Planar Separator Theorem

A Geometric Approach

based on

A Framework for ETH-Tight Algorithms
and Lower Bounds in Geometric Intersection Graphs

by

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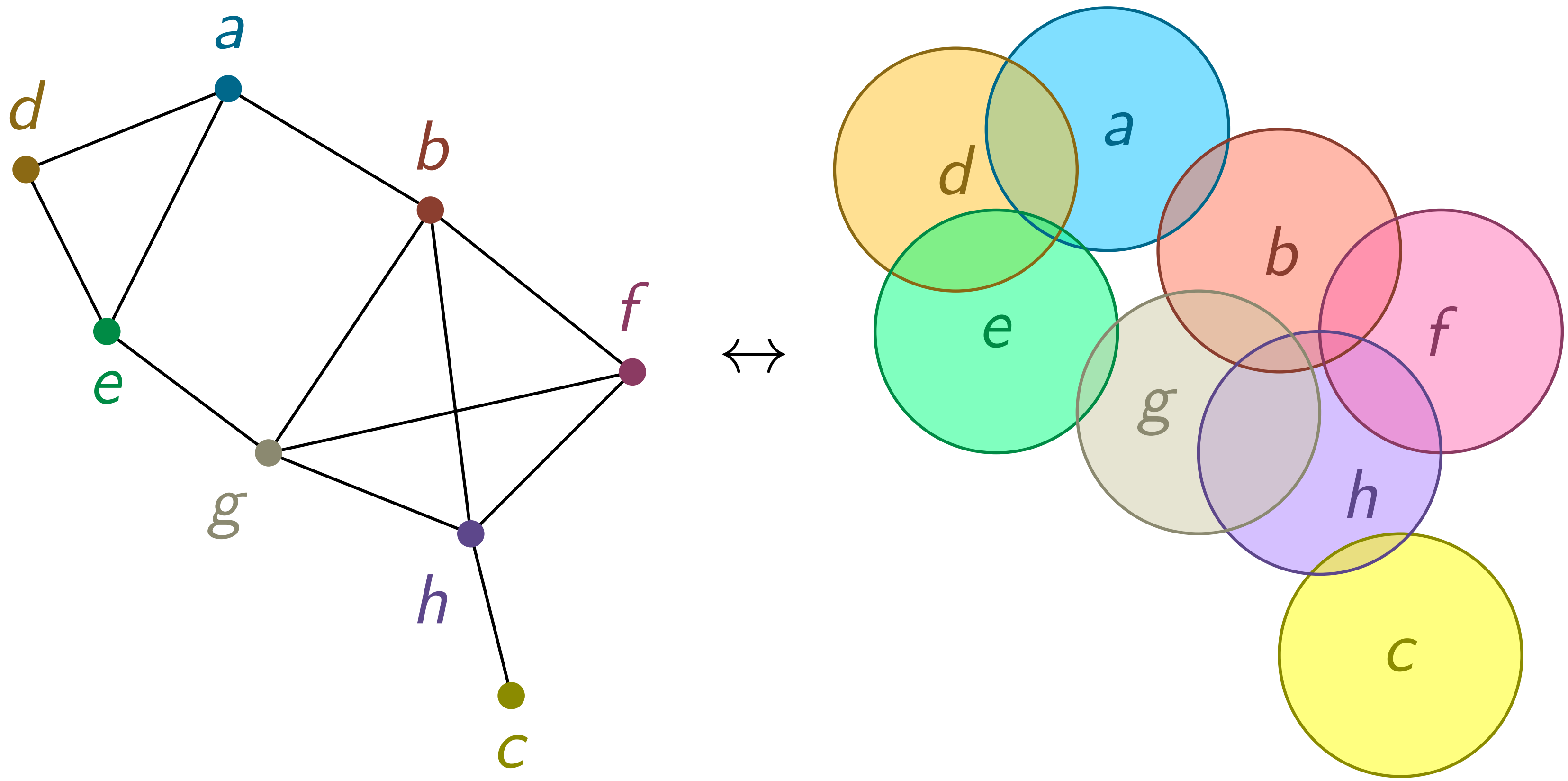
Unit Disk Graphs

- represent vertices as *unit disks*, i.e., disks with diameter 1
- disks intersect iff corresponding vertices are adjacent



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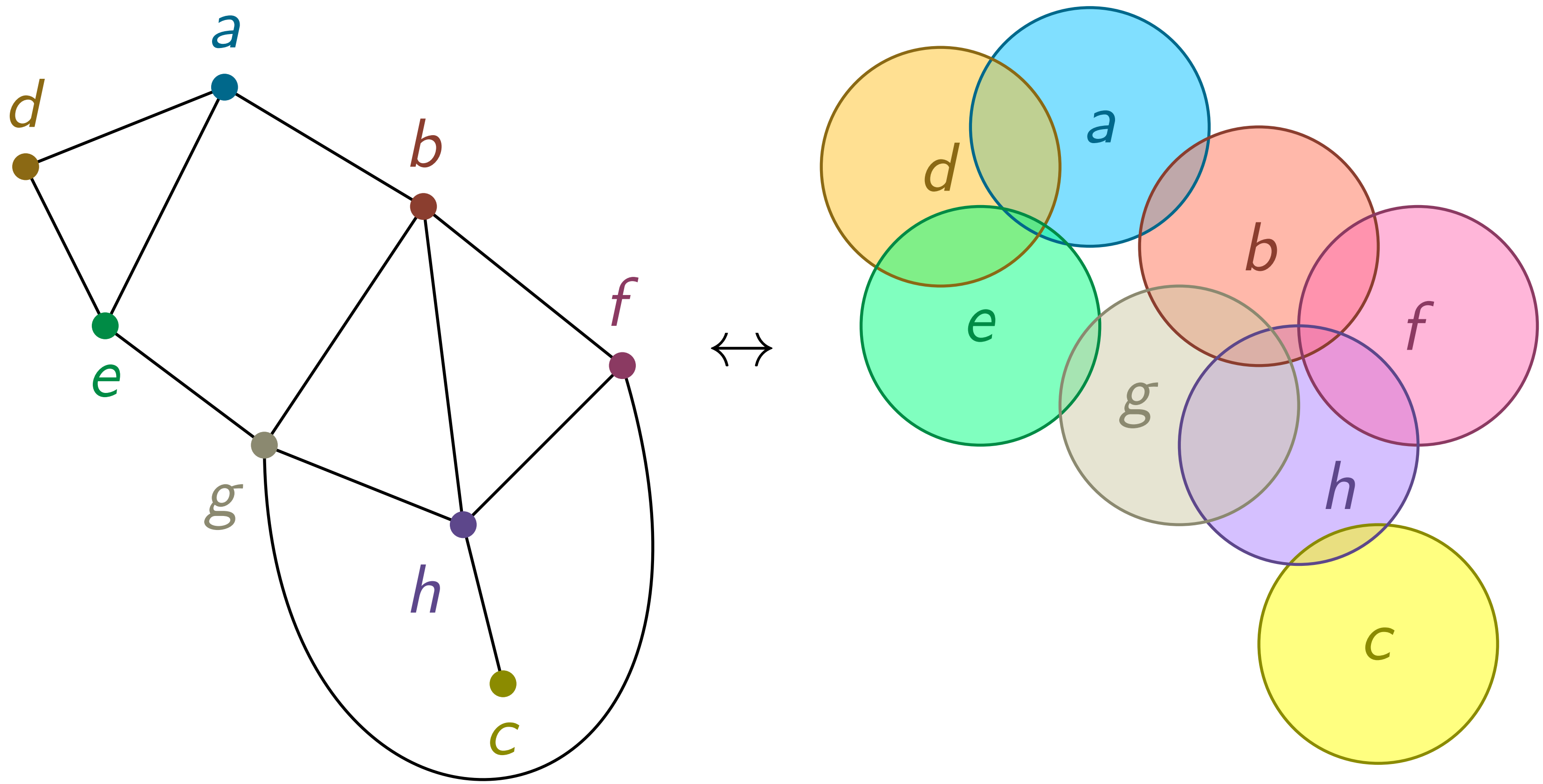
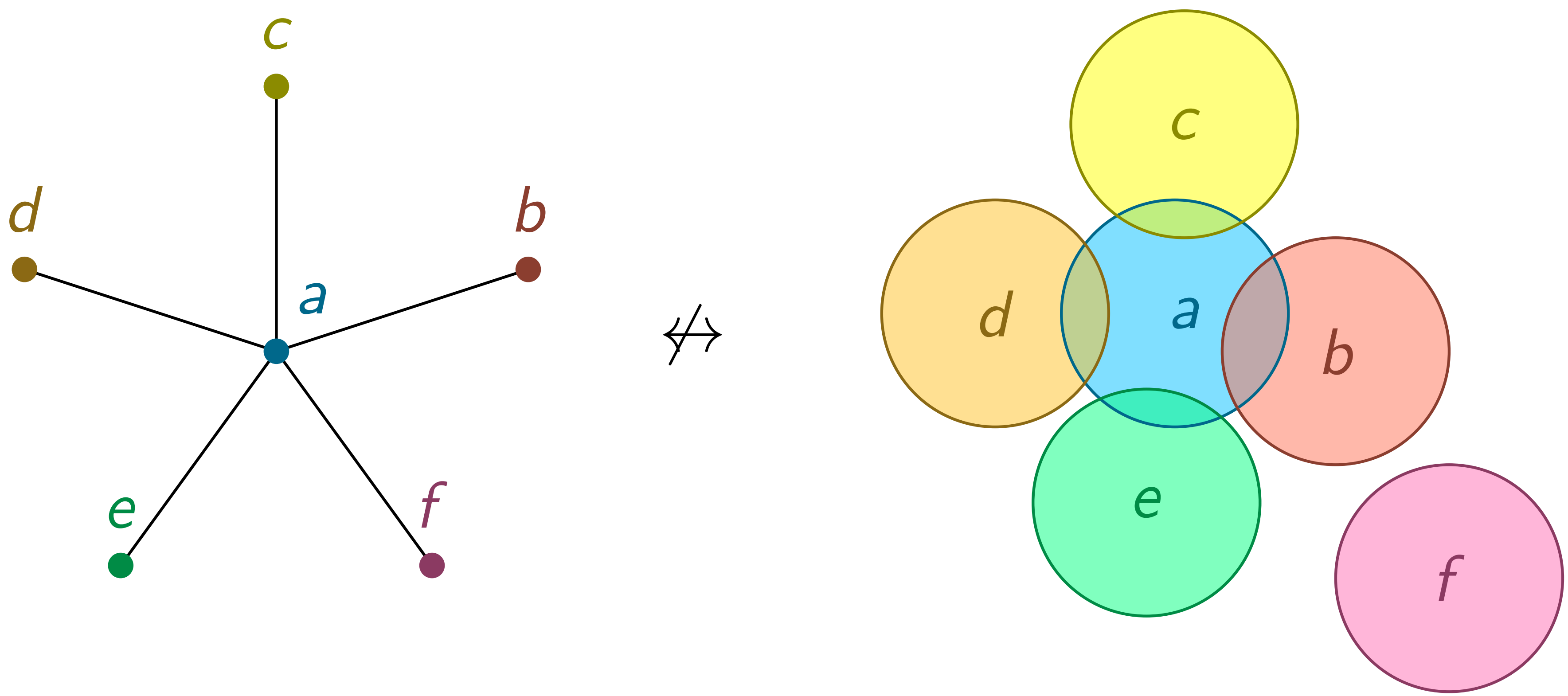
- some unit disk graphs are planar

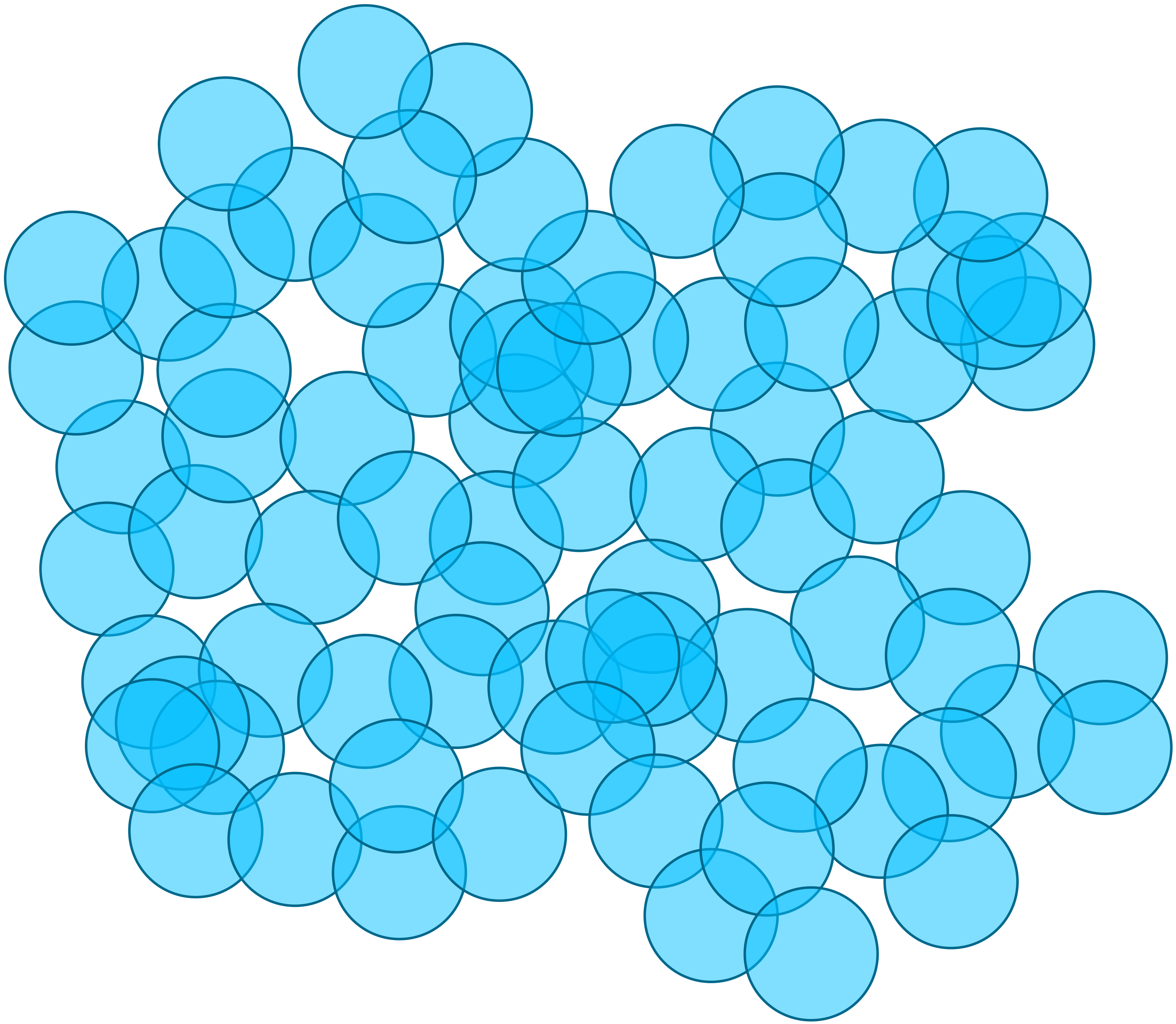


Unit Disk Graphs

- represent vertices as *unit disks*, i.e., disks with diameter 1
- disks intersect iff corresponding vertices are adjacent
- some unit disk graphs are planar
- some planar graphs are not unit disk graphs



Geometric Separators



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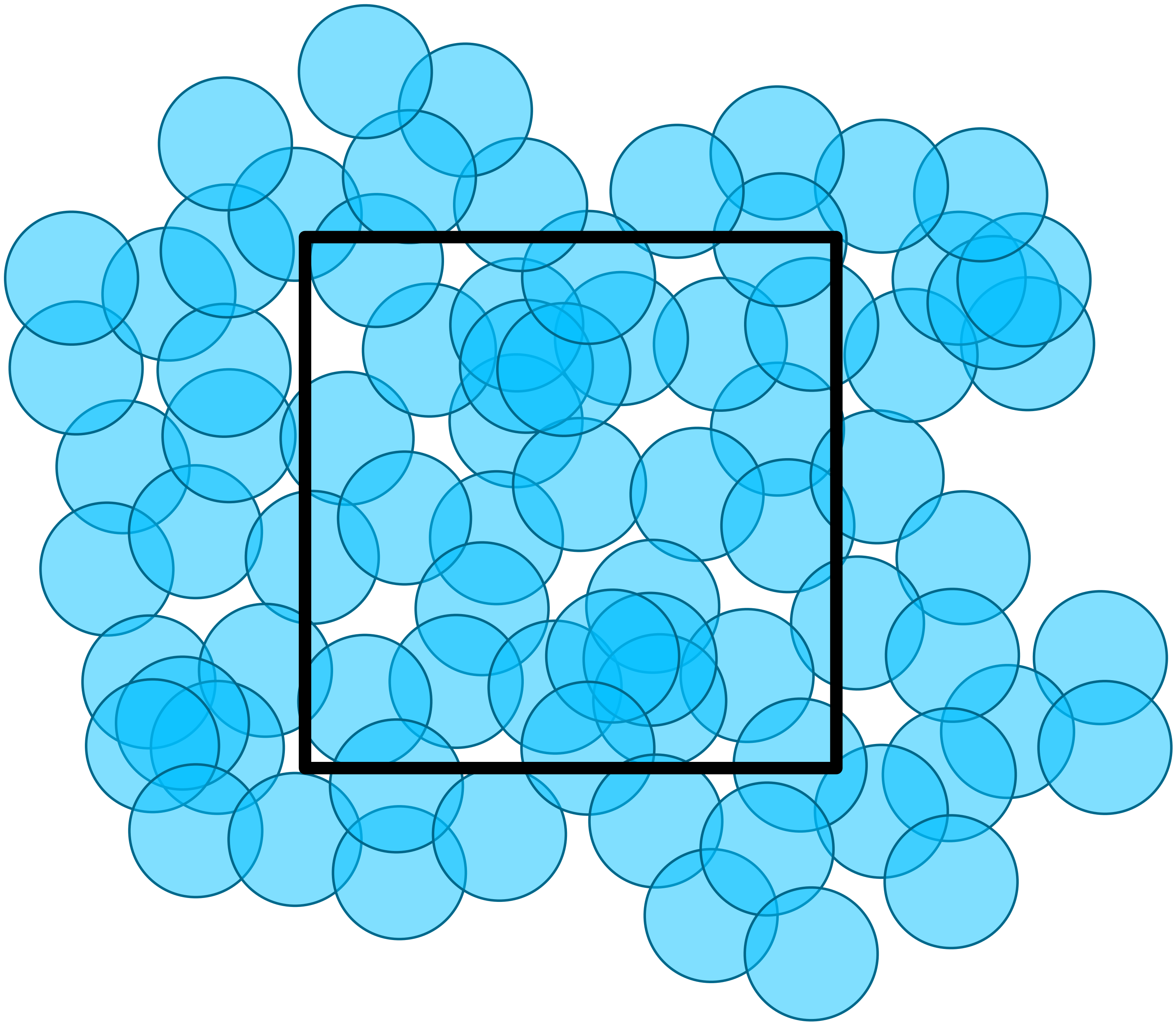

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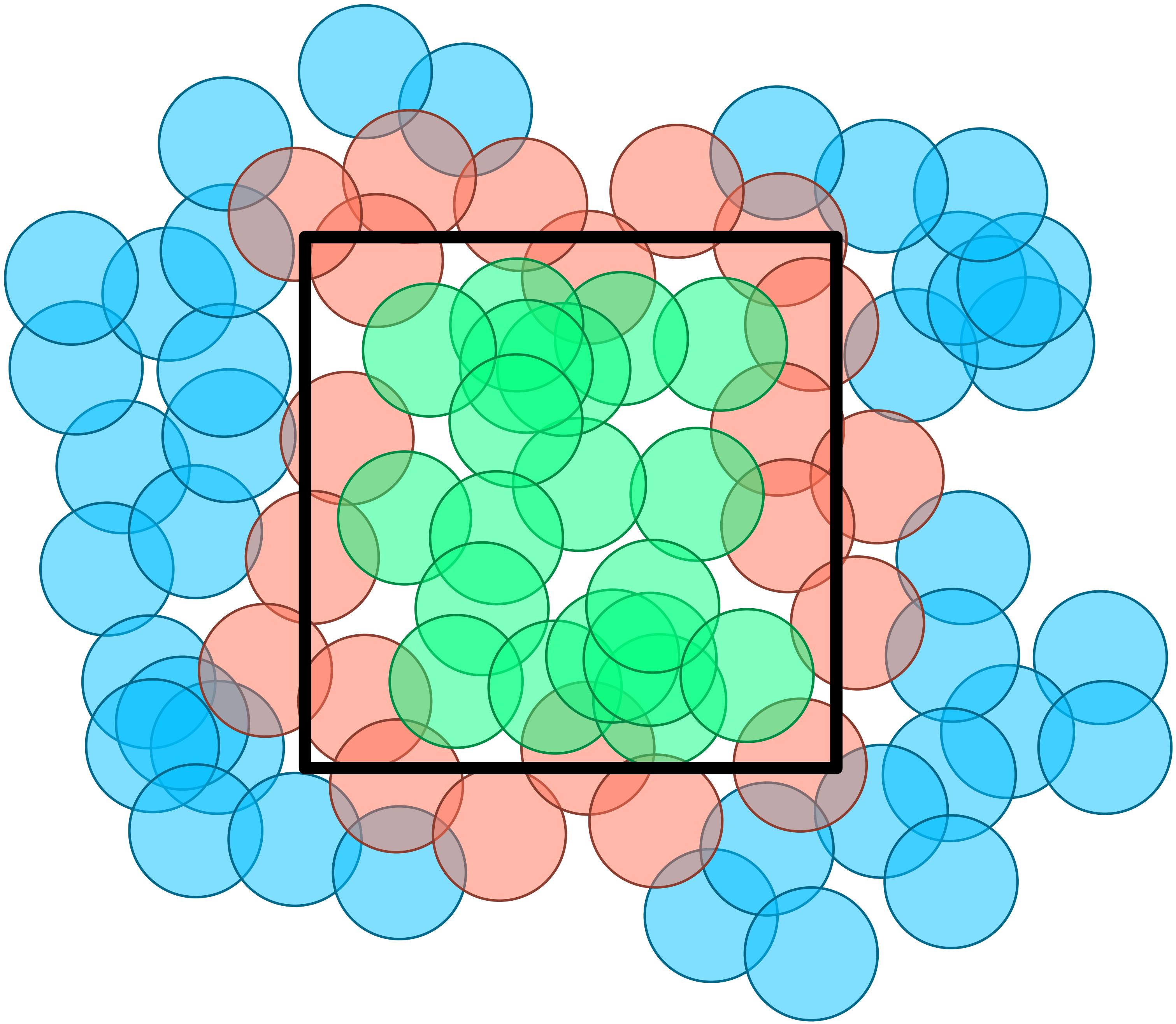

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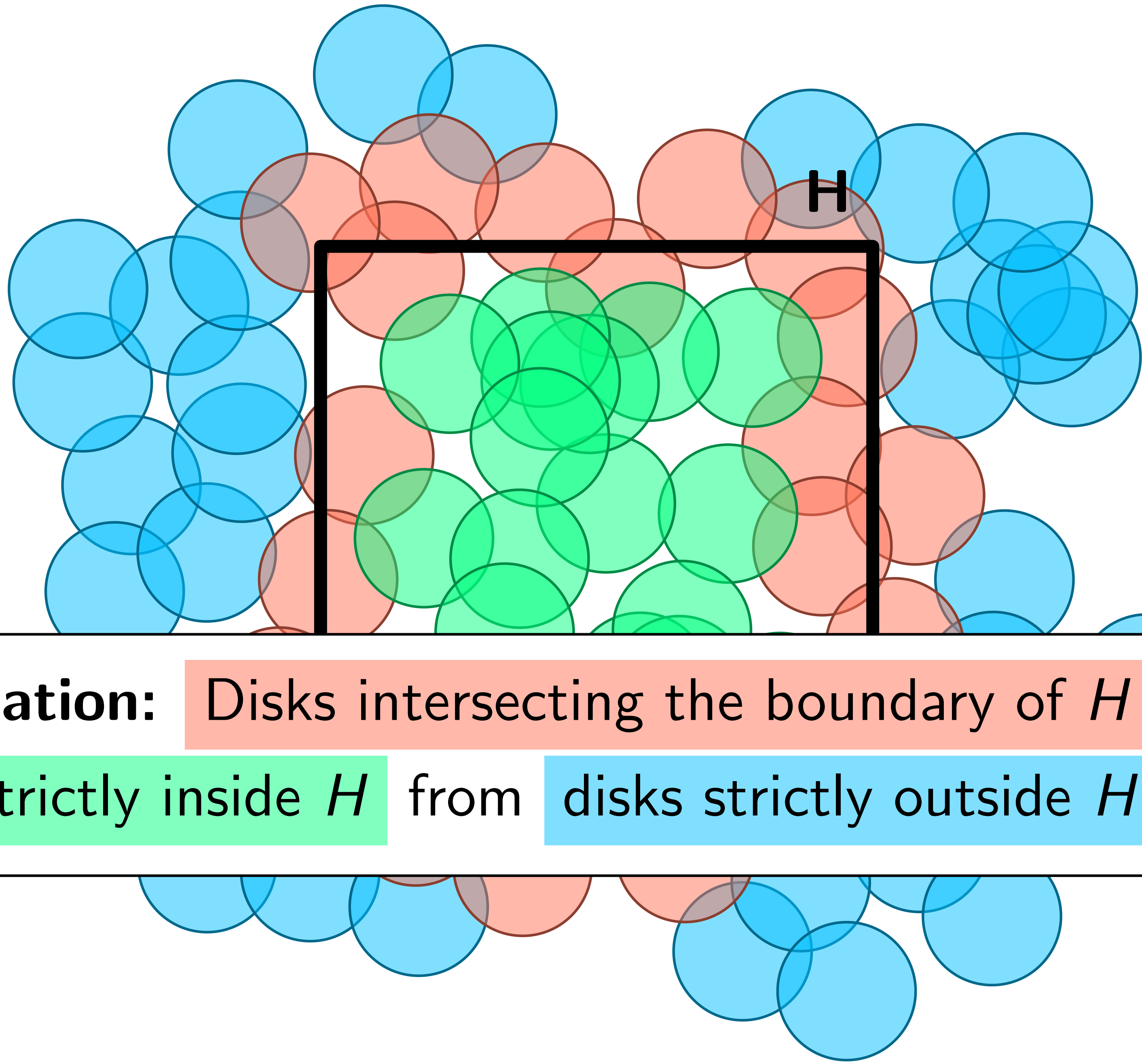


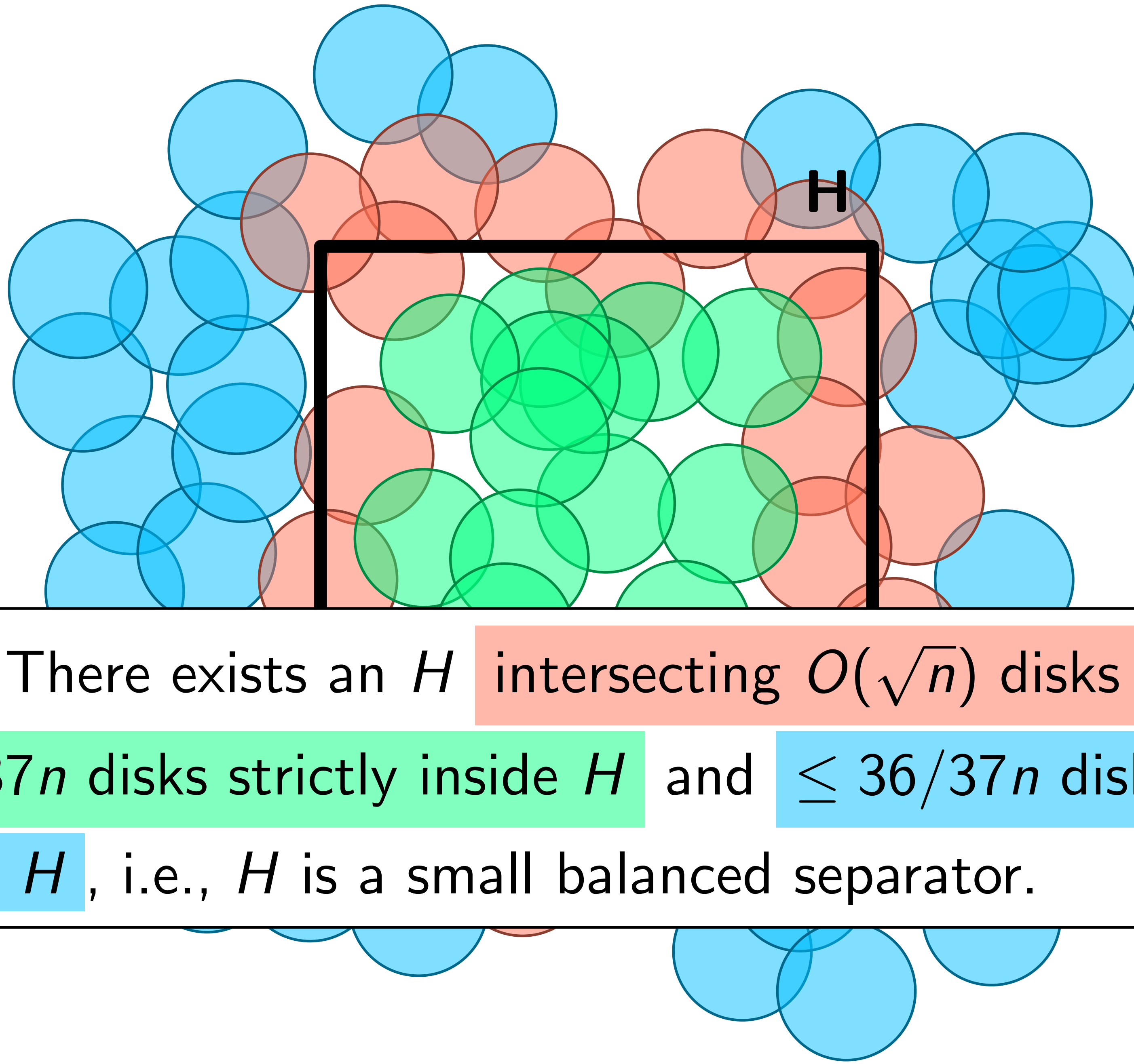
Observation: Disks intersecting the boundary of H separate disks strictly inside H from disks strictly outside H.

Small Balanced Separators

Let $G = (V, E)$ be a graph. $H \subseteq V$ is

- a *separator* if there is a partition H, V_1, V_2 of V so that no edge in E has one endpoint in V_1 and one endpoint in V_2,
- *small* if $|H| \in O(\sqrt{n})$,
- *balanced* if $|V_1|, |V_2| \leq \beta n$ for some constant β

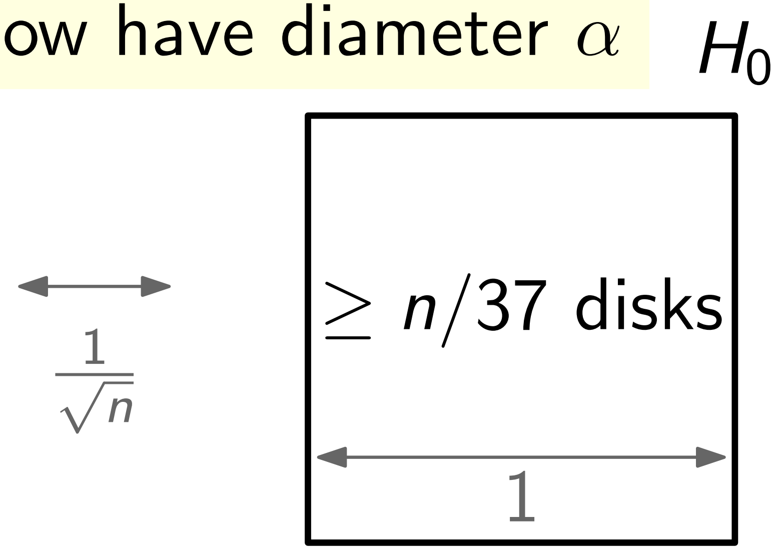
Small Balanced Geometric Separators



Claim: There exists an H intersecting $O(\sqrt{n})$ disks with $\leq 36/37n$ disks strictly inside H and $\leq 36/37n$ disks strictly outside H, i.e., H is a small balanced separator.

Separator Candidates

- choose H_0 as smallest square that contains $\geq n/37$ disks
- resize disk representation so that H_0 has diameter 1
- all disks now have diameter α



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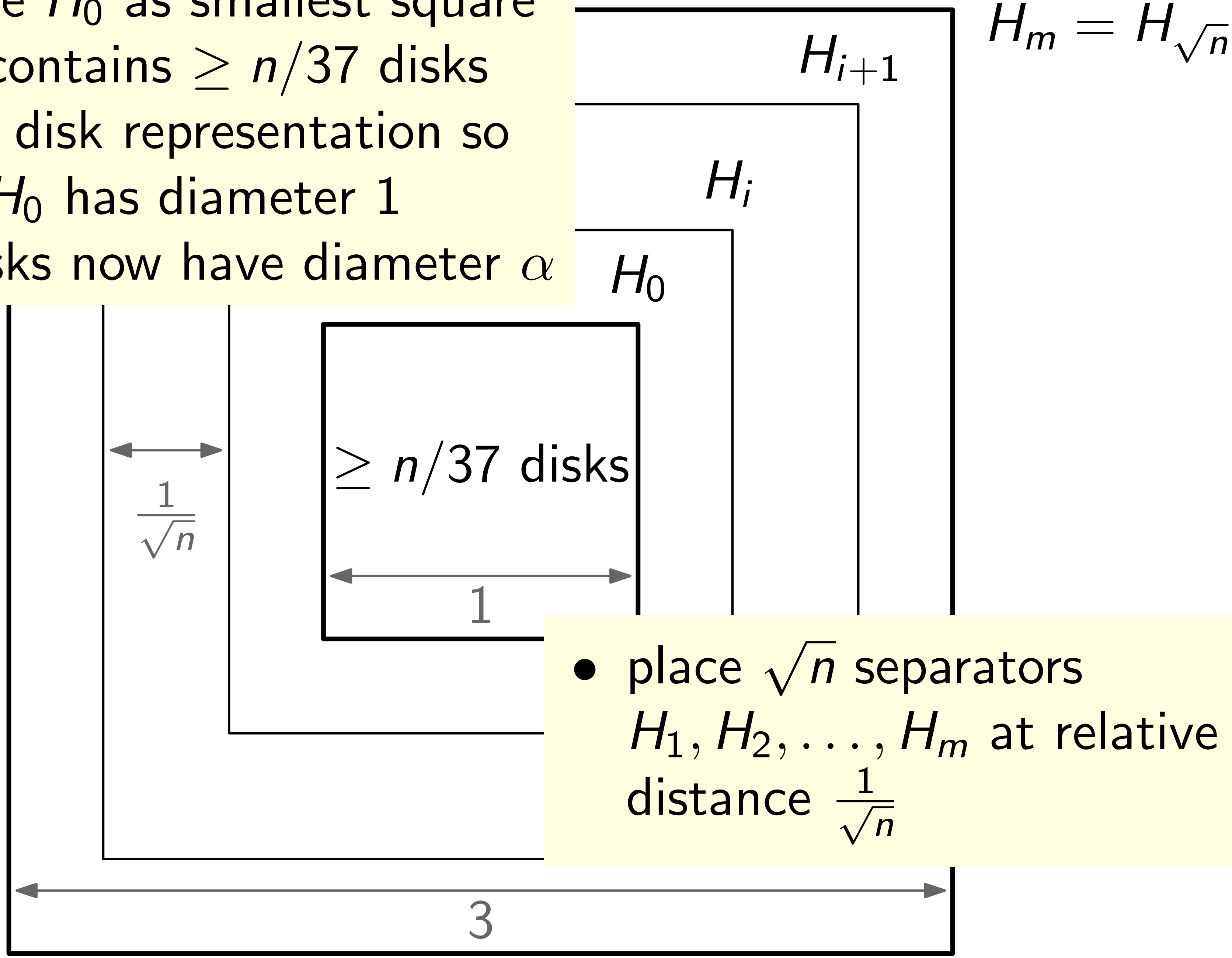

- place $\sqrt{n}$ separators $H_1, H_2, \ldots, H_m$ at relative distance $\frac{1}{\sqrt{n}}$

H_i Is Balanced

Bound number of disks strictly outside H_i:

$$|H_i^{\text{out}}| = n - |H_i^{\cap}| - |H_i^{\text{in}}| \leq n - |H_i^{\text{in}}| \leq n - \frac{1}{37}n = \frac{36}{37}n$$

H_i contains H_0 and H_0 contains $\geq n/37$ disks

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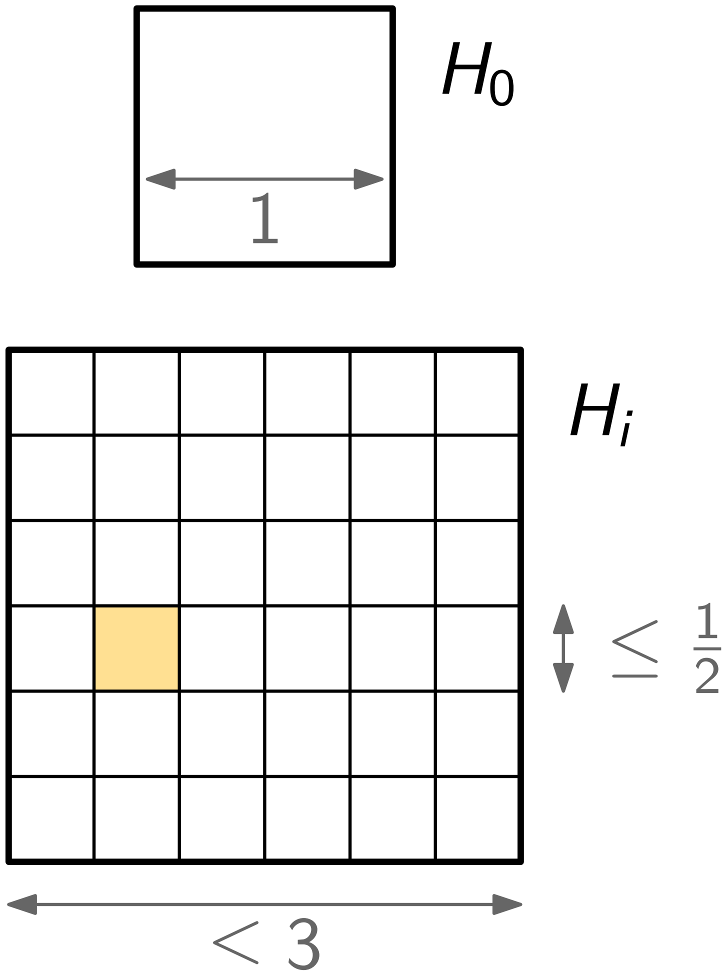

Bound number of disks strictly inside H_i for $\alpha < 1/4$:

- H_0 is *smallest* square that contains $\geq \frac{1}{37}n$ disks
- subdivide H_i into $6^2 = 36$ subsquares with edge length $\leq 1/2$
- every subsquare H_{sub} of H_i touches $< 1/37n$ disks
- $|H_i^{\text{in}}| \leq 36 \cdot \frac{1}{37}n$

H_i Is Balanced

Bound number of disks strictly outside H_i:

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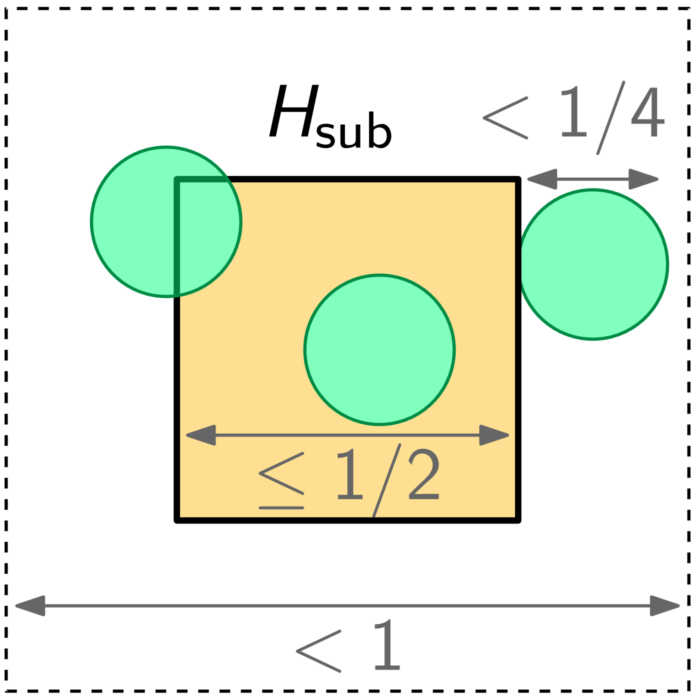

disks that touch H_{sub} are contained inside the dashed square with edge length < 1

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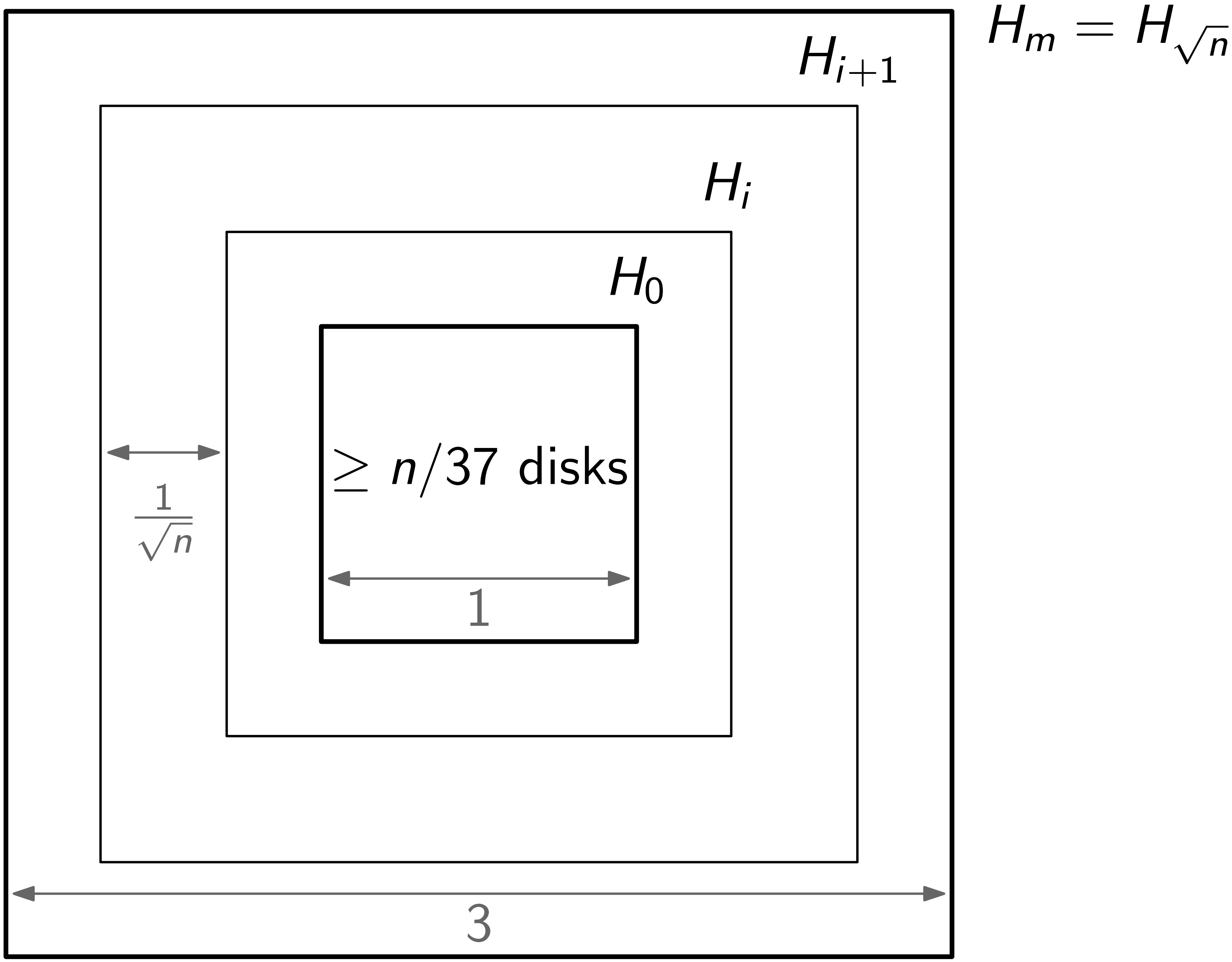
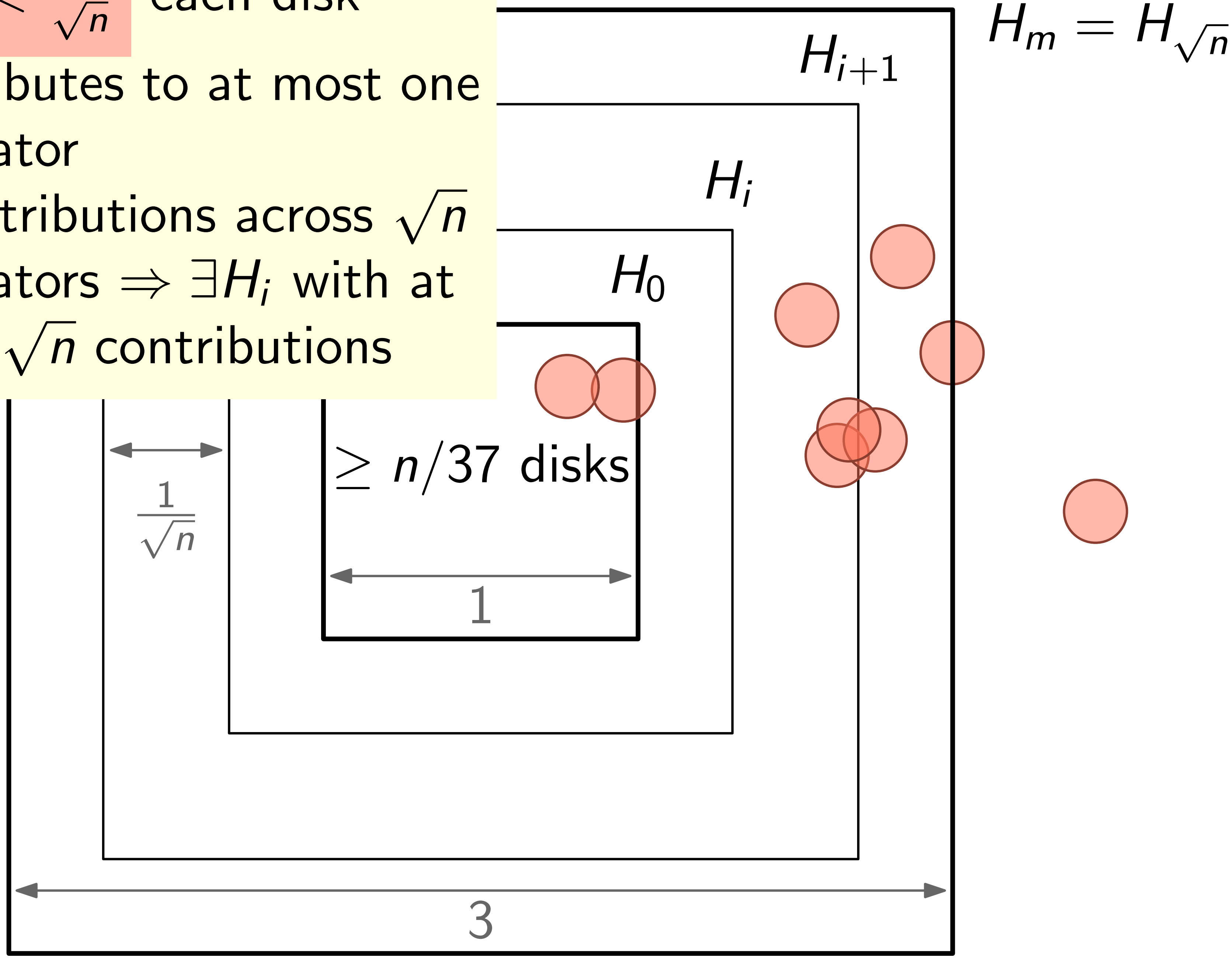
- every subsquare H_{sub} of H_i touches $< 1/37n$ disks
- $|H_i^{\text{in}}| \leq 36 \cdot \frac{1}{37}n$

At Least One H_i Is Small



At Least One H_i Is Small

- if $\alpha < \frac{1}{\sqrt{n}}$ each disk contributes to at most one separator
- n contributions across $\sqrt{n}$ separators $\Rightarrow \exists H_i$ with at most $\sqrt{n}$ contributions



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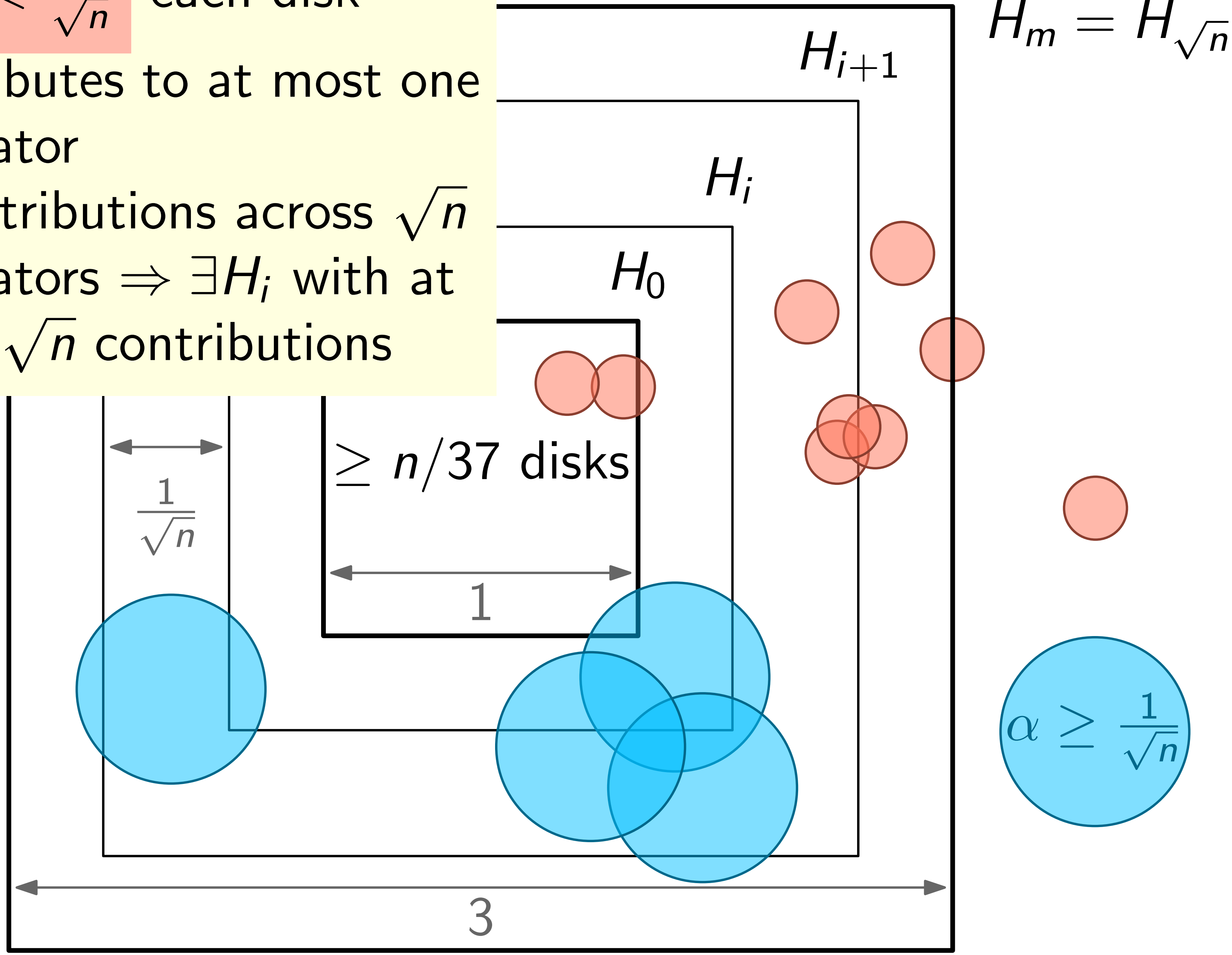


At Least One H_i Is Small

Suppose $\alpha \geq \frac{1}{\sqrt{n}}$. H_m (and therefore each H_i) contains at most

$$4 \cdot \frac{3^2}{\pi(\alpha/2)^2} \in O(\frac{1}{\alpha^2}) \text{ disks.}$$

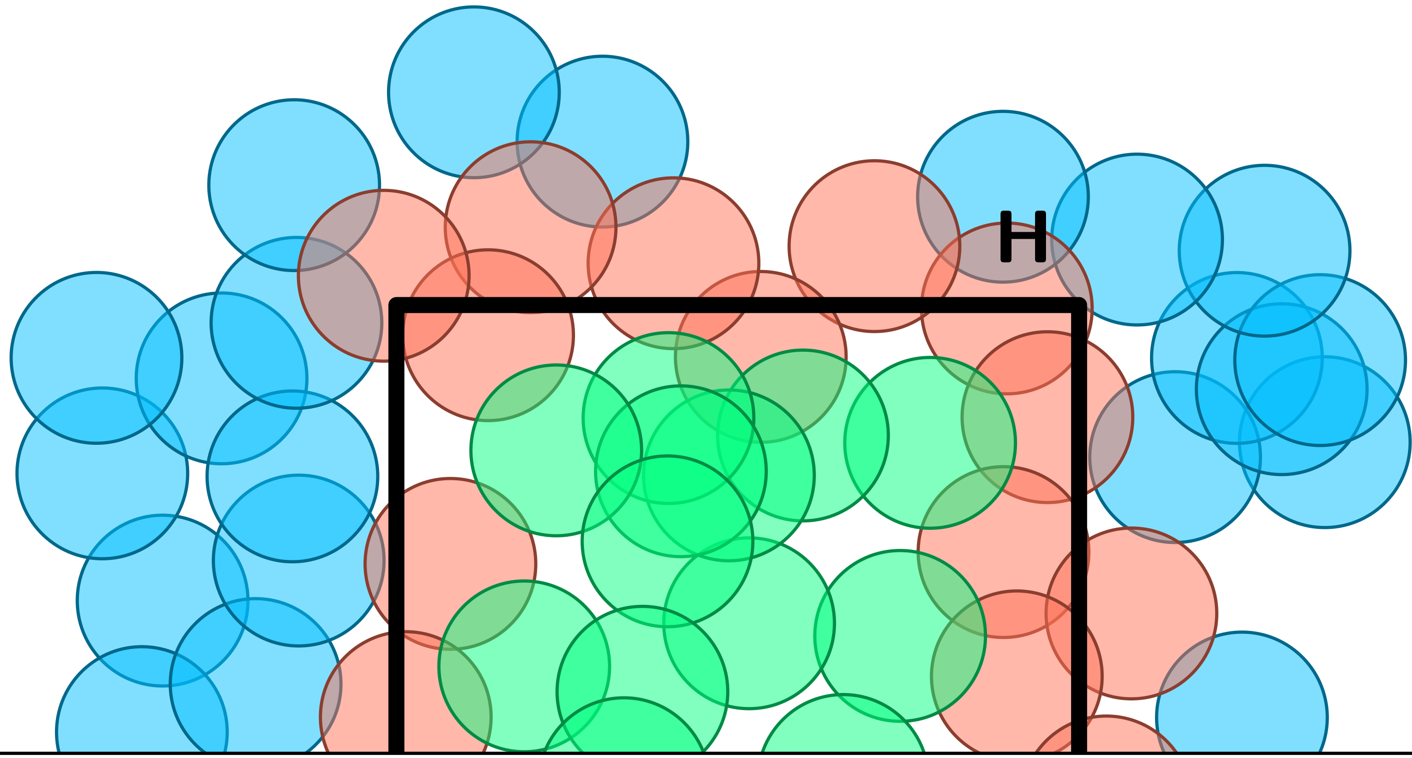
G is planar, i.e., K_5-free

Each disk contributes to at most $1 + \frac{\alpha}{1/\sqrt{n}}$ separators, which bounds the sum of all contributions by

$$O(\frac{1}{\alpha^2}) \cdot (1 + \frac{\alpha}{1/\sqrt{n}}) = O(\frac{1}{\alpha^2} + \frac{\sqrt{n}}{\alpha}) = O(n + n) = O(n)$$

Because we have $\sqrt{n}$ separator candidates, at least one of these candidates has size $O(\sqrt{n})$.

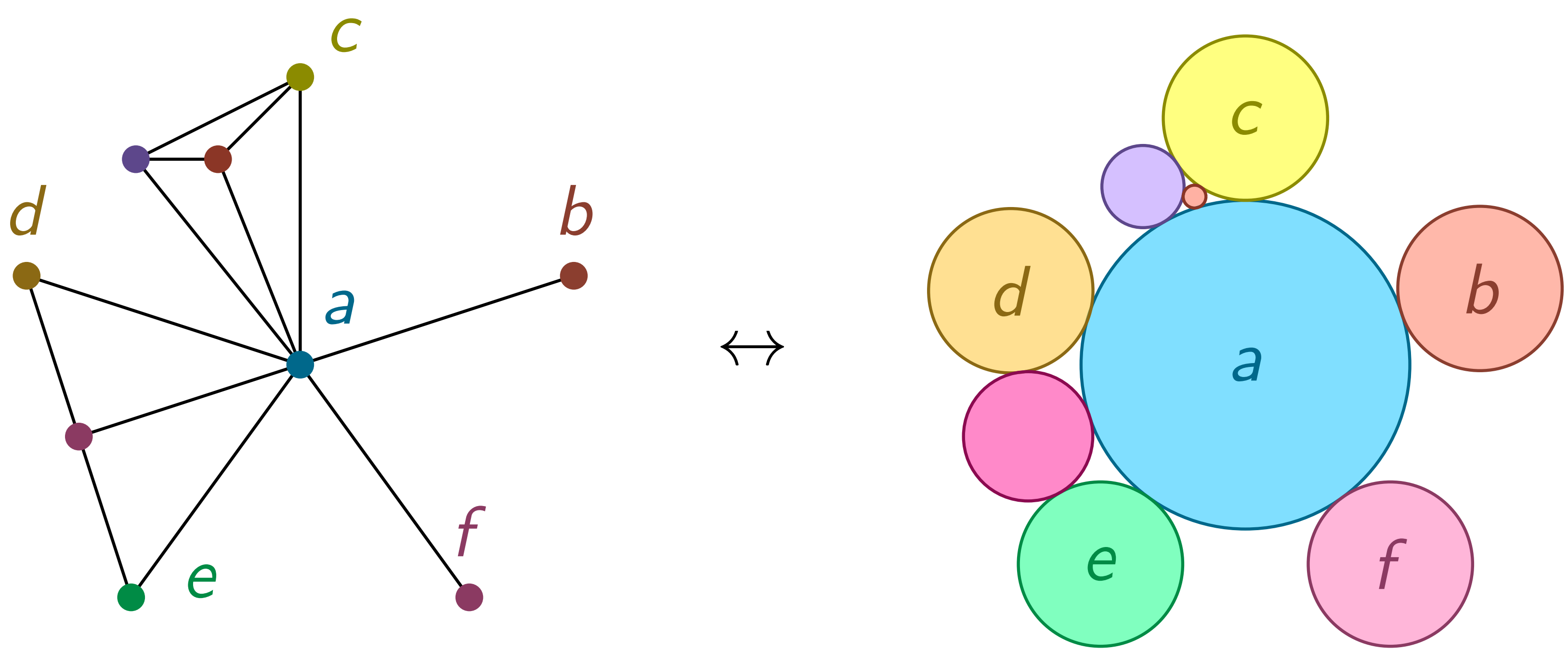
Unit Disk Graphs



Theorem: There exists an H intersecting $O(\sqrt{n})$ disks with $\leq 36/37n$ disks strictly inside H and $\leq 36/37n$ disks strictly outside H, i.e., H is a small balanced separator.

(Non-Unit) Disk Graphs

- represent vertices as *disks with arbitrary diameter*
- disks intersect iff corresponding vertices are adjacent



Circle Packing Theorem: The intersection graphs of non-crossing circles are exactly the planar graphs.

[Koebe 1936]

At Least One H_i Is Small

For $s = 1, 2, \ldots$ define *size class*

$$D_s = \{d : \frac{2^{s-1}}{\sqrt{n}} \leq \text{diameter of } d < \frac{2^s}{\sqrt{n}}\}.$$

- Disks with diameter $\geq \frac{1}{\sqrt{n}}$ are partitioned into size classes.

Disks with diameter $< \frac{1}{\sqrt{n}}$ contribute to at most one separator.

At Least One H_i Is Small

For $s = 1, 2, \ldots$ define *size class*

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- Disks with diameter $\geq \frac{1}{\sqrt{n}}$ are partitioned into size classes.
- H_m contains at most

$$4 \cdot \frac{3^2}{\pi \left(\frac{\frac{2^{s-1}}{\sqrt{n}}}{2} \right)^2} \in O\left(\frac{n}{2^{2s}}\right)$$

G is planar, i.e., K_5-free

upper bound on area of H_i

lower bound on area of disk in D_s

disks in D_s.

At Least One H_i Is Small

For $s = 1, 2, \ldots$ define *size class*

$$D_s = \{d : \frac{2^{s-1}}{\sqrt{n}} \leq \text{diameter of } d < \frac{2^s}{\sqrt{n}}\}.$$

- Disks with diameter $\geq \frac{1}{\sqrt{n}}$ are partitioned into size classes.
- H_m contains at most $O(\frac{n}{2^{2s}})$ disks in D_s.
- One disk in D_s contributes to at most

$$1 + \frac{\frac{2^s}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} \in O(2^s)$$

upper bound on diameter of disk in D_s

distance between consecutive separators

separators.

At Least One H_i Is Small

For $s = 1, 2, \ldots$ define *size class*

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- Disks with diameter $\geq \frac{1}{\sqrt{n}}$ are partitioned into size classes.
- H_m contains at most $O(\frac{n}{2^{2s}})$ disks in D_s.
- One disk in D_s contributes to at most $O(2^s)$ separators.

Bound the number of contributions:

$$\sum_{s=1,2,\ldots} O(\frac{n}{2^{2s}}) \cdot O(2^s) = O(n \cdot \sum_{s=1,2,\ldots} \frac{1}{2^s}) = O(n)$$