Algorithmic Graph Theory

Problem Session 7

Laura Merker and Samuel Schneider, July 30, 2025

Problems

(1) Prove that every cograph is a permutation graph.

(2) Let G be a split graph that is not a complete graph. Prove or disprove that it is always possible to add an edge e such that $G + e$ is a split graph.

(3) Read the arXiv newsletter of July 14, 2025 (usually sent by email):
https://i11www.iti.kit.edu/teaching/sommer2025/algorithmic_graph_theory/newsletter
Read all titles, some abstracts, only open papers if you think they are relevant for you

Keep in mind: articles on arXiv are not (necessarily) reviewed!

(4) We aim to partition the edges of a split graph into as few complete bipartite graphs as possible. What is the smallest/largest number of complete bipartite graphs needed for n-vertex split graph in terms of n? Also find graphs that neither reach the minimum nor maximum.

(5) How many maximal cliques can a split graph or its complement have at most? Give your answers in terms of $\omega, \alpha, \kappa, \chi$, resp. the number of vertices, or prove that this is not possible. Which of your results generalize to chordal graphs?

Cographs are permutation graphs

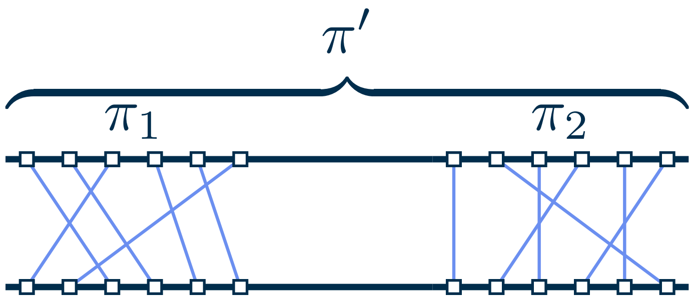
Prove that every cograph is a permutation graph.

Every cograph can be constructed as follows:
- K_1 is a cograph.
- The complement of a cograph is a cograph.
- The disjoint union of two cographs is a cograph.

- K_1 is clearly a permutation graph.

Complement:
- G is permutation graph $\iff$ G and $\overline{G}$ are comp. graphs $\iff$ $\overline{G}$ is permutation graph

Disjoint union:
- Let G_1 and G_2 be permutation graphs with respective permutations π_1 and π_2.
- "concatenate"[1] π_1 and π_2 to obtain π'
- Two segements cross in π' if and only if they cross in π_1 or π_2.
- π' is permutation for $G_1 \cup G_2$.

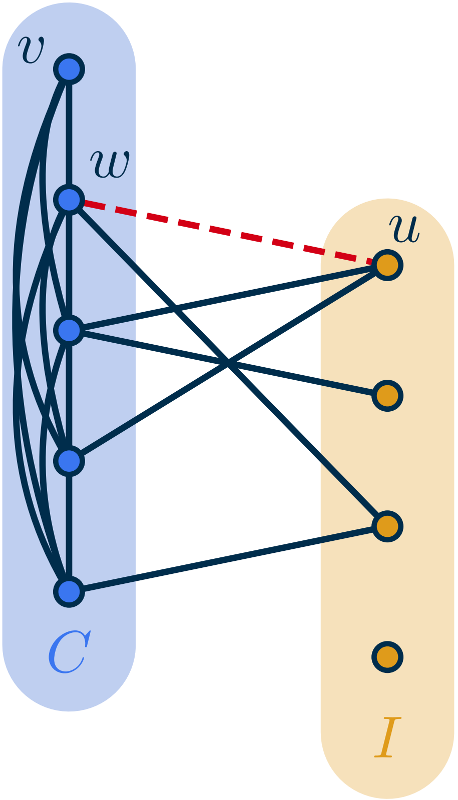


[1] : For this to be a permutation we have to add $|V(G_1)|$ to every number in π_2.

Adding edges to split graphs

Let G be a split graph that is not a complete graph. Prove that it is always possible to add an edge e such that $G + e$ is a split graph.

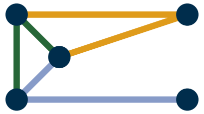
- Let G be a split graph with a partition into a clique C and an independet set I.
- If there is a vertex $v \in I$ with $N(v) = C$ we can add it to C.
- Thus, assume w. l. o. g. that C is a maximal clique.
- Let $u \in I$. By maximality of C there is a vertex $w \in C$ s. t. $uw \notin E$.
- C is clique and I is independent set in $G + uw$
  $\implies G + uw$ is split graph.



Bipartition number of split graphs

Def: ***Bipartition number***

bp(G) = min k such that $E(G)$ can be partitioned into k complete bipartite graphs



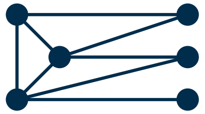
bp$(G) = 3$

What is the max/min bipartition number of n-vertex split graphs?

- Minimal: mc$(E_n) = 0$, mc(star) $= 1$

Babu, Jacob, 2025, arXiv:2507.08114

- Maximal: Abstract / Thm 2 (Graham, Pollak, 1972): bp$(K_n) = n - 1$ Is this the worst case?
  - bp$(G) \leq n - 1$ for ***every*** graph: cover edges with stars
- More interesting examples?
  - Main theorem: bp(G) = mc$(G^c) - 1$, where mc denotes the number of maximal cliques
  - Alternative with deeper reading: Lemma 8: bp$(G) \approx \omega(G)$ for split graphs G



half of the vertices in the clique and independent set each

Number of maximal cliques

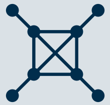
Let G be a split graph (chordal), then for the number mc(G) of maximal cliques we have ...

- mc$(G) \leq \alpha(G) + 1$ $(= \kappa(G) + 1)$.

  Babu, Jacob, 2025, arXiv:2507.08114: bp$(H) =$ mc$(\overline{H}) - 1$, bp$(H) \leq \omega(H)$

  choose $H = \overline{G}$: mc$(G) =$ bp$(\overline{G}) + 1 \leq \omega(\overline{G}) + 1 = \alpha(G) + 1$

  This is tight:

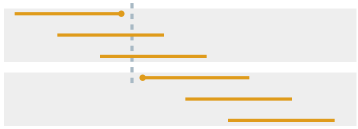
  

  $\alpha = k$

  mc $= k + 1$

  k-clique plus leaves

  mc is unbounded in α $(= \kappa)$:

  clique of size $n/2$

  clique of size $n/2$, mc $= n/2$

  $\}\ \alpha = \kappa = 2$

- mc$(G) \leq n$.

  Case $\alpha(G) < n$: mc$(G) \leq \alpha(G) + 1 \leq n$

  Case $\alpha(G) = n$: $G = E_n$, mc$(E_n) = n = \alpha(E_n)$

  We need a new argument!

  Note: for general graphs, mc can be exponential in n

  PES:

  appending next vertex v to the left:

  $\rightarrow$ v is in only one maximal clique

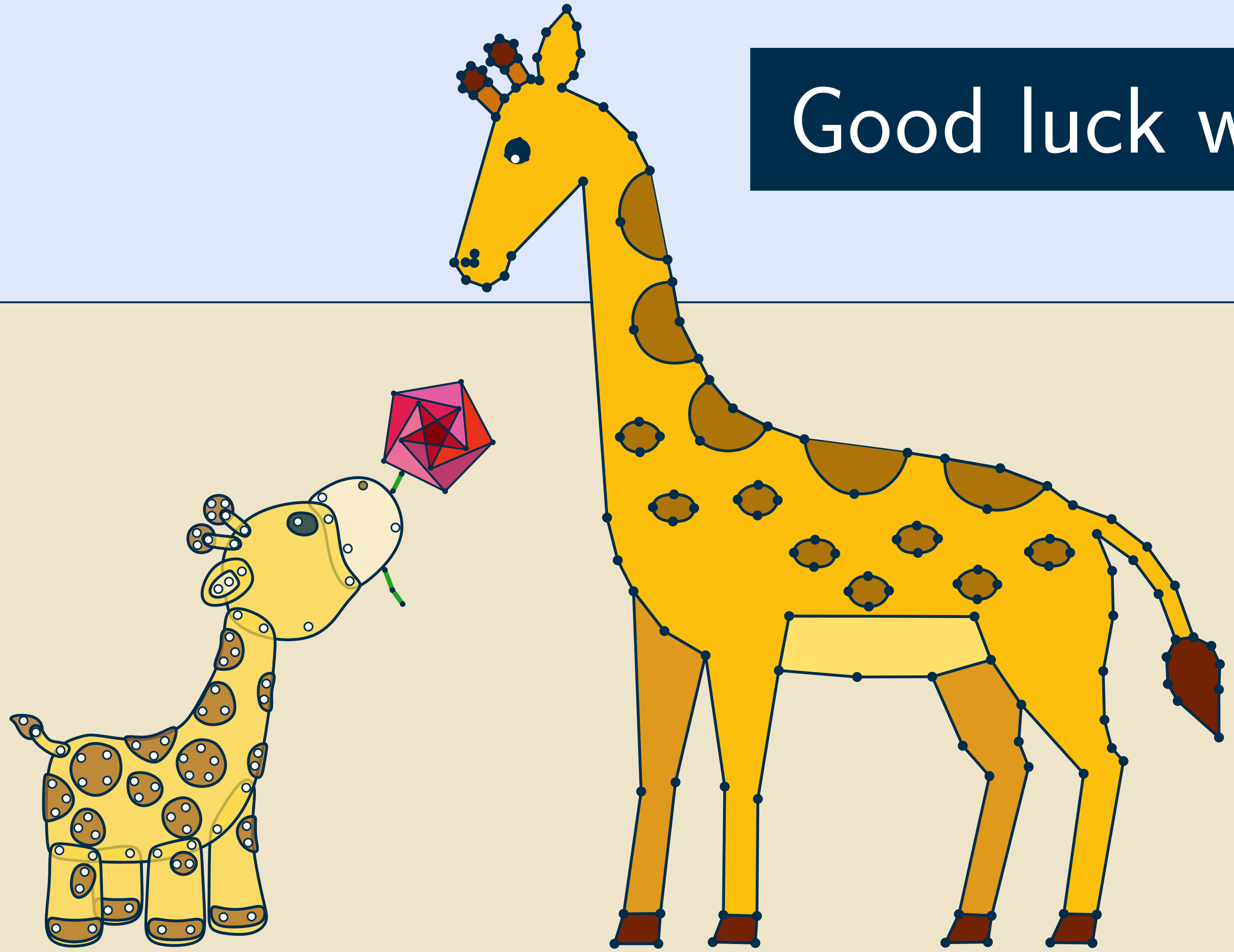
  $\rightarrow$ no new maximal cliques that do not contain v

- mc(G) is unbounded in $\omega(G)$ $(= \chi(G))$, i.e., there is no function f s.t. mc$(G) \leq f(\omega(G))$.

  $\omega(E_n) = 1$ but mc$(E_n) = n$, "unbounded" is stronger for a subclass $\Longrightarrow$ chordal graphs ✓

  hence for every f, we have mc$(E_{f(1)+1}) = f(1) + 1 > f(1) = f(\omega(E_n))$

Good luck with the exam!



Hyper-Giraph

Giraph

Artwork by Elly Schmidt