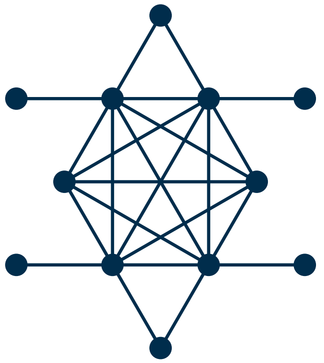
Algorithmic Graph Theory

Solution Sheet 3

Laura Merker and Samuel Schneider, June 4, 2025

Exercise Sheet 3

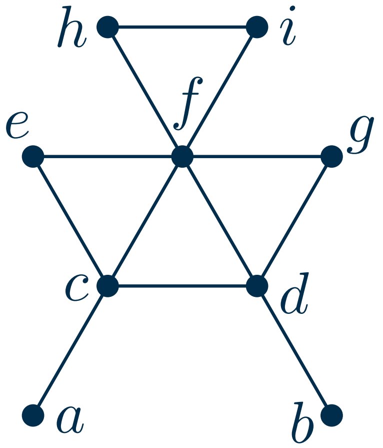


(1) List all simplicial vertices in the graph on the right.

(2) Let S be a minimal vertex separator in a chordal graph $G = (V, E)$.
Prove that every component of G_{V-S} contains a vertex c such that $S \subseteq N(c)$.

(3) Let G be an interval graph. Give two different proofs for the chordality of G by proving the following statements:

- **(a)** G has a perfect elimination scheme.
- **(b)** Every minimal vertex separator of G is a clique.



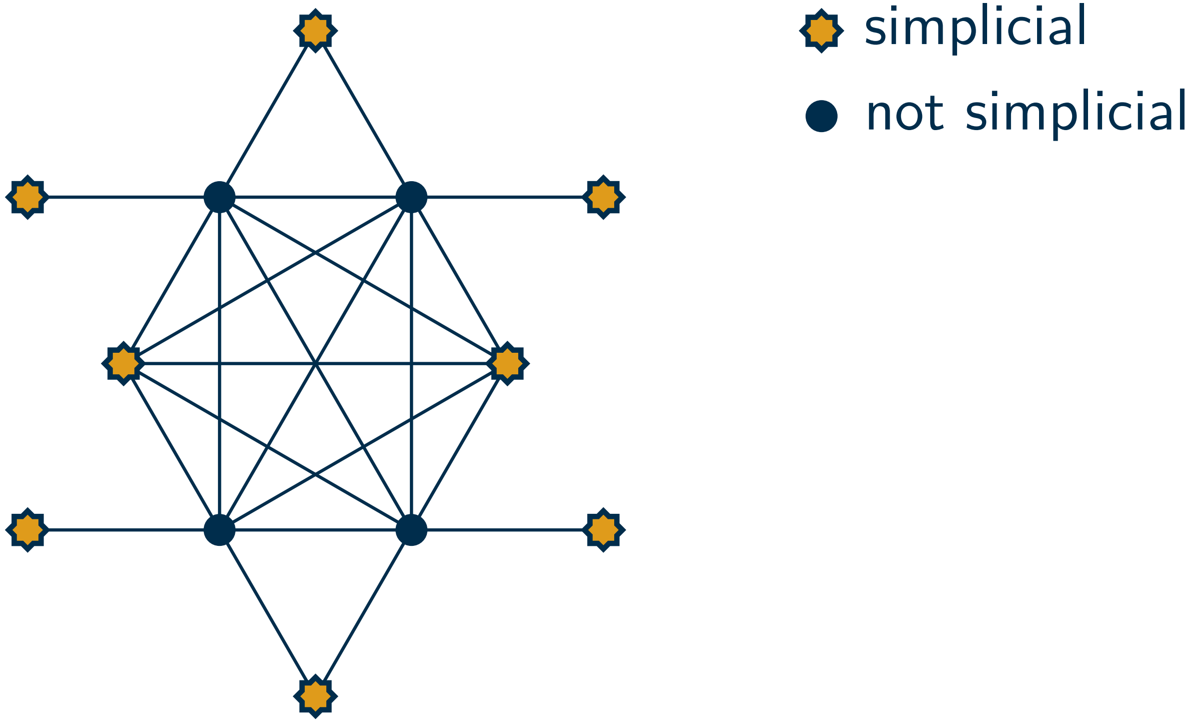
(4) Run a lexicographic BFS on the graph on the right.

(5) Give a graph with a perfect elimination scheme σ such that σ cannot be computed using a lexicographic BFS.

(6) Prove that k-trees are chordal. Give a chordal graph that is not a k-tree.

Exercise 1

List all simplicial vertices in the given graph.

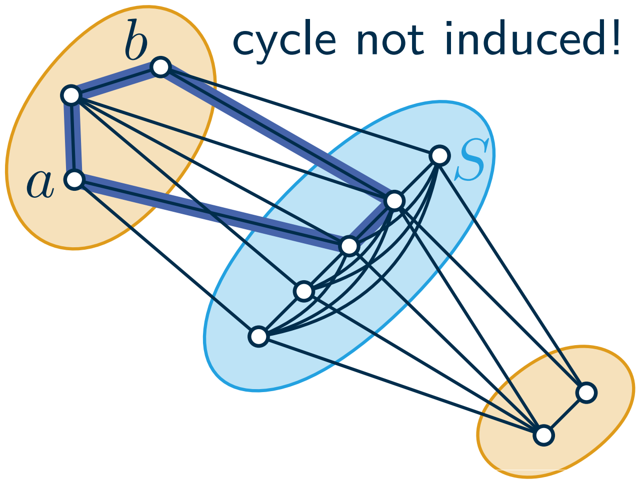


Exercise 2

Let S be a minimal vertex separator in a chordal graph $G = (V, E)$.
Prove that every component of G_{V-S} contains a vertex c such that $S \subseteq N(c)$.

Recall (from lecture):

Let S be a minimal vertex separator in a chordal graph $G = (V, E)$.
Then S is a clique.



New: certified by a single vertex

Arguments used in the proof:

- every component of G_{V-S} has an edge to **every** vertex of S
- take shortest path between different neighbors a, b of S
- find induced cycle of length ≥ 4

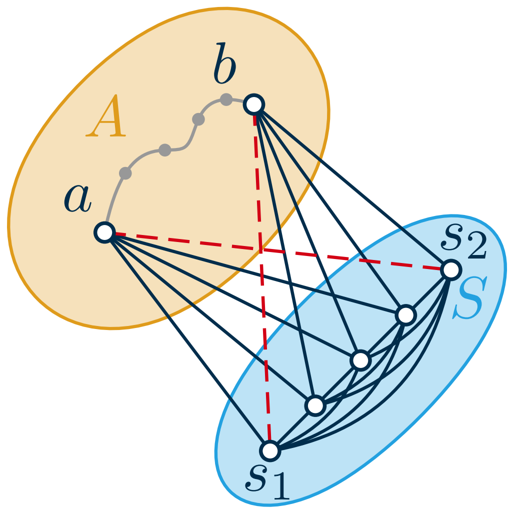
choose a, b carefully!

Does the same argument work again? $\rightarrow$ **No!**

Exercise 2

Let S be a minimal vertex separator in a chordal graph $G = (V, E)$.
Prove that every component of G_{V-S} contains a vertex c such that $S \subseteq N(c)$.

- Let A component of G_{V-S} and $a \in A$ maximizing $N_S(a)$
- Every vertex in S is connected to some vertex in A. Assume not all to the same.
- Let $P := (a, \ldots, b)$ min. path in G_A s.t. $N_S(a)$ and $N_S(b)$ uncomparable i.e. there are $s_1 \in N_S(a) \setminus N_S(b)$ and $s_2 \in N_S(b) \setminus N_S(a)$



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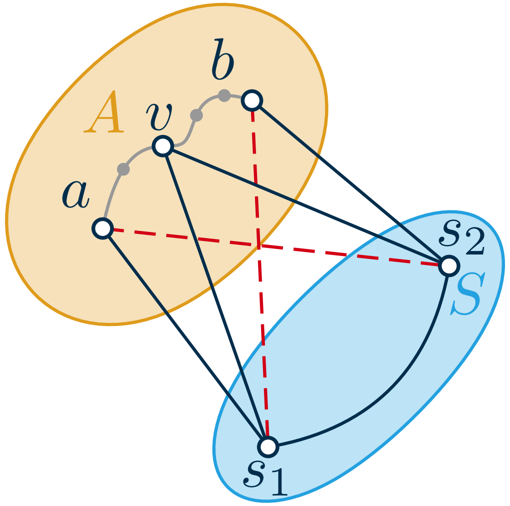

Claim: No vertex of P is adjacent to s_1 and s_2

- Assume there is such a vertex $v \in V(P)$.
- By minimality of P the sets $N_S(a)$ and $N_S(v)$ are comparable.
- This contradicts the maximality of $N_S(a)$. ⚡

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Prove that every component of G_{V-S} contains a vertex c such that $S \subseteq N(c)$.

- Let A component of G_{V-S} and $a \in A$ maximizing $N_S(a)$
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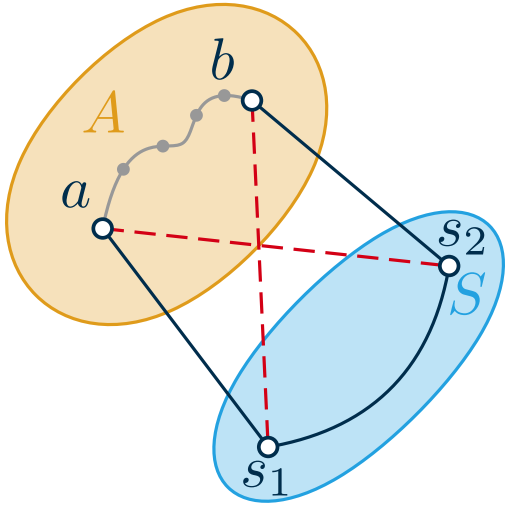

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- There is subpath $P' := (a', \ldots, b')$ of P s.t. only a' is adjacent to s_1 and only b' is adjacent to s_2
$\Rightarrow$ there is an induced cycle of length at least 4.
- For all $w \in A$ we have that $N_S(w)$ and $N_S(a)$ are comparable $\Rightarrow s_2 \notin N(A)$ ↯

Exercise 3

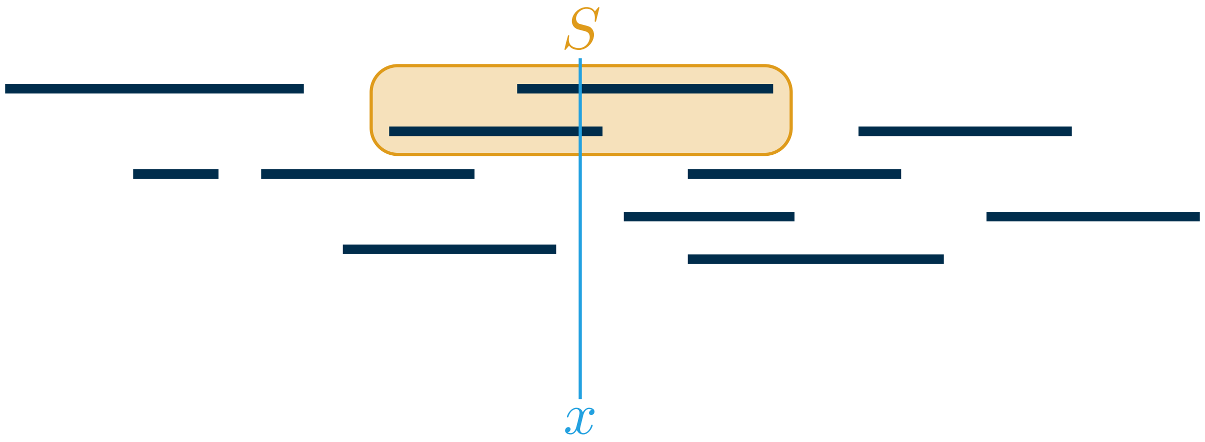
Prove that every interval graph has a perfect elemination scheme.

- We construct a PES using induction.
- Let G be the interval graph corresponding to the intervalls $I_v = [a_v, b_v]$ with $v \in V(G)$.
- Let $v_1 \in V(G)$ such that b_{v_1} is minimal under all right endpoints.
- For all $u \in N(v_1)$ it holds that $a_u \leq b_{v_1}$ und $b_{v_1} \leq b_u \Rightarrow b_{v_1} \in \bigcap_{u \in N_G(v_1)} I_u$
  $\Rightarrow N_G(v_1)$ is a clique $\Rightarrow v_1$ is simplicial in G.
- Call induction on $G - v_1$ and prepend v_1 to the resulting PES.

Exercise 3

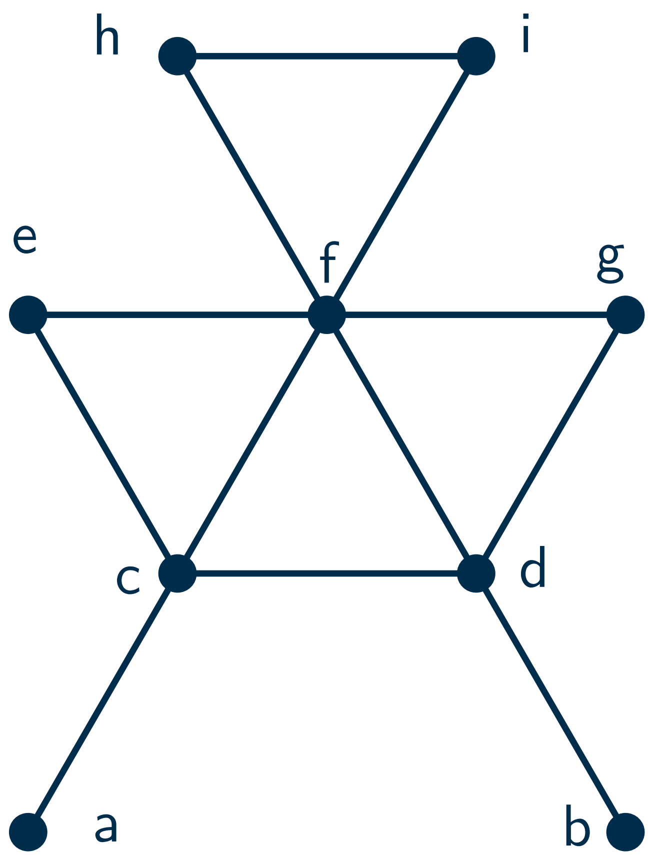
Prove that every minimal vertex separator of an interval graph is a clique.

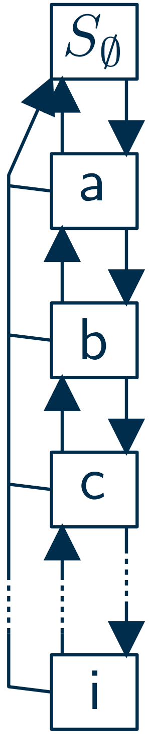
- Let G be an interval graph and let S be a minimal vertex separator



- Then there is a position x only intersecting intervals in S.
  - Otherwise $G - S$ is connected.
- Clearly the set S' of intervals intersecting x is a separator and a clique.
- As S is minimal and $S' \subseteq S$ we have $S' = S$.

Exercise 4





$\sigma = [\qquad\qquad\qquad\qquad]$

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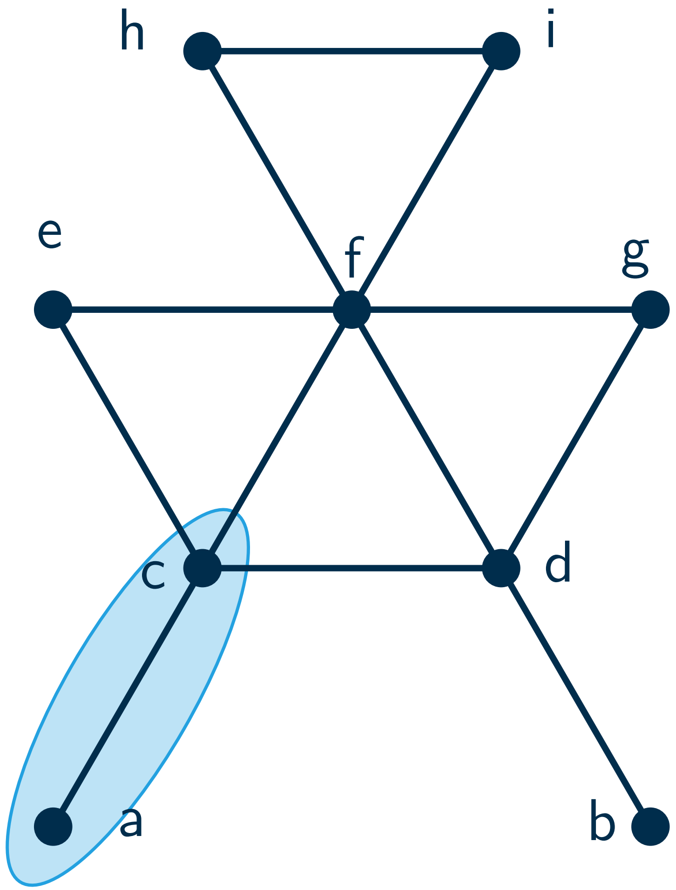

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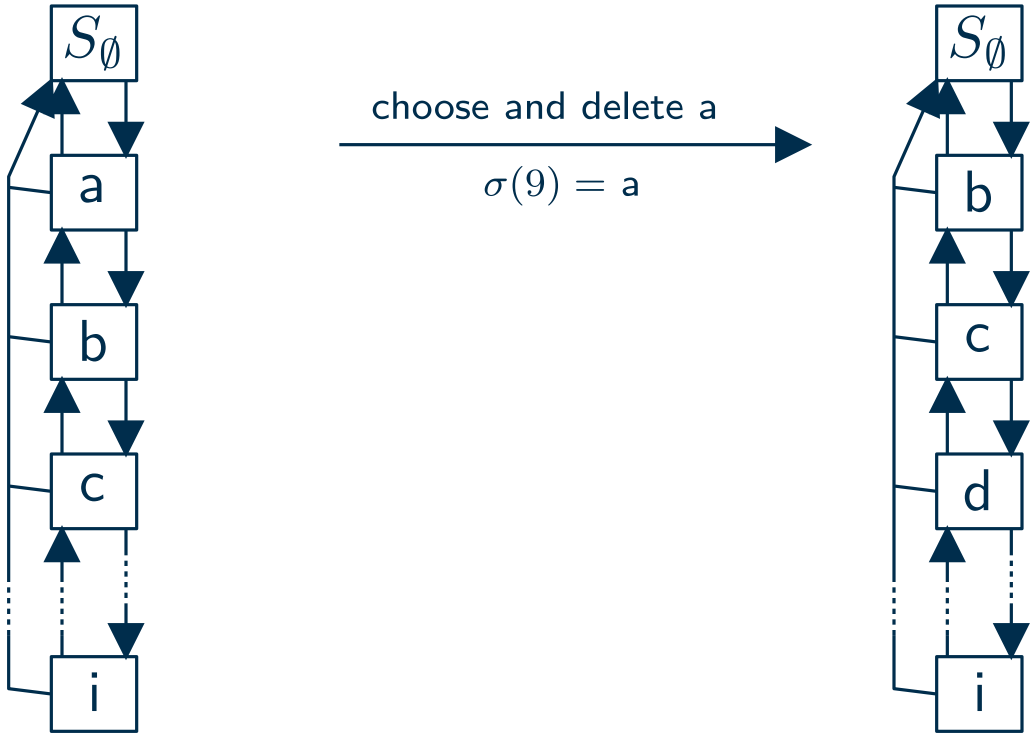

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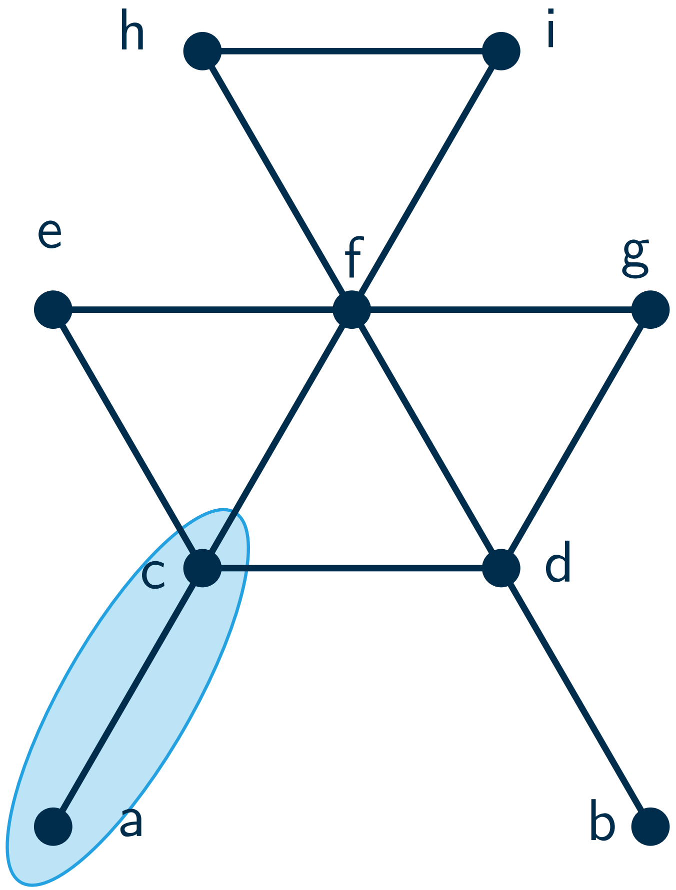

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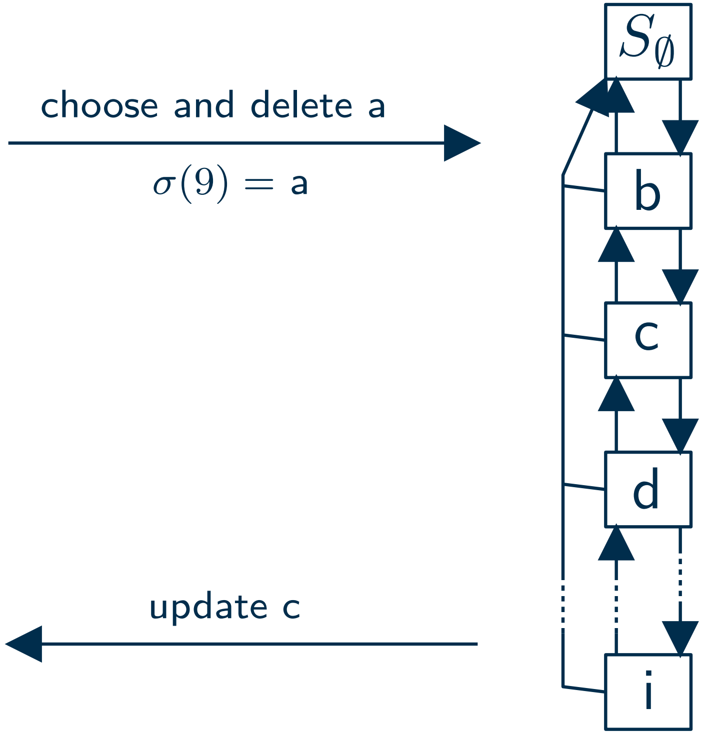

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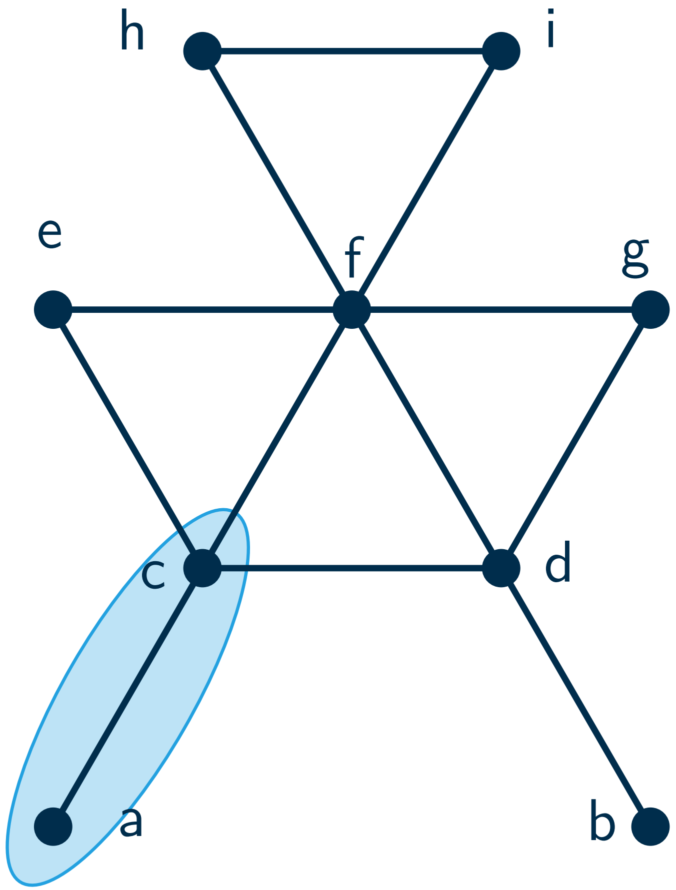

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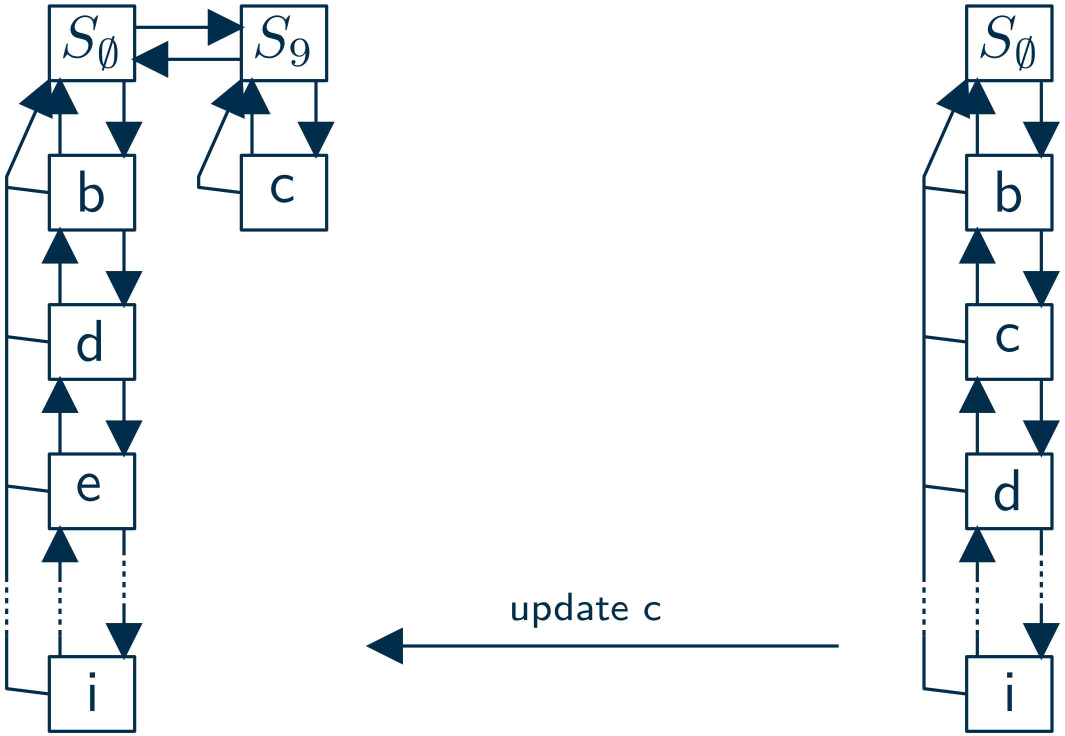

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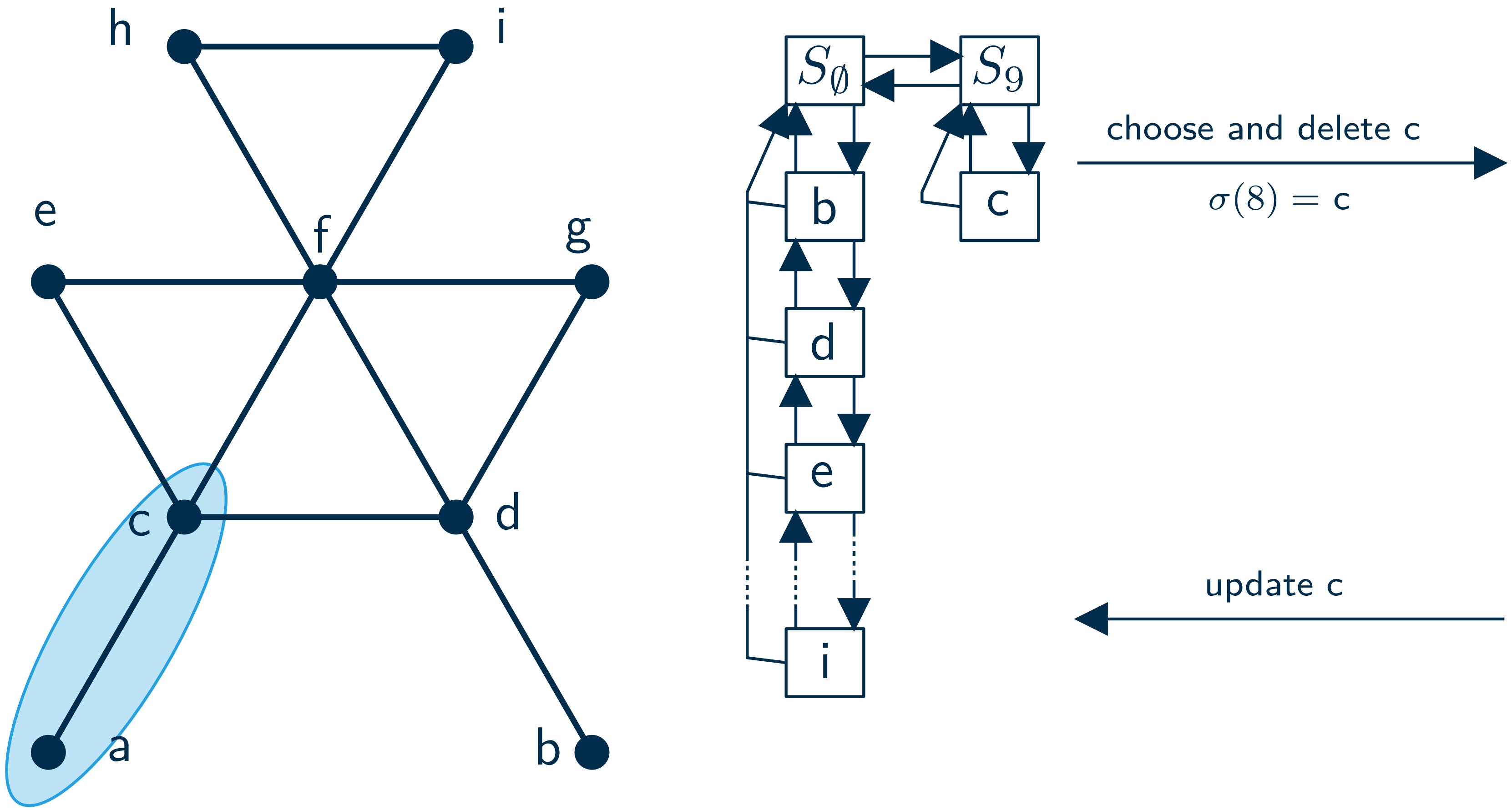

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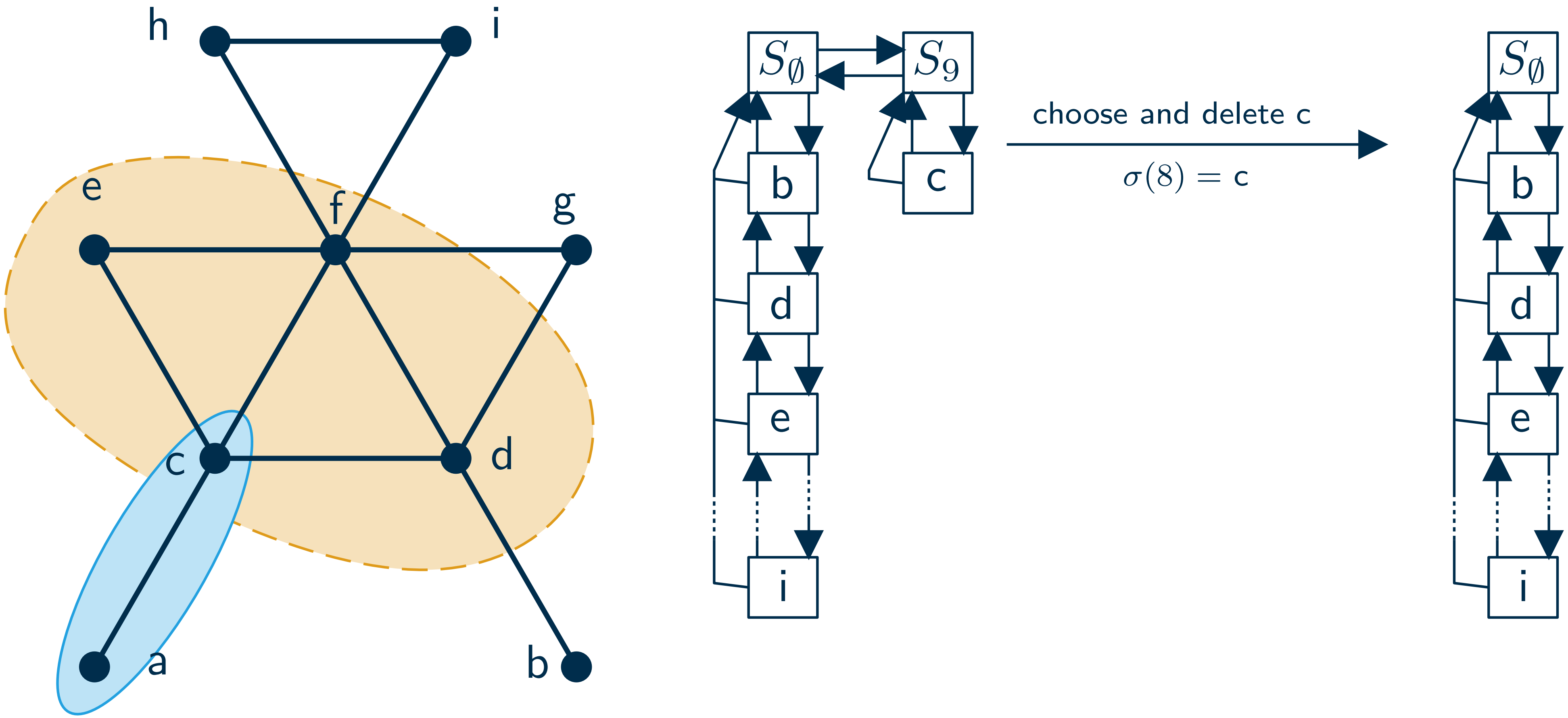

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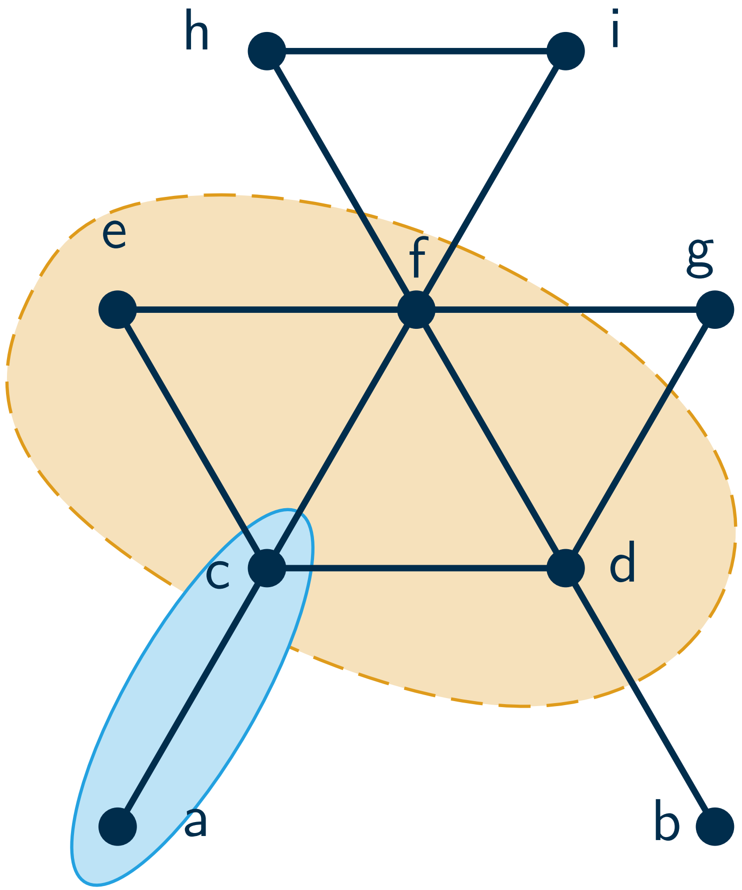
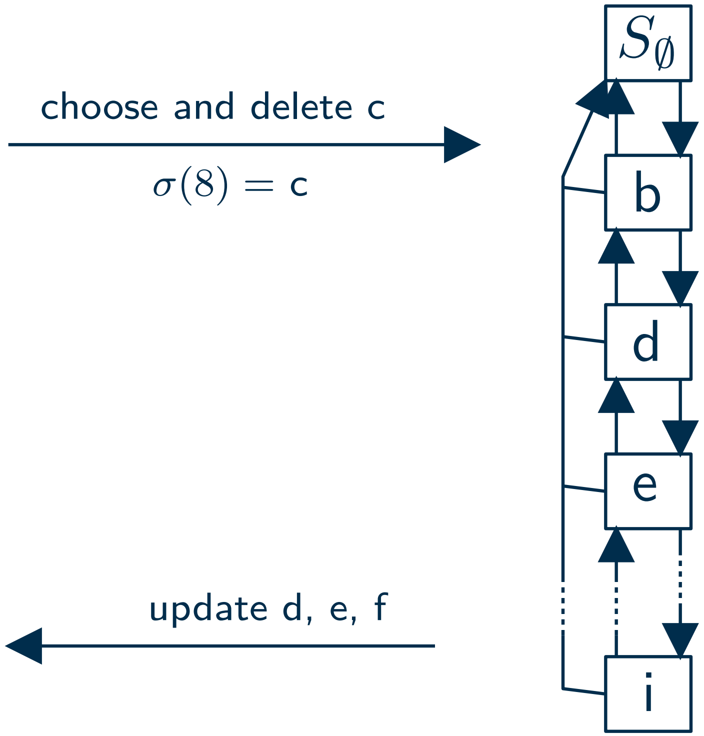


$\sigma = [$ c, a $]$



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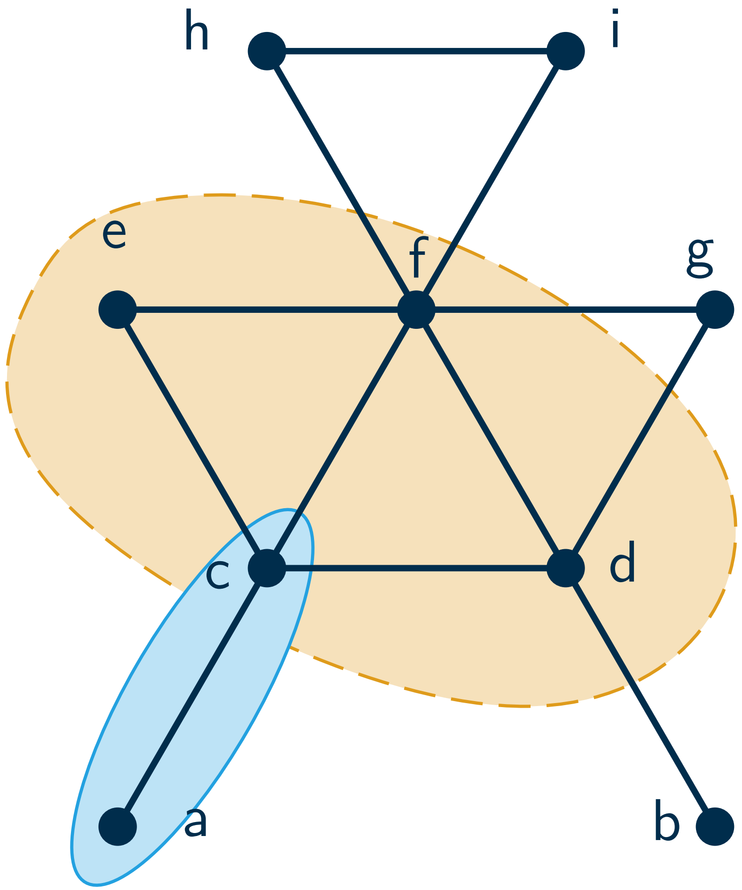

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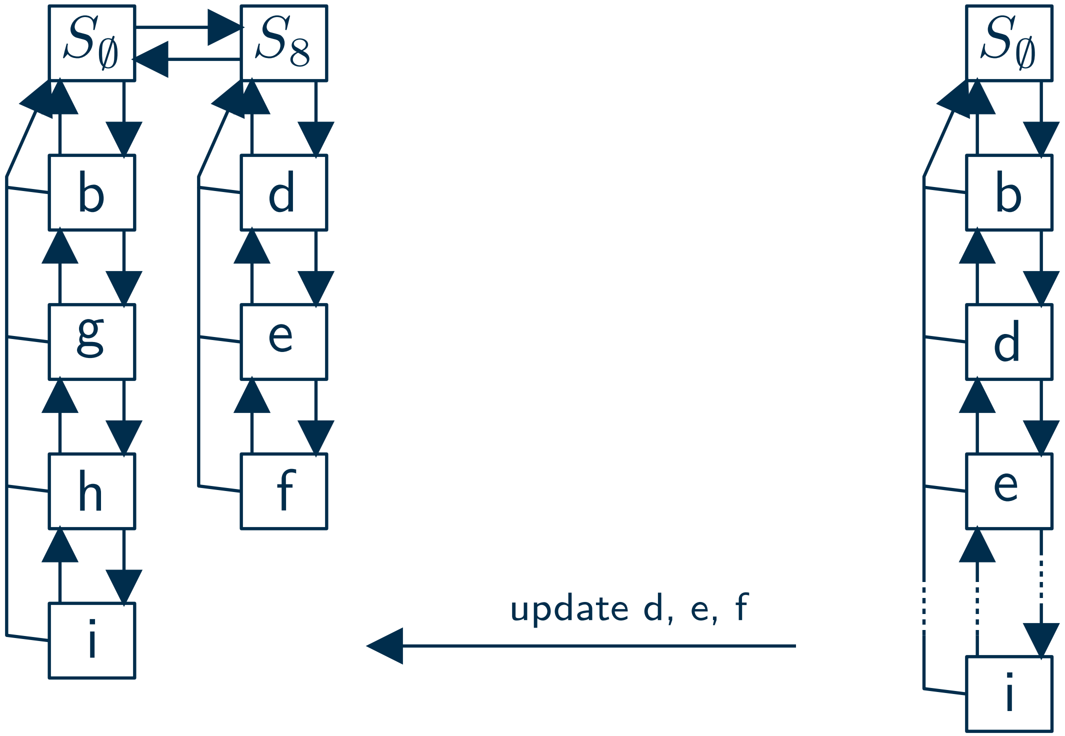

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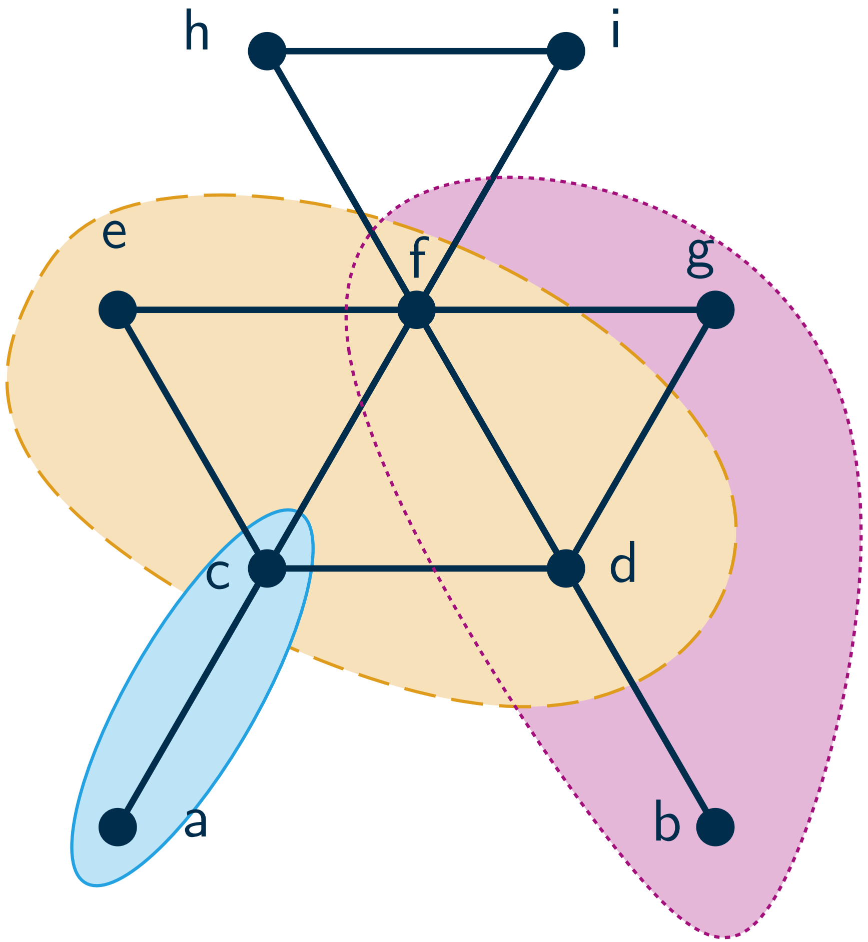

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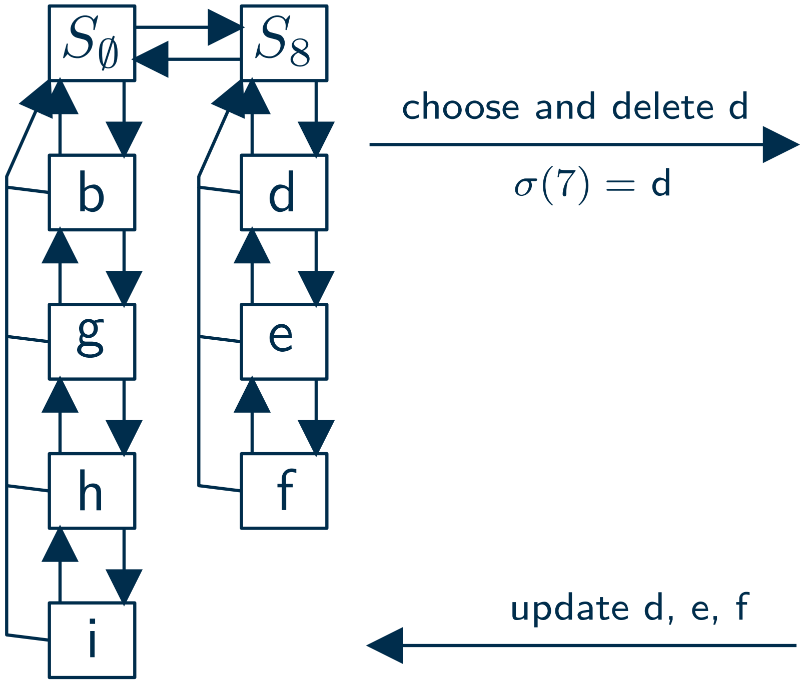

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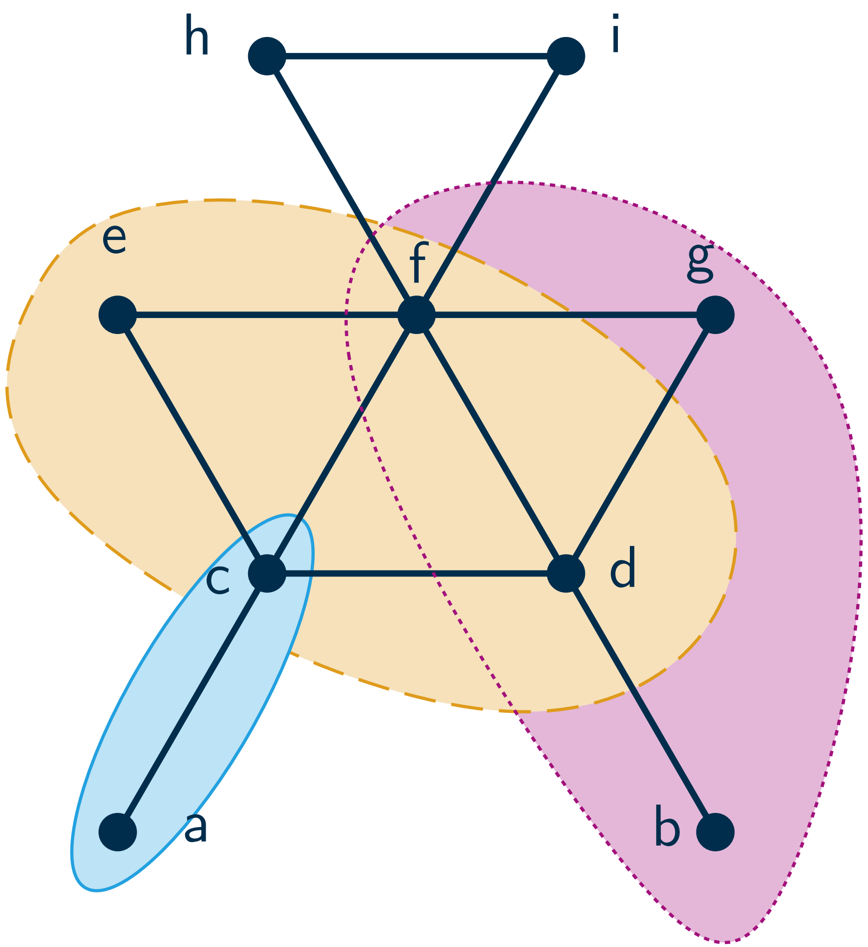

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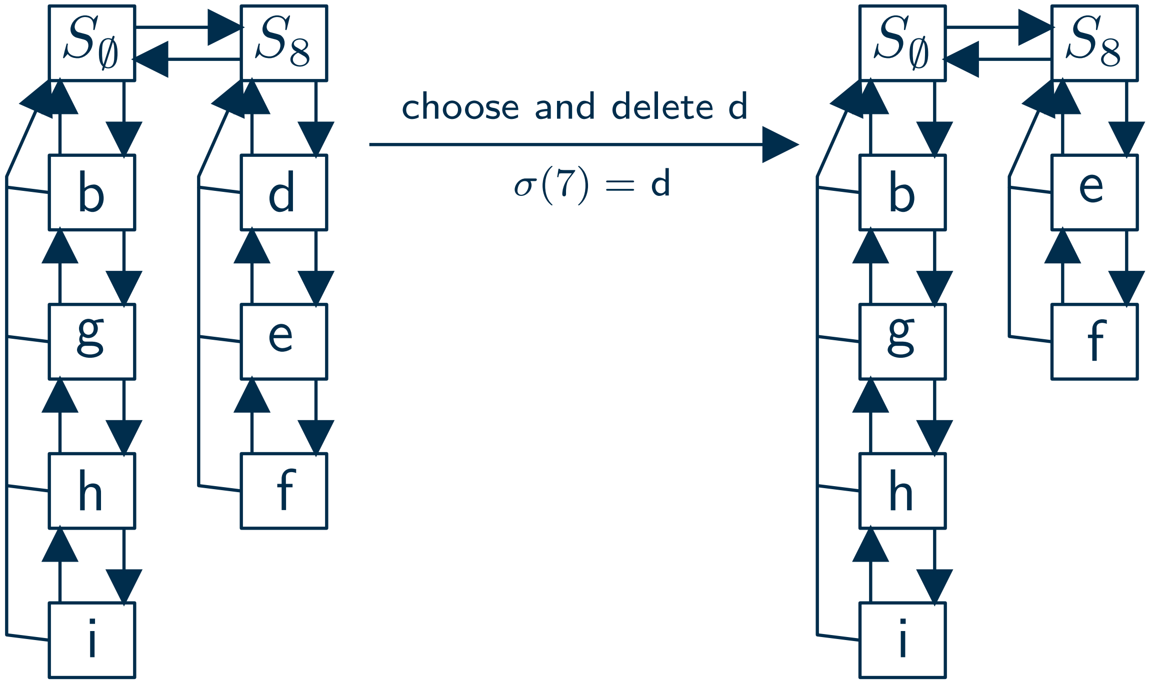

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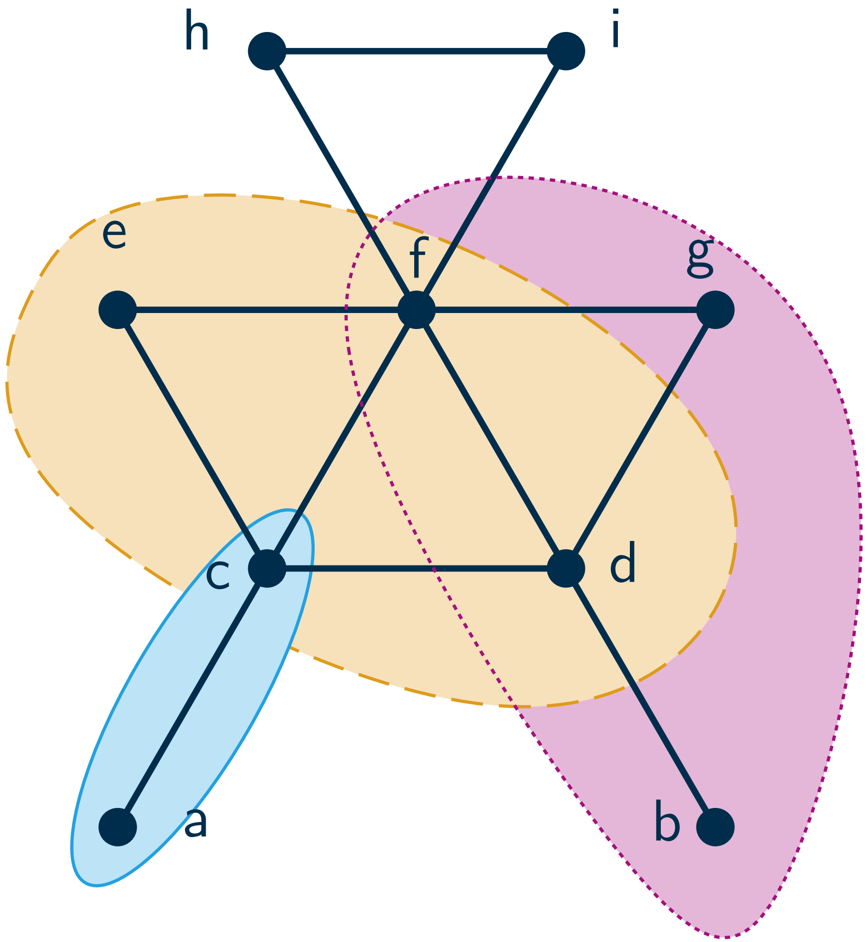

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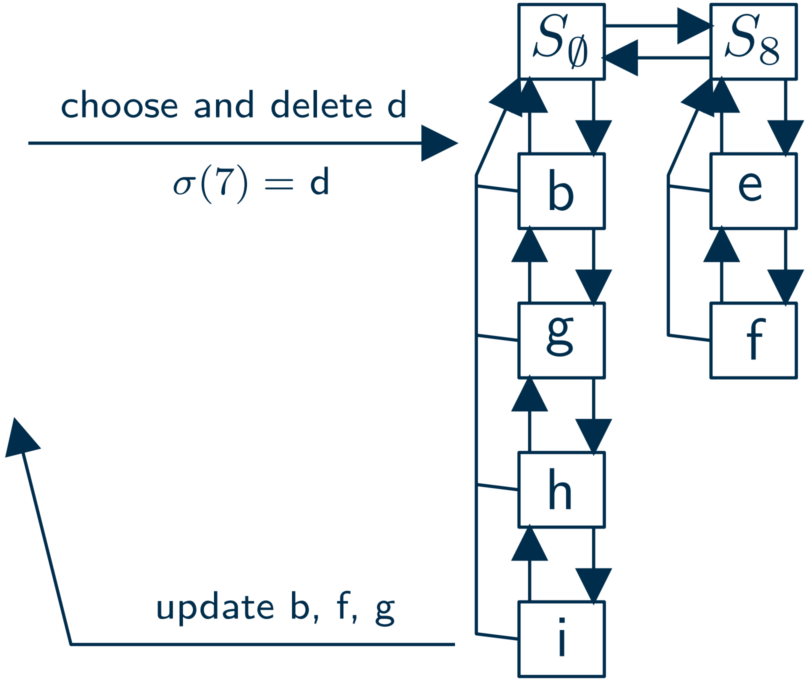

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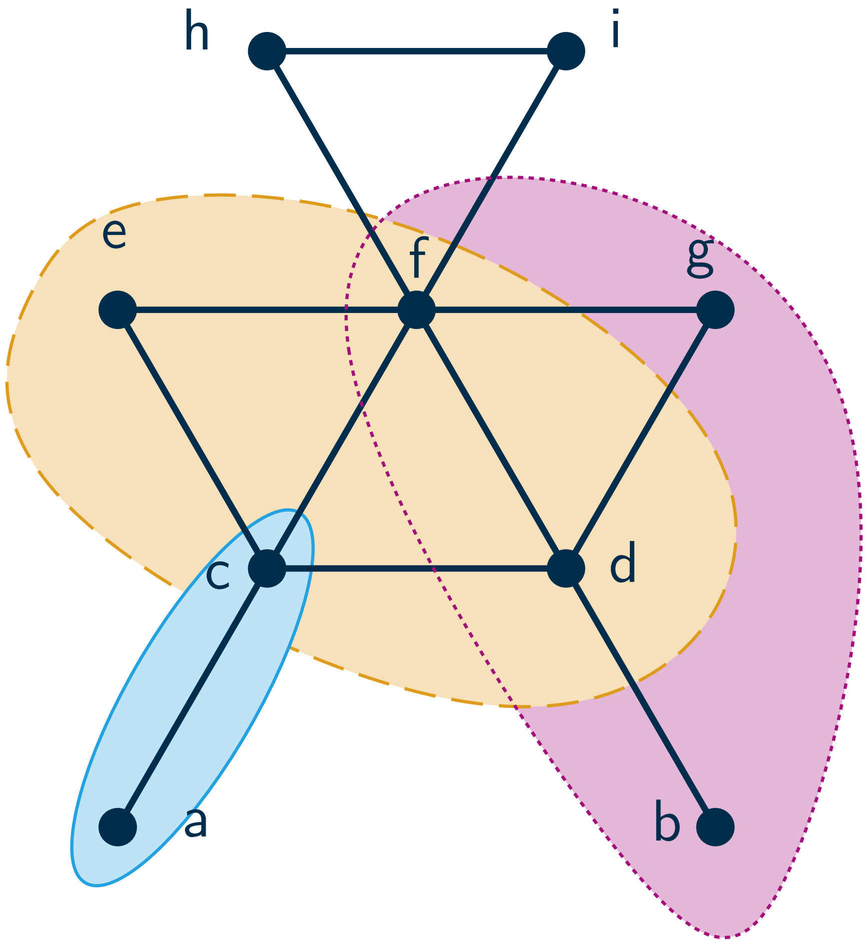

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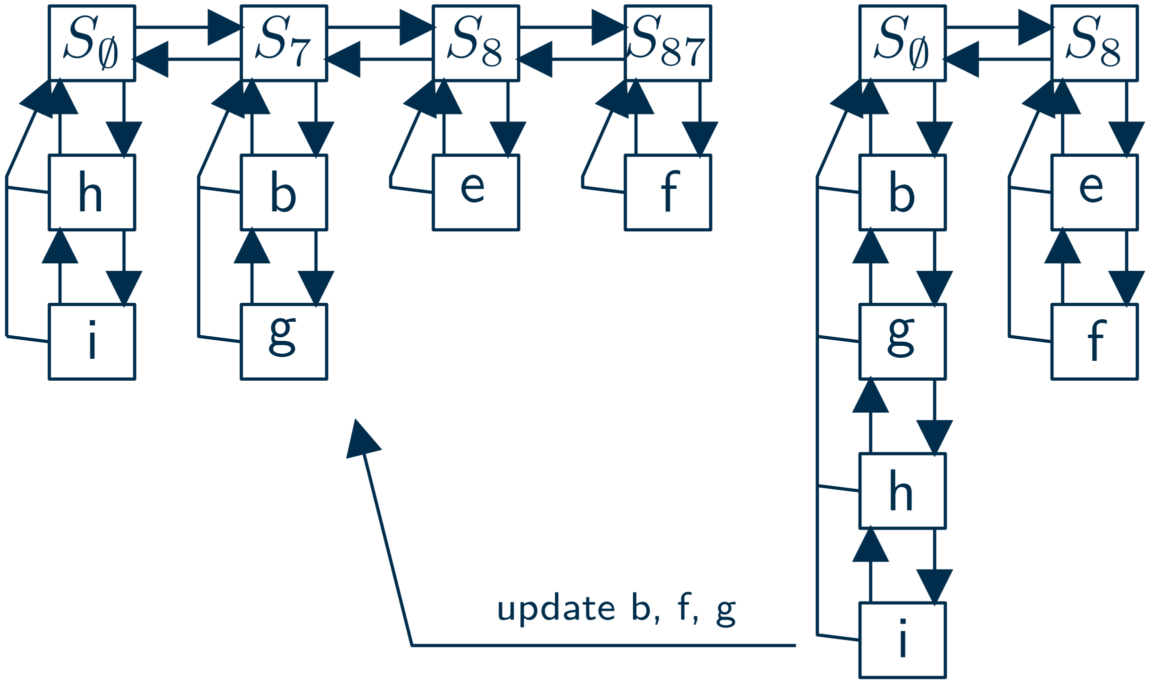

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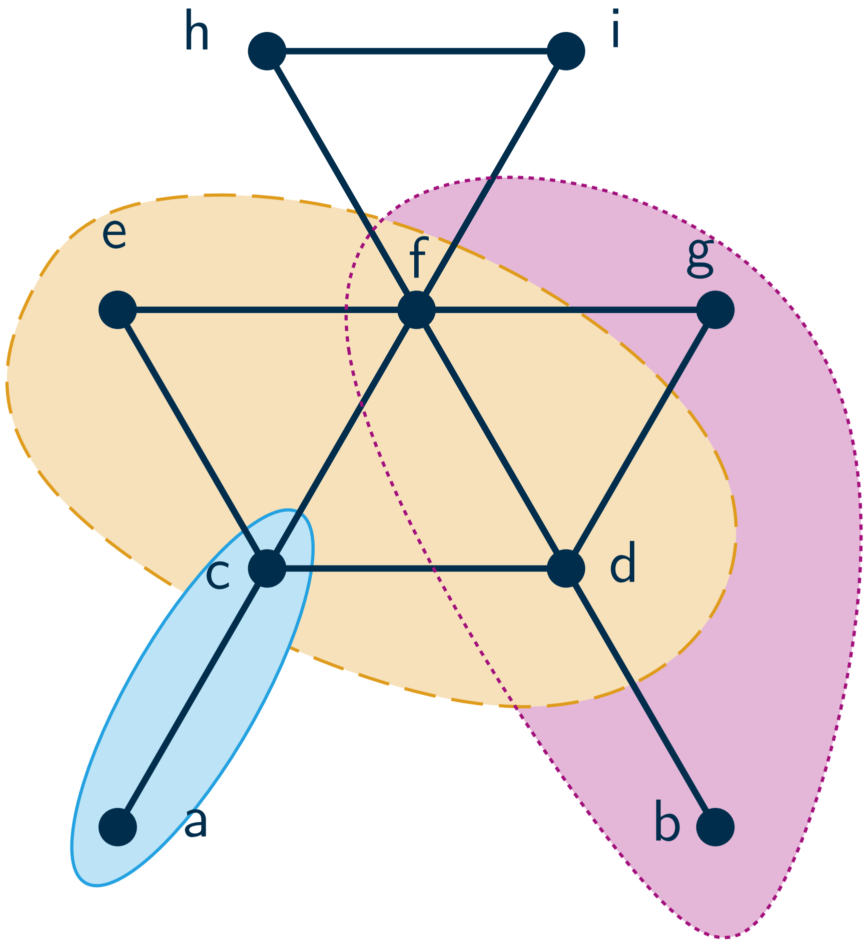

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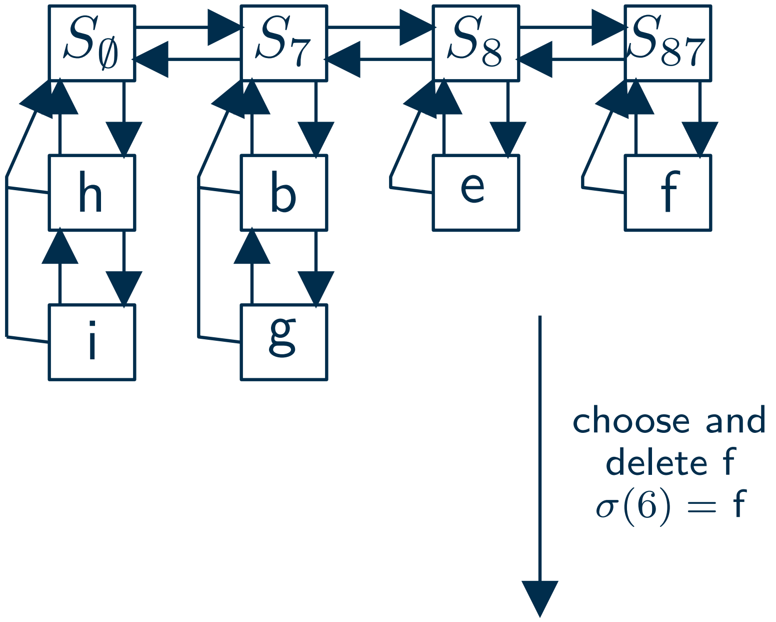

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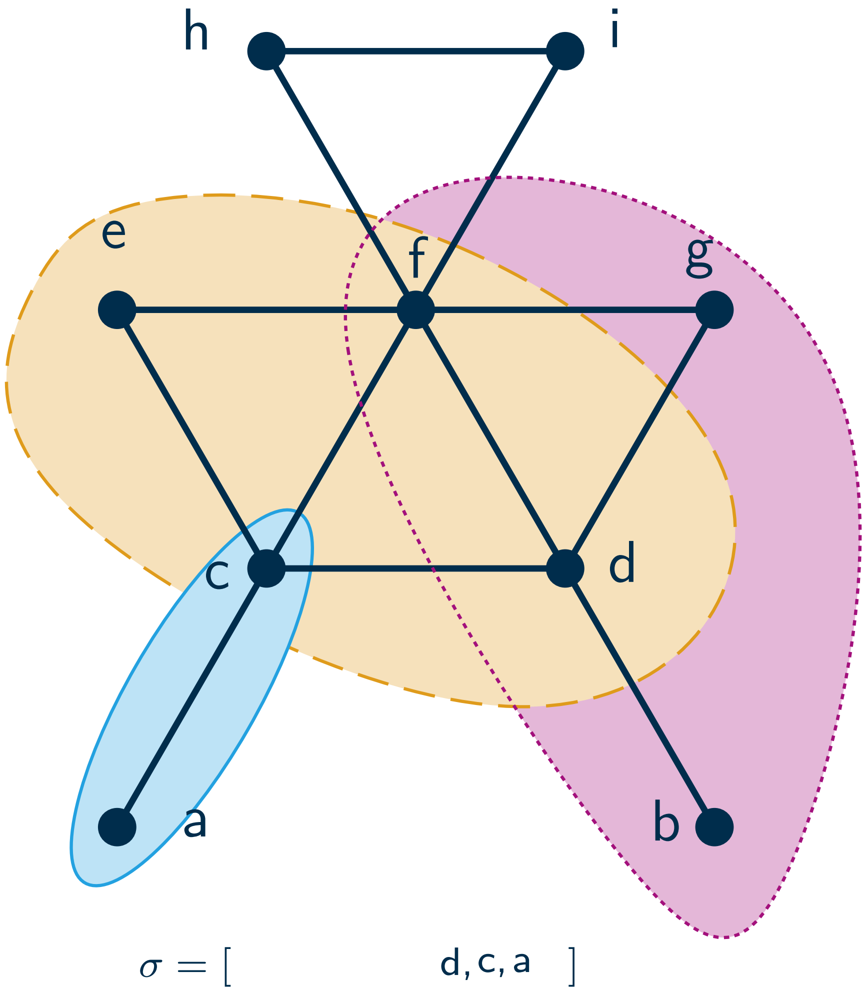
h
i
e
f
g
c
d
a
b
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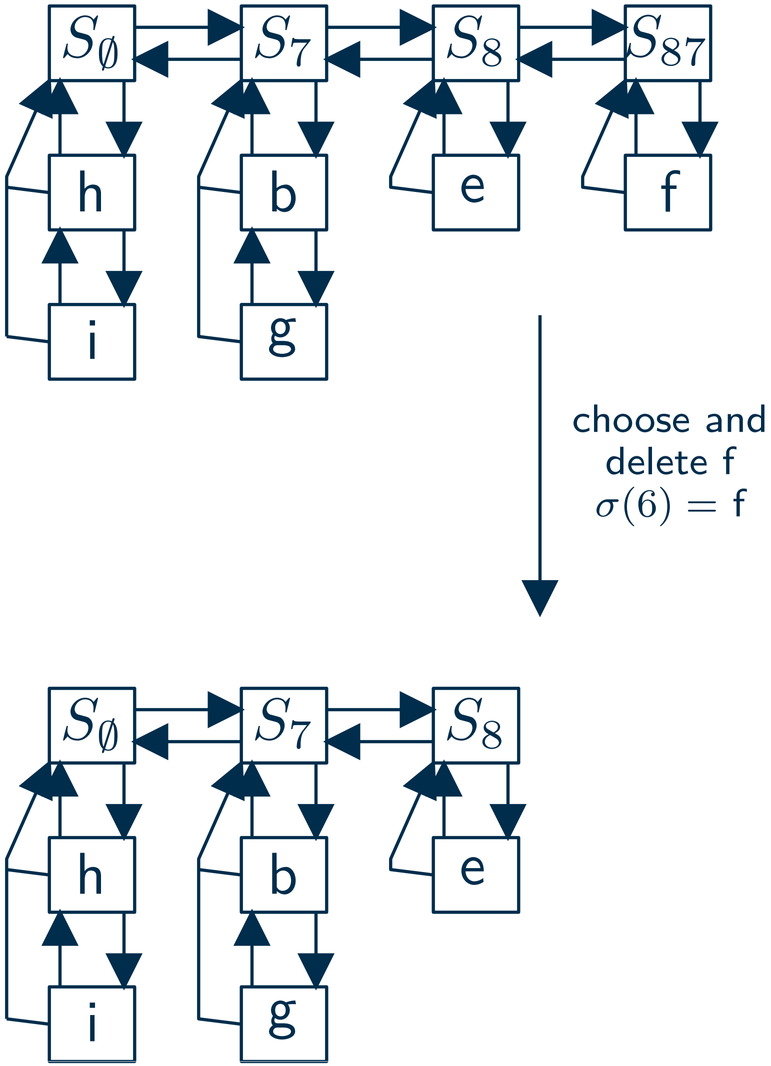
$S_\emptyset$
S_7
S_8
S_{87}
h
b
e
f
i
g
choose and delete f
$\sigma(6) = f$
$S_\emptyset$
S_7
S_8
h
b
e
i
g

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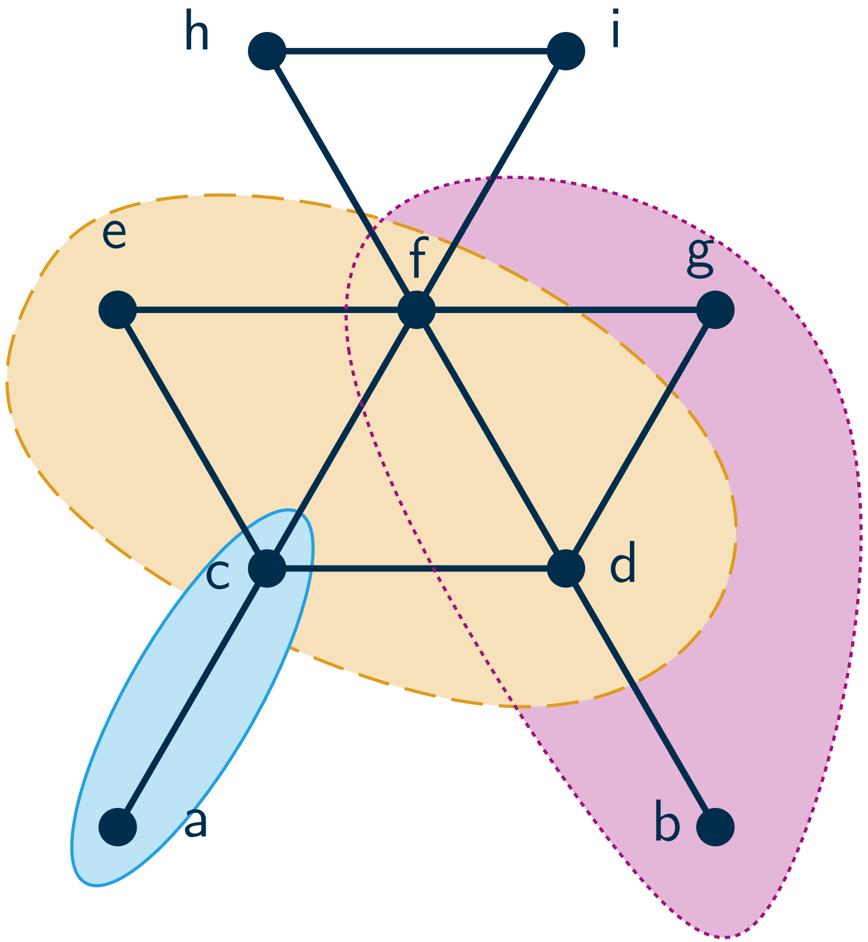
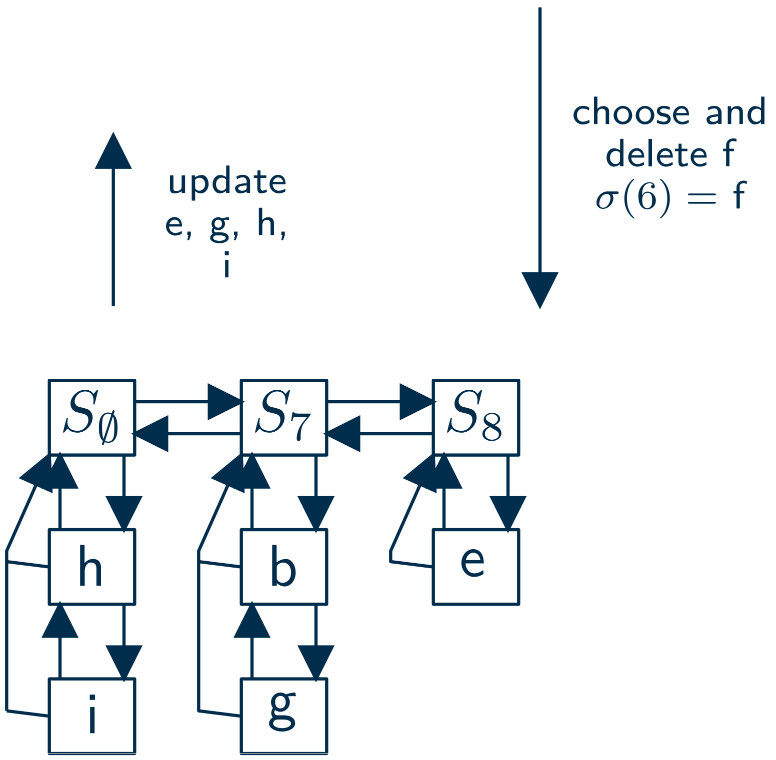


$\sigma = [$ f,d, c, a $]$



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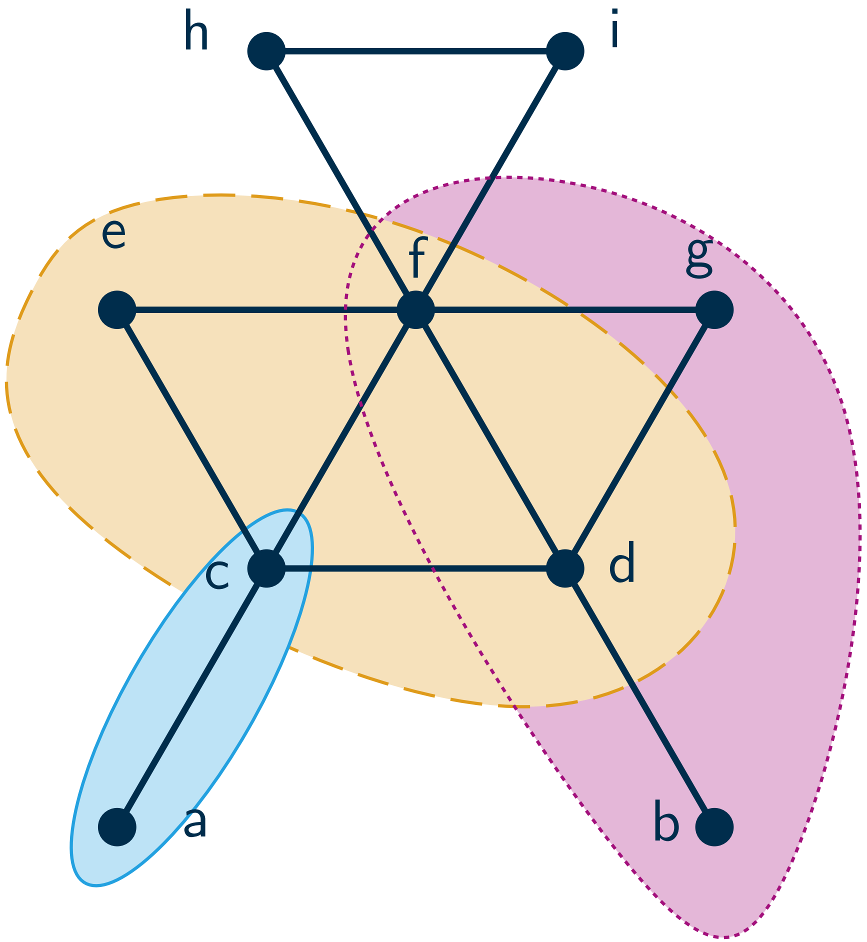

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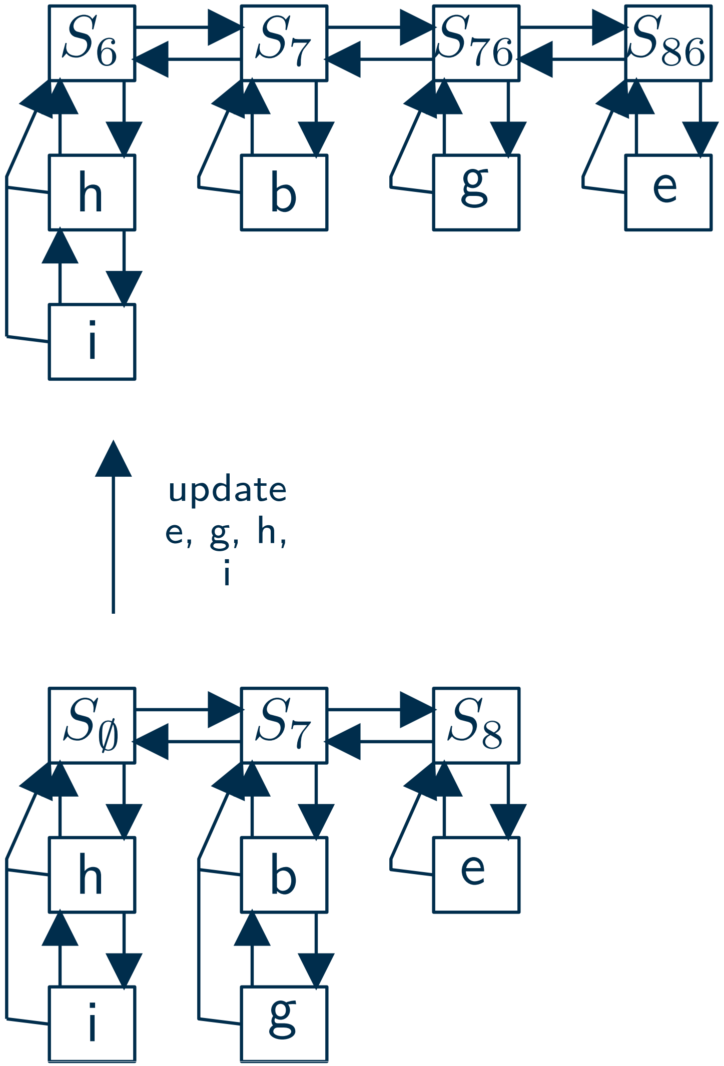

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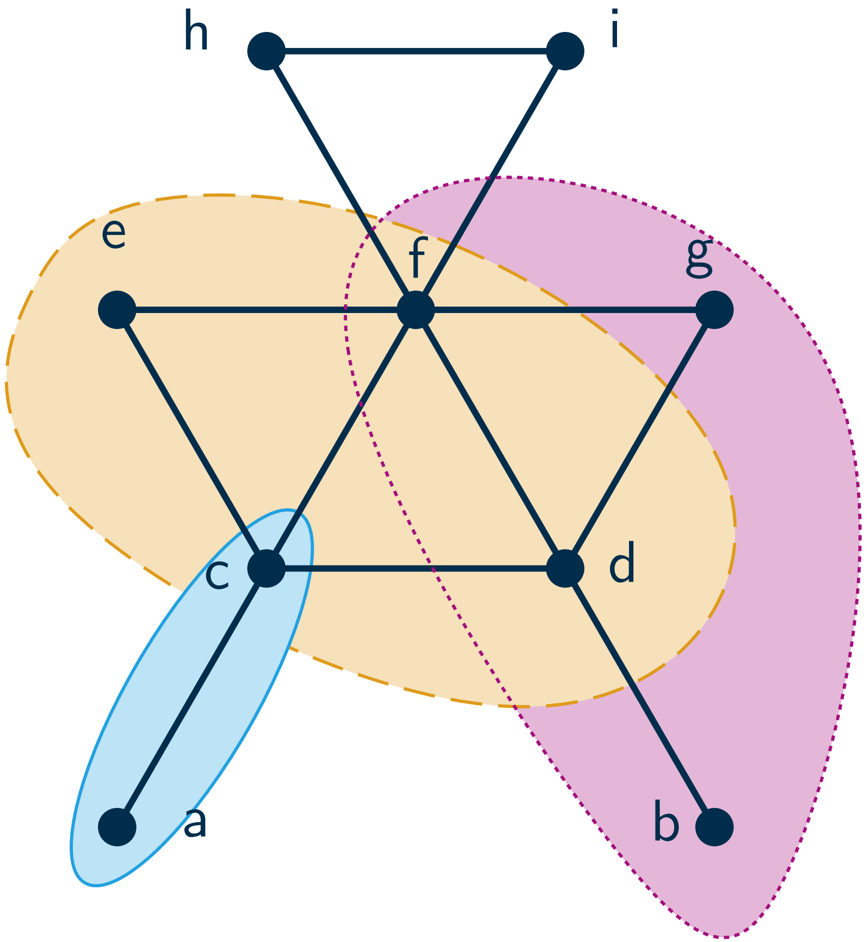

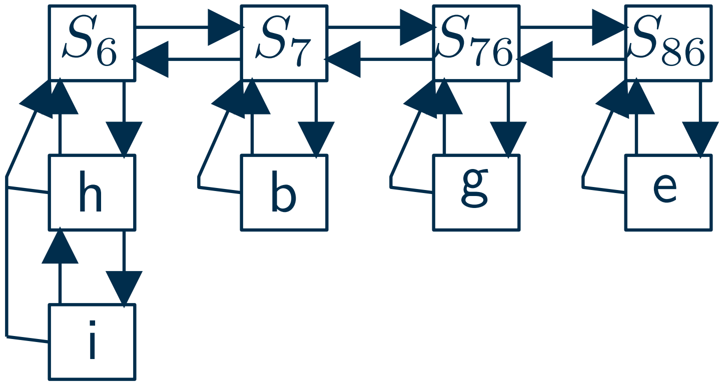

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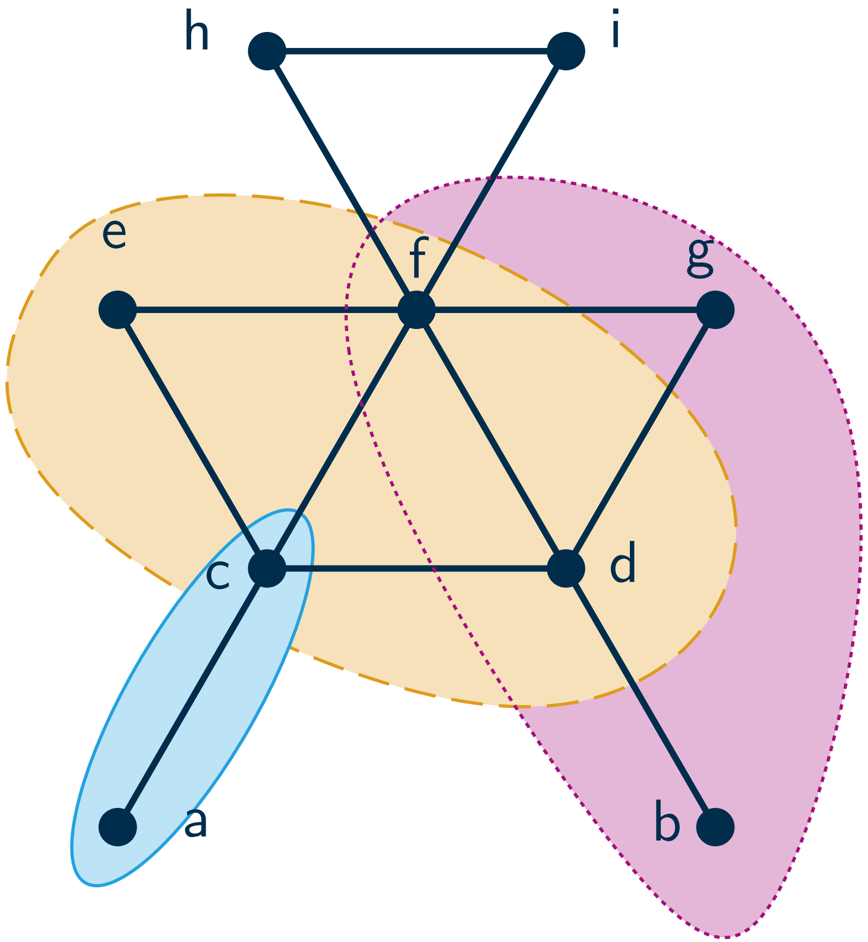

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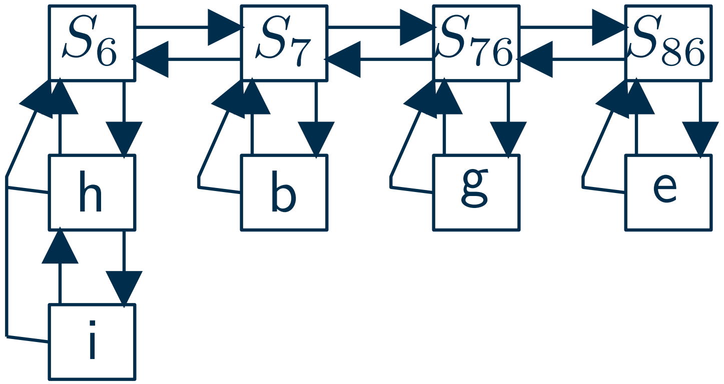

Only deletions left (except for an update at the very end for vertices h and i).

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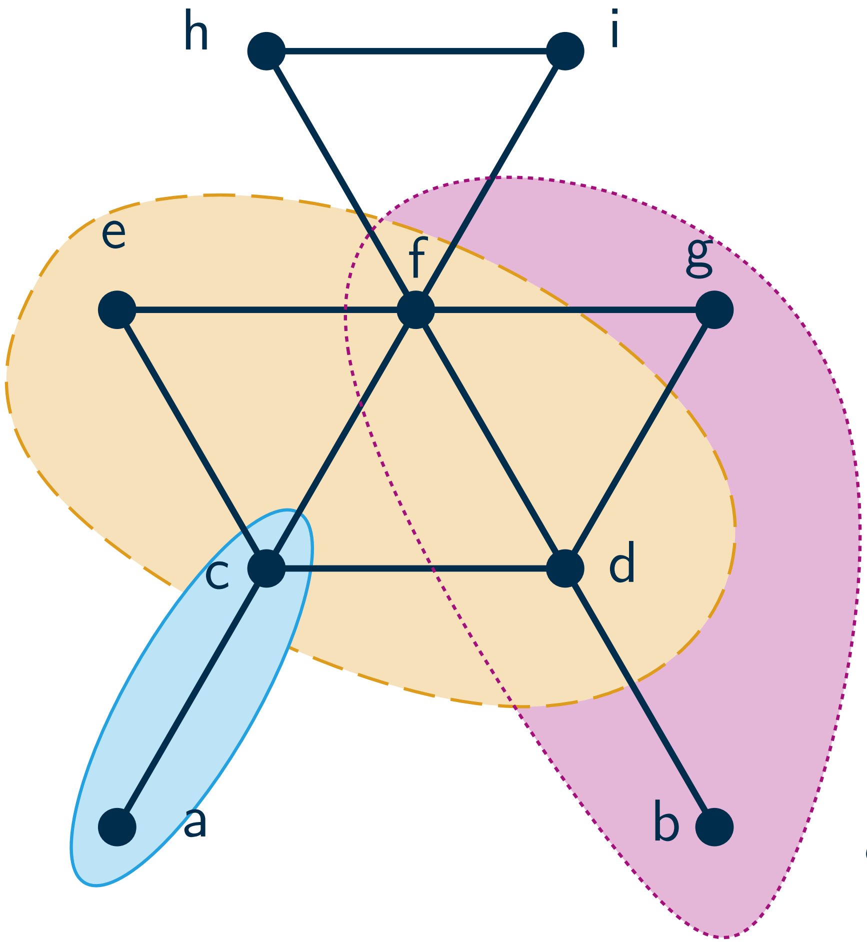

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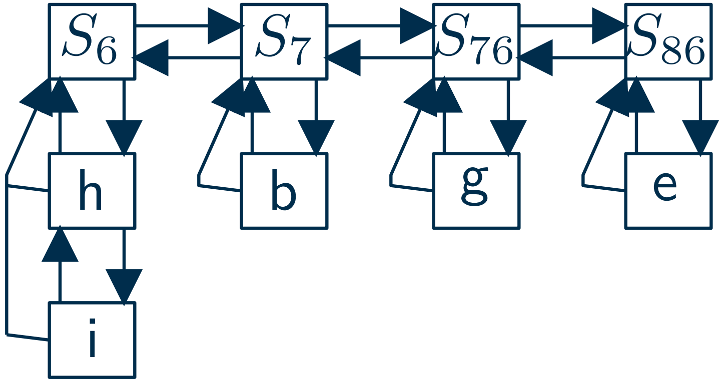

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$\sigma = [$i, h, b, g, e, f, d, c, a$]$

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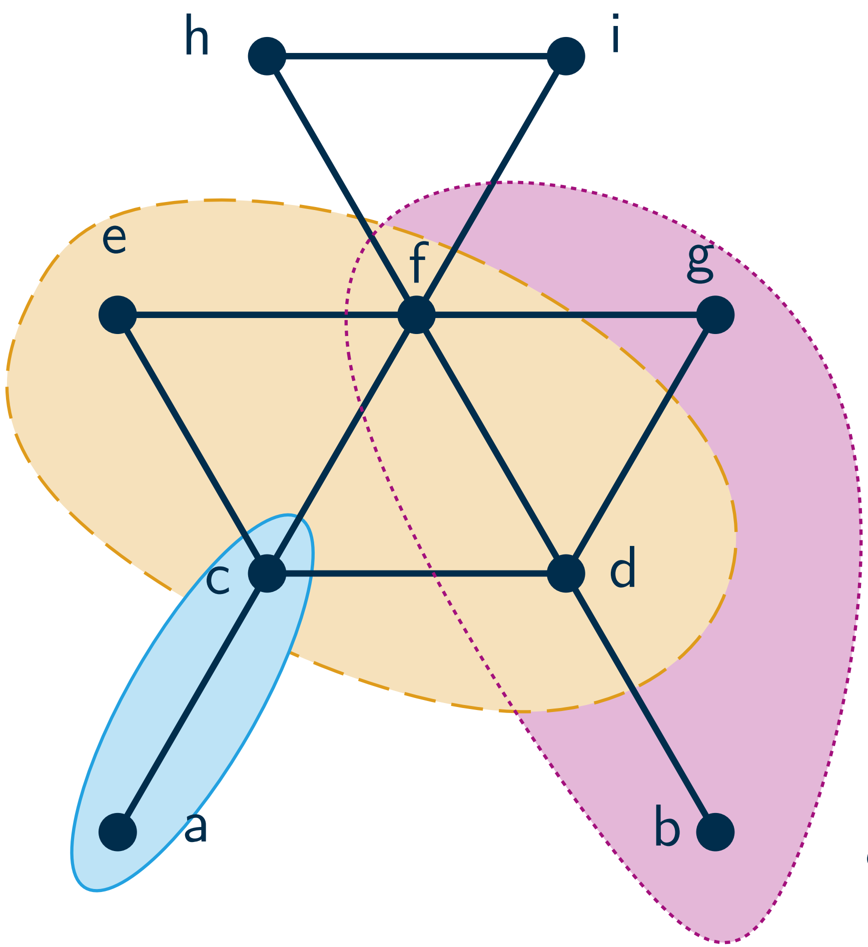

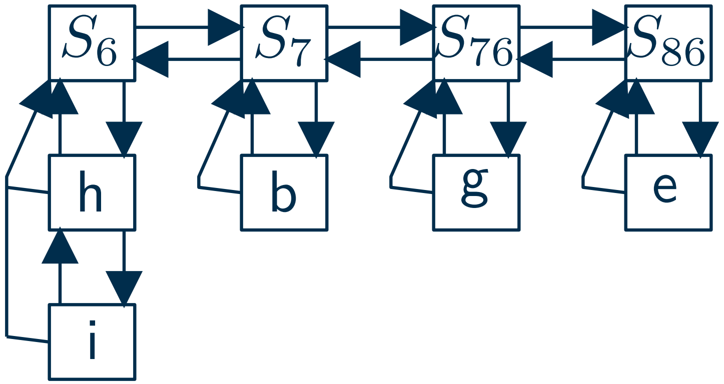

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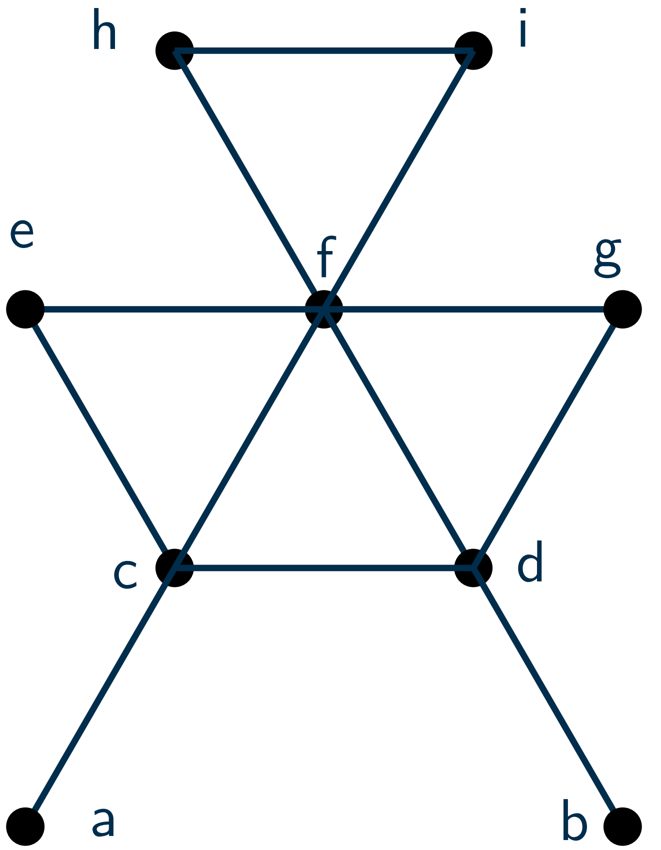
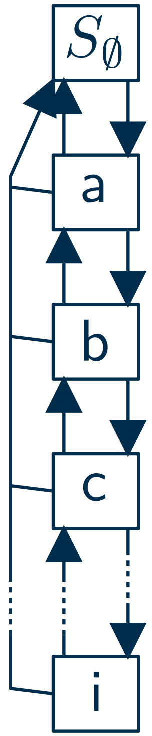
PES? ✓

Exercise 4 – LexDFS





$\sigma = [\quad]$

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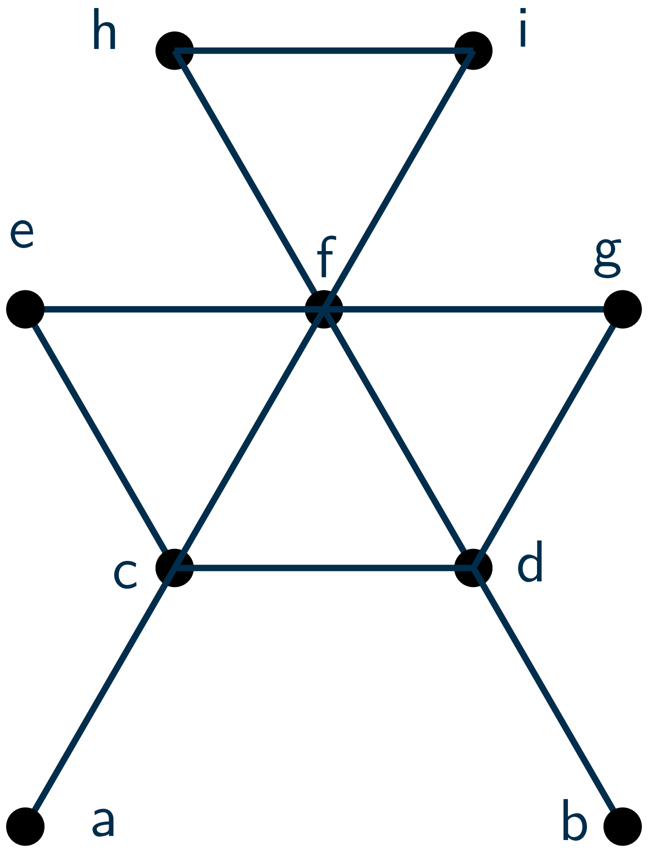

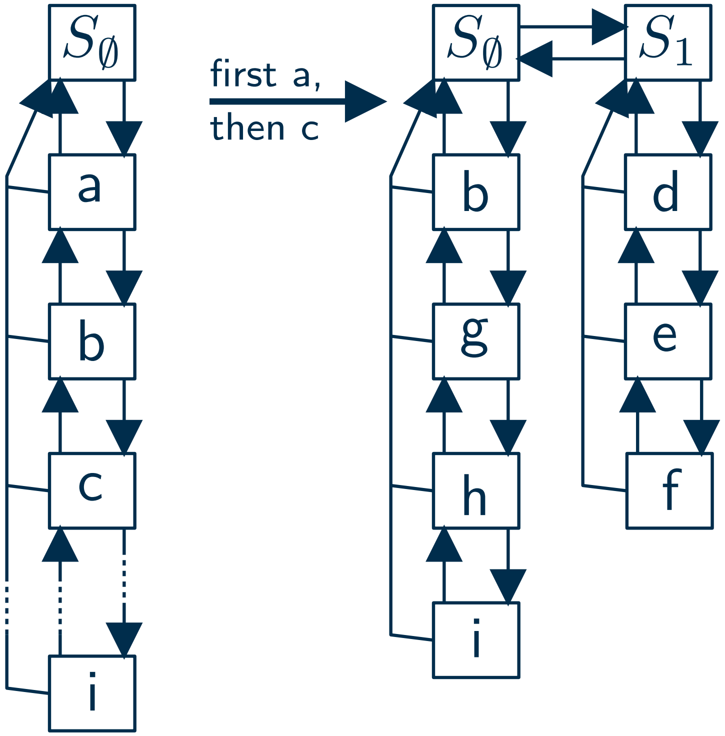

$\sigma = [$ c, a $]$

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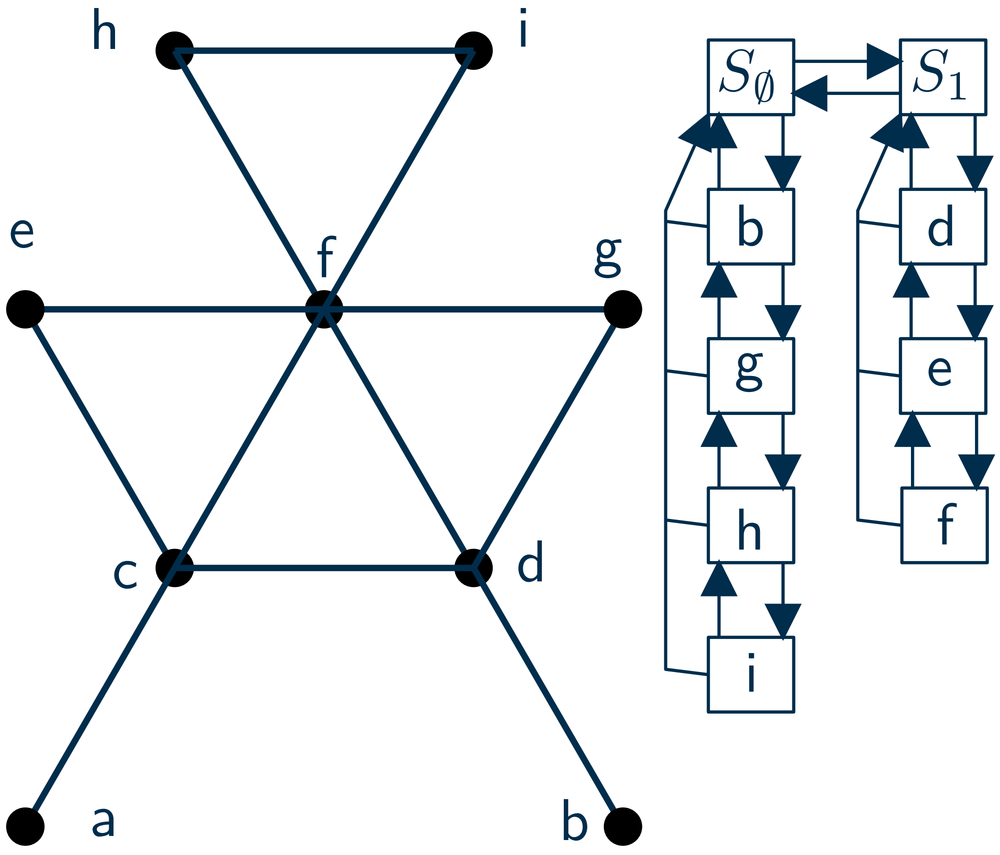

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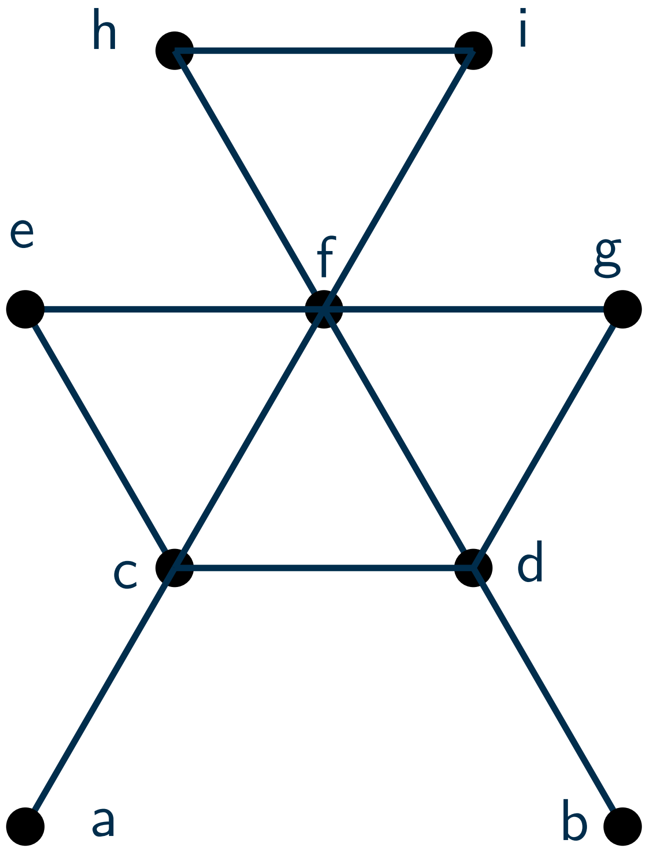

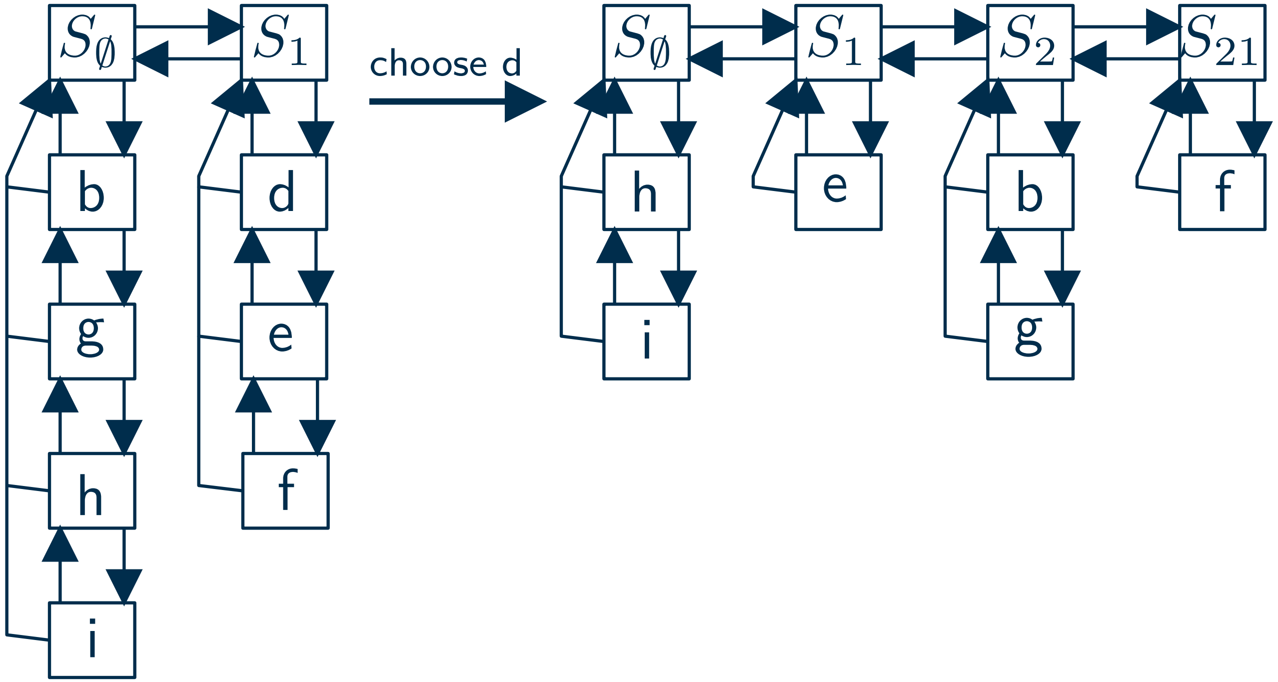

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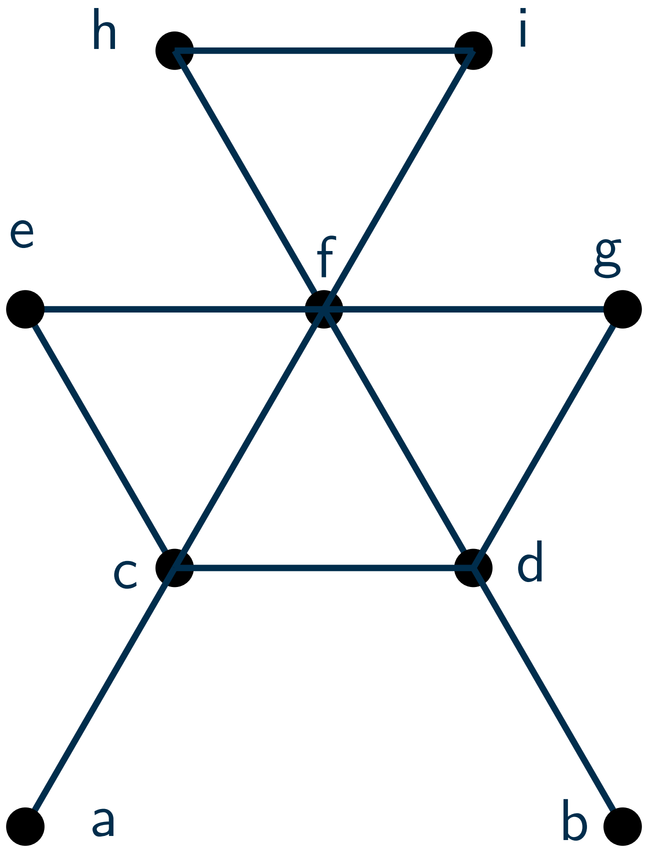

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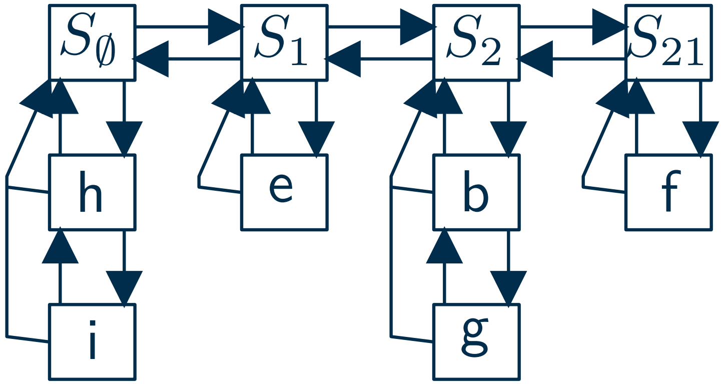

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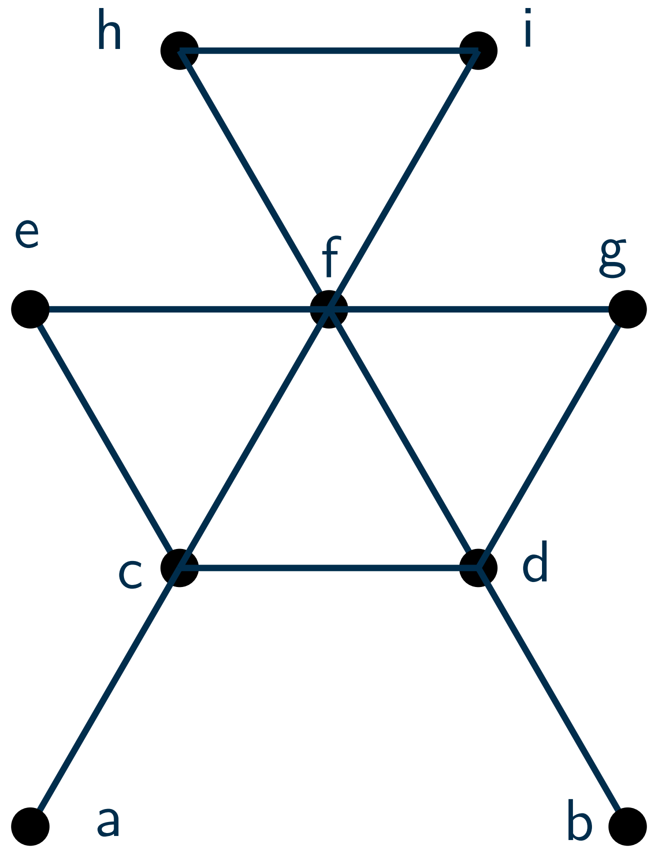
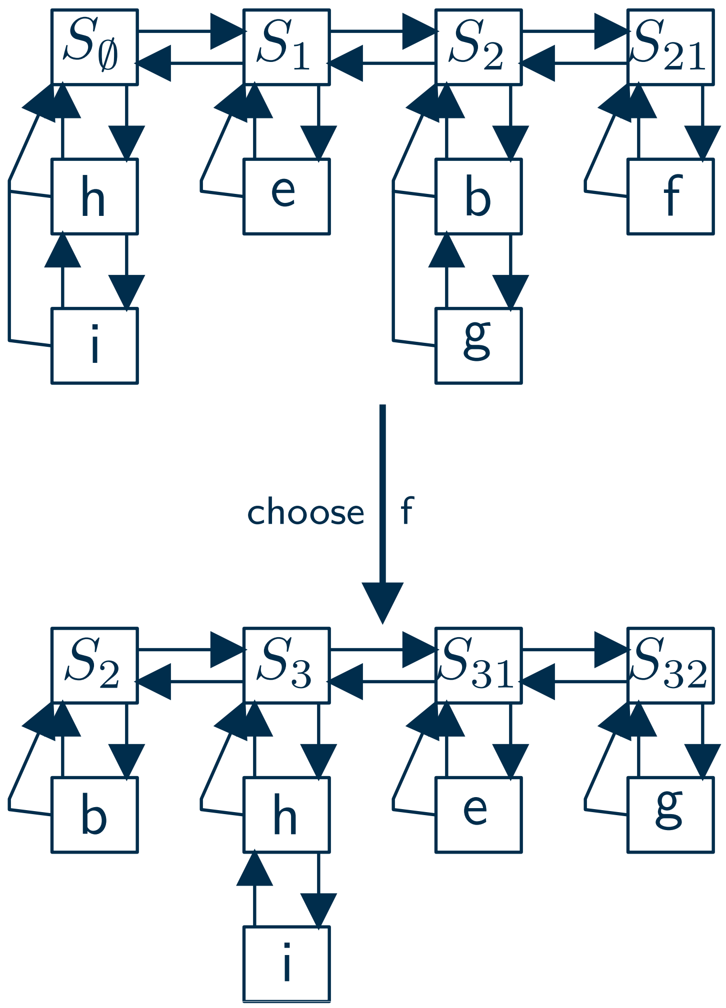
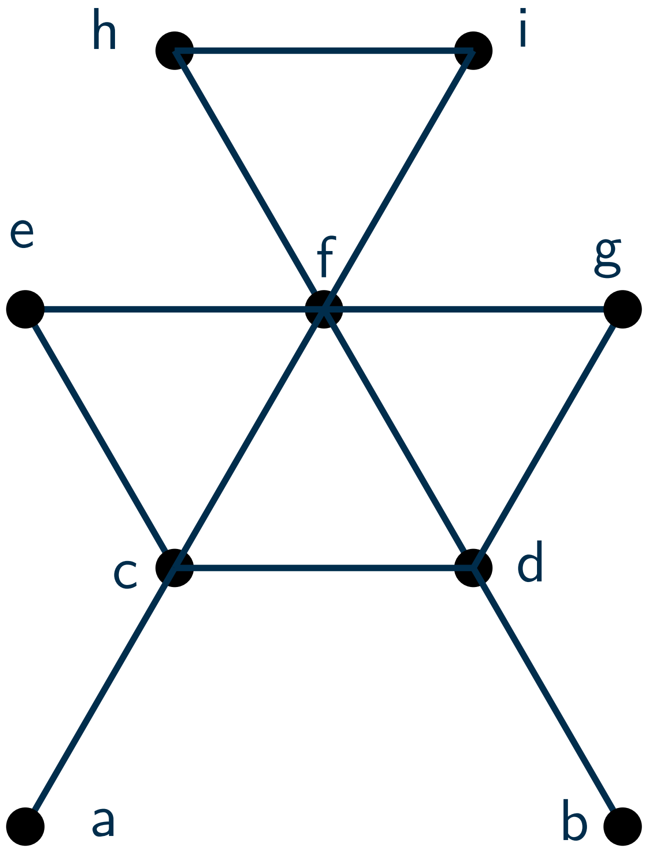
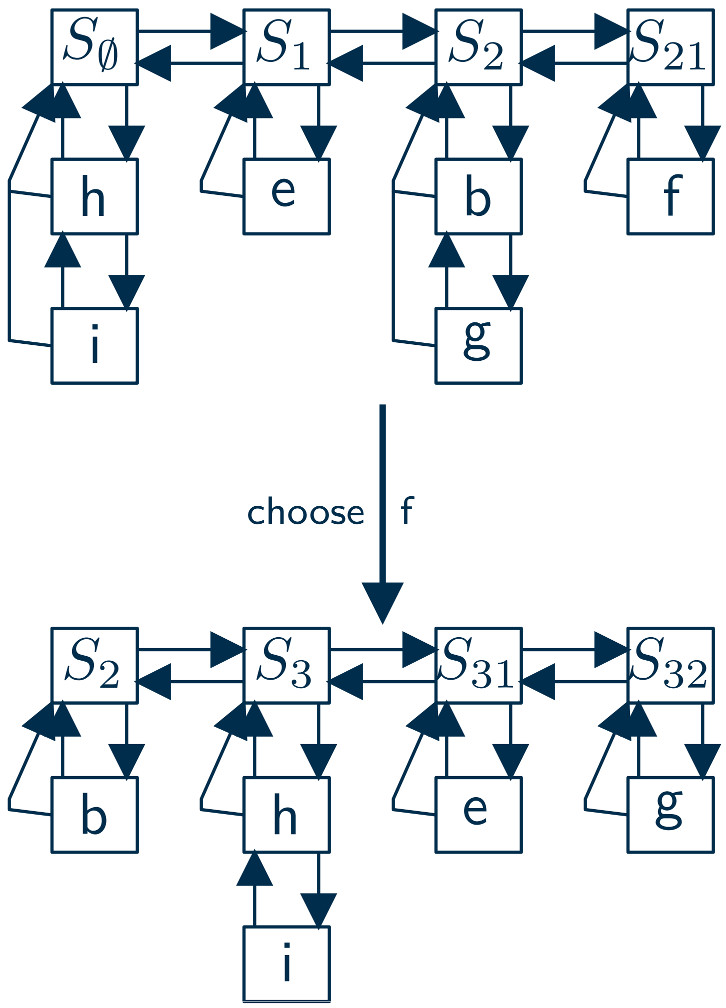


$\sigma = [$ f, d, c, a $]$



Exercise 4 – LexDFS

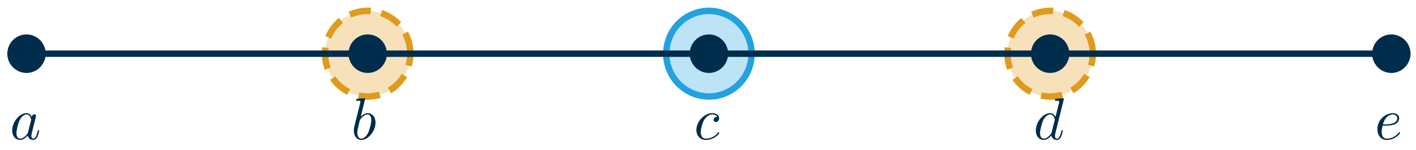




The rest is straight forward

PES: $\sigma = $ [b, i, h, e, g, f, d, c, a]

Exercise 5



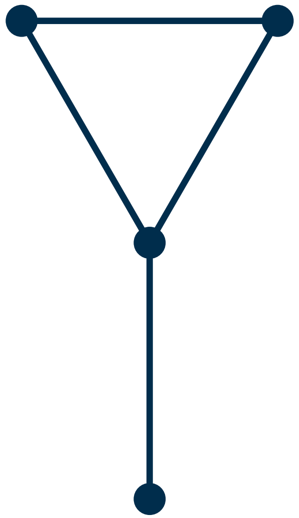
- A vertex of a path is simplicial if and only if it is a leaf.
  $\Rightarrow \sigma = [a, b, e, d, c]$ is a PES.
- In order to produce σ a LexBFS c has to start with c.
- The vertices b and d have the highest possible labels and are thus chosen next.
- Therefore, e cannot be chosen third and σ cannot be the result of a LexBFS.

Exercise 6

Let G be a k-tree. **Goal:** Construct PES for G.

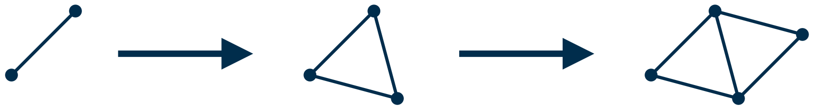
- If G is a clique every vertex ordering is a PES.
- Otherwise consider the vertex v that was added last. By construction v is simplicial.
- Call induction on $G - v$ and prepend v to the resulting PES.

Chordal graph G that is not a k-tree:

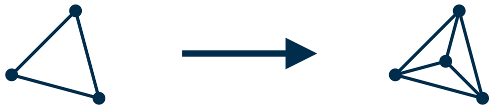


Chordal: ✓

- 1-trees are exactly trees. $\Rightarrow G$ is not a 1-tree
- There is only one 2-tree with four vertices: $\Rightarrow G$ is not a 2-tree



- There is only one 3-tree with four vertices: $\Rightarrow G$ is not a 3-tree



- Every k-tree with $k > 3$ contains a K_4 $\Rightarrow G$ is not a k-tree