Algorithmic Graph Theory

Problem Session 6

Laura Merker and Samuel Schneider, July 16, 2025

Problems

(1) Let T be a rooted tree whose edges are oriented away from the root.

Prove that the height of T equals the clique number of the transitive closure of T.

(2) Let T be tree. Prove that there is an orientation of T such that its transitive closure is chordal.

Disprove that the transitive closure of T is chordal for every orientation of T.

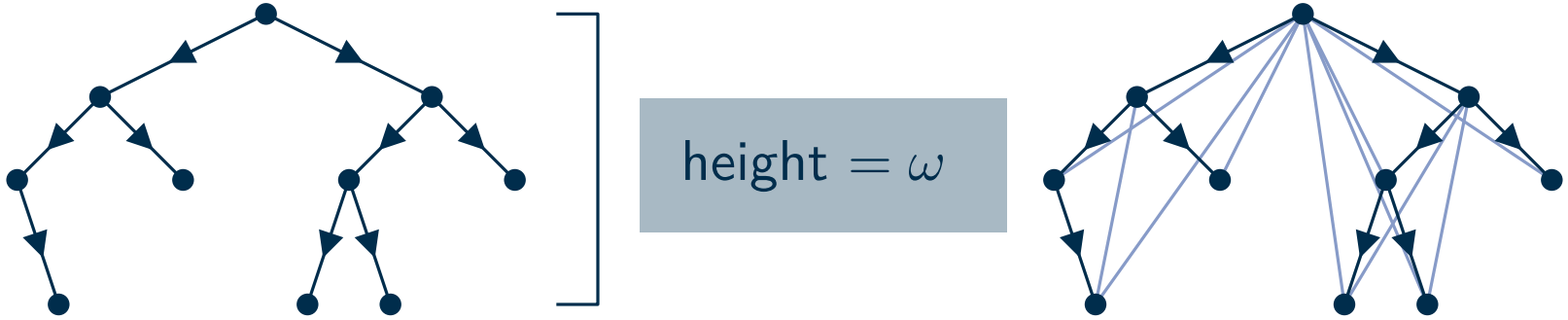
(3) Let G be a comparability graph. **Disprove** that if the transitive reduction of G is chordal, then G is chordal.

(4) Prove that this is not necessary.

(5) A skew partition of a graph G is a partition of its vertex set into sets A and B such that A induces a disconnected graph and B induces the complement of a disconnected graph.

Many connected comparability graphs admit a skew partition – but not all.
Find and prove a nice list of exceptions.

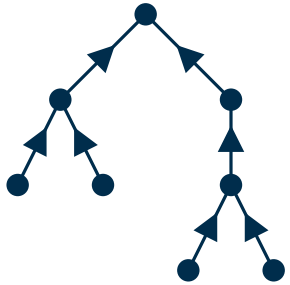
Transitive Closure of Trees



"$\leq$" every directed path implies a clique on the same number of vertices

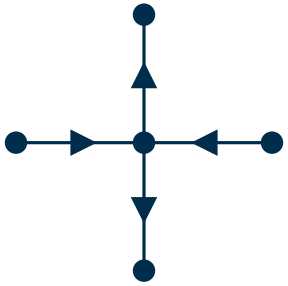
"$\geq$" every clique contains a directed path on the same number of vertices

Transitive Closure of Trees

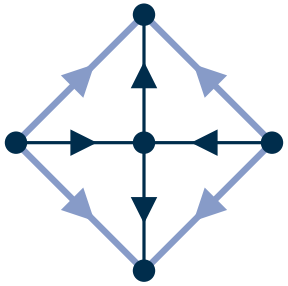


orientation towards to the root
$\rightarrow$ bottom-to-top ordering gives perfect eleminiation scheme of the closure
$\rightarrow$ chordal

General orientation:

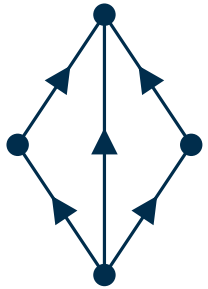


tree (chordal)

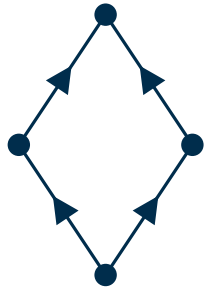


not chordal

General chordal comparability graphs:



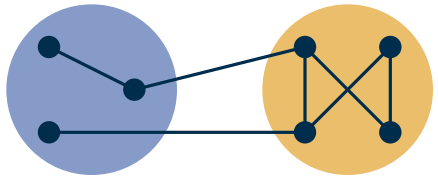
chordal



not chordal

Skew Partitions

Definition: Skew partition



disconnected

complement is disconnected

Every connected comparability graph G admits a skew paritition or it is . . .

Assume $|G| \geq 4$
(parts both need at least two vertices)

Consider a transitive orientation.
Observation: Let u be a vertex that is neither a source nor a sink, let v be incomparable to u
Then:

- $\overline{N(u)}$ is disconnected
- $G - N(u)$ is disconnected

$\rightarrow (N(u), V(G) - N(u))$ is a skew parition

yes if not bipartite

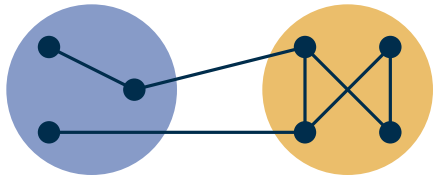
yes if not a complete graph

Does such a u always exist?

List of possible exceptions: complete graphs, bipartite graphs, graphs on at most three vertices

Skew Partitions

Definition: Skew partition



disconnected complement is disconnected

Every connected comparability graph G admits a skew paritition or it is complete or bipartite.

List of possible exceptions: complete graphs✓, bipartite graphs?, graphs on at most three vertices~

$\rightarrow$ are bipartite or cliques

Positive examples (if not too small)**:** complete bipartite graphs, $N(v) \subseteq N(w)$ for some v, w

Negative examples: even cycles, complete bipartite graph minus a matching

$\rightarrow$ keep bipartite graphs in the list of exceptions

$\rightarrow$ more details: Reed, 2008, https://doi.org/10.1016/j.dam.2007.05.054