Algorithmic Graph Theory
Summer Term 2025
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Exercise Sheet 1

Discussion: 7 May 2025

Exercise 1

For each $p \in \{\chi, \omega, \alpha, \kappa\}$, give $p(G \dot\cup G')$ depending on $p(G)$ and $p(G')$, where $\dot\cup$ is the disjoint union. Prove your claims.

Exercise 2

Let G be a graph and x, y two of its vertices. Prove that $(G \circ x) - y = (G - y) \circ x$.

Exercise 3

Let G be a graph with vertices $x_1, \dots, x_n$ and let $h = (h_1, \dots, h_n) \in \mathbb{N}_0^n$ be a vector. What does the algorithm to the right compute? What is the run time? Can H be computed faster?

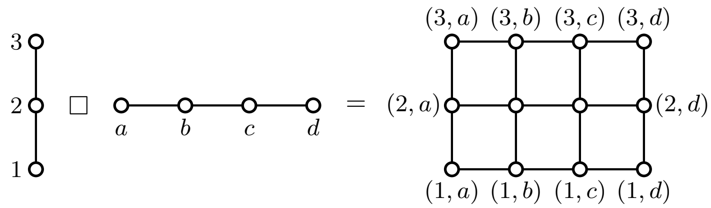
$H \leftarrow G$
for $i \leftarrow 1$ **to** n **do**
    if $h_i = 0$ **then** $H \leftarrow H - x_i$
    else
        while $h_i > 1$ **do**
            $H \leftarrow H \circ x_i$
            $h_i \leftarrow h_i - 1$

Exercise 4

Show that each $G \in \{C_n, \overline{C_n} \colon n \geq 5 \text{ odd}\}$ is minimally imperfect, that is, G is not perfect but every proper (i.e. strictly smaller) induced subgraph is perfect.

Exercise 5

The *Cartesian product* $G_1 \square G_2$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is defined as $G_1 \square G_2 = (V_1 \times V_2, E)$, where $E = \{\{(v_1, v_2), (v_1', v_2')\} \mid v_1 = v_1' \text{ and } \{v_2, v_2'\} \in E_2 \text{ or } v_2 = v_2' \text{ and } \{v_1, v_1'\} \in E_1\}$, see figure below.



Let G be a graph on n vertices. Show that $\chi(G) \leq r$ holds if and only if $\alpha(G \,\square\, K_r) = n$.

Conclude that Independent Set is NP-hard (given that Graph Coloring is NP-hard).