Computational Geometry – WSPD

WSPD

LEHRSTUHL FÜR ALGORITHMIK I · INSTITUT FÜR THEORETISCHE INFORMATIK · FAKULTÄT FÜR INFORMATIK

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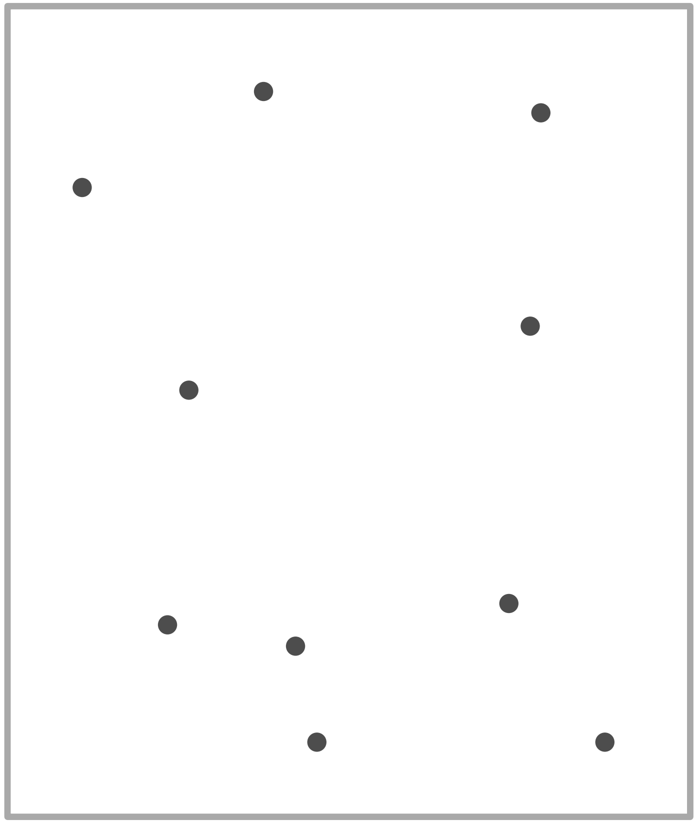
06.07.2018

Motivation: Spanners



Task:

A set of cities shall be connected by a new road network.

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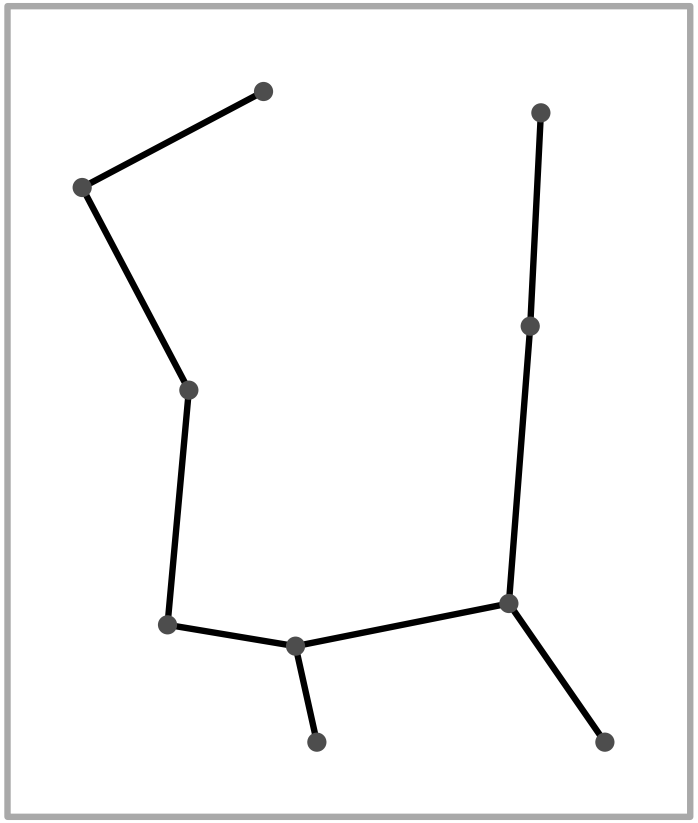
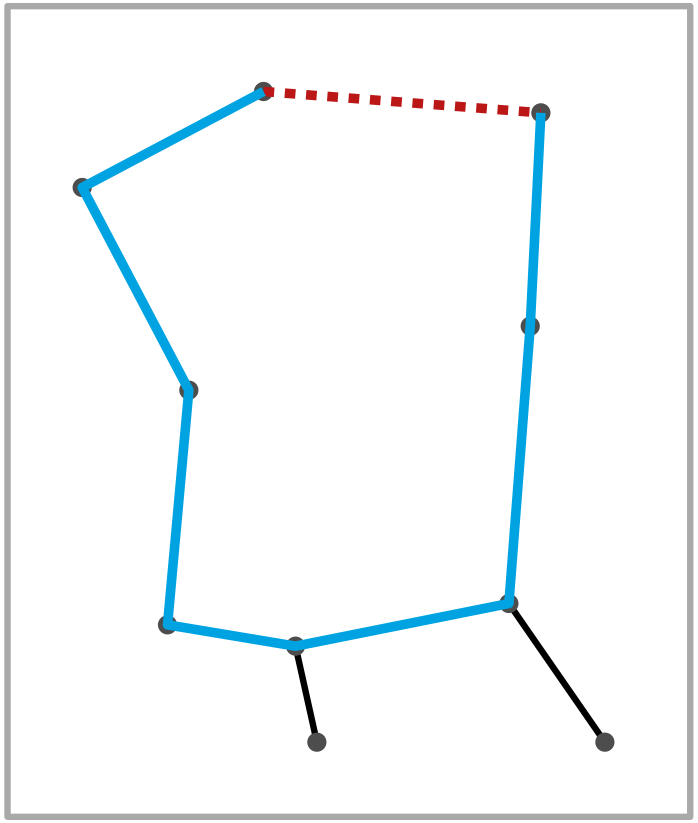


Idea 1: Euclidean minimum spanning tree

Motivation: Spanners



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But for no pair (x, y) the path length in the road network should be much larger than the distance $||xy||$.

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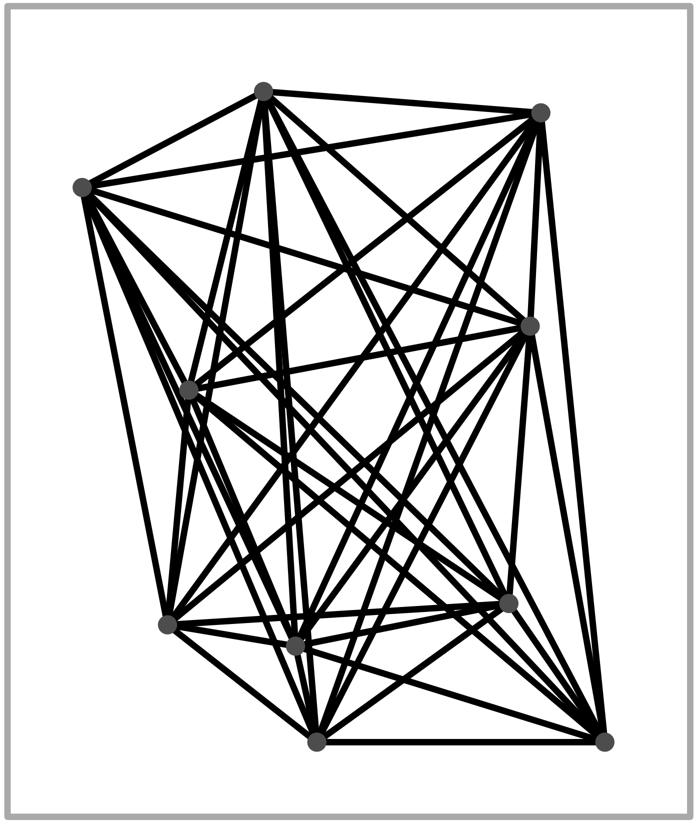

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Idea 1: Euclidean minimum spanning tree
Idea 2: complete graph

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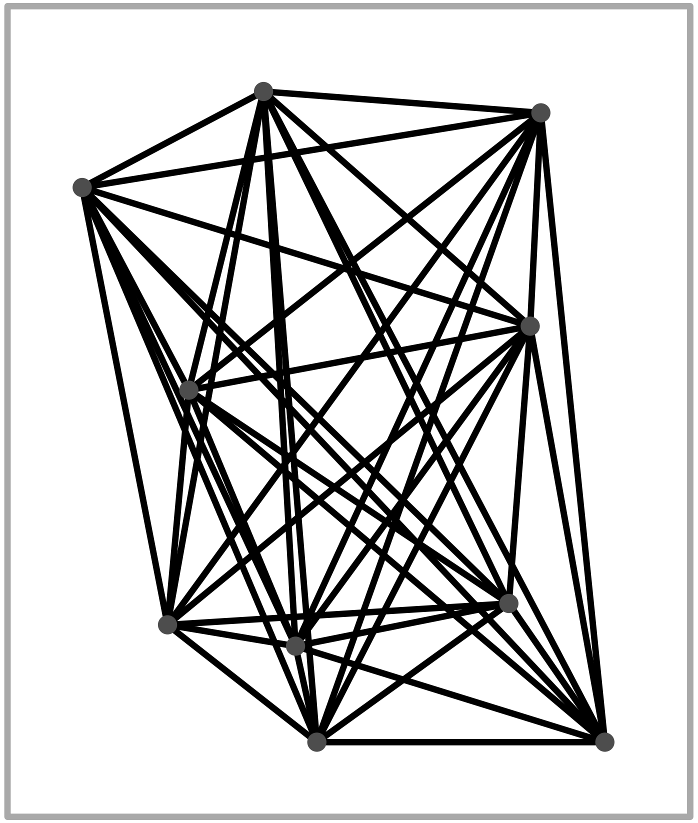

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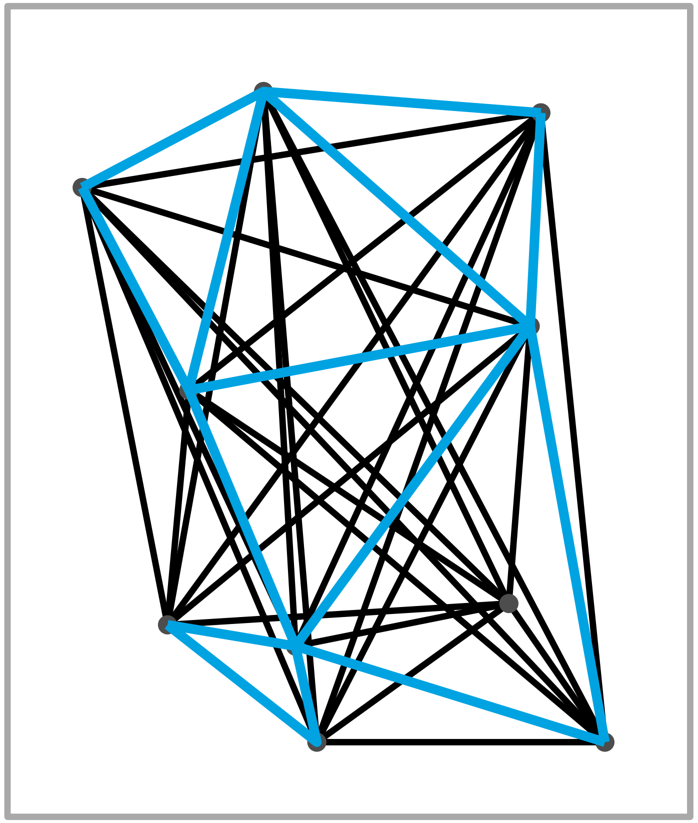

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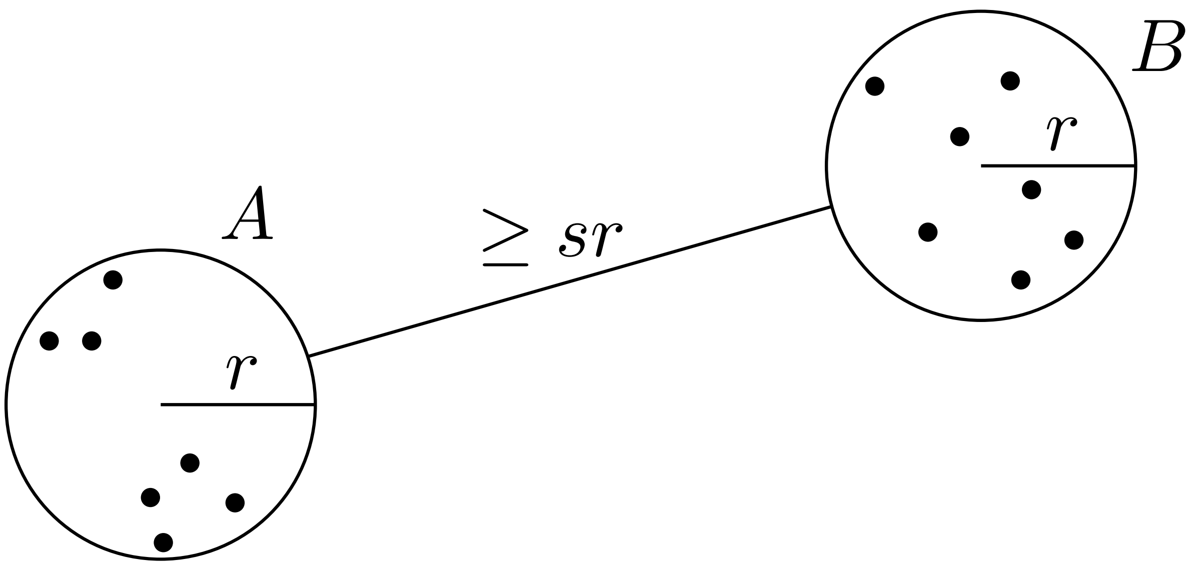
Idea 3: sparse t-spanner

Well-Separated Pairs

Def: A pair of disjoint point sets A and B in $\mathbb{R}^d$ is called s-**well separated** for some $s > 0$, if A and B can each be covered by a ball of radius r whose distance is at least sr.



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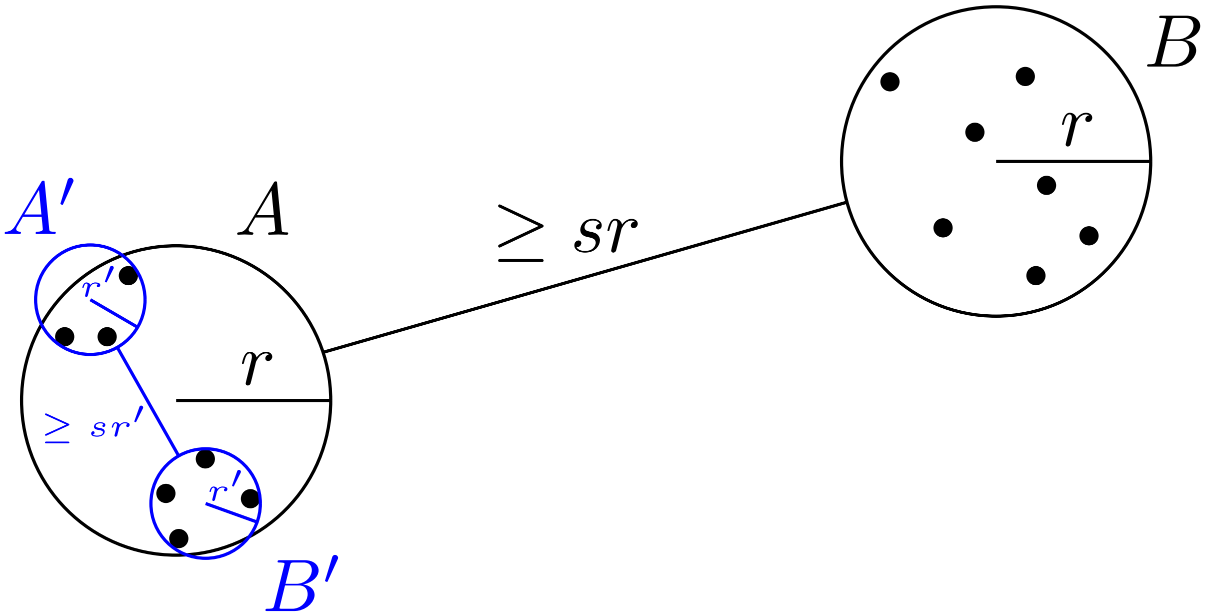

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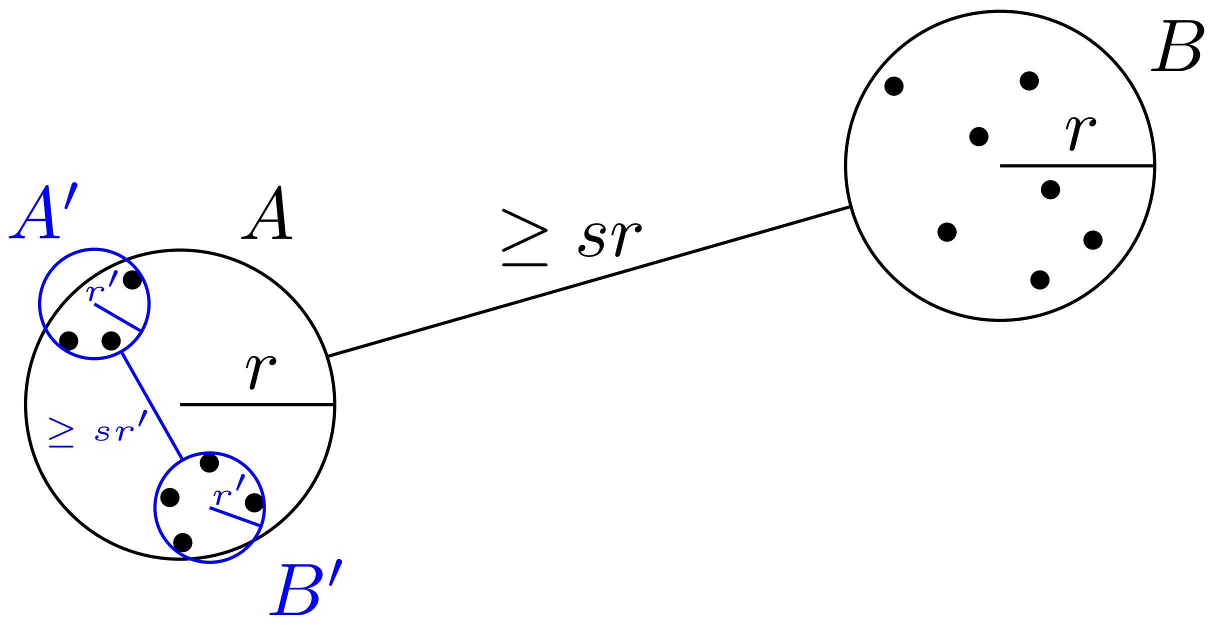

Obs:
- s-well separated $\Rightarrow$ s'-well separated for all $s' \leq s$
- singletons $\{a\}$ and $\{b\}$ are s-well separated for all $s > 0$

Well-Separated Pair Decomposition (WSPD)

For well-separated pair $\{A, B\}$ we know that the distance for all point pairs in $A \otimes B = \{\{a, b\} \mid a \in A, b \in B, a \neq b\}$ is similar.

Goal: $o(n^2)$-sized data structure that approximates the distances of all $\binom{n}{2}$ pairs of points in a set $P = \{p_1, \ldots, p_n\}$.

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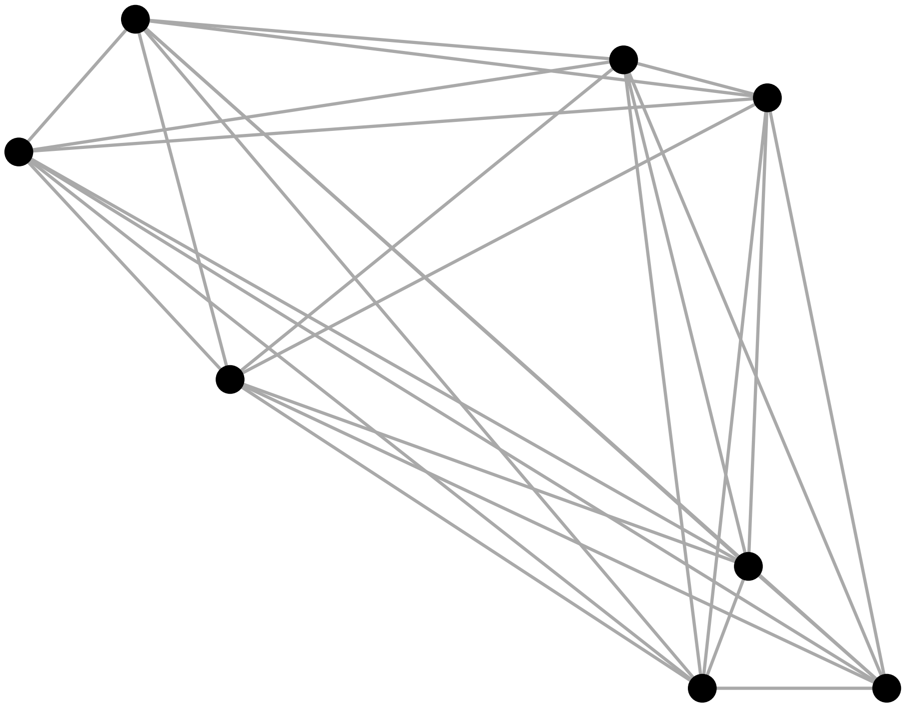
Def: For a point set P and some $s > 0$ an s-**well separated pair decomposition** (s-WSPD) is a set of pairs $\{\{A_1, B_1\}, \ldots, \{A_m, B_m\}\}$ with

- $A_i, B_i \subset P$ for all i
- $A_i \cap B_i = \emptyset$ for all i
- $\bigcup_{i=1}^{m} A_i \otimes B_i = P \otimes P$
- $\{A_i, B_i\}$ s-well separated for all i

Example



28 point pairs

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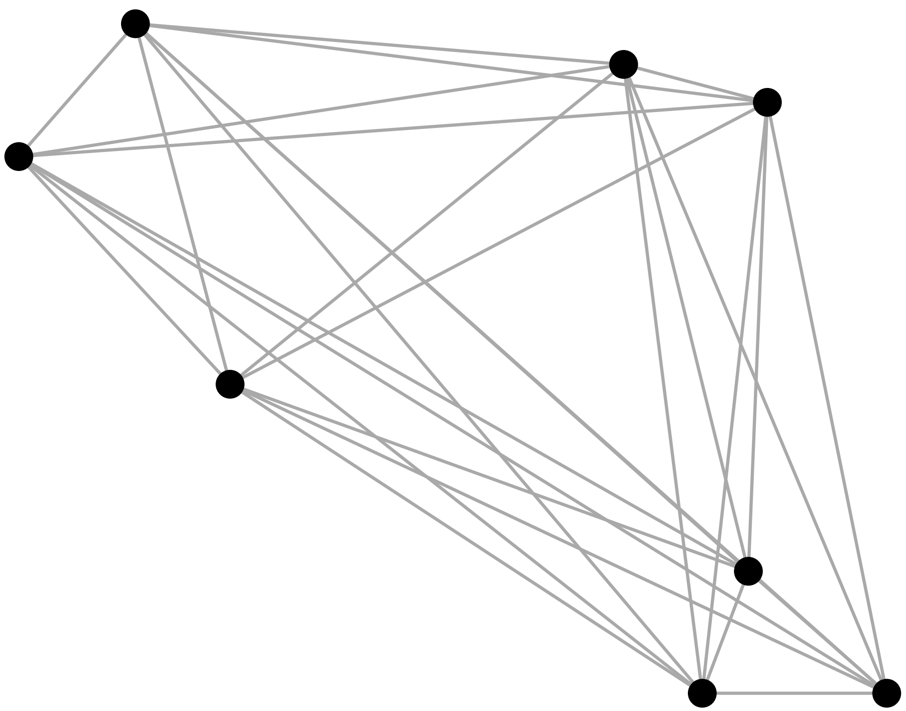

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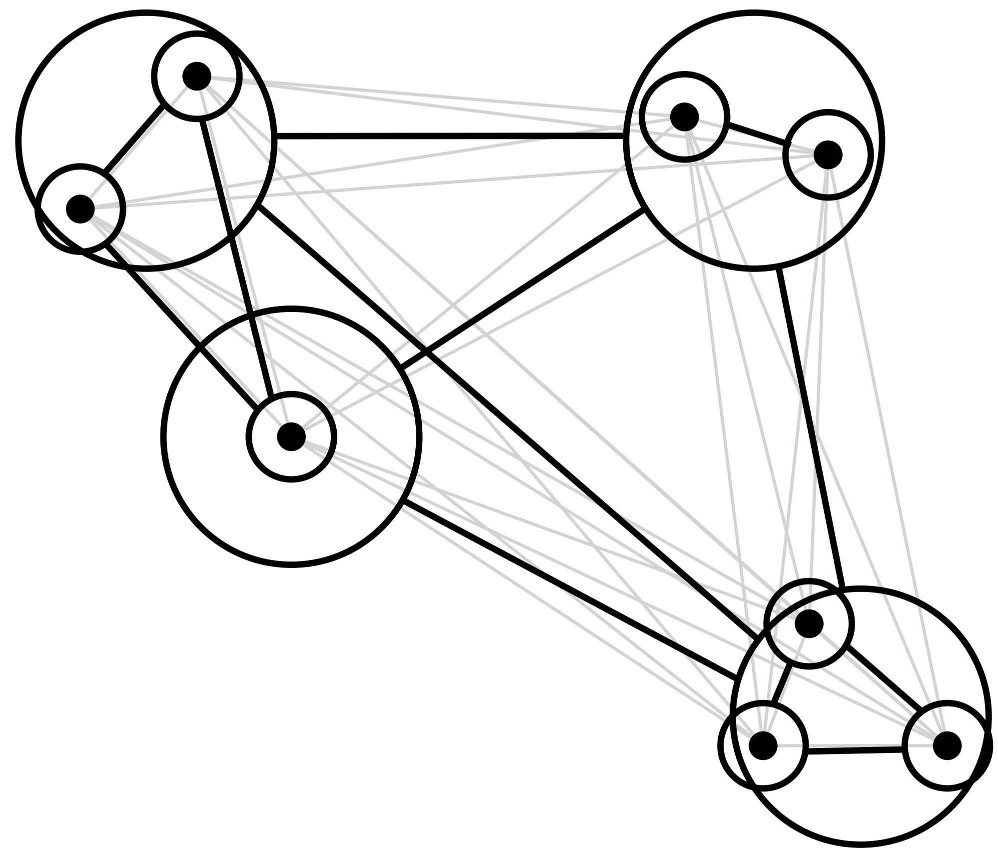

12 s-well separated pairs

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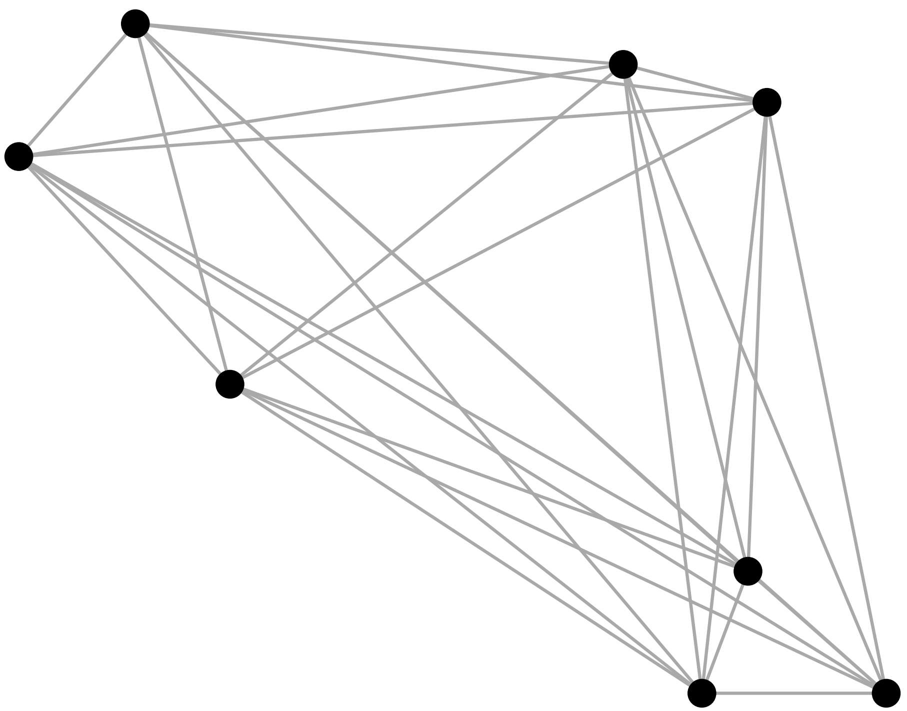

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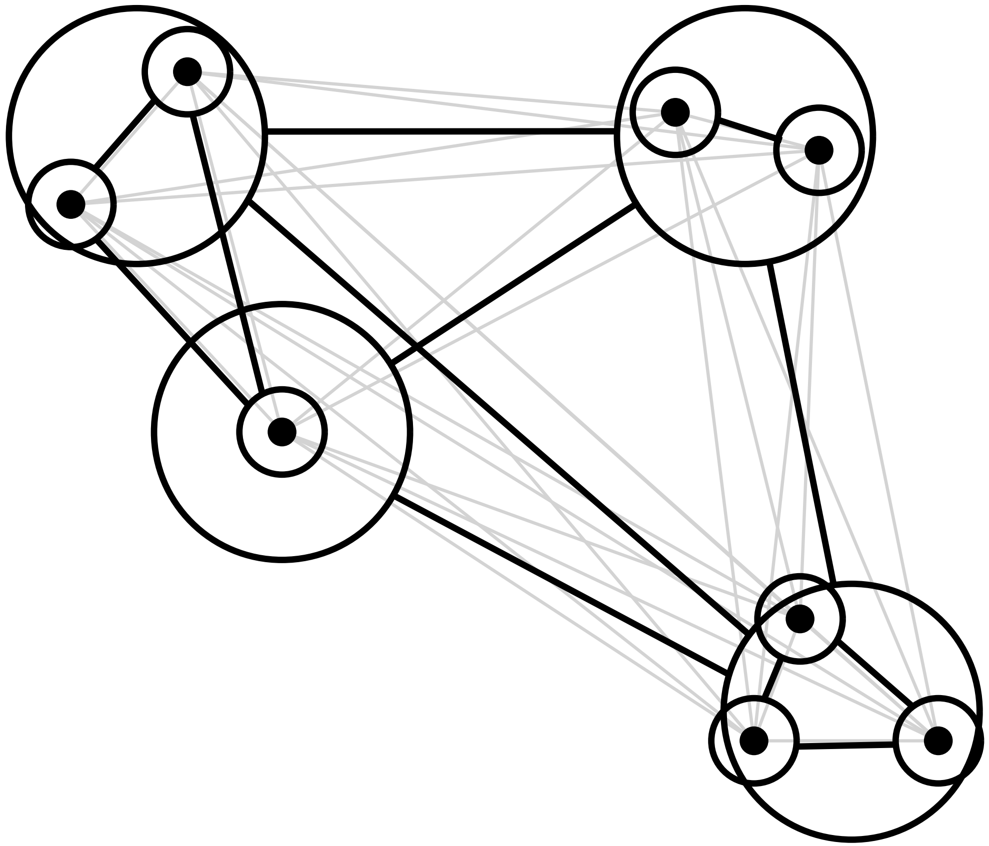


12 s-well separated pairs

Thm 3: Given a point set P in $\mathbb{R}^d$ and $s \geq 1$ we can construct an s-WSPD with $O(s^d n)$ pairs in time $O(n \log n + s^d n)$.

Exercise 5

- x := 2/s + 1
- $S := \{x^i \mid 0 \leq i \leq n-1\}$

$\mathcal{W} = \{A_j, B_j\}_{1 \leq j \leq m}$ arbitrary s-WSPD for S $(s > 0)$

Show:

$$\sum_{j=1}^{m}(|A_j| + |B_j|) = \binom{n}{2} + m$$

Hint: Show that for each j at least one of both sets A_j and B_j is a singleton.

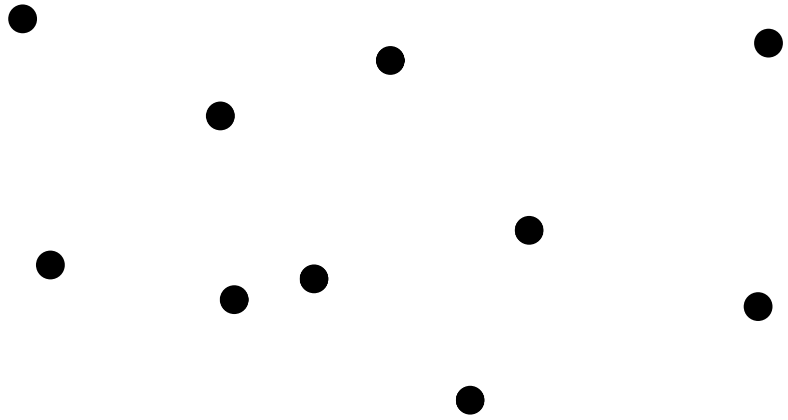
Alternative Definition

Def.: For a point set P and some $s > 0$ an s-**well separated pair decomposition** (s-WSPD) is a set of pairs $\{\{A_1, B_1\}, \ldots, \{A_m, B_m\}\}$ with

- $A_i, B_i \subset P$ for all i
- $\{A_i, B_i\}$ s-well separated for all i
- for two distinct points $p, q \in P$ there is exactly one index i with $1 \leq i \leq m$ such that
  - $p \in A_i$ and $q \in B_i$, or
  - $q \in A_i$ and $p \in B_i$.

Exercise 6/7

- P: n Punkte aus dem $\mathbb{R}^d$



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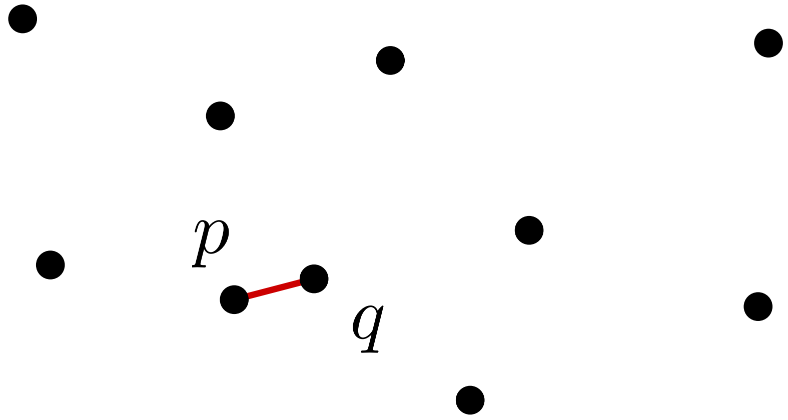
- $p, q \in P$ and q is the next neighbor of p



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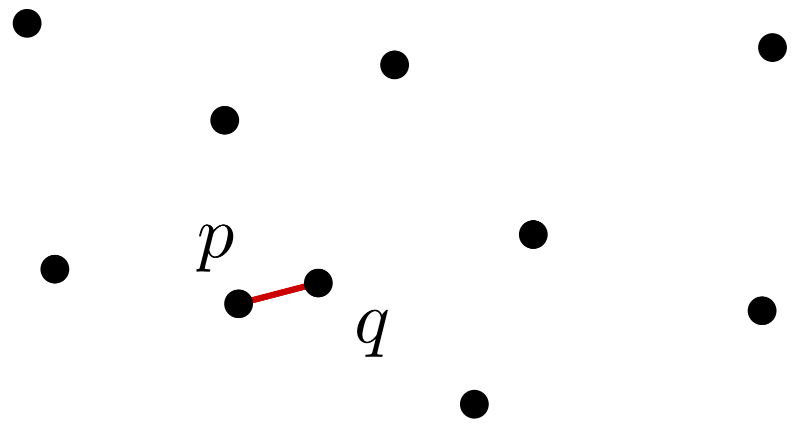

Given: s-WSPD $\mathcal{W}$ for P with $s > 2$
Let $\{A, B\} \in \mathcal{W}$ with $p \in A$ and $q \in B$

Show that:

- A is a singleton.
- size of $\mathcal{W}$ is at least $n/2$.
- if p, q have minimal distance among all pairs, then $\{\{p\}, \{q\}\}$ lies in WSPD.