Karlsruhe Institute of Technology

Algorithmic Graph Theory

Problem Class 2 | 7 May 2025

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Exercise Sheet 1

(1) For each $p \in \{\chi, \omega, \alpha, \kappa\}$, give $p(G \,\dot\cup\, G')$ depending on $p(G)$ and $p(G')$, where $\dot\cup$ is the disjoint union. Prove your claims.

(2) Let G be a graph and x, y two of its vertices. Prove that $(G \circ x) - y = (G - y) \circ x$

(3) Let G be a graph with vertices $x_1, \ldots, x_n$ and $h = (h_1, \ldots, h_n) \in \mathbb{N}_0^n$ be a vector.

- What does the algorithm to the right compute?
- What is the run time?
- Can H be computed faster?

$H \leftarrow G$
for all $i \leftarrow 1, \ldots, n$ **do**
  if $h_i = 0$ **then**
    $H \leftarrow H - x_i$
  else
    while $h_i > 1$ **do**
      $H \leftarrow H \circ x_i$
      $h_i \leftarrow h_i - 1$

(4) Show that each $G \in \{C_n, \overline{C_n} \colon n \geq 5 \text{ odd}\}$ is minimally imperfect, that is, G is not perfect but every proper (i. e. strictly smaller) induced subgraph is perfect.

(5) Let G be a graph on n vertices. Show that $\chi(G) \leq r$ holds iff $\alpha(G \,\square\, K_r) = n$. Conclude that Independent Set is NP-hard. (given that Coloring is NP-hard).

Exercise 1

Claim: $\chi(G \dot\cup G') = \max\{\chi(G), \chi(G')\}$

"$\leq$": Color components with $\chi(G)$ and $\chi(G')$ colors, respectively.

"$\geq$": G and G' are induced subgraphs of $G \dot\cup G'$.

Claim: $\omega(G \dot\cup G') = \max\{\omega(G), \omega(G')\}$

"$\geq$": Every clique in G or G' is clique in $G \dot\cup G'$.

"$\leq$": No edges between any vertices $u \in V(G)$ and $v \in V(G')$.

Claim: $\alpha(G \dot\cup G') = \alpha(G) + \alpha(G')$

"$\leq$": No edges between vertices $u \in V(G)$ and $v \in V(G')$.

$\Rightarrow$ Combine maximum independent sets in G and G'.

"$\geq$": Independet set in $G \dot\cup G'$ can be partitioned in independet sets in G and G'.

Claim: $\kappa(G \dot\cup G') = \kappa(G) + \kappa(G')$

"$\leq$": Cover components with $\kappa(G)$ and $\kappa(G')$ cliques, respectively.

"$\geq$": Clique cover of $G \dot\cup G'$ can be partitioned in clique covers of G and G'.

Exercise 2

Refer to the solution of *Exercise 2* of the second problem class.

Exercise 3

$H \leftarrow G$
for all $i \leftarrow 1, \ldots, n$ **do**
  if $h_i = 0$ **then**
    $H \leftarrow H - x_i$ //delete edges incident to x_i $\Rightarrow \deg(x_i) + \sum_{j<i} h_j$ operations
  else
    while $h_i > 1$ **do**
      $H \leftarrow H \circ x_i$ //copy edges incident to x_i $\Rightarrow \deg(x_i) + \sum_{j<i} h_j$ operations
      $h_i \leftarrow h_i - 1$

How many edges do we have to delete/copy?

Define: $\hat{h} = \max\{h_1, \ldots, h_n\}$ and $h'_i = \max\{h_i, 1\}$

Run time – upper bound:

$$\mathcal{O}\Big(\sum_{i=1}^{n} h'_i \cdot \Big(\deg(x_i) + \sum_{j<i} h_j\Big)\Big)$$
$$\subseteq \mathcal{O}\big(n \cdot \max(\hat{h}, 1) \cdot (n + n\hat{h})\big)$$
$$\subseteq \mathcal{O}\big(n^2 \cdot \max^2(\hat{h}, 1)\big)$$

Run time – lower bound:

- Let $G = K_n$ and $h_i = h_j > 1$ for all $i, j \in [n]$
- $G \circ h$ has $n \cdot (h_1 - 1)$ new vertices
- $\frac{1}{2} \cdot n \cdot (h_1 - 1) \cdot (n - 1) \cdot h_1 \in \Theta(n^2 h_1^2)$ new edges
  $\Rightarrow$ upper bound is tight

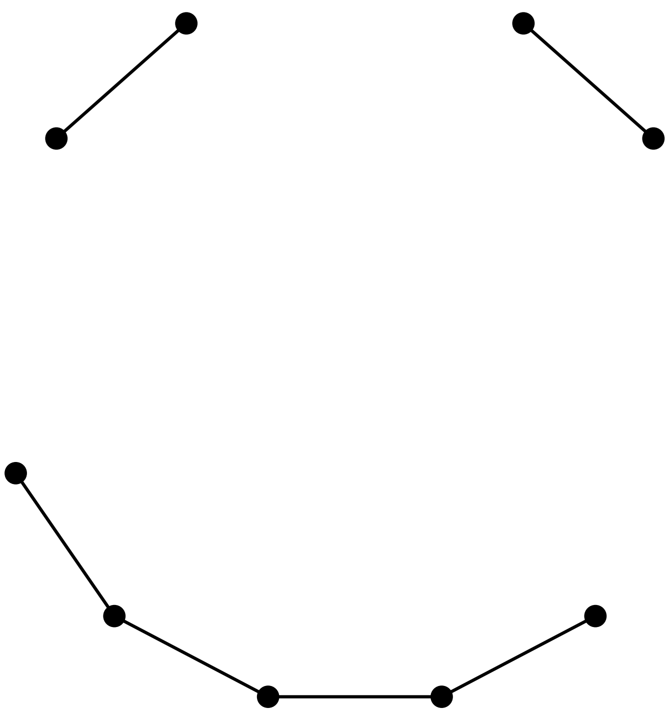
Exercise 4

Let $n \geq 5$ odd

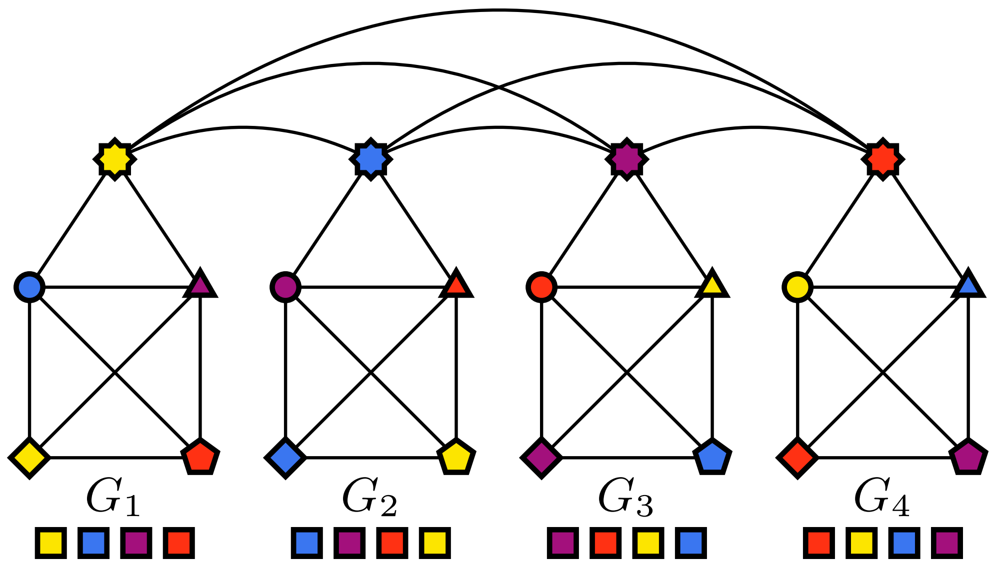
- C_n is not perfect as $\omega(C_n) = 2 < \chi(G) = 3$.
- $\overline{C_n}$ is not perfect as C_n is not perfect.
- For every $A \subsetneq V(C_n) : C_n[A]$ is perfect $\Leftrightarrow \overline{C_n[A]}$ is perfect

Goal: Show that every $H \subsetneq_{\text{ind}} C_n$ is perfect.

- H is a disjoint union of paths.
- Paths are perfect $\Rightarrow H$ is perfect.



Exercise 5



$\chi(G) \leq r \Rightarrow \alpha(G \,\square\, K_r) = n$

- Consider r-coloring of $G \,\square\, K_r$
- There is a color class with at least $\frac{|V(G \,\square\, K_r)|}{r} = \frac{n \cdot r}{r} = n$ vertices
- Every color class is an independent set

$\Rightarrow \alpha(G \,\square\, K_r) \geq n$

- But also: $\alpha(G \,\square\, K_r) \leq \kappa(G \,\square\, K_r) \leq n$

$\chi(G) \leq r \Rightarrow \chi(G \,\square\, K_r) = r$

- $\chi(G \,\square\, K_r) \geq r$ because $K_r \subseteq G \,\square\, K_r$
- color $G_1, \ldots, G_r$ with the same r colors but cyclically permuted

$\Rightarrow$ r colors suffice for $G \,\square\, K_r$

$\chi(G) \leq r \Leftarrow \alpha(G \,\square\, K_r) = n$

- Let I be a maximum independent set of $G \,\square\, K_r$

$\Rightarrow \forall v \in V(G)$ there is exactly one $(v, w) \in I$

- For every $w \in V(K_r)$ let $V_w := \{v \mid (v, w) \in I\}$

Claim: Every V_w is an independent set

- Assume not: $\exists u, v \in V_w : uv \in E(G)$

$\Rightarrow (u, w)(v, w) \in E(G \,\square\, K_r)$ ⚡

$\Rightarrow$ one color for each V_w gives r-coloring of G