Computation Geometry – Exercise

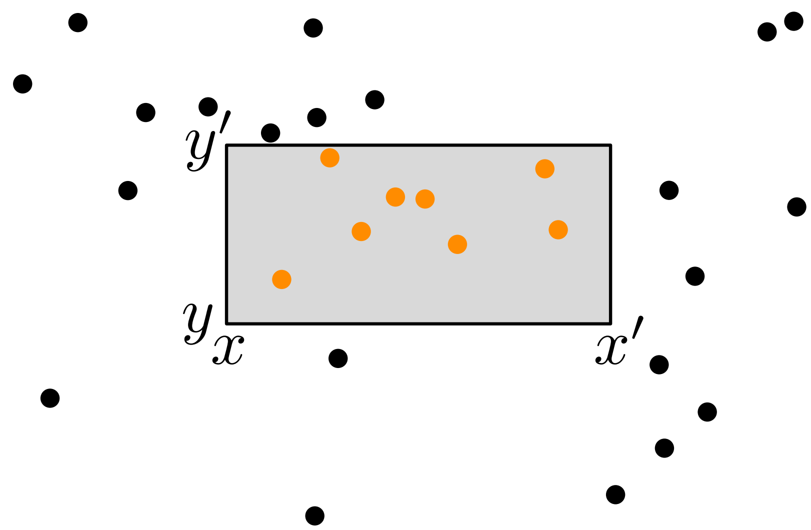
Range Searching II

LEHRSTUHL FÜR ALGORITHMIK I · INSTITUT FÜR THEORETISCHE INFORMATIK · FAKULTÄT FÜR INFORMATIK

Guido Brückner

20.07.2018

Object types in range queries



Setting so far:

- Input: set of points P (here $P \subset \mathbb{R}^2$)
- Output: all points in $P \cap [x, x'] \times [y, y']$
- Data structures: kd-trees or range trees

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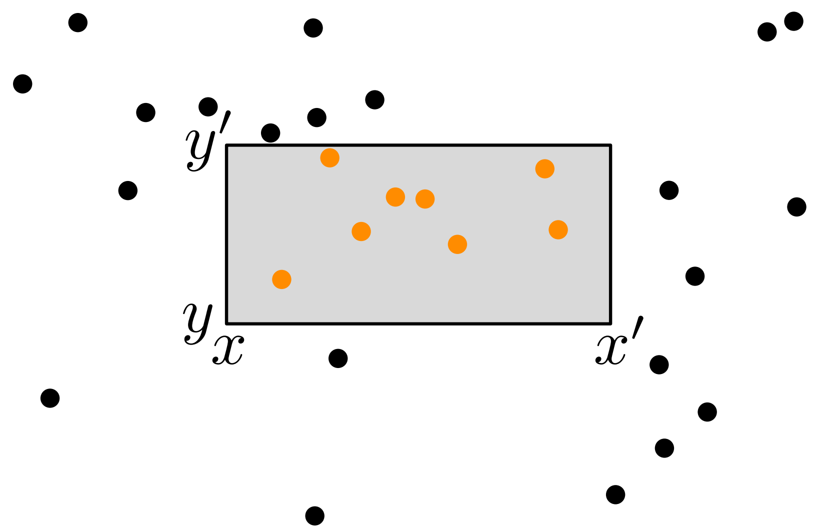

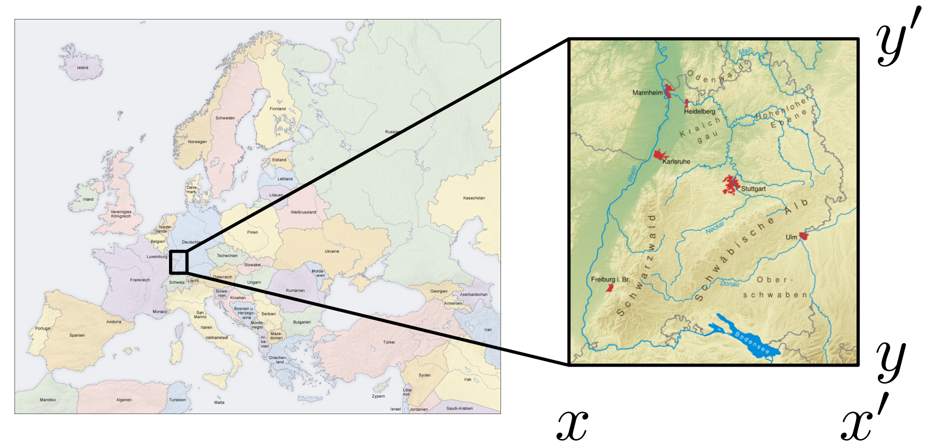

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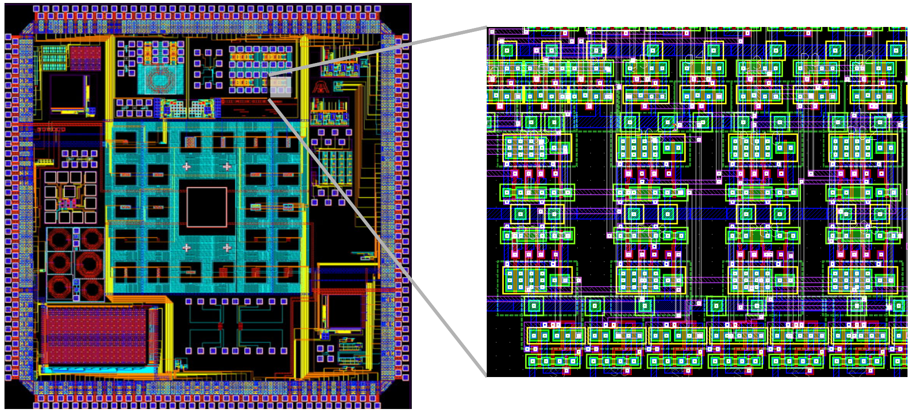
Further variant

- Input: set of line segments S (here in $\mathbb{R}^2$)
- Output: all segments in $S \cap [x, x'] \times [y, y']$
- Data structures: ?

Axis-parallel line segments



special case (e.g., in VLSI design): all line segments are axis-parallel

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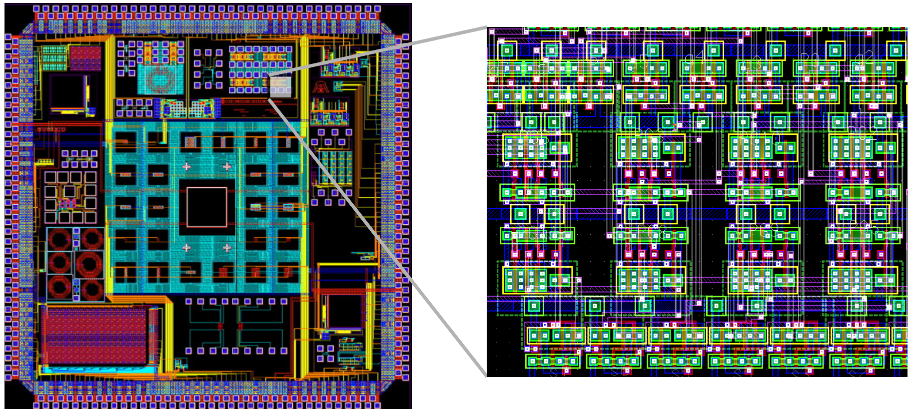

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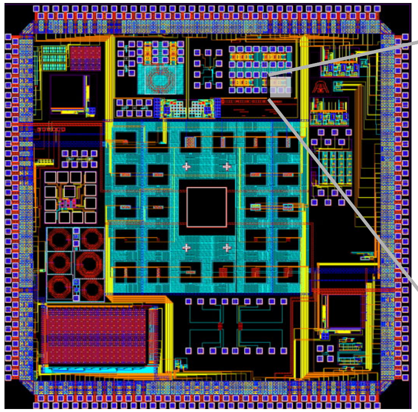
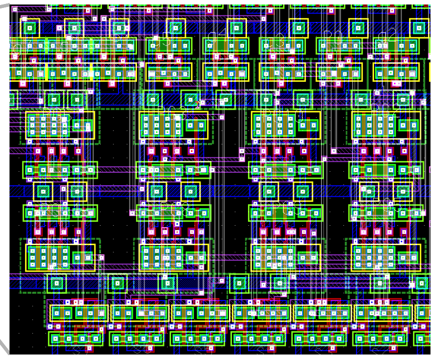
Problem:

Given n vertical and horizontal line segments and an axis-parallel rectangle $R = [x, x'] \times [y, y']$, find all line segments that intersect R.

Axis-parallel line segments





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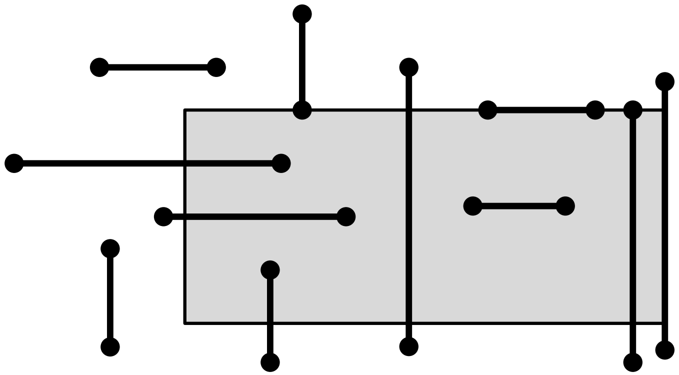
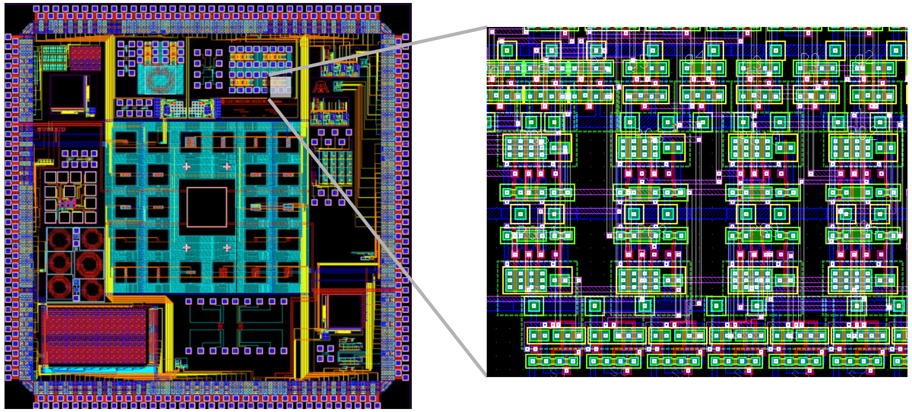


How to approach this case?

Axis-parallel line segments



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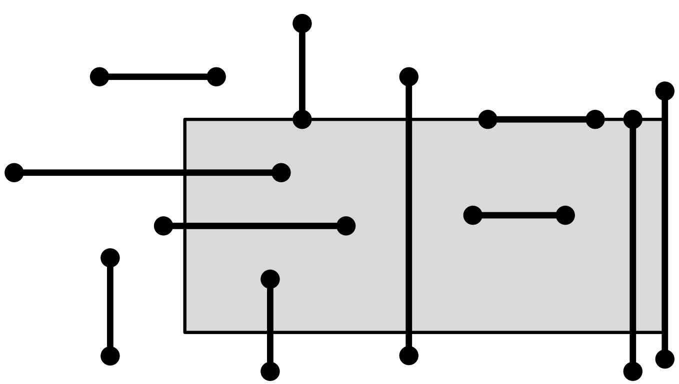


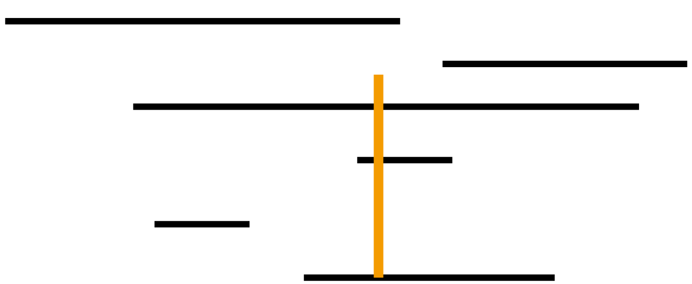
Case 1: ≥ 1 endpoint in R
$\rightarrow$ use range tree

Case 2: both endpoints $\notin R$
$\rightarrow$ intersect left or top edge of R

Case 2 in detail

Problem:

Given a set H of n horizontal line segments and a vertical query segment s, find all line segments in H that intersect s. (Vertical segments and a horizontal query are analogous.)



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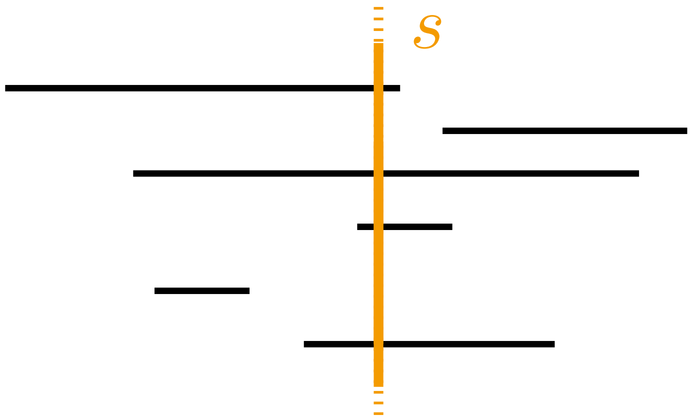
One level simpler: vertical line $s := (x = q_x)$

Given n intervals $I = \{[x_1, x'_1], [x_2, x'_2], \ldots, [x_n, x'_n]\}$ and a point q_x, find all intervals that contain q_x.



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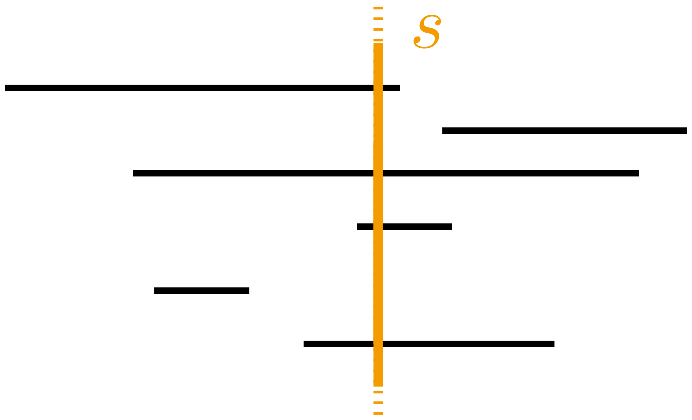


What do we need for an appropriate data structure?

Interval Trees

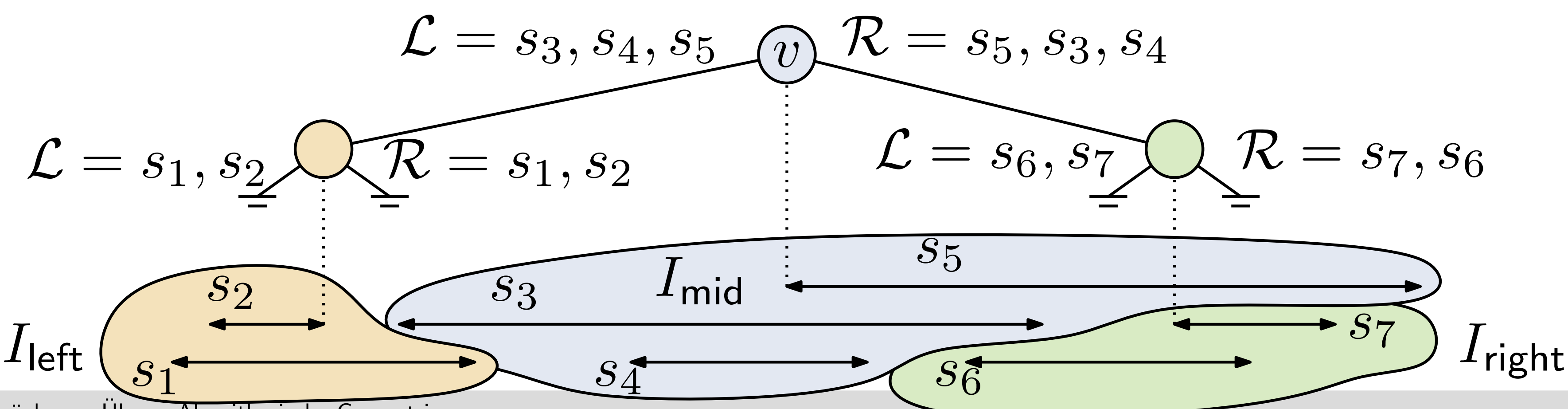
Construction of an interval tree $\mathcal{T}$

- if $I = \emptyset$ then $\mathcal{T}$ is a leaf
- else let x_{mid} be the median of the endpoints of I and define

$$\begin{aligned} I_{\text{left}} &= \{[x_j, x'_j] \mid x'_j < x_{\text{mid}}\} \\ I_{\text{mid}} &= \{[x_j, x'_j] \mid x_j \leq x_{\text{mid}} \leq x'_j\} \\ I_{\text{right}} &= \{[x_j, x'_j] \mid x_{\text{mid}} < x_j\} \end{aligned}$$

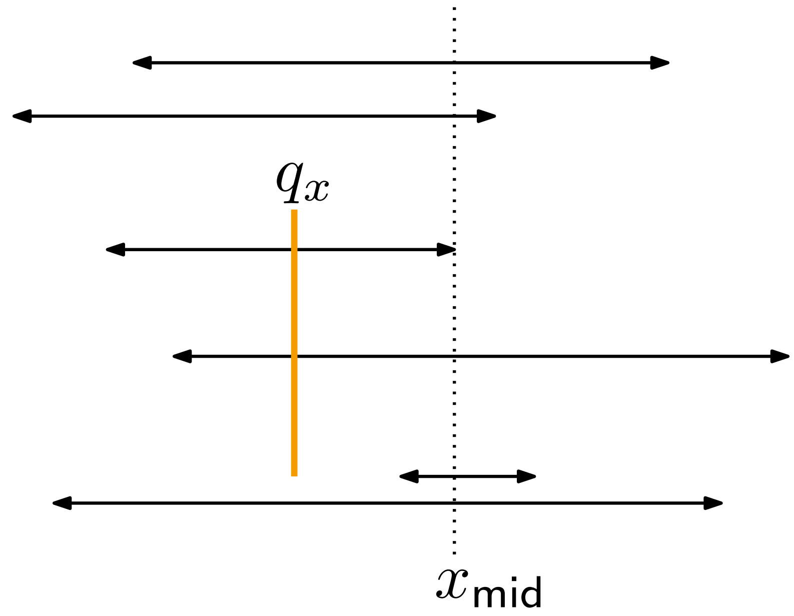
$\mathcal{T}$ consists of a node v for x_{mid} and

- lists $\mathcal{L}(v)$ and $\mathcal{R}(v)$ for I_{mid} sorted by left and right interval endpoints, respectively
- left child of v is an interval tree for I_{left}
- right child of v is an interval tree for I_{right}



From lines to line segments

How can we adapt the idea of an interval tree for query segments $q_x \times [q_y, q'_y]$ instead of a query line $x = q_x$?



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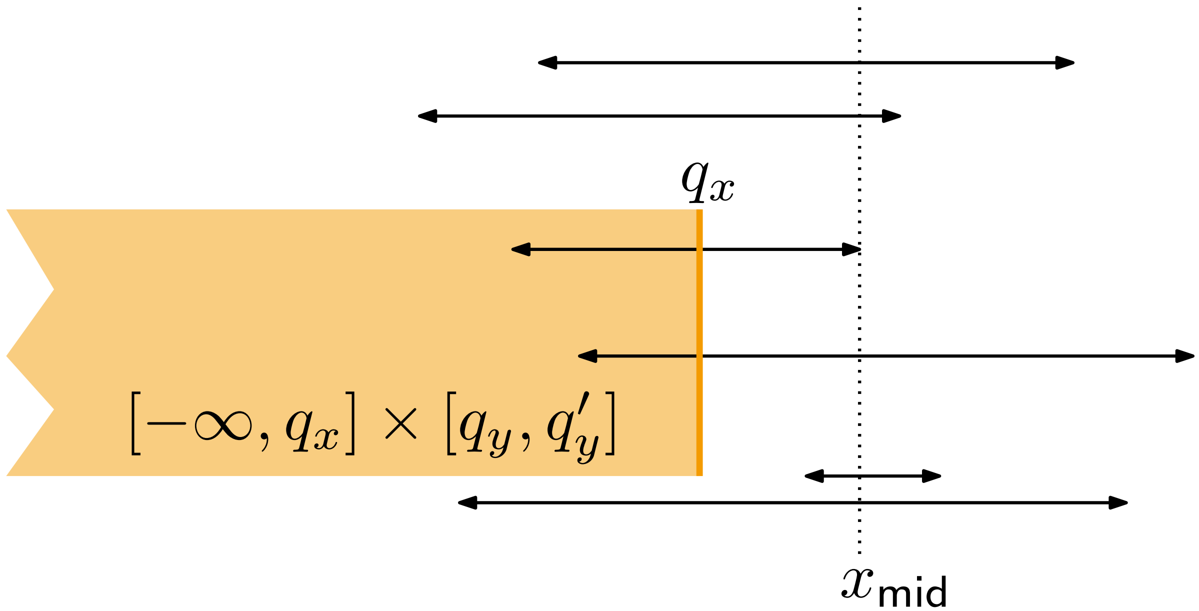

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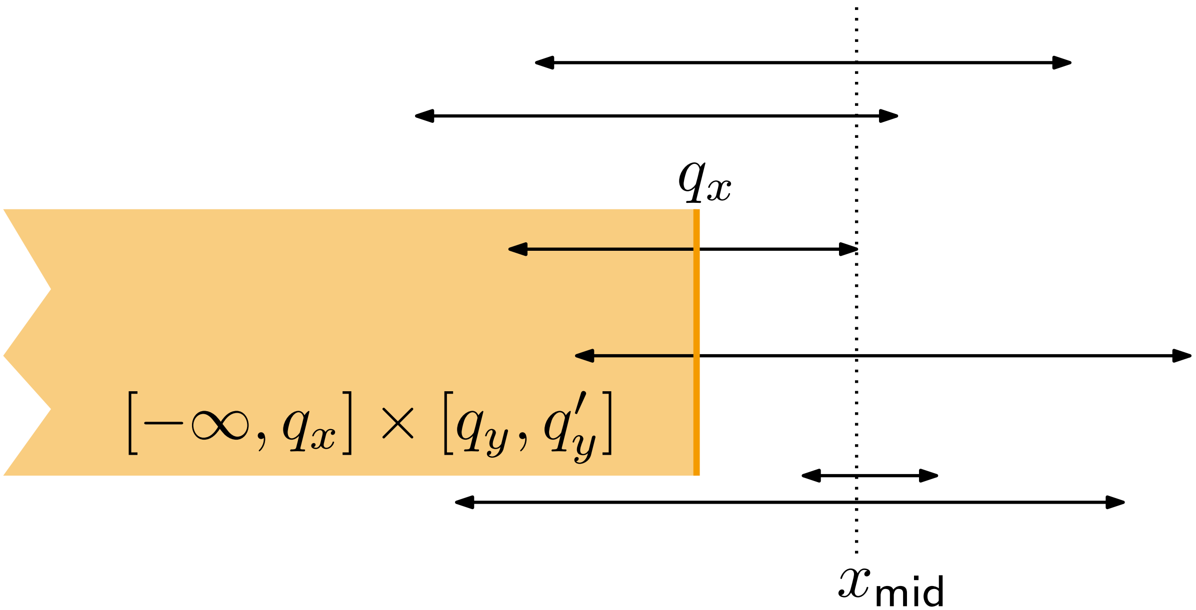

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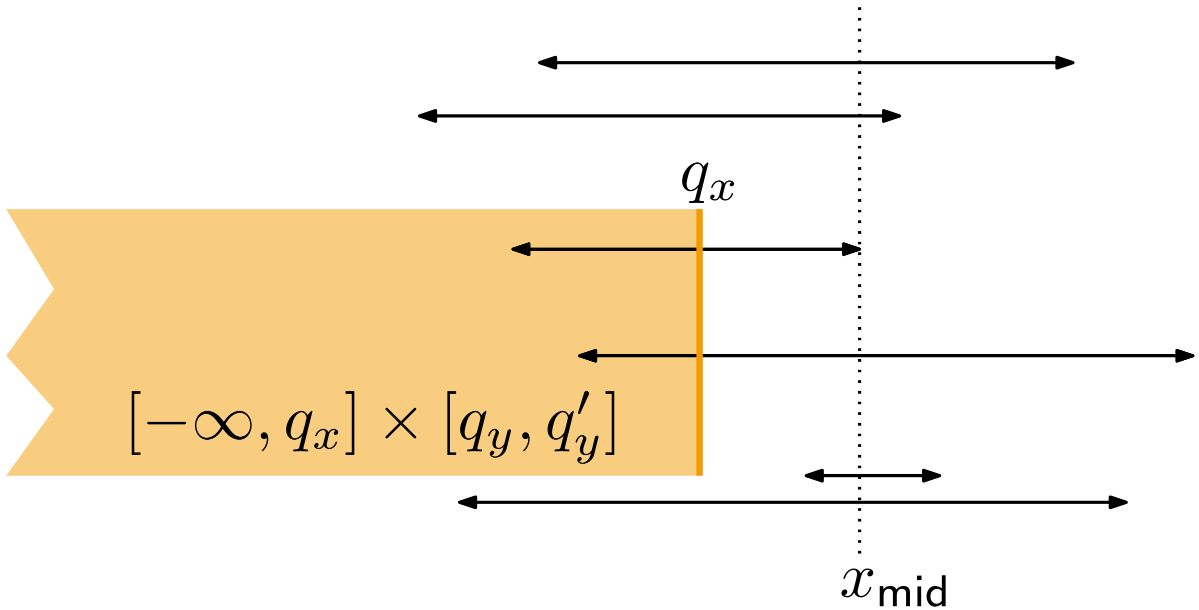

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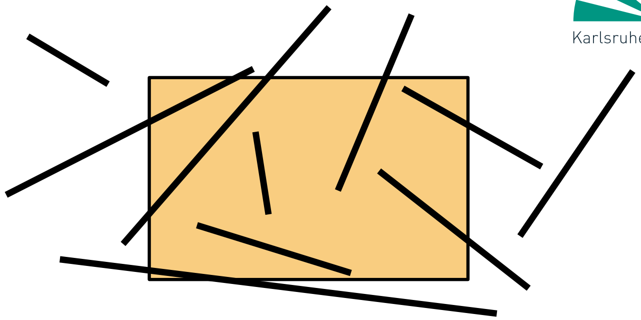
Theorem 1: Let S be a set of horizontal (axis-parallel) line segments in the plane. All k line segments that intersect a vertical query segment (an axis-parallel rectangle R) can be found in $O(\log^2(n) + k)$ time. The data structure requires $O(n \log n)$ space and construction time.

Arbitrary line segments

Map data often contain arbitrarily oriented line segments.



Problem:

Given n disjoint line segments and an axis-parallel rectangle $R = [x, x'] \times [y, y']$, find all line segments that intersect R.

Exercise 5: How to use interval trees?

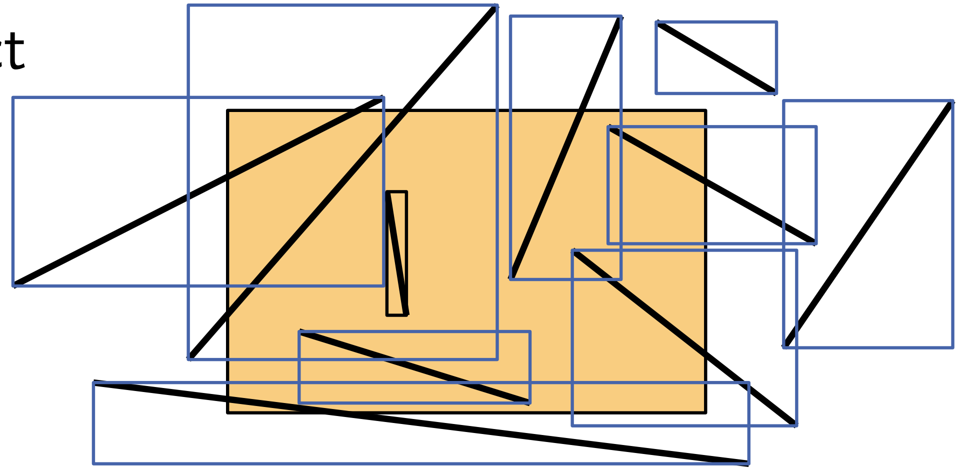
Solution

Use Bounding Box of Segments

1. Interval trees on segments of bounding-boxes.

2. If segments of bounding-box intersect query range:

   Check whether contained segment intersects query range.



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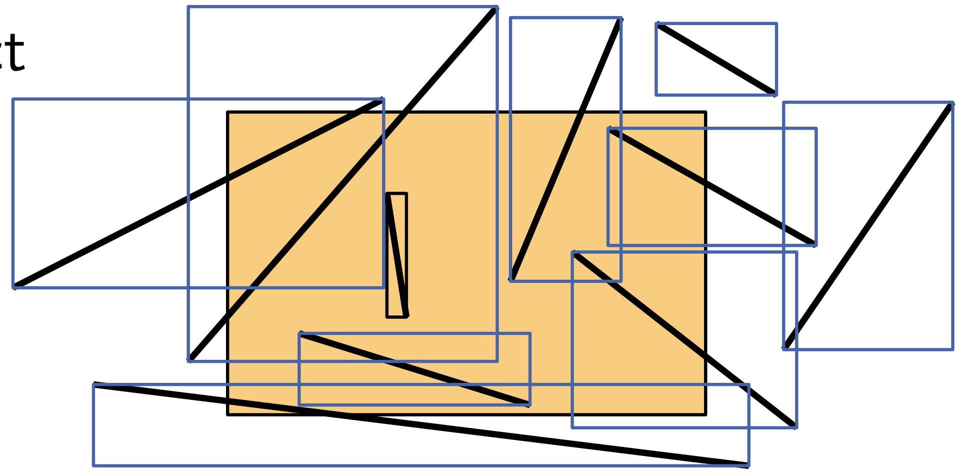

Correct, because:

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Karlsruhe Institute of Technology

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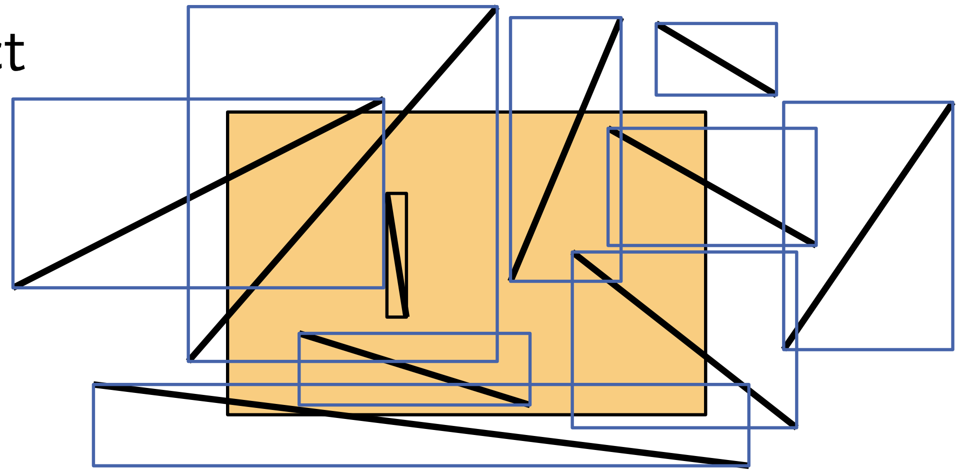

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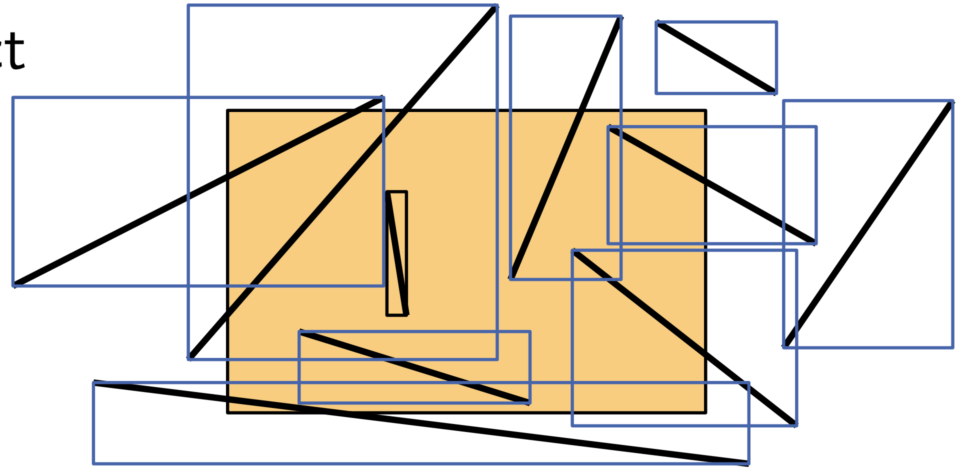

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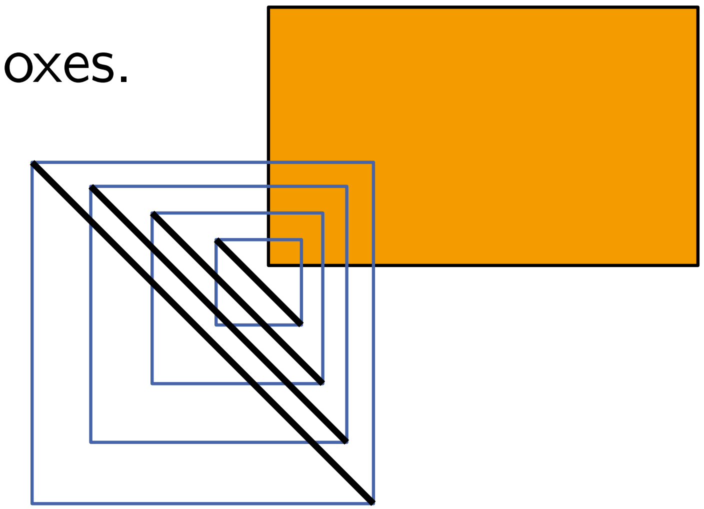

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Problem: More segments may be considered than necessary.

because it is not true that

~~If the bounding-box intersects the query range, then the contained segment intersects the query range.~~

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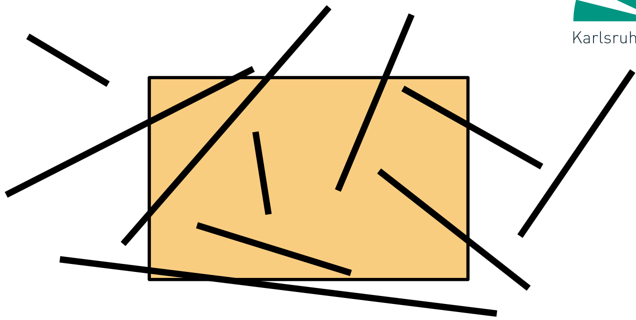

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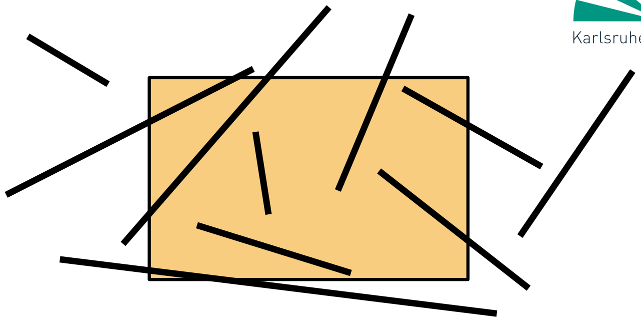

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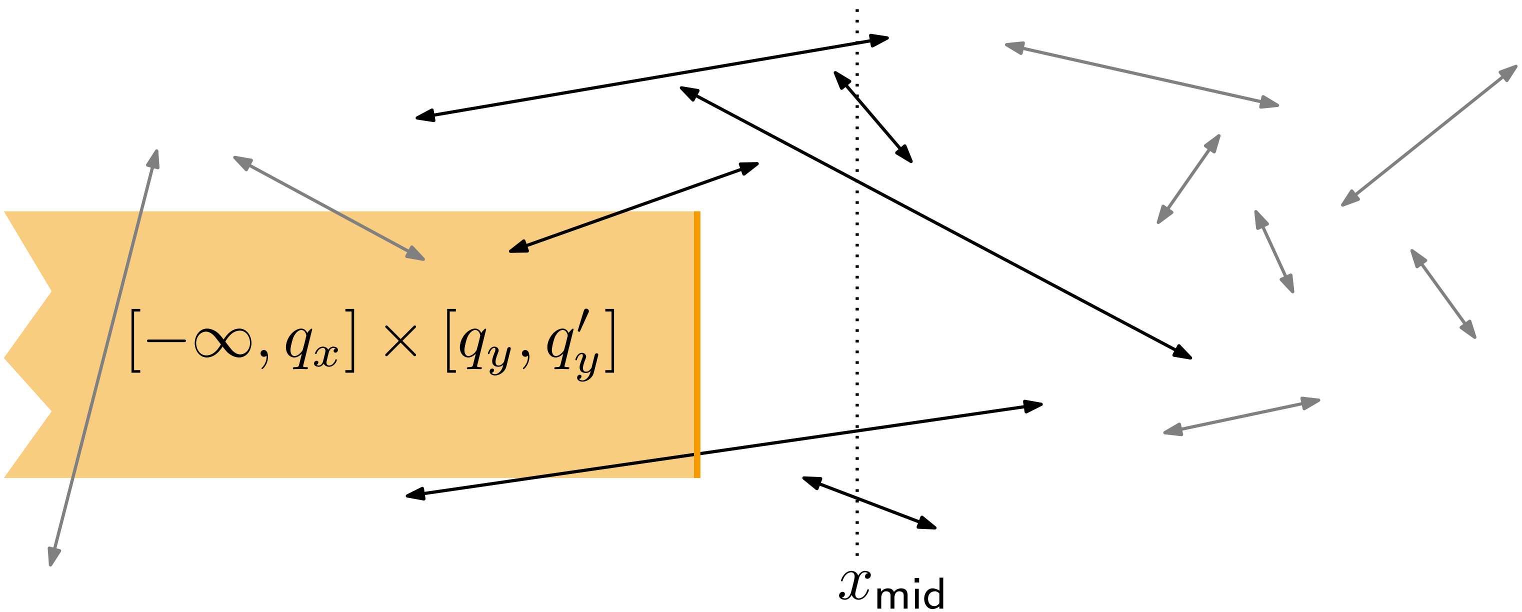
Decomposition into elementary intervals

Interval trees don't really help here



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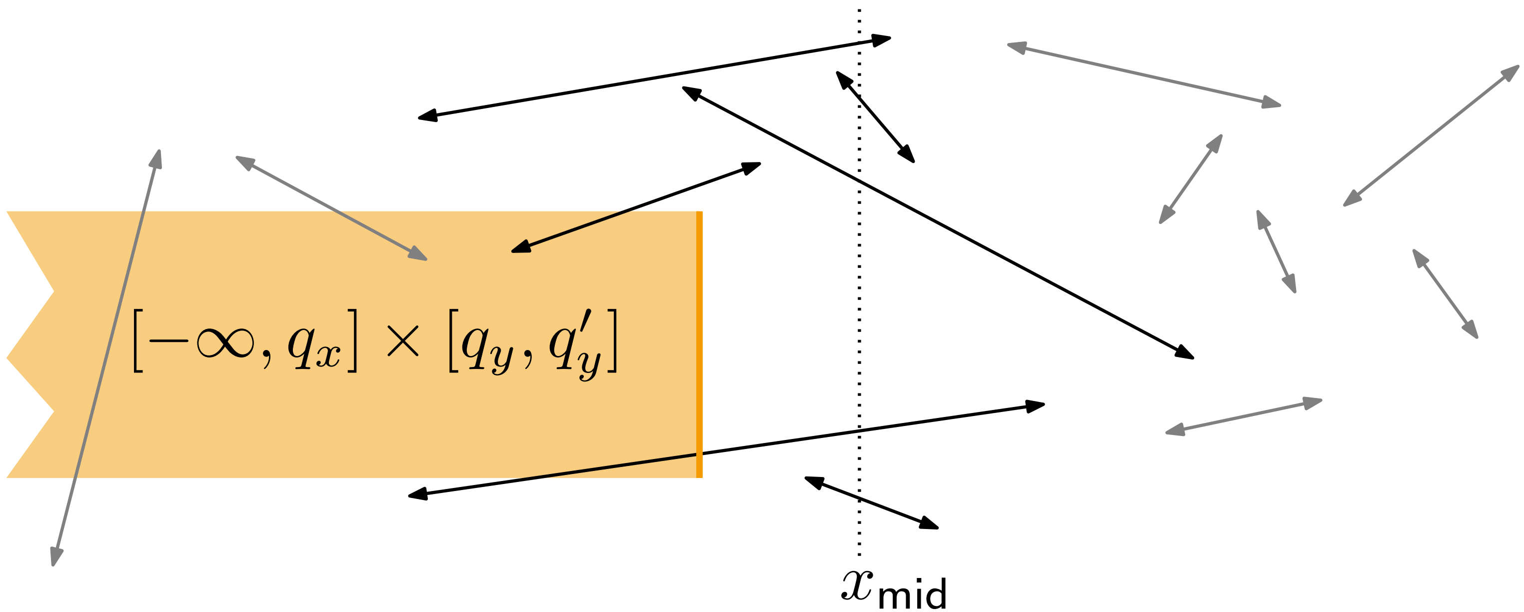

Identical 1d base problem:

Given n intervals $I = \{[x_1, x_1'], [x_2, x_2'], \ldots, [x_n, x_n']\}$ and a point q_x, find all intervals that contain q_x.

- sort all x_i and x_i' in list $p_1, \ldots, p_{2n}$
- create sorted elementary intervals $(-\infty, p_1), [p_1, p_1], (p_1, p_2), [p_2, p_2], \ldots, [p_{2n}, p_{2n}], (p_{2n}, \infty)$

Segment trees

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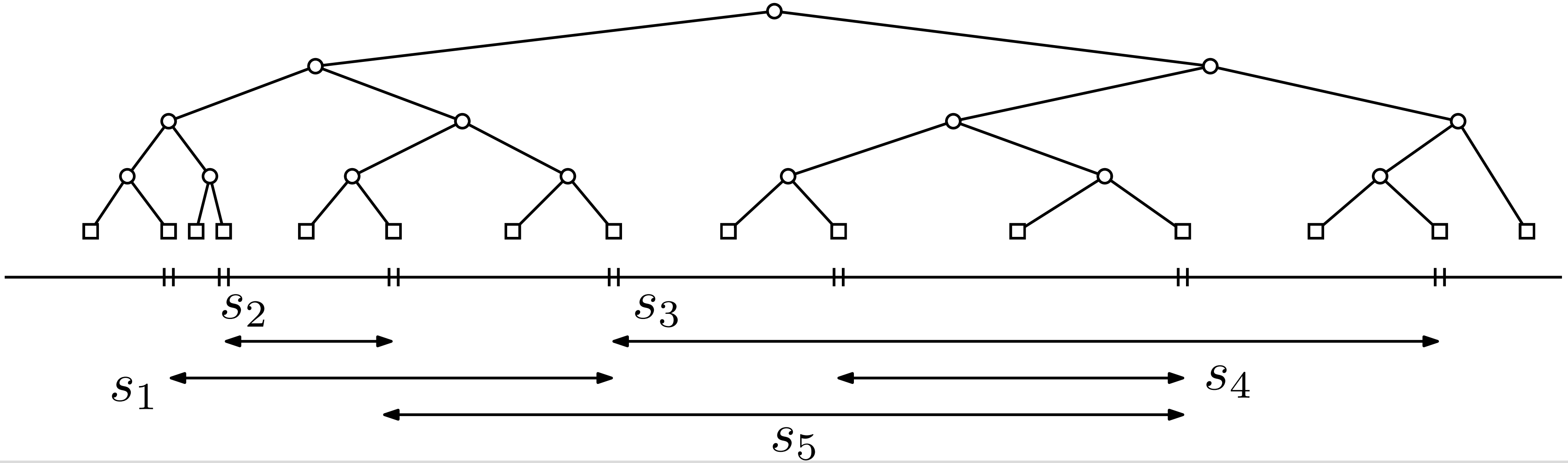
Idea for data structure:

- create binary search tree with elementary intervals in leaves
- for all points q_x in the same elementary interval the answer is the same
- leaf μ for elementary interval $e(\mu)$ stores interval set $I(\mu)$
- query requires $O(\log n + k)$ time



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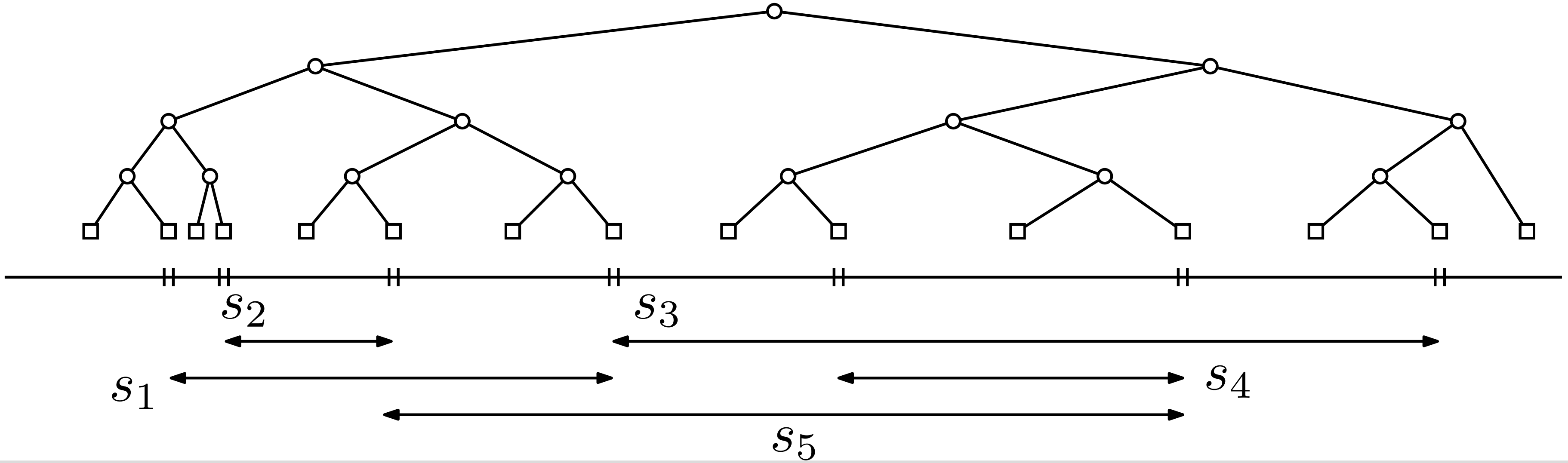
Store intervals as high up in the tree as possible

- node v represents interval $e(v) = e(lc(v)) \cup e(rc(v))$
- input interval $s_i \in I(v) \Leftrightarrow e(v) \subseteq s_i$ and $e(\text{parent}(v)) \not\subseteq s_i$



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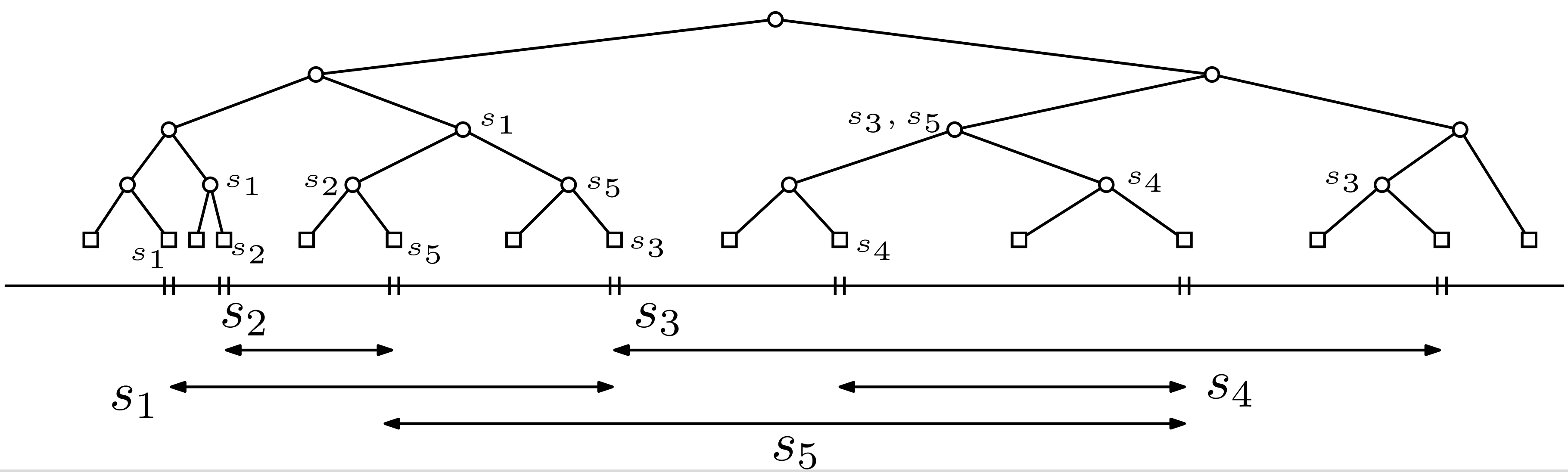
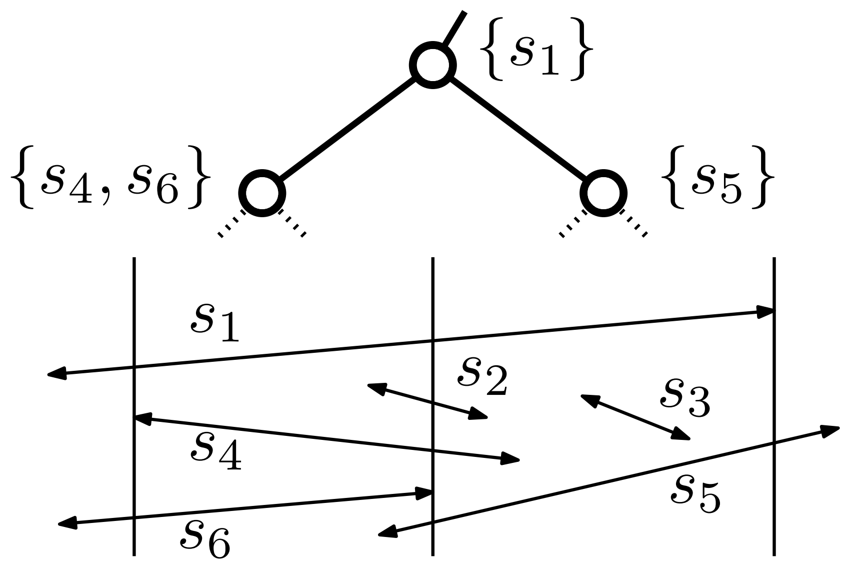


Back to arbitrary line segments

- create segment tree for the x intervals of the line segments
- each node v corresponds to a vertical strip $e(v) \times \mathbb{R}$
- line segment s is in $I(v)$ iff s crosses the strip of v but not the strip of parent(v)
- at each node v on the search path for the vertical segment $s' = q_x \times [q_y, q'_y]$ all segments in $I(v)$ cover the x-coordinate q_x



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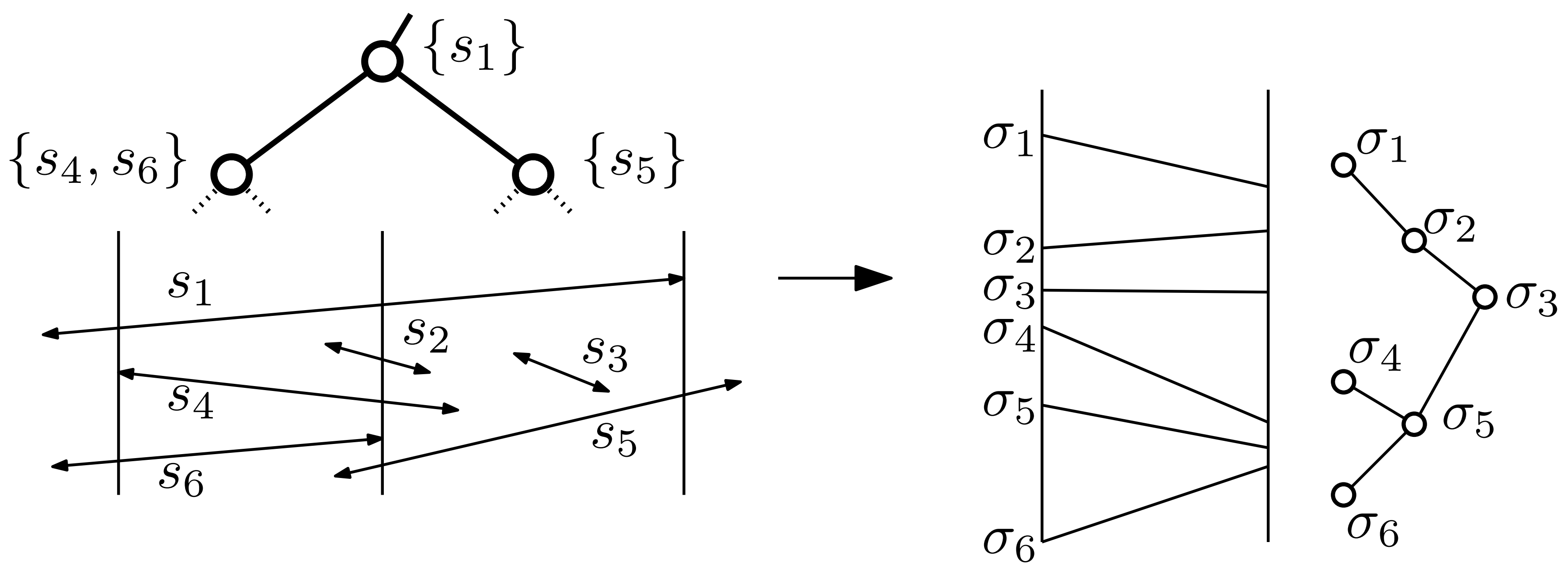
- at each node v on the search path for the vertical segment $s' = q_x \times [q_y, q'_y]$ all segments in $I(v)$ cover the x-coordinate q_x
- find segments in the strip that cross s' using a vertically sorted auxiliary binary search tree



Summary

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Theorem 2: Let S be a set of interior-disjoint line segments in the plane. All k segments that intersect a vertical query segment (an axis-parallel query rectangle R) can be found in time $O(k + \log^2 n)$. The corresponding data structure uses $O(n \log n)$ space and $O(n \log^2 n)$ construction time.

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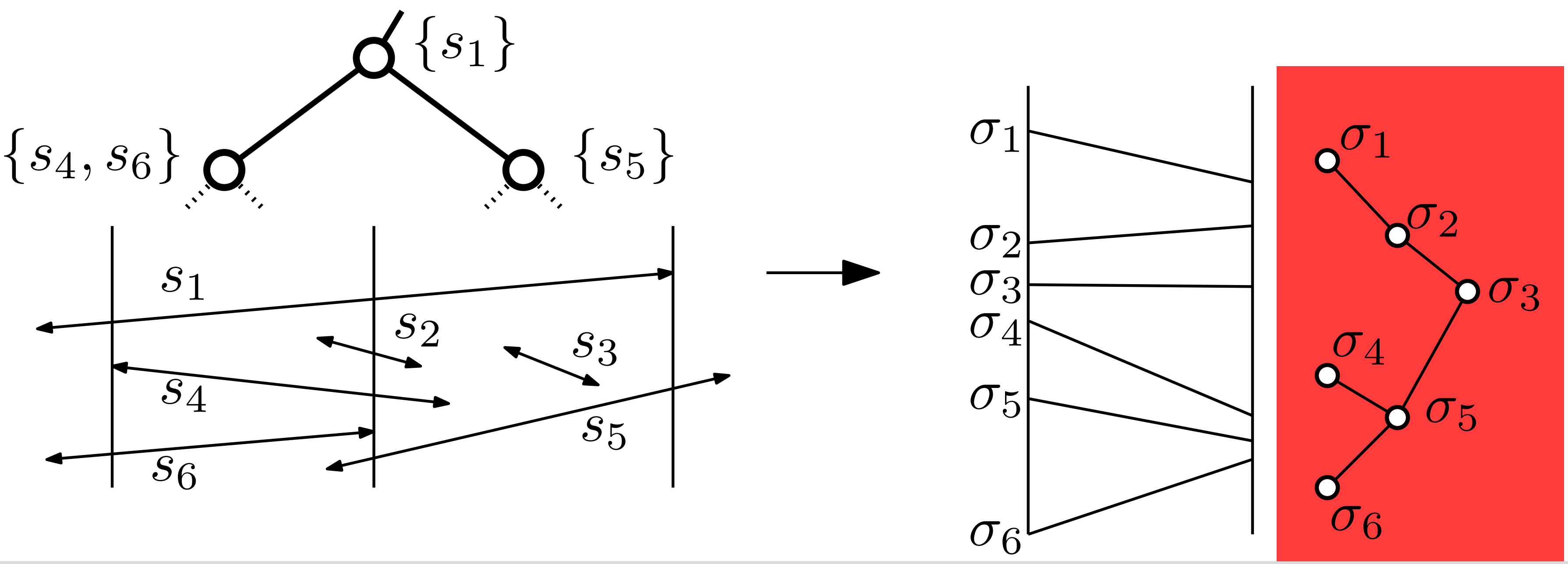
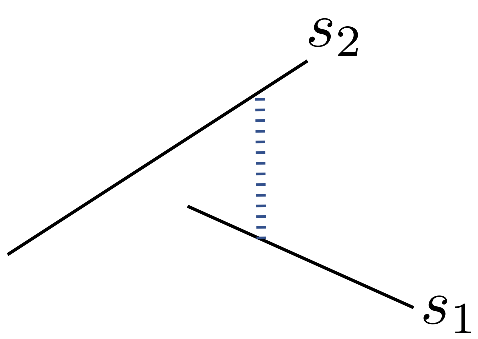
Problem: Construction of auxiliary tree.



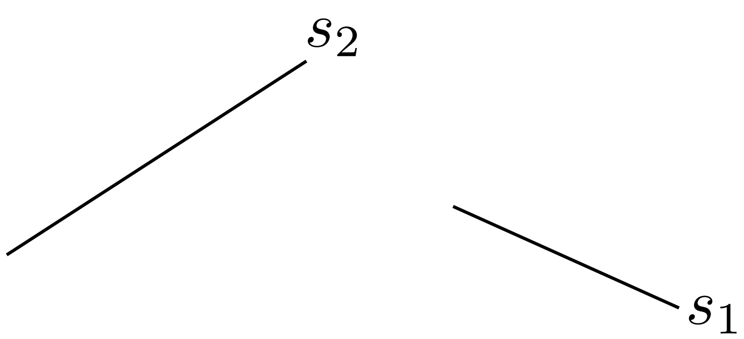
Solution

Let s_1, s_2 be two segments.

s_1 lies *below* s_2 ($s_1 \prec s_2$), if there is a point $p_1 \in s_1$ and $p_2 \in s_2$ with $x(p_1) = x(p_2)$ and $y(p_1) < y(p_2)$.



$s_1 \prec s_2$



$s_1 \not\prec s_2$

1. Show that relation $\prec$ on S is acyclic.

   ⟶ There exists a topological orderding.

2. Compute topological ordering S
3. Use topological ordering to construct help trees.

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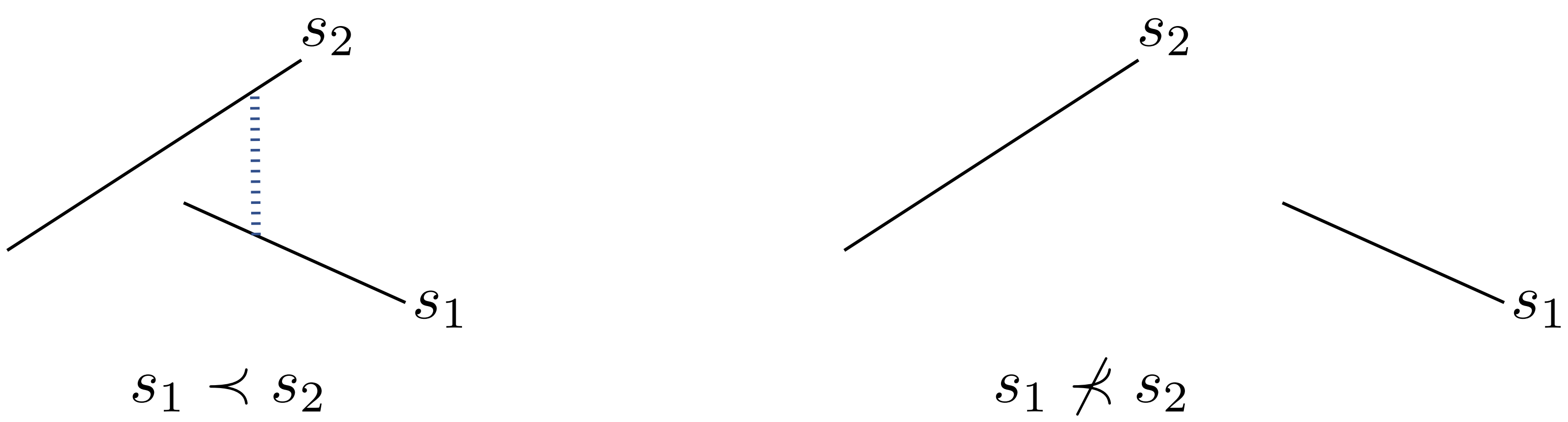

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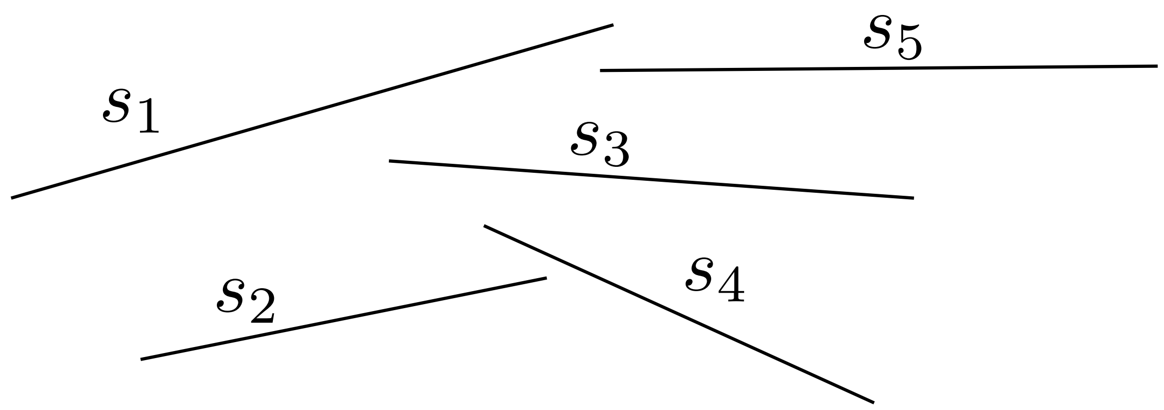

Verical sweep-line from left to right to obtain ordering T:

Events: Endpoints of segments.

Sweep-Line-State: Segments that intersect sweep-line (binary tree S representation)

Handling event p:

p ist left end point of segment s_i: insert s_i into S.

Insert s_i into T correspondingly to its neighbors in S.

p is right end point of segment s_i: s_i is removed from S

Computation of Topological Ordering

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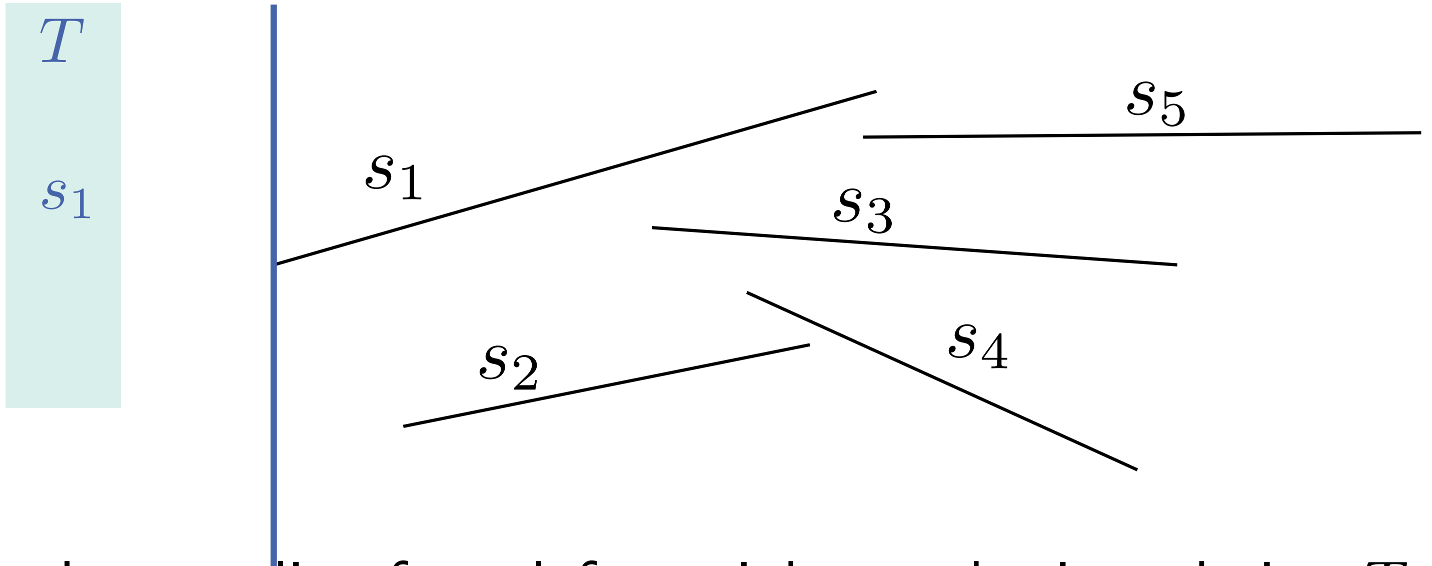

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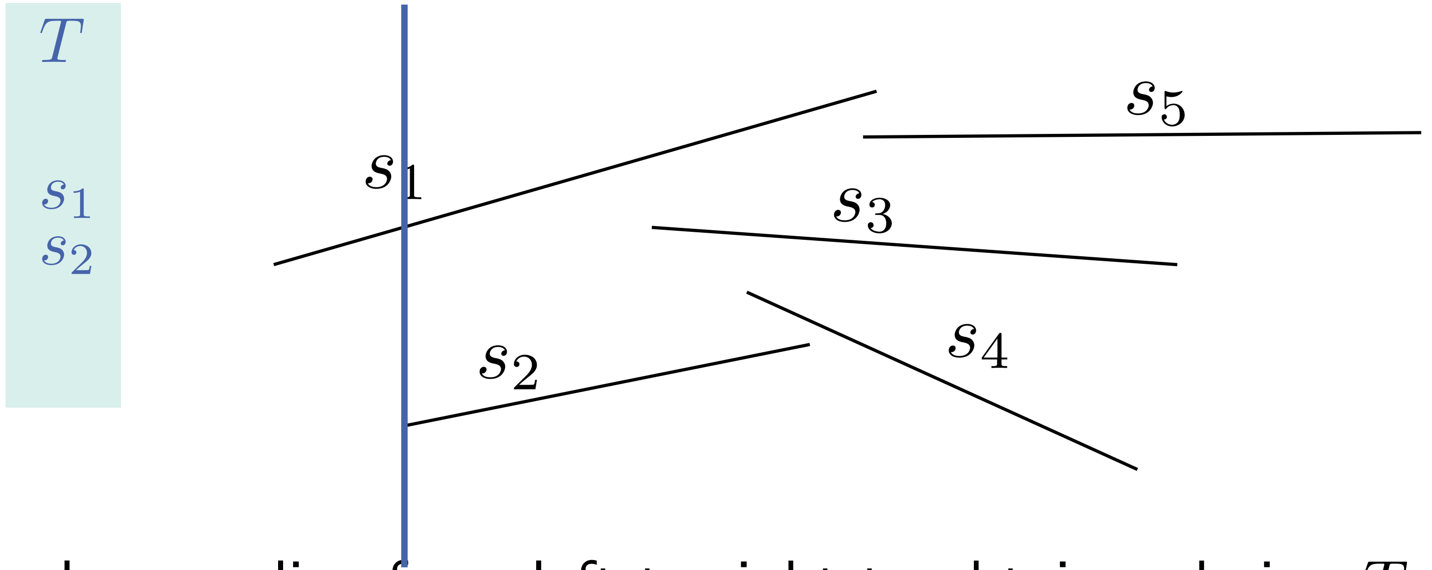

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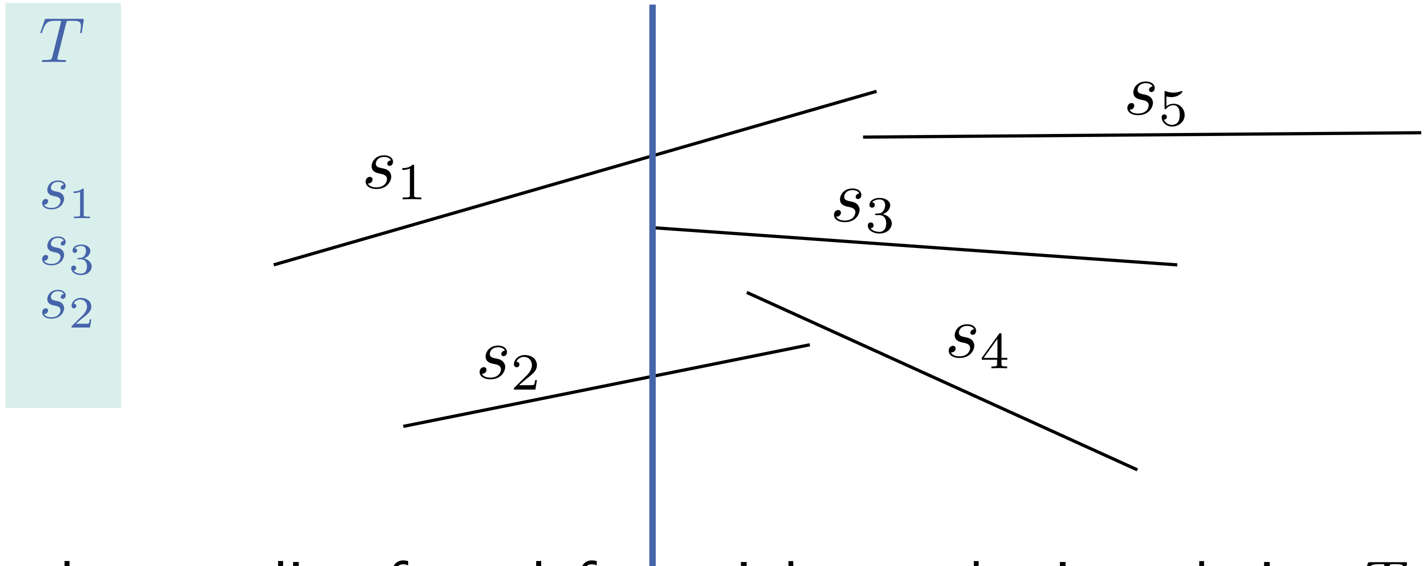
Computation of Topological Ordering



Verical sweep-line from left to right to obtain ordering T:

Events: Endpoints of segments.

Sweep-Line-State: Segments that intersect sweep-line (binary tree S representation)

Handling event p:

p ist left end point of segment s_i: insert s_i into S.

Insert s_i into T correspondingly to its neighbors in S.

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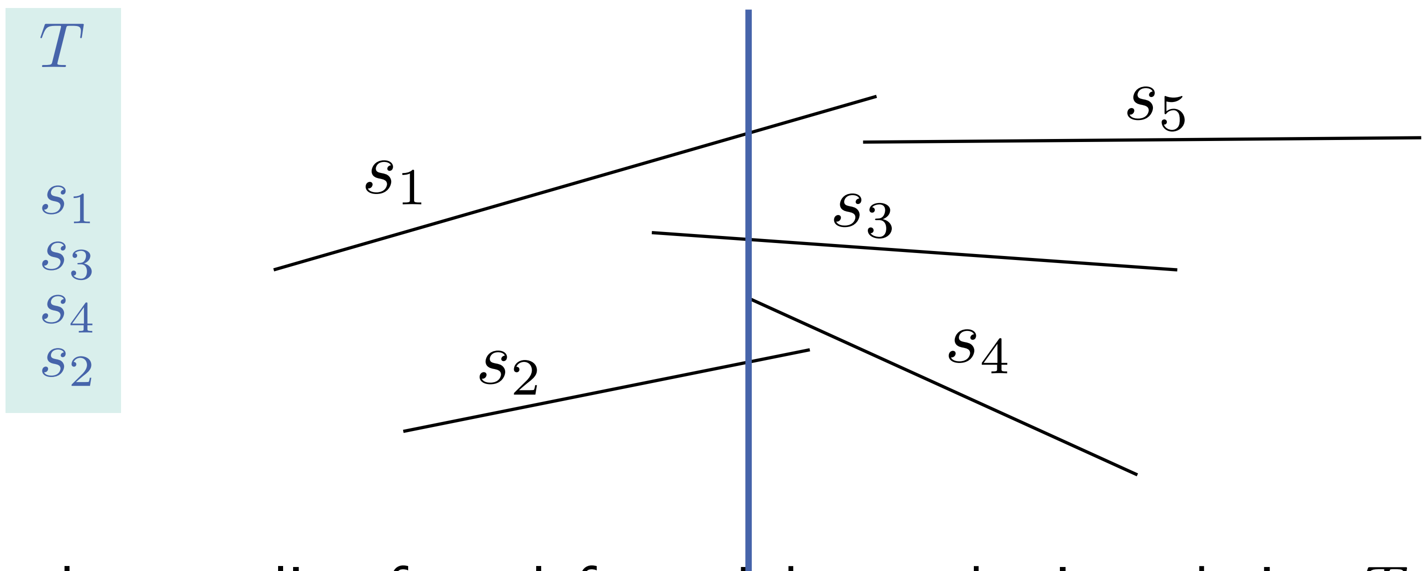

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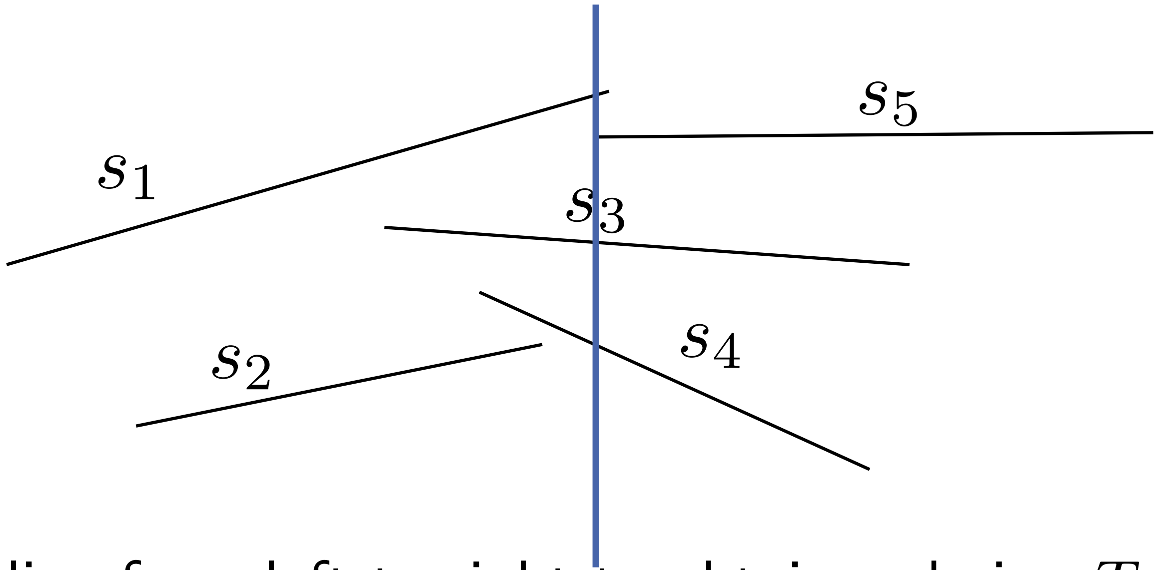
Computation of Topological Ordering

T

s_1
s_5
s_3
s_4
s_2



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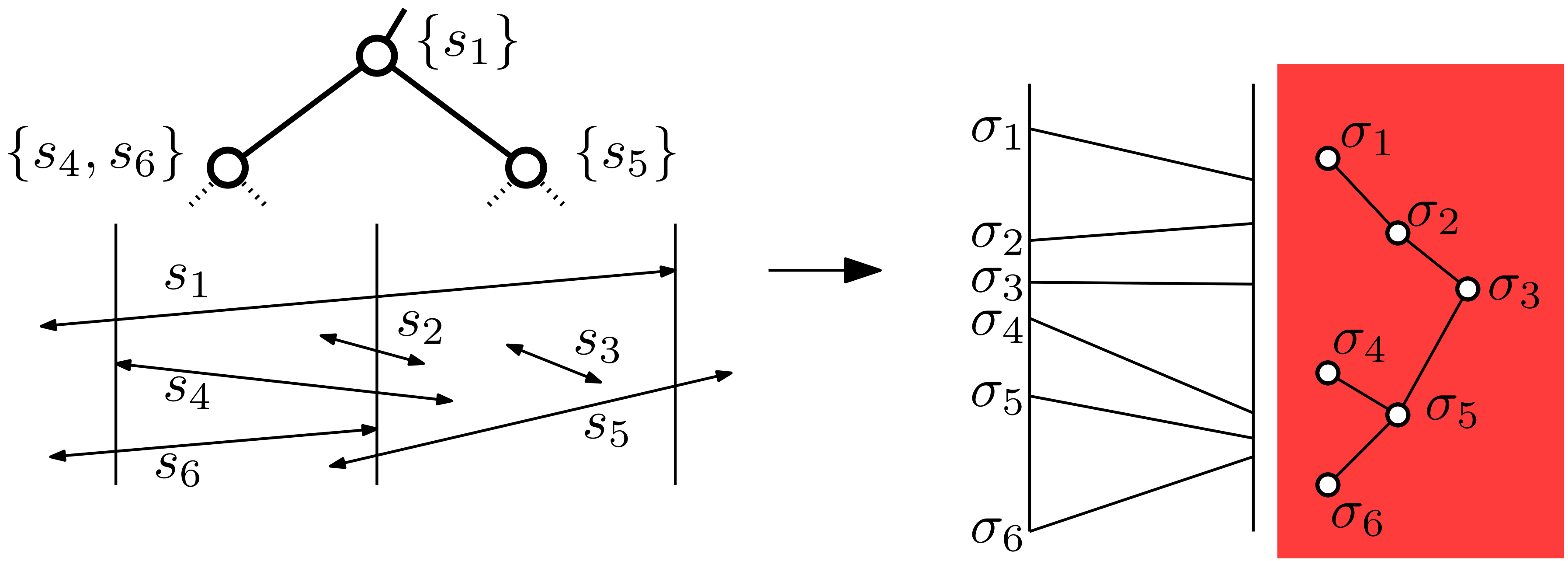
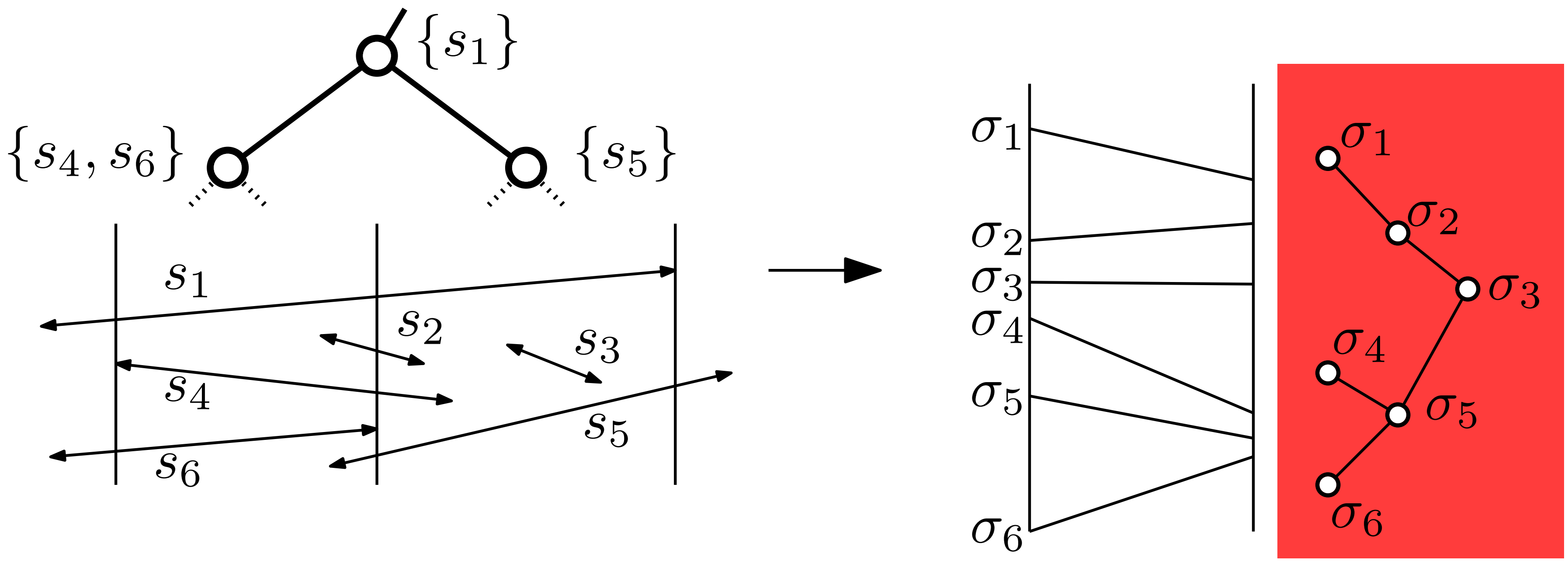
Construction of Trees

Apply topological ordering.



Construction of Trees

Apply topological ordering.



In each strip the topological ordering corresponds with the vertical ordering.

→ Insert segments into $I(v)$ w.r.t. topological ordering.

→ Construct binary tree based on $I(v)$ in $|I(v)|$ time.

→ $O(n)$ time in total.

Exercise 3

given: Set I of n intervals

find: In how many intervals is a point $p \in \mathbb{R}$ contained? Datastructure!

Datastrucrue is based on interval trees:

Interval Trees

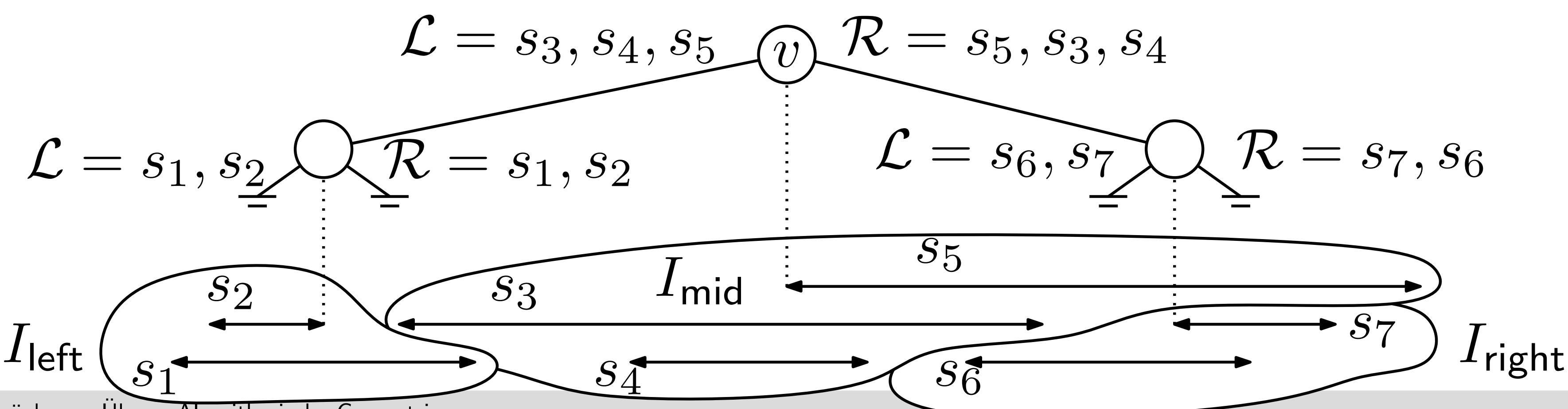
Construction of an interval tree $\mathcal{T}$

- if $I = \emptyset$ then $\mathcal{T}$ is a leaf
- else let x_{mid} be the median of the endpoints of I and define

$$\begin{aligned} I_{\text{left}} &= \{[x_j, x'_j] \mid x'_j < x_{\text{mid}}\} \\ I_{\text{mid}} &= \{[x_j, x'_j] \mid x_j \le x_{\text{mid}} \le x'_j\} \\ I_{\text{right}} &= \{[x_j, x'_j] \mid x_{\text{mid}} < x_j\} \end{aligned}$$

$\mathcal{T}$ consists of a node v for x_{mid} and

- lists $\mathcal{L}(v)$ and $\mathcal{R}(v)$ for I_{mid} sorted by left and right interval endpoints, respectively
- left child of v is an interval tree for I_{left}
- right child of v is an interval tree for I_{right}



Exercise 3

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Data structure is based on interval trees:

QIT(v, q_x)

if v is not Blatt **then**
  if $q_x < x_{\mathsf{mid}}(v)$ **then**
    return QIT$(lc(v), q_x)$+Number of intervals in $\mathcal{L}$ that contain q_x
  else
    return QIT$(rc(v), q_x)$+Number of intervals in $\mathcal{R}$ that contain q_x
return 1

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binary search tree with $O(\log n)$ query time.

Running Time: $O(\log^2 n)$

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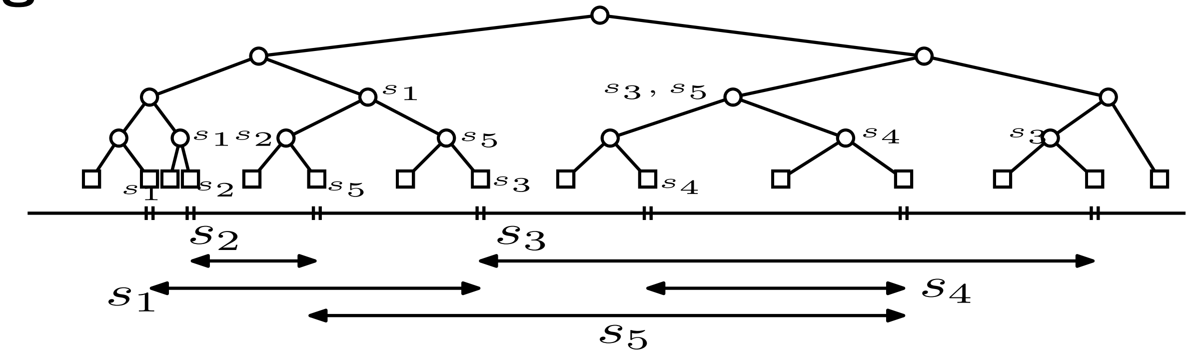
Data structure based on segment trees:



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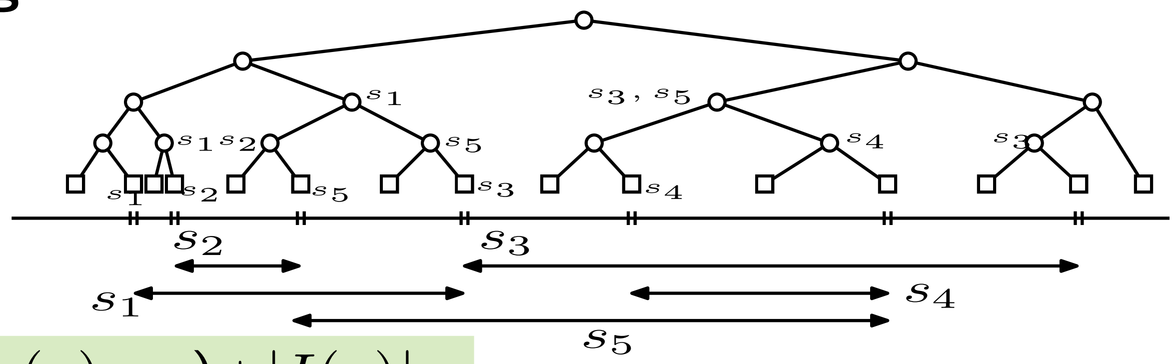

QuerySegmentTree(v, q_x)

if v is not leaf **then**

  if $q_x \in e(lc(v))$ **then**

    QuerySegmentTree($lc(v), q_x$)$+|I(v)|$

  else

    QuerySegmentTree($rc(v), q_x$)$+|I(v)|$

return 1

Store $|I(v)|$ instead of $I(v)$

$O(\log n)$ time and $O(n)$ storage.

Exercise 3

given: Set I of n intervals

find: In how many intervals is a point $p \in \mathbb{R}$ contained? Datastructure!

Data structure based on binary tree.

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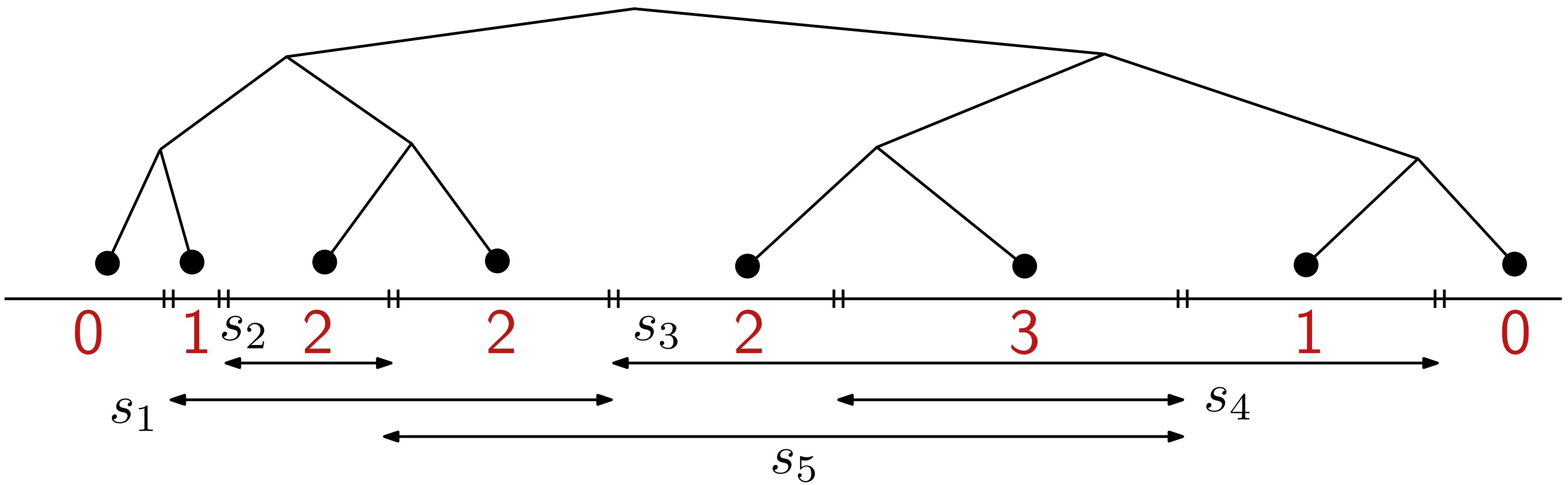


1. Split intervals into elementary intervals.
2. Store for each elem. interval, in how many intervals it is contained.
3. Construct binary tree based on borders of elem. intervals.

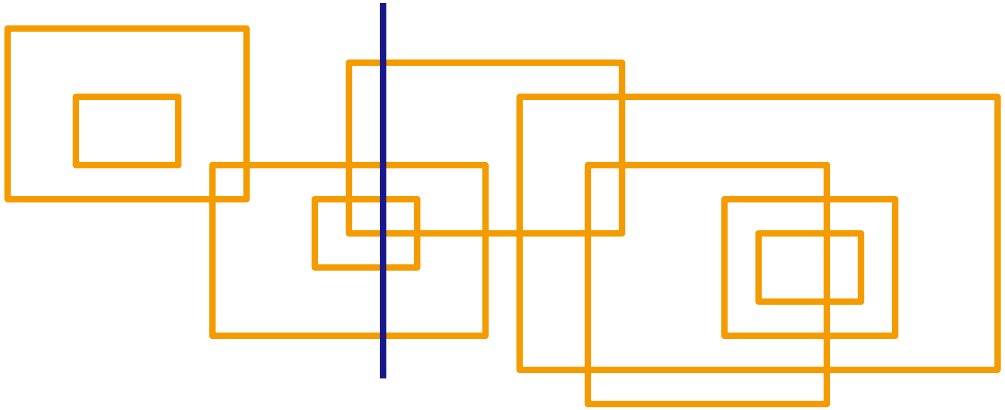
→ Time $O(\log n)$, Storarge $O(n)$

Exercise 4

Given: Set $\mathcal{R}$ of axis-aligned rectangles.

Find: Algorithm that computes $\max_{p \in \mathbb{R}} w_{\mathcal{R}}(p)$ in $O(n \log n)$ time.

For $p \in \mathbb{R}$, $w_{\mathcal{R}}(p)$ is the number of rectangles in $\mathcal{R}$ that contain p.



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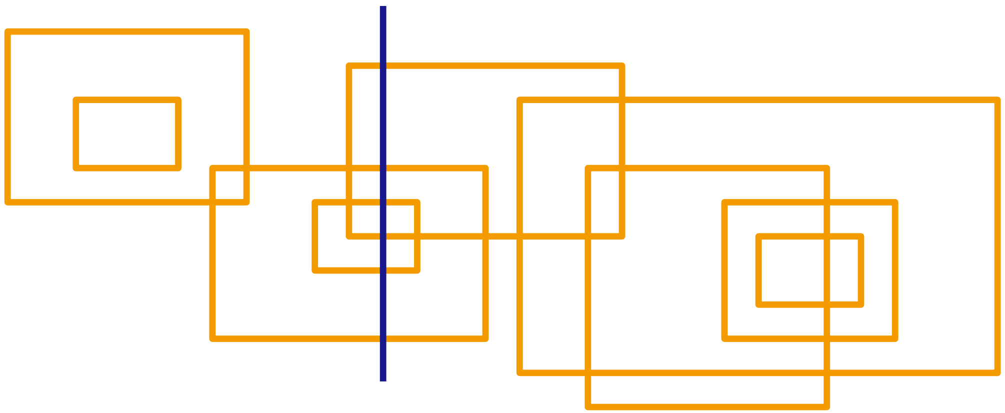

Sweep-Line: from left to right

SL-State: segment tree T that stores vertical edges as intervals.

Events: vertical edges of rectangles.

left vert. edge $\overline{pq}$: 1. determine the number or intervals in T intersecting $[y(p), y(q)]$.

→ update $\max w_{\mathcal{R}}(p)$

2. Insert $[y(p), y(q)]$ into T.

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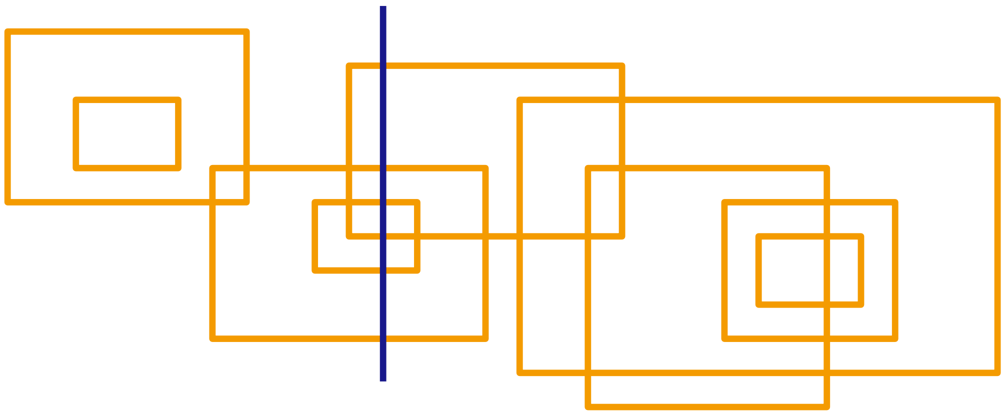

Sweep-Line: from left to right

SL-State: segment tree T that stores vertical edges as intervals.

Events: vertical edges of rectangles.

right vert. edge $\overline{pq}$: delete interval $[y(p), y(q)]$.