $H = (V(H), E(H))$ where $V(H) := E(G)$ and $E(H) := \{N_E(v) : v \in V(G)\}$. Figure 17 shows the transformation from the Frucht-graph G to its dual graph H.

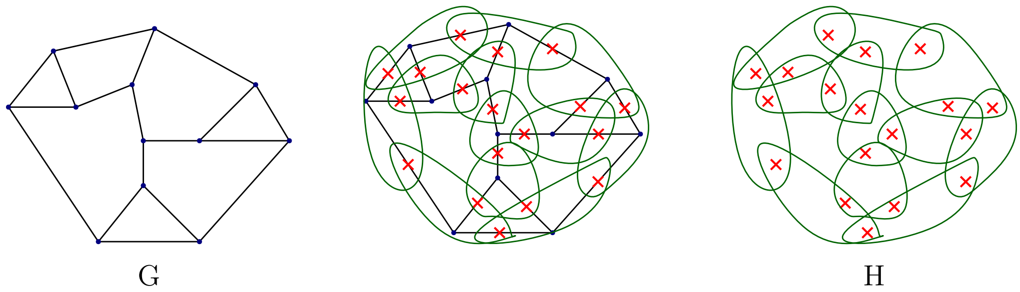


Figure 17: The transformation of the Frucht-graph G to is dual H

Lemma 3.6. If $G = (V(G), E(G))$ is an r-regular graph for $r \in \mathbb{N}$, $r \geq 3$ then its dual $H = (V(H), E(H))$ is an 2-regular r-graph.

Proof. Let $G = (V(G), E(G))$ be a r-regular graph and $H = (V(H), E(H))$ its dual. Let $e \in E(G)$ be an arbitrary edge of G, and note that e is also a vertex of H, so we refer to e_G and e_H respectively to denote the object we are looking at. There are exactly two distinct vertices $v_1, v_2 \in V(G)$ with $v_1, v_2 \in e_G$, thus $deg(e_H) = 2$ and we know that H is 2-regular. On the other hand for every $v \in V(G)$, $|N_E(v)| = r$ and thus H is indeed a r-graph. □

Lemma 3.7. Let $G = (V(G), E(G))$ be a graph. If there is an automorphism $\Phi\colon V(G) \mapsto V(G)$ with $\Phi(v) \neq v$ for a vertex $v \in V(G)$ with $deg(v) \geq 2$ then there also is a non-trivial edge-automorphism for G.

Proof. We see in definition 3.4 that if the condition mentioned above is not met, i.e. if all automorphisms restricted to vertices of degree ≥ 2 are the identity then the statement does not hold.

So let G be a graph with an automorphism that suffices to the condition. Now we give an non-trivial edge-autmorphism $\Theta\colon E(G) \mapsto E(G)$. Let $e = \{u, v\} \in E(G)$ be an arbitrary edge. We know that $e' = \{\Phi(u), \Phi(v)\} \in E(G)$ because Φ is an automorphism, thus we just define $\Theta(e) = e'$. Now let $v \in V(G)$ be an arbitrary vertex, then $x \in N_E(\Phi(v))$ if and only if there is an edge x' with $x' \in N_E(v)$ and $\Theta(x') = x$. Thus Θ is indeed an edge-automorphism. It remains to show that Θ is not the identity. Let $v \in V(G)$ be a vertex with $deg(v) \geq 2$ and $\Phi(v) \neq v$. Then there is an edge e with $e \in N_E(v), e \notin N_E(\Phi(v))$, and thus $\Theta(e) \neq \Theta$. □

Proposition 3.8. *Let $G = (V(G), E(G))$ be an r-regular graph for $r \in \mathbb{N}$ with $r \geq 3$, then G is asymmetric if and only if G is edge asymmetric.*

Proof. We prove the first direction by contraposition. So let $G = (V(G), E(G))$ be a graph that is not edge-asymmetric. Then there exists a non-trivial edge-automorphism $\Theta : E(G) \mapsto E(G)$. Now we give a non-trivial automorphism $\Phi : V(G) \mapsto V(G)$. Let $\{u, v\} = e \in E(G)$ be an arbitrary edge with $\Theta(e) = \{u', v'\}$. Because Θ is an edge-automorphism, there is a vertex $w \in V(G)$ such that $N_E(w) = \{\Theta(f) : f \in N_E(v)\}$. G is an r-regular graph and thus $N_E(u') \neq N_E(v')$. Therefore $w = u'$ or $w = v'$. In the first case define $\Phi(v) = u'$ and in the latter $\Phi(v) = v'$. If there are two edges e, e' that intersect in a vertex $v \in V(G)$ then the procedure above gives for both edges the same vertex v' for v to be mapped to. Second let $e = \{u, v\} \in E(G)$ be an arbitrary edge and let $w, w' \in V(G)$ be vertices such that $N_E(w) = \{\Theta(f) : f \in N_E(u)\}$ and $N_E(w') = \{\Theta(f) : f \in N_E(v)\}$. Then $N_E(w) \cap N_E(w') \neq \emptyset$ and thus there is a edge $\{w, w'\} = \{\Phi(u), \Phi(v)\} \in E(G)$ which shows that Φ is indeed a proper automorphism. It remains to show that Φ is not the identity. Since Θ is not the identity there is an edge $\{u, v\} = e \in E(G)$ with $\Theta(e) = e', e \neq e'$. Then the procedure defined above tells us that either $\Phi(u) \neq u$ or $\Phi(v) \neq v$.

The other direction follows by contraposition directly of Lemma 3.7 because every vertex of G has degree ≥ 3.

□

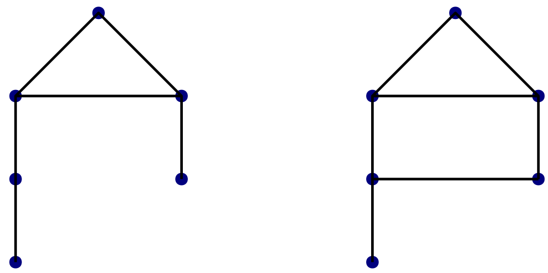
Lemma 3.9. If $G = (V(G), E(G))$ is an edge-asymmetric r-regular graph for $r \geq 3$ then its dual $H = (V(H), E(H))$ is asymmetric.

Proof. Let $\Theta \colon V(H) \mapsto V(H)$ be a non-trivial automorphism on H. By definition 3.5 the dual of a hypergraph transforms the edges $E(G)$ to vertices $V(H)$ and adds edges $E(H)$ corresponding to the vertices $V(G)$, so to the intersection points of the edges in $E(G)$. Thus Θ induces an non-trivial edge-automorphism $\Theta' \colon E(G) \mapsto E(G)$ on G with $\Theta'(e) = \Theta(e)$ for each edge $e \in E(G)$. Let $u \in V(G)$ be an arbitrary vertex, then because G is r-regular, there are distinct edges $e_1, \ldots, e_r \in N_E(u)$. Then there is an edge $u' \in E(H)$ with $\Theta\{e_1\}, \ldots, \Theta\{e_r\} \in N_E(u')$. If we now view u' as a vertex in G then we know that $N_E(u') = \{\Theta'(e) : e \in N_E(u)\}$ and thus Θ' is indeed a edge-automorphism on G. Furthermore Θ' is non-trivial as Θ already was non-trivial which proves the statement. □

Now we can finally prove Theorem 3.1.

Proof. Baron and Imrich [6] showed the existence of infinitely many r-regular asymmetric graphs for every $r \in \mathbb{N}$ with $r \geq 3$. By Proposition 3.8 these graphs are also edge-asymmetric and thus by Lemma 3.9 the dual of those graphs are asymmetric. By construction the duals of two distinct regular graphs are again distinct and Lemma 3.6 tells us that the dual of an r-regular graph is a 2-regular r-graph which completes the proof that there are infinitely many 2-regular k-graphs for every $k \in \mathbb{N}$ with $k \geq 3$. □

Now that we have shown the existence of regular asymmetric k-graph the next natural question to ask is if there are minimal regular asymmetric k-graphs. For $k = 2$ the answer is no because there are only 18 minimal asymmetric graphs [13] and none of them is regular. But already the case of 2-regular 3-graphs seems hard. The techniques in this section sadly can not be applied to this problem as minimal asymmetric k-graphs are in general not minimal edge-asymmetric as the following example shows.



(a) Graph G_1 (b) Graph G_2

Figure 18: Two of the 18 minimal asymmetric graphs

In Figure 18 we see two of the minimal asymmetric graphs and also that G_1 can be obtained by deleting an edge of G_2, so G_2 is indeed not minimal edge asymmetric Nevertheless this leads to the following conjecture.

Conjecture 3.10. *There is no minimal asymmetric r-regular k-graph for any $r, k \in \mathbb{N}$ with $k \geq 3$*

We were able to show the following interim result.

Lemma 3.11. Let $G = (V(G), E(G))$ be a minimal asymmetric 2-regular k-graph. Then G is linear.

Proof. Assume there are two intersecting edges $e_1, e_2 \in E(G)$ with $e_1 \cap e_2 = \{v_1, v_2\}$ for two vertices $v_1, v_2 \in V(G)$. Then there is a non-trivial automorphism Φ on G with $\Phi(v_1) = v_2, \Phi(v_2) = v_1$ and $\Phi(w) = w$ for all other vertices $w \in V(G)$. $\square$

4 Random Asymmetric k-graphs

In this Section we will consider the degree of asymmetry of random k-graphs. Remember that the degree of asymmetry $A(G)$ of a k-graph G, introduced by Erdős and Rényi [12] for graphs, is the minimum number of edge deletions and additions to the edge set of G, such that the resulting graph G' is symmetric. Erdős and Rényi [12] also showed the following two results.

Theorem 4.1 (Erdős and Rényi [12]). *Let $G = (V(G), E(G))$ be a random graph in $\mathcal{G}(V(G), 0.5, 2)$ with an infinite countable vertex set $V(G)$. Then G is, with a probability 1, symmetric.*

Theorem 4.2 (Erdős and Rényi [12]). *Let $G = (V(G), E(G))$ be a graph with $|V(G)| = n$. The degree of asymmetry $A(G)$ of G can be bounded by $A(G) \leq \frac{n-1}{2}$.*

Here we will use their techniques and extend them to show similar results for k-graphs with $k \in \mathbb{N}$ and $k \geq 3$. To prove the next result we need the following lemma formulated by Henze [11]. Originally the lemma was proven by Borel [2] and Cantelli [3] independently.

Lemma 4.3. Lemma of Borel-Cantelli

Let $(A_n)_{n\geq 1}$ be an arbitrary sequence of events in some probability space $(\Omega, \mathcal{A}, \mathcal{P})$ then:

1. If $\sum\limits_{n=1}^{\infty} \mathcal{P}(A_n) < \infty$ then $\mathcal{P}(\limsup_{n\to\infty} A_n) = 0$.
2. Let the events $A_1, A_2, \ldots$ be independent.

   Then if $\sum\limits_{n=1}^{\infty} \mathcal{P}(A_n) = \infty$ then $\mathcal{P}(\limsup_{n\to\infty} A_n) = 1$.

Theorem 4.4. *Let $k \in \mathbb{N}$ with $k \geq 3$ and let $G = (V(G), E(G))$ be a random k-graph in $\mathcal{G}(V(G), 0.5, k)$ with an infinite countable vertex set $V(G)$. Then G is with a probability 1 symmetric.*

Proof. We will modify the procedure given in section 3 of [12] to build up a non-trivial automorphism for k-graphs.

Let $\mathcal{G}(V, 0.5, k)$ be the probability space of k-graphs with vertex set $V = \{v_i : i \in \mathbb{N}\}$ and edge probability 0.5. Furthermore let $G = (V(G), E(G)) \in \mathcal{G}$. Then we start defining a non-trivial function $\Phi : V(G) \mapsto V(G)$. We will show that Φ is almost surely a non-trivial automorphism and moreover an involution. Let $\Phi(v_1) := v_2$, $\Phi(v_2) := v_1$ as well as $\Phi(v_i) = v_i$ for $i = 3, ..., k$. Because the k-graph G' induced by $\{v_1, ..., v_k\}$ has only one possible edge consisting of all its vertices all vertex permutations are automorphisms. Thus Φ restricted G' to it is indeed a non-trivial automporphism. Also by definition of the restriction of Φ the only vertices that are not fixed by it are v_1 and v_2, thus it is also an involution.

Now we iteratively assign a value to the next "free" vertex. So let I be an index set such that $\Phi(v_i)$ is already defined for every $i \in I$ and let $j \in \mathbb{N}$ be the smallest index not in I. For any vertex $v_\alpha \in V(G), \alpha \notin I$ let $G_I^{v_\alpha}$ be the k-graph induced by $\bigcup_{i\in I} v_i \cup v_\alpha$. Let furthermore $L^{G_I^{v_\alpha}}(v_\alpha)$ be the link of v_α in $G_I^{v_\alpha}$. We call $L_{good}^{G_I^{v_\alpha}}(v_\alpha) := \{\ell \in L^{G_I^{v_\alpha}}(v_\alpha) : \Phi(\ell) = \ell\}$ the set of good links of v_α and $L_{bad}^{G_I^{v_\alpha}}(v_\alpha) := \{\ell \in L^{G_I^{v_\alpha}}(v_\alpha) : \Phi(\ell) \neq \ell\}$ the set of bad links of v_α in $G_l^{v_\alpha}$. Now we want to find suitable vertex $v_{j'}$ for v_j to be mapped to by Φ. Thus we need a vertex $v_{j'} \in V(G)$ with $j' \neq j, j' \notin I$ as well as $L_{good}^{v_j} = L_{good}^{v_{j'}}$ and $L_{bad}^{v_j} = \Phi(L_{bad}^{v_{j'}})$. Moreover we need to restrict ourselves to vertices $v_{j'}$ such that in the k-graph G' induced by $\bigcup_{i\in I} v_i \cup v_j \cup v_{j'}$ there is no edge $e \in E(G')$ with $v_j, v_{j'} \in e$. We then define $\Phi(v_j) = \Phi(v_{j'})$ and $\Phi(v_{j'}) = \Phi(v_j)$.

Now let G' be the subgraph of G induced by $\bigcup_{i\in I} v_i \cup v_j \cup v_{j'}$. Then we need to check if Φ restricted to G' is indeed an involution. By construction we know that if it is an automorphism then it is already an involution. Let $v \in V(G')$ be an arbitrary vertex and $e \in E(G')$ be an arbitrary edge with $v \in e$. If $v_j, v_{j'} \notin e$ then $\phi(e) \in E(G')$ because we know inductively that Φ restricted to the k-graph induced by $\bigcup_{i\in I} v_i$ is an automorphism. So without loss of generality let $v_j \in e$, the case $v_{j'} \in e$ works similarly. If $e \setminus v_j \in L_{good}^{G_I^{v_j}}(v_j)$ then also by definition $e \setminus v_j \in L_{good}^{G_I^{v_{j'}}}(v_{j'})$ and thus $\Phi(e) \in E(G')$. Else $e \setminus v_j \in L_{bad}^{G_I^{v_j}}(v_j)$, but then $\Phi(e \setminus v_j) \in L_{bad}^{G_I^{v_{j'}}}(v_{j'})$ and thus $\Phi(e) \in E(G')$. Then Φ is indeed an involution on G'.

The only thing left to do is to show the existence of such a "swapping vertex". So let us assume we are in the setting above and already have defined Φ for some n vertices, so $= |I| = n$ for a $n \geq k \in \mathbb{N}$. Let v_j again be the vertex to be "swapped" and let $J' = \{x \in \mathbb{N} : x \notin \bigcup_{i\in I} v_i \cup v_j\}$. Because there are $\alpha := \binom{n}{k-1}$ many possibilities to pick $k-1$ vertices out of those defined by I we know that $|L_{good}^{v_j}|, |L_{bad}^{v_j}| \leq \alpha$. Furthermore there are $\beta := \binom{n}{k-2}$ many possibilities to pick a $k-2$ set out of I. Thus let $p_{j'}$ be the probability of the event $A_{j'}$ that a vertex $v_{j'}$ with $v_{j'} \notin \bigcup_{i\in I} v_i \cup v_j$ satisfies all our conditions. Then $p_j \geq \frac{1}{2}^\alpha \cdot \frac{1}{2}^\alpha \cdot \frac{1}{2}^\beta = \frac{1}{2}^{2\alpha+\beta}$. It is clear that the events $A_{j'}$ are independent for each vertex $v_{j'}$. Also because $\frac{1}{2}^{2\alpha+\beta} > 0$ is a constant we know that $\sum_{j'\in J'} p_{j'} = \infty$. Thus all conditions for the Lemma of Borell-Cantelli are satisfied and we know that $\limsup_{x\to\infty} A_{j'} = 1$. So in every step there are almost surely infinite many vertices that our current vertex can be swapped with and thus indeed our algorithm produces, with probability 1 an inversion.

□

Theorem 4.5. *For a k-graph G with $k \in \mathbb{N}$, $k \geq 2$ and $|V(G)| = n$ the degree of asymmetry $A(G)$ of G can be bounded by $A(G) \leq \binom{n-1}{k-1}\frac{n-1}{2(n-k+1)}$.*

Note that a trivial upper bound $A(G) \leq \binom{n-1}{k-1}$ can be achieved by knowing that there are two vertices $u, v \in V(G)$ with either $deg(u), deg(v) \leq \binom{n-1}{k-1} \cdot \frac{1}{2}$ or $deg(u), deg(v) \geq$

$\binom{n-1}{k-1} \cdot \frac{1}{2}$. So if we either remove all edges incident to u or v or add all possible edges then we can define an automorphism on this modified k-graph that just swaps u and v and thus $A(G) \leq \binom{n-1}{k-1}$.

Proof. By potential relabeling of vertices we may assume that $V(G) = \{v_1, \ldots v_n\}$. If two vertices have the same link it is clear that there is automorphism that just interchanges those vertices. Therefore we can bound $A(G)$ by the smallest number of edges that can be removed such that in the resulting graph G' there are two vertices $u, v \in V(G'), u \neq v$ with $L_u = L_v$. We call this number $\Delta_{u,v}$.

We define the good and bad link of a pair of vertices $u, v \in G$ as

$$L_{good}(u, v) := \{\ell : (\ell \in L(u) \text{ and } (\ell \in L(v) \text{ or } v \in \ell)) \text{ or } \ell \in L(v) \text{ and } u \in \ell\}$$

and

$$L_{bad}(u, v) := L(u) \cup L(v) \setminus L_{good}(u, v).$$

Note that $L_{good}(u, u) = L(u)$ and $L_{bad}(u, u) = \emptyset$ for every $u \in V(G)$.

For $u, v \in V(G)$ and $\ell \in L(u)$ it is clear that $\ell \in L_{good}(u, v)$ or $\ell \in L_{bad}(u, v,)$ and in reverse if $\ell' \in L_{bad}(u, v)$ then either $\ell' \in L(u)$ or $\ell' \in L(v)$. Moreover $L_{good}(u, v)$ and $L_{bad}(u, v)$ are symmetric so $L_{good}(i, j) = L_{good}(j, i)$ and $L_{bad}(i, j) = L_{bad}(j, i)$. This leads us to conclude $|L(u)| + |L(v)| - 2|L_{good}(u, v)| = |L_{bad}(u, v)|$ for every vertex $u, v \in V(G)$. On the other hand we can make the following estimate

$$A(G) \leq \min_{u,v \in V(G), u \neq v} |L(u)| + |L(v)| - 2L_{good}(u, v),$$

because a symmetrization of minimal size that equalizes the link of two vertices would not have to modify edges that contribute to good links of those vertices.

Ergo

$$A(G) \leq \min_{u \neq v} |L_{bad}(u, v)|$$

and by forming the average and because $L_{bad}(w, w) = \emptyset$ for all $w \in V(G)$

$$A(G) \leq \min_{u,v \in V(G)} |L_{bad}(u, v)| \leq \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} |L_{bad}(v_i, v_j)|}{n(n-1)}.$$

Now define $l_1, \ldots, l_{\binom{n}{k-1}}$ to be all the $(k-1)$-size subsets of $V(G)$ and for $s \in [\binom{n}{k-1}]$ let $\alpha(l_s) := |\{e : e \in E(G), l_s \subset e\}|$ be the number of edges in $E(G)$ that have l_s as a subset.

Then by a double counting argument the following equation holds

$$\sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} L_{bad}(v_i, v_j) = 2 \sum_{s=1}^{\binom{n}{k-1}} \alpha(l_s)(n - 1 - \alpha(l_s)).$$

By definition of the bad link of two vertices the first part of the equation counts the triples $(v_i, \{w_1, \ldots, w_{k-1}\}, v_j)$ such that either $\{v_i, w_1, \ldots, w_{k-1}\} \in E(G)$ or $\{v_j, w_1, \ldots, w_{k-1}\} \in E(G)$. In the second part we fix an $k-1$ element subset $l_s \subset V(G)$. Then there are $\alpha(l_s)$ many possibilities to choose a vertex v_j such that $l_s \cup v_j \in E(G)$ and $(n-1-\alpha(l_s))$ many possibilities to choose a vertex v_i such that there is no edge with $l_s \cup v_i \in E(G)$. In the end we have to double the value because in our triple we can interchange the first and last vertex.

Furthermore

$$\alpha(l_s)(n-1-\alpha(l_s)) = \left(\frac{n-1}{2}\right)^2 - \left(\frac{n-1}{2}\right)^2 + (n-1)\alpha(l_s) - (\alpha(l_s))^2$$

$$= \left(\frac{n-1}{2}\right)^2 - \left(\alpha(l_s) - \frac{n-1}{2}\right)^2$$

Now because $\left(\alpha(l_s) - \frac{n-1}{2}\right)^2 \geq 0$

$$2 \cdot \sum_{s=1}^{\binom{n}{k-1}} \alpha(l_s)(n-1-\alpha(l_s)) \leq \binom{n}{k-1} \frac{(n-1)^2}{2}.$$

Dividing by $n(n-1)$ yields

$$\begin{aligned} A(G) &\leq \binom{n}{k-1} \frac{(n-1)^2}{2n(n-1)} \\ &= \frac{n!}{(k-1)!(n-k+1)!} \frac{(n-1)}{2n} \\ &= \frac{(n-1)!}{(k-1)!(n-k+1)!} \frac{(n-1)}{2} \\ &= \frac{(n-1)!}{(k-1)!(n-1-k+1)!} \frac{(n-1)}{2(n-k+1)} \\ &= \binom{n-1}{k-1} \frac{(n-1)}{2(n-k+1)}. \end{aligned}$$

We see that this bound is only an improvement compared to the trivial bound $\binom{n-1}{k-1}$ if $k \leq \frac{n}{2}$. But for $k << n$, i.e. if n is large compared to k we get an improvement of almost $\frac{1}{2}$. Also note that in the case $k = 2$ we get bound of Theorem 4.2, i.e. we can bound the degree of asymmetry for every 2-graph G by $A(G) \leq \frac{n-1}{2}$. □

Theorem 4.6. *(Erdős Rènyi [12]) Let $G \in \mathcal{G}([n], p)$ for $p \in (0,1)$ and let $\epsilon > 0$. We call P_n the probability that G can be made symmetric by changing no more then $\frac{n(1-\epsilon)}{2}$ edges of G. Than*

$$\lim_{n \to \infty} P_n(\epsilon) = 0.$$

This result tells us that not only almost every graph is asymmetric but also that it has a high degree of asymmetry as $\frac{n-1}{2}$ is an upper bound for the degree of asymmetry of a graph. It is in stark contrast to Theorem 4.2 where it is shown that almost every graph with countable infinite vertices is symmetric. To prove it Erdős and Rényi used the following estimate:

$$P_n(\epsilon) \leq \sum_{q=2}^{n} \frac{A_{n,q} \cdot B_{n,q} \cdot C_{n,q}}{2^{\binom{n}{2}}}$$

where $A_{n,q}$ is the number of permutations $S_{n,q}$ on $[n]$ leaving exactly $n-q$ elements invariant. $B_{n,q}$ is an upper bound for the number of graphs where such permutation is a proper automorphism and $C_{n,q}$ is an upper bound for the number of graphs G that can be transformed into a graph G' such that there is a permutation $p \in S_{n,q}$ that is a proper automorphism on G' where the transformation from G to G' consists of no more then $\frac{n(1-\epsilon)}{2}$ edge additions or deletions. It is clear that this indeed is an estimate because if G is a graph with degree of asymmetry $A(G) \leq \frac{n(1-\epsilon)}{2}$ then there is a non-trivial automorphism on a modified graph G' that is the identity for some $q' \in \{0, \dots, n-2\}$ vertices. Note that there is no automorphism that changes just one vertex and any automorphism that is the identity on all vertices is trivially just the identity. They then go on to look at those terms individually.

Within the limits of this thesis we were not able to show the equivalent statement for k-graphs but we conjecture the following.

Conjecture 4.7. *Almost every k-graph is asymmetric.*

5 Computing Hypergraph Automorphism

In the following we will discuss a paper due to Luks [5]. We aim to solve the following problem.

Problem 5.1. `hypergraph automorphism`
Input: Hypergraph $G = (\Sigma, E)$.
Output: The set of automorphisms on G, $Aut(G)$.

Luks showed the following theorem.

Theorem 5.2. *Luks [5]* `hypergraph automorphism` *can be solved in* $\mathcal{O}(c^{|\Sigma|})$ *time for a constant* c.

The Author solves the problem `hypergraph automorphism` by first solving the problem `graph isomorphism`. But even testing if two graphs are isomorphic is not trivial at all. Thus `graph isomorphism` gets reduced to another Problem: `coset-intersection`.

Problem 5.3. `graph isomorphism`
Input: Two graphs G_1 and G_2.
Output: The isomorphism group between G_1 and G_2 or $\emptyset$.

5.1 Group theoretical notation

In order to understand Luks proof we will need to dive into Group Theory so here are some basic definitions.

Definition 5.4. (group)
A *group* $\mathcal{G} = (G, \cdot)$ is tuple where G is a set and " $\cdot$ " is a binary relation on the elements of G satisfying the following group axioms.

Associativity: For every $a, b, c \in G$: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.

Existence of Neutral element: There is an $e \in G$ with $a \cdot e = a$ for every a in G.

Existence of inverse element: For every $a \in G$ there is an $a' \in G$ with $a \cdot a' = e$ for the neutral element e. We denote $a^{-1} := a'$.

Example: For a set Σ we call $Sym(\Sigma)$ the group of all permutations of Σ. If $\Sigma = [n]$ for some $n \in \mathbb{N}$ we denote $S_n := Sym([n])$. The operation $\cdot$ in this case is the composition of two permutations. For instance let $n = 6$ and consider the permuations group S_6 as above. If we take two elements $\pi_1, \pi_2 \in$ in cyclic notation e.g. $\pi_1 = (123456)$, $\pi_2 = (132465)$ then $\pi_1 \cdot \pi_2 = (14)(2)(36)(5)$. In the following we often omit the operation " $\cdot$ ".

Definition 5.5. (subgroup)
Let $\mathcal{G} = (G, \cdot)$ be a group. We say that $\mathcal{H} = (H, \cdot)$ is a *subgroup* of $\mathcal{G}$ if $H \subseteq G$ and for all $h_1, h_2 \in H$, $h_1 \cdot h_2^{-1} \in H$. If $\mathcal{H}$ is a subgroup of $\mathcal{G}$ we denote this by $\mathcal{H} \leq \mathcal{G}$.

Example: Let $\mathcal{G}_6$ be the group as above. Then $\mathcal{H} = (H, \cdot')$ with $H = \{(123456), (135)(246), (14)(25)(36), (153)(264), (1654321)\}$ is a subgroup of $\mathcal{G}$.So Informally we say $\mathcal{H}$ consist of all shifts of 123456, so all operation that shifts every digit the same distance to the left or right.

Definition 5.6. (coset)
Let G be a group and H be a subgroup of G with $g \in G$ then we define the right-coset of H with g in G: $Hg = \{h \cdot g : h \in H\}$. The left-coset are defined similarly.

Example: Let $\mathcal{G}$ and $\mathcal{H}$ as above and $g = (12)(3)(4)(5)(6) \in G$. Then $\mathcal{H}g = \{h \cdot (12)(3)(4)(5)(6) : h \in H\}$ is the right-coset of $\mathcal{H}$ in $\mathcal{G}$ with g that consists of all shifts of 123456 that afterwards swap the first and and second digit.

Let Σ be a set. For $X \subseteq \Sigma$ and $\pi \in Sym(\Sigma)$ we denote the image of X under π as $X^\pi := \{a^\pi : a \in X\}$. So if $\Sigma = \{[5]\}$ and $\pi = (15243)$ then $3^\pi = 1$ and $\{1, 2, 3\}^\pi = \{5, 4, 1\}$.

Definition 5.7. (stabiliser)
Let Σ be a set and $A \subseteq Sym(\Sigma)$. For any $\Delta \in \Sigma$ the set stabiliser of Δ in A is $A_\Delta := \{a \in A : \Delta^a = \Delta\}$.

Example: Let $A = \{(123)(456), (142635), (1)(564)(23)\} \subset Sym([6])$ and $\Delta = \{1, 3, 5\}$. Then the set-stabilizer A_Δ consist of all elements in A that leave $\{1, 2, 3\}$ invariant so $A_\Delta = \{(123)(456), (1)(564)(23)\}$.

Definition 5.8. (diagonal of a set)
For any set Σ define the *diagonal* of Σ as $diag(\Sigma) := \{(x, x) : x \in \Sigma\}$

5.2 Solving `graph isomorphism`

We will reduce `graph isomorphism` to the following problem.

Problem 5.9. `coset-intersection`
Let Σ be a set.
Input: $G, H \leq Sym(\Sigma)$ and $\pi_1, \pi_2 \in Sym(\Sigma)$
Output: $G\pi_1 \cap H\pi_2$.

Luks [5] showed the following theorem. To not immerse too deeply into group theory we only provide a sketch of the proof. The interested reader is referred to [5].

Theorem 5.10. *Luks [5]* `coset-intersection` *can be solved in $\mathcal{O}(c^{|\Sigma|})$ time for a constant c.*

Babai [9] improved on this bound and showed that `coset-intersection` can be solved in $\exp(\mathcal{O}(\sqrt{n}\log n))$ time.

In order to solve `coset-intersection`, we first state two generalisations of it.

Problem 5.11. `partial coset`

Let Γ and Δ be two sets.

Input: $L \leq Sym(\Gamma) \times Sym(\Delta)$, $z \in Sym(\Gamma \times \Delta)$, $\Pi \subseteq \Gamma \times \Delta$

Output: $(Lz)_\Pi := \{x \in Lz : \Pi^x = \Pi\}$.

To have an efficient algorithm we need to do two more things. First we need to assume that both Γ and Δ are sets with a size of a power of 2. This can be done by extending both sets and letting our group operations act trivially on the extended part. Second, we need to refine our problem to be able to solve it recursively, thus:

Problem 5.12. `partial coset recursion`

Let Γ and Δ be two sets.

Input: $L \leq Sym(\Gamma) \times Sym(\Delta)$, $z \in Sym(\Gamma \times \Delta)$, $\Pi \subseteq \Gamma \times \Delta$, $\Theta := \Phi \times \Psi \subseteq \Gamma \times \Delta$ such that $L_\Theta = L$ and with $|\Theta|$ being a power of 2.

Output: $(Lz)_\Pi[\Theta] := \{x \in Lz : (\Pi \cup \Theta)^x = \Pi \cup \Theta^x\}$

Note that with $\Theta = \Gamma \times \Delta$ we have a instance of the `partial coset` problem.

Lemma 5.13. Luks [5] `partial coset` can be solved in $\mathcal{O}(c^{|\Gamma|+|\Delta|})$ time for a constant c.

Here we omit the proof of Lemma 5.13.

Proof of Theorem 5.10. To solve coset-intersection we now use `partial coset` in the following setting. Let $L = G \times H, z = (x, y), \Gamma = \Delta = \Sigma$, $\Pi = diag(\Sigma \times \Sigma)$. Then for $g \in G, h \in H$ with $(gx, hy) \in (Sym(\Sigma) \times Sym(\Sigma))$ it follows that if $gx = hy$ then $(g, h) \in (Lz)_\Pi$. Thus the first coordinate projection of our special case of `partial coset` is the solution to `coset-intersection`. $\square$

Theorem 5.14. *Luks [5]* `graph isomorphism` *can be solved in $c^{|V(G_1)|}$ time for some constant c.*

Proof. Assume That $V(G_1) \cap V(G_2) = \emptyset$. If not just rename the vertices of one of the graphs. Secondly assume $|V(G_1)| = |V(G_2)|$ if not then there obviously can not be any any isomorphism between G_1 and G_2.

We set $\Sigma := V(G_1) \cup V(G_2)$ and $\Pi := \{e = (\sigma, r) : e \in E(G_1) \cup E(G_2)\}$ as well as $\mathcal{G} = Sym(\Sigma)_{V(G_1)}$, so $\mathcal{G}$ is the subgroup of $Sym(\Sigma)$ that leaves $V(G_1)$ invariant. Let $t \in Sym(\Sigma)$ be a fixed permutation that transposes $V(G_1)$ and $V(G_2)$. Then $\mathcal{G}t$ contains all permutations that transpose $V(G_1)$ and $V(G_2)$ because any such permutation can be split in a part that transposes the two vertex sets, which is done by t, and a part that is just a permutation on each vertex set, and those are just in $\mathcal{G}$.

Now we consider a special case of `coset-intersection` with $\Gamma = \Delta = \Sigma$, $L = diag(\mathcal{G}, \mathcal{G})$, $z = (t, t)$ and Π as above. As we noted above any permutation that transposes $V(G_1)$ and $V(G_2)$ is in $\mathcal{G}t$ and thus any bijection $f : V(G_1) \mapsto V(G_2)$ induces a permutation $\pi \in \mathcal{G}t$ such that $\pi_{V(G_1)} = f$ and $\pi_{V(G_2)} = f^{-1}$. If f is an isomorphism then for its induced permutation π it must hold that $\pi \in Lz_\Pi$ because this guarantees that previously connected vertices are still connected after the permutation. On the other hand if $(xt, xt) \in Lz_\Pi$ then the corresponding bijection $xt_{V(G_1)}$ is an isomorphism. Thus $(Lz)_{V(G_1)}$ already contains all isomorphism if we restricted the elements to $V(G_1)$. $\square$

5.3 Solving `hypergraph automorphism`

Now we can finally prove the equivalent statement for hypergraphs.

Problem 5.15. `hypergraph isomorphism`
Input: Two hypergraphs G and H.
Output: isomorphism group between G and H or $\emptyset$

Any hypergraph G has a natural bijection to a bipartite graph B where one partite set represents the vertices of the original hypergraph and the other one represents the edges so $V(B) = V(G) \cup E(G)$, and the edges of B are getting added by inclusion so $E(B) := \{(v, e) : v \in V(H), e \in E(H) \text{ with } v \in e\}$.

But bluntly solving the isomorphism problem for the appropriate bipartite graph would yield an explosion in the running time due to there being potentially up to $2^{|V(G)|}$ many edges in G and thus $|V(G)| + 2^{|V(G)|} \notin \mathcal{O}(|V(G)|)$ many vertices in B. Thus we need a more clever way to solve the problem.

Due to `hypergraph isomorphism` being reducible to `hypergraph automorphism` [5] it suffices to solve the latter. To minimize running time we use a dynamic programming scheme. For this define for every hypergraph and subsets of vertices a corresponding bipartite graphs the following way. For any $\Delta \subseteq \Sigma$ define

$$E^\Delta := \{\Theta \cap \Delta : \Theta \in E, \Theta \cup \Delta = \Sigma\},$$

and for any $\Delta, \Gamma \subseteq \Sigma$ let $B_\Gamma^\Delta = (V(B), \mathcal{E}(B))$ be the bipartite graph with

$$V(B) := \Sigma \cup \mathcal{P}(\Sigma)$$

where $\mathcal{P}(\Sigma)$ is the power set of Σ and

$$\mathcal{E}(B_\Gamma^\Delta) := \{(\lambda, \Phi) : \Phi \in E^\Delta, \lambda \in \Phi \cap \Gamma\}.$$

Note that for $\Delta = \Gamma = \Sigma$ and the deletion of some vertices corresponding to non-existent edges we get the natural bipartite graph corresponding to our hypergraph G.

Lemma 5.16. Luks [5]

(i) For $\gamma \in \Delta : \mathcal{E}(B^{\Delta}_{\{\gamma\}}) = \{(\gamma, \{\gamma\} \dot{\cup} \Phi) : \Phi \in E^{\Delta \setminus \{\gamma\}}\}$

(ii) $\mathcal{E}(B^{\Delta}_{\Sigma}) = \{(\sigma, \Phi) : \sigma \in \Phi \in \mathcal{E}^{\Delta}\}$

Proof. (i) Let $\Phi \in E^{\Delta}$ then either $\gamma \cap \Phi = \emptyset$ or $\gamma \cap \Phi = \gamma$, thus each element $E \in \mathcal{E}(B^{\Delta}_{\{\gamma\}})$ must contain γ. Then each $E \in \mathcal{E}(B^{\Delta}_{\{\gamma\}})$ has the form $E = \{\gamma\} \cup E'$ for a determined later E'. Note that $E' \in E^{\Delta \setminus \gamma}$ because for any $\Phi \in E$ with $\Phi \cup \Delta = \Sigma$ and $\gamma \in \Phi \cap \Delta$ we also know that $\Phi \cup (\Delta \setminus \gamma) = \Sigma$ and $\Phi \cap (\Delta \setminus \gamma) = E'$. On the other hand if we take any $E' \in E^{\Delta \setminus \{\gamma\}}$ then it follows directly that $E' \cup \{\gamma\} \in E^{\Delta}$.

(ii) This follows directly by definition because $\Phi \cap \Sigma = \Phi$. □

Proof of Theorem 5.2. As before we assume that $|\Sigma|$ is a power of 2 by extending G and looking at the subgroup of the solution that fixes the added points. In the following we construct a table of sub-solutions and use dynamic programming to reduce calculation overhead. Let $\Delta, \Delta', \Gamma, \Gamma' \subseteq \Sigma$ with $|\Delta| = |\Delta'|$ and $|\Gamma| = |\Gamma'|$ both being a power of 2. Let additionally $Iso(\Gamma, \Delta, \Gamma', \Delta')$ be a subset of $Sym(\Sigma)$ mapping Γ to Γ' and $\mathcal{E}^{\Delta}_{\Gamma}$ to $\mathcal{E}^{\Delta'}_{\Gamma'}$. Now we show that $Iso(\Gamma, \Delta, \Gamma, \Delta)$ is a subgroup of $Sym(\Sigma)$ and that $Iso(\Gamma, \Delta, \Gamma', \Delta')$ is a right coset of this subgroup. Let $\mathcal{S} = Iso(\Gamma, \Delta, \Gamma, \Delta)$ then it is clear that the identity is in $\mathcal{S}$. Let $s_1, s_2 \in \mathcal{S}$ and let $E \in \mathcal{E}^{\Delta}_{\Gamma}$ then $s_2(E) \in \mathcal{E}^{\Delta}_{\Gamma}$, additionally there is an edge $E' \in \mathcal{E}^{\Delta}_{\Gamma}$ with $s_2(E') = E$ implying that $s_2^{-1}(E) = E'$ and therefore we know that $s_2^{-1} \in S$. So S is by Definition 5.5 a subgroup of $Sym(\Sigma)$. If there is no $\pi \in Sym(\Sigma)$ with $\Gamma^{\pi} = \Gamma'$ with $\pi \in Iso(\Gamma, \Delta, \Gamma', \Delta')$, then $Iso(\Gamma, \Delta, \Gamma', \Delta') = \emptyset$. So let us assume it is non-empty and $\pi \in Iso(\Gamma, \Delta, \Gamma', \Delta')$. Let again $s_1 \in Iso(\Gamma, \Delta, \Gamma, \Delta)$, $E \in \mathcal{E}^{\Delta}_{\Gamma}$ and $x = s_1\pi$. We know that $s_1(E) = E'$ for an $E' \in \mathcal{E}^{\Delta}_{\Gamma}$, thus $x(E) = \pi(E') \in \mathcal{E}'^{\Delta'}_{\Gamma}$. Then $Iso(\Gamma, \Delta, \Gamma', \Delta')$ is indeed a right coset of $Iso(\Gamma, \Delta, \Gamma, \Delta)$.

Now we fill the lookup-table with all of those cosets in order of increasing Δ and for each Δ in order of increasing Γ. The full table contains in particular $Iso(\Sigma, \Sigma, \Sigma, \Sigma)$, the solution of `hypergraph automorphism`.

Computation of $Iso(\Gamma, \Delta, \Gamma', \Delta')$:

First note that for $\Delta = \emptyset$ the entry of $Iso(\Gamma, \Delta, \Gamma', \Delta')$ contains just all permutations of $Sym(\Sigma)$ that map Γ to Γ' which can be calculated in polynomial time. For $\Delta \neq \emptyset$ we distinguish between the two following cases.

Case 1: $|\Gamma| = 1$.

Let $\Gamma = \{\gamma\}$ and $\Gamma' = \{\gamma'\}$. If exactly one of $\mathcal{E}(B^{\Delta}_{\{\gamma\}}), \mathcal{E}(B^{\Delta}_{\{\gamma'\}})$ is empty then there obviously can not exist an isomorphism between them and if both are empty then by a previous remake $Iso(\{\gamma\}, \Delta, \{\gamma'\}, \Delta')$ contains all isomorphism mapping γ to γ'. So let us assume from now on that both are non-empty.

Then $\gamma \in \Delta$ and $\gamma' \in \Delta'$. Thus by Lemma 5.16 (i) $Iso(\{\gamma\}, \Delta, \{\gamma'\}, \Delta')$ contains those permutations that map γ to γ' and $\mathcal{E}^{\Delta \setminus \{\gamma\}}$ to $\mathcal{E}^{\Delta' \setminus \{\gamma'\}}$. Then Lemma 5.16 (ii) tells us that the edges of $B_\Sigma^{\Delta\setminus\gamma}$, $B_\Sigma^{\Delta'\setminus\gamma'}$ are self contained and thus the permutations that map $\mathcal{E}^{\Delta\setminus\{\gamma\}}$ to $\mathcal{E}^{\Delta'\setminus\{\gamma'\}}$ are precisely those that induce isomorphism from $B_\Sigma^{\Delta\setminus\gamma}$ to $B_\Sigma^{\Delta'\setminus\gamma'}$ so $Iso(\Sigma, \Delta \setminus \{\gamma\}, \Sigma, \Delta' \setminus \gamma')$. Those again were already calculated and can be just read from our lookup-table.

Case 2: $\Gamma > 1$.
We use a simple divide and conquer approach to solve this case. For each $\Gamma_1 \subset \Gamma$ with $|\Gamma_1| = \frac{|\Gamma|}{2}$ determine $Iso(\Gamma_1, \Delta, \Gamma_1', \Delta') \cap Iso(\Gamma \setminus \Gamma_1, \Delta, \Gamma' \setminus \Gamma_1', \Delta')$. The subisomorphism can just be read off the lookup-table and the coset-intersection gets solved by the Theorem 5.10 in exponential time. Now we just form the union over the results for all of those Γ_1. □

Now that we have an algorithm that computes the automorphism group $\mathcal{G}$ of an hypergraph G we can also decide if G is asymmetric by considering the order of $\mathcal{G}$. Within the limits of this thesis were not able to extend the dynamic programming scheme to decide if a given hypergraph is minimal asymmetric. Instead we have to execute the algorithm above on every induced subgraph of G.

Theorem 5.17. *Let $G = (V(G), E(G))$ be a hypergraph. For some constant c' it can be decided in $\mathcal{O}(c'^{|V(G)|})$ time whether G is minimal asymmetric.*

Proof. There are $2^{|V(G)|}$ induced subgraphs of G. We can bound the computation time of `hypergraph automorphism` of every subgraph by the computation time of `hypergraph automorphism` of G. Thus we can bound the computation time of deciding whether G is minimal asymmetric by $2^{|V(G)|}c^{|V(G)|} = c'^{|V(G)|}$ for some constant c' where c is the constant of Theorem 5.2. □

6 Concluding Remarks

In Section 2 we gave a construction for minimal asymmetric linear k-graphs with maximum degree 2. These are optimal in the sense that there can not be a minimal asymmetric or even an asymmetric k-graph with smaller degree. Another related property, that we did not consider here, is the sparsity of a minimal asymmetric graph. So one could ask what is the minimal number $\frac{|E(G)|}{\binom{|V(G)|}{k}}$ for a minimal asymmetric k-graph $G = (V(G), E(G))$.

The third Section concluded in Theorem 3.1, where we showed that for every $k \in \mathbb{N}$ there are infinitely many asymmetric 2-regular k-graphs. We believe it is possible to extend this result and construct infinitely many asymmetric r-regular k-graphs for every $r \geq 3$, $k \geq 3$. We also stated a more difficult problem in Conjecture 3.10 where we conjecture that there is no minimal asymmetric r-regular k-graph for any $r, k \in \mathbb{N}$ with $k \geq 3$.

In the forth Section we then studied the degree of asymmetry of k-graphs. We generalised some results on graph by Erdős and Rényi [12] to k-graphs. Probably the most well known result of their paper is Theorem 4.6, where they showed that almost every graph is asymmetric. Here we also conjectured that this result can be generalised to k-graphs.

In the fifth Section we studied an algorithm by Luks [5] that calculates the automorphism group of an given hypergraph G in exponential running time. We then gave a slight extension of said algorithm to also decide in $c^{|V(G)|}$ running time, for a constant c, if a given hypergraph G is minimal asymmetric. Here and in Luks [5] it remains open to minimize the base c.

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Erklärung

Ich versichere wahrheitsgemäß, die Arbeit selbstständig verfasst, alle benutzten Hilfsmittel vollständig und genau angegeben und alles kenntlich gemacht zu haben, was aus Arbeiten anderer unverändert oder mit Abänderungen entnommen wurde, sowie die Satzung des KIT zur Sicherung guter wissenschaftlicher Praxis in der jeweils gültigen Fassung beachtet zu haben.

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