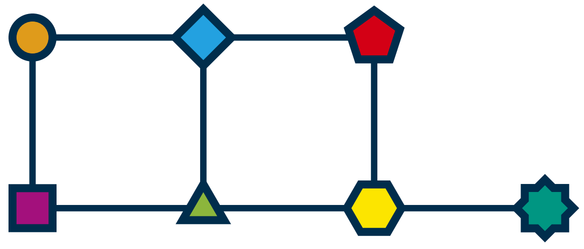
Algorithmic Graph Theory

Problem Session 4

Laura Merker and Samuel Schneider, June 4, 2025

Pathwidth

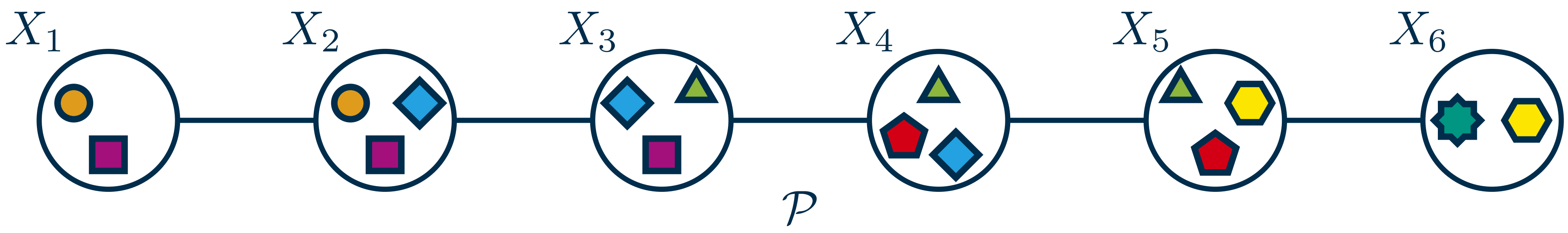
Let G be a graph.



Path decomposition:
A path $\mathcal{P} \coloneqq (v_1, \dots, v_r)$ with ***bags*** $X_1, \dots, X_r \subseteq V(G)$ such that:

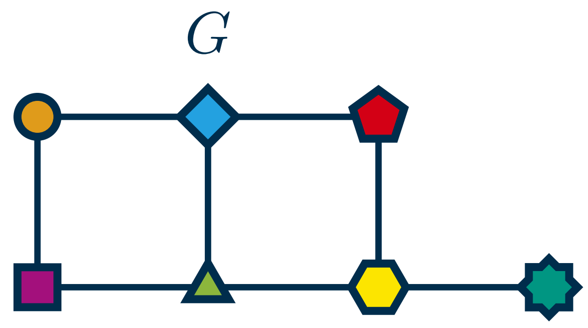
(i) $X_1 \cup \dots \cup X_r = V(G)$
(ii) $uv \in E(G) \Rightarrow u, v \in X_i$ for at least one bag X_i
(iii) for every vertex $v \in V(G)$ the graph induced by the bags containing v is connected

Width of a path decomposition: $\max\{|X_i| : i \in [r]\} - 1$
Pathwidth of G: minimal width of all path decompositions of G



~~Pathwidth~~ Treewidth

Let G be a graph.



Tree
~~Path~~ decomposition:

A ~~path $\mathcal{P}$~~ tree $\mathcal{T}$ $:= (v_1, \ldots, v_r)$ with ***bags*** $X_1, \ldots, X_r \subseteq V(G)$ such that:

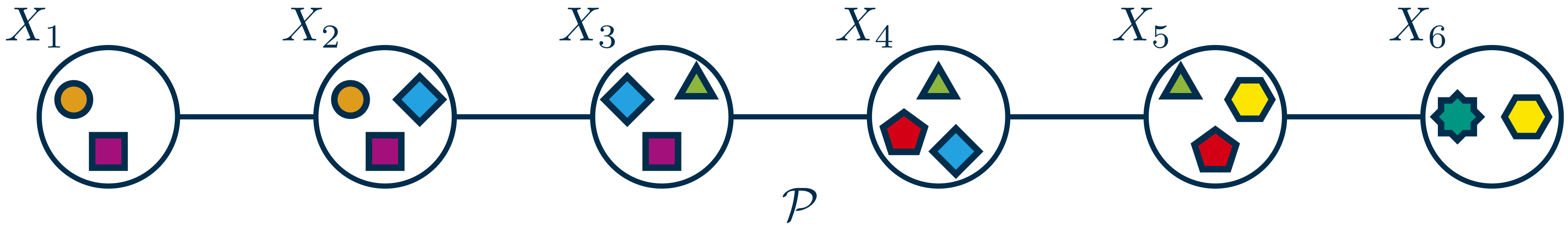
(i) $X_1 \cup \cdots \cup X_r = V(G)$

(ii) $uv \in E(G) \Rightarrow u, v \in X_i$ for at least one bag X_i

(iii) for every vertex $v \in V(G)$ the graph induced by the bags containing v is connected

Width of a ~~path~~ tree decomposition: $\max\{|X_i| : i \in [r]\} - 1$

~~Pathwidth~~ Treewidth of G: minimal width of all ~~path~~ tree decompositions of G



Evaluation

https://onlineumfrage.kit.edu/evasys/online.php?p=L3ZK6

- Do you like the format of the excercise class?
  - Would you prefer less/more focus on discussing the excercise sheets?
- Would you prefer easier/harder problems?

Problems

(1) Prove that a graph G is an interval graph if and only if it admits a path decomposition such that each bag is a clique in G.

(2) Find an algorithm for computing the treewidth of chordal graphs.

(3) Prove that the treewidth of a graph is at most k if and only if it is a partial k-tree.

subgraph of a k-tree

(4) Find fun facts on (tw, ω)-boundedness: definition, examples, sufficient conditions, related parameters, related exercises, algorithmic implications, open questions, ...

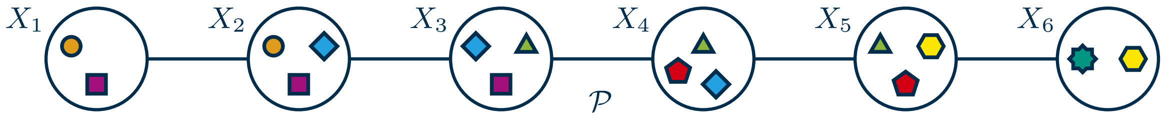
Path (tree) decomposition:

A path (tree) $\mathcal{P} := (v_1, \ldots, v_r)$ with ***bags*** $X_1, \ldots, X_r \subseteq V(G)$ such that:

(i) $X_1 \cup \cdots \cup X_r = V(G)$

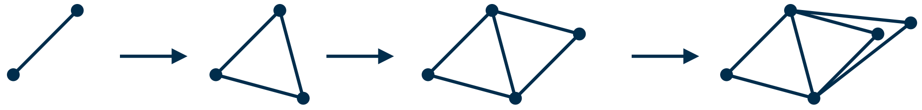
(ii) $uv \in E(G) \Rightarrow u, v \in X_i$ for at least one bag X_i

(iii) for every vertex $v \in V(G)$ the graph induced by the bags containing v is connected



Width of decomposition: $\max\{|X_i| : i \in [r]\} - 1$ **Pathwidth (treewidth) of G:** min. over all decompositions

k-tree: obtained from a K_k by iteratively adding degree-k vertices to some k-clique



Interval graphs and path decompositions

Prove that a graph G is an interval graph if and only if it admits a path decomposition such that each bag is a clique in G.

"$\Rightarrow$":

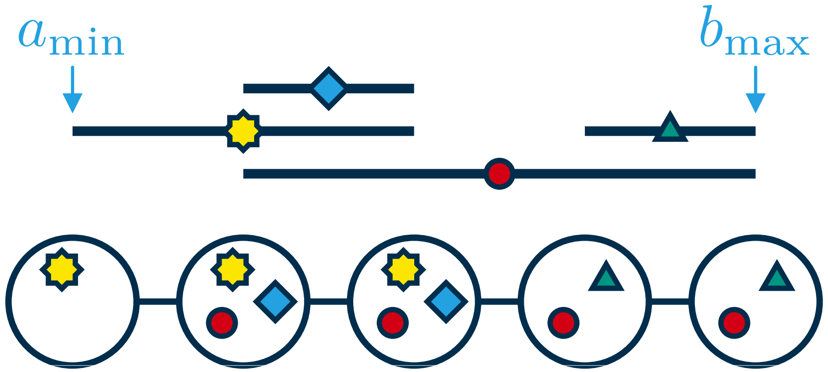
- Let G be the interval graph corresponding to the intervalls $I_v = [a_v, b_v] \subset \mathbb{N}$ with $v \in V(G)$.
- Let $a_{\min} = \min\{a_v : v \in V(G)\}$ and $b_{\max} = \max\{b_v : v \in V(G)\}$.
- We construct a path decomposition $\mathcal{P} := (v_{a_{\min}}, \dots, v_{b_{\max}})$ with bags $X_{a_{\min}}, \dots, X_{b_{\max}}$
- Let $v \in V(G)$ with $I_v = [a_v, b_v]$. Then, we add v to every bag X_i with $a_v \leq i \leq b_v$.

Claim: Every bag is a clique.

- Let $u, v \in X_i$ for some $a_{\min} \leq i \leq b_{\max}$.
- Then, $i \in I_u \cap I_v$ and thus $uv \in E(G)$.

Claim: Every $uv \in E(G)$ is in some bag X_i*

- There is some $i \in I_u \cap I_v$ and thus $u, v \in X_i$



*this is property (ii) of path decompositions, property (i) and (iii) clearly hold.

Interval graphs and path decompositions

Prove that a graph G is an interval graph if and only if it admits a path decomposition such that each bag is a clique in G.

"$\Leftarrow$":

- Let $\mathcal{P} := (v_1, \ldots, v_r)$ with bags $X_1, \ldots, X_r$ be a path decomposition of G with all X_i cliques.
- We construct intervals $I_v = [a_v, b_v]$ for all $v \in V(G)$.
- For every vertex $v \in V(G)$ let $X_a^v, \ldots, X_b^v$ be the bags that contain v. We set $I_v = [a, b]$.
  - Note that $[a, b]$ is an interval as the bags containing v induce a connected graph in $\mathcal{P}$

Claim: G is the interval graph of these intervals.

- As every bag is a clique two distinct vertices share a bag if and only if they are adjacent.
- Two intervals I_u and I_v intersect if and only if u and v share a bag.

$\Rightarrow$ $uv \in E(G)$ if and only if I_u and I_v intersect.

Treewidth in chordal graphs

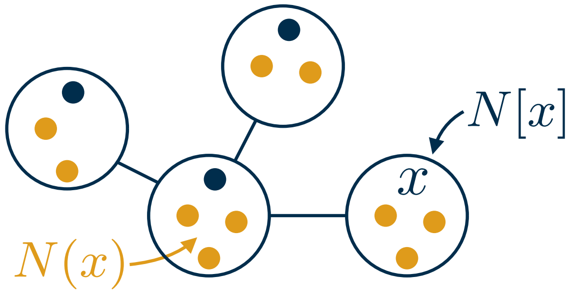
Claim: For every chordal graph G, we have $\mathrm{tw}(G) = \omega(G) - 1$.

$\mathrm{tw}(K_n) = n - 1 \implies \mathrm{tw}(G) \geq \omega(G) - 1$ for all graphs G (chordality not needed)

By induction: $\mathrm{tw}(G) \leq \omega(G) - 1$ and for every clique C there is a bag containing C
Base case: $\mathrm{tw}(K_1) = 0$

Recall (lecture): Every chordal graph has a simplicial vertex x.
$G - x$ admits a tree decomposition of width $\omega(G - x) - 1$ by induction
$N(x)$ is a clique (since x is simplicial) $\implies$ there is a bag B containing $N(x)$ by induction

Case 1: $\omega(G) > \omega(G - x) \to$ add x to B $\to$ increases width by 1 ✓
Case 2: $\omega(G) = \omega(G - x)$
$\to$ add new bag $N[x]$ of size $\omega(G)$ adjacent to B $\to$ width $\omega(G) - 1$



In both cases: For all cliques C containing x: $C \subseteq N[x] \to$ contained in new/extended bag

Treewidth in chordal graphs

Claim: For every chordal graph G, we have $\operatorname{tw}(G) = \omega(G) - 1$.

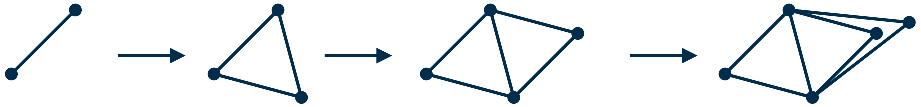
Algorithm to compute the treewidth in chordal graphs:

compute clique number (LexBFS, PES), subtract 1

Treewidth and k-trees

Prove that the treewidth of a graph is at most k if and only if it is a partial k-tree.

k-tree: obtained from a K_k by iteratively adding degree-k vertices to some k-clique



"$\Leftarrow$"

Observation: Treewidth is ***monotone***, i.e., $G \subseteq H \implies \mathrm{tw}(G) \leq \mathrm{tw}(H)$
$\rightarrow$ we only need to consider non-partial k-trees

- Sheet 3, Exercise 6: k-trees are chordal
- Just shown (previous exercise): G chordal $\implies \mathrm{tw}(G) = \omega(G) - 1$
- $\omega(k\text{-tree}) \leq k + 1 \implies \mathrm{tw}(k\text{-tree}) \leq k$

Treewidth and k-trees

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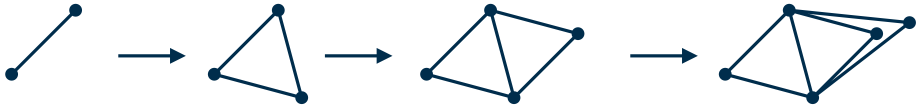


"$\Rightarrow$"

Find construction sequence by induction on the tree:

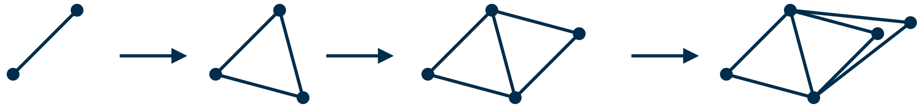
- **Base case:** If $|V(G)| \leq k+1$ we are done as $G \subseteq K_{n+1}$ and K_{n+1} is a k-tree
- Assume $|V(G)| > n+1$ and let $\mathcal{T}$ be a tree decomposition with bags $X_1, \ldots, X_r$ and width k.
- We assume w. l. o. g. that $|X_i| = k+1$ for all X_i and that every X_i is a clique in G.

adding edges allowed since we only need a ***partial*** k-tree

Treewidth and k-trees

Prove that the treewidth of a graph is at most k if and only if it is a partial k-tree.

k-tree: obtained from a K_k by iteratively adding degree-k vertices to some k-clique



"$\Rightarrow$"

Find construction sequence by induction on the tree:

- **Base case:** If $|V(G)| \leq k+1$ we are done as $G \subseteq K_{n+1}$ and K_{n+1} is a k-tree
- Assume $|V(G)| > n+1$ and let $\mathcal{T}$ be a tree decomposition with bags $X_1, \ldots, X_r$ and width k.
- We assume w. l. o. g. that $|X_i| = k+1$ for all X_i and that every X_i is a clique in G.
- Let X_i be a leaf bag, X_j be its parent and $v \in X_i - X_j$. $\Rightarrow N[v] = X_i$
- By induction $G - v$ is a partial k-tree.
- As $N(v) = X_i - v$ is a k-clique, G is a partial k-tree.

(tw, ω)-Boundedness

Definition:

A graph class $\mathcal{G}$ is called $(\mathbf{tw}, \boldsymbol{\omega})$-***bounded*** if $\exists$ function f s.t. $\mathrm{tw}(G) \leq f(\omega(G))$ for all $G \in \mathcal{G}$

Some relevant papers:

[1] Dallard, Milanič, Štorgel, 2021, https://doi.org/10.1137/20M1352119

[2] Dallard, Milanič, Štorgel, 2023, https://doi.org/10.1016/j.jctb.2023.10.006

[3] Chaplick, Töpfer, Voborník, Zeman, 2021, https://doi.org/10.1007/s00453-021-00846-3

. . . and follow-up papers (use “cited by”-function by Journals / Google Scholar)

Examples / sufficient conditions:

- chordal graphs: $\mathrm{tw}(G) \leq \omega(G) - 1$ (see Exercise **(3)**)
- graphs with bounded tree-independence number [2]
- line graphs of bounded-treewidth graphs [2]

Algorithmic implications:

- k-clique, k-list coloring polynomially solvable [3]

Open Problems:

Is Independent Set solvable in polynomial time for (tw, ω)-bounded graph classes?