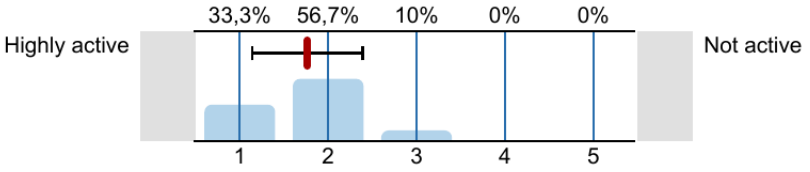
Algorithmic Graph Theory

Problem Session 5

Laura Merker and Samuel Schneider, July 4, 2025

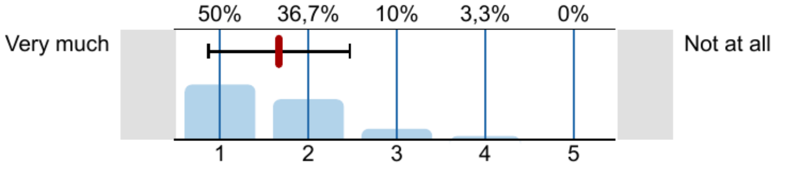
Evaluation

1.1) How do you assess the participation of your fellow students in this course?



n=30
mw=1,77
s=0,63

1.2) How much do you like attending this course?



n=30
mw=1,67
s=0,8

1.3) Why do you attend this course?

n=30

Mandatory/mandatory elective subject	53,3%
Personal interest	90%
For repetition/specialization	30%

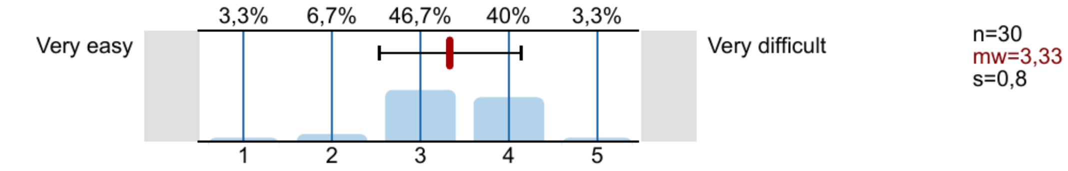
1.4) My commitment to this course is reflected by (multiple choice):

n=30

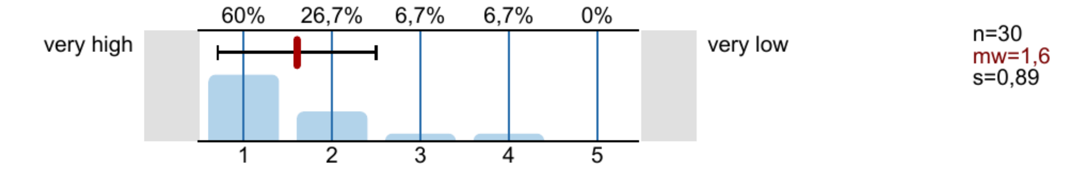
Regular attendance	96,7%
Constant taking of notes	53,3%
Active contributions	26,7%
Regular follow-up work	30%
Participation in a study group	16,7%
Reading of recommended literature	13,3%
Work on scripts/materials	40%
Internet research	26,7%

Evaluation

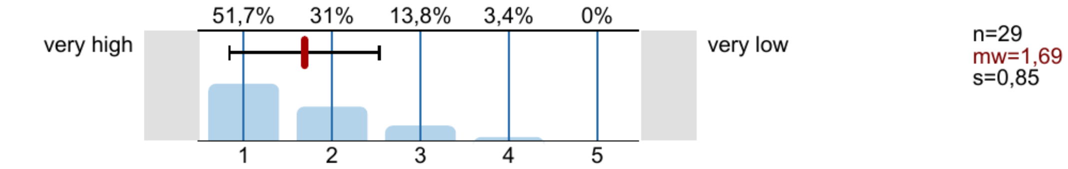
1.6) Contents



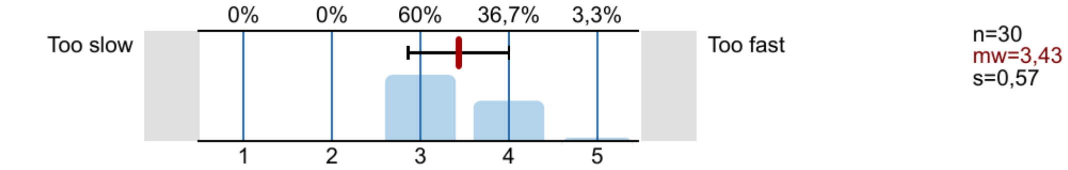
1.7) The matching with the lecture



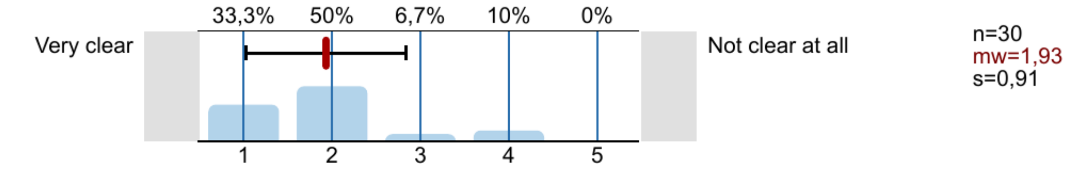
1.8) Knowledge gain



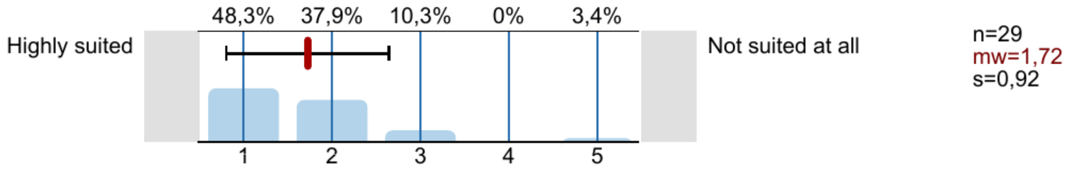
1.9) Speed



1.10) Clarity (using helpful examples)

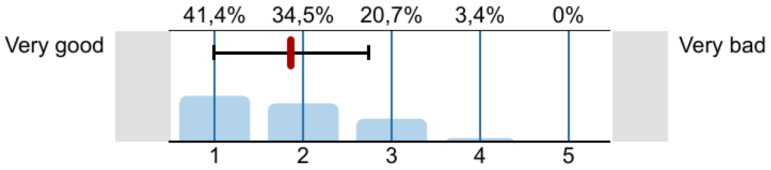


1.11) Quality of learning material



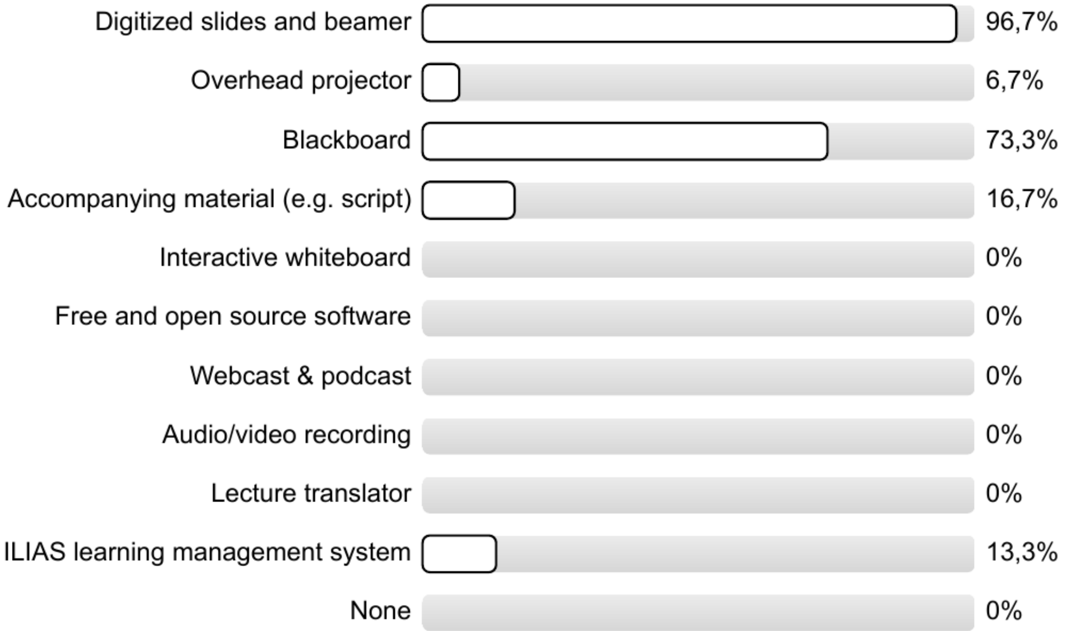
Evaluation

1.5) Coordination of the content with that of other courses of my studies plan is …



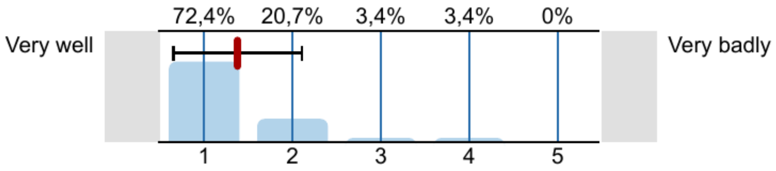
n=29
mw=1,86
s=0,88

1.12) Which aids (media) does the lecturer use to support teaching and learning? (multiple choice)



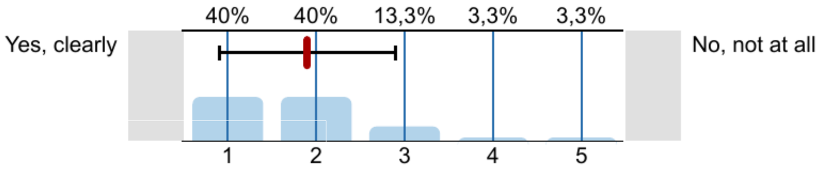
n=30

1.14) Did the lecturer know how to use the aids?



n=29
mw=1,38
s=0,73

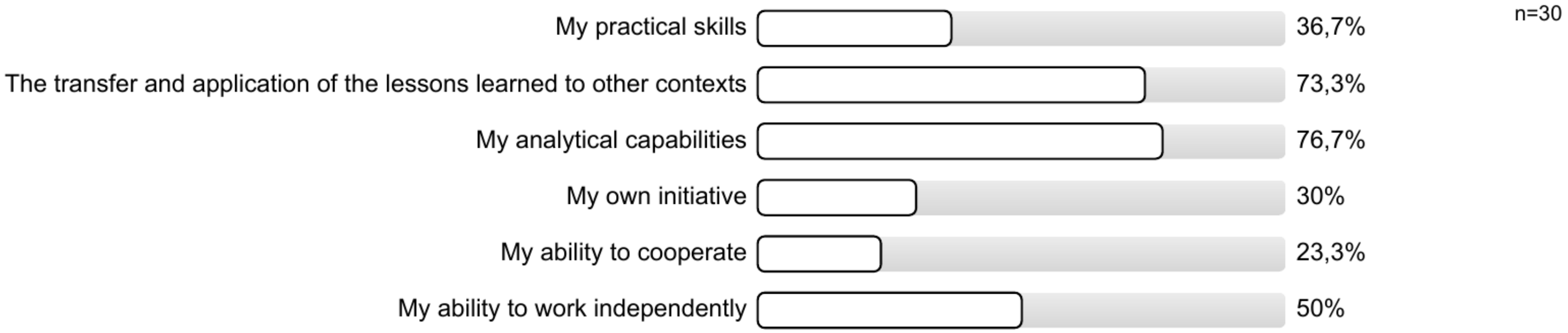
1.15) Do you see the relevance of the teaching content to your further studies?



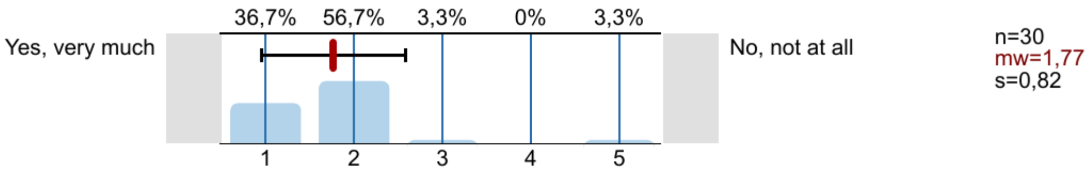
n=30
mw=1,9
s=0,99

Evaluation

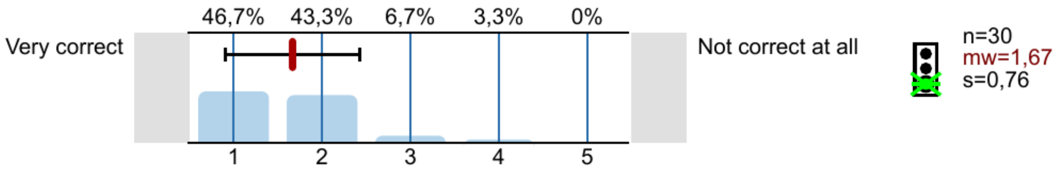
1.16) The course supports (multiple choice)



1.17) Do you benefit from the lecture?



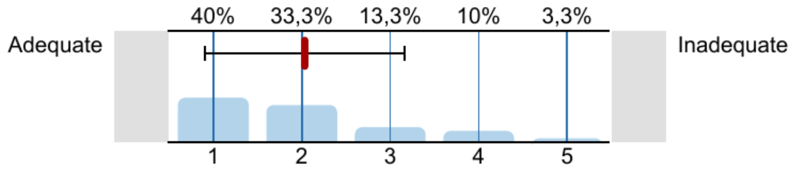
1.18) I learn a lot in this course.



Evaluation

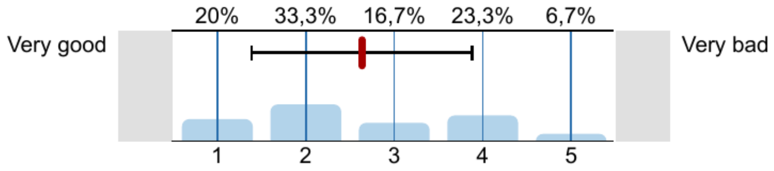
2. Questions Concerning the Venue

2.1) In comparison to the number of participants, the size of the room is



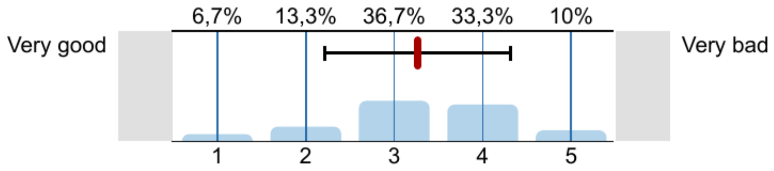
n=30
mw=2,03
s=1,13

2.2) Acoustic conditions are



n=30
mw=2,63
s=1,25

2.3) Visibility conditions are

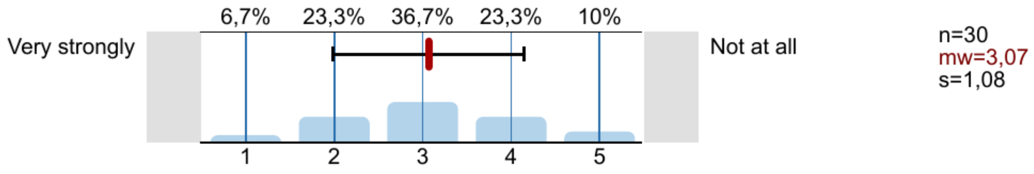


n=30
mw=3,27
s=1,05

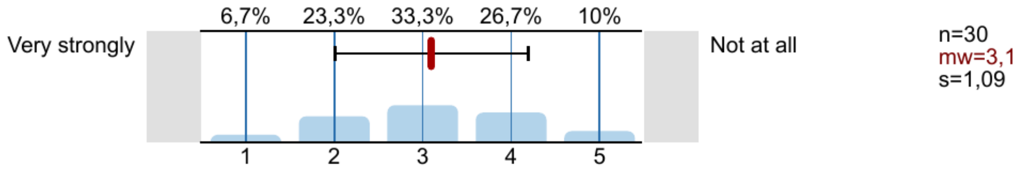
Evaluation

3. Questions Concerning the Lecturer

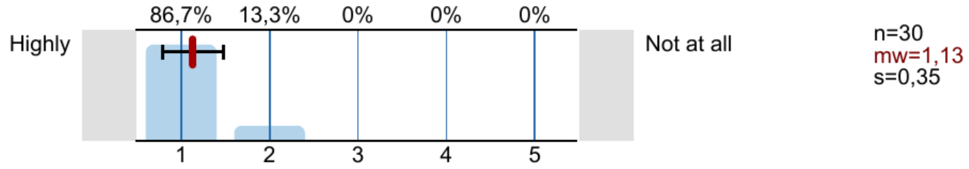
3.1) Does the lecturer refer to latest research activities?



3.2) Does the lecturer refer to correlations between theory and practice?

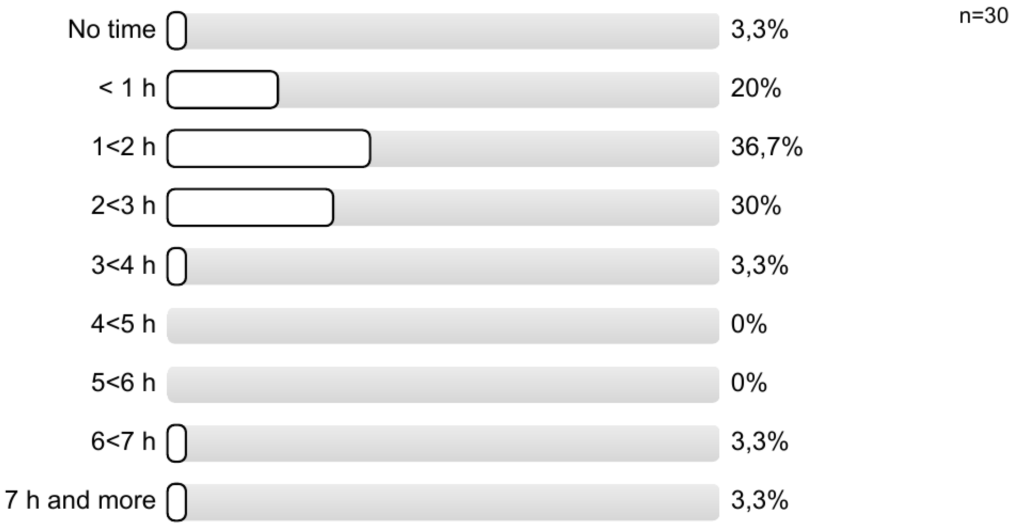


3.3) Does the lecturer appear competent during the exercise?



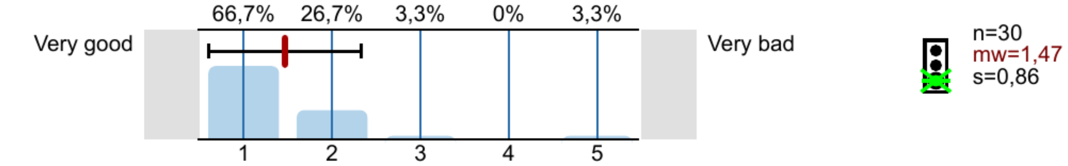
Evaluation

[4.5] How much time have you spent on the average per week for the preparation and follow-up of this course **(so far!)**?

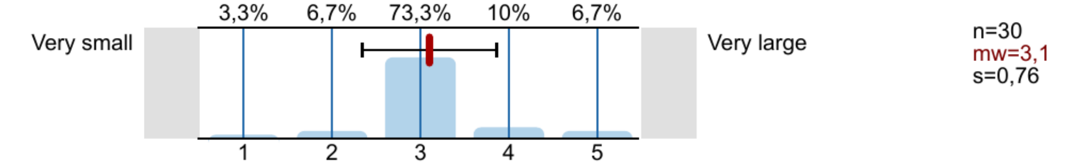


Evaluation

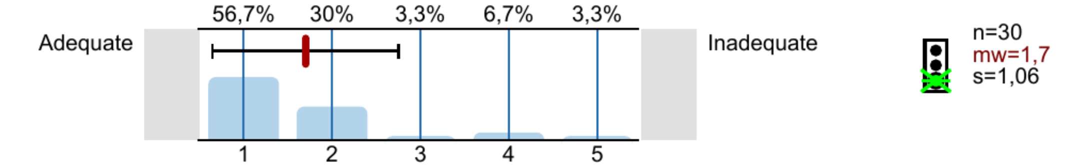
5.1) Please rate the course as a whole



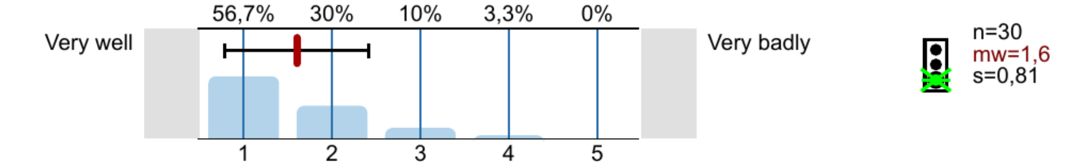
5.2) How large is the amount of work for this course?



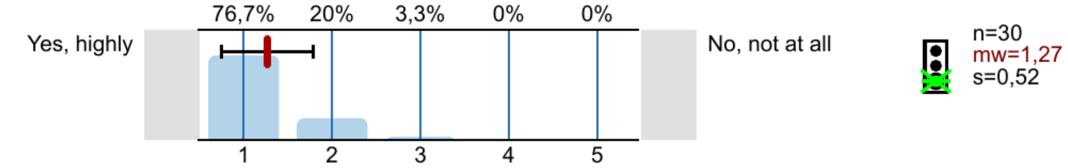
5.3) The amount of work required for this course is...



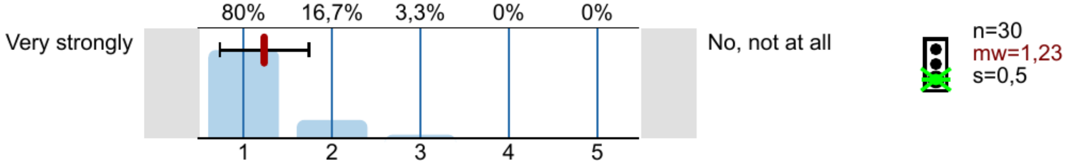
5.4) How is the course structured?



5.5) Does the lecturer appear dedicated and motivated during the course?



5.6) Is the lecturer responsive to questions and concerns of the students?



Evaluation

What you liked most:

- ++++++++++++ format of the exercise class, i. e., new problems in class, brief discussion of exercise sheet
- +++ reading a paper

What you did not like at all:

- -- the room
- -- exercises are too easy
- - no learning materials
- - problem class harder than sheet
- - course language
- - exercise sheet + new problems in class is too much
- - unclear how many problems one should be able to solve
- - exercises are too hard, presentation of solutions too short
- - irregular uploads of the exercise sheets
- - some exercises unrelated to lecture

Problems

(1) Prove that trees are comparability graphs.

(2) Prove that a graph has chromatic number at most k if and only if it admits a linear vertex order $\prec$ without a monotone path on $k+1$ vertices.


Path $(v_1, \ldots, v_{k+1})$ with $v_1 \prec \cdots \prec v_{k+1}$

(3) Let G be a graph. Recall: $\min\{\omega(G') \mid G' \supseteq G, G' \text{ is chordal}\} = \operatorname{tw}(G)$.

Find a characterization for $\min\{\omega(G') \mid G' \supseteq G, G' \text{ is comparability graph}\} = ?$.

(4) Look at the following paper: `https://dmtcs.episciences.org/14441/pdf`

- What is a poset and how is it related to comparability graphs?
- Familiarize yourselves with C-I graphs: definition, examples, non-examples, interesting properties
- Characterize the trees that are C-I graphs.
- Relate this paper to Problem Session 3.

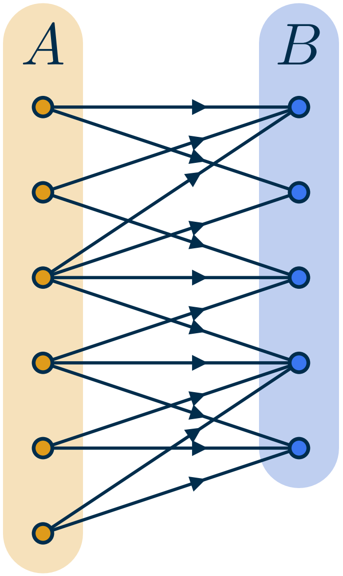
Trees and Comparability

bipartite graphs

Prove that ~~trees~~ are comparability graphs.

- Let G be a bipartite graph with bipartitions A and B.
- Set $F = E \cap (A \times B)$, i. e. orient every edge from A to B.
- Let $ab \in F$. Then, for all $v \in V(G)$ we have $bv \notin F$.

$\implies$ F is transitive and G is comparability graph

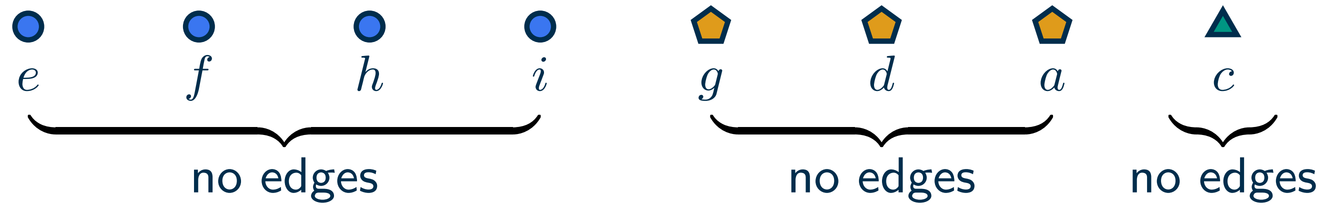


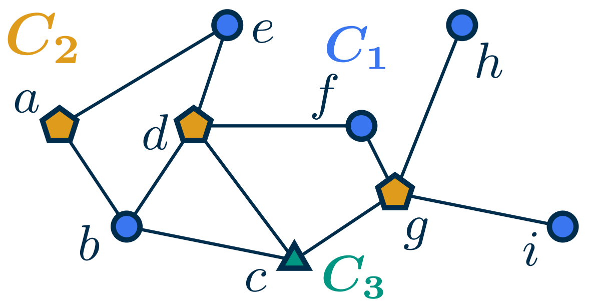
Monotone Paths and the Chromatic Number

Prove that for every graph G it holds that:
$\chi(G) \leq k \iff G$ admits vertex order σ s.t. the longest monotone path has length at most k

"$\Rightarrow$": Consider a partition $V(G) = C_1 \dot\cup \ldots, \dot\cup C_k$ into color classes (independent sets).

- Let σ be some ordering with $C_1 \prec_\sigma \cdots \prec_\sigma C_k$.





- Every path $(v_1, \ldots, v_\ell)$ in σ with $v_1 \prec_\sigma \cdots \prec_\sigma v_\ell$ contains at most one vertex of each C_i.

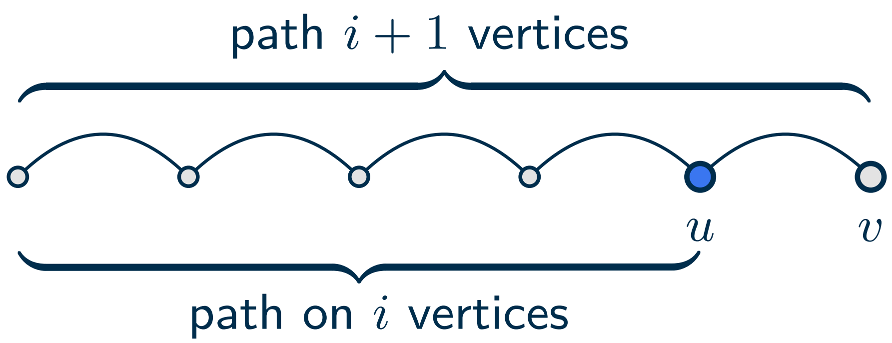
Monotone Paths and the Chromatic Number

Prove that for every graph G it holds that:
$\chi(G) \leq k \iff G$ admits vertex order σ s.t. the longest monotone path has length at most k

"$\Leftarrow$": Let σ be a such that the longest monotone path has length at most k

- Define the color of v as the length of the longest monotone path ending in v
- Clearly this uses at most k colors.
- Let $uv \in E(G)$, $u \prec_\sigma v$ with u being assigned color i.
- Then, v has color at least $i+1$.

$\implies$ coloring is proper.

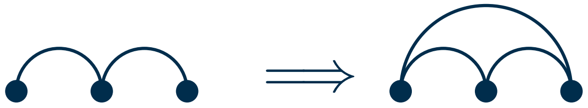


Comparability Supergraphs

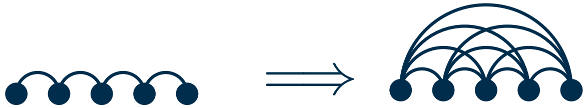
Let G be a graph. Recall: $\min\{\omega(G') \mid G' \supseteq G \text{ is chordal}\} = \operatorname{tw}(G)$.

Find a characterization for $\min\{\omega(G') \mid G' \supseteq G, G' \text{ is comparability graph}\} = ?$.

Comparability graphs admit vertex orderings such that



Thus monotone paths imply cliques



Note: "$\Longleftarrow$" also holds since cliques contain paths

We conclude: Let $\prec$ be a vertex ordering certifying that H is a comparability graph. Then the length of a longest monotone path in $\prec$ equals $\omega(H)$.

Does this sound familiar?

Recall: $\chi(H) \leq k \iff H$ admits vertex order s.t. longst monotone path has k vertices.

Idea: Prove that $\min\{\omega(G') \mid G' \supseteq G, G' \text{ is comparability graph}\} = \chi(G)$

Comparability Supergraphs

Idea: Prove that $\min\{\omega(G') \mid G' \supseteq G, G' \text{ is comparability graph}\} = \chi(G)$

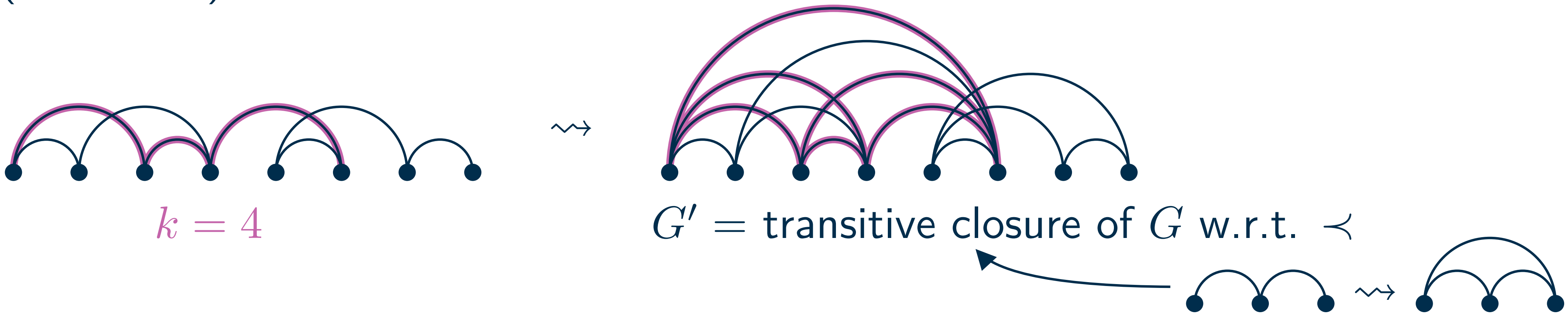
"$\geq$" Let
- G be a graph
- G' be comparability supergraph of G that minimizes $\omega(G')$

Then: $\chi(G) \leq \chi(G') = \omega(G')$ since G' is perfect

Comparability Supergraphs

Idea: Prove that $\min\{\omega(G') \mid G' \supseteq G, G' \text{ is comparability graph}\} = \chi(G)$

"$\leq$" Let $\prec$ be a vertex order of G such that the longest monotone path has length $k = \chi(G)$ (Exercise 2)



only transitive edges added $\implies$ maximal monotone paths in G' same as in G

clique on c vertices contains monotone path on c vertices $\implies \omega(G') \leq k = \chi(G)$

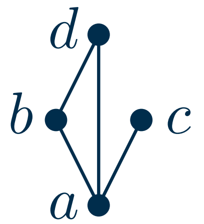
Posets and C-I Graphs

Section 2: Preliminaries
$\rightarrow$ contains all necessary definitions, usually brief and technical
$\rightarrow$ too be read only as needed (skip and come back if necessary)

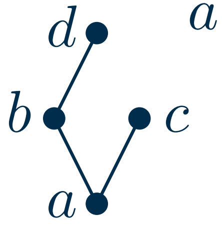
Poset $(P, \leq)$: Partially ordered set,
i.e., a (finite) set together with a partial order $\leq$

Example: $P = \{a, b, c, d\}$; $a \leq b \leq d$; $a \leq c$

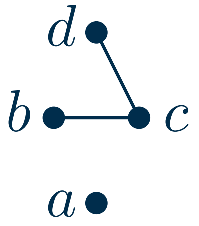
Comparability graph:
vertex set $= P$, distinct x, y adjacent iff x, y comparable, i.e., $x \leq y$ or $y \leq x$



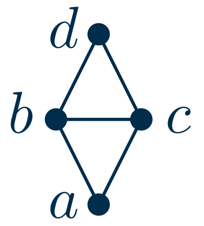
Cover graph $G(P)$: transitive reduction of the comparability graph,
i.e., remove transitive edges



Incomparability graph: complement of the comparability graph,
i.e., x, y adjacent iff incomparable

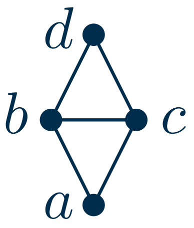


C-I graph: Union of the cover graph and the incomparability graph of a poset,
i.e., only the transitive edges are missing



C-I Graphs

C-I graph: Union of the cover graph and the incomparability graph of a poset, i.e., only the transitive edges are missing



- Lemmas 1,2,3, Theorem 1
- not hereditary
- connections to different subclasses of perfect graphs → introduction

C-I Trees

Claim: A tree is a C-I graph if and only if it is a path.

Theorem 3: For G chordal: G C-I graph $\iff$ G has at most 2 independent simplicial vertices.

- trees are chordal
- leaves are simplicial and independent
- the trees with at most two leaves are exactly the paths

$\rightarrow$ **Graph that is not a C-I graph:** tree with ≥ 3 leaves

C-I cographs

- Recall from Problem Session 3:
  Bretscher, Corneil, Habib, and Paul (2008) give a simplified LexBFS-based algorithm for recognizing cographs.
- used for recognizing C-I cographs