Karlsruhe Institute of Technology

# Algorithmic Graph Theory

Problem Class 2 | 7 May 2025

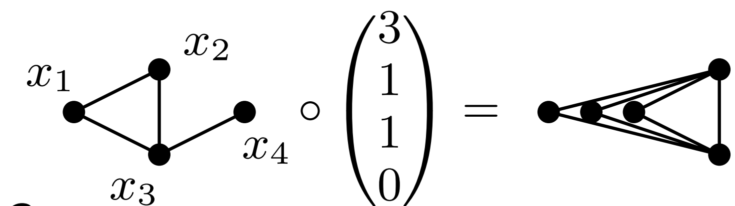
Laura Merker, Samuel Schneider

# Update from the faculty

- exams may be taken in English or German

Graph $G$ on vertices $x_1, \dots, x_n$, tuple $h = (h_1, \dots, h_n)$ of nonnegative integers
Recall $H = G \circ h$: $\quad V(H) = \{x_i^1, \dots, x_i^{h_i} \mid i \in [n]\}$, $\;E(H) = \{x_i^a x_j^b \mid x_i x_j \in E(G), a \in [h_i], b \in [h_j]\}$

**New definition:** $H = G \ominus h$ with
$V(H) = \{x_i^1, \dots, x_i^{h_i} \mid i \in [n]\}$,
$E(H) = \{x_i^a x_j^1 \mid x_i x_j \in E(G), a \in [h_i], h_j > 0\}$




**(1)** What is the difference between $\circ$ and $\ominus$?

**(2)** Can $\circ$, resp. $\ominus$, be realized by elementary operations?
I.e., is there a sequence $h^1, h^2, \dots$ of tuples, each with all entries 1 except for one 0 or 2, such that $G * h = G * h^1 * h^2 * \dots$ for each $* \in \{\circ, \ominus\}$? If so, does the order matter?

**(3)** Find a largest cycle, a largest induced cycle, $\omega, \chi, \alpha$, and $\kappa$ of $K_n \ominus (2, \dots, 2)$.

**(4)** Prove or disprove: If $G$ is perfect, then ...

- $\omega(G \ominus h) = \chi(G \ominus h)$ (follow a proof of the lecture)
- $G \ominus h$ is perfect

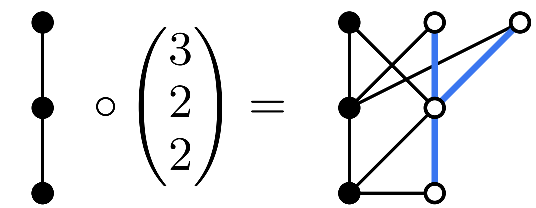
Let $G_0 = K_2$ and $H_i = G_{i-1} \ominus (2, \dots, 2)$, $G_i = H_i + u + \{uv \mid v \in V(H_i) - V(G_{i-1})\}$, where $u$ is a new vertex, $i \geq 1$

**(5)** Prove that $G_1, G_2, \dots$ are not perfect.

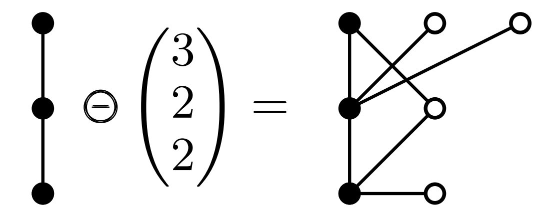
**(6)** Prove that $G_0, G_1, \dots$ are "far from perfect". For this, find $\omega(G_i)$ and $\chi(G_i)$, $i \geq 0$.

# What is the difference?

Graph $G$ on vertices $x_1, \dots, x_n$, tuple $h = (h_1, \dots, h_n)$ of nonnegative integers
Recall: $H = G \circ h$ defined as
$V(H) = \{x_i^1, \dots, x_i^{h_i} \mid i \in [n]\}$
$E(H) = \{x_i^a x_j^b \mid x_i x_j \in E(G), a \in [h_i], b \in [h_j]\}$



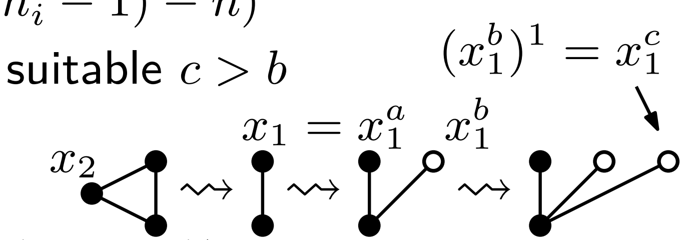
**New definition:** $H = G \ominus h$ with
$V(H) = \{x_i^1, \dots, x_i^{h_i} \mid i \in [n]\}$
$E(H) = \{x_i^a x_j^1 \mid x_i x_j \in E(G), a \in [h_i], h_j > 0\}$



**Claim:** $(G \circ h) \circ h' = G \circ (h + h' - (1, \dots, 1))$ (padding $h'$ with 1's as necessary)

so $G \circ (3, 0, 1) = G \circ (2, 0, 1) \circ (2, 1) = G \circ (1, 0, 1) \circ (2, 1) \circ (2, 1, 1)$

- same number of vertices: $n + \sum(h_i - 1) + \sum(h_i' - 1) = n + (\sum(h_i + h_i' - 1) - n)$
- Identify vertices: Denote twins (due to $h'$) of $x_i^b \in V(G \circ h)$ by $x_i^c$ with suitable $c > b$
- edges with both endpoints in $G \circ h$: definitions coincide
- $x_i^a x_i^c \in E(G \circ h \circ h')$ with $x_j^c$ twin of some $x_j^b$ in $G \circ h$
  $\implies x_i^a x_j^b \in E(G \circ h) \implies x_i x_j \in E(G) \implies x_i^a x_j^c \in E(G \circ h + h' - (1, \dots, 1))$
- other direction analogous

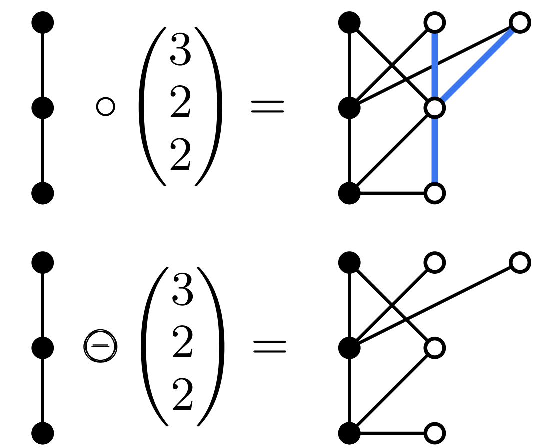



$\implies$ elementary operations $G \circ x_i$ and $G - x_i$ suffice (in any order)

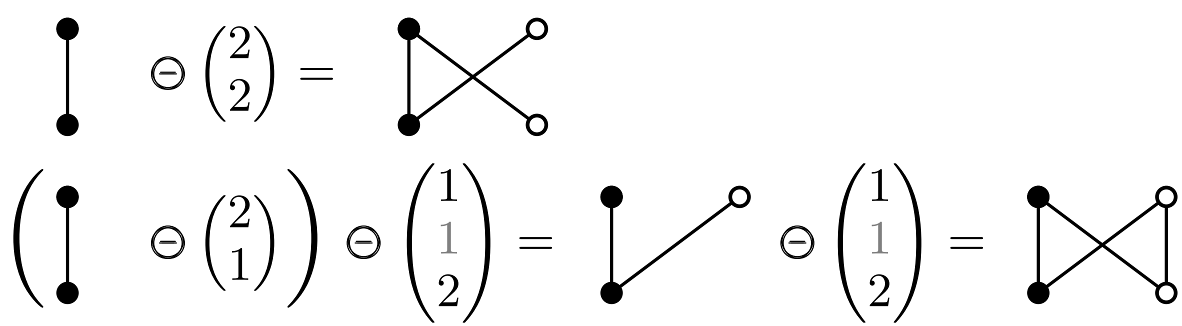
# What is the difference?

Graph $G$ on vertices $x_1, \ldots, x_n$, tuple $h = (h_1, \ldots, h_n)$ of nonnegative integers
Recall: $H = G \circ h$ defined as
$V(H) = \{x_i^1, \ldots, x_i^{h_i} \mid i \in [n]\}$
$E(H) = \{x_i^a x_j^b \mid x_i x_j \in E(G), a \in [h_i], b \in [h_j]\}$

**New definition:** $H = G \ominus h$ with
$V(H) = \{x_i^1, \ldots, x_i^{h_i} \mid i \in [n]\}$
$E(H) = \{x_i^a x_j^1 \mid x_i x_j \in E(G), a \in [h_i], h_j > 0\}$



Does the same work for $\ominus$? $\rightarrow$ **No!**
What about other sequences?



What in the proof for $\circ$ fails for $\ominus$?

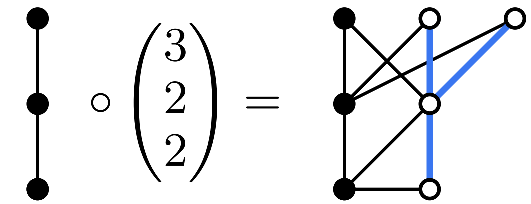
**Claim:** The graphs obtained form $K_2$ by elementary $\ominus$-replications are exactly the complete bipartite graphs. $\rightarrow$ no $P_4$

**Therefore:** single operation for all new vertices necessary for new vertices with $\ominus$

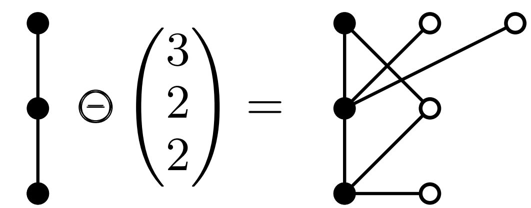
# What is the difference?

Karlsruhe Institute of Technology

Graph $G$ on vertices $x_1, \dots, x_n$, tuple $h = (h_1, \dots, h_n)$ of nonnegative integers
Recall: $H = G \circ h$ defined as
$V(H) = \{x_i^1, \dots, x_i^{h_i} \mid i \in [n]\}$
$E(H) = \{x_i^a x_j^b \mid x_i x_j \in E(G), a \in [h_i], b \in [h_j]\}$



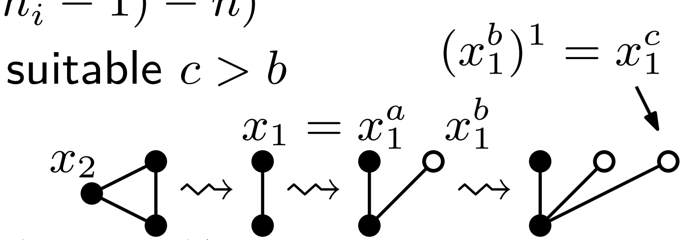
**New definition:** $H = G \ominus h$ with
$V(H) = \{x_i^1, \dots, x_i^{h_i} \mid i \in [n]\}$
$E(H) = \{x_i^a x_j^1 \mid x_i x_j \in E(G), a \in [h_i], h_j > 0\}$



Does the same work for $\ominus$?
What about other sequences?

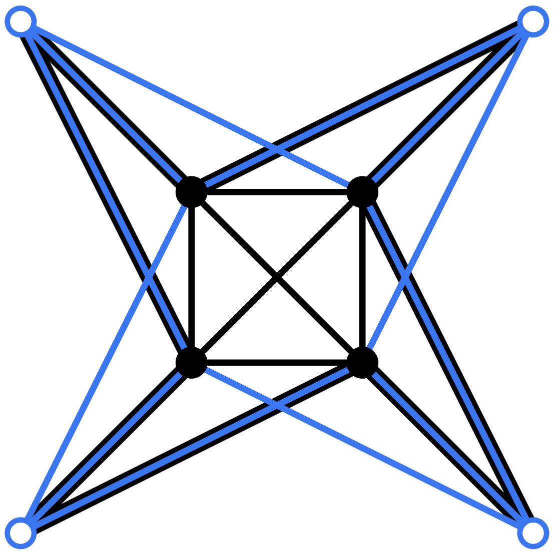
What in the proof for $\circ$ fails for $\ominus$?

- same number of vertices: $n + \sum(h_i - 1) + \sum(h_i' - 1) = n + (\sum(h_i + h_i' - 1) - n)$
- Identify vertices: Denote twins (due to $h'$) of $x_i^b \in V(G \circ h)$ by $x_i^c$ with suitable $c > b$
- edges with both endpoints in $G \circ h$: definitions coincide
- $x_i^a x_i^c \in E(G \circ h \circ h')$ with $x_j^c$ twin of some $x_j^b$ in $G \circ h$
  $\implies x_i^a x_j^b \in E(G \circ h)$ $\boxed{\not\Leftarrow}$ $x_i x_j \in E(G) \implies x_i^a x_j^c \in E(G \circ h + h' - (1, \dots, 1))$
- other direction analogous




$\implies$ elementary operations $G \circ x_i$ and $G - x_i$ suffice (in any order)

# $K_n \ominus (2, \ldots, 2)$



- Hamilton cycle (length $2n$)
- largest induced cycle: triangle
  - larger induced cycle contains $\leq 2$ black vertices (clique)
  - all black vertices are consecutive (clique)
  - blue vertices not adjacent $\implies$ only one blue vertex
- $\omega = \chi = \alpha = \kappa = n$

# Perfect?

**Lemma (lecture):** If $G$ is perfect, then $G \circ h$ is perfect.

**Goal:** adapt proof for $\ominus$

**Observation:** If $h$ is a 0-1-tuple, then $G \ominus h = G \circ h \subseteq_{\text{ind}} G$

Let $h'$ be such that $h'_i = \begin{cases} 1 & \text{if } h_i > 0 \\ 0 & \text{if } h_i = 0 \end{cases}$ and $G' = G \ominus h' \subseteq_{\text{ind}} G$

Since $G$ is perfect, we have $\omega(G') = \chi(G')$.

- new vertices in $G \ominus h$ form an independent set $\implies \leq 1$ new vertex in every clique
  If new vertex $x_i^a$ in largest clique $C$, then $C - x_i^a + x_i^1 \subseteq G'$ is clique of same size.
  $\implies \omega(G \ominus h) = \omega(G')$
- Observe: $N(x_i^a) \subseteq N(x_i)$ for each twin $x_i^a$ of $x_i$
  Let $c' : V(G') \to [\chi(G')]$ be a proper coloring.
  Now $c(x_i^a) = c'(x_i)$ is a proper coloring of $G \ominus h$.
  $\implies \chi(G \ominus h) = \chi(G')$

So, $\omega(G \ominus h) = \omega(G') = \chi(G') = \chi(G \ominus h)$

# Perfect?

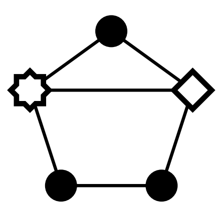
**Lemma (lecture):** If $G$ is perfect, then $G \circ h$ is perfect.

**Goal:** adapt proof for $\ominus$

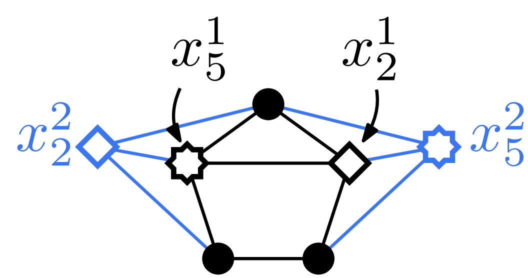
**Observation:** If $h$ is a 0-1-tuple, then $G \ominus h = G \circ h \subseteq_{\text{ind}} G$

**Have:** $\omega(G \ominus h) = \chi(G \ominus h)$

**But what about induced subgraphs?**



$\ominus \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \\ 2 \end{pmatrix} =$

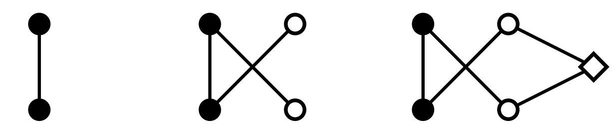



perfect

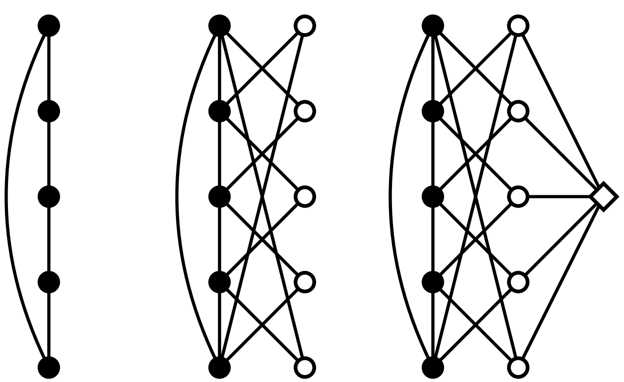
not perfect

# Far from perfect (Mycielski 1955)

Let $G_0 = K_2$ and $H_i = G_{i-1} \ominus (2, \dots, 2)$, $G_i = H_i + u + \{uv \mid v \in V(H_i) - V(G_{i-1})\}$, where $u$ is a new vertex, $i \geq 1$

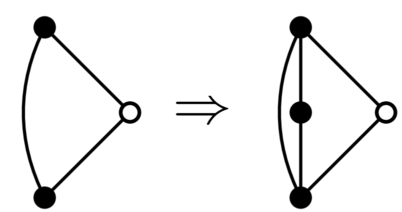


$G_0$ $H_1$ $G_1$



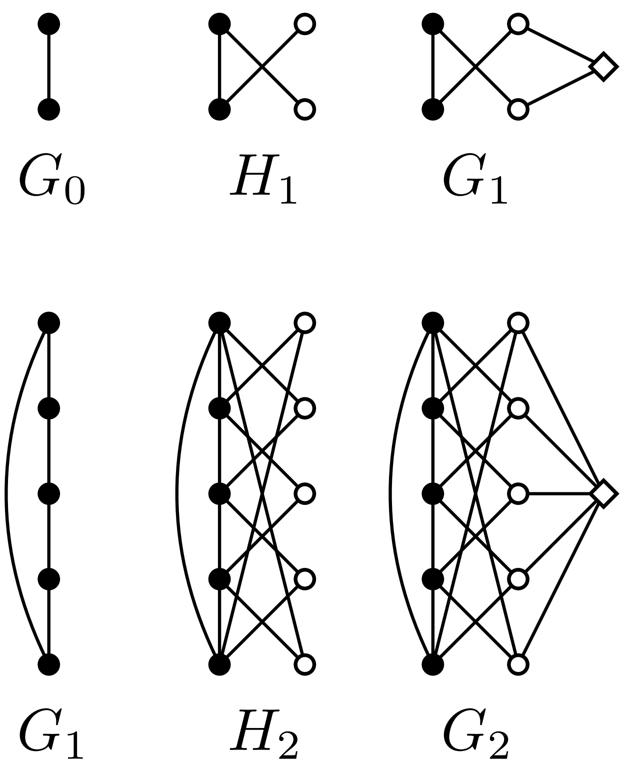
$G_1$ $H_2$ $G_2$

- $G_0 = K_2$ is perfect
- $G_i, i > 0$ contain induced $C_5 \implies$ not perfect
- $\omega(G_i) = 2$ for all $i$ by induction:
  - triangle does not contain $\diamond$
  - only one $\circ$
  - so at least two $\bullet$, but they form triangle with twin of chosen $\circ$
  - contradiction: $\bullet$ do not form a triangle by induction



# Far from perfect (Mycielski 1955)

Let $G_0 = K_2$ and $H_i = G_{i-1} \ominus (2,\dots,2)$, $G_i = H_i + u + \{uv \mid v \in V(H_i) - V(G_{i-1})\}$, where $u$ is a new vertex, $i \geq 1$




- $\chi(G_i) \geq i+2$ by induction:
  - let $k = i+1$
  - let $c$ be a $k$-coloring that minimizes the number of vertices colored in the same color as $\diamond$
  - w. l. o. g $c(\diamond) = k$
  - If there is a $\bullet$-vertex $x_i$ colored with $k$, then its neighborhood contains all other colors (choice of $c$)
  - thus, the neighborhood of its twin $x_i^2$ contains all colors $1,\dots,k-1$ and $c(x_i^2) = k$
  - not a proper coloring
  - so the $\bullet$-vertices admit a $(k-1)$-coloring, contradiction to induction
- $\chi(G_i) \leq i+2$: copy colors for $\circ$ and use new color for $\diamond$