1. Vorlesung

23.04.2025

Def: A graph $\mathcal{G}=(V,E)$ consists of $V\neq\emptyset$ finite set of vertices and E a set of edges with $E\subseteq\{\{u,v\}\mid u,v\in V, u\neq v\}=\binom{V}{2}$ Notation
(undirected, simple, no loops, no parallel edges)

Notation: $\{u,v\}=uv$. We have $uv=vu$.

Notation: $V(\mathcal{G})$ denotes the vertex set of $\mathcal{G}$, $E(\mathcal{G})$ the edge set

Important graphs:

- complete graphs: $n\geq 1$, $K_n=(\{1,\dots,n\}, \binom{[n]}{2})$ $([0]=\emptyset)$
- complete bipartite graph: $n,m\geq 1$, $K_{n,m}=(\{a_1,\dots,a_n\}\cup\{b_1,\dots,b_m\}, \{a_ib_j \mid i\in[n], j\in[m]\})$
- cycle : $n\geq 3$ $C_n=([n], \{\{i,i+1\}\mid i\in[n-1]\}\cup 1n\})$
- path $n\geq 1$ $P_n=([n], \{\{i,i+1\}\mid i\in[n-1]\})$ $(P_n=C_n-1n \ (n\geq 3))$
- empty graph $n\geq 1$ $E_n=([n],\emptyset)$

Def: For a graph $\mathcal{G}=(V,E)$ and a vertex subset $A\subseteq V$, the induced subgraph $\mathcal{G}_A$ is defined as $V(\mathcal{G}_A)=A$,
$E(\mathcal{G}_A)=\{uv\in E(\mathcal{G})\mid u,v\in A\}$ ‼ A can be an empty set.

Notation: $\mathcal{G}_A\subseteq\mathcal{G}$

The 4 important parameters

Def: Let $\mathcal{G}=(V,E)$ graph $A\subseteq V$.

- A is clique, if $\mathcal{G}_A$ is a complete graph
- A is an independent set if $\mathcal{G}_A$ is an empty graph (i-set)
- $\omega(\mathcal{G})=\max\{|A|: A\subseteq V(\mathcal{G}) \text{ is a clique}\}$ clique number
- $\alpha(\mathcal{G})=\max\{|A|: A\subseteq V(\mathcal{G}) \text{ is an i-set}\}$ independance number

Note: every graph contains small cliques.
Note: a single vertex is both a clique and i-set so $|V|\geq\alpha(\mathcal{G}),\omega(\mathcal{G})\geq 1$

Notation: disjoint union of sets: $A+B=A\cup B$ with $A\cap B=\emptyset$

Def: A partition into parts of a set V is $V_1+\dots+V_t=V$ with $t\geq 1$,

- $V_1+\dots+V_t$ is a coloring if V_i is a i-set $\forall i\in[t]$.
- $V_1+\dots+V_t$ is a clique cover if V_i is a clique $\forall i\in[t]$.
- $\chi(\mathcal{G})=\min\{t:\exists \text{ coloring } V_1+\dots+V_t \text{ of } \mathcal{G}\}$ chromatic number
- $\kappa(\mathcal{G})=\min\{t:\exists \text{ clique cover } V_1+\dots+V_t \text{ of } \mathcal{G}\}$ clique cover number

Note: There is always a clique cover and a coloring for every graph by taking $|V_i|=1$ $\forall i\in[t]$.
Here: $t=|V|$
so, $1\leq\chi(\mathcal{G}),\kappa(\mathcal{G})\leq|V|$

	$\omega(\mathcal{G})$	$\alpha(\mathcal{G})$	$\chi(\mathcal{G})$	$\kappa(\mathcal{G})$
K_n	n	1	n	1
$K_{n,m}$	2	$\max\{m,n\}$	2	$\max\{m,n\}$
C_n	$\begin{cases}3, & n=3\\ 2, & n\geq 2\end{cases}$	$\lfloor\frac{n}{2}\rfloor$	$\begin{cases}2, & \text{even}\\ 3, & \text{odd}\end{cases}$	$\begin{cases}\lceil\frac{n}{2}\rceil & n\geq 4\\ 1, & n=3\end{cases}$
P_n	$\begin{cases}2, & n\geq 1\\ 1, & n=1\end{cases}$	$\lceil\frac{n}{2}\rceil$	$\begin{cases}1, & n=1\\ 2, & n\geq 2\end{cases}$	$\lceil\frac{n}{2}\rceil$
E_n	1	n	1	n

coincide coincide

Note: $\mathcal{G}$ is bipartite $\Leftrightarrow\chi(\mathcal{G})=2\Leftrightarrow$ 2-colorable

2. Vorlesung

28.04.2025

Observation: For every graph $\mathcal{G}$ we have

- $\chi(\mathcal{G})\geq\omega(\mathcal{G})$
- $\kappa(\mathcal{G})\geq\alpha(\mathcal{G})$

proof

If $I\subseteq V_{\mathcal{G}}$ is independent and $C\subseteq V_{\mathcal{G}}$ is a clique then $|I\cap C|\leq 1$.

Hence for any coloring $V_1+\dots+V_t=V_{\mathcal{G}}$ and any clique C, we have $|C\cap V_i|\leq 1$ for $\forall i\in[t]$. If $|C|=\omega(\mathcal{G})$ then $t\geq|C|$. Thus $\chi(\mathcal{G})\geq\omega(\mathcal{G})$

For any clique cover $V_1+\dots+V_t=V_{\mathcal{G}}$ and any i-set I, we have $|I\cap V_i|\leq 1$, $\forall i\in[t]$. If $|I|=\alpha(\mathcal{G})$ then $t\geq|I|$. Thus $\kappa(\mathcal{G})\geq\alpha(\mathcal{G})$.

Main Question of AGT: When is $\chi(G)=\omega(G)$ and $\kappa(G)=\alpha(G)$

boring answer: For: $\omega(G)=\chi(G)$

Let G be a graph. Construct G' by adding a clique of size $\chi(G)$ next to G

Extention of the question: Equality should also hold for every induced subgraph.:

(P1) $\forall A\subseteq V_G: \chi(G_A)=\omega(G_A)$

(P2) $\forall A\subseteq V_G: \kappa(G_A)=\alpha(G_A)$

Properties hold:
K_n: P1, P2
E_n: P1, P2
$K_{n,m}$: P1, P2
P_n: P1, P2

Cycles only hold P1, P2 if they are even

Observation: if G,H are vertex disjoint:
- $\alpha(G+H)=\alpha(G)+\alpha(H)$
- $\kappa(G+H)=\kappa(G)+\alpha(H)$

Def: A graph G is called *perfect* if G has P1 and P2.

See: Every graph with $|V|\leq 4$ is perfect.
$\rightsquigarrow C_5$ is the smallest graph that is not perfect.

Def: For a graph $G=(V,E)$ the *complement of G* is the graph $\bar{G}=(V,\bar{E})$, with $\bar{E}=\binom{V}{2}-E$

Relation:

G	Clique	coloring $V_1+\dots+V_t=V_G$	P1 for G is P2 for $\bar{G}$
	$\omega(G)=\alpha(\bar{G})$	$\kappa(\bar{G})=\chi(G)$	and the other way around.
$\bar{G}$	i-set	clique cover $V_1+\dots+V_t=V_G$	P1 for $\bar{G}$ is P1 for G

(Def: G is perfect, if P1 for G and $\bar{G}$ or P2 for G and $\bar{G}$)

Theorem: (Weak Perfect Graph Theorem WPGT)
For every graph G it holds: G has (P1) $\Leftrightarrow$ G has (P2)

Warning: $\forall A\subseteq V_G: \chi(G_A)=\omega(G_A) \Leftrightarrow \kappa(G_A)=\alpha(G_A)$ is not true, because P1, P2 could break on different subsets.

To prove WPGT we consider

(P3): $\forall A\subseteq V_G: \omega(G_A)\cdot\alpha(G_A)\geq |A|$

See: Always satisfied for $|V_G|\leq n$. (P3) does not hold for C_5

We will prove that

G has (P1) $\Leftrightarrow$ G has (P3)
WPGT $\Updownarrow$
G has (P2) $\Leftrightarrow$

Def: For a graph $G=(V,E)$ and $h: V\to \mathbb{N}_0$ We define $G\circ h$ as the graph on vertex set $V(G\circ h)=\bigcup_{v\in V_G}\{v^1,\dots,v^{h(v)}\}$ and edge set $E(G\circ h)=\{u^iv^j \mid uv\in E\ \forall i\in[h(u)], j\in[h(v)]\}$ called *vertex replication / repetition* of G.

Def: Let $\mathbb{1}$ be $\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}\in\mathbb{N}^V$ (e.g. $h(v)=1\ \forall v\in V$), graph $G=(V,E)$, $i\in[|V|]$ $e_i=\begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}$, with 1 on ith coordinate. Vertex $v\in V$ define

$G\circ v$ as $G\circ h$ with $h(x)=\begin{cases}1 & x\neq v\\ 2 & x=v\end{cases}$, $h=\mathbb{1}+e_i$, with v on ith coordinate. and

$G-v$ as $G\circ h$ with $h=\mathbb{1}-e_i$ with v be the ith coordinate.

This are called *elementary operations*

Observation: Every $G\circ h$ can be done from G by a sequence of elementary operations.
[proof in exercise]

Lemma 2.6 For G and $H=G\circ h$ we have:
(i) (P1) for G $\Rightarrow$ (P1) for H
(ii) (P2) for G $\Rightarrow$ (P2) for H

Proof:
(i)
wlog. $H=G\circ v$ or $H=G-v$

Case: $H=G-v$ then $H=G_{(V-v)}$ hence (P1) for $G\Rightarrow$ (P1) for H.

Case: $H=G\circ v$ $v\in V_G \rightsquigarrow v^1,v^2\in V_H$. We have $H-v_1\cong G$ and $H-v_2\cong G$. Take any $A\subseteq V_H$.

If $|A\cap\{v^1,v^2\}|<2$, then $A\subseteq V_G$. Hence $\chi(H_A)=\chi(G_A)=\omega(G_A)=\omega(H_A)$

Let $v_1,v_2\in A$. consider $A'=A-v^1\subseteq V_G$ by (P1) for G $\chi(H_A)\leq\chi(G_{A'})\overset{\text{(P1) for }G}{=}\omega(G_{A'})\leq\omega(H_A)\overset{\text{observation}}{\leq}\chi(H_A)$. So, (P1) holds for H.

3. Vorlesung

5.5.2025

(ii) Let $\mathcal{G}$ have (P2).
Without loss of generality, let $H = \mathcal{G} \circ x$. Let x, x' be the two copies of x in H.
Wlog. A' contains x, x'. Let $A = A' - x' \subseteq V_{\mathcal{G}}$.

Since (P2) holds for $\mathcal{G}$ it follows $\chi(\mathcal{G}_A) = \alpha(\mathcal{G}_A) \Rightarrow V_1 + \dots + V_t$ clique cover of $\mathcal{G}_A = H_A$ with $t = \alpha(H_A)$.
Every i-set I of $|I| = t$.

Case 1: $\exists I \subseteq A$ iset of H_A, $|I| = t$, $x \in I$
then $I + x'$ is an i-set in $H_{A'}$ $\Rightarrow \alpha(H_{A'}) \geq t+1$.
$V_1 + \dots + V_t + \{x'\}$ is a clique cover of $H_{A'}$. $\rightarrow \chi(H_{A'}) \leq t+1 \leq \alpha(H_{A'})$

Case 2: $\forall I \subseteq A$ iset in H_A: $|I| = t$: $x \notin I$
Let $C = V_1 - x$ $\Rightarrow H_{A-C}$ has $\alpha(H_{A-C}) \leq t-1$
Since (P2) holds for $\mathcal{G}$ $\Rightarrow \exists$ clique cover $V_1' + \dots + V_{t-1}'$ of $\mathcal{G}_{A-C} = H_{A-C}$ with at most $t-1$ cliques. Then
$V_1' + \dots + V_{t-1}' + \underbrace{C + x'}_{\text{clique}}$ is a clique cover of $H_{A'}$ $\Rightarrow \chi(H_{A'}) \leq t \leq \alpha(H_{A'}) \underset{\text{for all graphs}}{\leq} \chi(H_{A'})$

Lemma 2.7

If $H = \mathcal{G} \circ h$ then: $\left.\begin{array}{l}\text{(P2) } \forall \text{ ind. subgraphs of } \mathcal{G} \\ \text{(P3) for } \mathcal{G}\end{array}\right\} \Rightarrow$ (P3) for H $\quad \left(\forall A \subseteq V_{\mathcal{G}}, A \neq V_{\mathcal{G}} \Rightarrow \chi(\mathcal{G}_A) = \alpha(\mathcal{G}_A)\right)$

proof:
Assume for the sake of contradiction that (P3) does not hold for H.

Without loss of generality: $\forall A \subseteq V_H$, $A \neq V_H$: $\omega(H_A)\alpha(H_A) \geq |A|$ (otherwise we can take a smaller H).
but $\omega(H)\alpha(H) < |V_H|$. Let $\omega(H) =: w$, $\alpha(H) =: a$.

Some vertex s of $\mathcal{G}$ has $h(s) = h \geq 2$ $\rightsquigarrow$ in H: $S = \{s_1, \dots, s_h\}$. Consider $H - s_h$ has (P3).

$\Rightarrow |V_H| - 1 \leq \omega(H - s_h)\alpha(H - s_h) \leq \omega(H) \leq \alpha(H) = wa < |V_H| - 1 \Rightarrow wa = |V_H| - 1$, $\alpha(H - s_h) = a$, $\omega(H - s_h) = w$
It follows: $\alpha(H - S) = a$ because $\alpha(H) = a = \alpha(H - s_h)$.

$H - S$ is obtained from $\mathcal{G} - s$ by vertex multiplication. $\mathcal{G} - s$ has (P2) and by Lemma 2.6, $H - S$ has (P2).
$\rightsquigarrow$ We find a clique cover $V_1 + \dots + V_a$ of $H - S$. $|V_H - S| = |V_H| - h = wa - (h-1)$. We also observe $|S| = h \leq a = \alpha(H)$ since S is an i-set in H.
$\hookrightarrow$ at most $h-1$ of $V_1, \dots, V_a$ have size w. wlog. $|V_1| = \dots = |V_{a-(h-1)}| = w$.
Let $X = V_1 + \dots + V_{a-(h-1)} + s_1$. So $|X| = (a - (h-1)) \cdot w + 1$. $\omega(H_X) \leq \omega(H) = w$.
Since (P3) holds for H_X $\Rightarrow \alpha(H_X) \geq \left\lceil \frac{|X|}{w} \right\rceil = \left\lceil \frac{(a-(h-1))w+1}{w} \right\rceil = a - (h-1) + \left\lceil \frac{1}{w} \right\rceil = a - (h-1) + 1$. $\Rightarrow \exists I$ iset in H_X, $|I| = a - (h-1) + 1$
$s_1 \in I$ (becaus we have $a-(h-1)$ „independent cliques")
$\rightsquigarrow I + \{s_2, \dots, s_h\}$ is an i-set in H. $\Rightarrow \alpha(H) \geq a - (h-1) + h = a - h + 1 + h = a + 1 = \alpha(H) + 1$ ↯

Proof of WPGT:

Let $\mathcal{G}$ be a graph. We prove (P1) $\Leftrightarrow$ (P2) $\Leftrightarrow$ (P3) for $\mathcal{G}$ by induction on $|V(\mathcal{G})|$.
Base case: $|V(\mathcal{G})| = 1$ ✓
Inductive Step:
(P1) $\Rightarrow$ (P3)
Say (P1) holds for $\mathcal{G}$. Let $A \subseteq V(\mathcal{G})$. If $A \neq V_{\mathcal{G}}$ then (P1) holds for $\mathcal{G}_A$. By induction (P3) holds for $\mathcal{G}_A$ i.e. $\omega(\mathcal{G}_A)\alpha(\mathcal{G}_A) \geq |A|$.

Wlog. $A = V_{\mathcal{G}}$. We need to show that $\omega(\mathcal{G})\alpha(\mathcal{G}) \geq |V_{\mathcal{G}}|$

Since (P1) holds $\mathcal{G} \Rightarrow \exists$ coloring $V_1 + \dots + V_t = V_{\mathcal{G}}$ with $t = \omega(\mathcal{G})$. By definition of a coloring we have $\forall i: |V_i| \leq \alpha(\mathcal{G})$.

$\Rightarrow \omega(\mathcal{G})\alpha(\mathcal{G}) \geq |V(\mathcal{G})|$

(P3) $\Rightarrow$ (P1)

Let (P3) hold for $\mathcal{G}$. To show (P1) to show $\chi(\mathcal{G}) \leq \omega(\mathcal{G})$ because the subgraphs are clear by induction).

Consider all cliques of size $\omega(\mathcal{G})$.

Case 1: $\exists I$ i-set in $\mathcal{G}$ s.t. $\forall C$ clique, $|C| = \omega(\mathcal{G})$: $I \cap C \neq \emptyset$.

Consider $\mathcal{G} - I$. We observe $\omega(\mathcal{G} - I) \leq \omega(\mathcal{G}) - 1$ $\overset{\text{By IH.}}{\Longrightarrow}$ (P1) holds for $\mathcal{G} - I$. i.e. $\exists$ coloring $V_1 + \dots + V_t = V_{\mathcal{G}}$ with $t \leq \omega(\mathcal{G} - I) \leq \omega(\mathcal{G}) - 1$. Then: $V_1 + \dots + V_t + I$ is a coloring of $\mathcal{G}$ $\Rightarrow \chi(\mathcal{G}) \leq t + 1 \leq \omega(\mathcal{G}) - 1 + 1 = \omega(\mathcal{G})$.

Case 2: $\forall I$ i-set $\exists C(I)$ clique, $|C(I)| = \omega(\mathcal{G})$, $C(I) \cap I = \emptyset$.

4. Vorlesung

12.05.2025

Let $Y = \{I \subseteq V_{\mathcal{G}} : I \text{ i-set}\}$. Chose for every I a disjoint clique and count for every vertex how often they appear in such a clique: $h(v) = |\{I \in Y \mid C(I) \ni v\}|$. Consider $H = \mathcal{G} \circ h$.

(P2) holds for $\mathcal{G}_A$, $\forall A \subsetneq V_{\mathcal{G}}$ (by induction), (P3) holds for $\mathcal{G}$. with Lemma 2.7 we have (P3) for H. Thus:

$\omega(H)\alpha(H) \geq |V_H| = |X|$, say $X = |V_H|$

- $|X| = \sum_{v \in V(\mathcal{G})} h(v) = \omega(\mathcal{G}) \cdot |Y|$
- $\omega(H) \leq \omega(\mathcal{G})$ (because h could be 0)
- $\alpha(H) \leq \max_{I \in Y}\left(\sum_{v \in I} h(v)\right) = \max_{I \in Y}\left(\left\{\sum_{I' \in Y} \overbrace{|C(I') \cap I|}^{0 \text{ or } 1}\right\}\right) \leq |Y| \overset{\text{because } I \cap C(I) = \emptyset}{-1}$

$\Rightarrow \omega(\mathcal{G})(|Y| - 1) \geq \alpha(H)\omega(\mathcal{G}) \geq \omega(H)\alpha(H) \geq |X| \geq \omega(\mathcal{G})|Y|$ $\rightsquigarrow$ Contradiction to the assumption that $C(I)$ exists for all I.

$\rightsquigarrow$ Case II does not happen.

(P2) $\Leftrightarrow$ (P3)

(P2) for $\mathcal{G}$ $\Leftrightarrow$ (P1) for $\bar{\mathcal{G}}$ $\Leftrightarrow$ (P3) for $\bar{\mathcal{G}}$ $\Leftrightarrow$ (P3) for $\mathcal{G}$

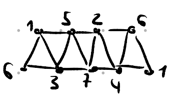
Equivalent:

- (P1) for $\mathcal{G}$
- (P2) for $\mathcal{G}$
- (P3) for $\mathcal{G}$
- $\mathcal{G}$ perfect
- $\bar{\mathcal{G}}$ perfect

Non perfect graphs:

- odd cycle C_t, $t \geq 5$
- complements of odd C_t, $t \geq 5$
- graph with induced odd C_t, $\bar{C_t}$ $t \geq 5$

Complete characterisation



Möbius-ladder ($\bar{C_7}$)

Theorem: (Strong Perfect Graph Theorem SPGT)

For every graph $\mathcal{G}$ it is equivalent:

- $C_t, \bar{C_t}$ for $t \geq 5$ odd no induced subgraph of $\mathcal{G}$
- $\mathcal{G}$ is perfect

Def: $\mathcal{S} = \{S(v) : v \in V\}$ collection of sets is an intersection representation of $\mathcal{G} = (V, E)$ if

$uv \in E \Leftrightarrow S(u) \cap S(v) \neq \emptyset$

- $\mathcal{G}$ is an interval graph if $\mathcal{G}$ has an intersection representation with intervals of $\mathbb{R}$. i.e. $\mathcal{I} = \{I(v) \, v \in V\}$ $\forall I(v) = [l_v, r_v] \subseteq \mathbb{R}$.

Def: For a graph $\mathcal{G}$, integer $t \geq 4$ a t-hole in $\mathcal{G}$ is an induced subgraph $\mathcal{G}_1 \cong C_t$.

A t-antihole in $\mathcal{G}$ is an induced subgraph $\mathcal{G}_1 \cong \bar{C_t}$

(SPGT: $\mathcal{G}$ perfect $\Leftrightarrow$ $\mathcal{G}$ has no odd hole nor antihole).

Lemma

If $\mathcal{G}$ is an intervalgraph $\Rightarrow$ no t-hole in $\mathcal{G}$. $t \geq 4$

proof:

Let $\mathcal{I}$ be an intervalrepresentation of $\mathcal{G}$. Assume for the sake of contradiction that there is a t-hole $C_t = [v_1, \dots, v_t]$, $t \geq 4$.

We have that: $I(v_i)$, $I(v_{i-1})$, $I(v_{i+1})$ $\rightarrow$ $I(v_{i-1}), I(v_{i+1})$ cover distinct endpoints of $I(v_i)$. $\Rightarrow I(v_1) \cap I(v_t) = \emptyset \Rightarrow v_1 v_t \notin E$ ↯

Lemma

$\mathcal{G}$ interval graph $\Rightarrow$ $\mathcal{G}$ perfect

Proof:

We use SPGT.

There is no odd t-hole, since prev. lemma.

There is no odd antihole:

- $C_5 = \overline{C_5}$ $\rightarrow$ previous lemma
- $\overline{C_t}$ $t \geq 7$: We find an induced 4-hole in $\overline{C_t}$ and with the previous lemma they don't appear.

What we showed: $\mathcal{G}$ interval graph $\Rightarrow$ $\mathcal{G}$ has no holes $\Rightarrow$ $\mathcal{G}$ is perfect.

5. Vorlesung

14.05.2025

3. Chapter: Chordal Graphs

Def.: $\mathcal{G} = (V, E)$ is chordal if $\mathcal{G}$ has no t-hole, $t \geq 4$

Equivalently, every cycle, $t \geq 4$ in $\mathcal{G}$ (subgraph, not induced) has a chord i.e. edge uv with u, v not consequtive on cycle.

$\rightarrow$ Adding an edge may turn an cordal graph in a non-cordal graph and vice versa.

Examples

- complete graphs
- paths, empty graphs, trees
- interval graphs
- ⋮

trees $\rightsquigarrow$ but think

- chordle and conquer
- tree have leaves

Def: $\mathcal{G} = (V, E)$ graph. A vertex $v \in V$ is simplicial if $Adj(v) = \{u \in V \mid uv \in E\}$ is a clique.

Neighbourhood

Goal: Every chordal $\mathcal{G}$ has at least one simplicial vertex

Lemma

$$\left.\begin{array}{l} v \text{ simplicial in } \mathcal{G} \\ \mathcal{G} - v \text{ perfect} \end{array}\right\} \Rightarrow \mathcal{G} \text{ is perfect}$$

proof: Verify (P1): $\forall A \subseteq V_{\mathcal{G}}$ $\chi(\mathcal{G}_A) = \omega(\mathcal{G}_A)$

- consider any fixed $A \subseteq V_{\mathcal{G}}$

case $v \notin A$:

then $A \subseteq V_{\mathcal{G}-v}$ and $\chi(\mathcal{G}_A) = \omega(\mathcal{G}_A)$ as $\mathcal{G} - v$ is perfect (has (P1))

case $v \in A$:

Consider $A' = A - v \subseteq V_{\mathcal{G}-v}$ $\Rightarrow \chi(\mathcal{G}_{A'}) = \omega(\mathcal{G}_{A'})$. Then we have a coloring $A' = V_1 + \dots + V_t$, $t = \omega(\mathcal{G}_{A'})$.

Case 1:

$|Adj(v) \cap A'| < t = \omega(\mathcal{G}_{A'})$. Then $\exists_i$ $V_i \cap (Adj(v) \cap A')$ $\rightarrow V_i = V_i + \{v\}$ extends the coloring

$\chi(\mathcal{G}_A) \leq \chi(\mathcal{G}_{A'}) = \omega(\mathcal{G}_{A'}) \leq \omega(\mathcal{G}_A) \leq \chi(\mathcal{G}_A)$

Case 2

$|Adj(v) \cap A'| \geq t = \omega(G_{A'})$. Since $Adj(v) \cap A'$ is a clique: $|Adj(v) \cap A'| = t = \omega(G_{A'})$.

$(Adj(v) \cap A') \cup \{v\}$ is a clique in G_A of size $t+1$. Then $\omega(G_A) \geq t+1 = \omega(G_{A'})+1 = \chi(G_{A'})+1 \geq \chi(G_A) \geq \omega(G_A)$

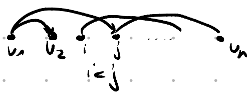
coloring $V_1 + \dots + V_t + \{v\}$ on $t+1$ colors

Observation: G chordal $\Rightarrow$ $\forall A \subseteq V_G$ is chordal in particular $G-v$ is chordal $\forall v \in V_G$

Def: $G=(V,E)$, $|V|=n$. A perfect elimination scheme (PES) of G is a vertex ordering $\sigma = [v_1, \dots, v_n]$ s.t. v_i is simplicial in $G_{\{v_i, \dots, v_n\}}$ $\forall i \in [n]$

From Lemma: graph with PES's are perfect.

Notation convention for vertex orderings $\sigma: [v_1, \dots, v_n]$



v_i is left/before v_j
v_j is right/after v_i

! σ PES $\Leftrightarrow$ every right neighborhood is a clique

Def.: $G=(V,E)$ graph $S \subseteq V$ is a separator if $G-S$ is disconnected i.e. G has at least two connected components.
If a, b non-adjacent vertices in G. S is an a,b-separator if a and b are in different components of $G-S$.

Goal: Find a separator S that is a clique in chordal G unless G is complete

Lemma 3.4 G chordal, $a, b \in V$ $ab \notin E$, $a \neq b$. $S \subseteq V_G$ is an $\underbrace{\text{inclusion-minimal}}_{A \subsetneq S \text{ is not a separator}}$ a,b-separator. Then:
S is a clique.

Proof:

If $|S| \leq 1$ then S is a clique.
So $|S| \geq 2$; take $x, y \in S$, $x \neq y$. We show that $xy \in E$.
We have that $S-x$ is not an a,b-separator. Let G_a, G_b the components with $a \in A = V(G_a)$, $b \in B = V(G_b)$.
We have that x has an edge to A and an edge to B (and so does y).
$\rightarrow$ We have a cycle $C = [x, a_1, \dots, a_p, y, b_1, \dots, b_q]$. Take C to be a shortest such cycle. C has at least 4 vertices. Since G chordal
$\Rightarrow C$ has a chord e.
Where is e?

- $e = a_i a_j$; no as C is shortest
- $e = b_i b_j$; no as C is shortest
- $e = a_i b_j$; no because S is an a,b-separator
- $e = x a_i / y a_i$ no as C is shortest
- $e = x b_i / y b_i$ no as C is shortest
- $e = xy$ yes

6. Vorlesung

19.05.2025

Lemma 3.6 G chordal. Then:

- G has a simplicial vertex
- If $G \neq K_n$ then G has two simplicial vertices

Proof:

By induction on $|V(G)| = n$

$n=1$: $G = K_1$ ✓
$n \geq 2$: Case $G \cong K_n$: every vertex is simplicial
So let $G \not\cong K_n$.
Let $a, b \in V_G$, $ab \notin E_G$. Let S be an inclusion minimal a,b-separator. Component G_A contains a and G_B contains b (of $G-S$).
· Apply induction on G_{A+S} and G_{B+S}

In each we have we get either all vertices are simplicial or there are two non-simplicial vertex in $\mathcal{G}_{S+A}$ (or $\mathcal{G}_{S+B}$)
In either case: $\exists$ simplicial vertex v in A (B). Then v is simplicial in $\mathcal{G}_{A+S}$ (S_{B+S}) and also in $\mathcal{G}$.
- $v_A v_B \notin E$.

Summary

(i) $\mathcal{G}$ is chordal i.e. every cycle length ≥ 4 has a chord
(ii) every induced cycle is a triangle (no ℓ-hole)
(iii) every inclusion minimal separator is clique
(iv) every induced subgraph has a simplicial vertex
(v) There is an PES (perfect elimination scheme)

proof of (*)

We have $\mathcal{G}$, PES σ, C cycle of length ≥ 4. Let v be the leftmost vertex of C in σ. Say $v = \sigma(i)$. We have $x, y \in Adj(v) \cap \{\sigma(i), \dots, \sigma(n)\}$. By Definition of PES, $xy \in E$. Thus C has a chord.

- (iv), (v) lead to a recognition algorithm with runtime $O(n \cdot$ time to find simplicial vertex$) = O(n^4)$

Algorithm: (LexBFS)

Input: undirected graph $\mathcal{G} = (V, E)$
Output: vertex ordering σ

- assign each vertex label $\emptyset$
- for $i = n$ to 1 do
  - choose a vertex v with no assigned number in σ with lexicographically largest label
  - $\sigma(i) = v$
- for every vertex $w \in Adj(v)$ with no assigned number in σ
  - append i to label(w).

Viewpoint 1:

- labels at vertices
- Strings over an alphabet $\{1, \dots, n\}$
- Lexicographic $1 < \dots < n$
  label(v) $= \alpha_1 \dots \alpha_k$
  label(w) $= \beta_1 \dots \beta_\ell$

$\alpha = \alpha_1 \dots \alpha_k < \beta_1 \dots \beta_\ell$ if
$\begin{cases} \alpha_1 < \beta_1 \text{ or} \\ \alpha = \emptyset \text{ or} \\ \alpha_1 = \beta_1 \text{ and } \alpha_2 \dots \alpha_k < \beta_2 \dots \beta_\ell \end{cases}$

Viewpoint 2:

- Q queue of all not-numbered vertices
  - $v \leftarrow$ first(Q)
- Elements of v (of same label) sorted lex in Q $\approx$ Bucketsort
- $Adj(v)$ splits each set X in Q into $Adj(v) \cap X$ and the others.

Plan:

- Given $\mathcal{G}$
- run LexBFS $\rightarrow$ obtain vertex ordering σ
  - prove that σ PES $\Leftrightarrow$ $\mathcal{G}$ chordal
  - Implement LexBFS in linear time
  - test if σ is PES in linear time

$\rightarrow$ Recognition algorithm.
- $\sigma \leftarrow$ LexBFS($\mathcal{G}$)
- test(σ) for PES
- Yes $\rightarrow$ $\mathcal{G}$ chordal, No $\rightarrow$ $\mathcal{G}$ not chordal

Lemma: $\sigma \leftarrow$ lex(BFS) then: $\forall a, b, c \in V(\mathcal{G})$ it holds:

L3 $\left\{ \begin{matrix} a <_\sigma b <_\sigma c \\ ac \in E_\mathcal{G},\ bc \notin E_\mathcal{G} \end{matrix} \right\} \exists d \in V_\mathcal{G} : c <_\sigma d \quad ad \notin E_\mathcal{G},\ bd \in E_\mathcal{G}$

7. Vorlesung

28.05.2025

proof:

Consider $a \quad b \quad c$ i.e. $a <_\sigma b <_c$ and $ac \in E$, $bc \notin E$.
When c is processed by LexBFS:
1) If label(a) = label(b)
   Then afterwards label(a) $>_{lex}$ label(b) and this will still hold, when b is processed. This contradicts choice of b ↯ (have to be 2))
2) label(a) $\neq$ label(b)
   Then label(b) $>_{lex}$ label(a). Consider the step when the first time label(a) $\neq$ label(b). This is when processing vertex d, $c <_\sigma d$. It follows $b \in Adj(d)$ but $a \notin Adj(d)$.

Theorem 3.9 $\mathcal{G}$ is chordal $\Leftrightarrow$ LexBFS outputs PES.

proof:

"$\Leftarrow$" Clear, since only chordal graphs have PES.

"$\Rightarrow$" Prove by contraposition, i.e., G is not a PES but LexBFS result $\Rightarrow \mathcal{G}$ is not chordal. $x_0\ x_1\ x_2$

If G is not a PES then $\exists x_0, x_1, x_2 \in V$, $x_0 <_G x_1 <_G x_2$, $x_0x_1, x_0x_2 \in E$, $bc \notin E$. Consider x_2 be the rightmost of such vertices.

By (L3) $\exists x_3$: $x_2 <_G x_3$ $x_3x_0 \notin E$ and $x_3x_1 \in E$, again pick x_3 as the right most such vertex

1) $x_2x_3 \in E$ Then we have that $x_0x_1x_2x_3$ induce a C_4. Hence $\mathcal{G}$ is not chordal.

2) $x_2x_3 \notin E$ Then $\exists x_4$ with $x_4 <_G x_3$ $x_4x_1 \notin E$, $x_4x_2 \in E$, by (L3). Choose x_4 again be the rightmost of such vertex.

- If $x_0x_4 \in E_{\mathcal{G}}$ Contradicts the choice of x_2 as rightmost. ↯ Thus $x_0x_4 \notin E$
  1) $x_3x_4 \in E$: Induced $C_5 \Rightarrow \mathcal{G}$ is not chordal.
  2) $x_3x_4 \notin E$: Then $\mathcal{G}[x_0, \dots, x_4] = P_5$ with endpoints x_3, x_4.
     $\rightarrow \exists x_5$ with $x_5x_3 \in E$, $x_5x_4 \notin E$, by (L3).
     - If $x_0x_5 \in E \rightarrow$ contradicts the choice of x_2 ↯ So $x_0x_5 \notin E$.
     - If $x_1x_5 \in E \rightarrow$ Contradicts the choice of x_3 ↯ so $x_1x_5 \notin E$.
     1) $x_4x_5 \in E \rightsquigarrow$ Induced $C_6 \Rightarrow \mathcal{G}$ not chordal.
     2) $x_4x_5 \notin E \rightarrow$ Induced P_6 and the argument continues.
        - vorletzter Knoten: contradicts x_{j-1}
        - alle anderen von rechts nach links: contradics x_2.

LexBFS in $\mathcal{O}(|V|+|E|)$

Q: queue of sets
- First(Q), as a doubly linked list.

S: set of vertices e.g. list but non-empty!
- Flag, whether it split already.

w: $S(w)$ set S with $w \in S$

List: Fixed list of sets that need clean up.

[Pseudocode auf Website] $\rightarrow$ The update step is in $\mathcal{O}(|Adj(v)| + |\text{Fixlist}|)$ $\leq |Adj(v)|$

$\rightsquigarrow$ LexBFS total: $\mathcal{O}(\underbrace{\sum |Adj(v)|}_{2|E|} + |V|) = \mathcal{O}(|E|+|V|)$.

Test whether given vertex ordering G is PES:

naive approach:
- test all triples for $\rightarrow \mathcal{O}(n^3)$.
- test right neighborhood of each vertex $\rightarrow \sum_u |Adj(u)|^2 = \mathcal{O}(n^3)$.

Idea: v tells it's leftmost right neighbor u set of vertices that should be pairwise adjacent.

7. Vorlesung

02.06.2025

Theorem: Algorithm 3 (test if G is PES) is correct

proof: We prove: Algo 3 returns true $\Leftrightarrow G$ is PES of $\mathcal{G}$ — equivalently one can show: Algo 3 returns false $\Leftrightarrow G$ is not an PES.

"$\Rightarrow$" (Algo 3 returns false $\Rightarrow G$ not an PES)

It exists a vertex u with $A(u) - Adj(u) \neq \emptyset$, say $w \in A(u) - Adj(u)$. Who put $w \in A(u) \rightsquigarrow$ some vertex v earlier

$\Rightarrow u$ leftmost in X_v, $w \in X_v - u$ $\Rightarrow$ $v\ u\ w$ $\Rightarrow G$ not PES.

"$\Leftarrow$"

Assume G is not PES. Take $v\ u\ w$ with v, u closest together.

Claim: u is the leftmost right neighbor of v.

If $a \in X_v$ $v < a < u$
- If $au \notin E$, then we have a contradiction of the choice of uvw. ↯
- If $aw \notin E$, then vaw is a better triple ↯
- If $aw \in E$ then auw is a better triple. ↯

So u is leftmost in X_v.

The algorithm puts w into $A(u)$. Later, when processing u, we have $w \in A(u) - Adj(u) \Rightarrow$ Algo returns false.

Theorem Algorithm 3 can be done in $O(|V|+|E|)$.

proof:

- for loop over each vertex v once.
- Lines 2-7 possible in $O(|Adj(v)|) \to$ line 7: append $X-\{v\}$ to $A(u)$ without checking for duplicate

total $O(\sum_v |Adj(v)|) = O(|V|+|E|)$

- Check line 8-10:

$A(v) - Adj(v) \neq \emptyset$? In $O(|A(v)| + |Adj(v)|)$

↳ Using Algorithm 4.

total $O(\sum_v |A(v)|) = O(\sum_v |Adj(v)|) = O(|V|+|E|)$

We can recognize in linear time wether G is chordal and

We compute $\chi(G), \omega(G), \alpha(G), k(G)$ for chordal graph(G) using a PES σ

→ Algo 5: $\omega(G)$ & $\chi(G)$ with clique C and coloring φ (optimal)

→ Algo 6: $\alpha(G)$ & $k(G)$ with optimal i-set U, clique cover ψ.

Convention for Algo 5:

coloring $V_1 + \dots + V_t = V$, V_i i-set $\iff \Phi: V \to [t]$ $\Phi(v) = i \iff v \in V_i$, if $\Phi(v) = 0$ then v is uncolored.

Algo 5, Line 6: first-fit coloring

Theorem Algorithm 5 computes a clique C, a coloring Φ with $|C| = \omega(G)$, and $\max_v(\Phi(v)) = \chi(G)$.

proof:

C is a clique

C is of the form $X_v + \{v\}$. As σ is a PES, X_v is a clique $\to C = X_v + \{v\}$ is a clique. We also have $|C| \leq \omega(G)$.

$\max_v(|X_v| + 1) = |C| \leq \omega(G)$

Φ is a coloring

We set color $\Phi(v)$ once and never change, $\Phi(v) \geq 1$.

Let $uv \in E_G$. Wlog. $u \in X_v$. It follows that we chose $\Phi(v)$ to be different from $\Phi(u)$.

$\Rightarrow \chi(G) \leq \max_v(\Phi(v))$.

C and Φ are optimal

- For every v: $\Phi(v) \leq |X_v| + 1$, hence $\chi(G) \leq \max_v(\Phi(v)) \leq \max_v(|X_v|+1) = |C| \leq \omega(G) \underset{\text{for every graph}}{\leq} \chi(G)$.

→ equality everywhere $\Rightarrow |C| = \omega(G)$, $\max_v(\Phi(v)) = \chi(G)$.

Theorem Algo 5 can be done in $O(|E|+|V|)$.

proof:

- The for-loop iteration for vertices v takes $O(|Adj(v)|)$.
- line 6 similar to Algorithm 4 in $O(|X_v|)$ $\to O(|V|+|E|)$.

Theorem Algorithm 6 computes i-set U, clique cover ψ with $|U| = \alpha(G)$, and $\max_v(\psi(v)) = k(G)$

proof:

U is an i-set

Invariant: $w \in U$, $v >_\sigma w$, $\psi(v) = 0 \Rightarrow vw \notin E_G$, equivalently $w \in U$, $v >_\sigma w$, $vw \in E_G \to \psi(v) \neq 0$

This invariant holds, because $v \in X_w$ gets assigned $\psi(v) \leftarrow |U| \neq 0$

$\Rightarrow |U| \leq \alpha(G)$.

ψ is a clique cover

Line 8: Set $\psi(w) \leftarrow |U|$ $\forall w \in \{v\} + X_v$ $\{v\} + X_v$ is a clique, since σ PES.

|U| is never assigned again.
$\to$ final $\psi: \{w \mid \psi(w)=1\} \subseteq \{v\} + X_v$ thus ψ is a clique.
$k(\mathcal{G}) \leq \max_v(\psi(v))$

U and ψ are optimal
$\chi(\mathcal{G}) \leq \max_v(\psi(v)) = \overset{\text{final}}{|U|} \leq \alpha(\mathcal{G}) \overset{\text{for every graph}}{\leq} k(\mathcal{G})$ Thus the inequations have to hold with equality

Theorem: Algo 6 is done in $\mathcal{O}(|V|+|E|)$.

proofsketch
Outer loop in $O(|V|)$, Inner loop in $\mathcal{O}(|Adj(v)|)$

8. Vorlesung

23.06.2025

Idea:
- underlying tree $T=(V_T, E_T)$
- Vertex of $\mathcal{G}$ corresponds to T_v of T (subtree)
- edge $e=uv$ correspond to intersecting (non-disjoint) subtrees ($uv \in E_{\mathcal{G}} \Leftrightarrow T_u \cap T_v \neq \emptyset$)

Plan:
chordal graphs = intersection graphs of subtrees of a tree.

Remark: interval graphs are interval graphs of subtrees of a path

main ingredient: **Helly Property**

Def.: a family $\{A_i\}_{i \in I}$ of sets has the Helly property if $\forall J \subseteq I: A_i \cap A_j \neq \emptyset \ \forall i,j \in J. \Rightarrow \bigcap_{j \in J} A_j \neq \emptyset$ This means: $\exists x: x \in A_j \ \forall j \in J$

Proposition 3.13 T tree $\Rightarrow \{T_i \subseteq T \mid T_i \text{ subtree}\}_{i \in I}$ has the Helly property

Theorem 3.14
For every graph $\mathcal{G}=(V,E)$ the following are equivalent:
(i) $\mathcal{G}$ is chordal
(ii) $\exists$ tree $T=(V_T, E_T), \{T_v \subseteq T \mid v \in V, T_v \text{ subtree}\}$ such that $uw \in E \Leftrightarrow T_u \cap T_w \neq \emptyset$
(iii) $\exists$ tree $T=(V_T, E_T)$ such that $V_T = \{X \subseteq V \mid X \text{ inclusion maximal cliques in } \mathcal{G}\}$ and $\forall v \in V$, $K_v = \{X \in V_T \mid v \in X\}$ induces a subtree

proof
(ii) $\Rightarrow$ (i):
Let $\mathcal{G}$ be an intersection graph of subtrees of tree T.
Let $C=[v_1, \dots, v_k]$ $k \geq 4$ be a cycle in $\mathcal{G}$. $T_1 := T_{v_1} \cup T_{v_2}$ is a subtree of T. $T_2 := T_{v_3} \cup T_{v_{k-1}}$ is a subtree of T. $T_3 := T_{v_4} \cup \dots \cup T_{v_k}$ is a subtree of T
To use the Helly property, we check that T_1, T_2, T_3 are always pairwise non-disjoint.
- $T_1 \cap T_2 \neq \emptyset$, since $v_2 v_3 \in E_{\mathcal{G}}$
- $T_2 \cap T_3 \neq \emptyset$, since $v_3 v_4 \in E_{\mathcal{G}}$ $\Rightarrow$ by Helly property (and prop 3.13) $\exists x \in V_T: x \in T_1, x \in T_2, x \in T_3$
- $T_1 \cap T_3 \neq \emptyset$, since $v_1 v_k \in E_{\mathcal{G}}$

Case 1: $x \in T_{v_1}$
Then v_1 is adjacent to v_j, $j \in \{3, \dots, k-1\}$ with $x \in T_{v_j} \subseteq T_2$ $\to v_1 v_j$ is a chord in $\mathcal{G}$
Case 2: $x \in T_{v_2}$
There is a chord $v_k v_2$ in $\mathcal{G}$.

(i) $\Rightarrow$ (iii)
Let $\mathcal{G}=(V,E)$ be chordal. Then we find tree T by induction on $|V_{\mathcal{G}}|$. Let $K(\mathcal{G}) = \{X \subseteq V_{\mathcal{G}} \mid X \text{ inclusion maximal clique in } \mathcal{G}\}$
Base: $|V_{\mathcal{G}}|=1$: $V_T = K(\mathcal{G})$ is one vertex.

Step $|V_\mathcal{G}| \geq 2$: Let v be simplicial in $\mathcal{G}$. We do induction on $\mathcal{G}-v$. By induction we get T' tree of $K(\mathcal{G}-v)$.

Case 1: $Adj(v) \in K(\mathcal{G}-v)$: $Adj(v)+v \in K(\mathcal{G})$ and $K(\mathcal{G}-v) - Adj(v) = K(\mathcal{G}) - (Adj(v)+\{v\})$ (every inclusion maximal set contains their connections to edges)

relabel $Adj(v)$ of T' to $Adj(v)+\{v\}$ in T. We observe that (ii) holds as $\forall u \neq v$ K_u did not change and $K_v = \{Adj(v)+\{v\}\}$

Case 2: $Adj(v) \notin K(\mathcal{G}-v)$

Let $X \in K(\mathcal{G}-v)$, $Adj(v) \subseteq X$. In T' there is a vertex for X. We add a new vertex $Adj(v)+\{v\}$ adjacent only to X in T. (ii) holds as $\forall w \in Adj(v): w \in X$.

(iii) $\Rightarrow$ (i) Let $T=(V_T, E_T)$ be tree with the properties by (ii). Take T_v as the subtree induced by K_v in T. Take $vw \in E_\mathcal{G}$. Then $\exists X \in K(\mathcal{G})$ with $v, w \in X$, and $X \in K_v$, $X \in K_w$. $\Leftrightarrow$ $\exists X \in V_T: X = T_v \cap T_w \Leftrightarrow T_v \cap T_w \neq \emptyset$.

4. Comparability graphs

Idea/Def:

- vertices are element
- edges are a "better-than" relation $\Rightarrow$ direct edge (u,v) $u \to v$ „v is better than u" $u \prec v$
- binary relation $\prec \subseteq V_\mathcal{G} \times V_\mathcal{G}$
  1. irreflexive: $v \nprec v$ $\forall v \in V_\mathcal{G}$
  2. transitivity: $\forall u,v,w$: if $u \prec v$ & $v \prec w \Rightarrow u \prec w$

is called a strict partial order

9. Vorlesung

25.06.2025

Notation for chapter 4:

only directed edges

- graph $\mathcal{G}=(V,E)$, V finite, $E \subseteq \{(u,v): u \in V, v \in V, u \neq v\}$ $u \to v$ (u,v)

[if not other specified, the underlying graph is undirected]

shorthand notation: uv for (u,v) i.e. $uv \neq vu$

- $\mathcal{G}$ is undirected $\forall u \neq v: uv \in E \Leftrightarrow vu \in E$

Def.: an orientation of $\mathcal{G}=(V,E)$ is $F \subseteq E$ s.t. $\forall uv \in E$: $uv \in F \Leftrightarrow vu \notin F$

Def.: For a subset $F \subseteq E$ of a graph $\mathcal{G}$ define
- $F^{-1} = \{vu : uv \in F\}$ as the reversal of F
- $\hat{F} = F \cup F^{-1} = \{uv : uv \in F \lor vu \in F\}$ is the (symmetric) closure of F.

Def.: Let $\mathcal{G}=(V,E)$ be a undirected, $F \subseteq E$ be an orientation. F is transitive if $\forall a,b,c \in V$: $(ab \in F \land bc \in F) \Rightarrow ac \in F$.

Def.: Let $\mathcal{G}=(V,E)$ be an undirected graph. $\mathcal{G}$ is an comparability graph if it admits a transitive orientations F.
We call $\mathcal{G}$ transitively orientations.

Examples: · complete graphs, paths

Observation: F is a transitive orientation $\Leftrightarrow$ F^{-1} is a transitive orientation

Theorem: $\mathcal{G}$ comparability graph $\Rightarrow$ $\mathcal{G}$ is perfect.

proof: (via SPGT)

- We observe that if $\mathcal{G}$ is a comparability graph then $\mathcal{G}_A$ is also a comparability graph $\forall A \subseteq V_\mathcal{G}$
- Hence we show that $C_t, \bar{C}_t$ t odd, $t \geq 5$ aren't transitively orientable.

Let C_t be an odd cycle, $t \geq 5$. Take any orientation F and assume transitivity. Because of the observation we can assume that $v_1 v_2 \in F$.

F transitive $\Rightarrow v_2 v_3 \notin F \Leftrightarrow v_3 v_2 \in F$ and $v_4 v_3 \notin F \Leftrightarrow v_3 v_4 \in F$.

In general: v_i : i even must be a sink and for i odd a source.

But t is odd $\Rightarrow v_t v_1 \in F$, $v_1 v_2 \in F$ but $v_t v_2 \notin F$ $\Rightarrow$ F is not transitive.

Let $\bar{C}_t$, $t \geq 5$ odd: wlog. $v_1 v_3 \in F$. Since $v_2 v_4 \notin E$, we have $v_1 v_4 \in F$ (for F not transitive). The same holds for v_5. Thus v_1 must be a source.

Because $v_1v_i \in F$, v_1 must be a sink for $i \in \{3,\dots,t-1\}$. (Because of symmetrie. If v_3 has an incoming edge, it has to be a source
$\Rightarrow v_3v_{t-1} \in F$ and $v_{t-1}v_3 \notin F$ (for $t \geq 6$)

Remark: C_t, t even or $t=3$ is comparability graph
More general: G bipartite $\Rightarrow G$ comparability graph

Observation:
If $F \subseteq E$ is a transitive orientation, $ab \in F$ and $ab' \in E$ and $bb' \notin E$ Then $ab' \in F$ or:
$ba \in F$ and $ab' \in E$ and $bb' \notin E$ Then $b'a \in F$

Def.: Gamma-Relation
For $ab \in E$, $a'b' \in E$ we define $ab \,\Gamma\, a'b'$ if $\begin{cases} a = a' \text{ and } bb' \notin E \\ b = b' \text{ and } aa' \notin E \end{cases}$

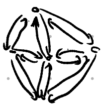
Observation: If F transitive, $ab \in F$, $ab \,\Gamma\, a'b' \Rightarrow a'b' \in F$. "ab implies/enforces a'b'"

Def.: a Γ-chain is $a_1b_1, a_2b_2 \dots a_kb_k$ sequence of edges (on non-necessarily distinct vertices) s.t. $a_ib_i \,\Gamma\, a_{i+1}b_{i+1}$ for $i = 1,\dots,k-1$
We say $a_1b_1 \,\Gamma^*\, a_kb_k$ (Γ^* is the transitive closure of Γ)

Observe: Γ^* is an equivalence relation on E
- $ab \,\Gamma^*\, a'b' \Leftrightarrow a'b' \,\Gamma^*\, ab$
- $a_1b_1 \,\Gamma^*\, a_2b_2$, $a_2b_2 \,\Gamma^*\, a_3b_3 \Rightarrow a_1b_1 \,\Gamma^*\, a_3b_3$

$\Rightarrow \Gamma^*$ splits E into equivalence classes $\mathcal{I}(G)$ Implication classes

Examples:
- $|\mathcal{I}(C_5)| = 1$
- $|\mathcal{I}(C_4)| = 2$
- $|\mathcal{I}(C_7)| = 6$
- $|\mathcal{I}(G)| = 6$

$G =$ 

Observe:
- G comparability graph $\Rightarrow |\mathcal{I}(G)|$ even
- if $A \in \mathcal{I}(G)$, $ab, ba \in A \Leftrightarrow G$ is not a comparability graph

Def.: If $A \in \mathcal{I}(G)$ then $\hat{A}$ is a called a color class of G. $\hat{\mathcal{I}}(G) = \{\hat{A} \mid A \in \mathcal{I}(G)\}$.

Observation: $A \in \mathcal{I}(G) \Rightarrow A^{-1} \in \mathcal{I}(G)$ since $ab \,\Gamma^*\, a'b' \Rightarrow ba \,\Gamma^*\, b'a'$

Theorem 4.1 $A \in \mathcal{I}(G)$, F transitive orientation of $G \Rightarrow F \cap \hat{A} = A$ or $F \cap \hat{A} = A^{-1}$

10. Vorlesung

30.06.2025

Observation: $ab \,\Gamma^*\, cd \Leftrightarrow cd \,\Gamma^*\, ab \Leftrightarrow ba \,\Gamma^*\, dc$ and $A \in \mathcal{I}(G) \Leftrightarrow A^{-1} \in \mathcal{I}(G)$

Proof of thm 4.1
Let $ab \in \hat{A}$, assume wlog. $ab \in A$.
Case 1: $ab \in F$
Then: $ab \in F \cap \hat{A}$. Take $cd \in A$. (We want to show that $cd \in F$) $ab \,\Gamma^*\, cd$. Along Γ-relation: $\left.\begin{matrix} a_1b_1 \,\Gamma\, a_2b_2 \\ a_1b_1 \in F \end{matrix}\right\} \Rightarrow a_2b_2 \in F$. From this follows $cd \in F$.
- So $A \subseteq F$
- We know that $F \cap F^{-1} = \emptyset$, since F is an orientation $\rightsquigarrow A^{-1} \cap F \neq \emptyset \rightsquigarrow F \cap \hat{A} = A$

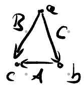
Case 2: $ba \in F$, $ba \in A^{-1}$
Similarly $A^{-1} \cap F \rightsquigarrow A \cap F = \emptyset \rightsquigarrow F \cap \hat{A} = A^{-1}$

Corollary: $\mathcal{G}$ comparability class. $A \in \mathcal{I}(\mathcal{G}) \Rightarrow A \cap A^{-1} = \emptyset$ (and not $A = A^{-1}$)

Proof: $\mathcal{G}$ comparability graph. Then we have: $\exists$ transitive orientation. With Thm 4.1: $F \cap \hat{A} = A$ or $F \cap \hat{A} = A^{-1}$ but $F \cap F^{-1} = \emptyset$. Hence: $A \cap A^{-1} = \emptyset$.

Triangle Lemma: Let $\mathcal{G}$ be an (undirected) graph. $A, B, C \in \mathcal{I}(\mathcal{G})$ and triangle abc in $\mathcal{G}$. Consider only the edges ab, ac, bc. We consider $ab \in C$, $ac \in B$, $bc \in A$.

Let $A \neq B$, $A \neq C^{-1}$



(i) Let also $b'c' \in A \Rightarrow ab' \in C$, $ac' \in B$

(ii) Let also $a'b' \in C$, $b'c' \in A \Rightarrow a'c' \in B$

Important: $A = C$, $B = C$ ... are possible, b', c', a' does not have to be new vertices!

Proof:

(i) (We do only one step of the gamma-chain, and the rest is sort of induction)

It is enough to consider one step in Γ-chain $bc \,\Gamma^*\, b'c'$. We have two cases: $b = b'$, $cc' \notin E$ or $c = c'$, $bb' \notin E$.

<u>First</u>: $c' \neq e$ since $cc' \notin E$. If $ac' \in E$ then $\underset{\rho_{C^{-1}}}{ba} \,\Gamma\, \underset{\rho_A}{bc'}$ thus $C^{-1} = A$. This is a contradiction to $C^{-1} \neq A$.

Hence $ac' \notin E$

$ac \,\Gamma\, ac'$, thus $ac' \in B$ Because ac & ac' must be in the same orientation class

We have $c = c'$, $bb' \notin E$. If $ab' \notin E$ then $\underset{\rho_A}{b'c} \,\Gamma\, \underset{\rho_B}{ac}$ this is a contradiction to $A \neq B$.

So $ab' \in E$ and $\underset{\rho_C}{ab} \,\Gamma\, ab'$ thus $ab' \in C$.

(ii) We first apply part (i) to $c = \bar{a}$ (and $a = \bar{b}$, $b = \bar{c}$, $\bar{A} = C$, $\bar{C} = B^{-1}$, $\bar{B} = A^{-1}$) We need to show that $\bar{A} \neq \bar{B}$, $\bar{A} \neq \bar{C}^{-1}$

- $(\bar{A})^{-1} = (C)^{-1} \neq A = (A^{-1})^{-1} = (\bar{B})^{-1} \Rightarrow \bar{A} \neq \bar{B}$
- $C = \bar{A} \neq \bar{C}^{-1} = (B^{-1})^{-1} = B$

case $B \neq C$: now apply part (i).

$\Rightarrow ab \in B$, $b'c \in A$. again part (i) to triangle $ab'c$ gives $a'c' \in B$

case $B = C$:

$A \neq B$, $A \neq C^{-1}$. Part (i) gives $ab' \in C$, $ac' \in B$. If $a'c' \notin E$ then $\underset{\rho_{C^{-1}}}{b'a'} \,\Gamma\, \underset{\rho_A}{b'c'}$ which is a contradiction to $A \neq C^{-1}$. Let $a'c' \in D \in \mathcal{I}$
assume $D \neq B$, otherwise we are done

Apply part (i) to c' in the triangle $a'b'c'$ with edges $c'a'$, ab', $c'b'$.

<u>Have to check</u>: $B^{-1} = C^{-1} \neq D^{-1}$, $C^{-1} \neq A$

We have $ba \in C^{-1}$. From this follows $ac' \in D$ but $B = D$.

11. Vorlesung

07.07.2025

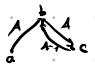
Theorem 4.4. $A \in \mathcal{I}(\mathcal{G}) \Rightarrow A = A^{-1}$ or $A \cap A^{-1} = \emptyset$ and A, A^{-1} are transitive

proof:

we know $A = A^{-1}$ or $A \cap A^{-1} = \emptyset$

<u>Case</u>: $A \cap A^{-1}$

We show that A is transitive



$\xrightarrow{\text{to show}} ac \in A$

If $ac \notin E$ then $\underset{\rho_A}{ab} \,\Gamma\, \underset{\rho_{A^{-1}}}{cb} \Rightarrow A = A^{-1}$ This is a contradiction to $A \cap A^{-1} = \emptyset$

So we have $ac \in E$ thus $ac \in B$, $B \in \mathcal{I}(\mathcal{G})$. We show that $A = B$.

<u>Assume</u> $B \neq A$ for the sake of contradiction

We use the triangle Lemma. Note that $A \neq B$ and $A \neq C^{-1} = A^{-1}$ We apply the lemma part (i) on $b'c' = ab$ as a new base.

So, we get $\left.\begin{array}{l} ac' \in B \\ \ddot{a}b \in A \end{array}\right\} \Rightarrow A = B$, wich is a contradiction to our assumption

Def.: $[B_1, B_2, \dots, B_k]$ is a $\mathcal{G}$-Decomposition if $\hat{B}_1 + \dots + \hat{B}_k = E$ ($B_1 \dots B_k$ are disjoint color classes) and $B_i \in \mathcal{I}(\hat{B}_i + \dots + \hat{B}_k)$ for $i = 1..k$

Note Algo 7 computes a $\mathcal{G}$-decomposition or stops at "not a comparability graph"

Def: a rainbow triangle is a subgraph on $\{a, b, c\}$ with $\tilde{bc} \in \tilde{A}, \tilde{ac} \in \tilde{B}, \tilde{ab} \in \tilde{C}$ with $\hat{A}, \hat{B}, \hat{C} \in \mathcal{I}(\mathcal{G})$ pairwise distinct

Theorem 4.6

$A \in \mathcal{I}(\mathcal{G})$ $D \in \mathcal{I}(E - \hat{A})$ <u>Then either (i) or (ii) hold:</u>

(i) $D \in \mathcal{I}(\mathcal{G})$ and $A \in \mathcal{I}(E - \hat{D})$ <u>OR</u>

(ii) $D = B + C$, $\hat{A}, \hat{B}, \hat{C}$ in a rainbow-triangle ($B, C \in \mathcal{I}(\mathcal{G})$, disjoint)

<u>Proof:</u>

Removing $\hat{A}$ introduces non-edges and possibly new Γ-relations. These new Γ-relations can cause implication classes to merge.

It follows: $D \in \mathcal{I}(\mathcal{G} - \hat{A})$ then D is the disjunt union of some previous implication classes.

<u>Case 1:</u> $D = B + C + \dots$ $B, C \in \mathcal{I}(\mathcal{G})$.

Then $\exists$ rainbow triangle $\hat{A}, \hat{B}, \hat{C}$

If D also merges with implication class X then $\exists$ rainbow triangle $\hat{A}, \hat{B}, \hat{X}$. With the second part of the triangle-lemma, we have $\hat{X} = \hat{C}$.

Thus $D = B + C$.

<u>Case 2:</u> $D \in \mathcal{I}(\mathcal{G})$

By case (1) every implication class of $\mathcal{I}(\mathcal{G} - \hat{D})$ is a union of ≤ 2 implication classes of $\mathcal{I}(\mathcal{G})$

If A merges with X in $\mathcal{G} - \hat{B}$ there is an rainbow-triangle $\hat{A}, \hat{B}, \hat{X}$. But then D merges with X or X^{-1} in $\mathcal{G} - \hat{A}$ ↯

$\Rightarrow A \in \mathcal{I}(\mathcal{G} - \hat{D})$

Theorem 4.7

The following are equivalent

(i) $\mathcal{G}$ comparability graph

(ii) $A \cap A^{-1} = \emptyset$ $\forall A \in \mathcal{I}(\mathcal{G})$

(iii) $\forall \mathcal{G}$-decomposition $[B_1, \dots, B_k]$ has $B_i \cap B_i^{-1} = \emptyset$ $\forall i \in [k]$

<u>Proof:</u>

(i) $\Rightarrow$ (ii) done by theorem 4.1)

(ii) $\Rightarrow$ (iii) Let $[B_1, \dots, B_k]$ be any $\mathcal{G}$-decomposition.

We use induction on k

$k = 1$. $B_1 \in \mathcal{I}(\mathcal{G})$ so $B_1 \cap B_1^{-1} = \emptyset$ by (ii)

<u>$k \geq 2$:</u> Again $B_1 \cap B_1^{-1} = \emptyset$ by (ii)

$[B_2, \dots, B_k]$ is a $\mathcal{G}$-decomposition of $\mathcal{G} - \hat{B}_1$. We have to verify that $D \cap D^{-1} = \emptyset$ $\forall D \in \mathcal{I}(\mathcal{G} - \hat{B}_1)$.

By Theorem 4.6 we have that $D \in \mathcal{I}(\mathcal{G})$ and $D \cap D^{-1} = \emptyset$ holds by (i) or $D = B + C$ for $B + C \in \mathcal{I}(\mathcal{G})$

$$D \cap D^{-1} = (B + C) \cap (B + C)^{-1} = (B + C) \cap (B^{-1} + C^{-1}) = (B \cap B^{-1}) \cup (B \cap C^{-1}) \cup (C \cap C^{-1}) \cup (C \cap B^{-1})$$

- $B \cap B^{-1}$, $C \cap C^{-1}$ are empty by assumption
- $B \cap C^{-1}$, $C \cap B^{-1}$ are empty since we have a rainbow triangle $\hat{B}_1, \hat{B}, \hat{C}$.

So (ii) holds for $\mathcal{G} - \hat{B}_1$ by induction $B_i \cap B_i^{-1} = \emptyset$ $\forall i \geq 2$

(iii) $\Rightarrow$ (i)

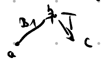
Induction on k: $[B_1, \dots, B_k]$

$k = 1$: $B_1 \cap B_1^{-1} = \emptyset$ By Thm 4.4 B_1 is a transitive orientation. ✓

$k \geq 2$: $[B_2, \dots, B_k]$ is $\mathcal{G}$-decomposition of $\mathcal{G} - \hat{B}_1$ and fulfills (ii) $\leadsto$ by induction $\mathcal{G} - \hat{B}_1$ has a transitive orientation T.

Claim: $B_1 + T$ is transitive orientation of $\mathcal{G}$.

(orientation part is clear)



If $ac \notin E \Rightarrow ab \underset{\in B_1}{\Gamma} cb \Rightarrow cb \in B_1$ ↯ $bc \in T$.

So $ac \in E$. Assume that $ca \in T \Rightarrow T$ is not transitive ↯

Assume that $ca \in B_1 \Rightarrow B_1$ is not transitive ↯ $(B_1 \cap B_1^{-1} = \emptyset \Rightarrow B_1$ transitive$)$

Thus we have that $ac \in T$ or $ac \in B_1$. (similar for ${}^{T}\!\wedge^{B_1}$)

Corollary: Algo 7 determines correctly whether $\mathcal{G}$ is a comparability graph in $\mathcal{O}(\Delta(\mathcal{G})(|E|+|V|))$.

12. Vorlesung

14.07.2025

Theorem: Algorithm 8 computes correctly $\chi(\mathcal{G}), \omega(\mathcal{G})$ for $\mathcal{G}$ is a comparability graph.

proof:

Claim 1: h is a proper coloring

$\forall uv \in E(\mathcal{G})$ to show that $h(u) \neq h(v)$.

Wlog let $uv \in F$. Hence $\sigma(u) < \sigma(v)$. Thus $h(u)$ is set already when $v = \sigma(i)$.

$h(v) = 1 + \max\{h(w) \mid wv \in F\} \geq 1 + h(u)$.

From Claim 1 follows: $\max_{v \in V}(h(v)) \geq \chi(\mathcal{G})$.

Claim 2: C is a clique in $\mathcal{G}$.

Let $C = \{w_\chi, w_{\chi-1}, \dots, w_1\}$ (w_i are the selected vertices in the respective iteration) Thus $\chi = h(w_\chi) = 1 + \max(\{h(v) \mid vw_\chi \in F\})$

From this follows $h(w_{\chi-1}) = \chi - 1$, $w_{\chi-1}w_\chi \in F$. In general it holds: $h(w_{\chi-i}) = \chi - i = 1 + \max(\{h(u) \mid uw_{\chi-i} \in F\})$. Thus $h(w_{\chi-i-1}) = \chi - i - 1$.

$\Rightarrow C$ is a directed path in F. By transitivity of F, C is a clique.

From Claim 2 follows: $\chi = |C| \leq \omega(\mathcal{G})$.

Claim 3: perfectness & correctness

$$\underset{\text{Claim 1}}{\chi(\mathcal{G}) \leq} \max_{v \in V}(h(v)) \underset{\text{Claim 1}}{=} \chi \underset{\text{Claim 2}}{=} |C| \underset{\text{Claim 2}}{\leq} \omega(\mathcal{G}) \underset{\text{general}}{\leq} \chi(\mathcal{G})$$

Corollary: runtime of algo 8 is in $\mathcal{O}(|V|+|E|)$.

Theorem: Algo 9 computes $\alpha(\mathcal{G}), k(\mathcal{G})$ for $\mathcal{G}$ comparability graph.

Special case: $\mathcal{G}$ is bipartite. (Every bipartite graph is a comparability graph)

Def.: $M \in E(\mathcal{G})$ is a *matching* if $\forall v \in V$ exists at most one $e \in M$: $v \in e$

It holds that

- $k(\mathcal{G}) \leq |V| - \max\{|M| : M$ matching$\}$

Def.: $S \subseteq V_\mathcal{G}$ is a *vertex cover* if $\forall e \in E$ $\exists v \in S$: $v \in e$.

Note: S is a vertex cover $\Leftrightarrow V - S$ is independent

- $\alpha(\mathcal{G}) = |V| - \min\{|S| : S$ vertex cover$\}$
- $k(\mathcal{G}) \geq \alpha(\mathcal{G})$

Theorem (König) for $\mathcal{G}$ bipartite it holds that $|V| - \max\{|M| : M$ matching$\} = k(\mathcal{G}) = \alpha(\mathcal{G}) = |V| - \min\{|S| : S$ vertex cover$\}$

proof: $\alpha(\mathcal{G}) = k(\mathcal{G})$ since $\mathcal{G}$ is a comparability graph and thus $\mathcal{G}$ is perfect. $k(\mathcal{G}) = |V| - \max\{|M| : M$ matching$\}$

Rule: v,w consecutive in clique cover $\overset{\text{Rule}}{\Leftrightarrow}$ $v'w'' \in M$

proof: clique cover $V_1 + \dots + V_k$ of G. We have k cliques in G that cover all vertices. Thus we find k directed paths in G_F $\Rightarrow$ We have $2k$ starts and ends of paths. Then we have $2k$ vertices of B (Bipartite graph ($B = V' + V'', E$), $V' = \{v' \mid v \in V\}$, $V'' = \{v'' \mid v \in V\}$) of B are unmatched.
$\Rightarrow |M| = |V| - k$. The other way around also holds.

It follows that $k(G) = |V_B| - \max\{|M| : M \text{ matching in } B\} \overset{\text{König}}{=} |V_B| - \min\{|S| : S \text{ vertex cover in } B\}$

We get a smallest vertex cover S of B. Then $\forall v \in S$: $S - v$ is not a vertex cover

Observation: $|S \cap \{v', v''\}| \leq 1$, $\forall v \in V_G$

proof: Assume not. Since $S - v'$ not a vertex cover: $\exists w \in V_G$, $w \notin S$, $vw \in F$. Since $S - v''$ not a vertex cover $\exists u \in V_G$, $u \notin S$, $uv \in F$.
$\Rightarrow uw \in F \Rightarrow u'w'' \in E_B$, $u'w''$ isn't covered in S ↯ to S being a vertex cover.

Hence $Y = \{v \in V_G : S \cap \{v', v''\} = \emptyset\}$ has exactly $|V| - |S| = |V| - |M| = k(G)$ elements.

Observation: Y is an independent set in G

proof:
$vw \in E_G$ wlog: $vw \in F \Rightarrow v'w'' \in E_B$ but $S \cap \{v', w''\} = \emptyset$ ↯ to S being a vertex cover
Hence $\alpha(G) \geq |Y| = k(G) \geq \alpha(G) \Rightarrow |Y| = \alpha(G)$
for all graphs

Algo G:

	runtime
· compute transitive orientation of F	$O(\lvert V \rvert + \lvert E \rvert^2)$
· compute bipartite graph B	$O(\lvert V \rvert + \lvert E \rvert)$
· compute max matching M	$O(\lvert E \rvert^{1,5})$
compute min vertex cover S	$O(\lvert V \rvert + \lvert E \rvert)$
· derive clique cover on $\lvert V \rvert - \lvert M \rvert$	$O(\lvert V \rvert + \lvert E \rvert)$
· derive i-set on $\lvert V \rvert - \lvert S \rvert$	$O(\lvert V \rvert + \lvert E \rvert)$

13. Vorlesung

21.07.2025

Def: a split graph is a graph, where G and $\bar{G}$ are chordal

Theorem 5.3 For every graph $G = (V, E)$ the following are equivalent:

(i) G is a splitgraph

(ii) $V = K + S$ with K is a clique and S is an independent set

(iii) $C_4, C_5 \notin_{ind} G$ and $C_4 \notin_{ind} \bar{G}$

proof:
(ii) $\Rightarrow$ (i)

$V = K + S$, K clique, S independent set. Let C be a cycle of length at least 4 in G

If $V(C) \cap S = \emptyset$ then C has a chord, since K is a clique. If $C = [v_1, v_2, v_3, \dots]$, $v_2 \in S$ then $v_1, v_3 \in K$ since S is independent we also have $v_1 v_3 \in E$ since K is a clique. Thus G is chordal. By the same argumentation $\bar{G}$ is chordal.
Thus G is a split graph.

(i) $\Rightarrow$ (iii) clear

(iii) $\Rightarrow$ (ii)

We have to find a split K, S. Choose K a maximum clique with $S = V - K$, G_S has the least number of edges.

Assume G_S has an edge xy for the sake of contradiction. We shall find C_4 or $C_5 \subseteq_{ind} G$ or $C_4 \subseteq_{ind} \bar{G}$.

Since K is maximum it exists a vertex $u \in K$ s.t. $ux \notin E$ and $v \in K$ s.t. $vy \notin E$.

If $u = v$ for all choices. Then we find a new clique $K - u + x + y$, that is larger as K. (Contradiction to maximum clique)

So $u \neq v$:

- If $uy, vx \in E$ then $C_4 \subseteq_{ind} G$ ✓
- If $vx, uy \notin E$ then $2K_2 \subseteq_{ind} G$ ✓ ($C_4 \subseteq_{ind} \bar{G}$)

So, wlog. $vx \notin E$, $uy \in E$.

Consider $K' = K - v + y$. (we want to show that K' is a clique.

If $w \in K - \{u, v\}$, $wy \notin E$.

- If $wx \notin E \Rightarrow G_{\{v,w,x,y\}} = 2K_2$ ✓
- If $wx \in E \Rightarrow G_{\{u,w,x,y\}} = C_4$ ✓

So $\forall w \in K - \{u, v\}$ $wy \in E$. In particular K' is a clique. $|K| = |K'|$.

We want to show that $G_{V-K'}$ has fewer edges than G_S.

To show: $|Adj(y) \cap S| > |Adj(v) \cap S|$. (we have $x \in Adj(y)$, $x \notin Adj(v)$)

Assume $t \in S$, $tv \in E$, $ty \notin E$.

Assume $tx \notin E$ then $G_{\{x,y,v,t\}} = 2K_2$

So $tx \in E$.

- If $tu \notin E$ then $G_{\{x,y,u,v,t\}} = C_5$ ✓
- If $tu \in E$ then $G_{\{x,y,u,t\}} = C_4$ ✓

So there is no such $t \in S$. $|Adj(y) \cap S| > |Adj(v) \cap S| \Rightarrow G_{V-K'}$ has fewer edges than G_{V-K}.

Contradiction to the choice of K.

Chapter 6 – Permutation graphs

Def.: $G = (V, E)$ is a permutation graph if G and $\bar{G}$ are comparability graphs.

Theorem: For every (undirected) graph $G = (V, E)$ the following are equivalent:

(i) G and $\bar{G}$ are comparability graph.

(ii) There exists a vertex ordering σ of G without  and 

(iii) There is an embedding $V \to \mathbb{R}^2$ such that $uv \in E \Leftrightarrow (u_x < v_x \Leftrightarrow u_y < v_y)$

proof:

(i) $\Rightarrow$ (ii)

G has a transitive orientation (V, F_1) and $\bar{G}$ has a transitive orientation (V, F_2). So $F = F_1 + F_2$ is an orientation of the complete graph on V.

Claim: F_1, F_2 transitive $\Rightarrow F = F_1 + F_2$ is transitive.

- F is an orientation of a complete graph is transitive iff F is acyclic
  (Not transitive $\rightsquigarrow$ directed cycle
  directed triangle $\rightsquigarrow$ not transitive)
- Let F has a directed triangle. If all edges of the triangle are in F_i then F_i is not transitive. Or two edges are in F_i then F_i is also not transitive.

Let σ be the topological ordering of $F = F_1 + F_2$.

$\not\exists$ F_1 transitive $\not\exists$ F_2 transitive

(ii) $\Rightarrow$ (iii)

Given σ without and . We obtain a transitive orientation F_1 of $\mathcal{G}$, F_2 of $\bar{\mathcal{G}}$ by orienting left-to-right

Take $\sigma_x = \sigma$ as order of x-coordinates of points for each vertex in V

$F_1 + F_2^{-1}$ is also a transitive orientation of the complete graph of V. $\rightsquigarrow$ We get a second ordering of σ_y as topological ordering of $F_1 + F_2^{-1}$

$uv \in E$ say w.l.o.g. $uv \in F_1$. Then $u_x < v_x$ and $u_y < v_y$

$uv \notin E$ say wlog. $uv \in F_2$. Then $u_x < v_x$ but $vu \in F_2^{-1}$: $u_y > v_y$.

14. Vorlesung

23.07.2025

(iii) $\Rightarrow$ (i)

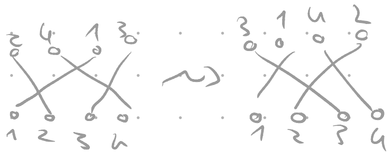
Given an embedding of V in the plane. Orientate an edge $uv \in E$ from u to v iff u is bottom-left of v. This is transitive, thus $\mathcal{G}$ is a comparability graph.

For the non-edge orientate them bottom-right. This is also transitive and thus $\bar{\mathcal{G}}$ is comparability graphs. (One can choose where to use $x_u \leq x_v$ or $y_u \leq y_v$).

Given an ordering π of $[n]$. $\mathcal{G} = \mathcal{G}_\pi$, $V(\mathcal{G}_\pi) = [n]$. $ij \in E \Leftrightarrow (i-j)(\pi_i - \pi_j) < 0$ is an permutation graph

We have a recognation algorithm and we can find $O(|V|+|E|)$ for comparability graphs. We also find the a faster algorithm $O(|V| \log(|V|))$

- For finding the complement, swap both axis.



(Example?)

- Given Intervals $I_1, \dots, I_n$, $I_i = (x_i, y_i)$ s.t. $x_1 \leq x_2 \leq \dots \leq x_k$

<u>Goal</u>: Find the number of translations s.t.
- $x_1 \leq x_2 \leq \dots \leq x_n$
- $y_i' < x_{i+1}'$ $\forall i \in [n-1]$

(Slides: L-graph, flip an axis)

Construction of a conflict graph $\mathcal{G}$ with $V(\mathcal{G}) = \{I_1, \dots, I_n\}$. $I_i I_j \Leftrightarrow x_j - y_i < \sum_{i<k\leq j} (y_k - x_k)$

- $\mathcal{G}$ is a permutation graph

Solution is a minimal set of intervalls that are moved $\Leftrightarrow$ maximal number of intervalls that aren't moved $\Leftrightarrow$ maximum independent set in $\mathcal{G}$.

Theorem 7.1

For every graph $\mathcal{G} = (V, E)$ the following are equivalent:

(i) $\mathcal{G}$ is an interval graph

(ii) $\exists$ vertex ordering σ without

(iii) $\mathcal{G}$ is chordal and $\bar{\mathcal{G}}$ is a comparability graph

(iv) $C_4 \not\subseteq_{ind} \mathcal{G}$ and $\bar{\mathcal{G}}$ is a comparability graph

(V) There exists an ordering $A_1, \dots, A_k$ of the inclusion maximal cliques in $\mathcal{G}$ such that $\forall v \in V$ the numbers in $\{i \mid v \in A_i\}$ are consecutive in $\{1, \dots, k\}$

Proof:

$(i) \Rightarrow (ii)$

$I = (x, y)$

Look at interval representation. W.l.o.g. all endpoints (y) are distinct. We define σ as the left-to-right ordering of the endpoints (y).

Let $u <_\sigma v <_\sigma w$ with $uw \in E \Rightarrow vw \in E$

$(ii) \Rightarrow (iii)$

Let σ a vertex ordering without . So in particular it does not have , so $\mathcal{G}$ is chordal.

It has not so $\bar{\mathcal{G}}$ doesn't contain , so $\bar{\mathcal{G}}$ is a comparability graph.

$(iv) \Rightarrow (v)$

We have that C_4 is not an induced subgraph of $\mathcal{G}$ and $\bar{\mathcal{G}}$ is a comparability graph. We also have $2K_2$ is not an induced subgraph of $\bar{\mathcal{G}}$.

Let F be the transitive orientation of $\bar{\mathcal{G}}$. Let A, B inclusion-maximal cliques. So we have $a \in A \setminus B, b \in B \setminus A$ s.t. $ab \notin E$. If $ab \in F$, then we say $A < B$ and vice versa.

__Claim:__ If $A < B$ then $B \nless A$ and vice versa.

1. Case: $ab \notin F$ and there is an edge $b'a' \notin F$. Since $\bar{S}$ has a transitive orientation $aa' \notin E$, but A is a clique, so $aa' \in E$ ↯
   - ↳ we also have the other orientation of this case
2. Case: Let $ab \notin F$ and $b'a' \notin F$

   Since we are not in case 1, we have $ab' \in E$, $a'b \in E$. Since A, B are cliques, $aa' \in E$, $bb' \in E$. So ab $a'b'$ form a $2K_2$, that is a contradiction.

$\rightsquigarrow <$ is well defined

__Goal: $<$ is acyclic__

Let $A < B < C$. __goal:__ $A < C$.

Let $a_1 b_1 \in F$, $a_1 \in A$, $b_1 \in B$. Let $a_2 b_2 \in F$ $a_2 \in B$, $b_2 \in C$

__Case:__ $b_2 \notin A$

If $b_1 = a_2$. Since F is transitive we also have $a_1 b_2 \in F$, so $A < C$.

__Assume:__ $b_1 \neq a_2$. Then we have $b_1 a_2 \in E$.

We assume $a_1 a_2 \in E$, $b_1 b_2 \in E$. C_4 is not an induced subgraph of $\mathcal{G} \Rightarrow a_1 b_2 \notin E$

By transitivity of F $a_1 b_2 \in F$.

There is a total ordering $A_1 < A_2 < \dots < A_k$ on maximal cliques.

Let $v \in A_i \cap A_k$ with $i < j < k$.

__Goal:__ $v \in A_j$.

Assume for the sake of contradiction that $v \notin A_j$, then there is $w \in A_j$ s.t. $vw \notin E$.

If $vw \in F$. Then we have $A_k < A_j$ ↯ to the assumption, otherwise $A_j < A_i$ which is also a contradiction.

$\Rightarrow v \in A_j$.

$(iii) \Rightarrow (iv)$ Since $\mathcal{G}$ is chordal we have $C_k \notin_{ind} \mathcal{G}$ for $k \geq 4$. i.e. $C_4 \notin_{ind} \mathcal{G}$.

$(v) \Rightarrow (i)$

Given $A_1 < \dots < A_x$ an inclusion-maximal cliques. $\{i \mid v \in A_i\}$ is an interval

Let I_v be the smallest interval s.t. $\{i \mid v \in A_i\} \subseteq I_v$. $vw \in E \Leftrightarrow \exists i \; v, w \in A_i \Leftrightarrow I_v \cap I_w \neq \emptyset$.

15. Vorlesung

28.07.2025

- Lecture + problem class + exercise sheets
  - → little change, more focus on the problem class
    - → small problem to solve