Karlsruhe Institute of Technology

Algorithmic Graph Theory

Problem Class 3 | 21 May 2025

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Exercise Sheet 2

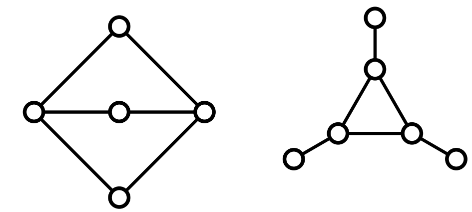
Karlsruhe Institute of Technology

(1) **(a)** Prove or disprove the following: **(i)** The class of all perfect graphs is hereditary.
(ii) The class of all perfect graphs is monotone.

(b) Give a smallest *hereditary* graph class that contains every complete graph.

(c) Give a smallest *monotone* graph class that contains every complete graph.

(2) Which of these graphs are circle graphs, circular arc graphs and interval graphs, respectively?



(3) Show that for every graph $G = (V, E)$ there exists a family $\mathcal{F}$ of subsets of E such that G is the intersection graph of $\mathcal{F}$.

(4) Prove the following for all graphs G_1 and G_2:

(a) $\chi(G_1 \boxtimes G_2) \geq \max\{\chi(G_1), \chi(G_2)\}$

(b) $\omega(G_1 \boxtimes G_2) = \omega(G_1) \cdot \omega(G_2)\}$

(c) $\alpha(G_1 \boxtimes G_2) \geq \alpha(G_1) \cdot \alpha(G_2)\}$

(d) $\kappa(G_1 \boxtimes G_2) \leq \kappa(G_1) \cdot \kappa(G_2)\}$

(5) Are the “bull head” and the complement of the “suspension bridge” interval graphs?

(6) Who is the thief?

Exercise 1

Definition: A graph class is **hereditary** / **monotone** if it is closed under taking **induced subgraphs** / **subgraphs**

(i) The class of all perfect graphs is hereditary.

- Let G be a perfect graph, then by defintion $\omega(H) = \chi(H)$ for all $H \subseteq_{\text{ind}} G$
- So all $H \subseteq_{\text{ind}} G$ are perfect

(ii) The class of all perfect graphs is **not** monotone.

- K_5 is perfect
- $C_5 \subseteq K_5$ is not perfect

(b) Give a smallest hereditary graph class that contains every complete graph.

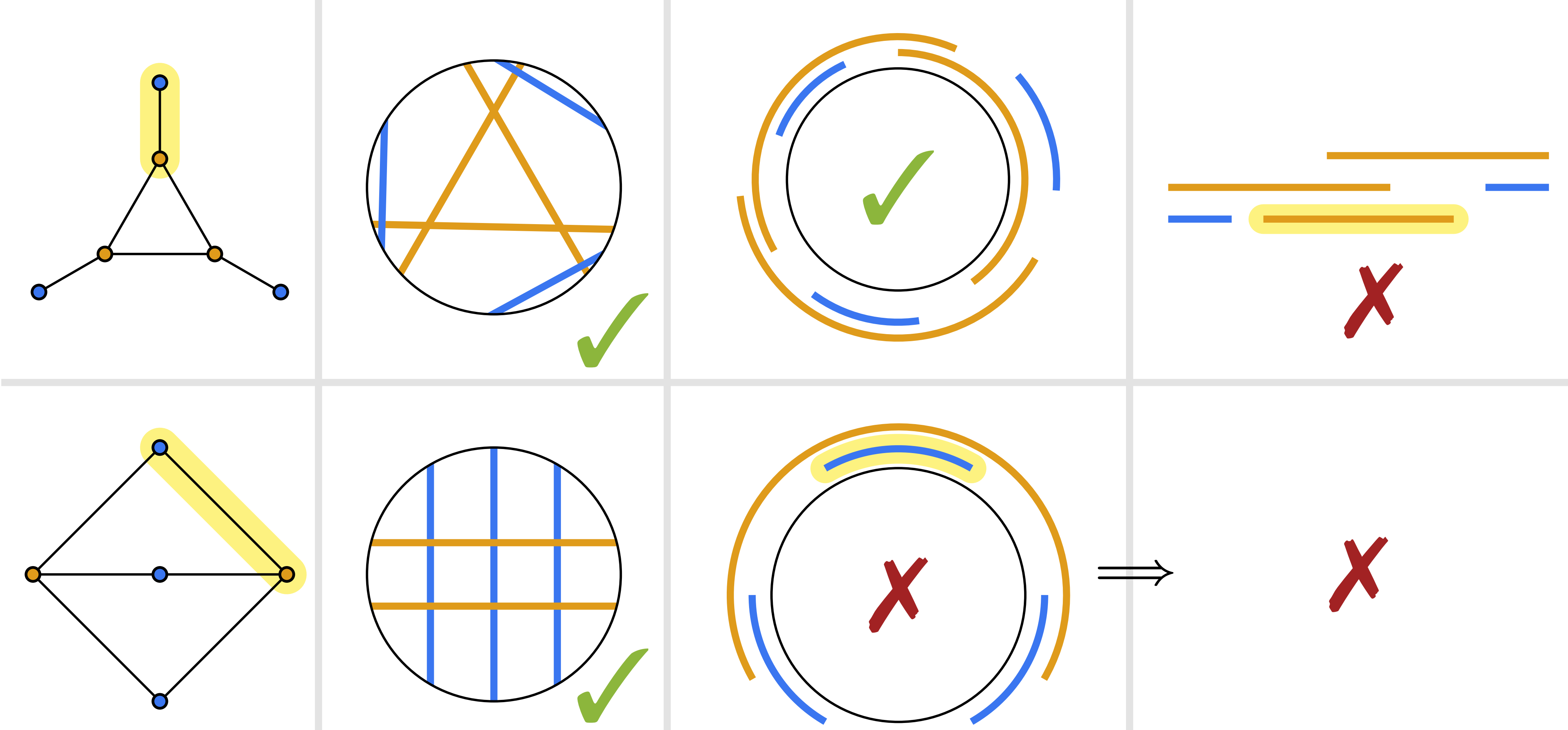
- Every induced subgraph of a complete graph is a complete graph
- So the smallest class is the set of all complete graphs

(c) Give a smallest monotone graph class that contains every complete graph.

- Every graph is a subgraph of a complete graph.
- The smallest class is the set of all graphs

Exercise 2

Circle graph: | **Circular-arc graph:** | **Interval graph:**



Exercise 3

Show that for every graph $G = (V, E)$ there exists a family $\mathcal{F}$ of subsets of E such that G is the intersection graph of $\mathcal{F}$.

Proof:

- For every vertex $v \in V$ we construct the set $F(v) = \{e \in E \mid v \in e\}$

Claim: G is the intersection graph $\mathcal{I}(\mathcal{F})$ of $\mathcal{F} = \{F(v) \mid v \in V\}$.

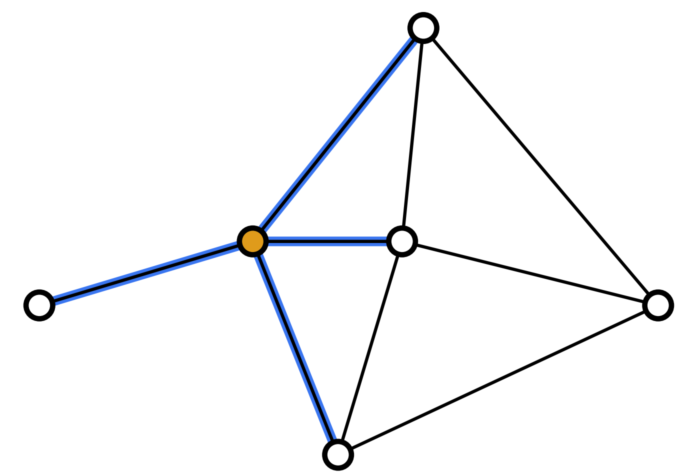
- Let $x, y \in V$ with $x \neq y$. Then:

$F(x)F(y)$ is an edge in the intersection graph

$\Leftrightarrow F(x) \cap F(y) \neq \emptyset$

$\Leftrightarrow$ there is an edge $e \in E$ such that $x \in e$ and $y \in e$

$\Leftrightarrow xy \in E$



Exercise 4

(a) $\chi(G_1 \boxtimes G_2) \geq \max\{\chi(G_1), \chi(G_2)\}$

"$\geq$": G_1 and G_2 are subgraphs of $G_1 \boxtimes G_2$.

"$=$": $K_1 \boxtimes G$

"$\neq$": $K_2 \boxtimes K_3$

(b) $\omega(G_1 \boxtimes G_2) = \omega(G_1) \cdot \omega(G_2)$

"$\geq$": Let $C_1 \subseteq V(G_1)$ and $C_2 \subseteq V(G_2)$ be cliques.

By definition of the strong product $C_1 \times C_2$ is a clique in $G_1 \boxtimes G_2$

$\Rightarrow \omega(G_1) \cdot \omega(G_2) = |C_1 \times C_2| \leq \omega(G_1 \boxtimes G_2)$

"$\leq$": Let C be a maximum clique in $G_1 \boxtimes G_2$

By definition of the strong product $\{u \mid (u,v) \in C\}$ and $\{v \mid (u,v) \in C\}$ cliques

$\Rightarrow \omega(G_1 \boxtimes G_2) = |C| \leq \omega(G_1) \cdot \omega(G_2)$

(c) $\alpha(G_1 \boxtimes G_2) \geq \alpha(G_1) \cdot \alpha(G_2)$

"$\geq$": Analogous to **(b)**

"$=$": $K_1 \boxtimes G$

"$\neq$": $C_5 \boxtimes C_5$

(d) $\kappa(G_1 \boxtimes G_2) \leq \kappa(G_1) \cdot \kappa(G_2)$

"$\leq$": Let $V_1^{(i)}, \ldots V_{\kappa(G_i)}^{(i)}$ be minimum clique covers of G_i with $i \in [2]$

Then, $\{V_j^{(1)} \times V_k^{(2)} \mid j \in [\kappa(G_1)], k \in [\kappa(G_2)]\}$ is a clique cover of $G_1 \boxtimes G_2$

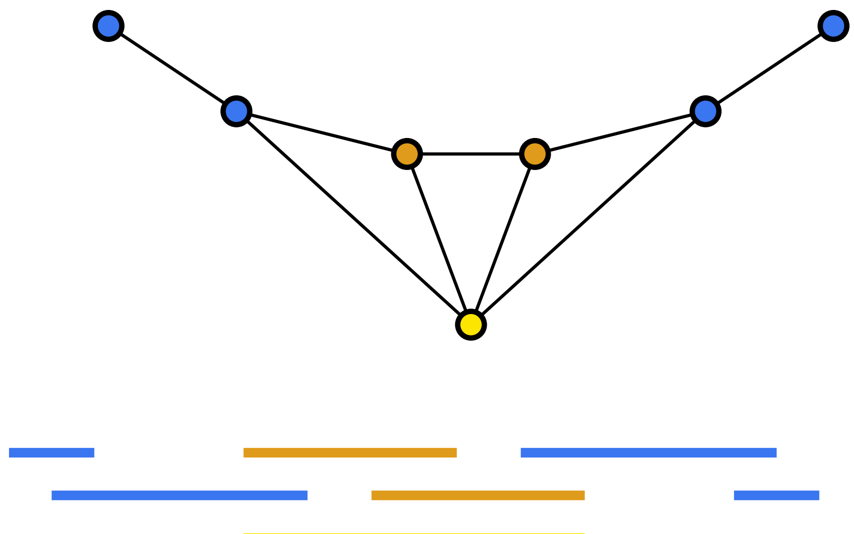
$\Rightarrow \kappa(G_1) \cdot \kappa(G_2) \geq \kappa(G_1 \cdot G_2)$

"$=$": $K_1 \boxtimes G$

"$\neq$": $C_5 \boxtimes C_5$

Exercise 5

Are the "bull head" and the complement of the "suspension bridge" interval graphs?



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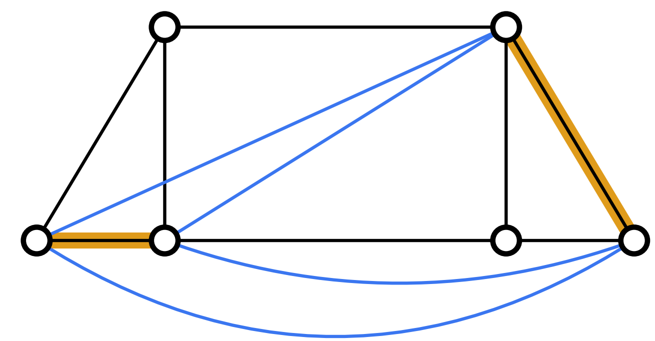


induced C_4

Lecture: Interval graphs are chordal

✗

Exercise 6

Six group members were in the office the day the coffee beans were stolen.
Every group member arrived at a certain time, stayed for some time and then left.
If two group members were present at the same time, at least one of them saw the other.

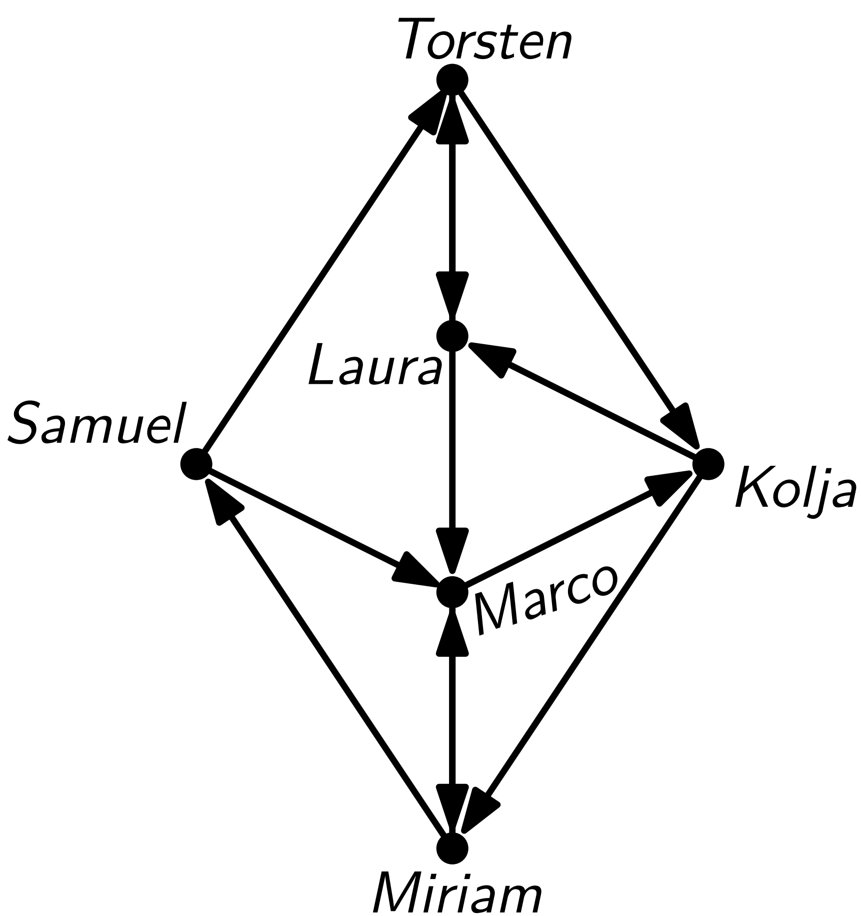
During questioning they say the following:

- *Torsten* claims to have seen *Laura* and *Kolja*,
- *Laura* claims to have seen *Torsten* and *Marco*,
- *Miriam* claims to have seen *Samuel* and *Marco*,
- *Samuel* claims to have seen *Torsten* and *Marco*,
- *Kolja* claims to have seen *Laura* and *Miriam*, and
- *Marco* claims to have seen *Miriam* and *Kolja*.

To deviously misdirect the suspicion to one of their colleagues, the thief claimed to have seen people that they did not see.
Who is the thief?

Exercise 6

Graph: Group member as vertices, “claims to have seen”-relation as edges



- *Torsten* claims to have seen *Laura* and *Kolja*,
- *Laura* claims to have seen *Torsten* and *Marco*,
- *Miriam* claims to have seen *Samuel* and *Marco*,
- *Samuel* claims to have seen *Torsten* and *Marco*,
- *Kolja* claims to have seen *Laura* and *Miriam*, and
- *Marco* claims to have seen *Miriam* and *Kolja*.

Exercise 6

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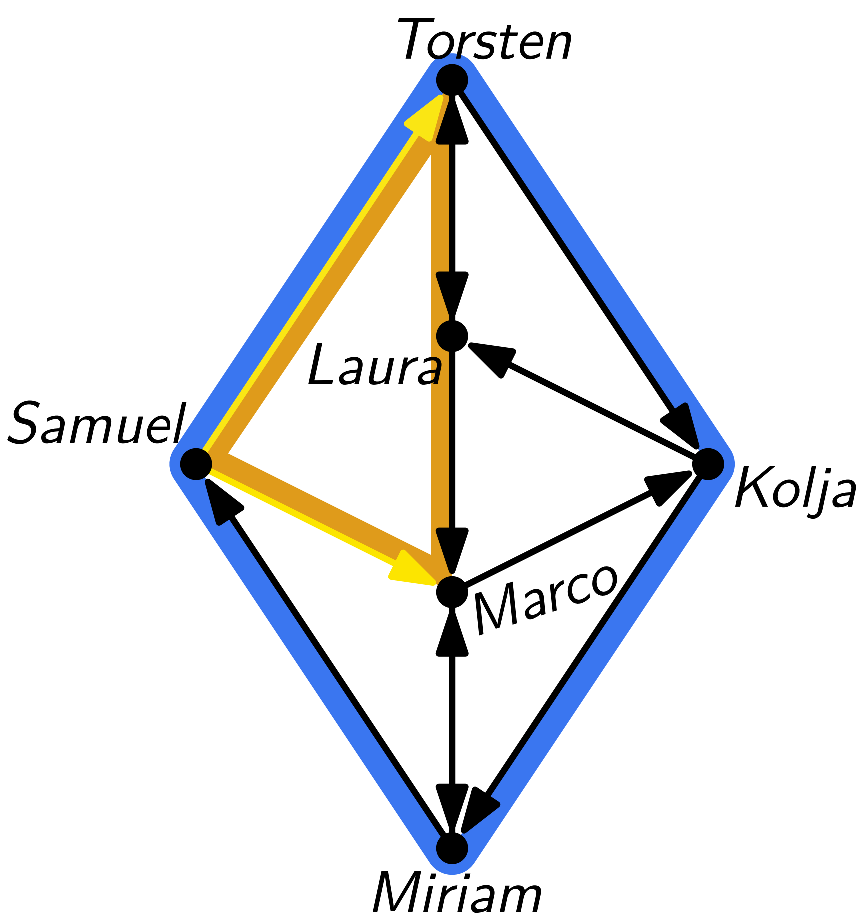

Underlying undirected graph is an interval graph if everybody says the truth.

This graph is not chordal!

Torsten and *Samuel* on both cycles, but only deletion of outgoing edges of Samuel yields chordal graph

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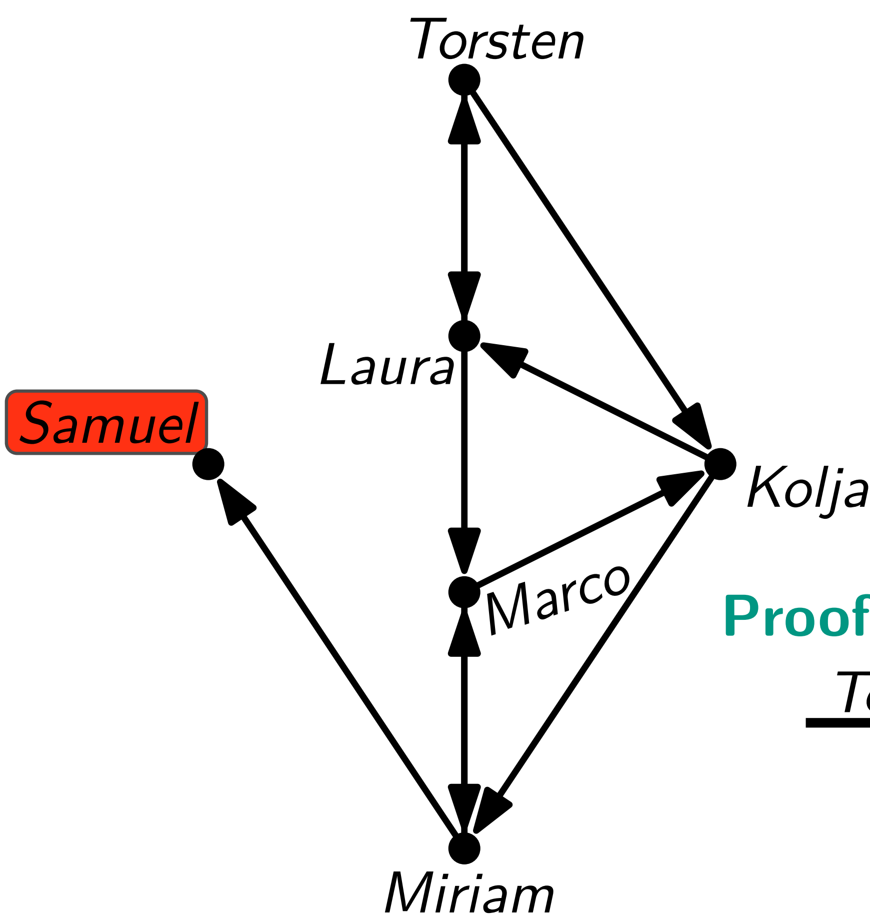
Graph: Group member as vertices, “claims to have seen”-relation as edges



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Proof of chordality: