Karlsruhe Institute of Technology

Algorithmic Graph Theory

Problem Class 1 | 30. April 2025

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Discord

Problem Classes

- problem solving in class (new problems not on exercise sheets)
- discussion of exercise sheets
  (published at least one week beforehand and to be solved at home):
  `https://i11www.iti.kit.edu/teaching/sommer2025/algorithmic_graph_theory`
- language: English
  (if you feel uncomfortable with English, you may ask questions in German)
- ~~the oral exam will be in English~~
  you can choose between German and English for the oral exam

How to contact us:

- use first names (in German “du”)
- `laura.merker2@kit.edu`, `samuel.schneider@kit.edu`
- Discord channel (optional): `https://discord.gg/pQ7WDfhwv8`

Any other suggestions / questions?

Show that...

(1) There exist graphs G and G' with: **(a)** $\alpha(G) = \kappa(G)$ but $\omega(G) < \chi(G)$
(b) $\omega(G') = \chi(G')$ but $\alpha(G') < \kappa(G')$

(2) For every graph G the following are equivalent:

- G is bipartite
- every cycle in G is even (i. e. the length is even)
- every induced cycle in G is even

(3) For every bipartite graph, the size of the largest matching equals the size of the smallest vertex cover. *Hint: Find a property of bipartite graphs connected to the lecture.*

Matching: subgraph of maximium degree 1, i.e. a set of independent edges
Vertex Cover: vertex set C such that every edge has at least one endpoint in C

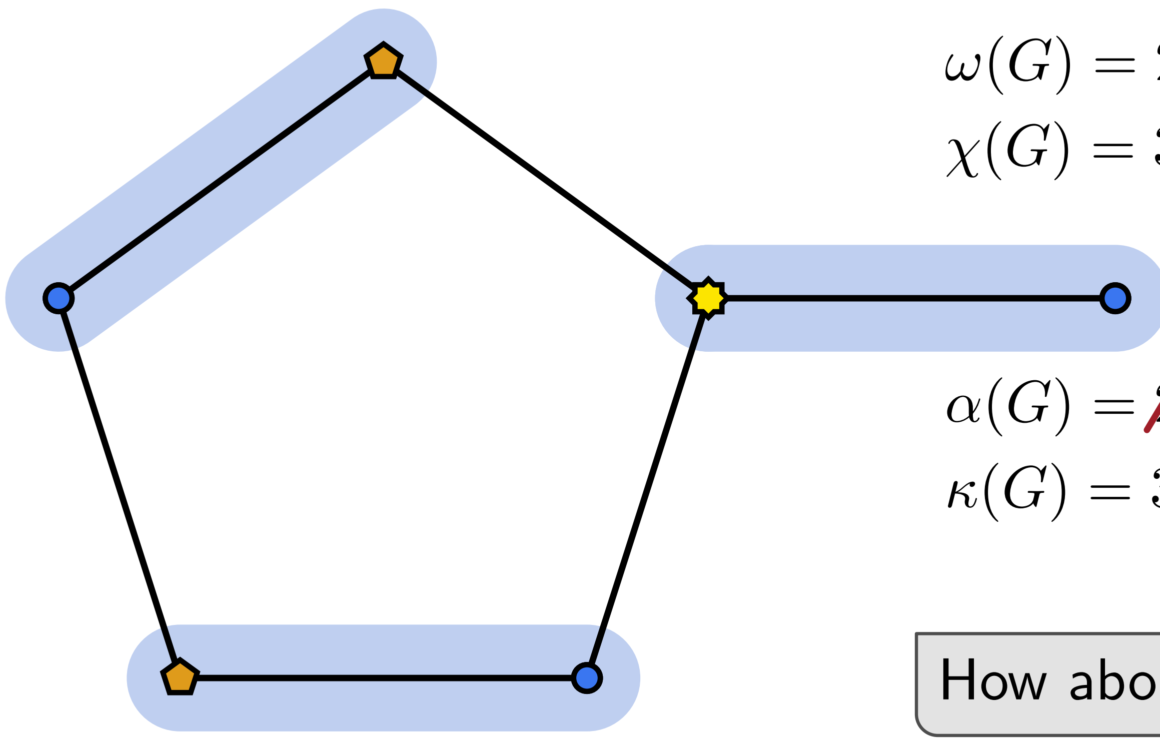
(4) For every $k \geq 1$, construct a bipartite graph on $2k$ vertices and a vertex ordering such that a greedy coloring uses at least k colors.

(5) Every connected graph G on n vertices with minimum degree δ contains a path on $\min(2\delta, n)$ vertices.

Somewhat perfect

We need a graph that is not perfect:

G



$\omega(G) = 2$ (no triangle)

$\chi(G) = 3$ (not 2, cycle is odd)

$\Rightarrow \omega(G) < \chi(G)$ ✓

$\alpha(G) =$ ~~2~~ 3 ~~(not 3 because $\alpha(K_2) = 1$)~~

$\kappa(G) = 3$ (not 2 because no triangle)

$\alpha(G) = \kappa(G)$ ✓

How about $\omega(G') = \chi(G')$, $\alpha(G') < \kappa(G')$?

→ $G' := \overline{G}$

Goal: increase independence number ✓

Doesn't this contradict the *Weak Perfect Graph Theorem*?

No: C_5 as induced subgraph with $\alpha(C_5) = 2 < 3 = \kappa(C_5)$

G is bipartite $\iff$ every cycle in G is even

"$\Rightarrow$": Let G be a bipartite graph. We show that G contains no odd cycle

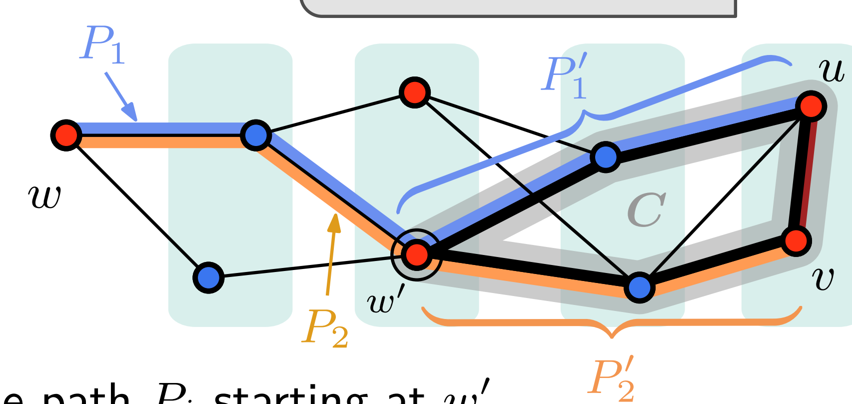
- Assume that G contains an odd cycle C
- C is not 2-colorable $\implies$ G is not 2-colorable and therefore not bipartite ↯

"$\Leftarrow$": Let G be a graph that contains no odd cycle. We show that G is bipartite.

- Assume w. l. o. g. that G is connected (otherwise we argue on every component of G).
- Choose a $w \in V$.
- Color all vertices with even distance to w red and all other vertices blue.
- Assume there exist two adjacent vertices u und v with the same color

We show: This coloring is a proper 2-coloring

Goal: Find an odd cycle in G for a contradiction



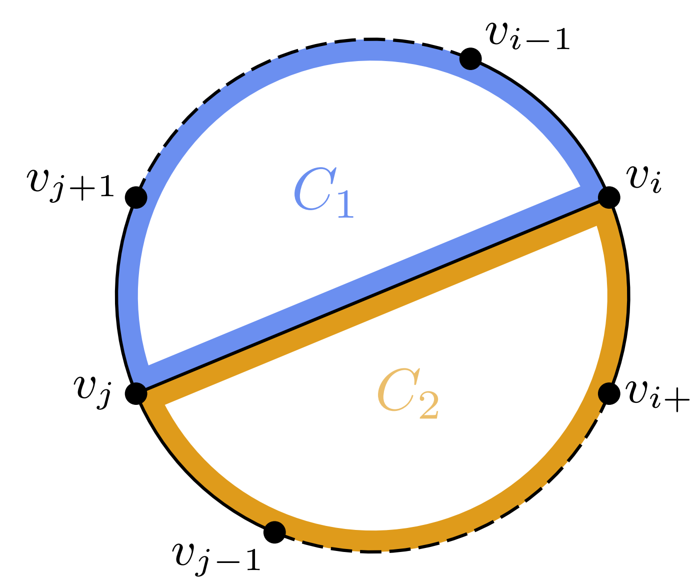
- Look at a shortest path P_1 from w to u and a shortest path P_2 from w to v.
- Let w' the last shared vertex of P_1 and P_2 and let P_i' be the path P_i starting at w'.
- Then, P_1', uv, P_2' is a cycle with length $\underbrace{|E(P_1')| + |E(P_2')|}_{\text{even}} + 1 \rightarrow$ odd ↯

all cycles even $\iff$ all induced cycles even

"$\Rightarrow$": obvious

"$\Leftarrow$": Let G be a graph with an odd cycle. We show that G contains an induced odd cycle.

- Let $C := (v_1, \dots, v_k)$ be a smallest odd cycle in G.
- Assume that C is not an induced cycle and let $v_i v_j$ be a chord of C
- Let $C_1 := (v_1, \dots, v_i, v_j, \dots, v_n)$ and $C_2 := (v_i, \dots, v_j)$ be the two induced cycles in $C + v_i v_j$
- Then, $|V(C_1)| + |V(C_2)| = |V(C)| + 2$
- As $|V(C)|$ is odd, one of $|V(C_1)|$ and $|V(C_2)|$ is odd
- This contradicts that C is a smallest odd cycle in G ↯



Kőnig's Theorem

Define: $\mathrm{M}(G)$: size of a maximum matching,
$\mathrm{vc}(G)$: size of a minimum vertex cover.

Intermediate steps:

Bipartite graphs are perfect

- Subgraphs of bipartite graphs are bipartite
- nonempty bipartite graph G: $\omega(G) = 2$ und $\chi(G) = 2$
- empty bipartite graph G: $\omega(G) = 1$ und $\chi(G) = 1$

$\mathrm{vc}(G) = n - \alpha(G)$

- Let $A \subseteq V(G)$ be a minimum vertex cover
  - no edge has both endpoints in $V(G) \setminus A$
  - $V(G) \setminus A$ is an independent set
  - $\alpha(G) \geq n - \mathrm{vc}(G)$
- Let $I \subseteq V(G)$ be a max. independent set
  - $V(G) \setminus I$ covers all edges
  - $\mathrm{vc}(G) \leq n - \alpha(G)$

Goal: For all bipartite graphs G: $\mathrm{vc}(G) = \mathrm{M}(G)$

- Only cliques of sizes 1 and 2. Use as many 2-cliques for the clique cover as possible.

$$\Rightarrow \kappa(G) = \mathrm{M}(G) + \underbrace{(n - 2 \cdot \mathrm{M}(G))}_{\text{vertices that are not covered}} = n - \mathrm{M}(G)$$

- As G is perfect we have $\alpha(G) = \kappa(G)$ and thus

$$\mathrm{vc}(G) = n - \alpha(G) = n - \kappa(G) = n - (n - \mathrm{M}(G)) = \mathrm{M}(G).$$

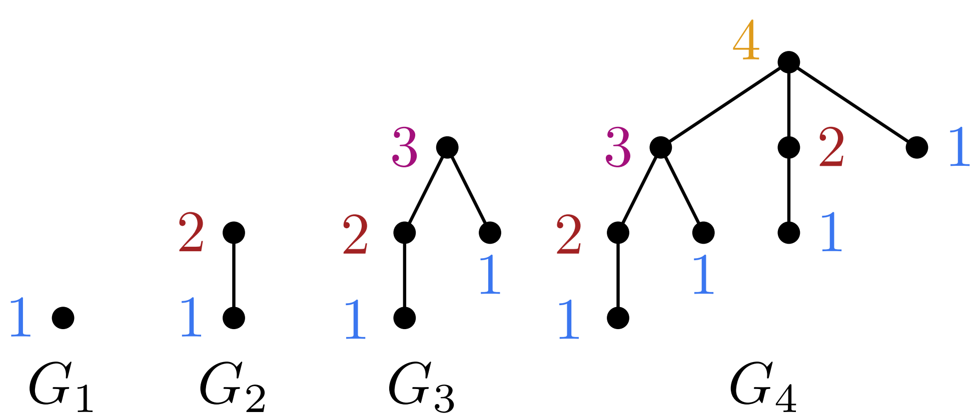
Bad greedy colorings

Goal: For every $k \geq 1$ find ...

- a bipartite graph on $2k$ vertices
- a vertex ordering

such that a greedy coloring uses k colors.

First: super-constant colors



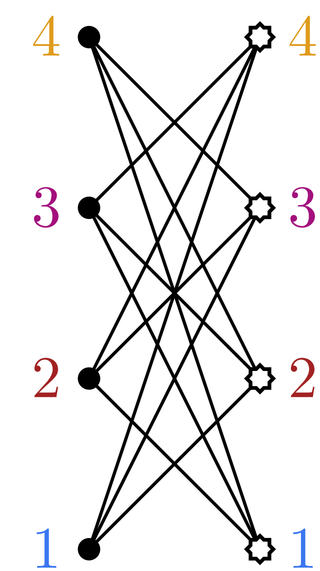
vertex ordering: from bottom to top

Now: k colors

In each step:

- connect new $\bullet$ to all ✪ before
- connect new ✪ to all $\bullet$ before

$\Rightarrow$ complete bipartite graph minus a perfect matching



vertex ordering: from bottom to top

Paths on 2δ vertices

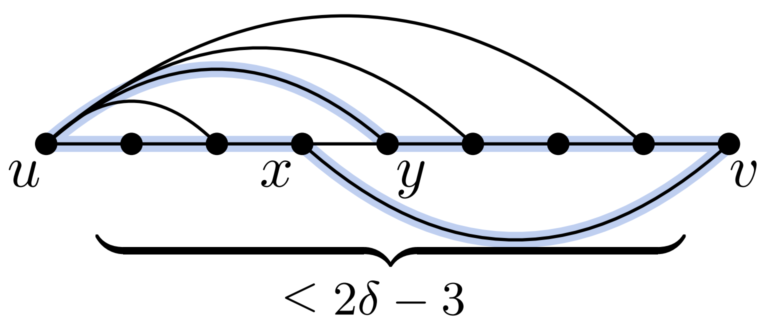
Claim: Let G be a connected graph G with minimum degree δ.
Then, G contains a path on $\min(2\delta, |V(G)|)$ vertices.

We prove a "stronger" statement (and assume w. l. o. g that $2\delta \leq |V(G)|$):

Goal: G contains a cycle or path on 2δ vertices.

- Let H be a largest cycle or path, where we choose a cycle if possible.
- Assume $|V(H)| < 2\delta$.

Case 1: H is a path



- u, v have only neighbors in H, otherwise H is not maximal
- there is a neighbor x of v that is followed (in H) by a neighbor y of u (counting argument).
- So, there is a cycle on $|V(H)|$ vertices.
$\implies$ Contradiction to the choice of H

Case 2: H is a cycle

- only neighbors in H, otherwise H not maximal
- since G is connected, H contains all vertices of G