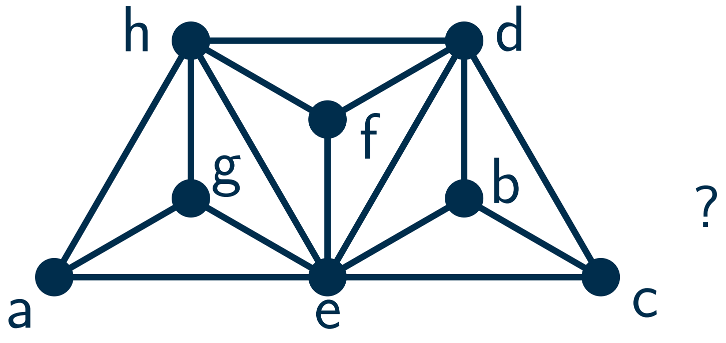
Algorithmic Graph Theory

Solution Sheet 4

Laura Merker and Samuel Schneider, July 2, 2025

Problems



(1) Is $\sigma = [a, b, c, d, e, f, g, h]$ a perfect elemenation scheme of

(2) Prove the following statements:

- **(i)** There are infinitly many triangulated graphs that are **not** chordal.
- **(ii)** There are infinitly many triangulated graphs that are chordal.
- **(iii)** Every triangulated graph that has a universal vertex is chordal.

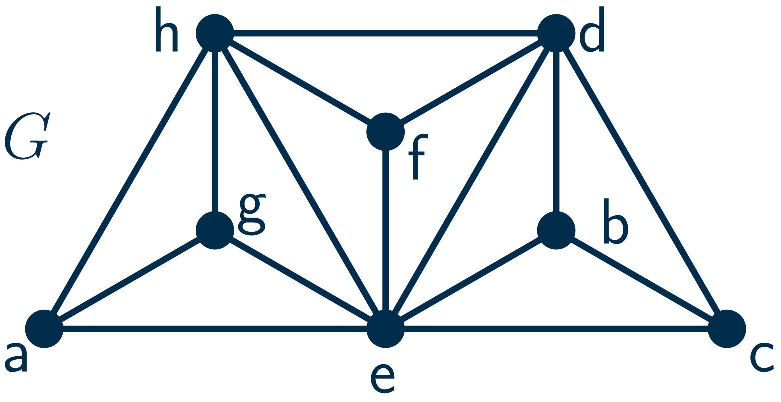
(3) Let σ be a PES and K_v be the clique consisting of v and its subsequent neighbors w. r. t. σ. Prove that K_v is a maximal clique $\Leftrightarrow$ there is no predecessor u of v such that $K_v \subseteq K_u$.

(4) Show that a minimum vertex cover can be computed efficiently on chordal graphs.

(5) Let G be a connected graph. Prove that G is a tree $\Leftrightarrow$ every family of paths in G fulfills the Helly property.

(6) Prove that if the line graph of G is chordal, then G is chordal. Show that the reverse does not hold.

Exercise 1



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3  |  v ← σ(i);
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5  |  if X = ∅ then go to line 8;
6  |  u ← argmin{σ(x) | x ∈ X};
7  |  add X − {u} to A(u);
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9  |  |  return false;
10 return true;
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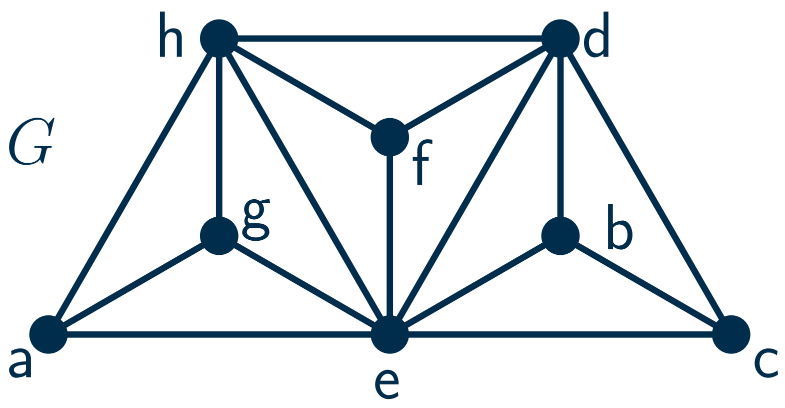

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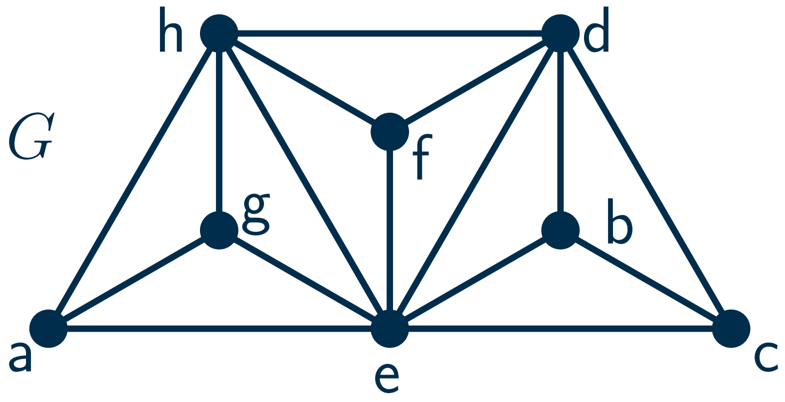

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$v = a, \ X_v = \{e, g, h\} \neq \emptyset$

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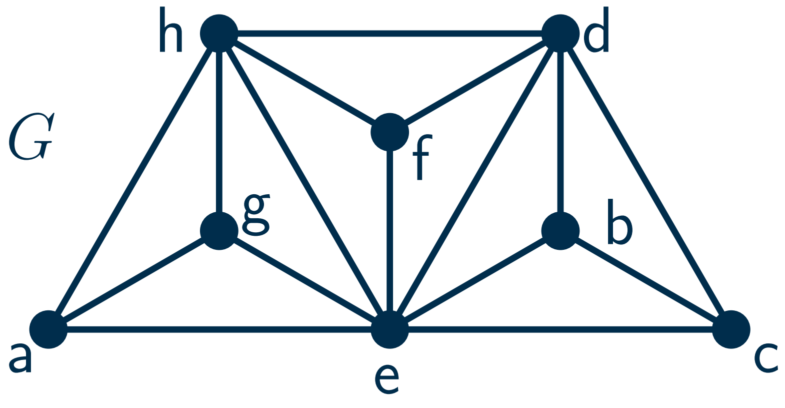

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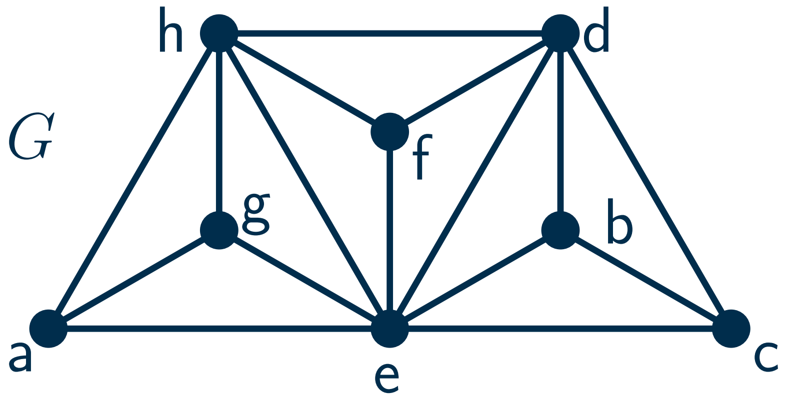

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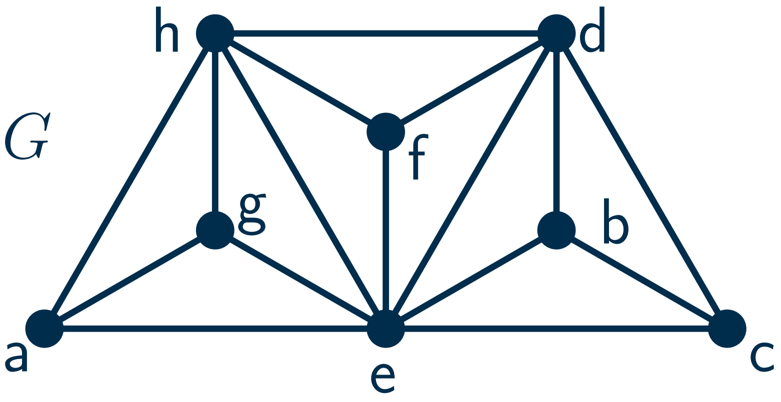

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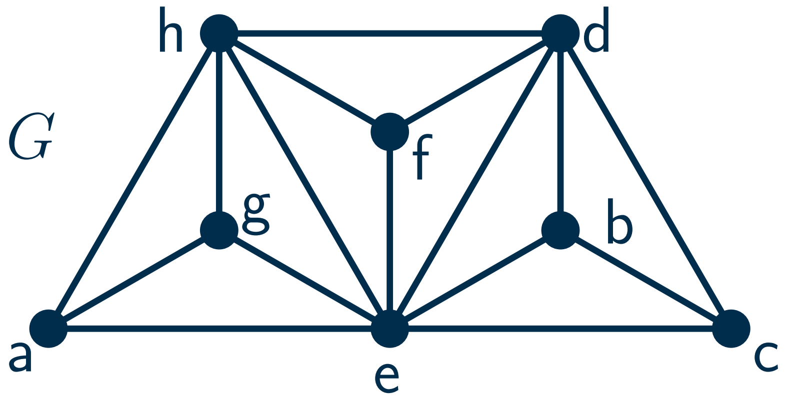

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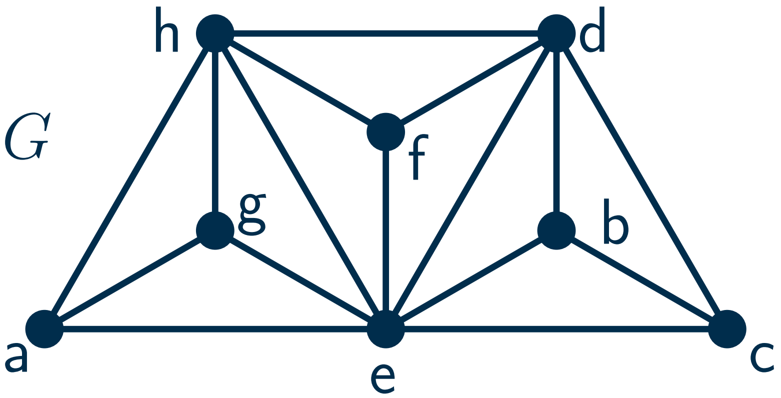

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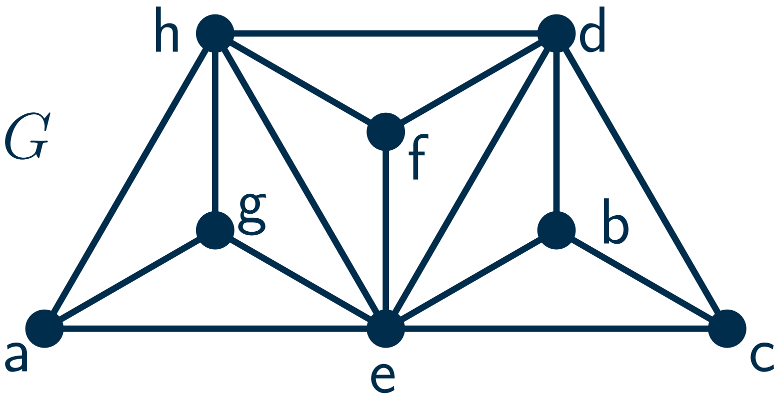

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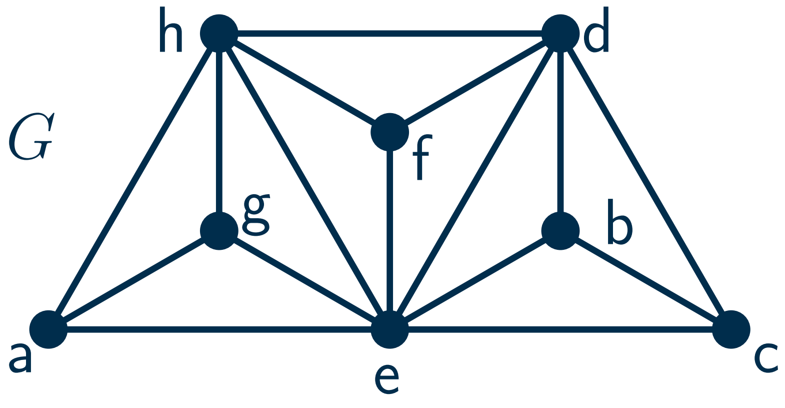

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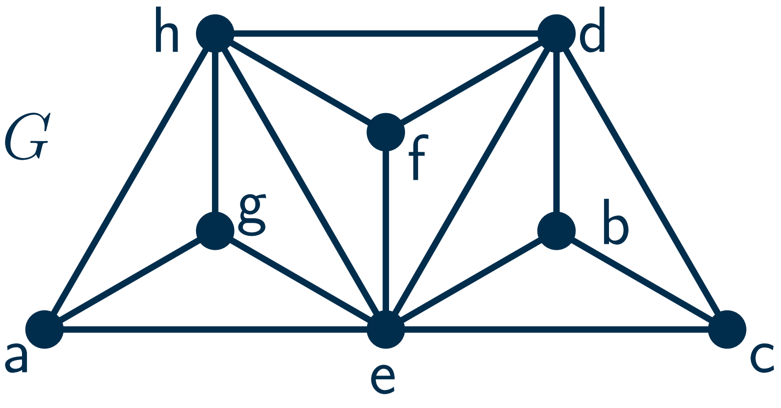

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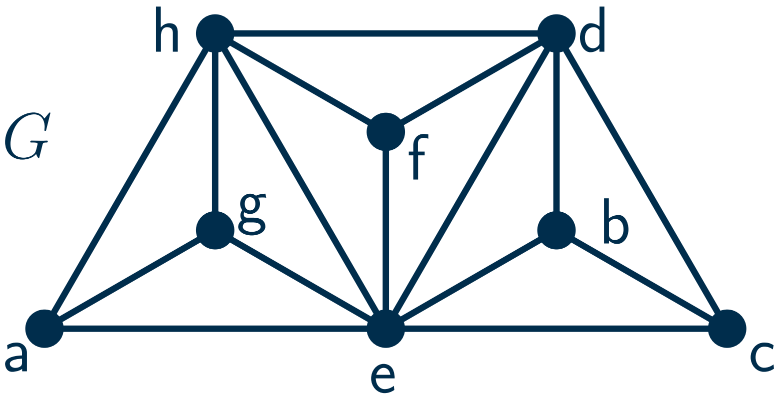

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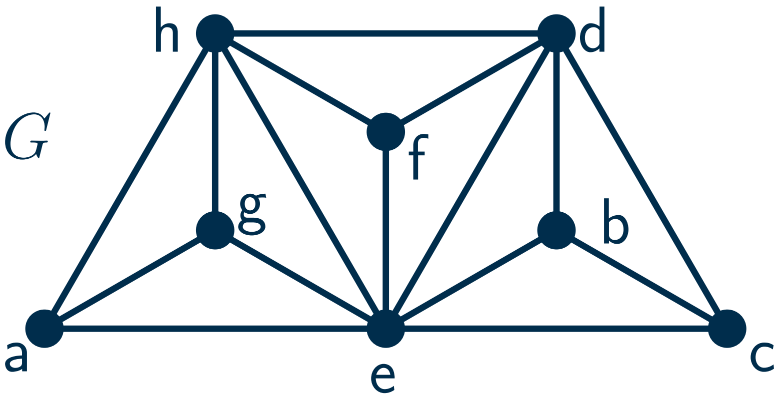

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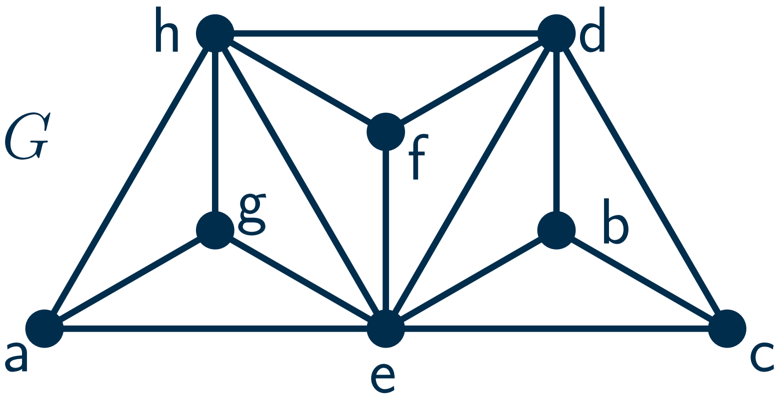

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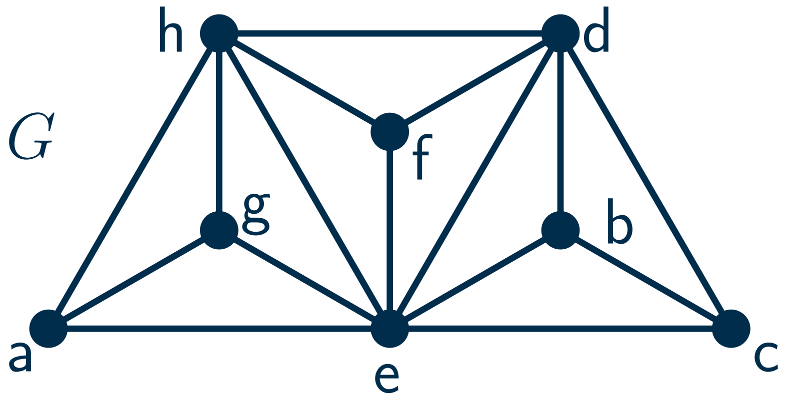

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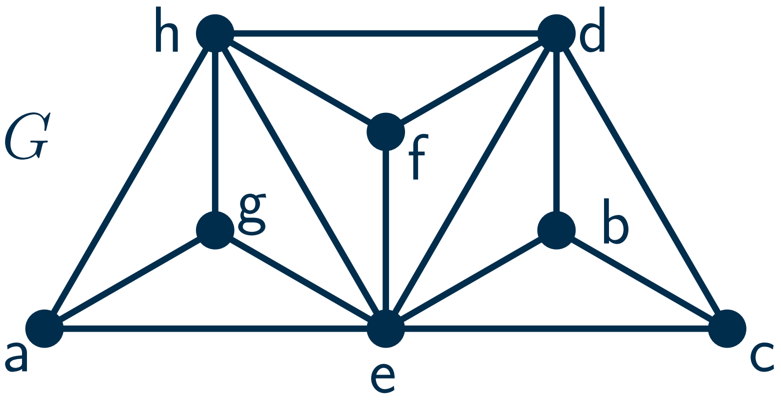

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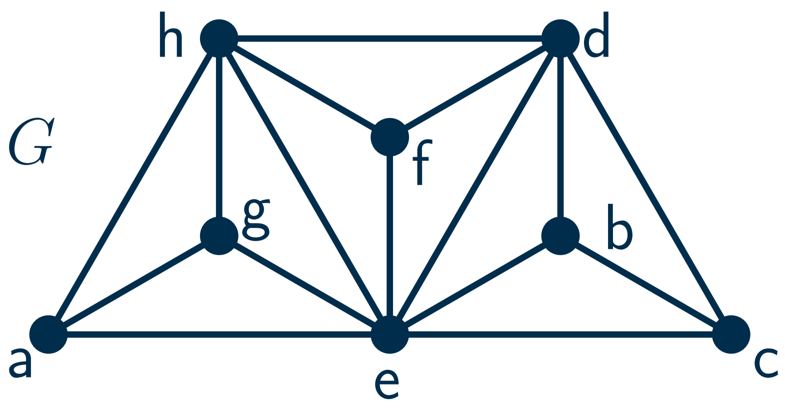
Line 8: $A(v) - \mathrm{Adj}(v) = \emptyset$

$v = d,\ X_v = \{e, f, h\} \neq \emptyset$

Line 8: $A(v) - \mathrm{Adj}(v) = \emptyset$

$v = e,\ X_v = \{f, g, h\} \neq \emptyset$

Exercise 1



$\sigma = [a, b, c, d, e, f, g, h]$

$i = 5$

$A(a) = \{\}$
$A(b) = \{\}$
$A(c) = \{d, e\}$
$A(d) = \{e\}$
$A(e) = \{f, g, h\}$
$A(f) = \{g, h\}$
$A(g) = \{\}$
$A(h) = \{\}$

1 **for** each vertex v **do** $A(v) \leftarrow \emptyset$;
2 **for** $i \leftarrow 1$ **to** $n - 1$ **do**
3     $v \leftarrow \sigma(i)$;
4     $X \leftarrow \{x \in \mathrm{Adj}(v) \mid \sigma(v) < \sigma(x)\}$;
5   **if** $X = \emptyset$ **then** go to line **8**;
6     $u \leftarrow \mathrm{argmin}\{\sigma(x) \mid x \in X\}$;
7   add $X - \{u\}$ to $A(u)$;
8   **if** $A(v) - \mathrm{Adj}(v) \neq \emptyset$ **then**
9   **return** false;
10 **return** true;

$v = a$, $X_v = \{e, g, h\} \neq \emptyset$
Line 8: $A(v) - \mathrm{Adj}(v) = \emptyset$

$v = b$, $X_v = \{c, d, e\} \neq \emptyset$
Line 8: $A(v) - \mathrm{Adj}(v) = \emptyset$

$v = c$, $X_v = \{d, e\} \neq \emptyset$
Line 8: $A(v) - \mathrm{Adj}(v) = \emptyset$

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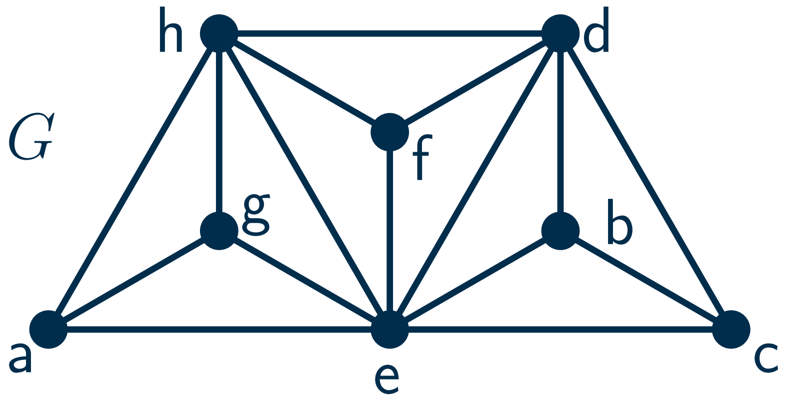
$v = e$, $X_v = \{f, g, h\} \neq \emptyset$
Line 8: $A(v) - \mathrm{Adj}(v) = \emptyset$

Exercise 1



$\sigma = [a, b, c, d, e, f, g, h]$

$i = 6$

$A(a) = \{\}$
$A(b) = \{\}$
$A(c) = \{d, e\}$
$A(d) = \{e\}$
$A(e) = \{f, g, h\}$
$A(f) = \{g, h\}$
$A(g) = \{\}$
$A(h) = \{\}$

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6      u ← argmin{σ(x) | x ∈ X};
7      add X − {u} to A(u);
8      if A(v) − Adj(v) ≠ ∅ then
9          return false;
10 return true;
```

$v = a,\ X_v = \{e, g, h\} \neq \emptyset$
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$v = b,\ X_v = \{c, d, e\} \neq \emptyset$
Line 8: $A(v) - \mathrm{Adj}(v) = \emptyset$

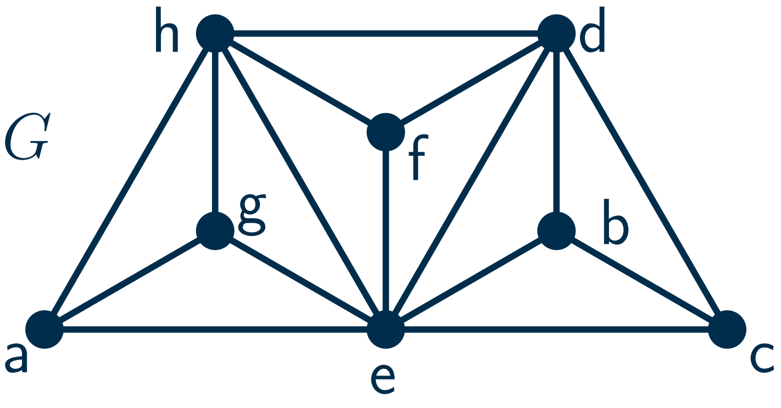
$v = c,\ X_v = \{d, e\} \neq \emptyset$
Line 8: $A(v) - \mathrm{Adj}(v) = \emptyset$

$v = d,\ X_v = \{e, f, h\} \neq \emptyset$
Line 8: $A(v) - \mathrm{Adj}(v) = \emptyset$

$v = e,\ X_v = \{f, g, h\} \neq \emptyset$
Line 8: $A(v) - \mathrm{Adj}(v) = \emptyset$

$v = f,\ X_v = \{h\} \neq \emptyset$

Exercise 1



$\sigma = [a, b, c, d, e, f, g, h]$

$i = 6$

$A(a) = \{\}$
$A(b) = \{\}$
$A(c) = \{d, e\}$
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$A(e) = \{f, g, h\}$
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7 $\quad$ add $X - \{u\}$ to $A(u)$;
8 $\quad$ **if** $A(v) - \mathrm{Adj}(v) \neq \emptyset$ **then**
9 $\quad\quad$ **return** false;
10 **return** true;

$v = a,\ X_v = \{e, g, h\} \neq \emptyset$
Line 8: $A(v) - \mathrm{Adj}(v) = \emptyset$

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Line 8: $A(v) - \mathrm{Adj}(v) = \emptyset$

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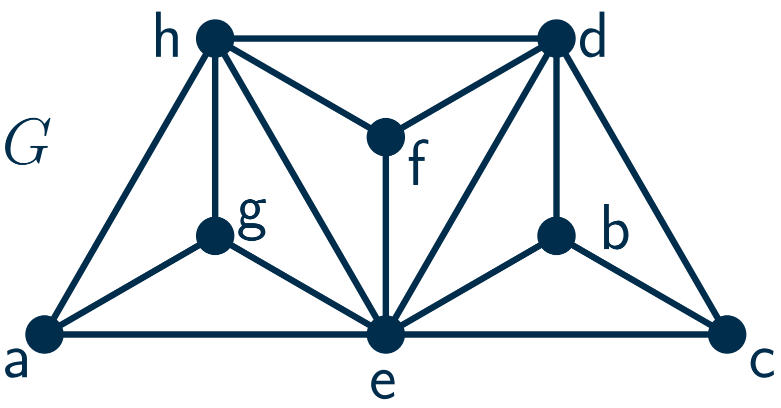
Line 8: $A(v) - \mathrm{Adj}(v) = \emptyset$

$v = d,\ X_v = \{e, f, h\} \neq \emptyset$
Line 8: $A(v) - \mathrm{Adj}(v) = \emptyset$

$v = e,\ X_v = \{f, g, h\} \neq \emptyset$
Line 8: $A(v) - \mathrm{Adj}(v) = \emptyset$

$v = f,\ X_v = \{h\} \neq \emptyset$

Exercise 1



$\sigma = [a, b, c, d, e, f, g, h]$

$i = 6$

$A(a) = \{\}$
$A(b) = \{\}$
$A(c) = \{d, e\}$
$A(d) = \{e\}$
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$A(f) = \{g, h\}$
$A(g) = \{\}$
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```

$v = a,\ X_v = \{e, g, h\} \neq \emptyset$

Line 8: $A(v) - \mathrm{Adj}(v) = \emptyset$

$v = b,\ X_v = \{c, d, e\} \neq \emptyset$

Line 8: $A(v) - \mathrm{Adj}(v) = \emptyset$

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Line 8: $A(v) - \mathrm{Adj}(v) = \emptyset$

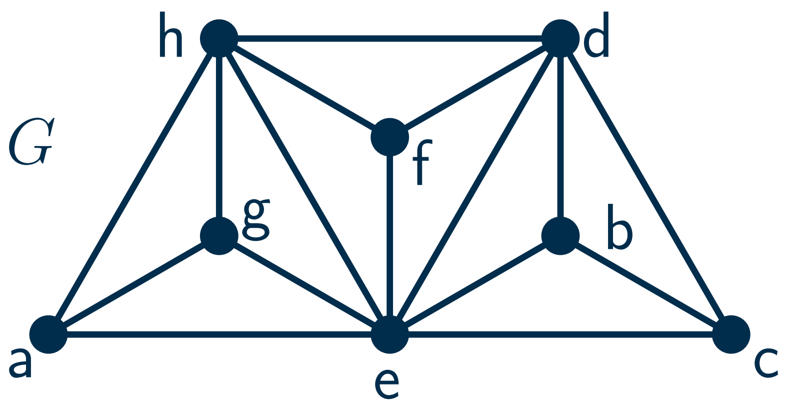
$v = e,\ X_v = \{f, g, h\} \neq \emptyset$

Line 8: $A(v) - \mathrm{Adj}(v) = \emptyset$

$v = f,\ X_v = \{h\} \neq \emptyset$

Line 8: $A(v) - \mathrm{Adj}(v) = \{g\} \neq \emptyset$

Exercise 1



$\sigma = [a, b, c, d, e, f, g, h]$

not a PES of G!

$A(a) = \{\}$
$A(b) = \{\}$
$A(c) = \{d, e\}$
$A(d) = \{e\}$
$A(e) = \{f, g, h\}$
$A(f) = \{g, h\}$
$A(g) = \{\}$
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8      if A(v) − Adj(v) ≠ ∅ then
9          return false;
10 return true;
```

$v = a,\ X_v = \{e, g, h\} \neq \emptyset$
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Line 8: $A(v) - \mathrm{Adj}(v) = \emptyset$

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Line 8: $A(v) - \mathrm{Adj}(v) = \emptyset$

$v = e,\ X_v = \{f, g, h\} \neq \emptyset$
Line 8: $A(v) - \mathrm{Adj}(v) = \emptyset$

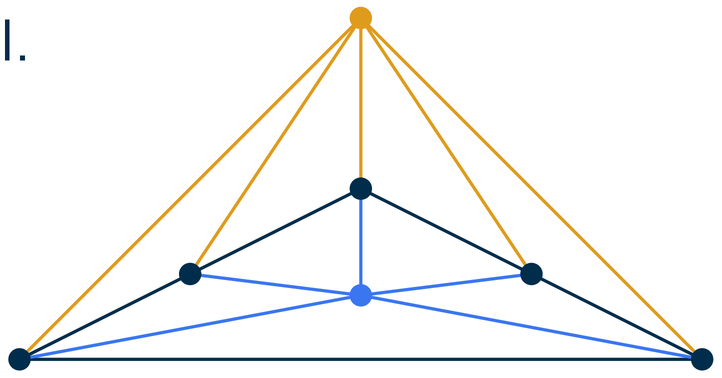
$v = f,\ X_v = \{h\} \neq \emptyset$
Line 8: $A(v) - \mathrm{Adj}(v) = \{g\} \neq \emptyset$

Exercise 2

(1) There are infinitly many triangulated graphs that are **not** chordal.

Idea: triangulate C_k with $k \geq 4$

- induced $C_k \implies$ not chordal

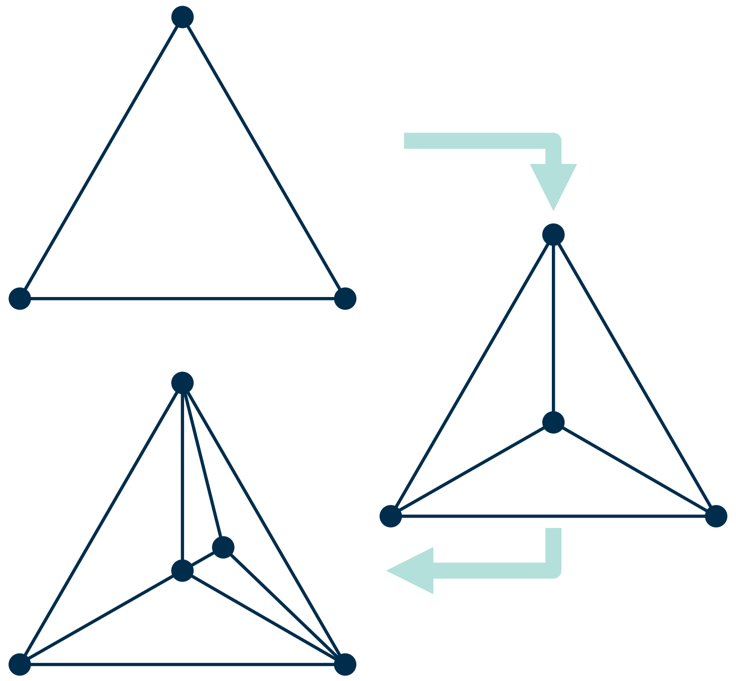


(2) There are infinitly many triangulated graphs that are chordal.

Idea: ***stacked triangulations***

- start with C_3
- repeatedly choose inner triangle t and add vertex v with $N(v) = t$
- resulting graph is a 3-tree $\implies$ chordal
- resulting graph is clearly triangulated

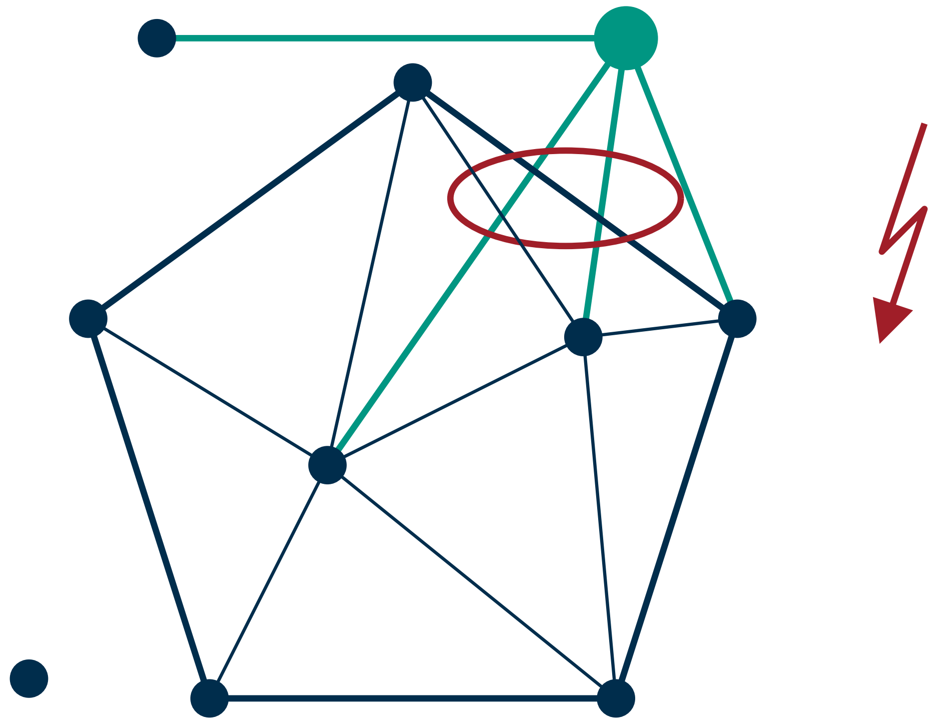
These graphs are also known as ***planar*** $\boldsymbol{3}$***-trees*** or ***Apollonian networks***.
Every triangulated planar graph with treewidth at most 3 is such a graph.



Exercise 2

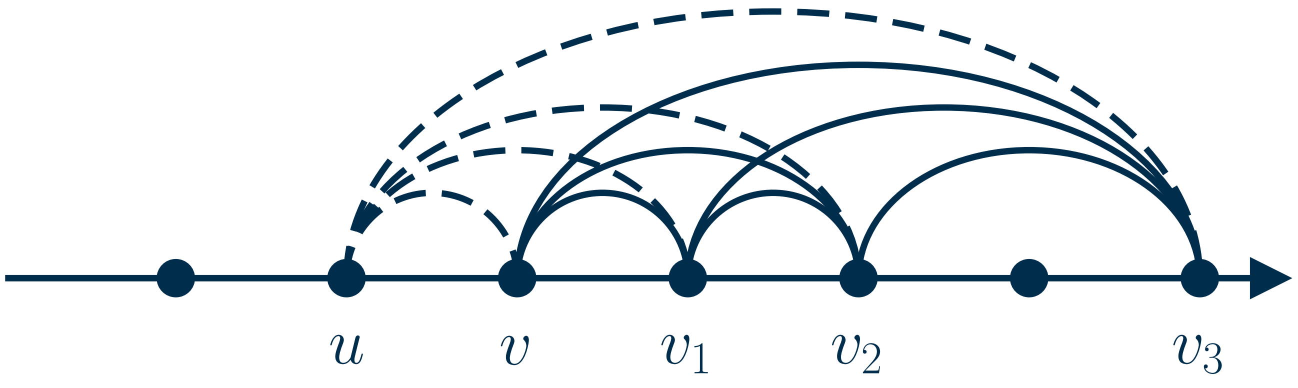
(3) Every triangulated graph that has a universal vertex is chordal.

- Look at induced $C_k (k \geq 4)$. It partitions the plane into two parts.
- No vertex on the cycle is universell (otherwise it would not be induced).
- There are vertices inside of C_k and outside of C_k because G is triangulated.



One vertex is universell and thus connected to the other part.

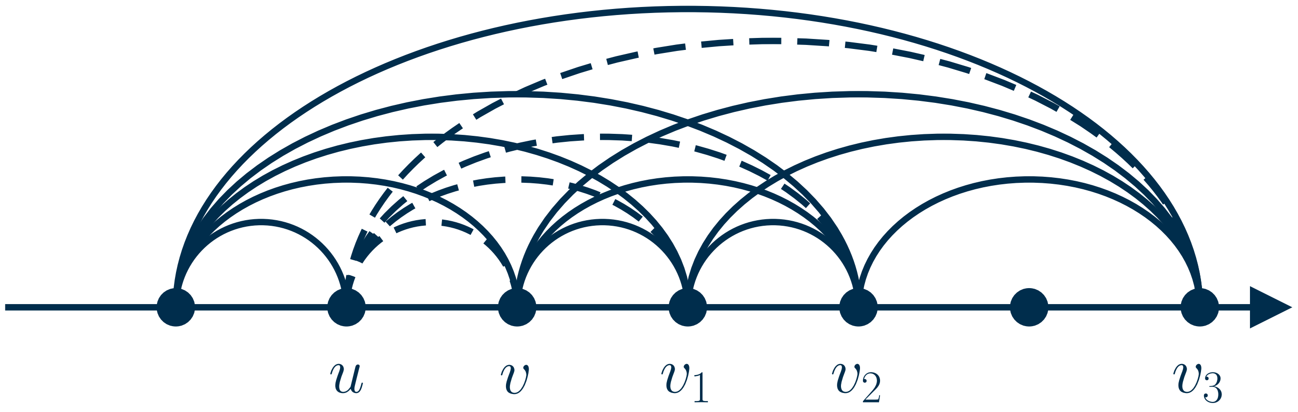
Exercise 3



Claim: K_v maximal $\Rightarrow \nexists$ predecessor u such that $K_v \subseteq K_u$

- If there is such a predecessor then obviously K_v is not maximal.

Exercise 3



Claim: K_v maximal $\Rightarrow \nexists$ predecessor u such that $K_v \subseteq K_u$

- If there is such a predecessor then obviously K_v is not maximal.

Claim: K_v maximal $\Leftarrow \nexists$ predecessor u such that $K_v \subseteq K_u$

- If K_v is not maximal then there is a clique C such that $K_v \subsetneq C$.
- Every $u \in C - K_v$ is left of v in σ. Thus, $K_v \subseteq K_u \subseteq C$.

Exercise 4

Show that a minimum vertex cover can be computed efficiently on chordal graphs.

Let $G = (V, E)$ be a chordal graph.

(i) Compute a maximum independent set. this takes $\mathcal{O}(n + m)$ time

(ii) Then, $V - I$ is a minimum vertex cover (exercise class 1, exercise 2) this takes $\mathcal{O}(n)$ time

Vertex cover can be solved in linear time on chordal graphs

Exercise 5

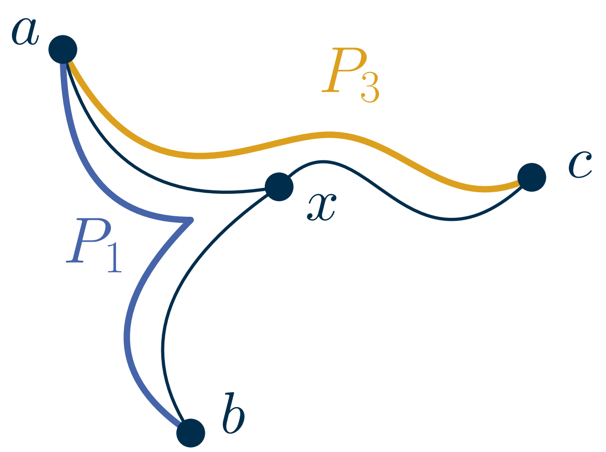
Lemma
Let T be a tree and let P_1, P_2, P_3 be paths in T such that

(i) $P_1 = (a, \ldots, b)$,
(ii) $P_2 = (b, \ldots, c)$,
(iii) $P_3 = (a, \ldots, c)$.

Then, there is a vertex $x \in P_1 \cap P_2 \cap P_3$.

Proof:

- Let x be the last vertex on P_3 that is on P_1.
- Then, $b \to P_1 \to x \to P_3 \to c$ is a path from b to c.
- Therefore, $b \to P_1 \to x \to P_3 \to c = P_2$.



Exercise 5

G is tree $\implies$ every family of paths fulfills the Helly property

Let $\{P_j \subseteq G \colon j \in J\}$ be paths with $P_i \cap P_j \neq \emptyset$ for all i, j.

Goal: $\bigcap_{j \in J} P_j \neq \emptyset$.

We do induction on $|J|$.

Base case: $|J| = 2$ ✓

- Let $|J| \geq 3$ and fix $j_1, j_2 \in J$.
- By induction there are $a \in \bigcap_{j \in J - j_1} P_j$, $b \in \bigcap_{j \in J - j_2} P_j$ and $c \in P_{j_1} \cap P_{j_2}$.

Let $P' = \bigcap_{j \in J - j_1 - j_2} P_j$. Then:

(i) P_{j_1} contains a path from b to c,
(ii) P_{j_2} contains a path from a to c and
(iii) P' contains a path from a to b.

- By the lemma there is $x \in P_{j_1} \cap P_{j_2} \cap P' \implies x \in \bigcap_{j \in J} P_j$.

Exercise 5

of subtrees

G is tree $\implies$ every family of ~~paths~~ fulfills the Helly property

Let $\{P_j \subseteq G \colon j \in J\}$ be ~~paths~~ with $P_i \cap P_j \neq \emptyset$ for all i, j.

subtrees

Goal: $\bigcap_{j \in J} P_j \neq \emptyset$.

We do induction on $|J|$.

Base case: $|J| = 2$ ✓

- Let $|J| \geq 3$ and fix $j_1, j_2 \in J$.
- By induction there are $a \in \bigcap_{j \in J - j_1} P_j$, $b \in \bigcap_{j \in J - j_2} P_j$ and $c \in P_{j_1} \cap P_{j_2}$.

Let $P' = \bigcap_{j \in J - j_1 - j_2} P_j$. Then:

(i) P_{j_1} contains a path from b to c,

(ii) P_{j_2} contains a path from a to c and

(iii) P' contains a path from a to b.

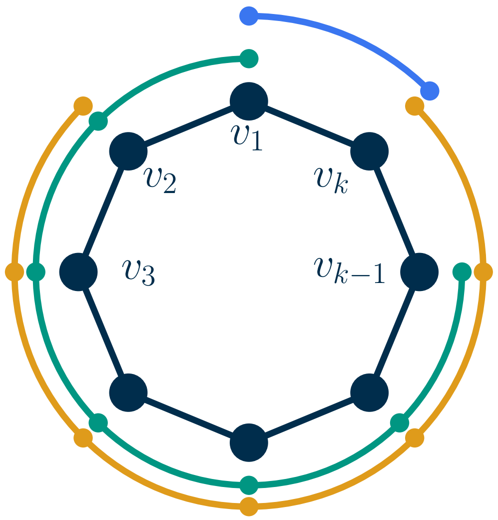
- By the lemma there is $x \in P_{j_1} \cap P_{j_2} \cap P' \implies x \in \bigcap_{j \in J} P_j$.

Exercise 5

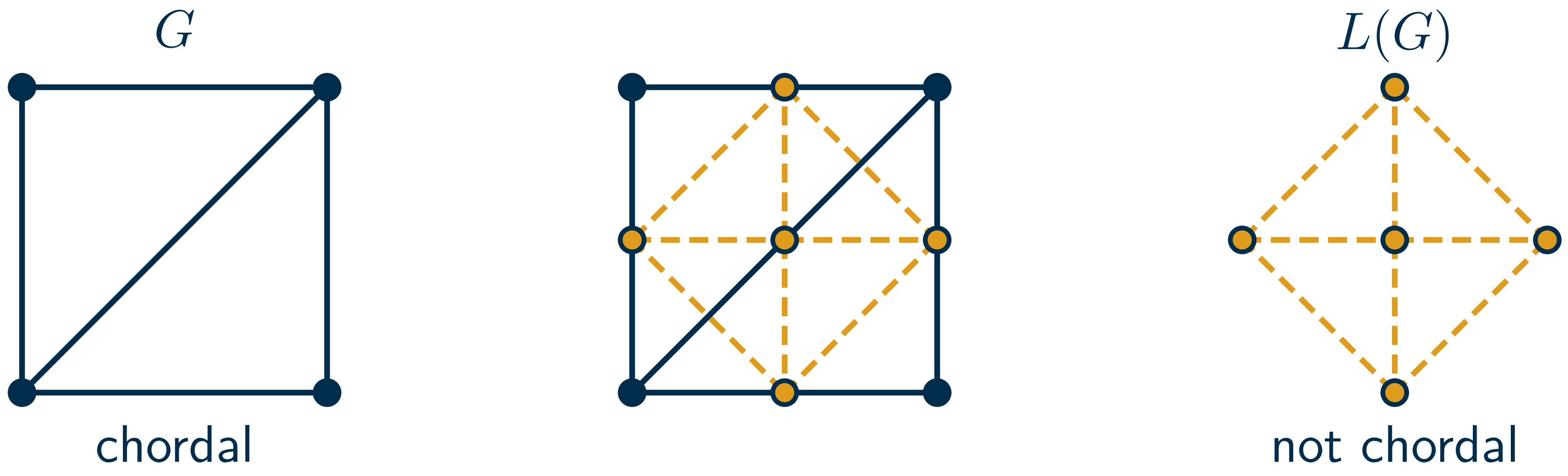
G is tree $\Longleftarrow$ every family of paths fulfills the Helly property

Contraposition: Let $(v_1, \ldots, v_k)$ be a cycle in G.

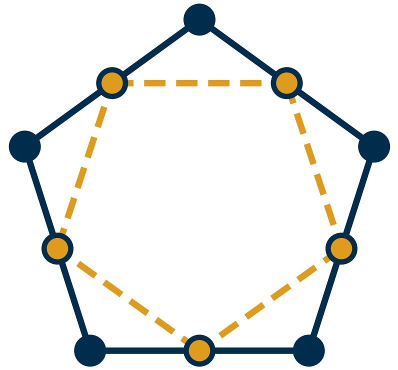
- Choose:
  - $P_1 = (v_1, v_2, \ldots, v_{k-1})$
  - $P_2 = (v_2, v_3, \ldots, v_k)$
  - $P_3 = (v_k, v_1)$
- $\{P_1, P_2, P_3\}$ does not fulfill the Helly property.



Exercise 6



$L(G)$ chordal $\Rightarrow G$ chordal



- Let $C = (v_1, \ldots v_k)$ an induced cycle in G.
- Then, $\{v_1v_2, v_2v_3, \ldots, v_kv_1\}$ induces a cycle of length k in $L(G)$.
- $L(G)$ chordal $\implies k = 3$.
- Thus, G has only induced cycles of length at most 3.

Exercise 7

Prove that a graph has treewidth at least three if and only if it contains K_4 as a topological minor.

Refer to: `https://doi.org/10.1016/0012-365X(90)90292-P` (the example on page 4)

Forbidden minors characterization of partial 3-trees [Arnborg, Proskurowski and Corneil 1990]

Example. We will show that the complete graph of 4 vertices, K_4, is the only forbidden minor of partial 2-trees. Partial 2-trees are easily recognizable by reducing a graph to an edge by application of the following "rewriting rules" (cf. Fig. 2(a)): remove vertices of degree 0 or 1, and contract 2-paths ("series reduction": replace by a single edge two edges incident with a common degree 2 vertex). Applications of these rewriting rules create minors of the original graph.

By absence of vertices of degree 2 or less (which would lead to a smaller minor through a rewriting rule), a minimal minor is cubic, since deletion of any edge must create two vertices of degree 2 or less (every 2-tree has at least two 2-leaves, which are present in partial 2-trees as vertices of degree at most 2). To create two vertices of degree 2 by contraction of any edge, every edge must be in at least two triangles: take such an edge (x, y) and consider two common neighbors of x and y, u and v. Since (x, u) must be in another triangle and x has already three neighbors, the third edge incident to u must lead to v giving a K_4.

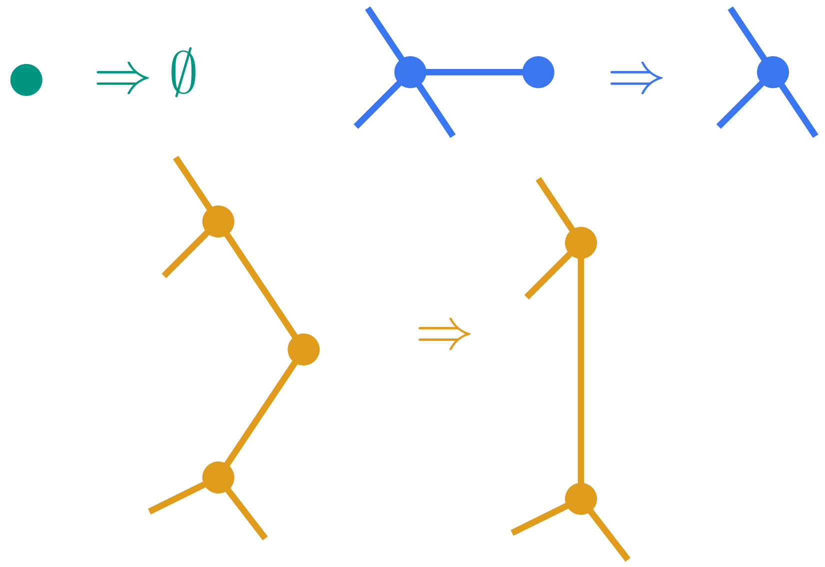


Figure 2(a): Rewriting rules for recognition of partial 2-trees