Algorithmic Graph Theory

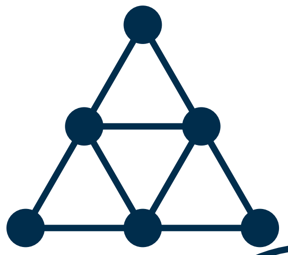
Solution Sheet 5

Laura Merker and Samuel Schneider, July 16, 2025

Exercises

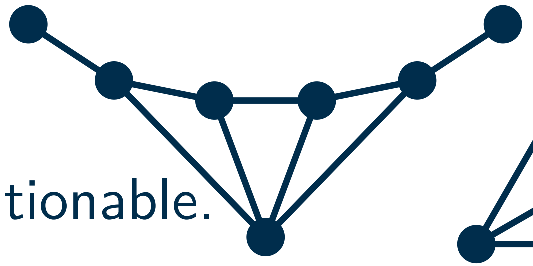
(1) Give an algorithm that computes for a given chordal graph G and a PES σ a set of subtrees of a tree such that G is the intersection graph of these subtrees.

Apply your algorithm to the given graph with $\sigma = [a, b, c, d, e, f]$.

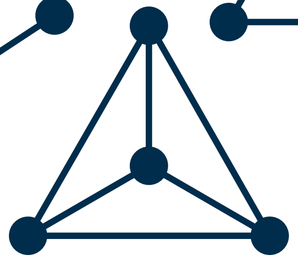


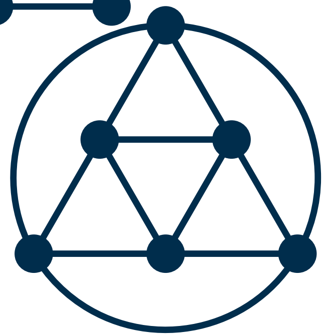
(2) Prove that: **(i)** $ab \;\Gamma\; a'b' \Leftrightarrow ba \;\Gamma\; b'a'$

(ii) $ab \;\Gamma^*\; a'b' \Leftrightarrow ba \;\Gamma^*\; b'a'$



(3) Prove that the “bull head” is transitively orientationable.





(4) Apply the algorithm for computing a transitive orientation to the given graphs.

(5) Show that the algorithm for computing a transitive orientation can be implemented to run in $\mathcal{O}(\Delta \cdot |E|)$ time and $\mathcal{O}(|V| + |E|)$ space with Δ denoting the maximum degree of a vertex.

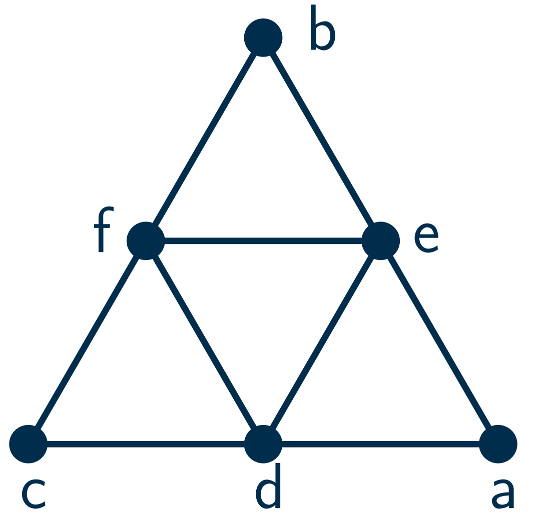
(6) Let $G = (V, E)$ be a graph. Prove the following statements.

(i) A vertex order σ is a perfect elimination scheme of G if and only if $a <_\sigma b <_\sigma c$ with $ab, ac \in E$ implies that $bc \in E$.

(ii) G is a comparability graph if and only if there is a vertex oder σ such that $a <_\sigma b <_\sigma c$ with $ab, bc \in E$ implies that $ac \in E$.

Bonus

Exercise 1



Give an algorithm that computes for a given chordal graph G and a PES σ a set of subtrees of a tree such that G is the intersection graph of these subtrees.
Apply your algorithm to the given graph with $\sigma = [a, b, c, d, e, f]$.

Theorem 3.14:
For all Graphs G the following are equivalent:

- G is chordal
- G is the intersection graph of subtrees of a tree
- There is a tree $T = (\mathcal{K}, \mathcal{E})$ whose vertices $\mathcal{K}$ are the maximal cliques of G such that for every $v \in V(G)$ the induced subgraphs of $T_{\mathcal{K}_v}$ are connected with $\mathcal{K}_v$ being the set of cliques in $\mathcal{K}$ that contain v.

The proof is constructive!

Exercise 1

Input: G chordal, PES σ von G

$T \leftarrow (\{\{\sigma(n)\}\}, \emptyset)$
for $i = n-1$ **to** 1 **do**
    $v \leftarrow \sigma(i)$
    $A \leftarrow \{v\} \cup (\mathrm{Adj}(v) \cap \{\sigma(i+1), \ldots, \sigma(n)\})$
    if $\mathrm{Adj}(v) \cap \{\sigma(i+1), \ldots, \sigma(n)\}$ is maximal clique $G_{\{\sigma(i+1),\ldots,\sigma(n)\}}$
        $B \leftarrow$ vertex of T corresponding to $\mathrm{Adj}(v) \cap \{\sigma(i+1), \ldots, \sigma(n)\}$
        $B \leftarrow B \cup \{v\}$
    else
        $B \leftarrow$ vertices of T with $\mathrm{Adj}(v) \cap \{\sigma(i+1), \ldots, \sigma(n)\} \subsetneq B$
        $V_T \leftarrow V_T \cup \{A\}$, $E_T \leftarrow E_T \cup \{AB\}$
$\forall v \in V(G)$: identify vertices of T that contain v.

Exercise 1

Input: G chordal, PES σ von G

$T \leftarrow (\{\{\sigma(n)\}\}, \emptyset)$
for $i = n-1$ **to** 1 **do**
  $v \leftarrow \sigma(i)$
  $A \leftarrow \{v\} \cup (\mathrm{Adj}(v) \cap \{\sigma(i+1), \ldots, \sigma(n)\})$
  if $\mathrm{Adj}(v) \cap \{\sigma(i+1), \ldots, \sigma(n)\}$ is maximal clique $G_{\{\sigma(i+1),\ldots,\sigma(n)\}}$
    $B \leftarrow$ vertex of T corresponding to $\mathrm{Adj}(v) \cap \{\sigma(i+1), \ldots, \sigma(n)\}$
    $B \leftarrow B \cup \{v\}$
  else
    $B \leftarrow$ vertices of T with $\mathrm{Adj}(v) \cap \{\sigma(i+1), \ldots, \sigma(n)\} \subsetneq B$
    $V_T \leftarrow V_T \cup \{A\}$, $E_T \leftarrow E_T \cup \{AB\}$
$\forall v \in V(G)$: identify vertices of T that contain v.

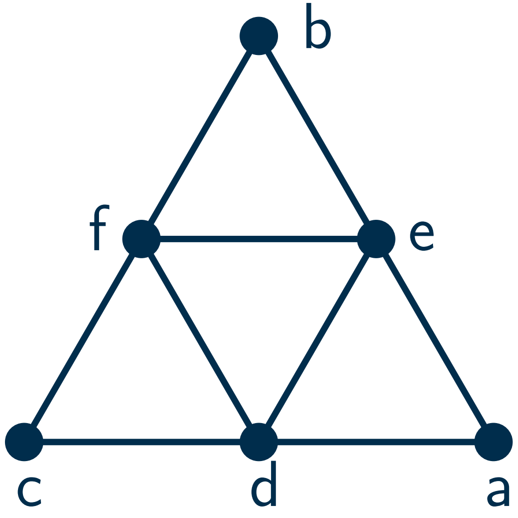
recall sheet 4, exercise 3

Correctness: proof of Theorem 3.14

Exercise 1

Input: G chordal, PES σ von G

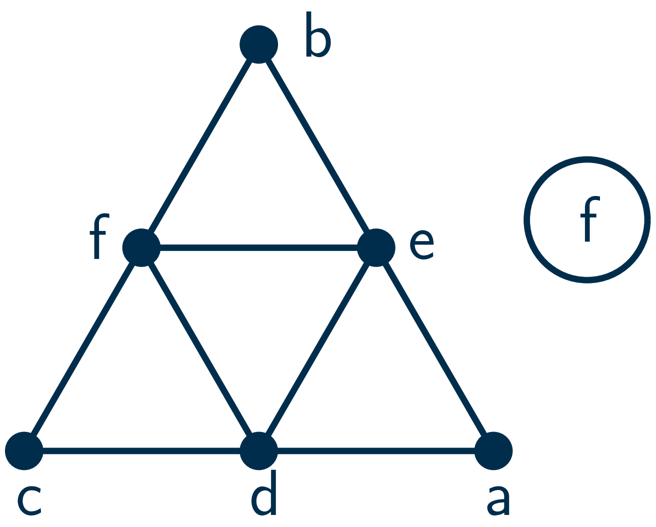
$T \leftarrow (\{\{\sigma(n)\}\}, \emptyset)$
for $i = n-1$ **to** 1 **do**
    $v \leftarrow \sigma(i)$
    $A \leftarrow \{v\} \cup (\mathrm{Adj}(v) \cap \{\sigma(i+1), \ldots, \sigma(n)\})$
    if $\mathrm{Adj}(v) \cap \{\sigma(i+1), \ldots, \sigma(n)\}$ is maximal clique $G_{\{\sigma(i+1),\ldots,\sigma(n)\}}$
        $B \leftarrow$ vertex of T corresponding to $\mathrm{Adj}(v) \cap \{\sigma(i+1), \ldots, \sigma(n)\}$
        $B \leftarrow B \cup \{v\}$
    else
        $B \leftarrow$ vertices of T with $\mathrm{Adj}(v) \cap \{\sigma(i+1), \ldots, \sigma(n)\} \subsetneq B$
        $V_T \leftarrow V_T \cup \{A\}$, $E_T \leftarrow E_T \cup \{AB\}$
$\forall v \in V(G)$: identify vertices of T that contain v.



Exercise 1

Input: G chordal, PES σ von G

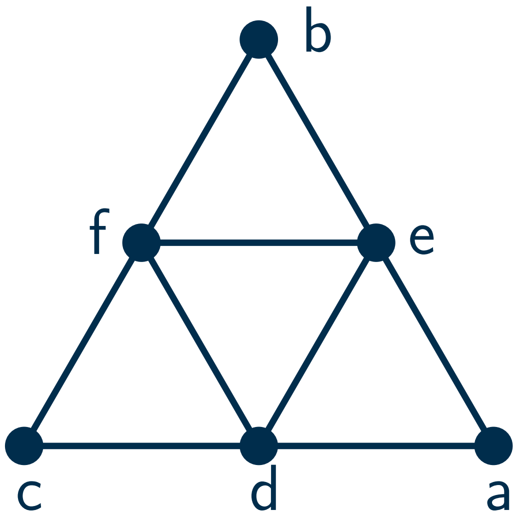
$T \leftarrow (\{\{\sigma(n)\}\}, \emptyset)$
for $i = n-1$ **to** 1 **do**
  $v \leftarrow \sigma(i)$
  $A \leftarrow \{v\} \cup (\mathrm{Adj}(v) \cap \{\sigma(i+1), \ldots, \sigma(n)\})$
  if $\mathrm{Adj}(v) \cap \{\sigma(i+1), \ldots, \sigma(n)\}$ is maximal clique $G_{\{\sigma(i+1), \ldots, \sigma(n)\}}$
    $B \leftarrow$ vertex of T corresponding to $\mathrm{Adj}(v) \cap \{\sigma(i+1), \ldots, \sigma(n)\}$
    $B \leftarrow B \cup \{v\}$
  else
    $B \leftarrow$ vertices of T with $\mathrm{Adj}(v) \cap \{\sigma(i+1), \ldots, \sigma(n)\} \subsetneq B$
    $V_T \leftarrow V_T \cup \{A\}$, $E_T \leftarrow E_T \cup \{AB\}$
$\forall v \in V(G)$: identify vertices of T that contain v.



Exercise 1

Input: G chordal, PES σ von G

$T \leftarrow (\{\{\sigma(n)\}\}, \emptyset)$
for $i = n-1$ **to** 1 **do**
    $v \leftarrow \sigma(i)$
    $A \leftarrow \{v\} \cup (\mathrm{Adj}(v) \cap \{\sigma(i+1), \ldots, \sigma(n)\})$
    if $\mathrm{Adj}(v) \cap \{\sigma(i+1), \ldots, \sigma(n)\}$ is maximal clique $G_{\{\sigma(i+1),\ldots,\sigma(n)\}}$
        $B \leftarrow$ vertex of T corresponding to $\mathrm{Adj}(v) \cap \{\sigma(i+1), \ldots, \sigma(n)\}$
        $B \leftarrow B \cup \{v\}$
    else
        $B \leftarrow$ vertices of T with $\mathrm{Adj}(v) \cap \{\sigma(i+1), \ldots, \sigma(n)\} \subsetneq B$
        $V_T \leftarrow V_T \cup \{A\}$, $E_T \leftarrow E_T \cup \{AB\}$
$\forall v \in V(G)$: identify vertices of T that contain v.

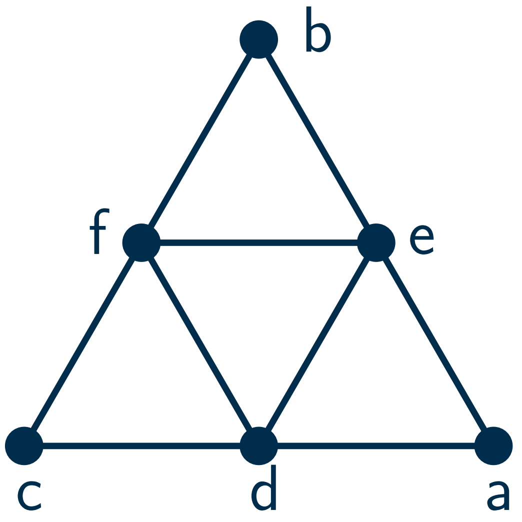


$i = 5$:

Exercise 1

Input: G chordal, PES σ von G

$T \leftarrow (\{\{\sigma(n)\}\}, \emptyset)$
for $i = n-1$ **to** 1 **do**
  $v \leftarrow \sigma(i)$
  $A \leftarrow \{v\} \cup (\mathrm{Adj}(v) \cap \{\sigma(i+1), \ldots, \sigma(n)\})$
  if $\mathrm{Adj}(v) \cap \{\sigma(i+1), \ldots, \sigma(n)\}$ is maximal clique $G_{\{\sigma(i+1),\ldots,\sigma(n)\}}$
    $B \leftarrow$ vertex of T corresponding to $\mathrm{Adj}(v) \cap \{\sigma(i+1), \ldots, \sigma(n)\}$
    $B \leftarrow B \cup \{v\}$
  else
    $B \leftarrow$ vertices of T with $\mathrm{Adj}(v) \cap \{\sigma(i+1), \ldots, \sigma(n)\} \subsetneq B$
    $V_T \leftarrow V_T \cup \{A\}$, $E_T \leftarrow E_T \cup \{AB\}$
$\forall v \in V(G)$: identify vertices of T that contain v.



f

$i = 5$:

e f

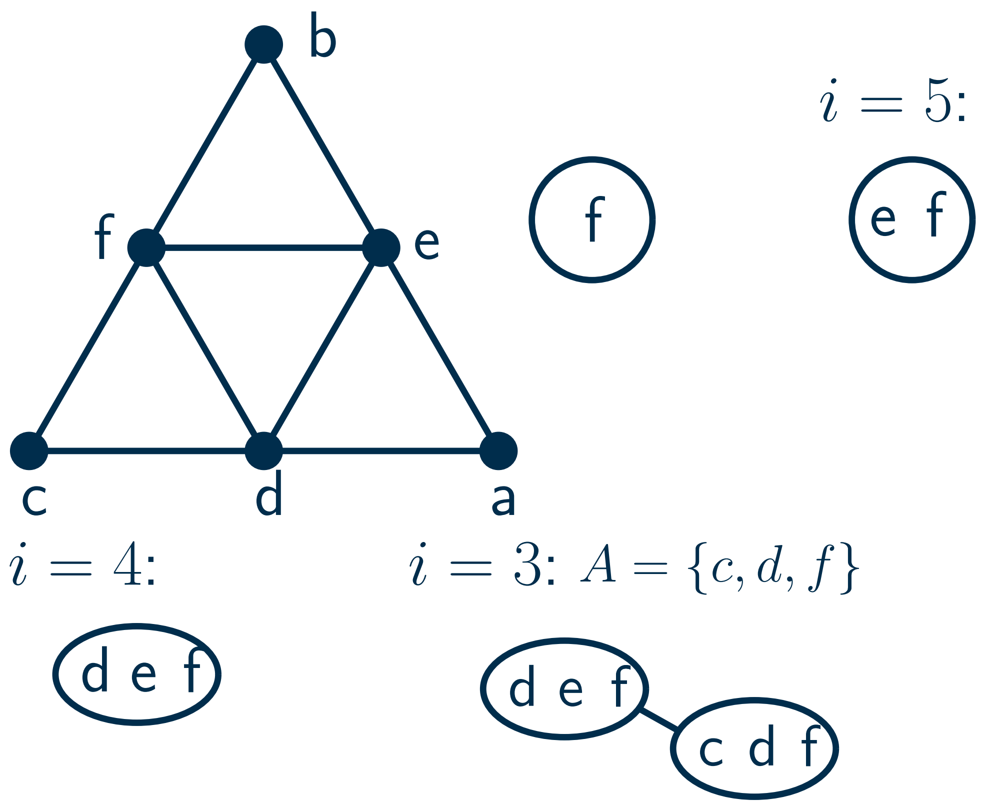
$i = 4$:

d e f

Exercise 1

Input: G chordal, PES σ von G

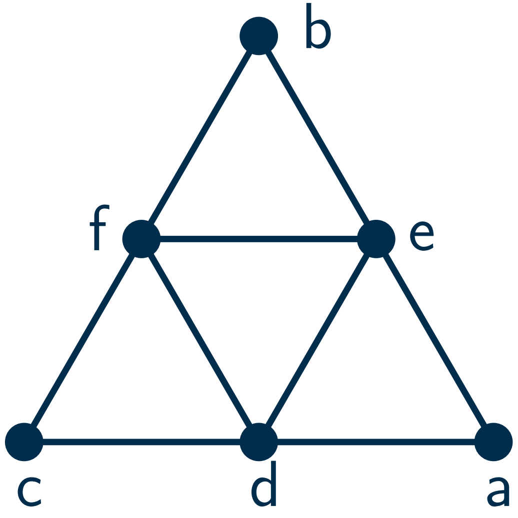
$T \leftarrow (\{\{\sigma(n)\}\}, \emptyset)$
for $i = n-1$ **to** 1 **do**
  $v \leftarrow \sigma(i)$
  $A \leftarrow \{v\} \cup (\mathrm{Adj}(v) \cap \{\sigma(i+1), \ldots, \sigma(n)\})$
  if $\mathrm{Adj}(v) \cap \{\sigma(i+1), \ldots, \sigma(n)\}$ is maximal clique $G_{\{\sigma(i+1),\ldots,\sigma(n)\}}$
    $B \leftarrow$ vertex of T corresponding to $\mathrm{Adj}(v) \cap \{\sigma(i+1), \ldots, \sigma(n)\}$
    $B \leftarrow B \cup \{v\}$
  else
    $B \leftarrow$ vertices of T with $\mathrm{Adj}(v) \cap \{\sigma(i+1), \ldots, \sigma(n)\} \subsetneq B$
    $V_T \leftarrow V_T \cup \{A\}$, $E_T \leftarrow E_T \cup \{AB\}$
$\forall v \in V(G)$: identify vertices of T that contain v.



Exercise 1

Input: G chordal, PES σ von G

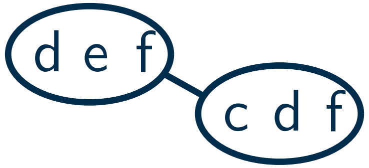
$T \leftarrow (\{\{\sigma(n)\}\}, \emptyset)$
for $i = n-1$ **to** 1 **do**
    $v \leftarrow \sigma(i)$
    $A \leftarrow \{v\} \cup (\mathrm{Adj}(v) \cap \{\sigma(i+1), \ldots, \sigma(n)\})$
    if $\mathrm{Adj}(v) \cap \{\sigma(i+1), \ldots, \sigma(n)\}$ is maximal clique $G_{\{\sigma(i+1),\ldots,\sigma(n)\}}$
        $B \leftarrow$ vertex of T corresponding to $\mathrm{Adj}(v) \cap \{\sigma(i+1), \ldots, \sigma(n)\}$
        $B \leftarrow B \cup \{v\}$
    else
        $B \leftarrow$ vertices of T with $\mathrm{Adj}(v) \cap \{\sigma(i+1), \ldots, \sigma(n)\} \subsetneq B$
        $V_T \leftarrow V_T \cup \{A\}$, $E_T \leftarrow E_T \cup \{AB\}$
$\forall v \in V(G)$: identify vertices of T that contain v.



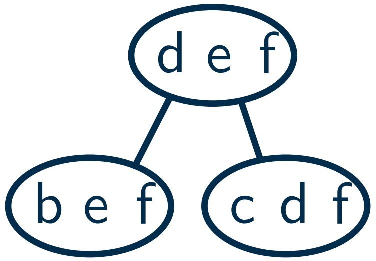
$i = 5$:

$i = 4$:

$i = 3$: $A = \{c, d, f\}$



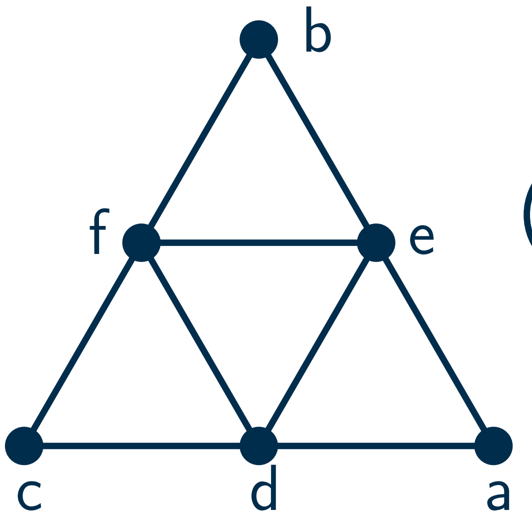
$i = 2$: $A = \{b, e, f\}$



Exercise 1

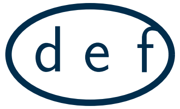
Input: G chordal, PES σ von G

$T \leftarrow (\{\{\sigma(n)\}\}, \emptyset)$
for $i = n-1$ **to** 1 **do**
  $v \leftarrow \sigma(i)$
  $A \leftarrow \{v\} \cup (\mathrm{Adj}(v) \cap \{\sigma(i+1), \ldots, \sigma(n)\})$
  if $\mathrm{Adj}(v) \cap \{\sigma(i+1), \ldots, \sigma(n)\}$ is maximal clique $G_{\{\sigma(i+1), \ldots, \sigma(n)\}}$
    $B \leftarrow$ vertex of T corresponding to $\mathrm{Adj}(v) \cap \{\sigma(i+1), \ldots, \sigma(n)\}$
    $B \leftarrow B \cup \{v\}$
  else
    $B \leftarrow$ vertices of T with $\mathrm{Adj}(v) \cap \{\sigma(i+1), \ldots, \sigma(n)\} \subsetneq B$
    $V_T \leftarrow V_T \cup \{A\}$, $E_T \leftarrow E_T \cup \{AB\}$
$\forall v \in V(G)$: identify vertices of T that contain v.

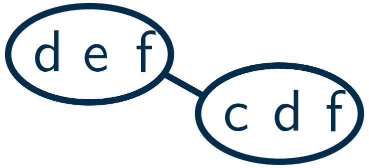


$i = 5$:

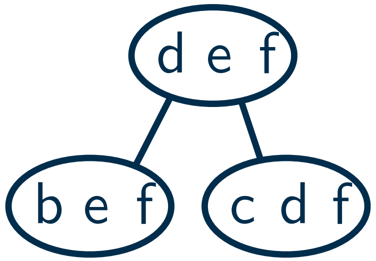
$i = 4$:



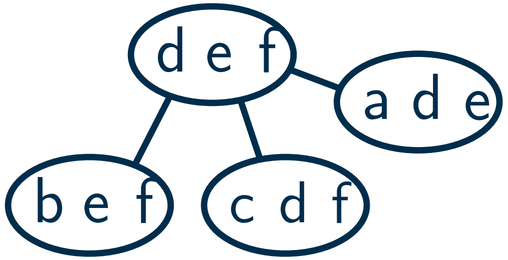
$i = 3$: $A = \{c, d, f\}$



$i = 2$: $A = \{b, e, f\}$



$i = 1$: $A = \{a, d, e\}$

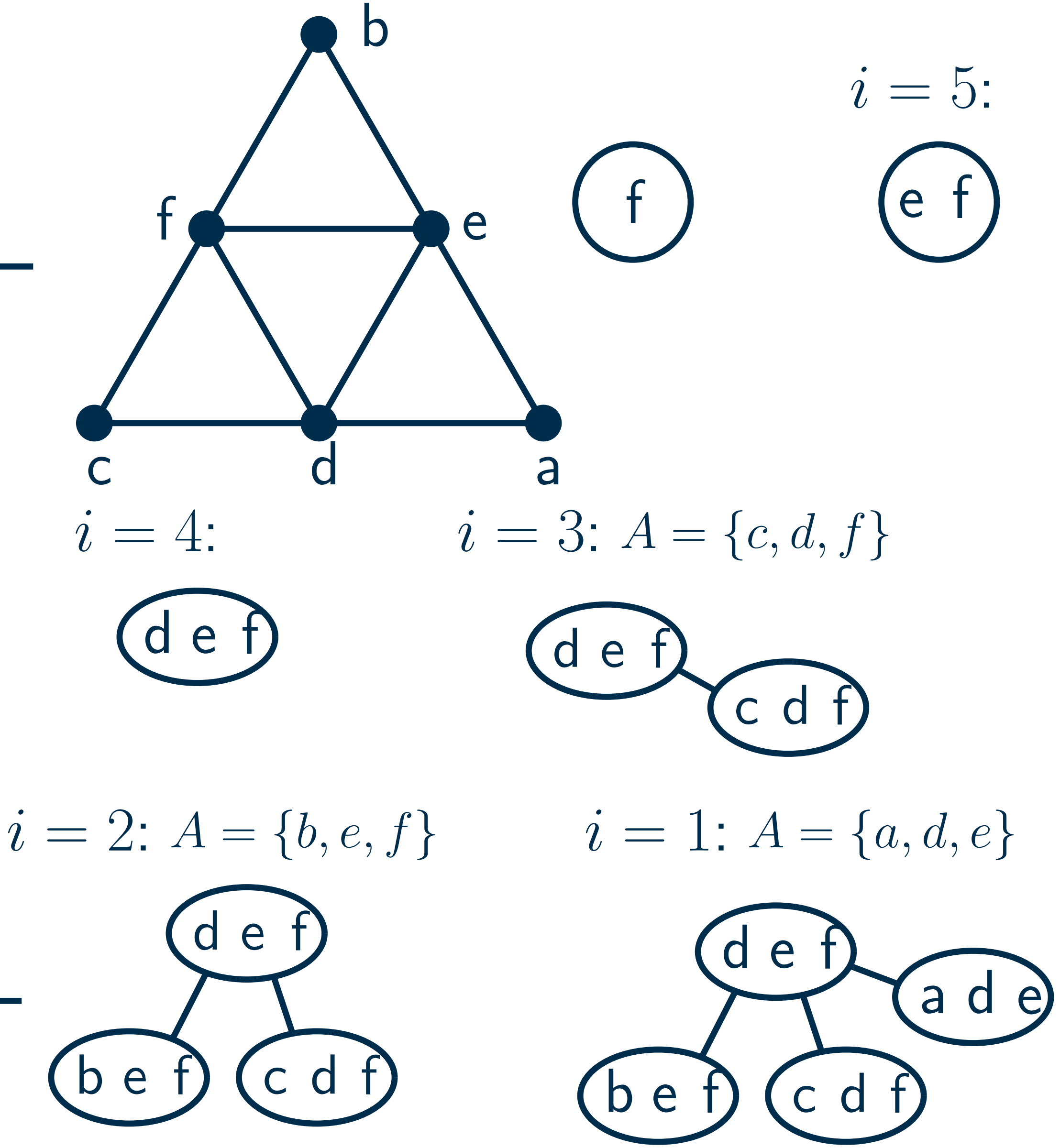


Exercise 1

Input: G chordal, PES σ von G

$T \leftarrow (\{\{\sigma(n)\}\}, \emptyset)$
for $i = n-1$ **to** 1 **do**
    $v \leftarrow \sigma(i)$
    $A \leftarrow \{v\} \cup (\mathrm{Adj}(v) \cap \{\sigma(i+1), \ldots, \sigma(n)\})$
    if $\mathrm{Adj}(v) \cap \{\sigma(i+1), \ldots, \sigma(n)\}$ is maximal clique $G_{\{\sigma(i+1),\ldots,\sigma(n)\}}$
        $B \leftarrow$ vertex of T corresponding to $\mathrm{Adj}(v) \cap \{\sigma(i+1), \ldots, \sigma(n)\}$
        $B \leftarrow B \cup \{v\}$
    else
        $B \leftarrow$ vertices of T with $\mathrm{Adj}(v) \cap \{\sigma(i+1), \ldots, \sigma(n)\} \subsetneq B$
        $V_T \leftarrow V_T \cup \{A\}$, $E_T \leftarrow E_T \cup \{AB\}$
$\forall v \in V(G)$: identify vertices of T that contain v.

Subtree for each $v \in V(G)$ is obtained by taking the tree vertices that contain v.



Exercise 2

Because of symmetry we only have to show one of the implications each.

(i) $ab \ \Gamma \ a'b' \Leftrightarrow ba \ \Gamma \ b'a'$

Exercise 2

Because of symmetry we only have to show one of the implications each.

(i) $ab \; \Gamma \; a'b' \Leftrightarrow ba \; \Gamma \; b'a'$

By definition:

$ab \; \Gamma \; a'b' \Leftrightarrow$ either $a = a'$ and $bb' \notin E(G)$ or $b = b'$ and $aa' \notin E(G)$

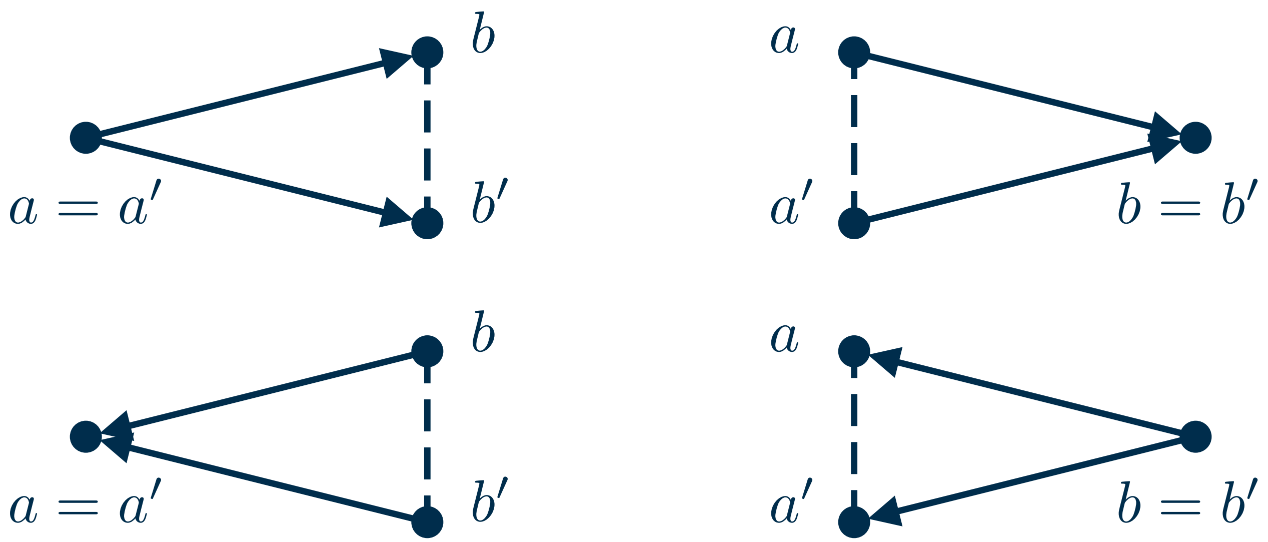
Exercise 2

Because of symmetry we only have to show one of the implications each.

(i) $ab\ \Gamma\ a'b' \Leftrightarrow ba\ \Gamma\ b'a'$

By definition:

$ab\ \Gamma\ a'b' \Leftrightarrow$ either $a = a'$ and $bb' \notin E(G)$ or $b = b'$ and $aa' \notin E(G)$



$\Leftrightarrow ba\ \Gamma\ b'a'$

Exercise 2

Because of symmetry we only have to show one of the implications each.

(i) $ab \ \Gamma \ a'b' \Leftrightarrow ba \ \Gamma \ b'a'$

(ii) $ab \ \Gamma^* \ a'b' \Leftrightarrow ba \ \Gamma^* \ b'a'$

Exercise 2

Because of symmetry we only have to show one of the implications each.

(i) $ab \; \Gamma \; a'b' \Leftrightarrow ba \; \Gamma \; b'a'$

(ii) $ab \; \Gamma^* \; a'b' \Leftrightarrow ba \; \Gamma^* \; b'a'$

$$ab \; \Gamma^* \; a'b' \Rightarrow ab = a_0b_0 \; \Gamma \; a_1b_1 \; \Gamma \; \dots \; \Gamma \; a_{k-1}b_{k-1} \; \Gamma \; a_kb_k = a'b'$$

Exercise 2

Because of symmetry we only have to show one of the implications each.

(i) $ab \; \Gamma \; a'b' \Leftrightarrow ba \; \Gamma \; b'a'$

(ii) $ab \; \Gamma^* \; a'b' \Leftrightarrow ba \; \Gamma^* \; b'a'$

$$\begin{aligned} ab \; \Gamma^* \; a'b' &\Rightarrow ab = a_0b_0 \; \Gamma \; a_1b_1 \; \Gamma \; \dots \; \Gamma \; a_{k-1}b_{k-1} \; \Gamma \; a_kb_k = a'b' \\ &\Rightarrow ba = b_0a_0 \; \Gamma \; b_1a_1 \; \Gamma \; \dots \; \Gamma \; b_{k-1}a_{k-1} \; \Gamma \; b_ka_k = b'a' \end{aligned}$$

Exercise 2

Because of symmetry we only have to show one of the implications each.

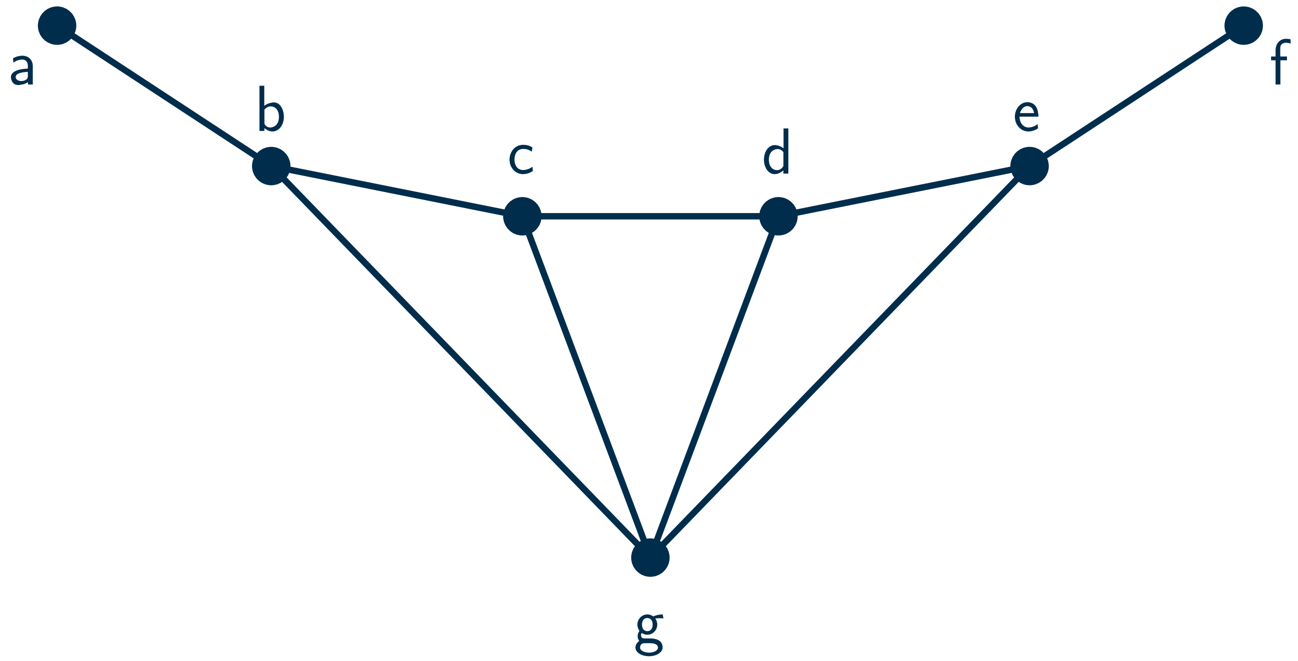
(i) $ab \; \Gamma \; a'b' \Leftrightarrow ba \; \Gamma \; b'a'$

(ii) $ab \; \Gamma^* \; a'b' \Leftrightarrow ba \; \Gamma^* \; b'a'$

$$\begin{aligned} ab \; \Gamma^* \; a'b' &\Rightarrow ab = a_0b_0 \; \Gamma \; a_1b_1 \; \Gamma \; \ldots \; \Gamma \; a_{k-1}b_{k-1} \; \Gamma \; a_kb_k = a'b' \\ &\Rightarrow ba = b_0a_0 \; \Gamma \; b_1a_1 \; \Gamma \; \ldots \; \Gamma \; b_{k-1}a_{k-1} \; \Gamma \; b_ka_k = b'a' \\ &\Rightarrow ba \; \Gamma^* \; b'a' \end{aligned}$$

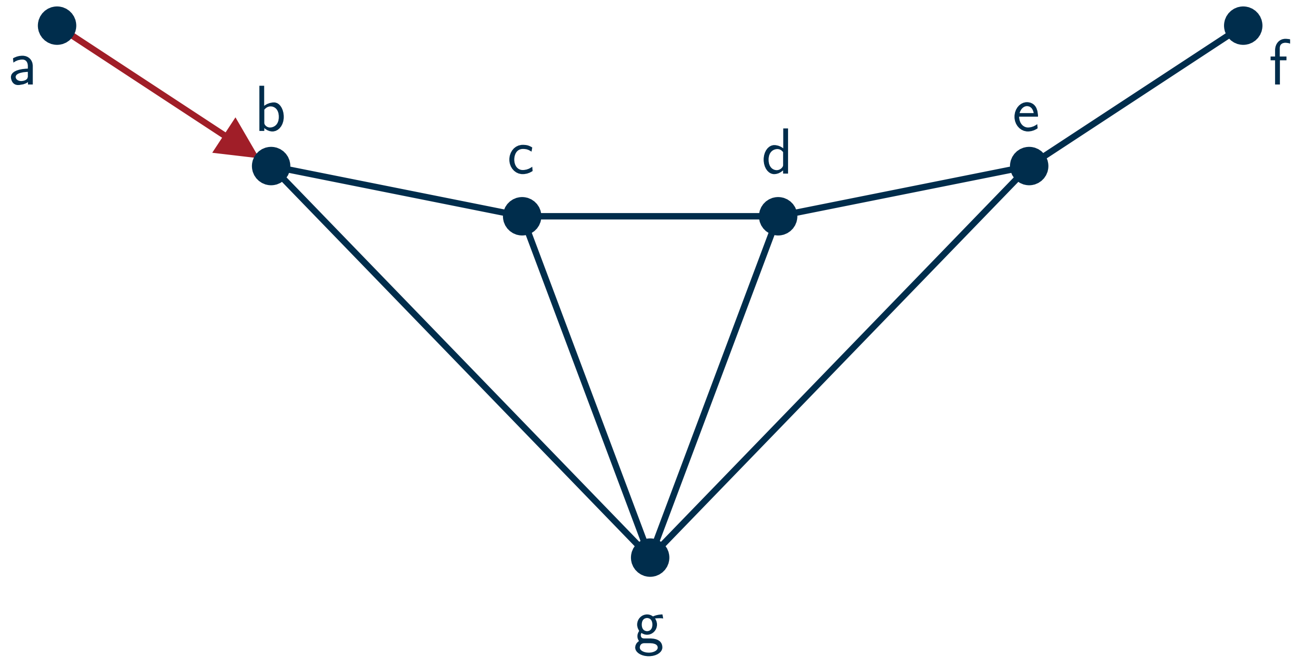
Exercise 3

Prove that the “bull head” is transitively orientationable. To do this prove that there is an implication class A such that $A = A^{-1}$.



Exercise 3

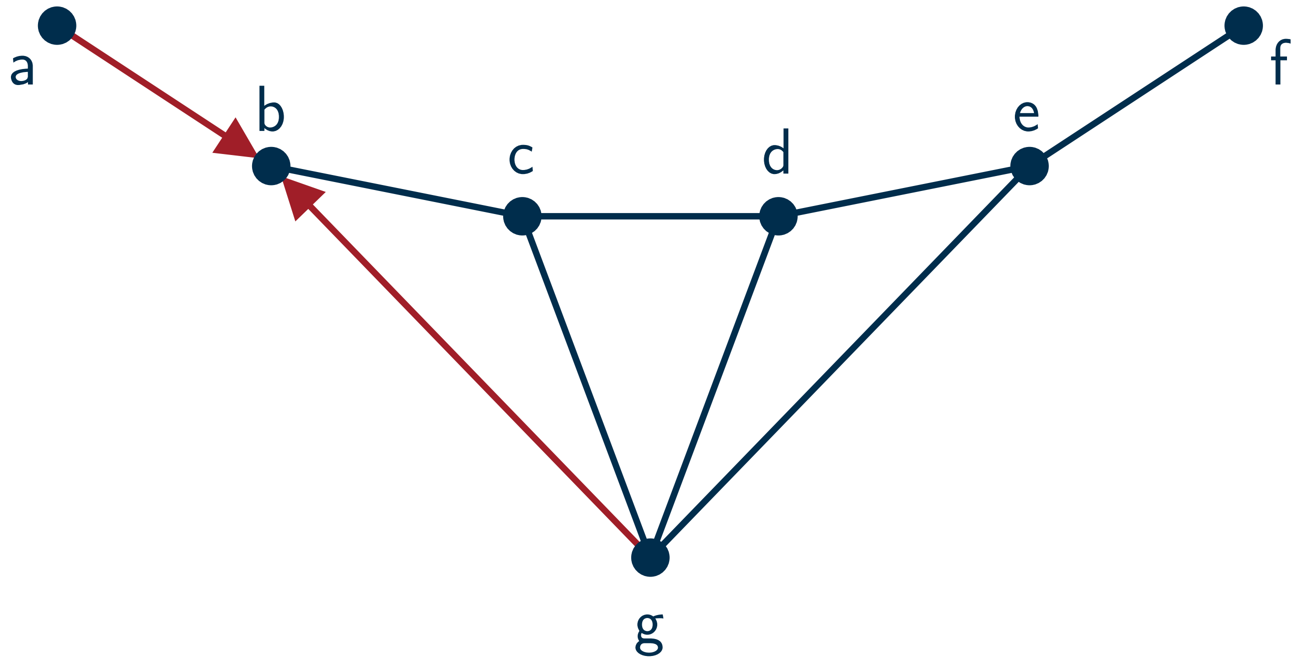
Prove that the "bull head" is transitively orientationable. To do this prove that there is an implication class A such that $A = A^{-1}$.



Let A be the implication class that contains ab.

Exercise 3

Prove that the “bull head” is transitively orientationable. To do this prove that there is an implication class A such that $A = A^{-1}$.

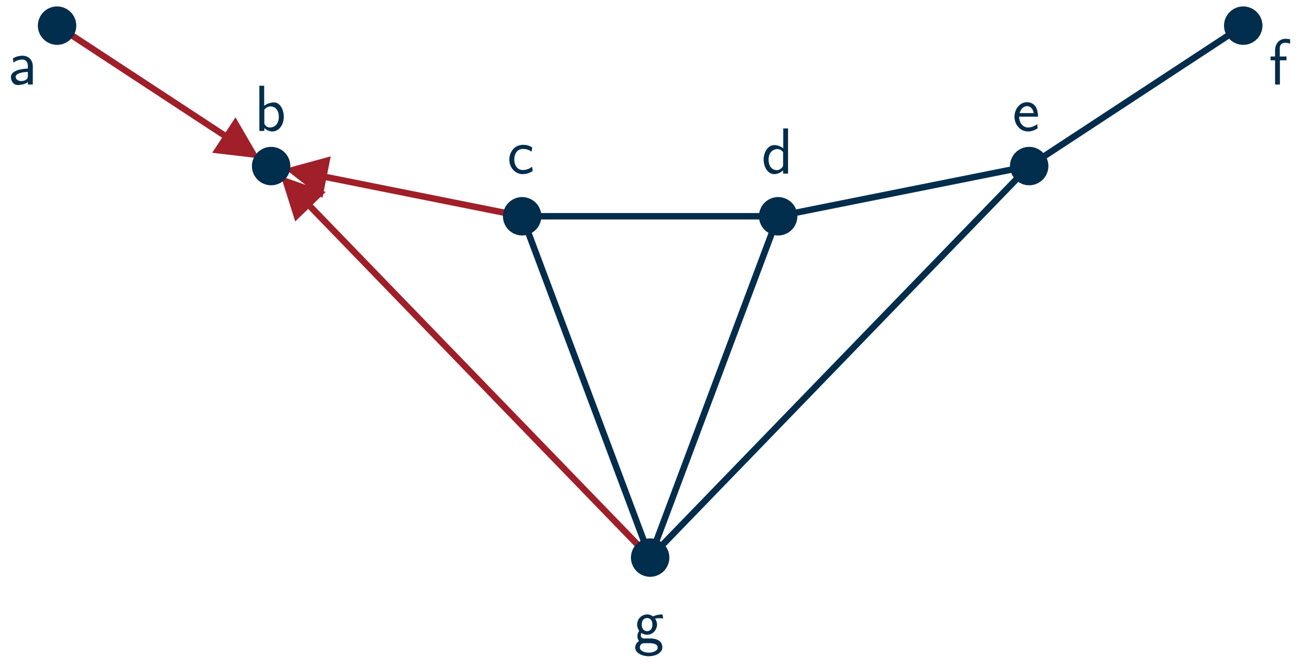


Let A be the implication class that contains ab.

$ab \; \Gamma \; gb$

Exercise 3

Prove that the "bull head" is transitively orientationable. To do this prove that there is an implication class A such that $A = A^{-1}$.



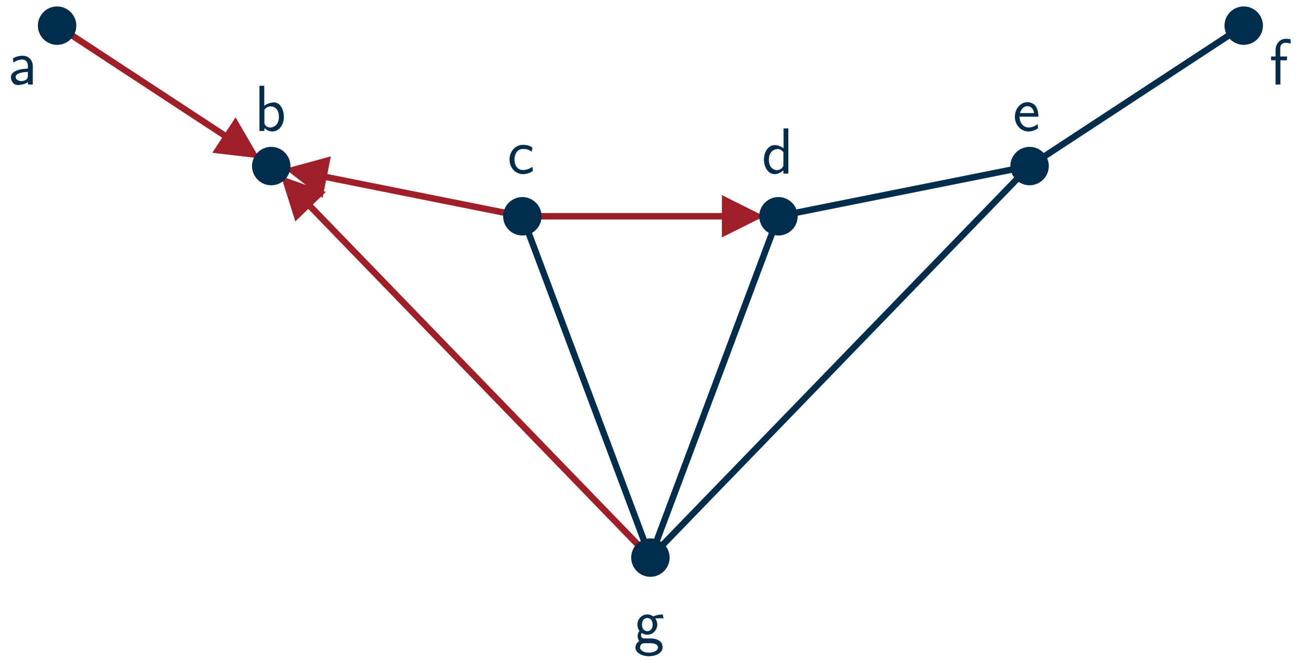
Let A be the implication class that contains ab.

$ab \; \Gamma \; gb$

$ab \; \Gamma \; cb$

Exercise 3

Prove that the "bull head" is transitively orientationable. To do this prove that there is an implication class A such that $A = A^{-1}$.



Let A be the implication class that contains ab.

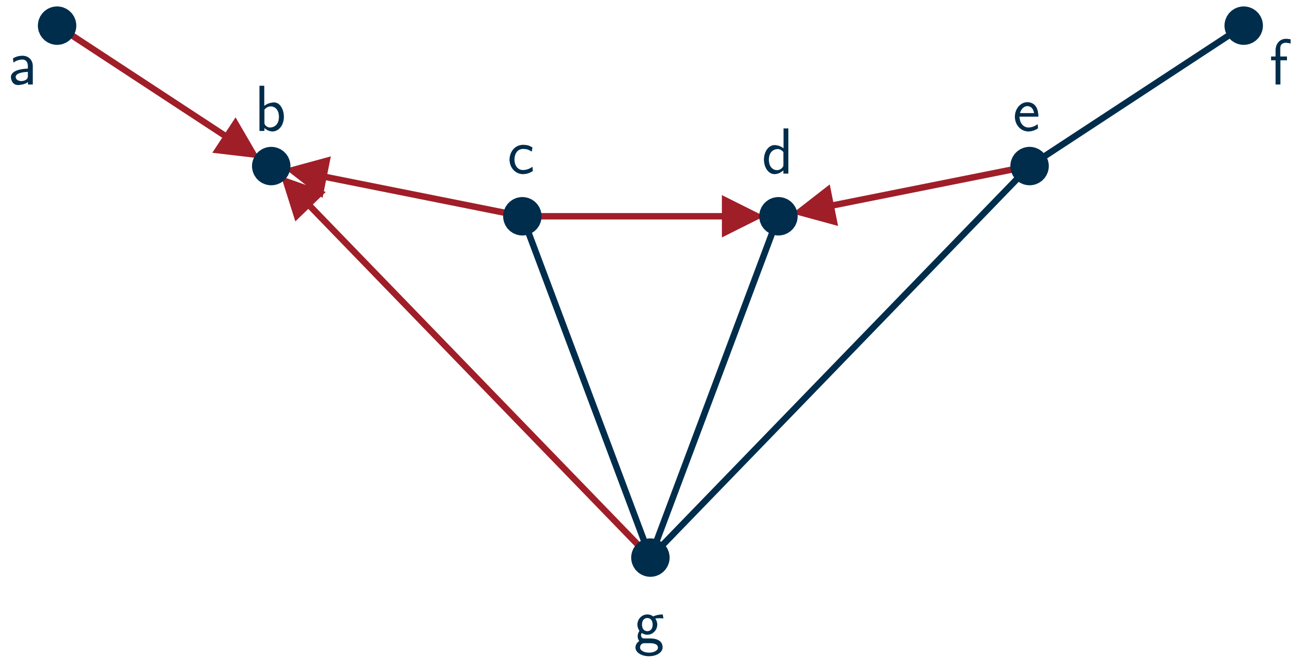
$ab \; \Gamma \; gb$

$ab \; \Gamma \; cb$

$cb \; \Gamma \; cd$

Exercise 3

Prove that the “bull head” is transitively orientationable. To do this prove that there is an implication class A such that $A = A^{-1}$.



Let A be the implication class that contains ab.

$ab \; \Gamma \; gb$

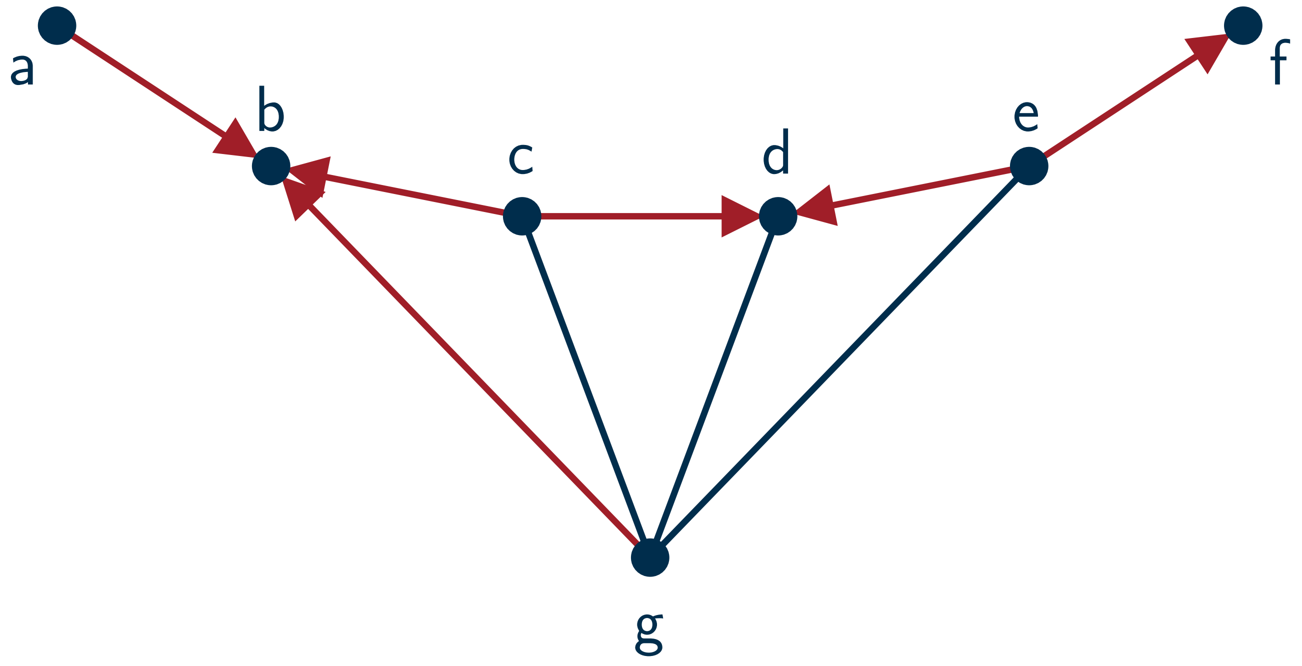
$ab \; \Gamma \; cb$

$cb \; \Gamma \; cd$

$cd \; \Gamma \; ed$

Exercise 3

Prove that the "bull head" is transitively orientationable. To do this prove that there is an implication class A such that $A = A^{-1}$.



Let A be the implication class that contains ab.

$ab \; \Gamma \; gb$

$ab \; \Gamma \; cb$

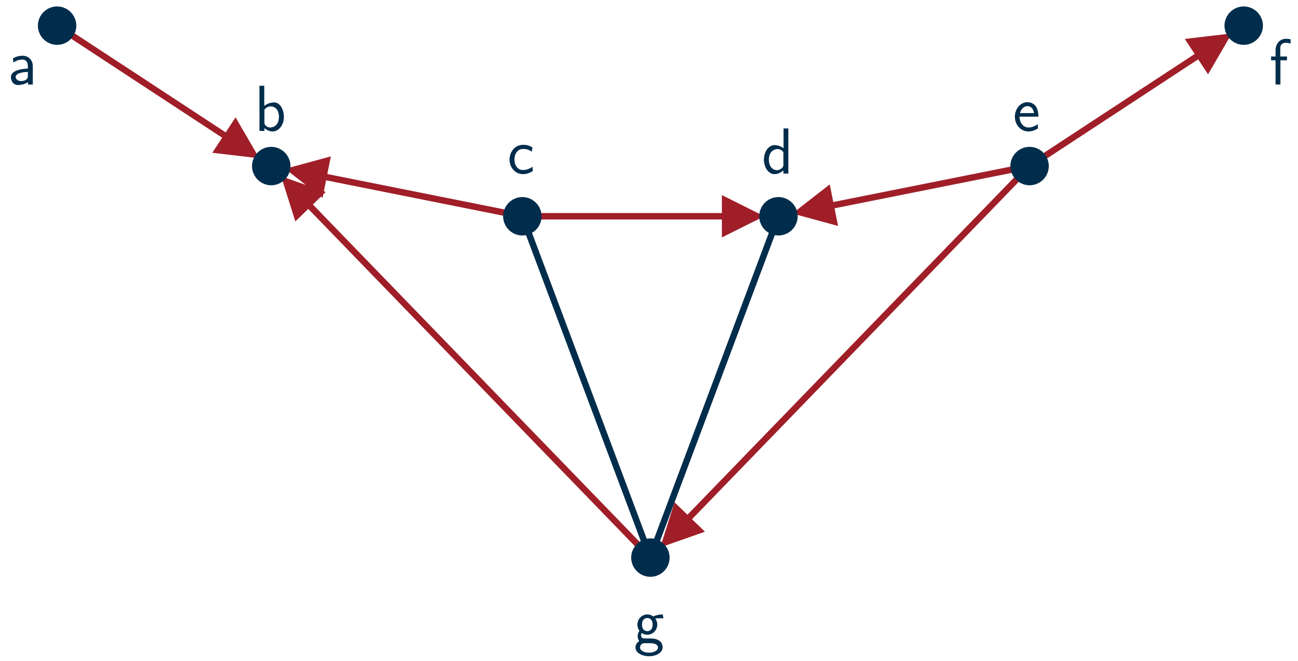
$cb \; \Gamma \; cd$

$cd \; \Gamma \; ed$

$ed \; \Gamma \; ef$

Exercise 3

Prove that the "bull head" is transitively orientationable. To do this prove that there is an implication class A such that $A = A^{-1}$.



Let A be the implication class that contains ab.

$ab \; \Gamma \; gb$ $\quad ed \; \Gamma \; ef$

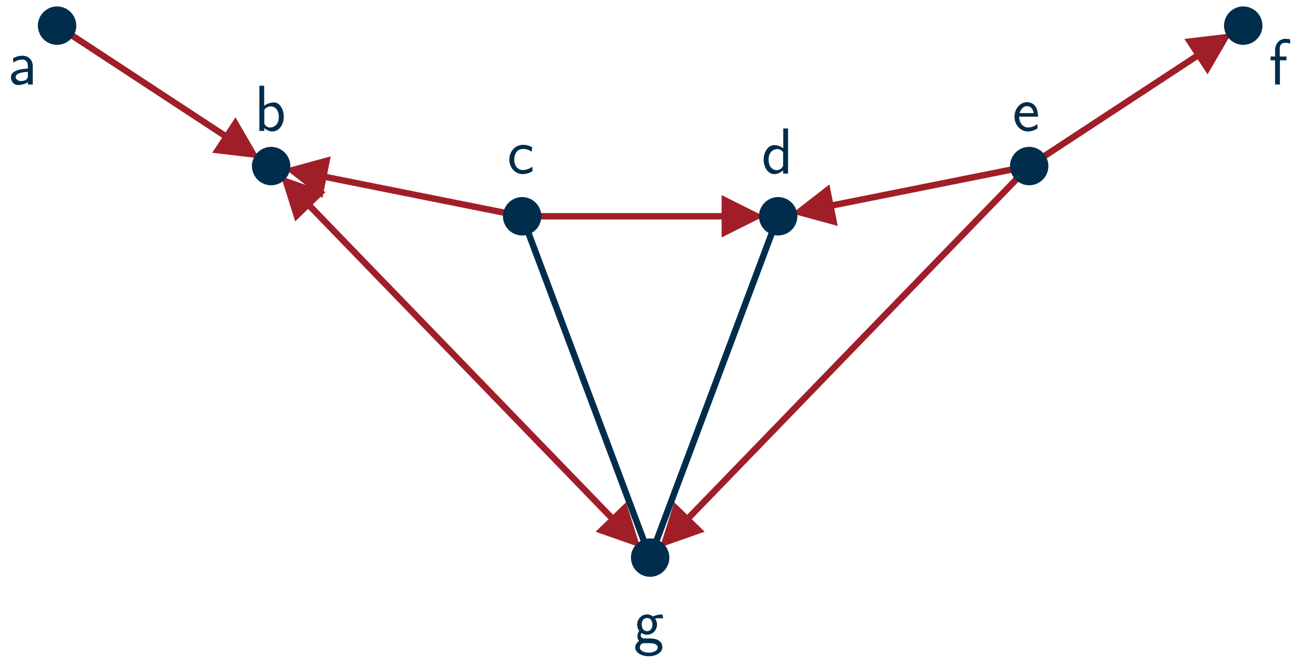
$ab \; \Gamma \; cb$ $\quad ef \; \Gamma \; eg$

$cb \; \Gamma \; cd$

$cd \; \Gamma \; ed$

Exercise 3

Prove that the “bull head” is transitively orientationable. To do this prove that there is an implication class A such that $A = A^{-1}$.



Let A be the implication class that contains ab.

$ab \; \Gamma \; gb$ $\quad$ $ed \; \Gamma \; ef$

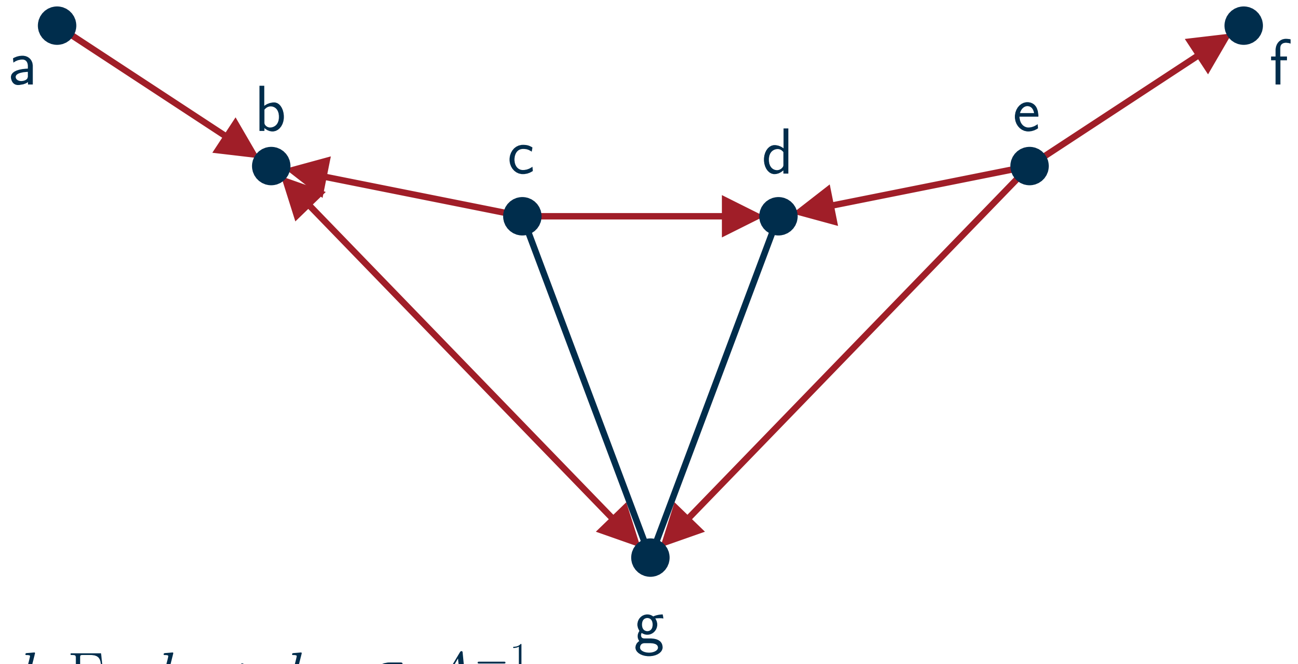
$ab \; \Gamma \; cb$ $\quad$ $ef \; \Gamma \; eg$

$cb \; \Gamma \; cd$ $\quad$ $eg \; \Gamma \; bg$

$cd \; \Gamma \; ed$

Exercise 3

Prove that the “bull head” is transitively orientationable. To do this prove that there is an implication class A such that $A = A^{-1}$.



$ab \; \Gamma \; gb \Rightarrow bg \in A^{-1}$

Let A be the implication class that contains ab.

$ab \; \Gamma \; gb$ $\quad$ $ed \; \Gamma \; ef$

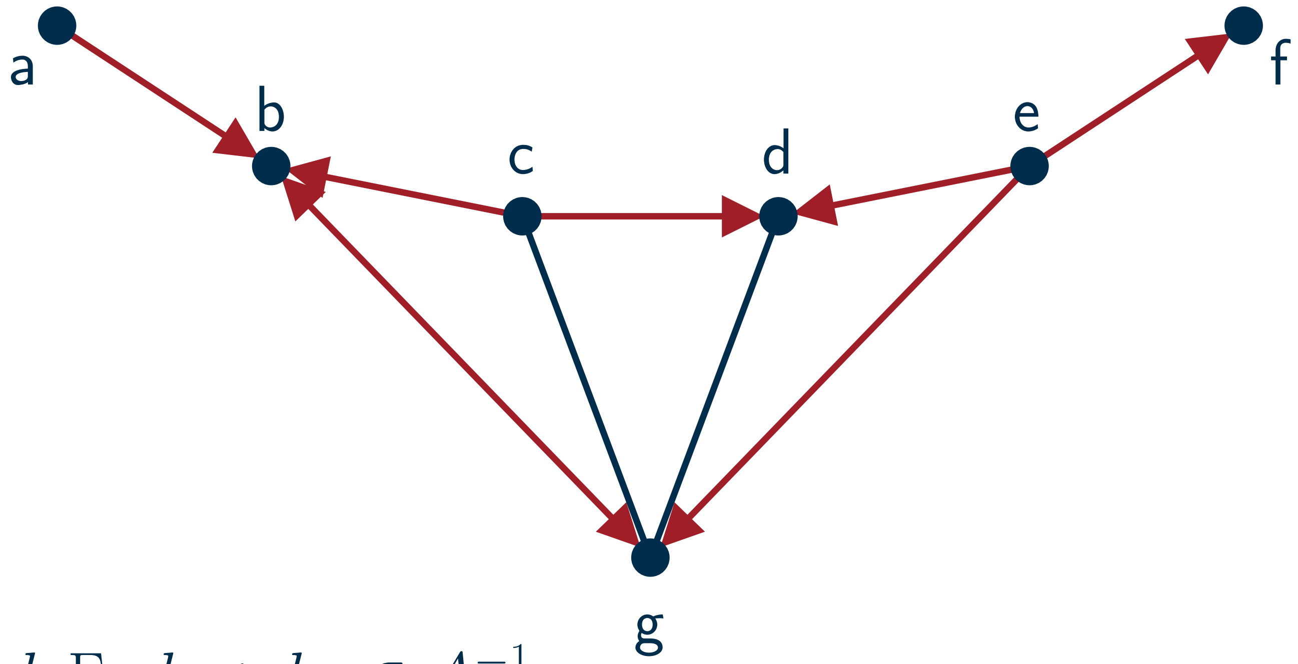
$ab \; \Gamma \; cb$ $\quad$ $ef \; \Gamma \; eg$

$cb \; \Gamma \; cd$ $\quad$ $eg \; \Gamma \; bg$

$cd \; \Gamma \; ed$

Exercise 3

Prove that the "bull head" is transitively orientationable. To do this prove that there is an implication class A such that $A = A^{-1}$.



$ab \; \Gamma \; gb \Rightarrow bg \in A^{-1}$

$ab \; \Gamma^* \; bg \Rightarrow bg \in A$

Let A be the implication class that contains ab.

$ab \; \Gamma \; gb$ $\quad$ $ed \; \Gamma \; ef$

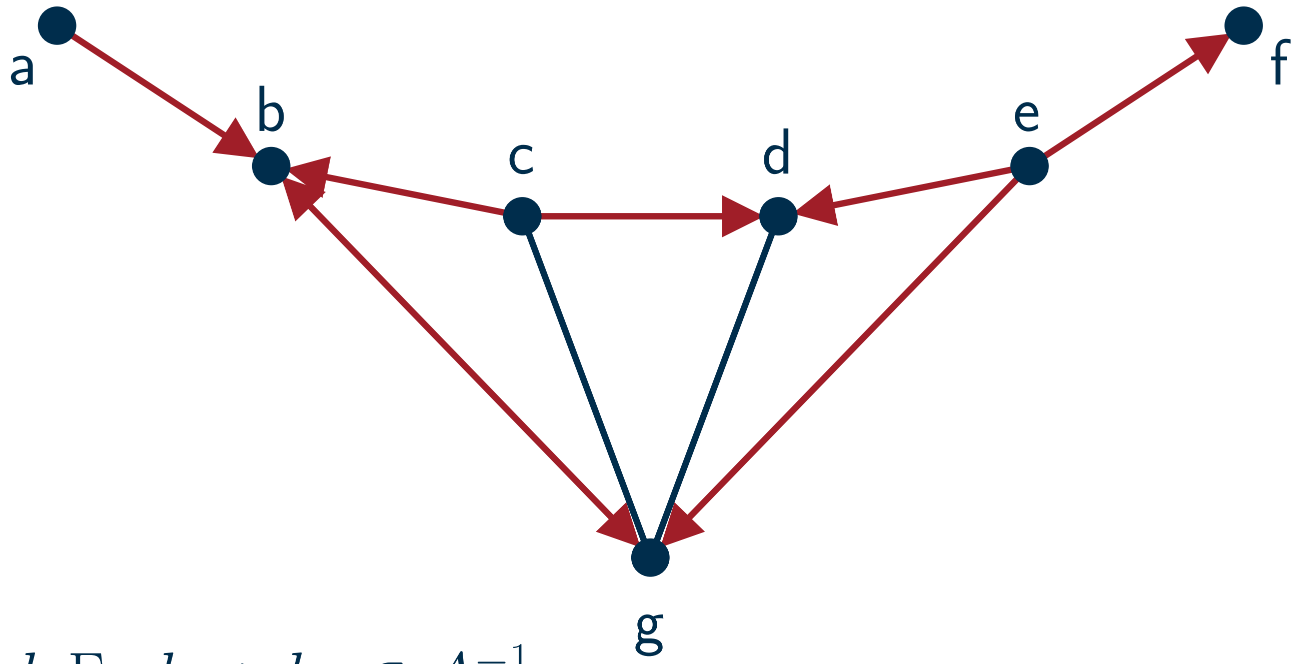
$ab \; \Gamma \; cb$ $\quad$ $ef \; \Gamma \; eg$

$cb \; \Gamma \; cd$ $\quad$ $eg \; \Gamma \; bg$

$cd \; \Gamma \; ed$

Exercise 3

Prove that the "bull head" is transitively orientationable. To do this prove that there is an implication class A such that $A = A^{-1}$.



Let A be the implication class that contains ab.

$ab \;\Gamma\; gb$ $\quad$ $ed \;\Gamma\; ef$

$ab \;\Gamma\; cb$ $\quad$ $ef \;\Gamma\; eg$

$cb \;\Gamma\; cd$ $\quad$ $eg \;\Gamma\; bg$

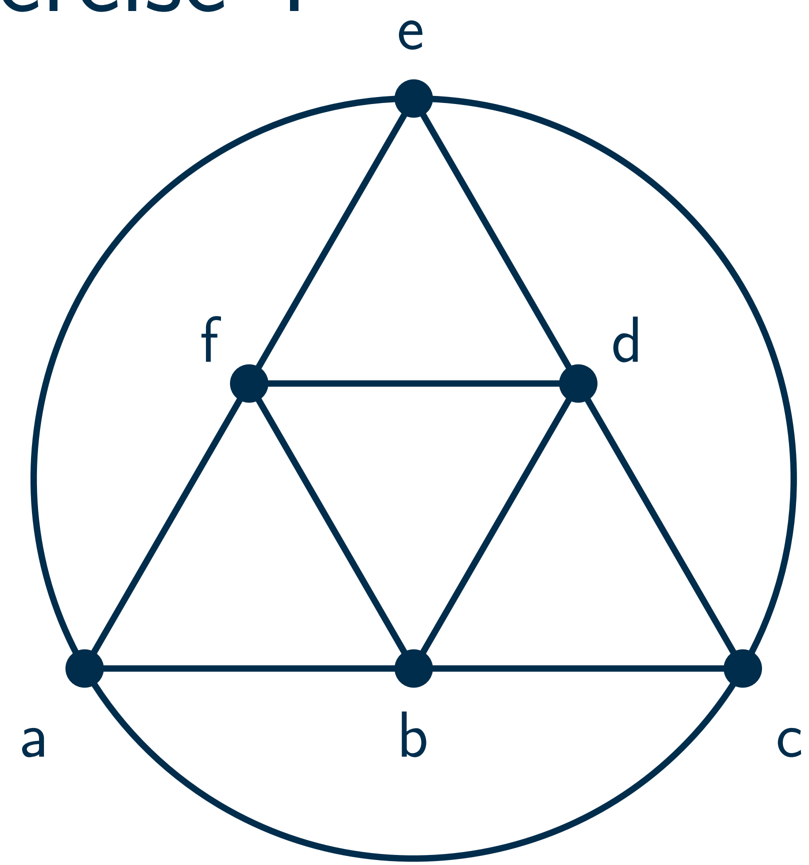
$cd \;\Gamma\; ed$

$ab \;\Gamma\; gb \Rightarrow bg \in A^{-1}$

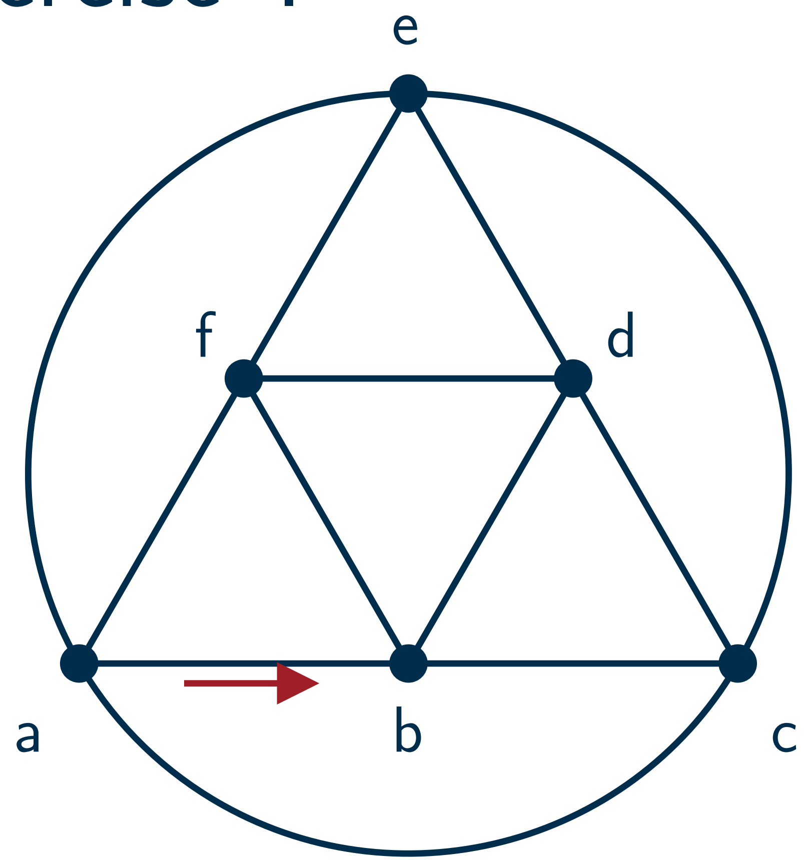
$ab \;\Gamma^*\; bg \Rightarrow bg \in A$

Therefore, we have $A \cap A^{-1} \neq \emptyset$ and thus $A = A^{-1}$.

Exercise 4

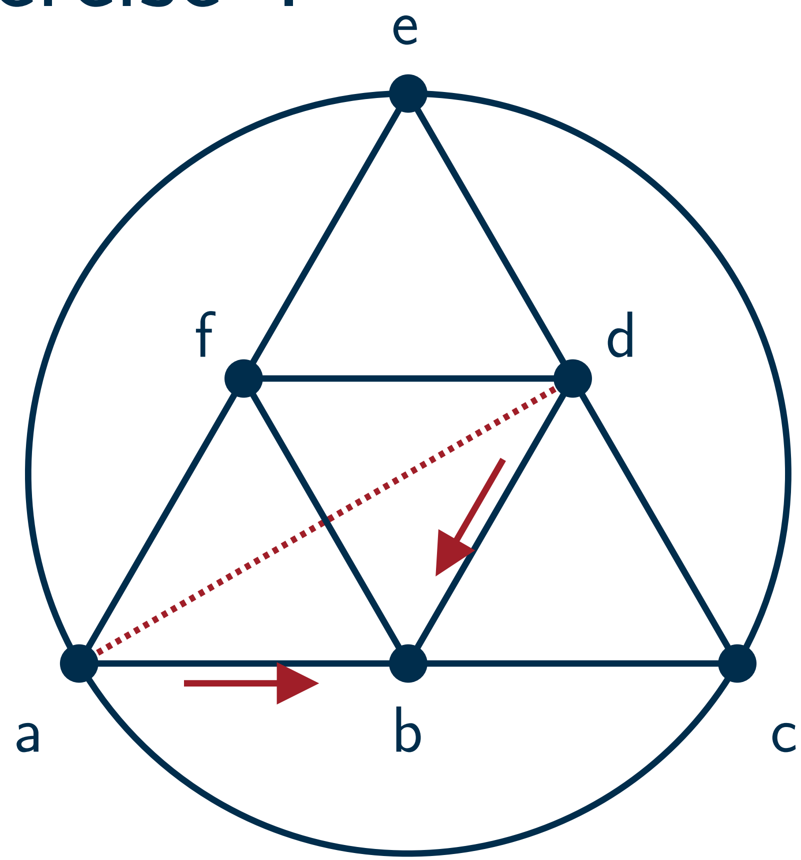


Exercise 4



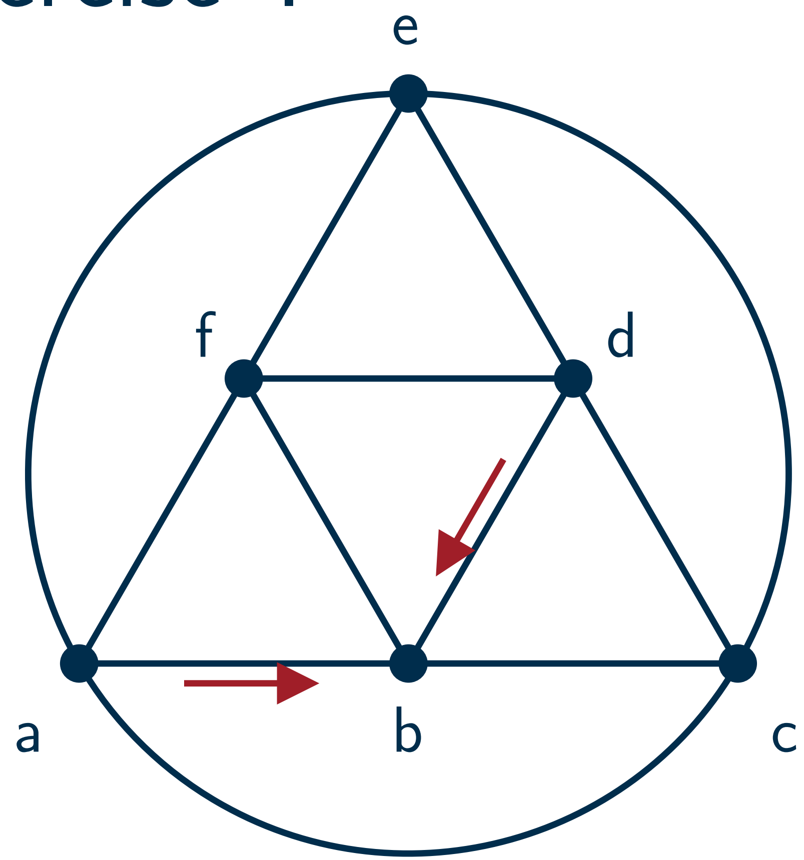
compute implication class of ab

Exercise 4



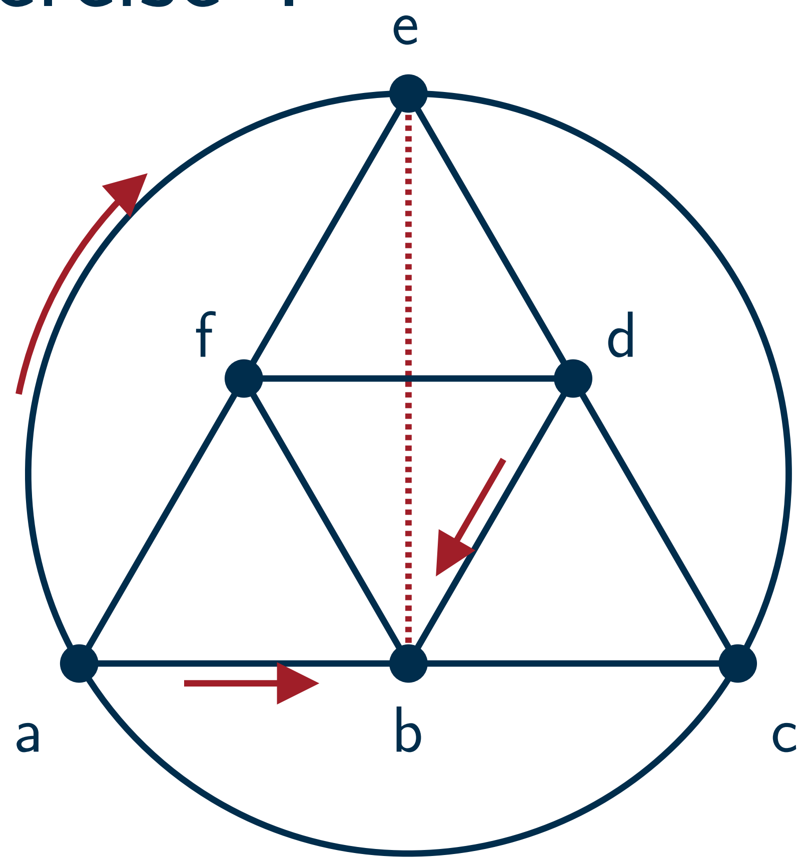
compute implication class of ab

Exercise 4



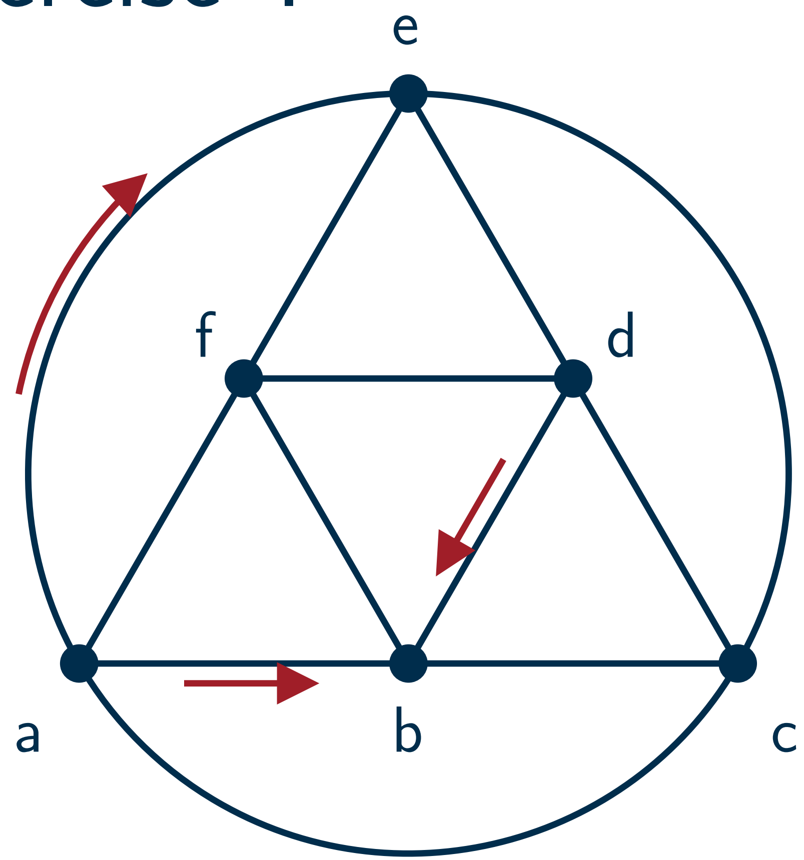
compute implication class of ab

Exercise 4



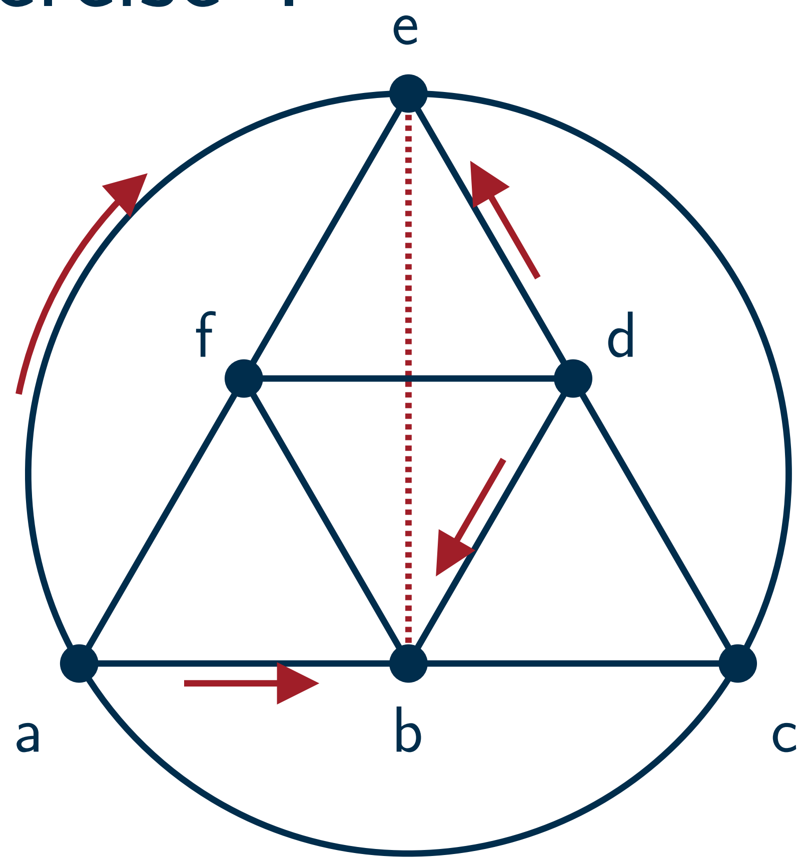
compute implication class of ab

Exercise 4



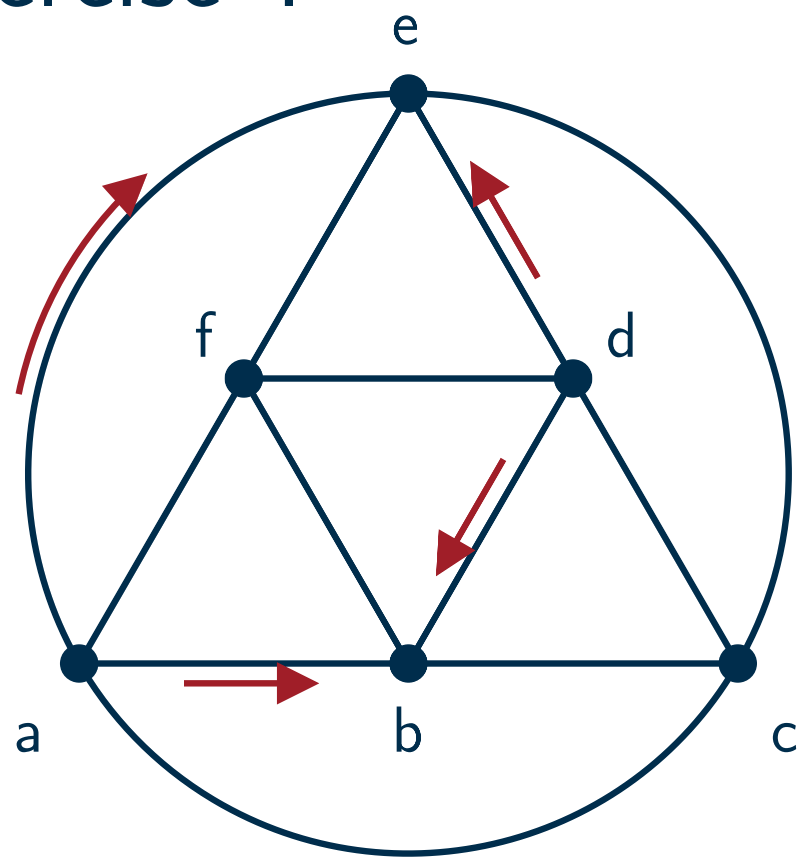
compute implication class of ab

Exercise 4



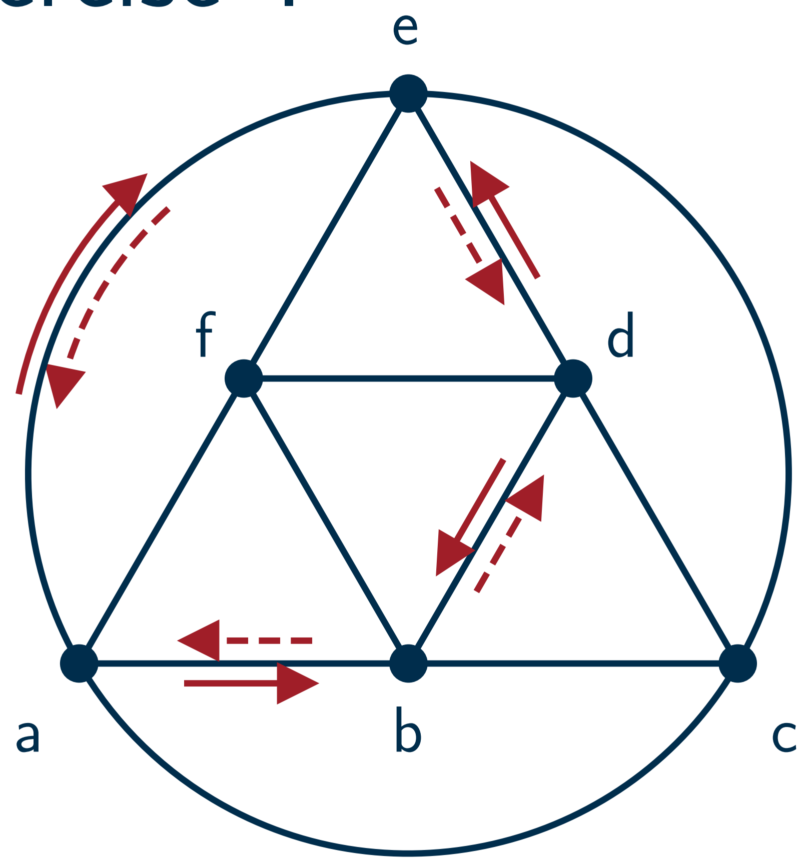
compute implication class of ab

Exercise 4



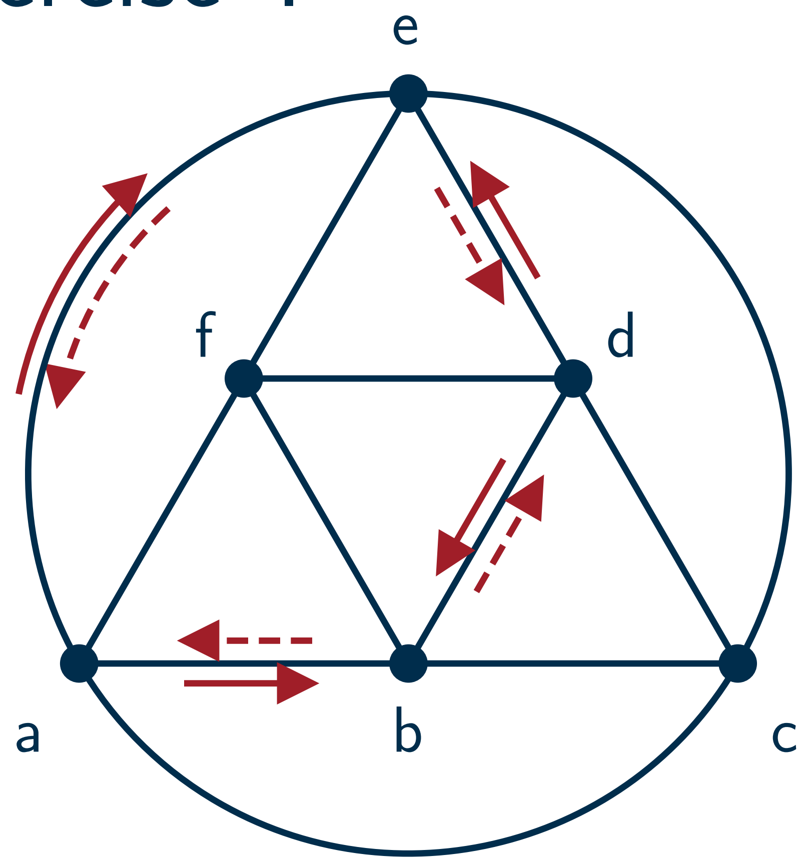
compute implication class of ab

Exercise 4



compute implication class of ab

Exercise 4

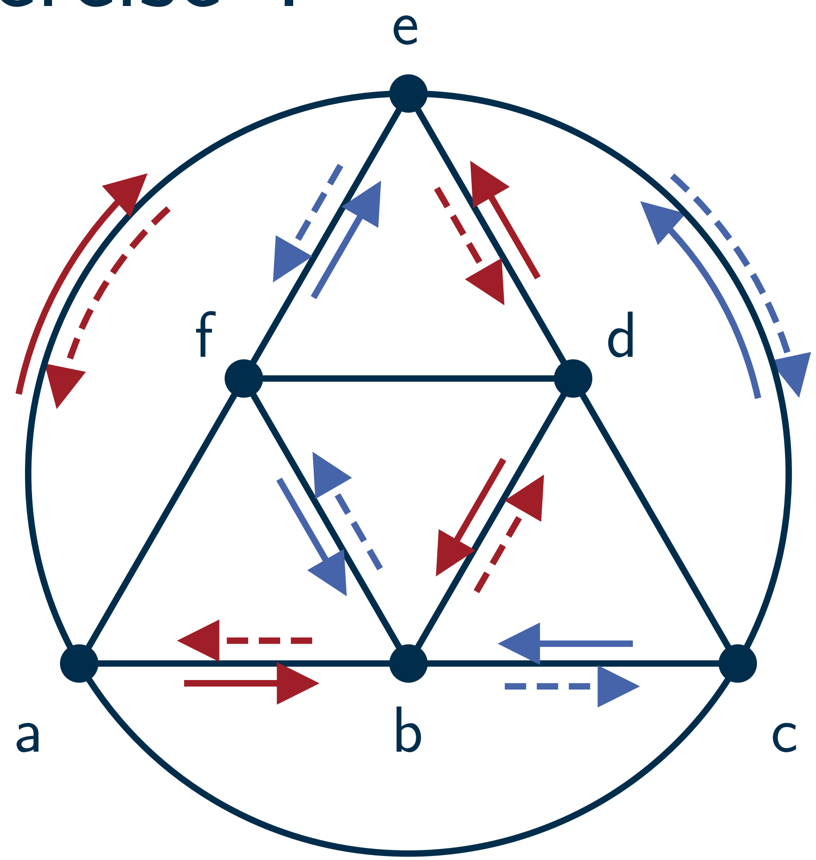


compute implication class of ab

$A_1 = \{ab, ae, db, de\}$

$A_1^{-1} = \{ba, ea, bd, ed\}$

Exercise 4



compute implication class of ab

$A_1 = \{ab, ae, db, de\}$

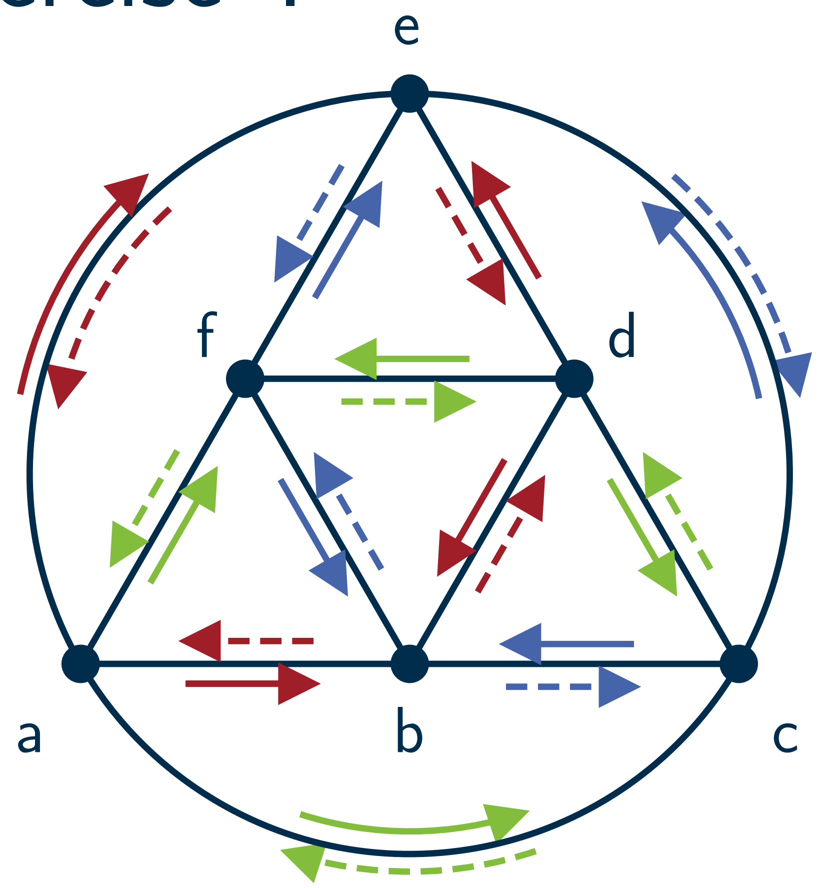
$A_1^{-1} = \{ba, ea, bd, ed\}$

by symmetry we get:

$A_2 = \{cb, ce, fe, fb\}$

$A_2^{-1} = \{bc, ec, ef, bf\}$

Exercise 4



compute implication class of ab

$A_1 = \{ab, ae, db, de\}$

$A_1^{-1} = \{ba, ea, bd, ed\}$

by symmetry we get:

$A_2 = \{cb, ce, fe, fb\}$

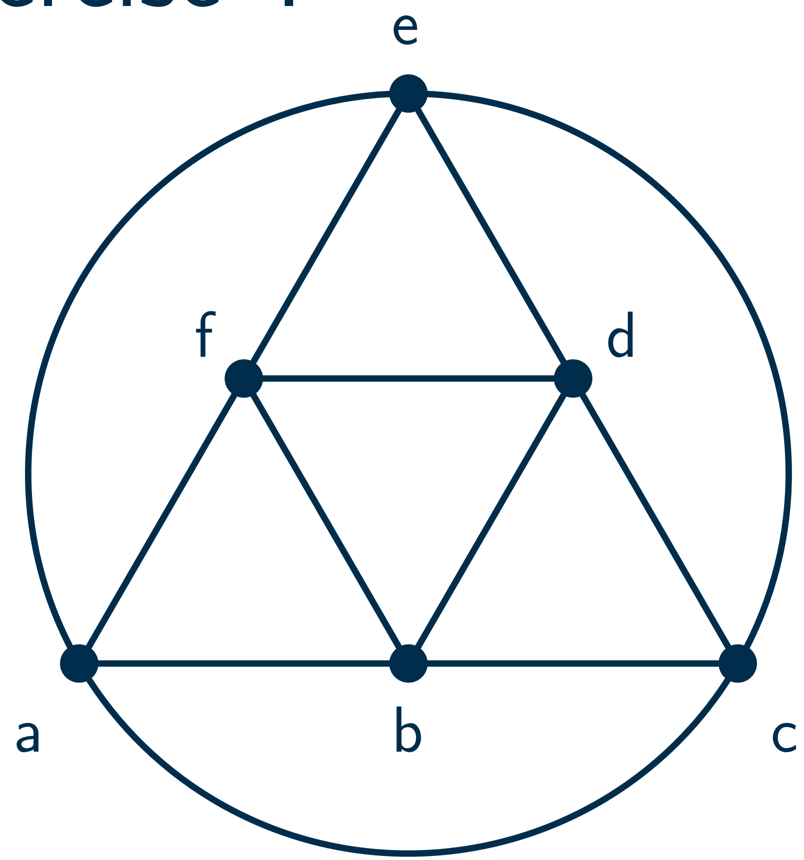
$A_2^{-1} = \{bc, ec, ef, bf\}$

and

$A_3 = \{ac, af, dc, df\}$

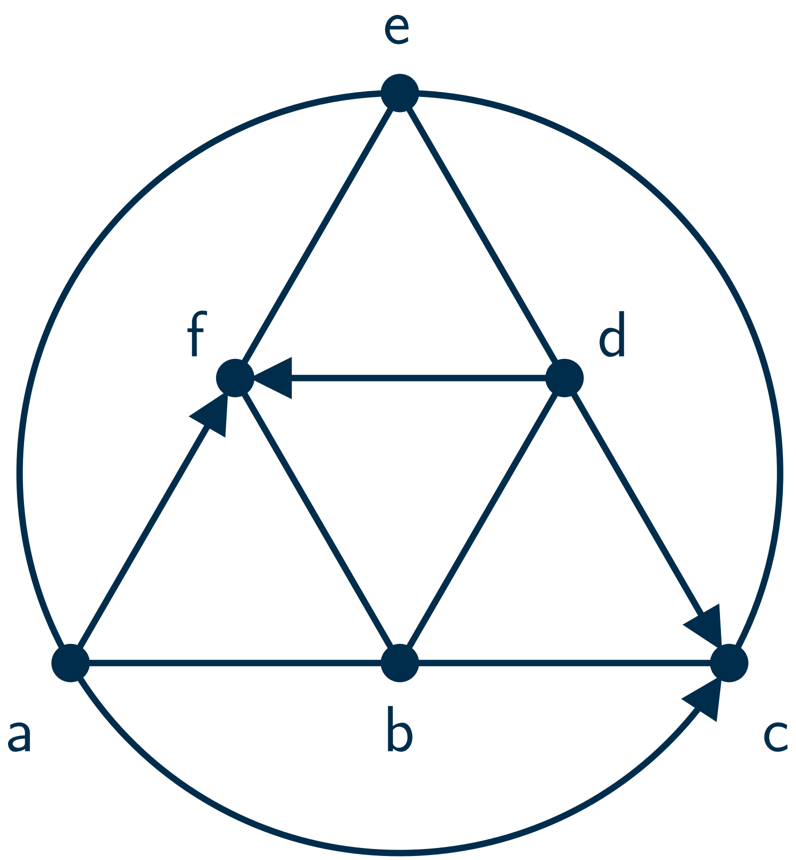
$A_3^{-1} = \{ca, fa, cd, fd\}$

Exercise 4

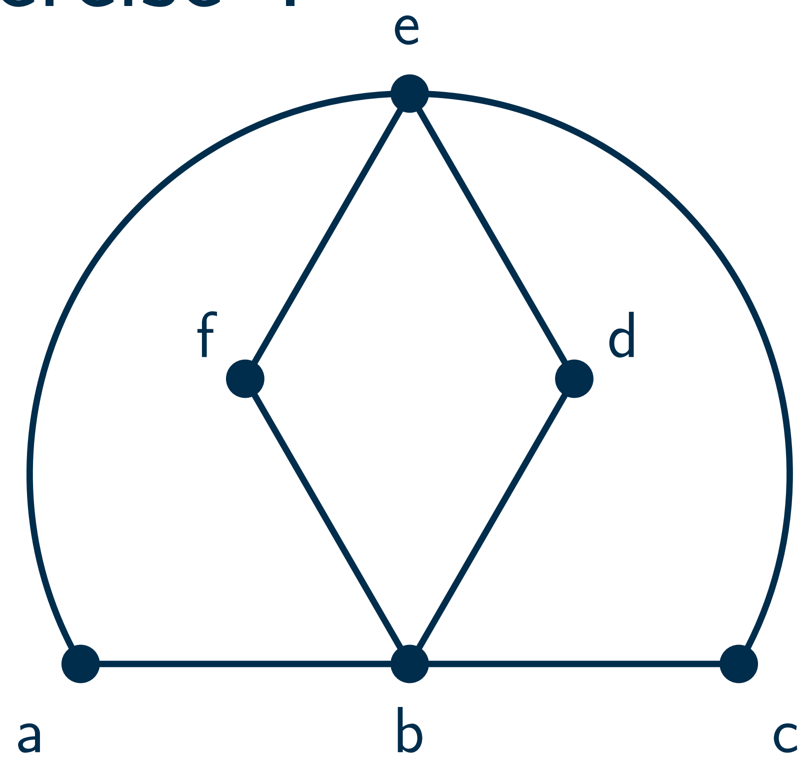


choose an implication class

$A_3 = \{ac, af, dc, df\}$

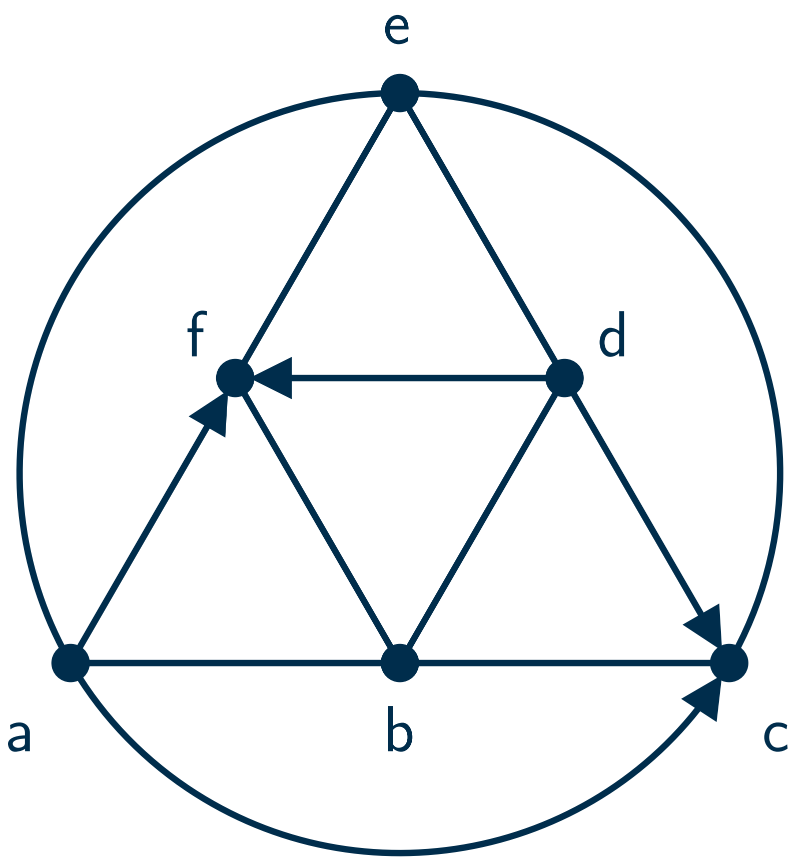


Exercise 4



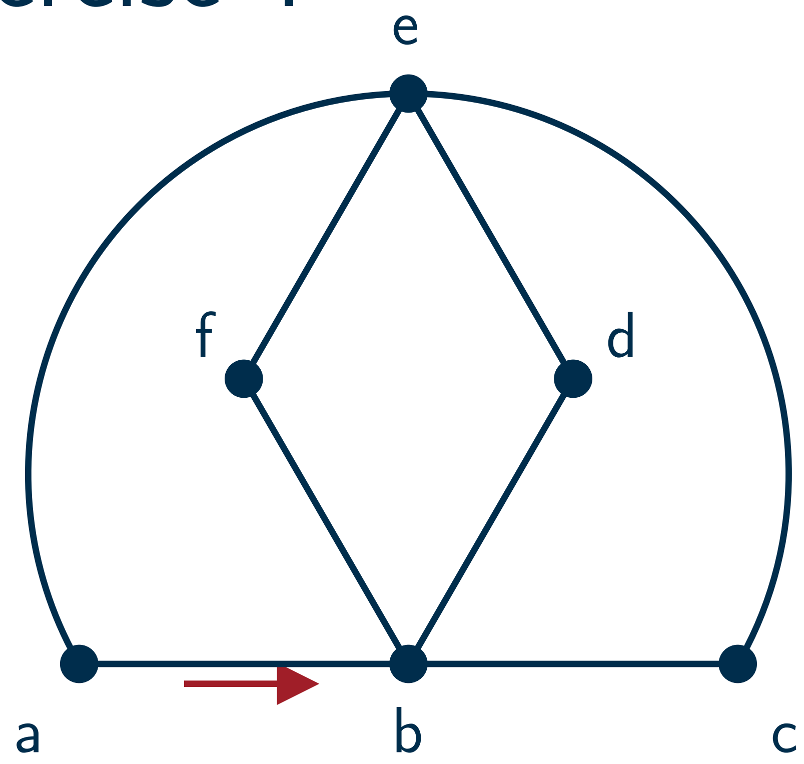
choose an implication class

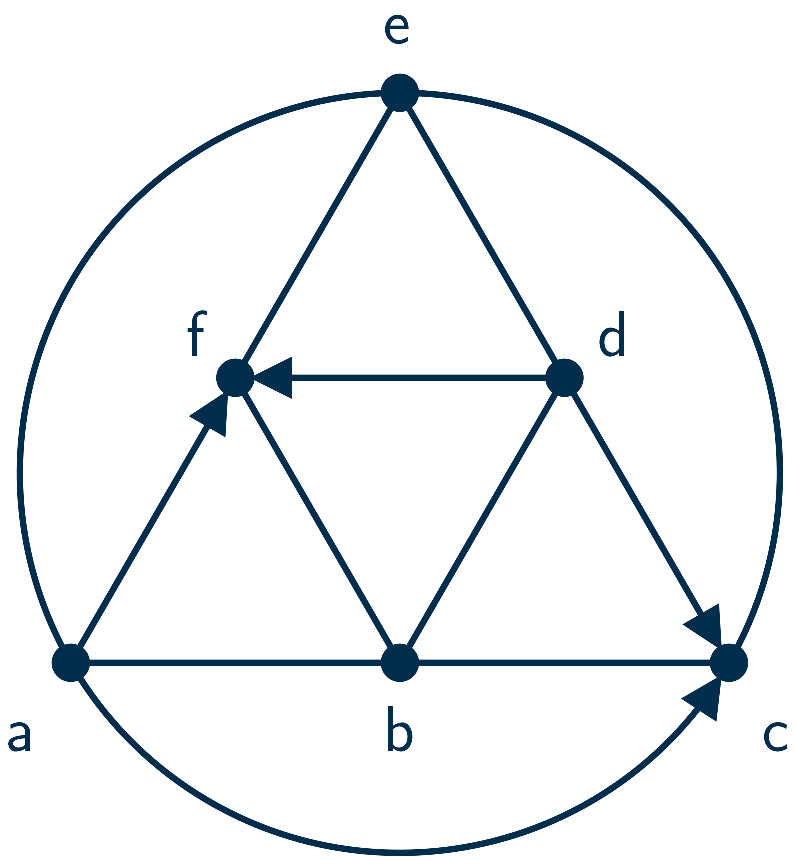
$A_3 = \{ac, af, dc, df\}$



Exercise 4

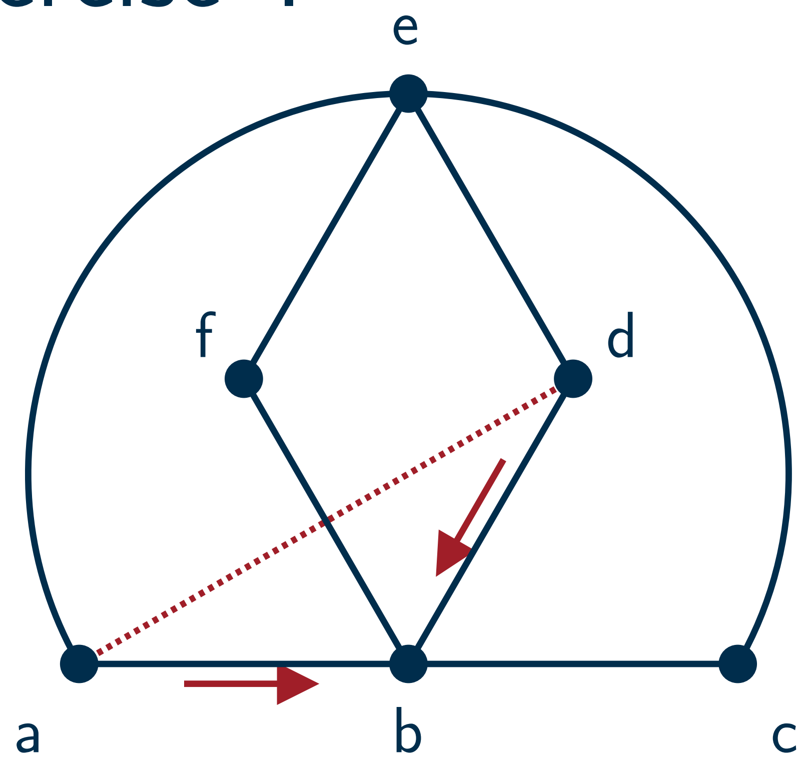
compute implication classes again

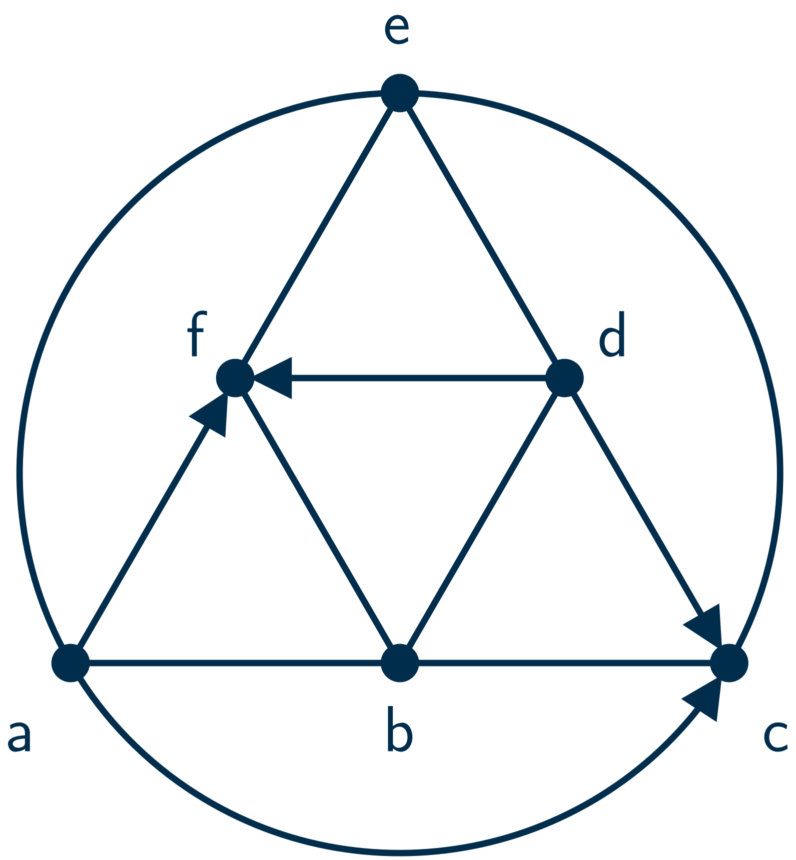




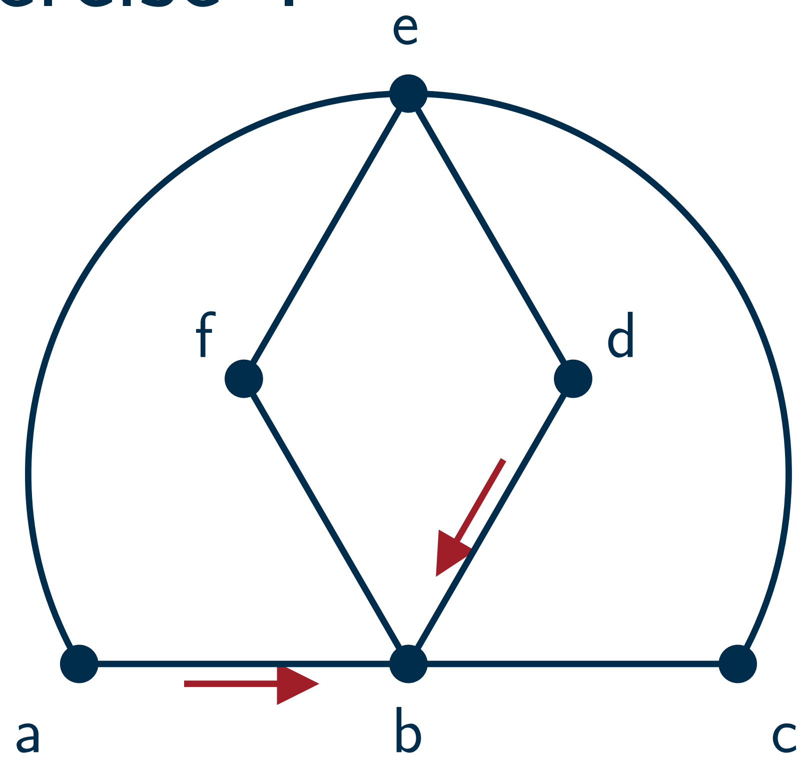
Exercise 4

compute implication classes again

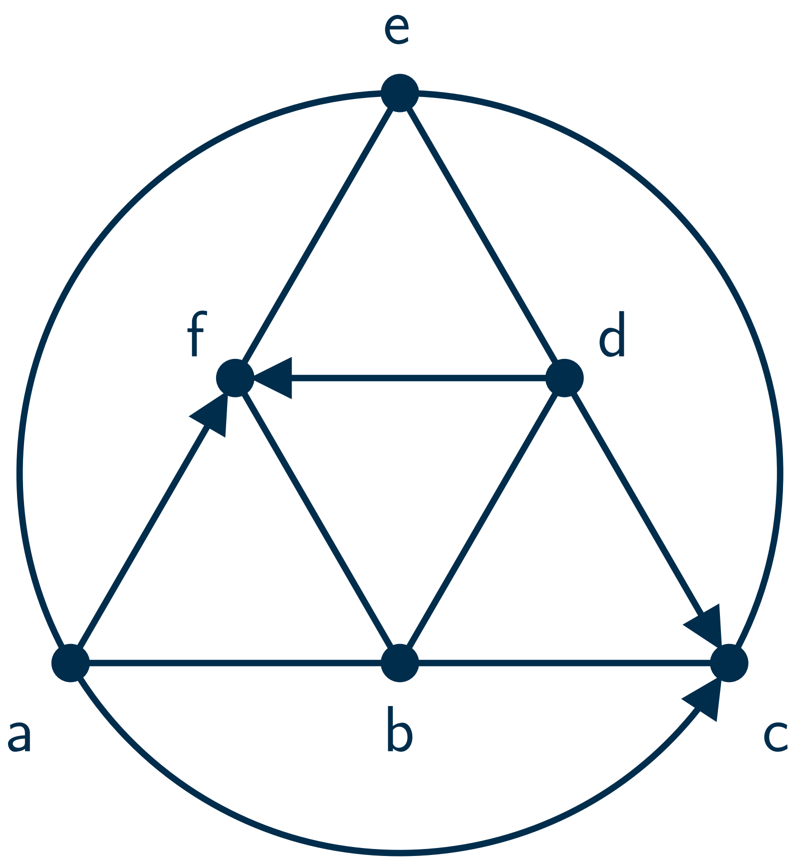




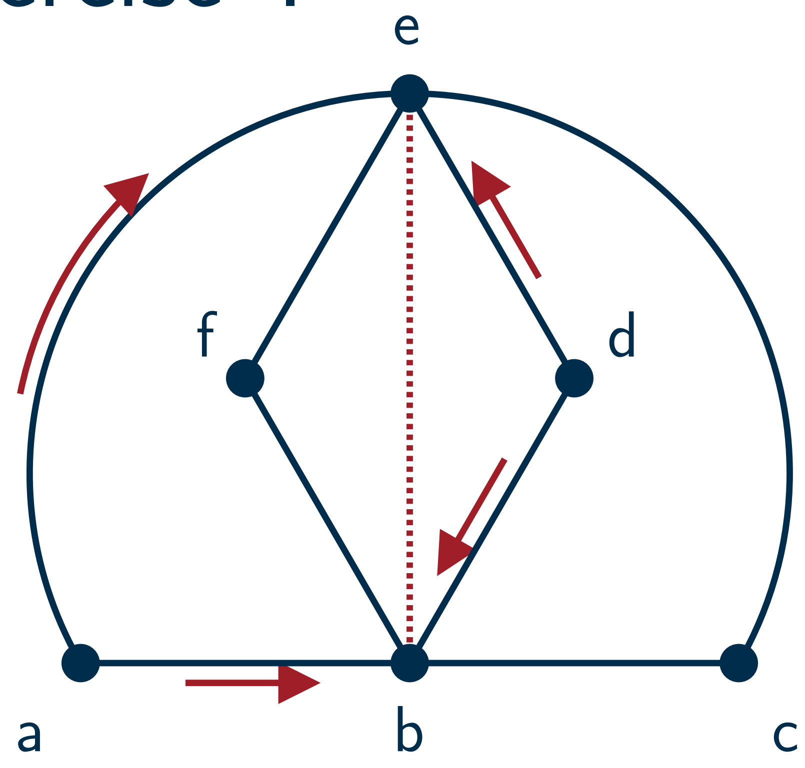
Exercise 4



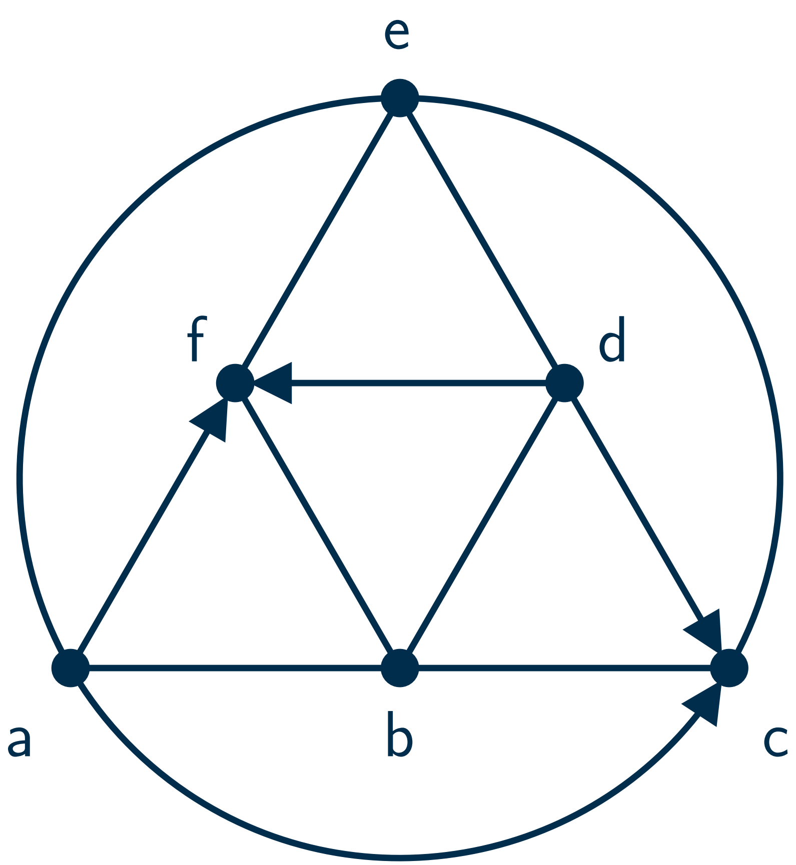
compute implication classes again



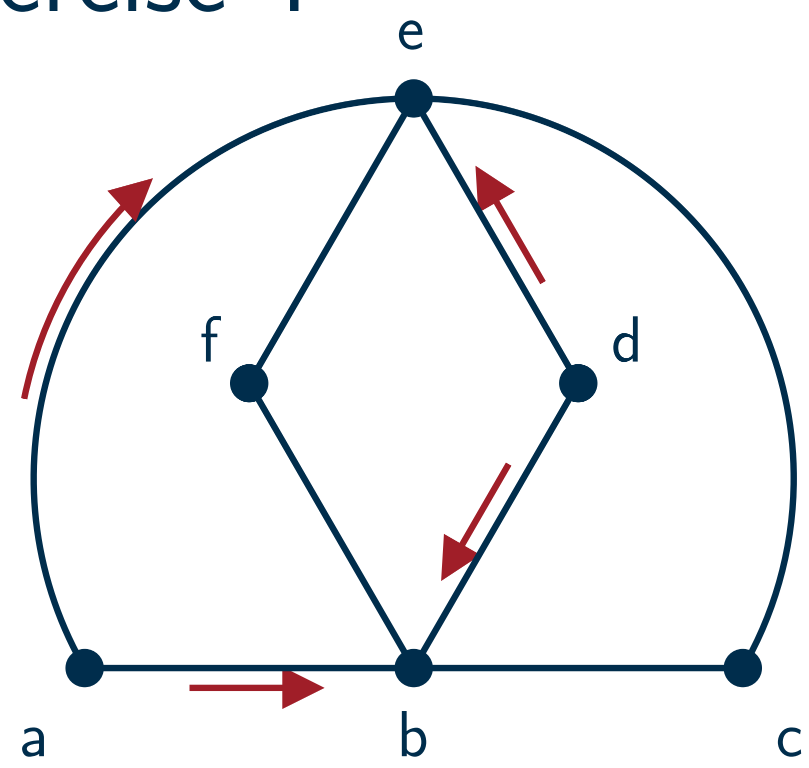
Exercise 4



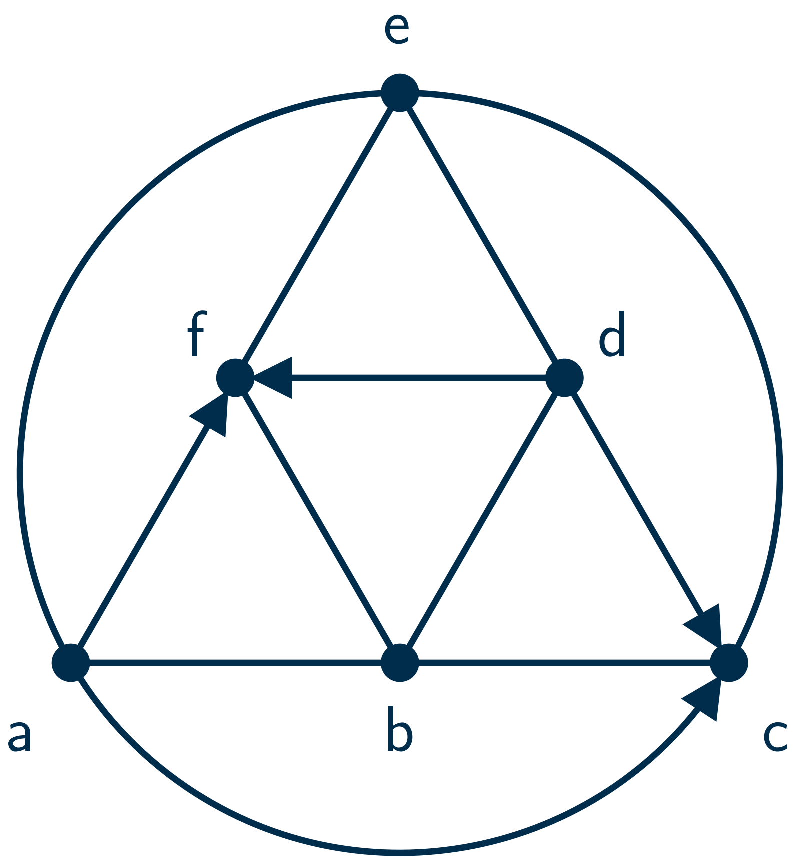
compute implication classes again



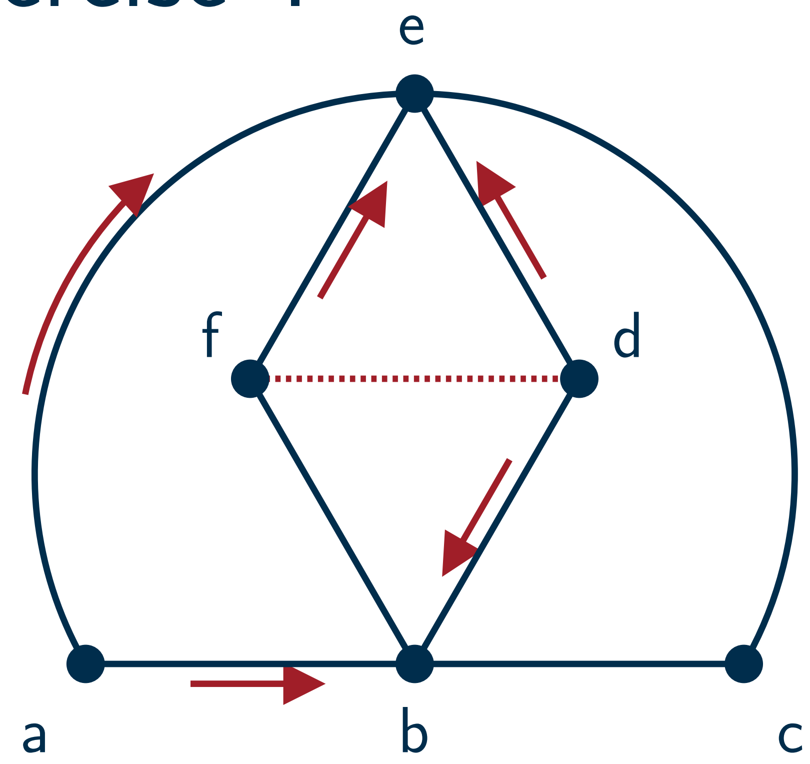
Exercise 4



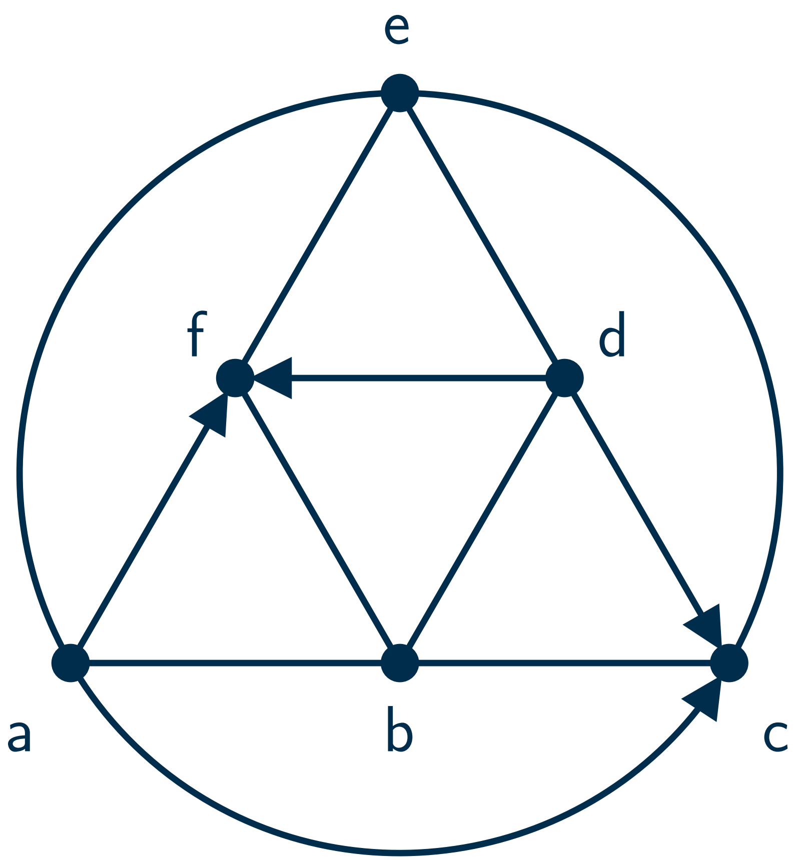
compute implication classes again



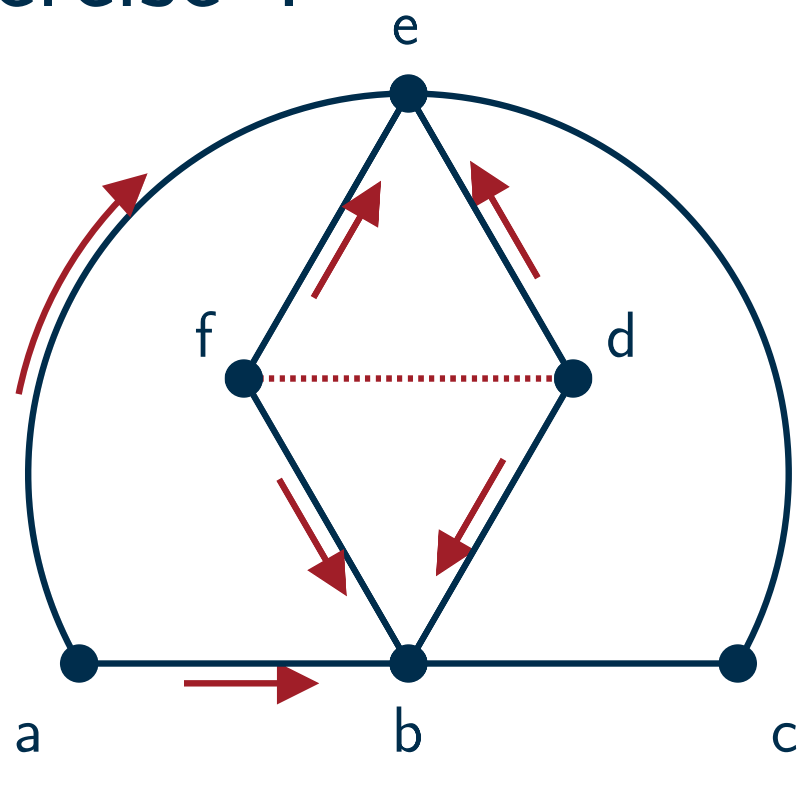
Exercise 4



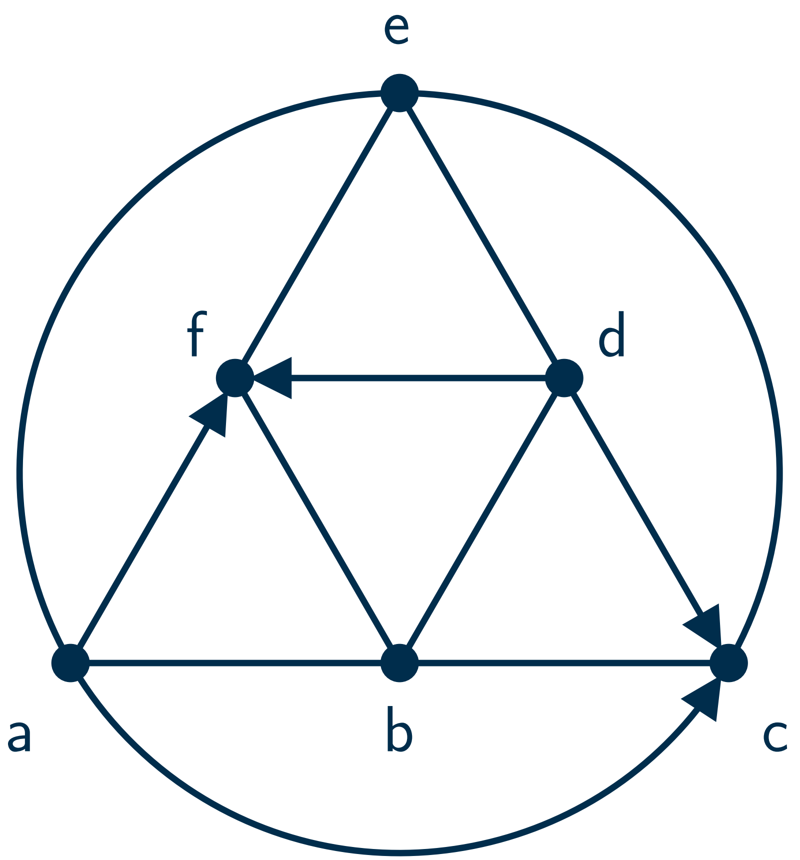
compute implication classes again



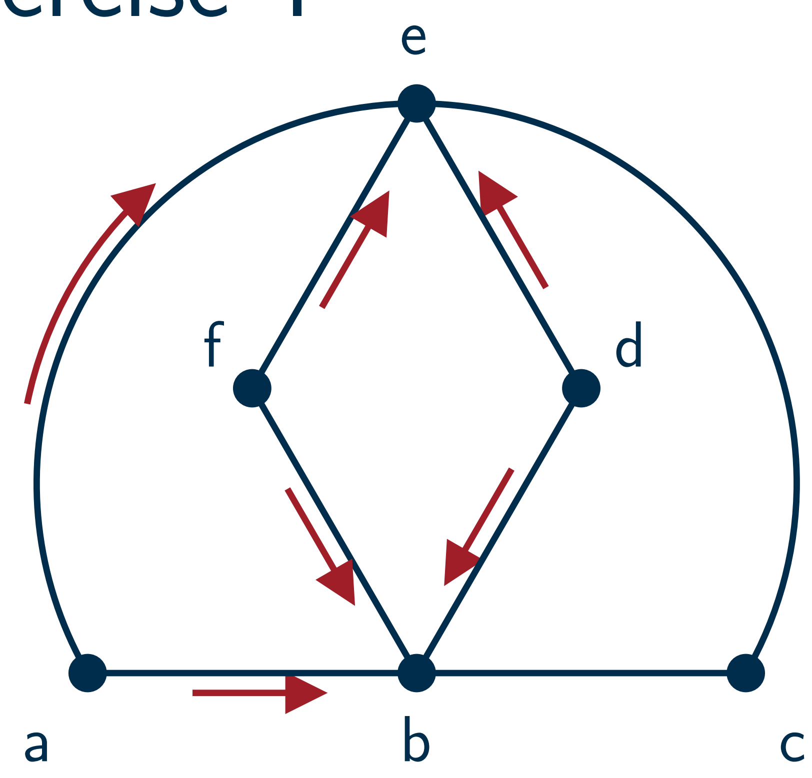
Exercise 4



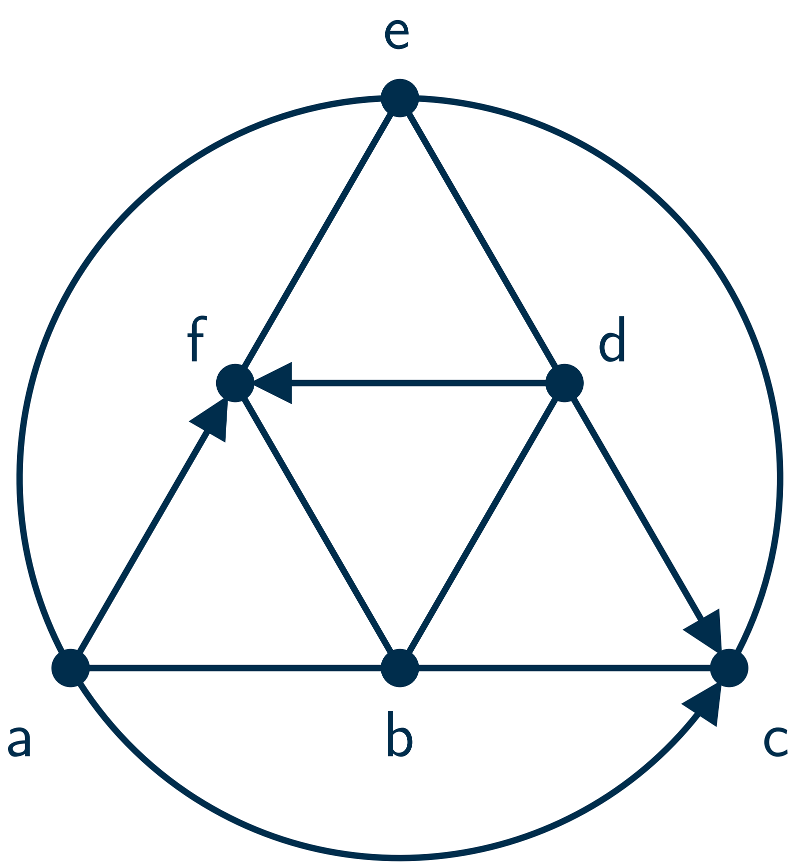
compute implication classes again



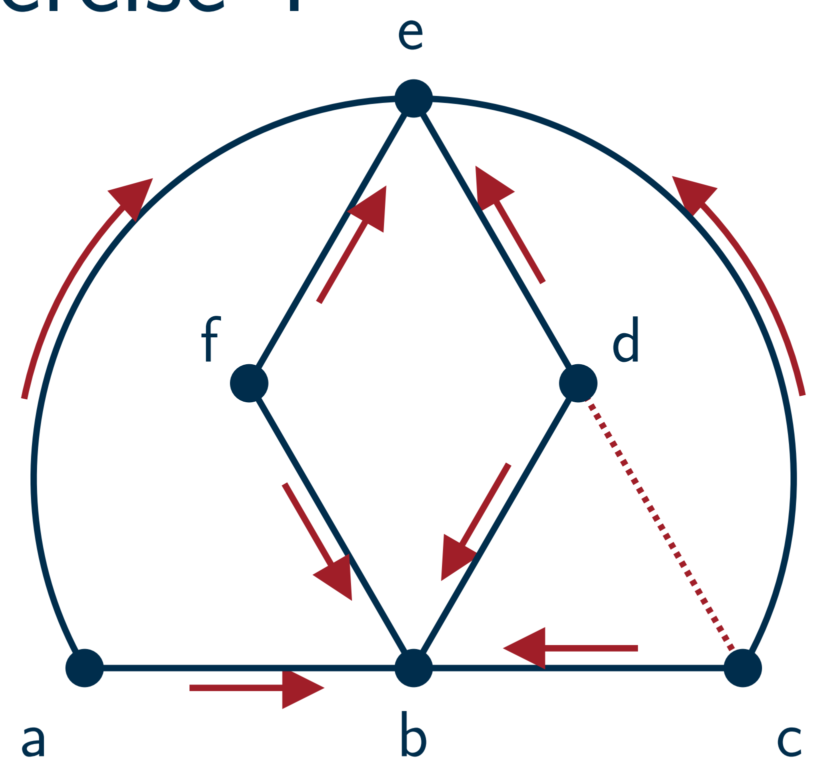
Exercise 4



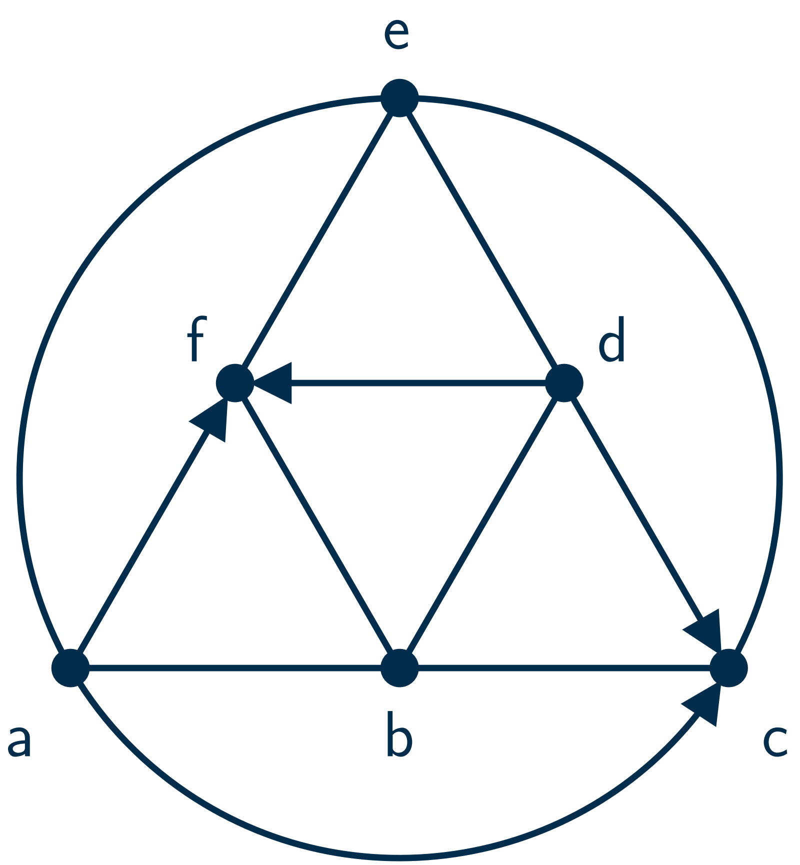
compute implication classes again



Exercise 4

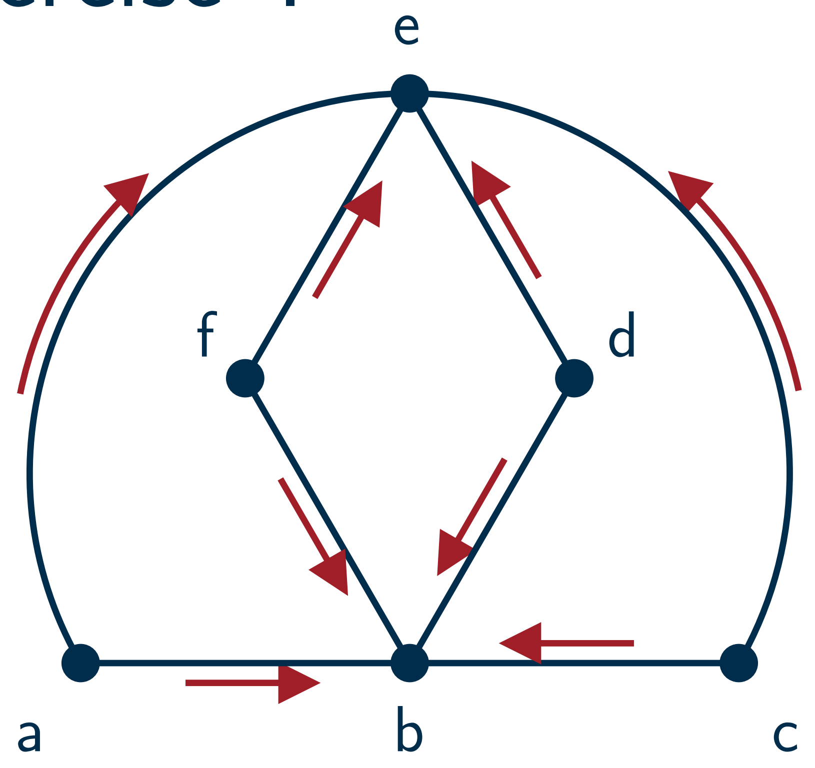


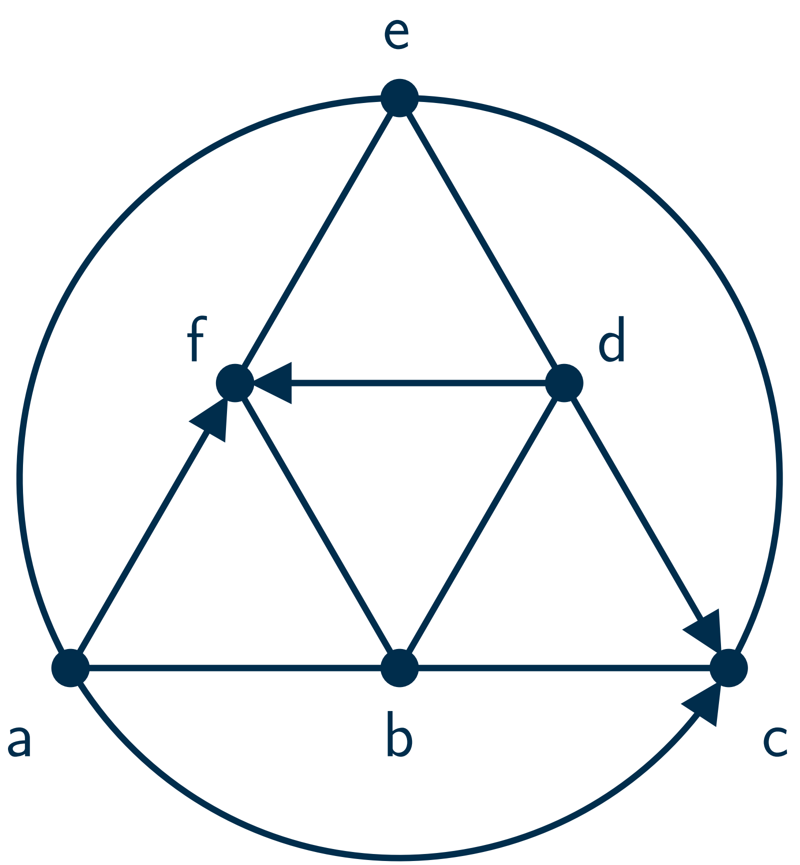
compute implication classes again



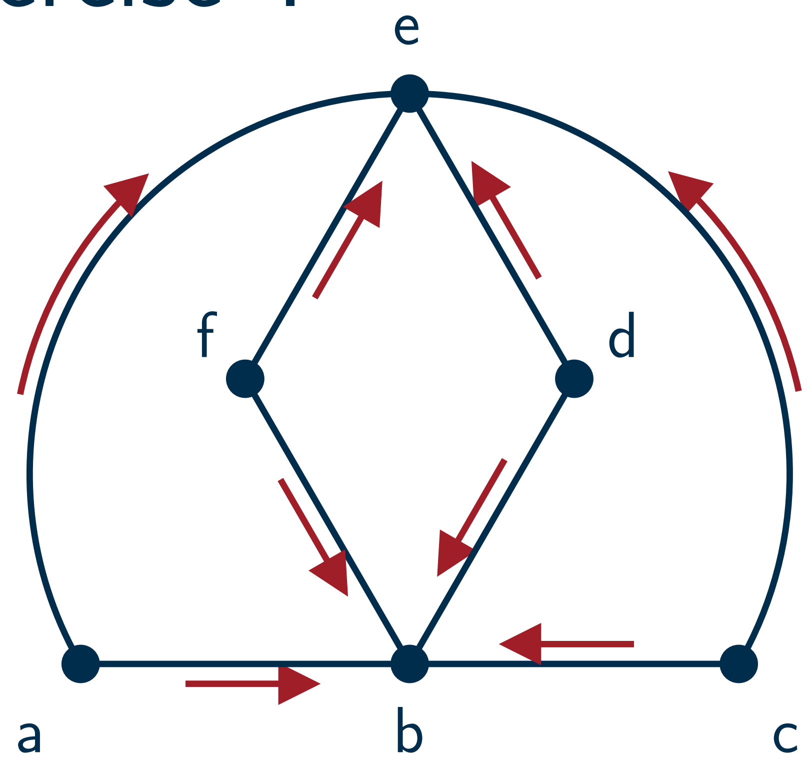
Exercise 4

compute implication classes again



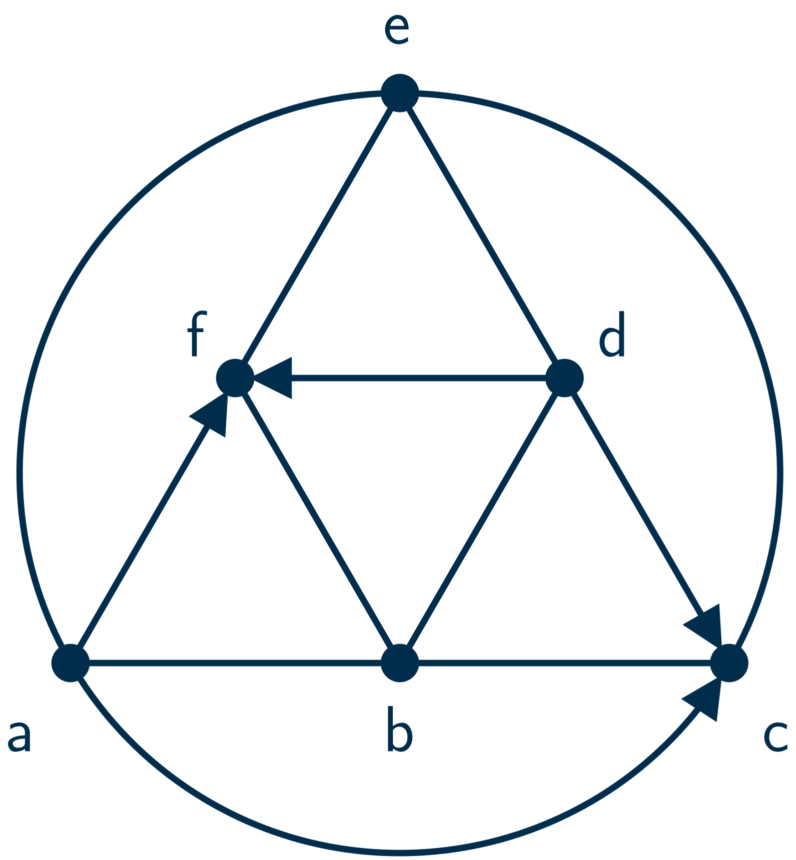


Exercise 4

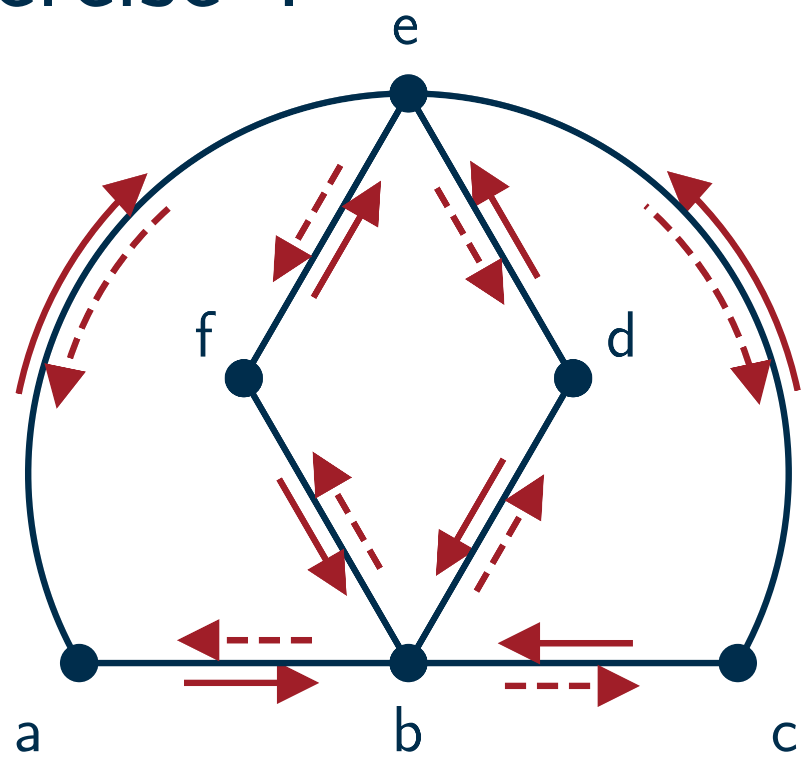


compute implication classes again

$B_1 = \{ab, ae, cb, ce, db, de, fb, fe\}$



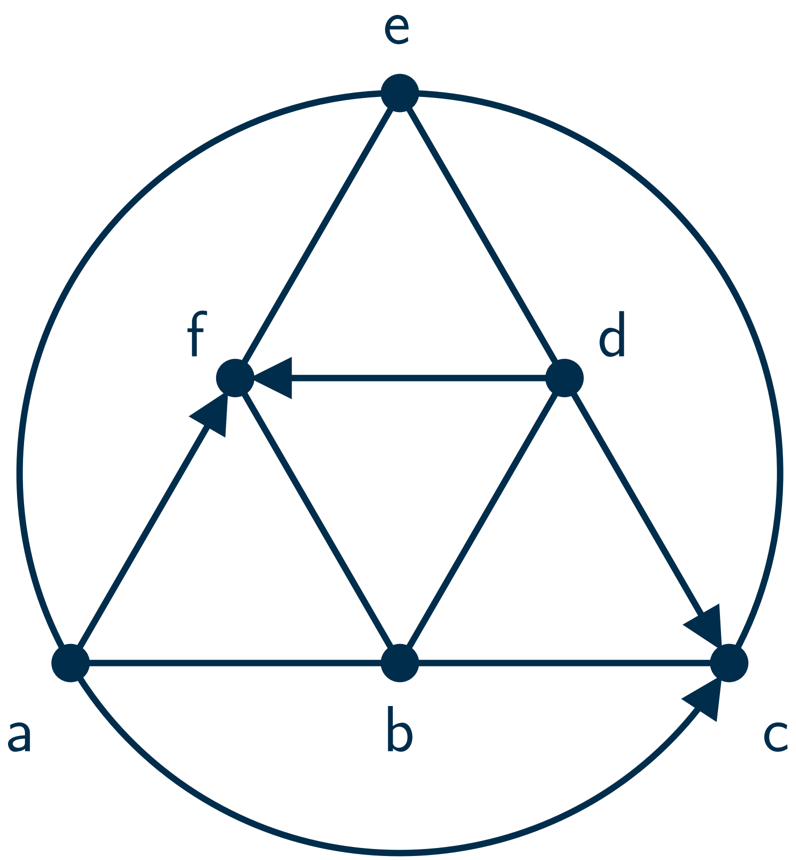
Exercise 4



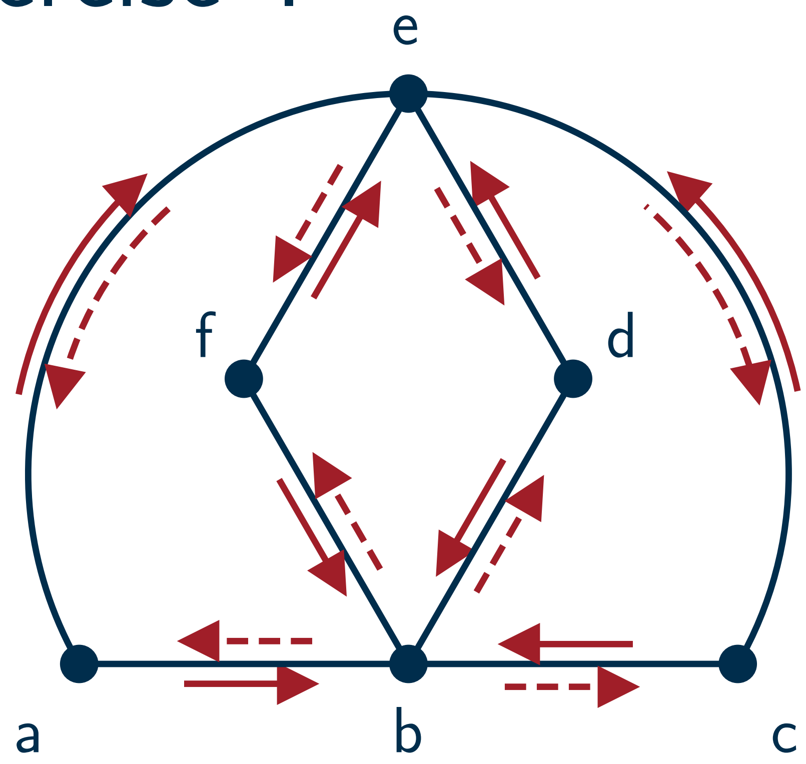
compute implication classes again

$B_1 = \{ab, ae, cb, ce, db, de, fb, fe\}$

$B_1^{-1} = \{ba, ea, bc, ec, bd, ed, bf, ef\}$



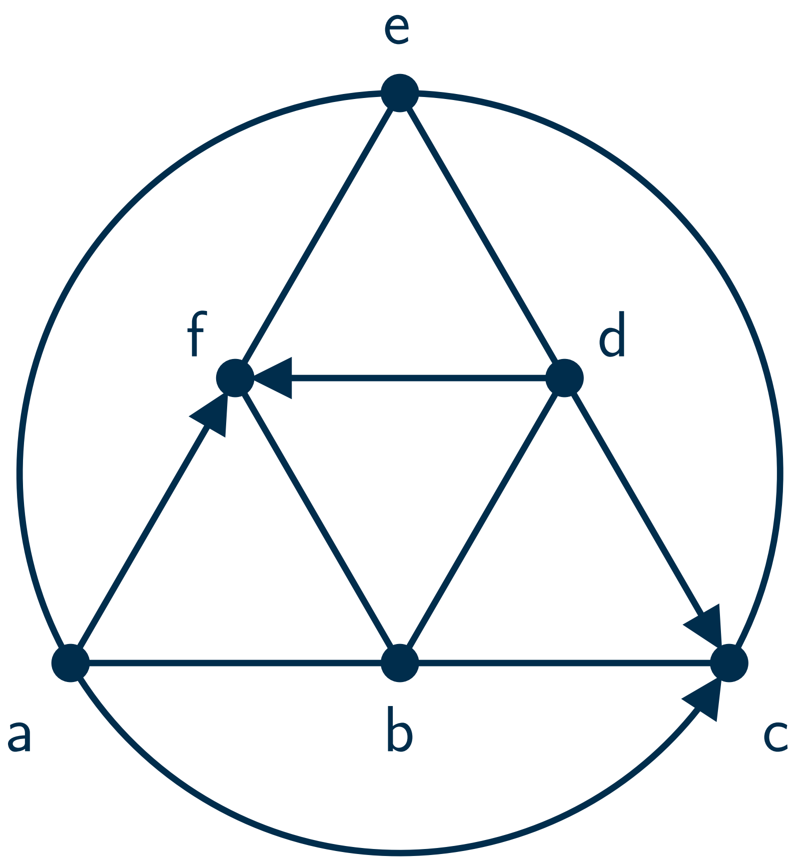
Exercise 4



compute implication classes again

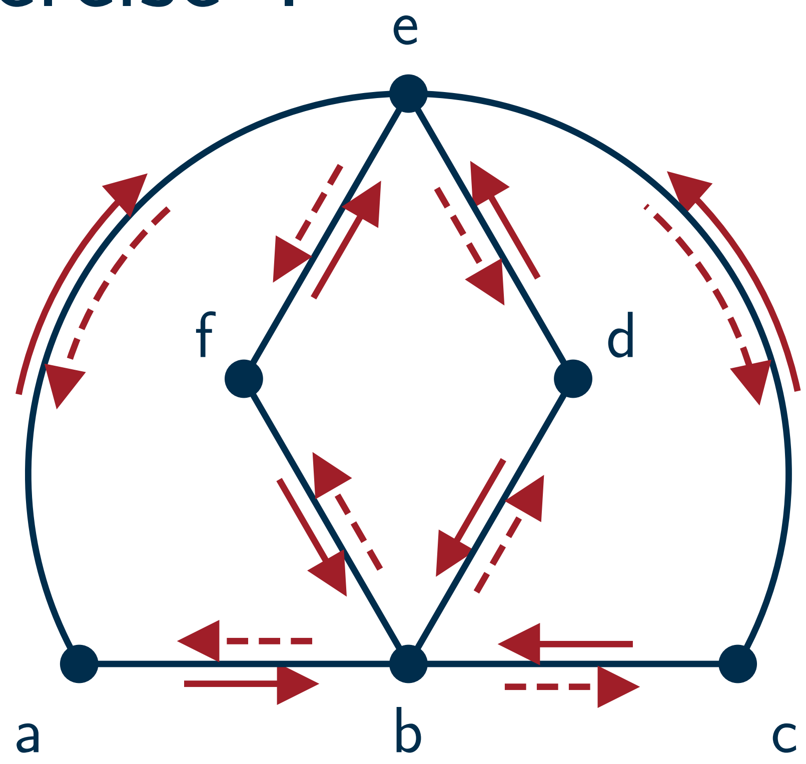
$B_1 = \{ab, ae, cb, ce, db, de, fb, fe\}$

$B_1^{-1} = \{ba, ea, bc, ec, bd, ed, bf, ef\}$



Deleting the green color class merges the red and blue farb classes

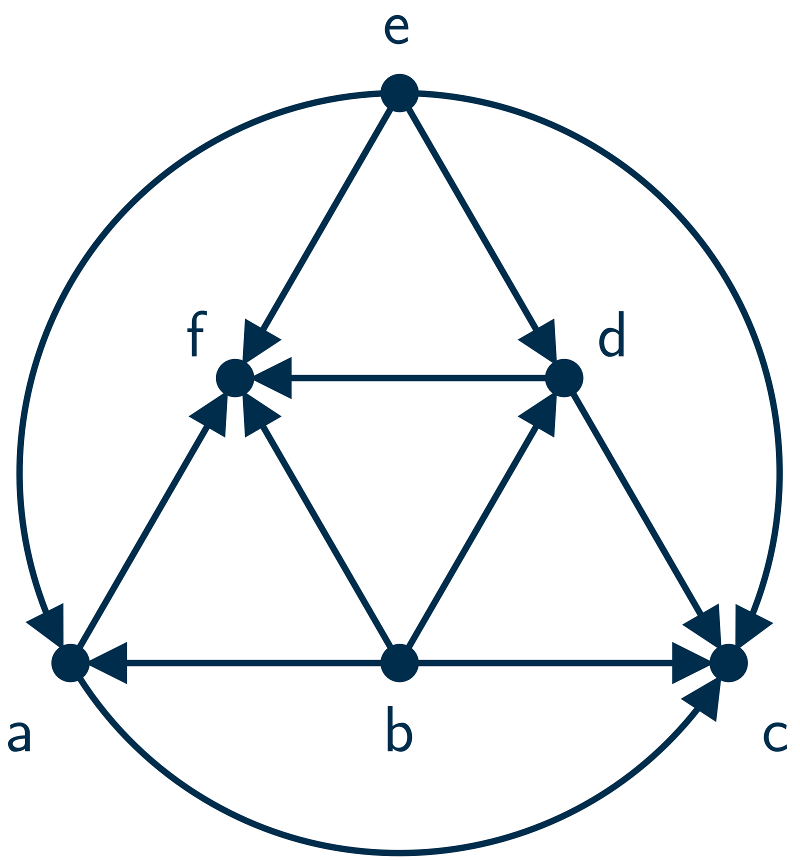
Exercise 4



compute implication classes again

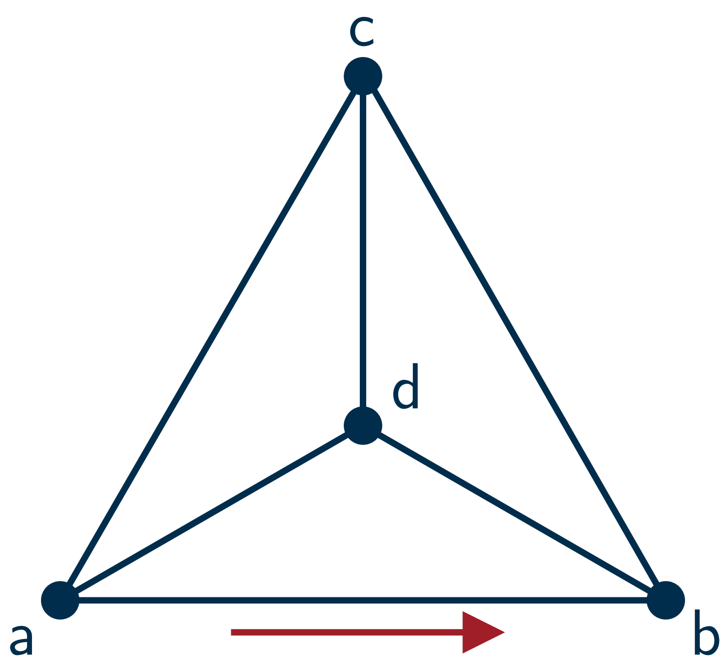
$B_1 = \{ab, ae, cb, ce, db, de, fb, fe\}$

$B_1^{-1} = \{ba, ea, bc, ec, bd, ed, bf, ef\}$



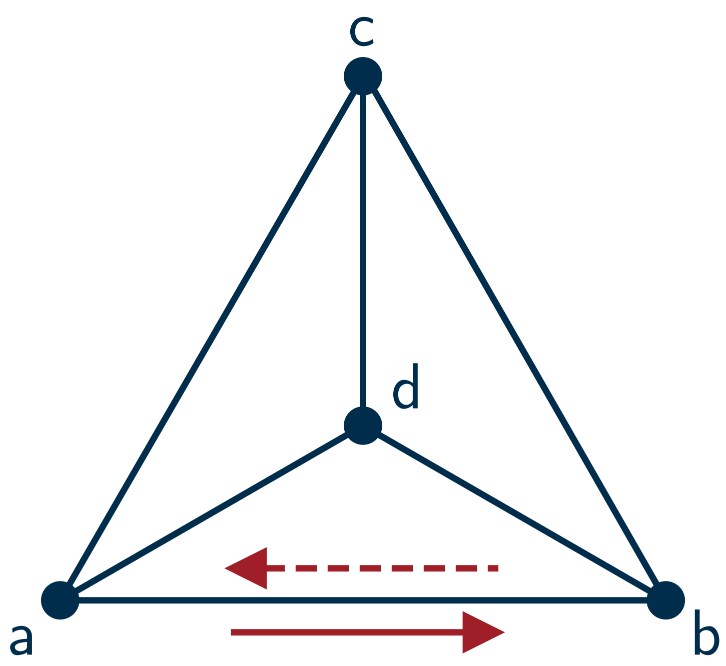
Deleting the green color class merges the red and blue farb classes

Exercise 4



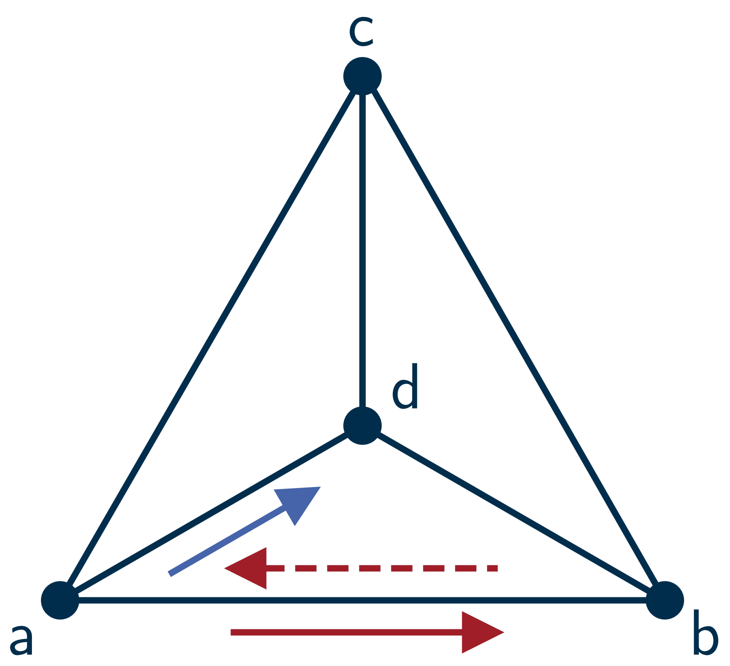
compute implication class of ab

Exercise 4



compute implication class of ab

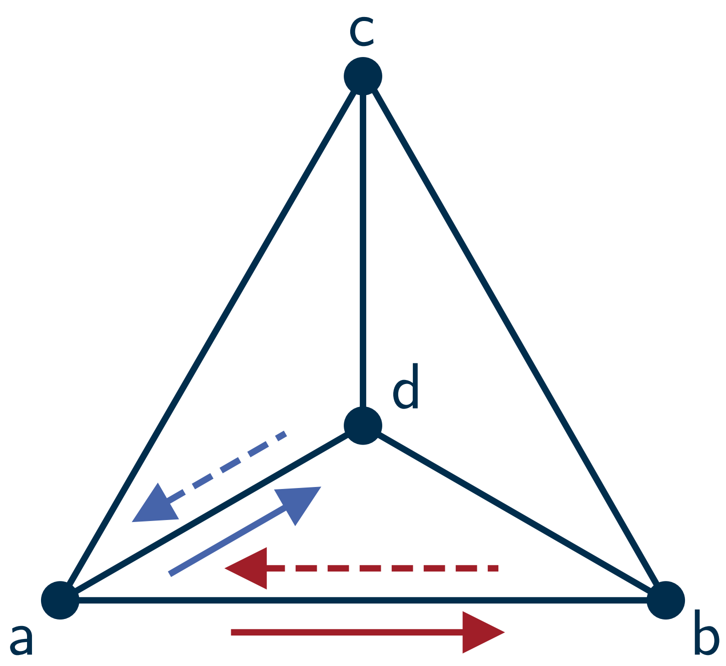
Exercise 4



compute implication class of ab

compute implication class of ad

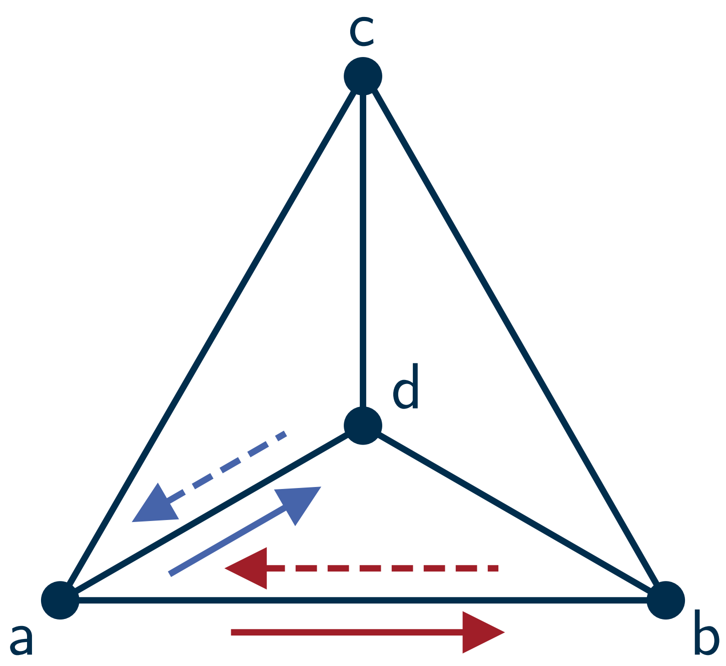
Exercise 4



compute implication class of ab

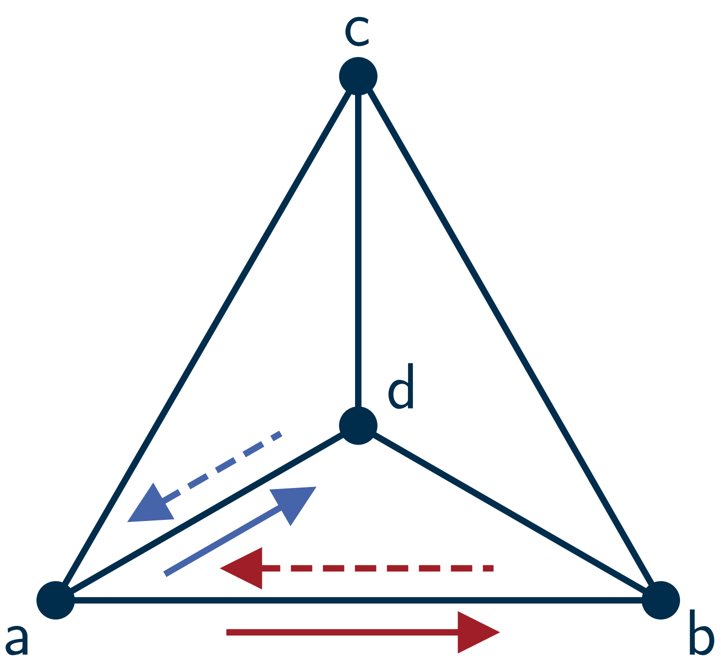
compute implication class of ad

Exercise 4



compute implication class of ab
compute implication class of ad
by symmetry every edge has its own implication class

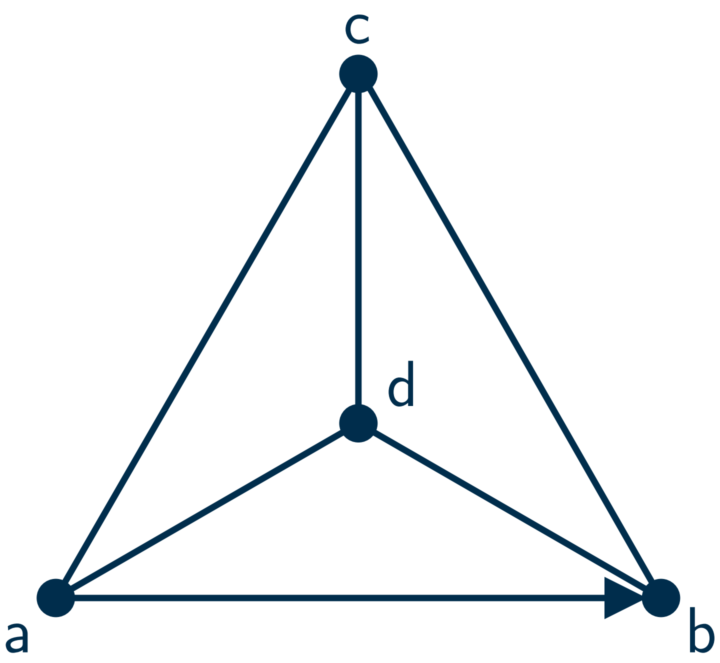
Exercise 4



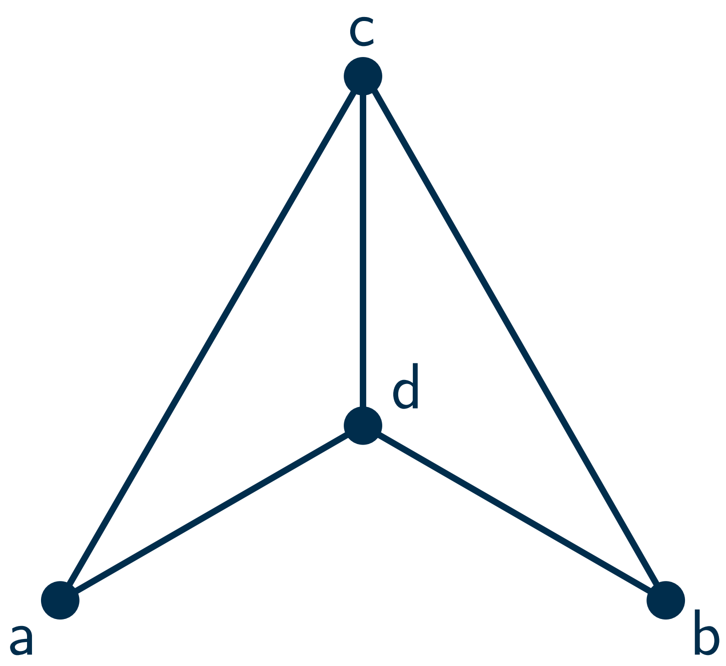
compute implication class of ab

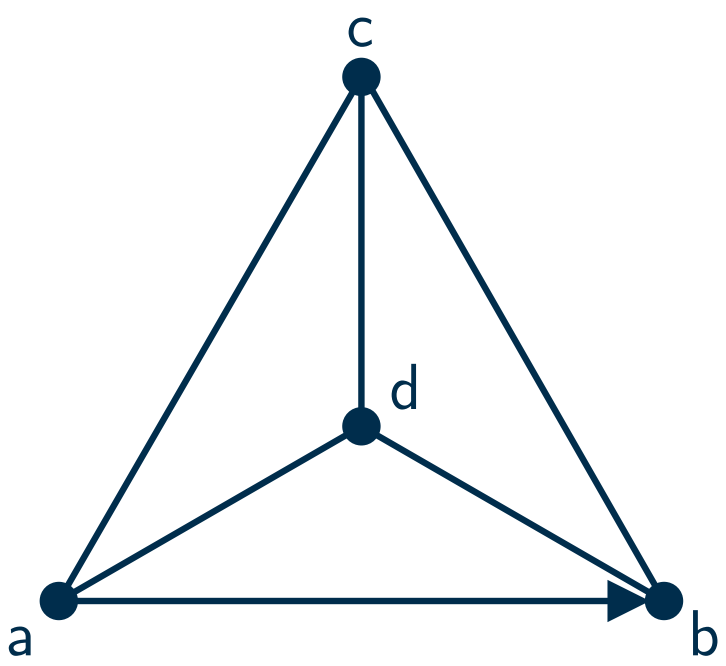
compute implication class of ad

by symmetry every edge has its own implication class



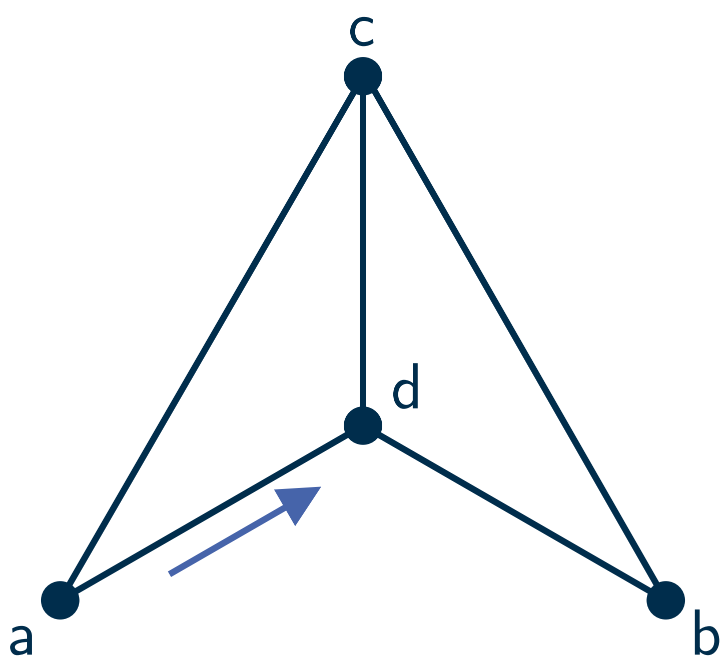
Exercise 4

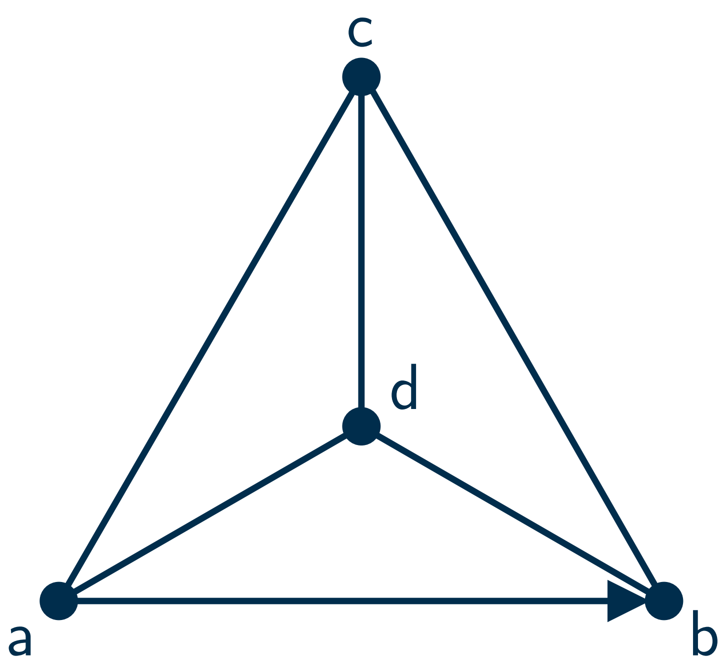




Exercise 4

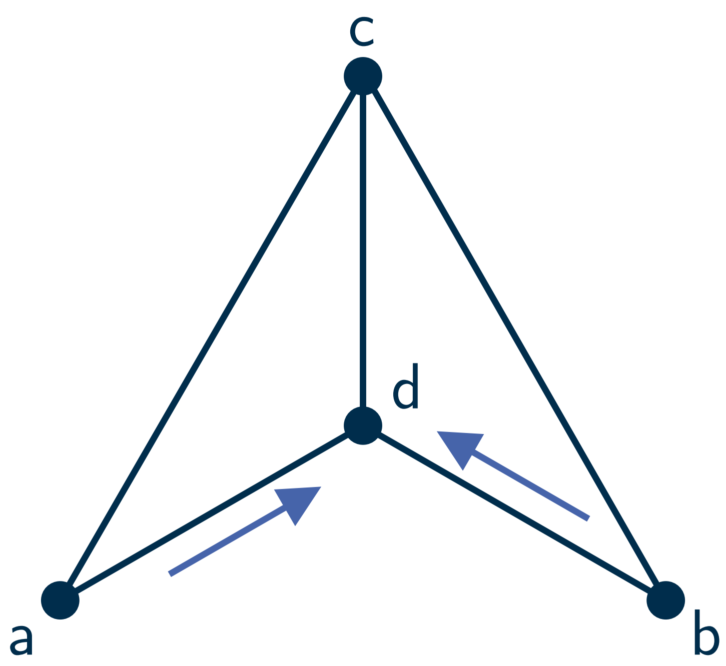
compute implication class of ad

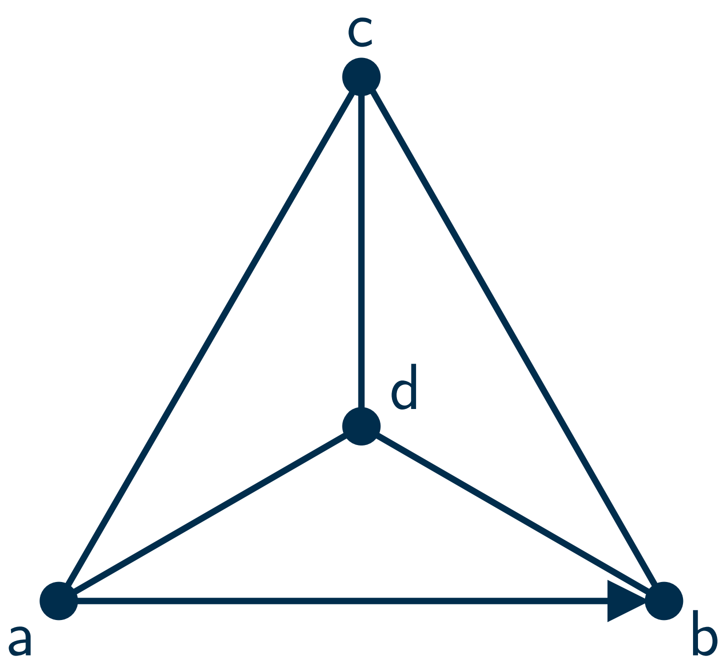




Exercise 4

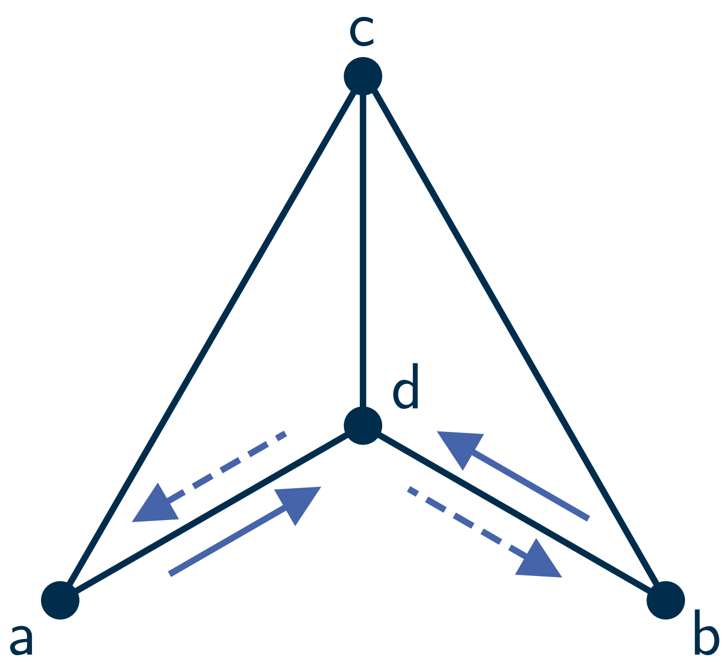
compute implication class of ad

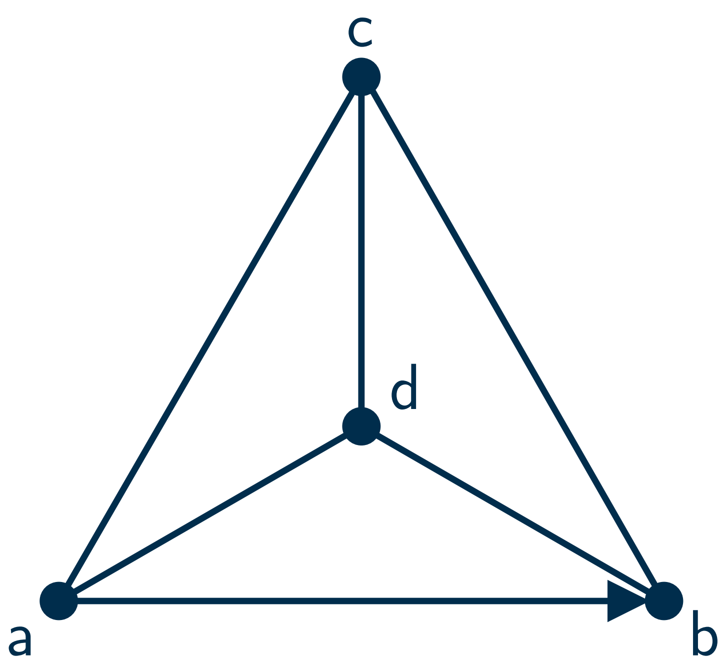




Exercise 4

compute implication class of ad

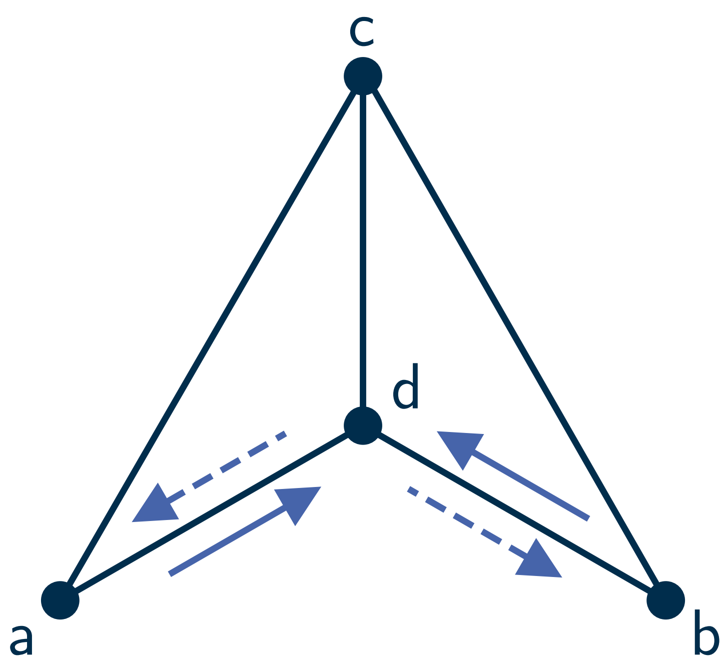


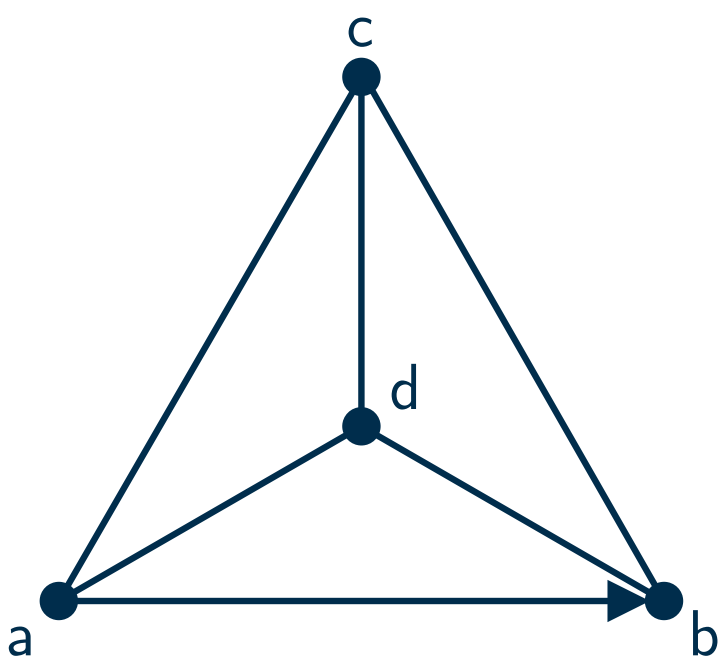


Exercise 4

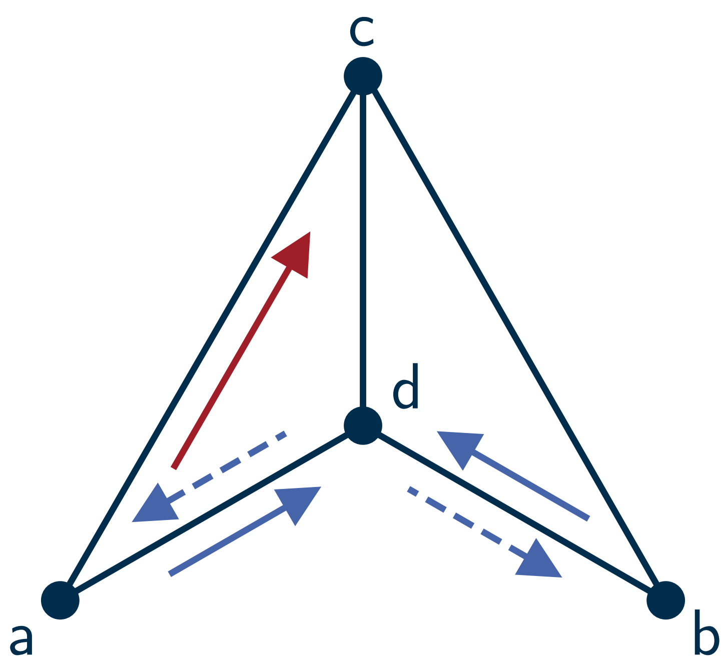
compute implication class of ad

$\{ad, bd\}$ and as inverse $\{da, db\}$





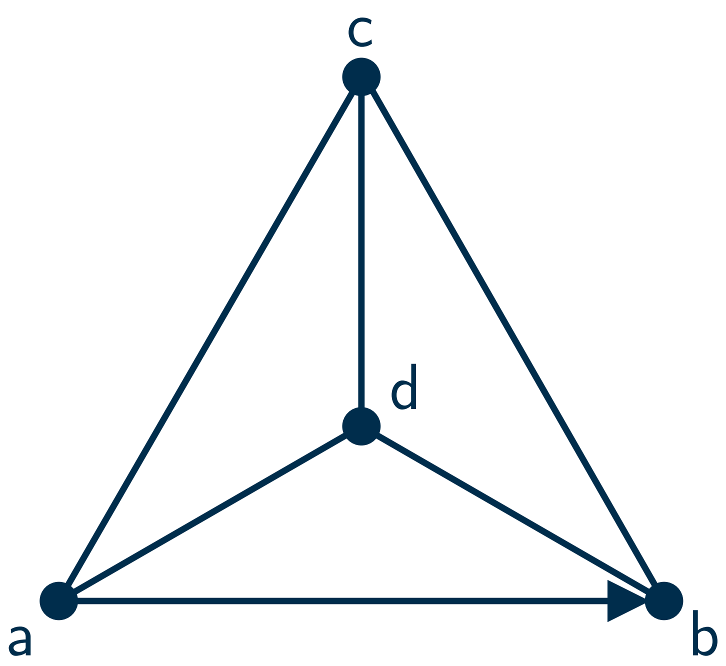
Exercise 4



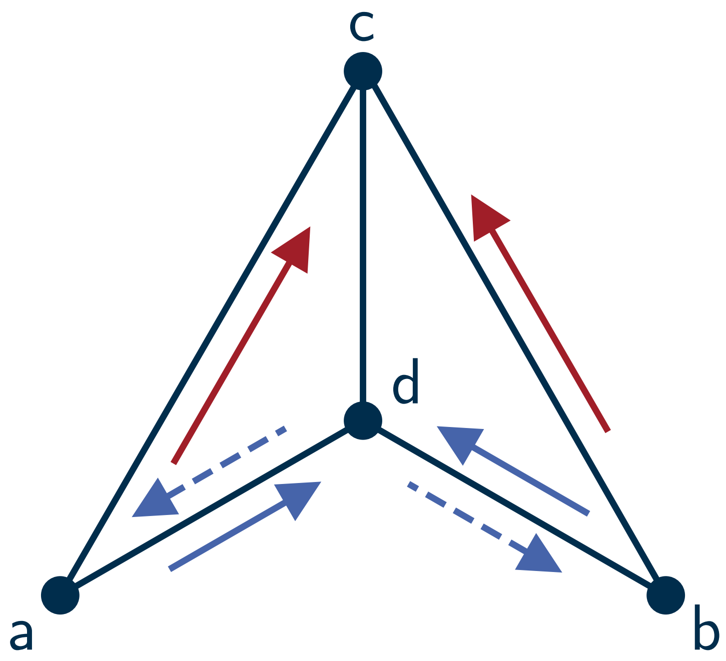
compute implication class of ad

$\{ad, bd\}$ and as inverse $\{da, db\}$

compute implication class of ac



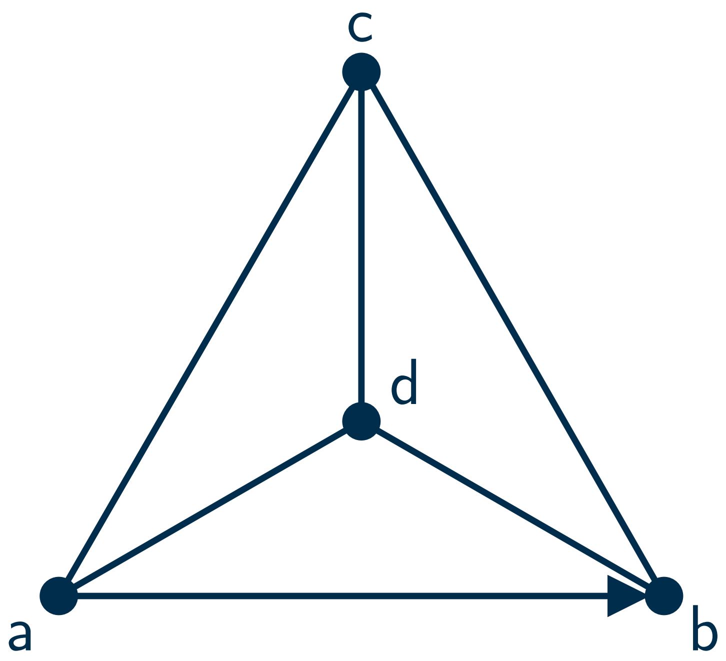
Exercise 4



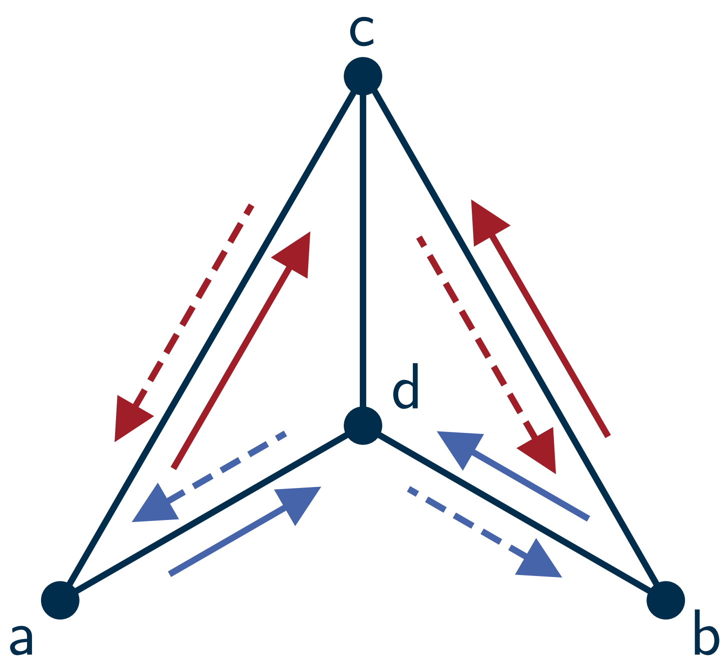
compute implication class of ad

$\{ad, bd\}$ and as inverse $\{da, db\}$

compute implication class of ac



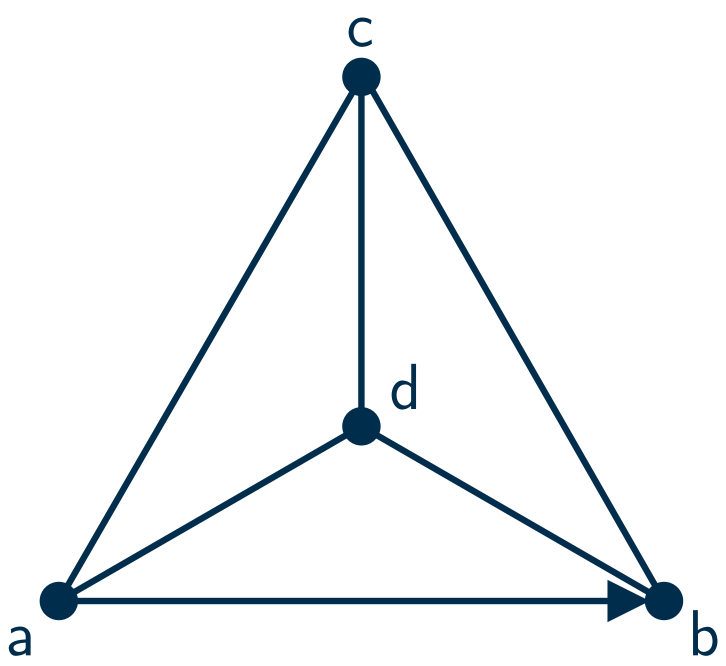
Exercise 4



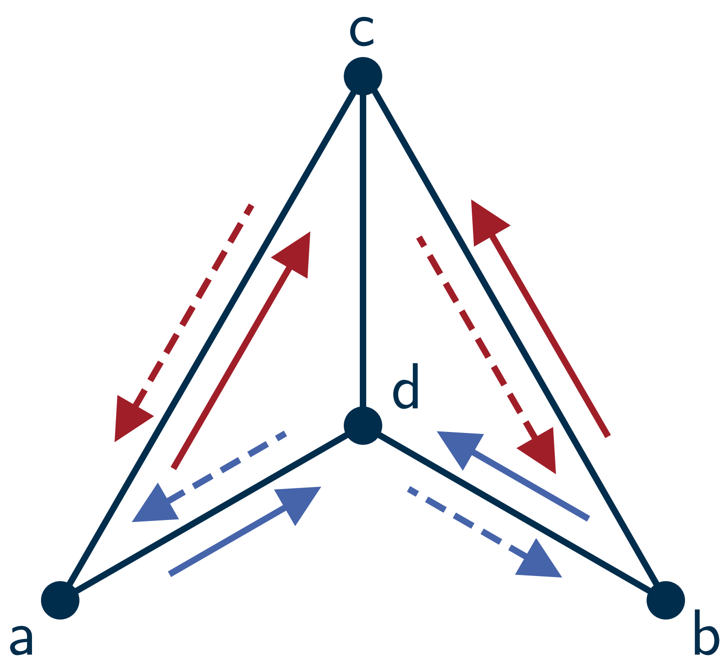
compute implication class of ad

$\{ad, bd\}$ and as inverse $\{da, db\}$

compute implication class of ac



Exercise 4

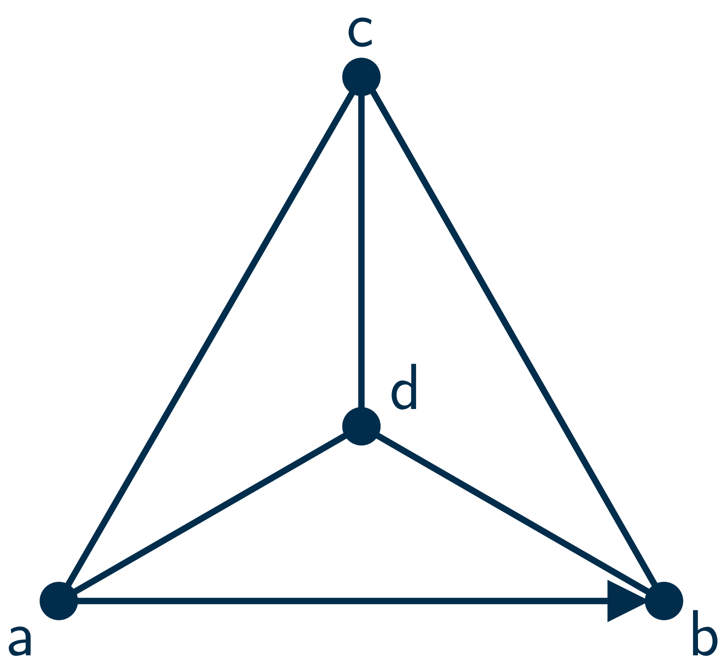


compute implication class of ad

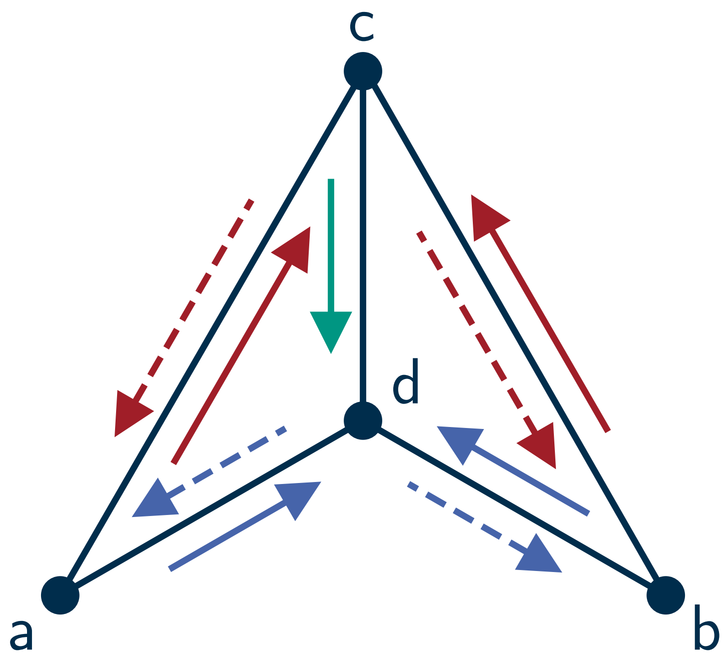
$\{ad, bd\}$ and as inverse $\{da, db\}$

compute implication class of ac

$\{ac, bc\}$ and as inverse $\{ca, cb\}$



Exercise 4



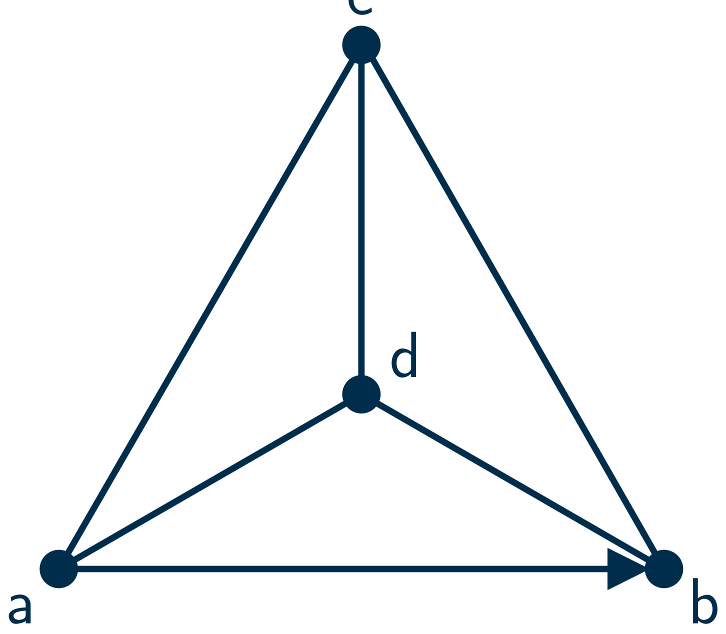
compute implication class of ad

$\{ad, bd\}$ and as inverse $\{da, db\}$

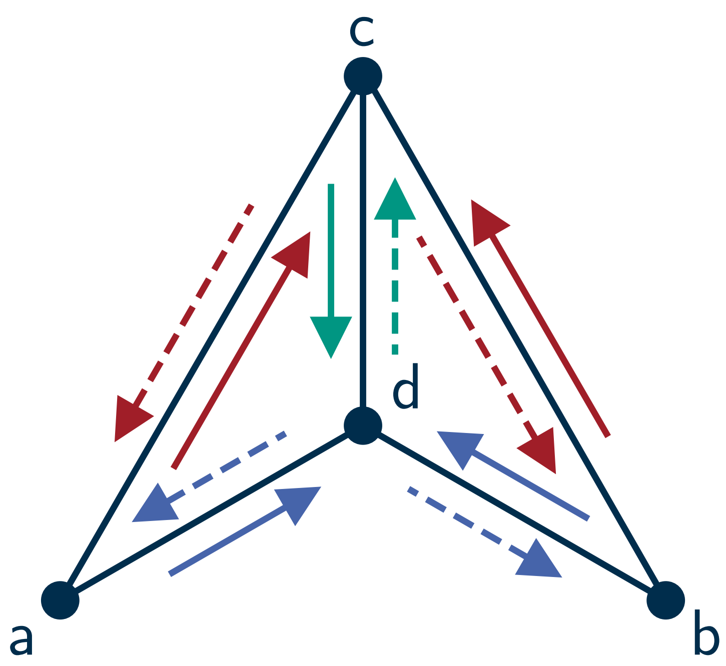
compute implication class of ac

$\{ac, bc\}$ and as inverse $\{ca, cb\}$

compute implication class of cd



Exercise 4



compute implication class of ad

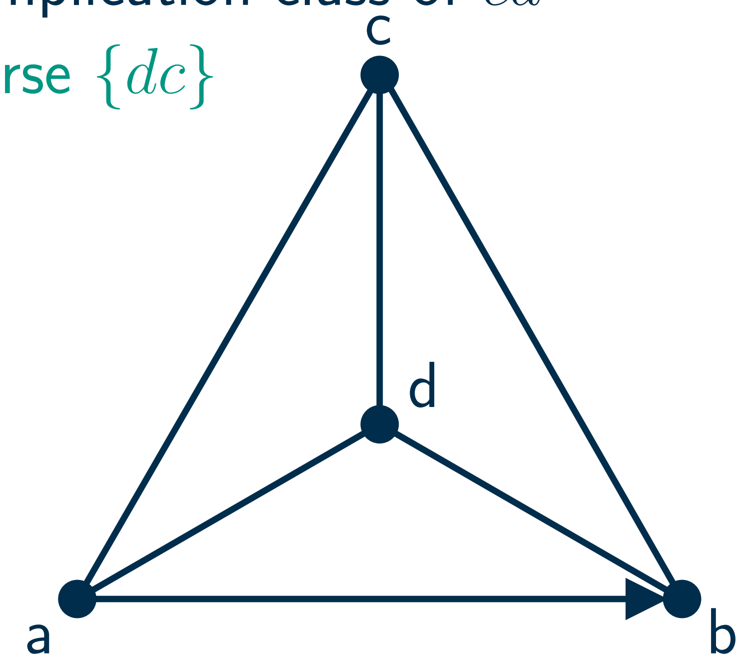
$\{ad, bd\}$ and as inverse $\{da, db\}$

compute implication class of ac

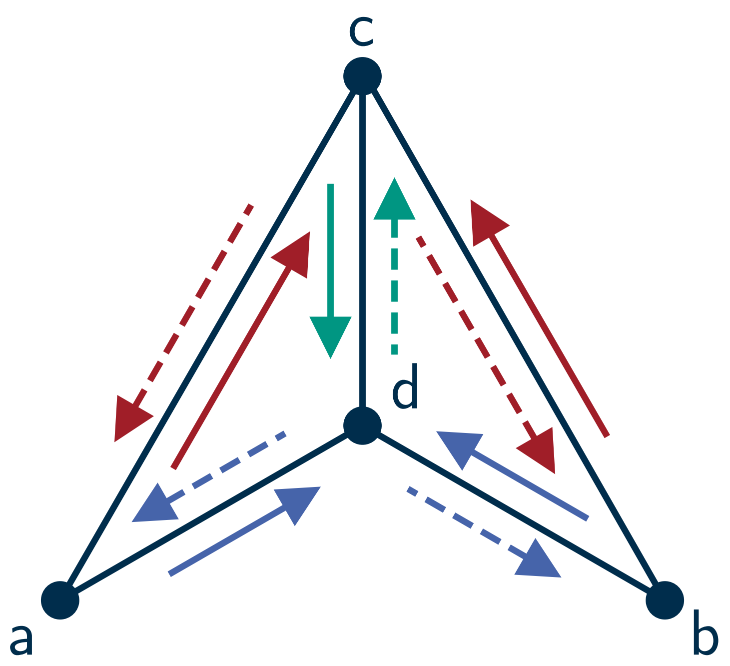
$\{ac, bc\}$ and as inverse $\{ca, cb\}$

compute implication class of cd

$\{cd\}$, inverse $\{dc\}$



Exercise 4



compute implication class of ad

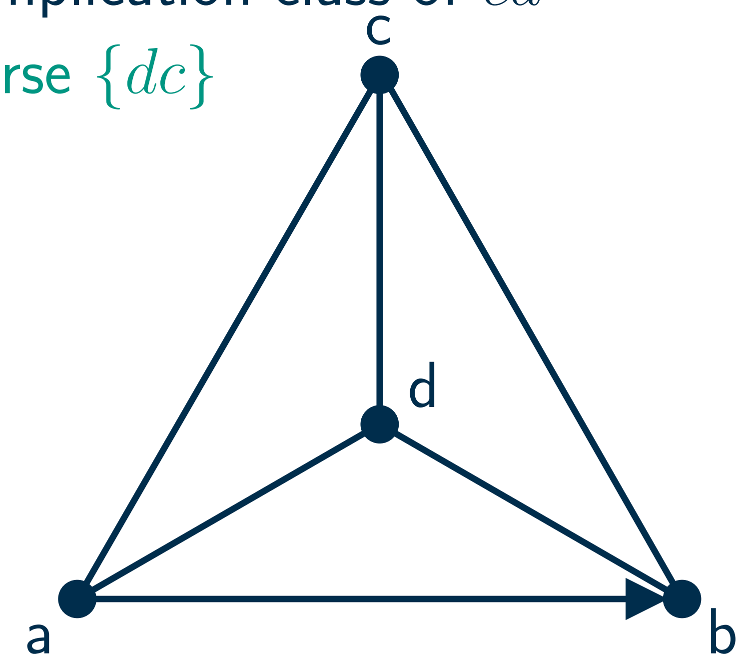
$\{ad, bd\}$ and as inverse $\{da, db\}$

compute implication class of ac

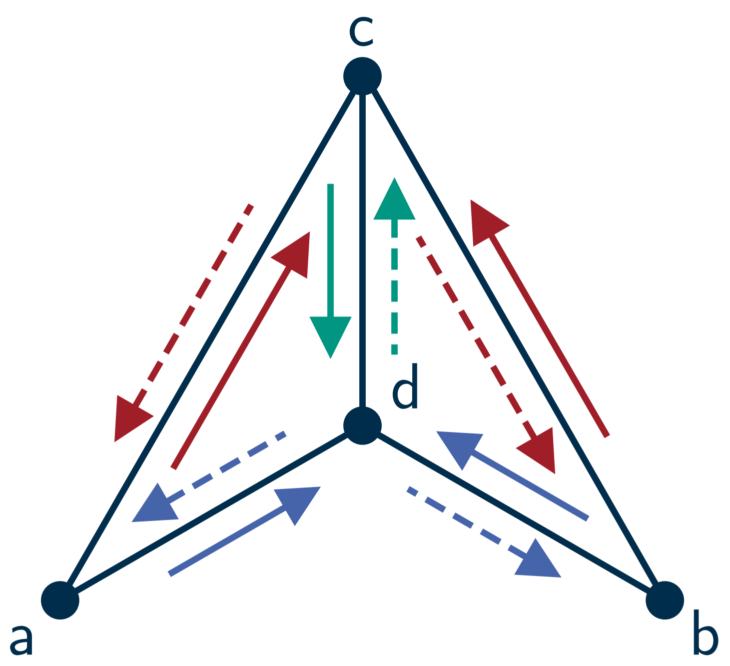
$\{ac, bc\}$ and as inverse $\{ca, cb\}$

compute implication class of cd

$\{cd\}$, inverse $\{dc\}$



Exercise 4



compute implication class of ad

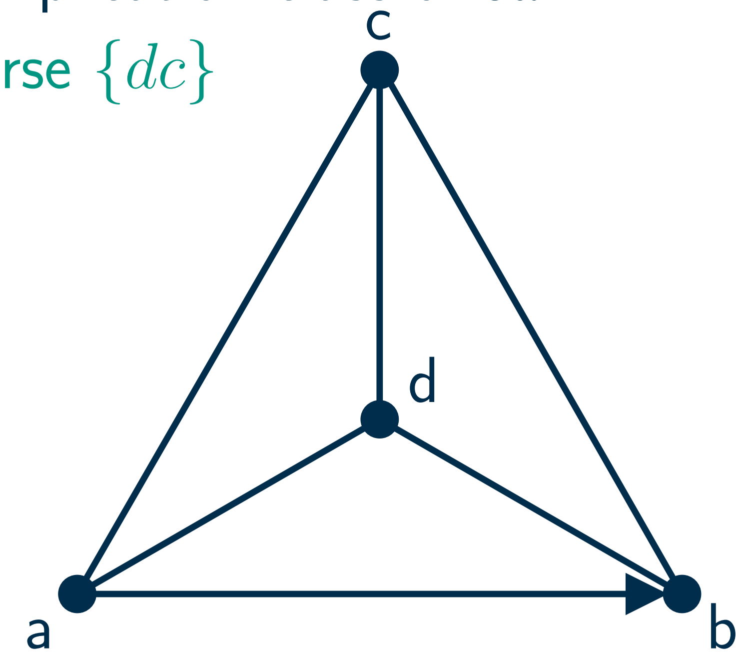
$\{ad, bd\}$ and as inverse $\{da, db\}$

compute implication class of ac

$\{ac, bc\}$ and as inverse $\{ca, cb\}$

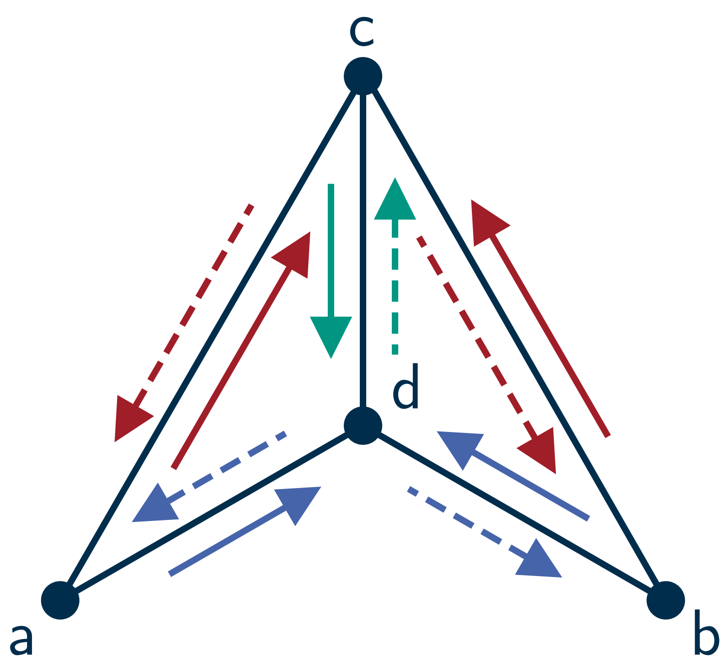
compute implication class of cd

$\{cd\}$, inverse $\{dc\}$



The color class of cd stays the same.
The color classes of ad and bd are merged.

Exercise 4



compute implication class of ad

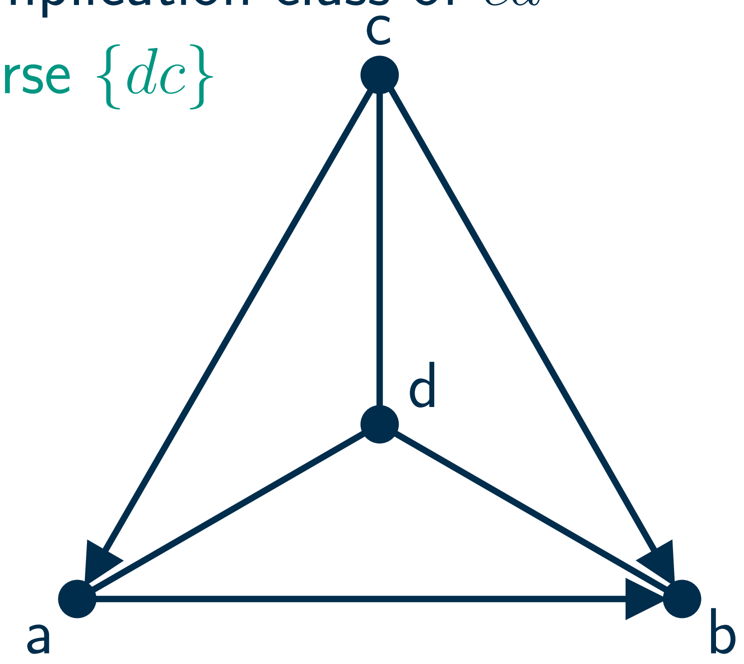
$\{ad, bd\}$ and as inverse $\{da, db\}$

compute implication class of ac

$\{ac, bc\}$ and as inverse $\{ca, cb\}$

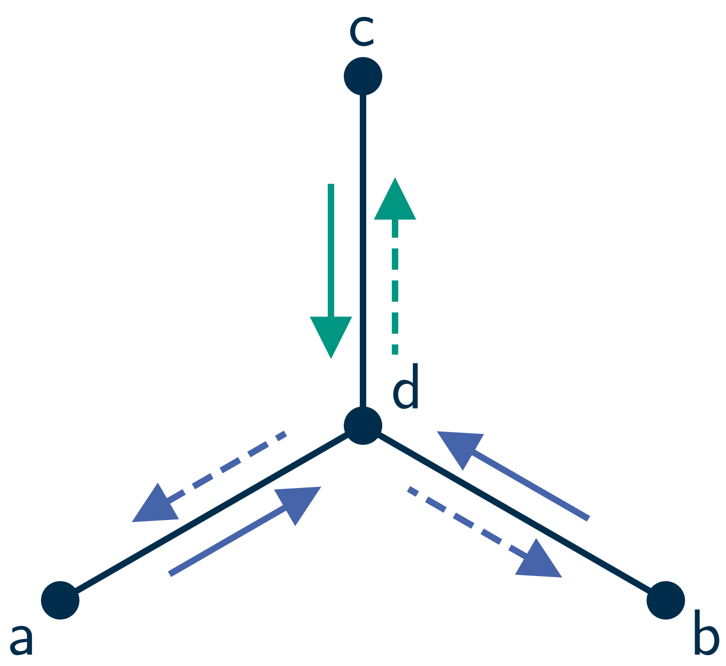
compute implication class of cd

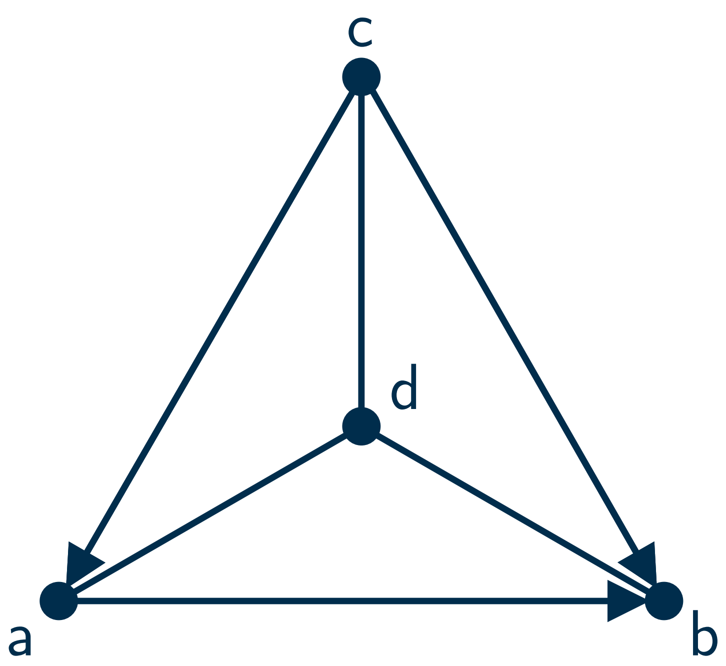
$\{cd\}$, inverse $\{dc\}$



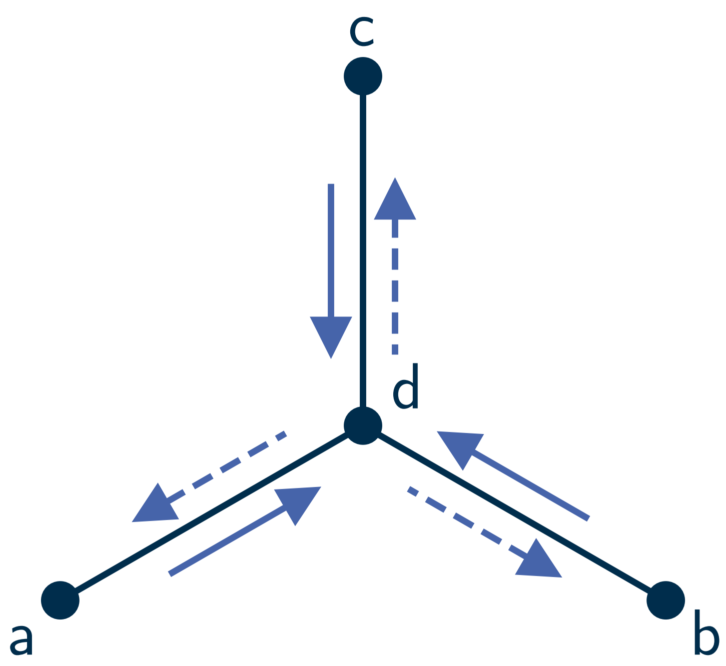
The color class of cd stays the same.
The color classes of ad and bd are merged.

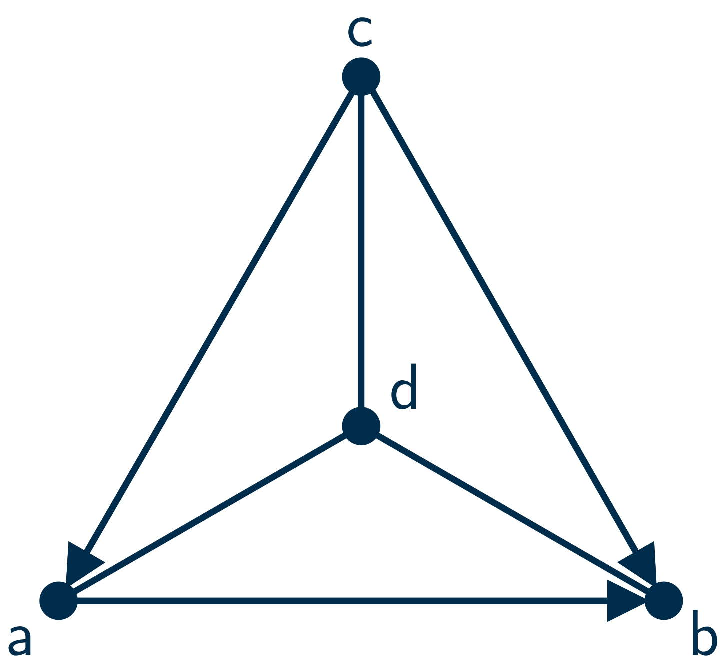
Exercise 4



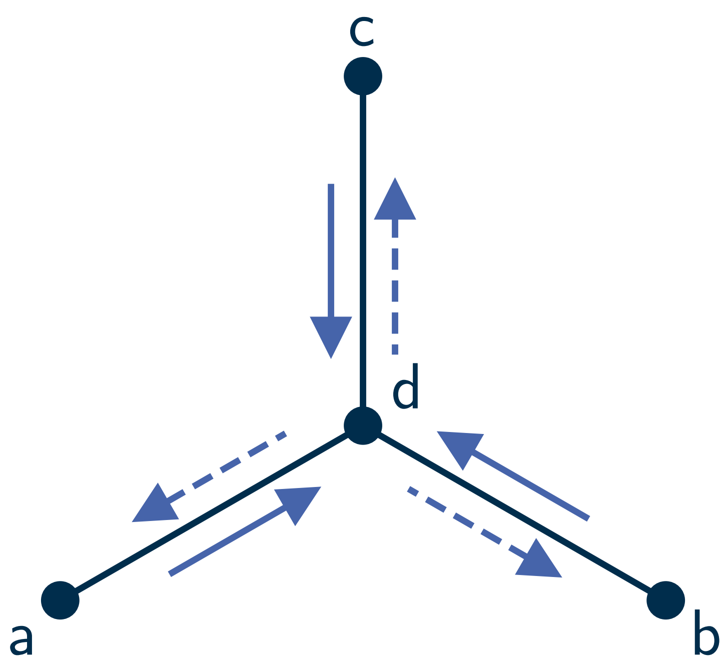


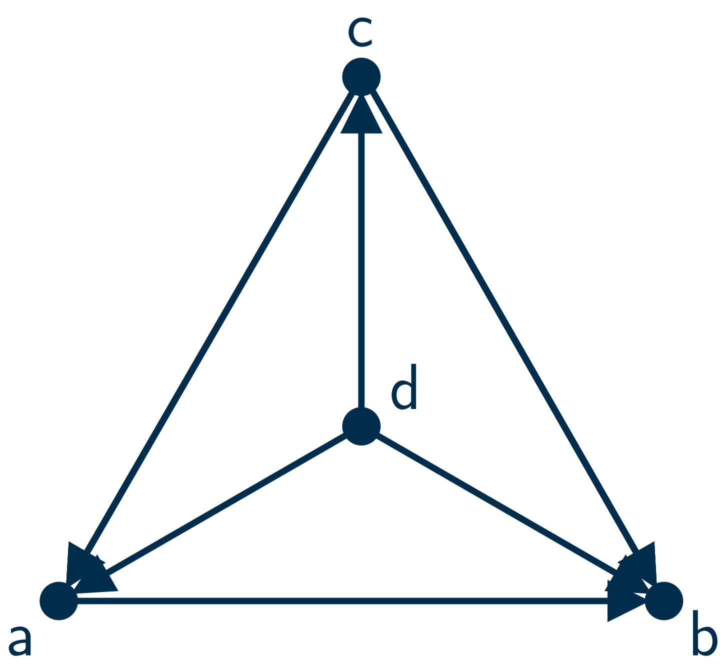
Exercise 4





Exercise 4





Exercise 5

Idea: computing the implication class B_i of $x_i y_i$ is possible in $\mathcal{O}(\Delta \cdot |B_i|)$ time.

Exercise 5

Idea: computing the implication class B_i of $x_i y_i$ is possible in $\mathcal{O}(\Delta \cdot |B_i|)$ time.

Test on $B_i = B_i^{-1}$ deleting $\hat{B}_i$ takes $\mathcal{O}(|B_i|)$ time $\implies$ in total $\mathcal{O}(\sum_i \Delta |B_i|) = \mathcal{O}(\Delta |E|)$.

Exercise 5

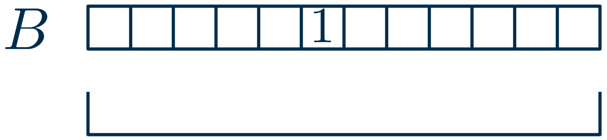
Idea: computing the implication class B_i of $x_i y_i$ is possible in $\mathcal{O}(\Delta \cdot |B_i|)$ time.

Test on $B_i = B_i^{-1}$ deleting $\hat{B}_i$ takes $\mathcal{O}(|B_i|)$ time $\implies$ in total $\mathcal{O}(\sum_i \Delta |B_i|) = \mathcal{O}(\Delta |E|)$.

Global vertex-ID-indexed arrays N_1, N_2, initialized with 0.

Global edge-ID-indexed array B, initialized with 0.

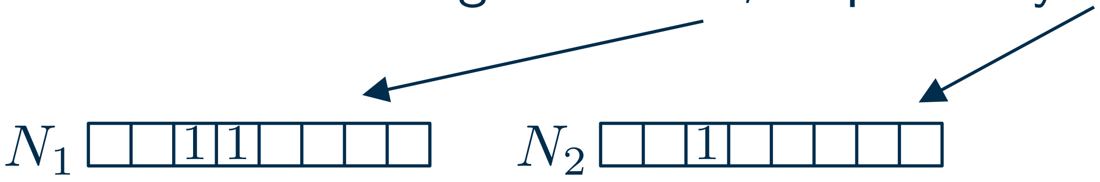
1 iff edge is in implication class
initialize with $x_i y_i$



one position for each edge

Current step with edge uv:

1 iff vertex is neighbor of u, respectively v



Exercise 5

Idea: computing the implication class B_i of $x_i y_i$ is possible in $\mathcal{O}(\Delta \cdot |B_i|)$ time.

Test on $B_i = B_i^{-1}$ deleting $\hat{B}_i$ takes $\mathcal{O}(|B_i|)$ time $\implies$ in total $\mathcal{O}(\sum_i \Delta |B_i|) = \mathcal{O}(\Delta |E|)$.

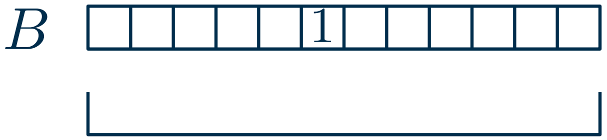
Global vertex-ID-indexed arrays N_1, N_2, initialized with 0.

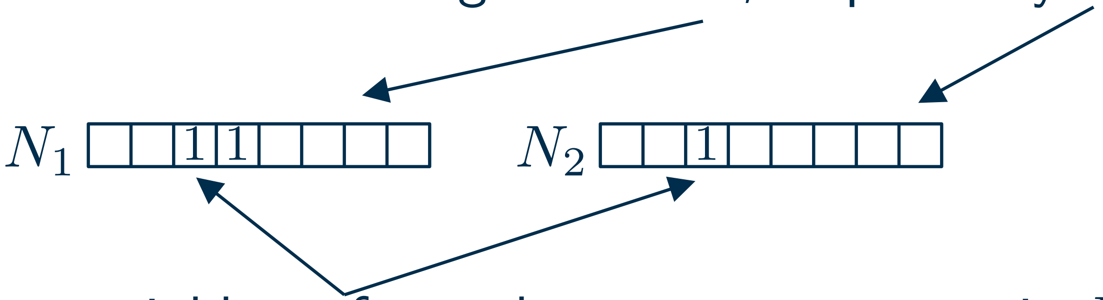
Global edge-ID-indexed array B, initialized with 0.

Current step with edge uv:

1 iff edge is in implication class
initialize with $x_i y_i$

1 iff vertex is neighbor of u, respectively v





one position for each edge

neighbor of u and v $\rightarrow$ no new entry in B

Exercise 5

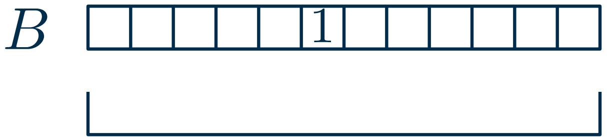
Idea: computing the implication class B_i of $x_i y_i$ is possible in $\mathcal{O}(\Delta \cdot |B_i|)$ time.

Test on $B_i = B_i^{-1}$ deleting $\hat{B}_i$ takes $\mathcal{O}(|B_i|)$ time $\implies$ in total $\mathcal{O}(\sum_i \Delta |B_i|) = \mathcal{O}(\Delta |E|)$.

Global vertex-ID-indexed arrays N_1, N_2, initialized with 0.

Global edge-ID-indexed array B, initialized with 0.

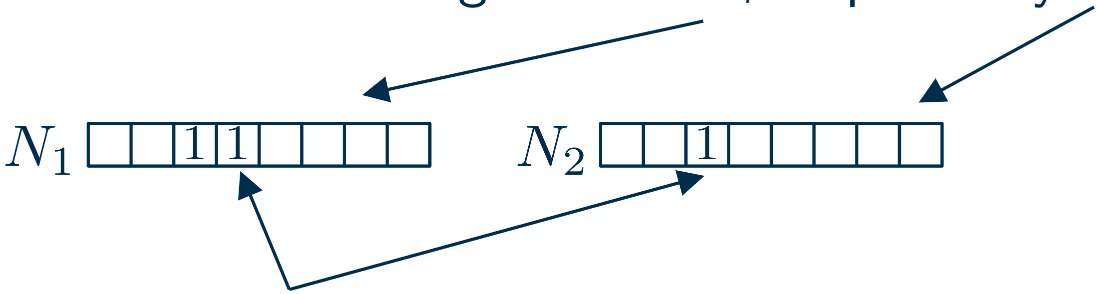
1 iff edge is in implication class
initialize with $x_i y_i$



one position for each edge

Current step with edge uv:

1 iff vertex is neighbor of u, respectively v



neighbor of u bot not of v
$\rightarrow$ add edge to B and orient away from u

Exercise 5

Idea: computing the implication class B_i of x_iy_i is possible in $\mathcal{O}(\Delta \cdot |B_i|)$ time.

Test on $B_i = B_i^{-1}$ deleting $\hat{B}_i$ takes $\mathcal{O}(|B_i|)$ time $\implies$ in total $\mathcal{O}(\sum_i \Delta|B_i|) = \mathcal{O}(\Delta|E|)$.

Global vertex-ID-indexed arrays N_1, N_2, initialized with 0.

Global edge-ID-indexed array B, initialized with 0.

B_i as list of edges with pointer on the current edge.

Exercise 5

Idea: computing the implication class B_i of x_iy_i is possible in $\mathcal{O}(\Delta \cdot |B_i|)$ time.

Test on $B_i = B_i^{-1}$ deleting $\hat{B}_i$ takes $\mathcal{O}(|B_i|)$ time $\implies$ in total $\mathcal{O}(\sum_i \Delta|B_i|) = \mathcal{O}(\Delta|E|)$.

Global vertex-ID-indexed arrays N_1, N_2, initialized with 0.

Global edge-ID-indexed array B, initialized with 0.

B_i as list of edges with pointer on the current edge.

Initialize $B[x_iy_i] = 1$ and set x_iy_i as the only element in B_i.

Let uv be the edge that is currently considered.

Exercise 5

Idea: computing the implication class B_i of $x_i y_i$ is possible in $\mathcal{O}(\Delta \cdot |B_i|)$ time.

Test on $B_i = B_i^{-1}$ deleting $\hat{B}_i$ takes $\mathcal{O}(|B_i|)$ time $\implies$ in total $\mathcal{O}(\sum_i \Delta |B_i|) = \mathcal{O}(\Delta |E|)$.

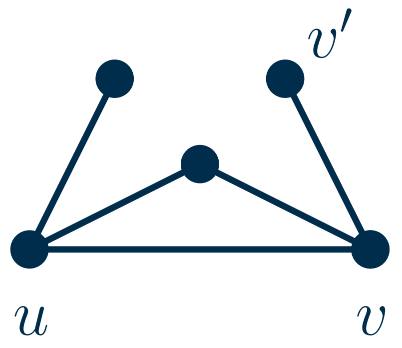
Global vertex-ID-indexed arrays N_1, N_2, initialized with 0.

Global edge-ID-indexed array B, initialized with 0.

B_i as list of edges with pointer on the current edge.

Initialize $B[x_i y_i] = 1$ and set $x_i y_i$ as the only element in B_i.

Let uv be the edge that is currently considered.



Set $N_1[v'] = 1 \forall v' \in N_G(v)$ & $N_2[u'] = 1 \forall u' \in N_G(u)$. $\Rightarrow \mathcal{O}(\Delta)$ time

Exercise 5

Idea: computing the implication class B_i of $x_i y_i$ is possible in $\mathcal{O}(\Delta \cdot |B_i|)$ time.

Test on $B_i = B_i^{-1}$ deleting $\hat{B}_i$ takes $\mathcal{O}(|B_i|)$ time $\implies$ in total $\mathcal{O}(\sum_i \Delta |B_i|) = \mathcal{O}(\Delta |E|)$.

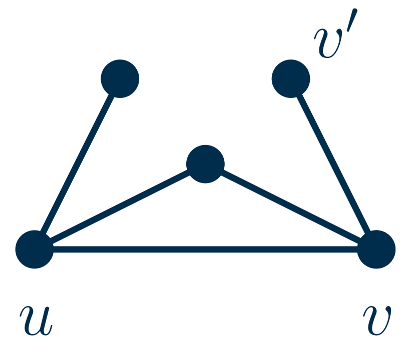
Global vertex-ID-indexed arrays N_1, N_2, initialized with 0.

Global edge-ID-indexed array B, initialized with 0.

B_i as list of edges with pointer on the current edge.

Initialize $B[x_i y_i] = 1$ and set $x_i y_i$ as the only element in B_i.

Let uv be the edge that is currently considered.



Set $N_1[v'] = 1 \forall v' \in N_G(v)$ & $N_2[u'] = 1 \forall u' \in N_G(u)$. $\Rightarrow \mathcal{O}(\Delta)$ time

Look at edges $v'v$. If $N_2[v'] = 0$ then $v'v \in B_i$. $\Rightarrow \mathcal{O}(\Delta)$ time

Exercise 5

Idea: computing the implication class B_i of $x_i y_i$ is possible in $\mathcal{O}(\Delta \cdot |B_i|)$ time.

Test on $B_i = B_i^{-1}$ deleting $\hat{B}_i$ takes $\mathcal{O}(|B_i|)$ time $\implies$ in total $\mathcal{O}(\sum_i \Delta |B_i|) = \mathcal{O}(\Delta |E|)$.

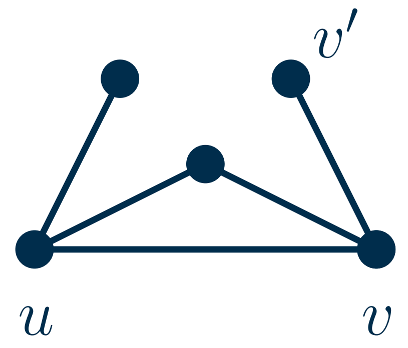
Global vertex-ID-indexed arrays N_1, N_2, initialized with 0.

Global edge-ID-indexed array B, initialized with 0.

B_i as list of edges with pointer on the current edge.

Initialize $B[x_i y_i] = 1$ and set $x_i y_i$ as the only element in B_i.

Let uv be the edge that is currently considered.



Set $N_1[v'] = 1 \forall v' \in N_G(v)$ & $N_2[u'] = 1 \forall u' \in N_G(u)$. $\Rightarrow \mathcal{O}(\Delta)$ time

Look at edges $v'v$. If $N_2[v'] = 0$ then $v'v \in B_i$. $\Rightarrow \mathcal{O}(\Delta)$ time

If $B[v'v] = 0$ set $B[v'v] = 1$ und append $v'v$ to B_i.

Exercise 5

Idea: computing the implication class B_i of $x_i y_i$ is possible in $\mathcal{O}(\Delta \cdot |B_i|)$ time.

Test on $B_i = B_i^{-1}$ deleting $\hat{B}_i$ takes $\mathcal{O}(|B_i|)$ time $\implies$ in total $\mathcal{O}(\sum_i \Delta |B_i|) = \mathcal{O}(\Delta |E|)$.

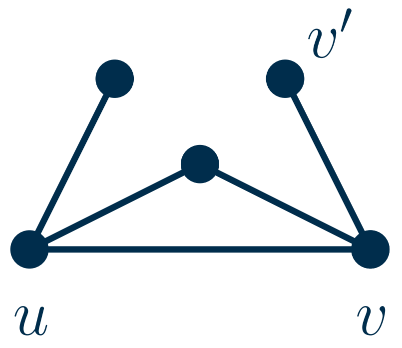
Global vertex-ID-indexed arrays N_1, N_2, initialized with 0.

Global edge-ID-indexed array B, initialized with 0.

B_i as list of edges with pointer on the current edge.

Initialize $B[x_i y_i] = 1$ and set $x_i y_i$ as the only element in B_i.

Let uv be the edge that is currently considered.



Set $N_1[v'] = 1 \forall v' \in N_G(v)$ & $N_2[u'] = 1 \forall u' \in N_G(u)$. $\Rightarrow \mathcal{O}(\Delta)$ time

Look at edges $v'v$. If $N_2[v'] = 0$ then $v'v \in B_i$. $\Rightarrow \mathcal{O}(\Delta)$ time

If $B[v'v] = 0$ set $B[v'v] = 1$ und append $v'v$ to B_i.

Do the same for edges uu'. Then reset N_1 and N_2. $\Rightarrow \mathcal{O}(\Delta)$ time

Exercise 5

Idea: computing the implication class B_i of x_iy_i is possible in $\mathcal{O}(\Delta \cdot |B_i|)$ time.

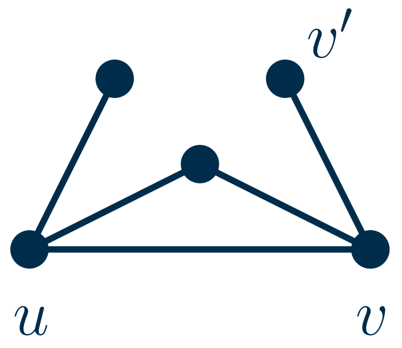
Test on $B_i = B_i^{-1}$ deleting $\hat{B}_i$ takes $\mathcal{O}(|B_i|)$ time $\implies$ in total $\mathcal{O}(\sum_i \Delta|B_i|) = \mathcal{O}(\Delta|E|)$.

Global vertex-ID-indexed arrays N_1, N_2, initialized with 0.

Global edge-ID-indexed array B, initialized with 0.

B_i as list of edges with pointer on the current edge.

Initialize $B[x_iy_i] = 1$ and set x_iy_i as the only element in B_i.



Let uv be the edge that is currently considered.

Set $N_1[v'] = 1 \forall v' \in N_G(v)$ & $N_2[u'] = 1 \forall u' \in N_G(u)$. $\Rrightarrow \mathcal{O}(\Delta)$ time

Look at edges $v'v$. If $N_2[v'] = 0$ then $v'v \in B_i$. $\Rrightarrow \mathcal{O}(\Delta)$ time

If $B[v'v] = 0$ set $B[v'v] = 1$ und append $v'v$ to B_i.

Do the same for edges uu'. Then reset N_1 and N_2. $\Rrightarrow \mathcal{O}(\Delta)$ time

Choose next edge from B_i and start from the beginning. At the end reset B. $\Rrightarrow \mathcal{O}(|B_i|)$ time

Exercise 5

Idea: computing the implication class B_i of $x_i y_i$ is possible in $\mathcal{O}(\Delta \cdot |B_i|)$ time.

Test on $B_i = B_i^{-1}$ deleting $\hat{B}_i$ takes $\mathcal{O}(|B_i|)$ time $\implies$ in total $\mathcal{O}(\sum_i \Delta |B_i|) = \mathcal{O}(\Delta |E|)$.

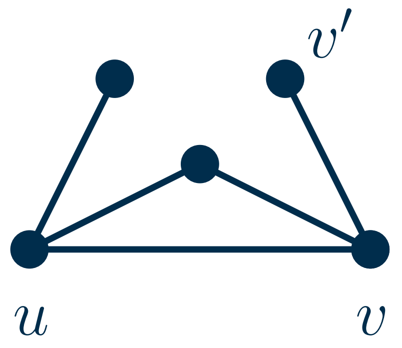
Global vertex-ID-indexed arrays N_1, N_2, initialized with 0.

Global edge-ID-indexed array B, initialized with 0.

B_i as list of edges with pointer on the current edge.

Initialize $B[x_i y_i] = 1$ and set $x_i y_i$ as the only element in B_i.

Let uv be the edge that is currently considered.



Set $N_1[v'] = 1 \forall v' \in N_G(v)$ & $N_2[u'] = 1 \forall u' \in N_G(u)$. $\Rightarrow \mathcal{O}(\Delta)$ time

Look at edges $v'v$. If $N_2[v'] = 0$ then $v'v \in B_i$. $\Rightarrow \mathcal{O}(\Delta)$ time

If $B[v'v] = 0$ set $B[v'v] = 1$ und append $v'v$ to B_i.

Do the same for edges uu'. Then reset N_1 and N_2. $\Rightarrow \mathcal{O}(\Delta)$ time

Choose next edge from B_i and start from the beginning. At the end reset B. $\Rightarrow \mathcal{O}(|B_i|)$ time

Space: N_1, N_2, B, B_i ✓ $\mathcal{O}(|V| + |E|)$ space

Exercise 6

A vertex order σ is a perfect elimination scheme of G $\iff$ $a <_\sigma b <_\sigma c$ with $ab, ac \in E$ implies that $bc \in E$.

Exercise 6

A vertex order σ is a perfect elimination scheme of G
$\iff$ $a <_\sigma b <_\sigma c$ with $ab, ac \in E$ implies that $bc \in E$.

"$\Rightarrow$" Let σ be a PES, let $ab, ac \in E$ with $a <_\sigma b <_\sigma c$. Let i such that $\sigma(i) = a$.

Exercise 6

A vertex order σ is a perfect elimination scheme of G $\iff$ $a <_\sigma b <_\sigma c$ with $ab, ac \in E$ implies that $bc \in E$.

"$\Rightarrow$" Let σ be a PES, let $ab, ac \in E$ with $a <_\sigma b <_\sigma c$. Let i such that $\sigma(i) = a$.

- σ PES $\implies$ right neighborhood of a is a clique

Exercise 6

A vertex order σ is a perfect elimination scheme of G
$\iff a <_\sigma b <_\sigma c$ with $ab, ac \in E$ implies that $bc \in E$.

"$\Rightarrow$" Let σ be a PES, let $ab, ac \in E$ with $a <_\sigma b <_\sigma c$. Let i such that $\sigma(i) = a$.

- σ PES $\implies$ right neighborhood of a is a clique
- b, c are in the right neighborhood of a, so $bc \in E$

Exercise 6

A vertex order σ is a perfect elimination scheme of G
$\iff$ $a <_\sigma b <_\sigma c$ with $ab, ac \in E$ implies that $bc \in E$.

"$\Rightarrow$" Let σ be a PES, let $ab, ac \in E$ with $a <_\sigma b <_\sigma c$. Let i such that $\sigma(i) = a$.

- σ PES $\implies$ right neighborhood of a is a clique
- b, c are in the right neighborhood of a, so $bc \in E$

"$\Leftarrow$" Assume for all a, b, c with $a <_\sigma b <_\sigma c$ we have $ab, ac \in E \Rightarrow bc \in E$.

Exercise 6

A vertex order σ is a perfect elimination scheme of G
$\iff a <_\sigma b <_\sigma c$ with $ab, ac \in E$ implies that $bc \in E$.

"$\Rightarrow$" Let σ be a PES, let $ab, ac \in E$ with $a <_\sigma b <_\sigma c$. Let i such that $\sigma(i) = a$.

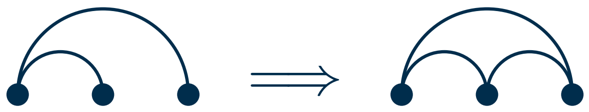
- σ PES $\implies$ right neighborhood of a is a clique
- b, c are in the right neighborhood of a, so $bc \in E$

"$\Leftarrow$" Assume for all a, b, c with $a <_\sigma b <_\sigma c$ we have $ab, ac \in E \Rightarrow bc \in E$.

- For each two right neighbors b, c of some vertex a, we have $bc \in E$

Exercise 6

A vertex order σ is a perfect elimination scheme of G
$\iff a <_\sigma b <_\sigma c$ with $ab, ac \in E$ implies that $bc \in E$.



"$\Rightarrow$" Let σ be a PES, let $ab, ac \in E$ with $a <_\sigma b <_\sigma c$. Let i such that $\sigma(i) = a$.

- σ PES $\implies$ right neighborhood of a is a clique
- b, c are in the right neighborhood of a, so $bc \in E$

"$\Leftarrow$" Assume for all a, b, c with $a <_\sigma b <_\sigma c$ we have $ab, ac \in E \Rightarrow bc \in E$.

- For each two right neighbors b, c of some vertex a, we have $bc \in E$

$\implies$ the right neighborhood of a is a clique

Exercise 6

G is a comparability graph
$\iff$ there is a vertex oder σ such that $a <_\sigma b <_\sigma c$ with $ab, bc \in E$ implies that $ac \in E$

Exercise 6

G is a comparability graph
$\iff$ there is a vertex oder σ such that $a <_\sigma b <_\sigma c$ with $ab, bc \in E$ implies that $ac \in E$

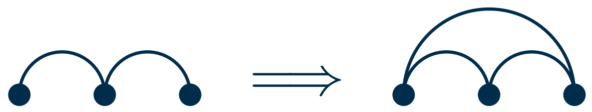
"$\Rightarrow$" Let F be a transitive orientation of G

Exercise 6

G is a comparability graph
$\iff$ there is a vertex oder σ such that $a <_\sigma b <_\sigma c$ with $ab, bc \in E$ implies that $ac \in E$

"$\Rightarrow$" Let F be a transitive orientation of G
Idea: any topological ordering works for σ.

Exercise 6



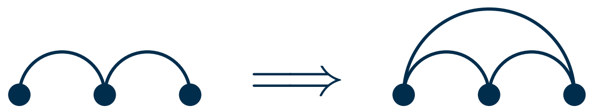
G is a comparability graph
$\iff$ there is a vertex oder σ such that $a <_\sigma b <_\sigma c$ with $ab, bc \in E$ implies that $ac \in E$

"$\Rightarrow$" Let F be a transitive orientation of G

Idea: any topological ordering works for σ.

Goal: show that there is a topological ordering, i.e., show that F is acyclic

Exercise 6



G is a comparability graph
$\iff$ there is a vertex oder σ such that $a <_\sigma b <_\sigma c$ with $ab, bc \in E$ implies that $ac \in E$

"$\Rightarrow$" Let F be a transitive orientation of G
Idea: any topological ordering works for σ.

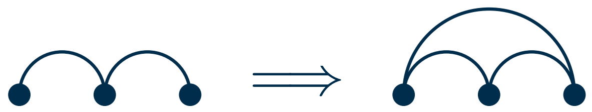
Goal: show that there is a topological ordering, i.e., show that F is acyclic

Consider shortest directed cycle

Case 1: length 3

Case 2: length > 3

Exercise 6



G is a comparability graph
$\iff$ there is a vertex oder σ such that $a <_\sigma b <_\sigma c$ with $ab, bc \in E$ implies that $ac \in E$

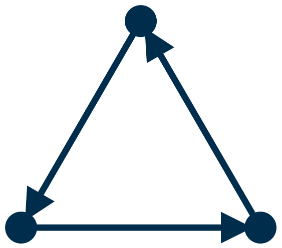
"$\Rightarrow$" Let F be a transitive orientation of G

Idea: any topological ordering works for σ.

Goal: show that there is a topological ordering, i.e., show that F is acyclic

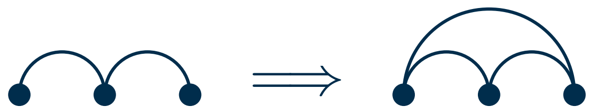
Consider shortest directed cycle

Case 1: length 3



Case 2: length > 3

Exercise 6



G is a comparability graph
$\iff$ there is a vertex oder σ such that $a <_\sigma b <_\sigma c$ with $ab, bc \in E$ implies that $ac \in E$

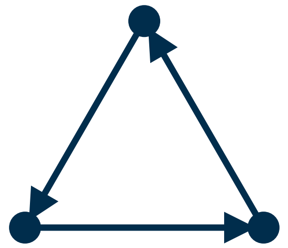
"$\Rightarrow$" Let F be a transitive orientation of G

Idea: any topological ordering works for σ.

Goal: show that there is a topological ordering, i.e., show that F is acyclic

Consider shortest directed cycle

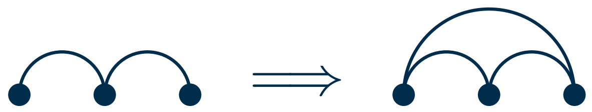
Case 1: length 3



not transitive

Case 2: length > 3

Exercise 6



G is a comparability graph
$\iff$ there is a vertex oder σ such that $a <_\sigma b <_\sigma c$ with $ab, bc \in E$ implies that $ac \in E$

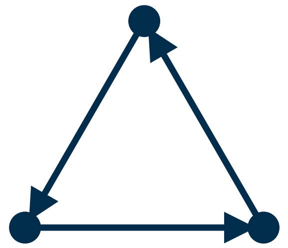
"$\Rightarrow$" Let F be a transitive orientation of G

Idea: any topological ordering works for σ.

Goal: show that there is a topological ordering, i.e., show that F is acyclic

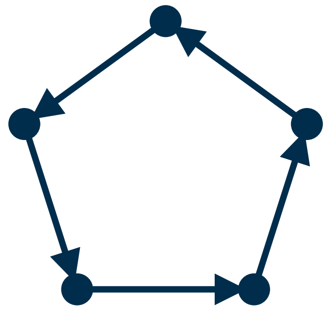
Consider shortest directed cycle

Case 1: length 3

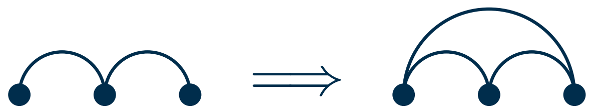


not transitive ↯

Case 2: length > 3



Exercise 6



G is a comparability graph
$\iff$ there is a vertex oder σ such that $a <_\sigma b <_\sigma c$ with $ab, bc \in E$ implies that $ac \in E$

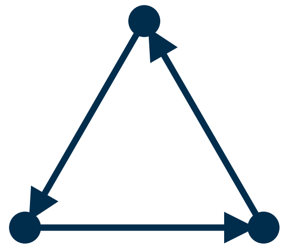
"$\Rightarrow$" Let F be a transitive orientation of G

Idea: any topological ordering works for σ.

Goal: show that there is a topological ordering, i.e., show that F is acyclic

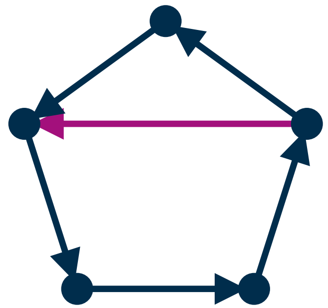
Consider shortest directed cycle

Case 1: length 3



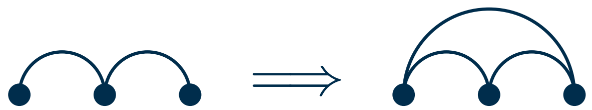
not transitive

Case 2: length > 3



transitive edge exists

Exercise 6



G is a comparability graph
$\iff$ there is a vertex oder σ such that $a <_\sigma b <_\sigma c$ with $ab, bc \in E$ implies that $ac \in E$

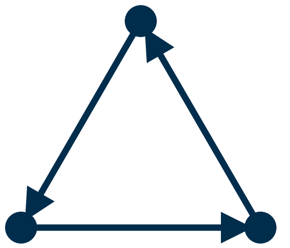
"$\Rightarrow$" Let F be a transitive orientation of G

Idea: any topological ordering works for σ.

Goal: show that there is a topological ordering, i.e., show that F is acyclic

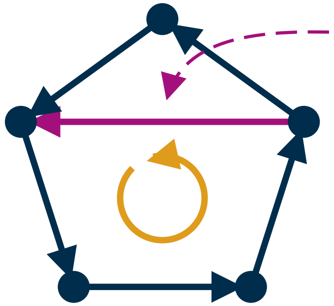
Consider shortest directed cycle

Case 1: length 3



not transitive

Case 2: length > 3



transitive edge exists

$\Rightarrow$ shorter directed cycle

Exercise 6

G is a comparability graph
$\iff$ there is a vertex oder σ such that $a <_\sigma b <_\sigma c$ with $ab, bc \in E$ implies that $ac \in E$

"$\Leftarrow$" Let $F = \{(u,v) \colon uv \in E \text{ and } u <_\sigma v\}$.

Exercise 6

G is a comparability graph
$\iff$ there is a vertex oder σ such that $a <_\sigma b <_\sigma c$ with $ab, bc \in E$ implies that $ac \in E$

"$\Leftarrow$" Let $F = \{(u, v)\colon uv \in E \text{ and } u <_\sigma v\}$.

Obs: F is orientation of G. Prove that F is transitive

Exercise 6

G is a comparability graph
$\iff$ there is a vertex oder σ such that $a <_\sigma b <_\sigma c$ with $ab, bc \in E$ implies that $ac \in E$

"$\Leftarrow$" Let $F = \{(u, v) \colon uv \in E \text{ and } u <_\sigma v\}$.

Obs: F is orientation of G. Prove that F is transitive

Let a, b, $c \in V$ with $ab, bc \in F$. Then $a <_\sigma b$ und $b <_\sigma c$.

Exercise 6

G is a comparability graph
$\iff$ there is a vertex oder σ such that $a <_\sigma b <_\sigma c$ with $ab, bc \in E$ implies that $ac \in E$

"$\Leftarrow$" Let $F = \{(u, v) \colon uv \in E \text{ and } u <_\sigma v\}$.

Obs: F is orientation of G. Prove that F is transitive

Let a, b, $c \in V$ with $ab, bc \in F$. Then $a <_\sigma b$ und $b <_\sigma c$.

By definition of σ we have $ac \in E$.

Exercise 6

G is a comparability graph
$\iff$ there is a vertex oder σ such that $a <_\sigma b <_\sigma c$ with $ab, bc \in E$ implies that $ac \in E$

"$\Leftarrow$" Let $F = \{(u,v) \colon uv \in E \text{ and } u <_\sigma v\}$.

Obs: F is orientation of G. Prove that F is transitive

Let a, b, $c \in V$ with $ab, bc \in F$. Then $a <_\sigma b$ und $b <_\sigma c$.

By definition of σ we have $ac \in E$.

With definition of F it follows that $ac \in F$, so G is a comparability graph

Exercise 7

Let G be a transitively oriented comparability graph with only one sink and one source.

There is a topological ordering σ of its vertex set $V(G)$ and a planar straight-line drawing of the transitive reduction of G such that $y(v) < y(w)$ if and only if $v <_\sigma w$.

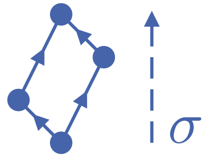
$$\Downarrow$$

There are two linear orders φ, φ' of $V(G)$ such that $(v, w) \in E(G)$ if and only if $v <_\varphi w$ and $v <_{\varphi'} w$.

Exercise 7

Let G be a transitively oriented comparability graph with only one sink and one source.

called ***upward planar*** or ***planar lattice***



There is a topological ordering σ of its vertex set $V(G)$ and a planar straight-line drawing of the transitive reduction of G such that $y(v) < y(w)$ if and only if $v <_\sigma w$.

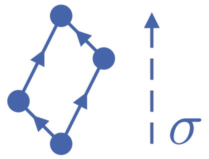
$$\Downarrow$$

There are two linear orders φ, φ' of $V(G)$ such that $(v, w) \in E(G)$ if and only if $v <_\varphi w$ and $v <_{\varphi'} w$.

Exercise 7

Let G be a transitively oriented comparability graph with only one sink and one source.

called ***upward planar*** or ***planar lattice***



There is a topological ordering σ of its vertex set $V(G)$ and a planar straight-line drawing of the transitive reduction of G such that $y(v) < y(w)$ if and only if $v <_\sigma w$.

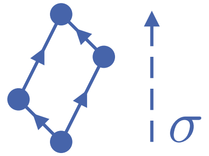
$\Downarrow$

There are two linear orders φ, φ' of $V(G)$ such that $(v, w) \in E(G)$ if and only if $v <_\varphi w$ and $v <_{\varphi'} w$. $\rightarrow$ called ***dominance drawing*** or ***2-dimensional***

Exercise 7

Let G be a transitively oriented comparability graph with only one sink and one source.

called ***upward planar*** or ***planar lattice***



There is a topological ordering σ of its vertex set $V(G)$ and a planar straight-line drawing of the transitive reduction of G such that $y(v) < y(w)$ if and only if $v <_\sigma w$.

$$\Downarrow$$

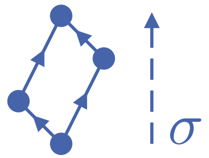
There are two linear orders φ, φ' of $V(G)$ such that $(v, w) \in E(G)$ if and only if $v <_\varphi w$ and $v <_{\varphi'} w$. $\rightarrow$ called ***dominance drawing*** or ***2-dimensional***

Proof sketch:
Choose φ with left-first DFS and φ' with right-first DFS.

Exercise 7

Let G be a transitively oriented comparability graph with only one sink and one source.

called ***upward planar*** or ***planar lattice***



There is a topological ordering σ of its vertex set $V(G)$ and a planar straight-line drawing of the transitive reduction of G such that $y(v) < y(w)$ if and only if $v <_\sigma w$.

$$\Downarrow$$

There are two linear orders φ, φ' of $V(G)$ such that $(v, w) \in E(G)$ if and only if $v <_\varphi w$ and $v <_{\varphi'} w$. $\rightarrow$ called ***dominance drawing*** or ***2-dimensional***

Proof sketch:

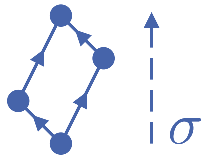
Choose φ with left-first DFS and φ' with right-first DFS.

If the transitive reduction is planar, then this yields an upward planar embedding ($\varphi \rightarrow$ x-coordinates, $\varphi' \rightarrow$ y-coordinates, rotate appropriately).

Exercise 7

Let G be a transitively oriented comparability graph with only one sink and one source.

called ***upward planar*** or ***planar lattice***



There is a topological ordering σ of its vertex set $V(G)$ and a planar straight-line drawing of the transitive reduction of G such that $y(v) < y(w)$ if and only if $v <_\sigma w$.

$$\Downarrow$$

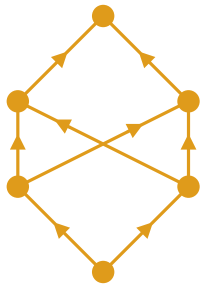
There are two linear orders φ, φ' of $V(G)$ such that $(v, w) \in E(G)$ if and only if $v <_\varphi w$ and $v <_{\varphi'} w$. $\to$ called ***dominance drawing*** or ***2-dimensional***

Proof sketch:

Non-planar counterexample for "$\not\Uparrow$"

Choose φ with left-first DFS and φ' with right-first DFS.

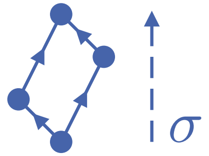
If the transitive reduction is planar, then this yields an upward planar embedding ($\varphi \to$ x-coordinates, $\varphi' \to$ y-coordinates, rotate appropriately).



Exercise 7

Let G be a transitively oriented comparability graph with only one sink and one source.

called ***upward planar*** or ***planar lattice***



There is a topological ordering σ of its vertex set $V(G)$ and a planar straight-line drawing of the transitive reduction of G such that $y(v) < y(w)$ if and only if $v <_\sigma w$.

$$\Downarrow$$

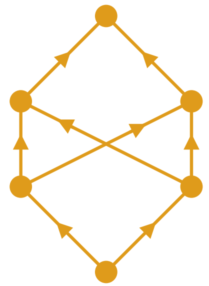
There are two linear orders φ, φ' of $V(G)$ such that $(v, w) \in E(G)$ if and only if $v <_\varphi w$ and $v <_{\varphi'} w$. $\rightarrow$ called ***dominance drawing*** or ***2-dimensional***

Proof sketch:

Non-planar counterexample for "$\not\Uparrow$"

Choose φ with left-first DFS and φ' with right-first DFS.

If the transitive reduction is planar, then this yields an upward planar embedding ($\varphi \rightarrow$ x-coordinates, $\varphi' \rightarrow$ y-coordinates, rotate appropriately).



https://en.wikipedia.org/wiki/Dominance_drawing#Planar_graphs