Computational Geometry – Exercise

Delaunay Triangulation

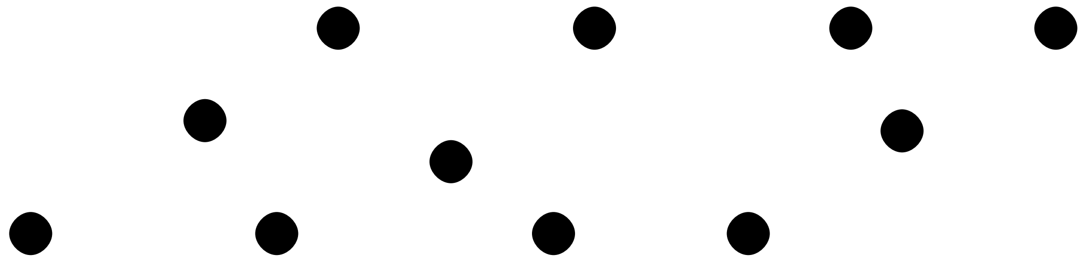
LEHRSTUHL FÜR ALGORITHMIK I · INSTITUT FÜR THEORETISCHE INFORMATIK · FAKULTÄT FÜR INFORMATIK

Guido Brückner

15.06.2018

Triangulation of a Point Set

Def.: A **triangulation** of a point set $P \subset \mathbb{R}^2$ is a maximal planar subdivision with a vertex set P.



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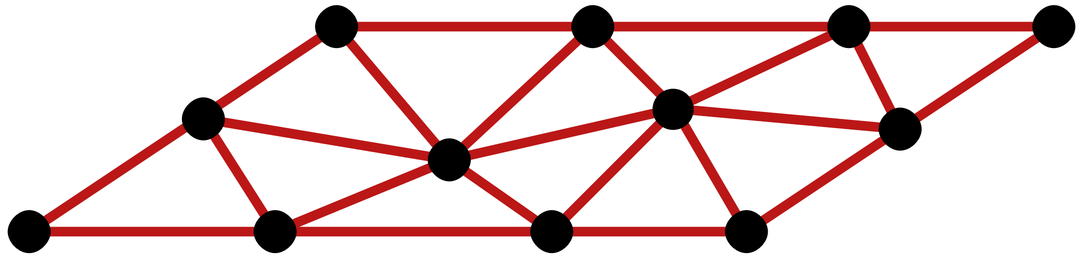

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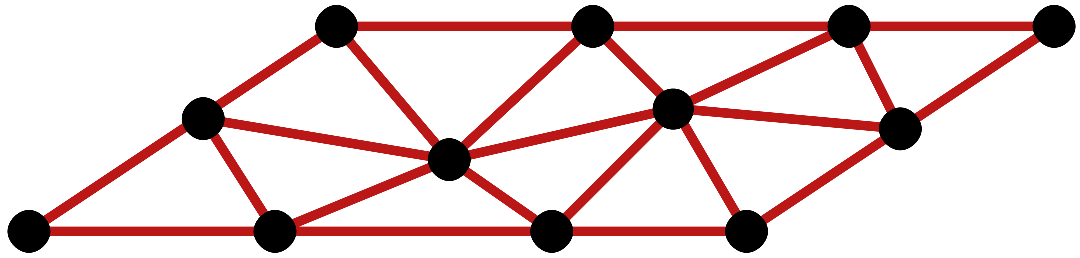

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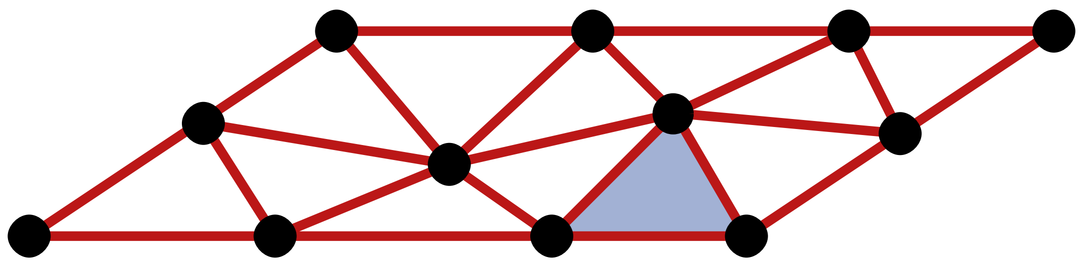

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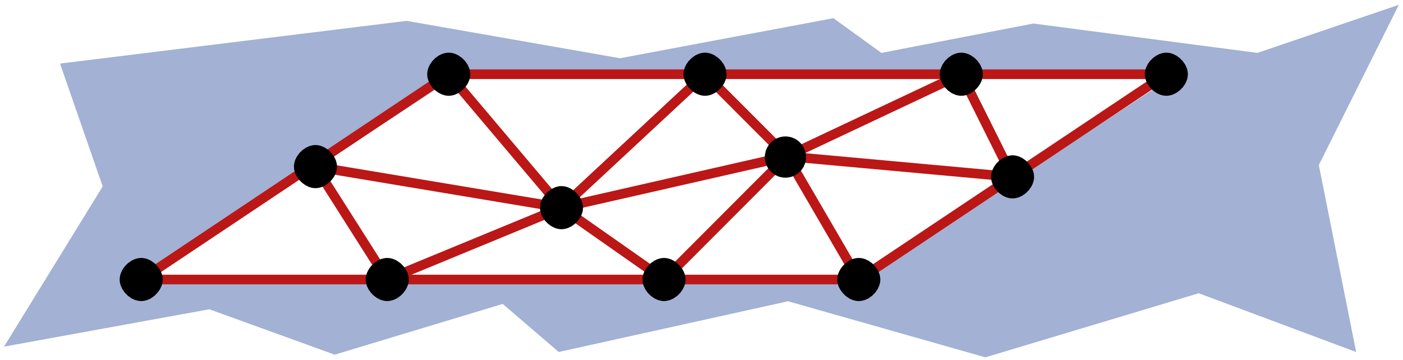

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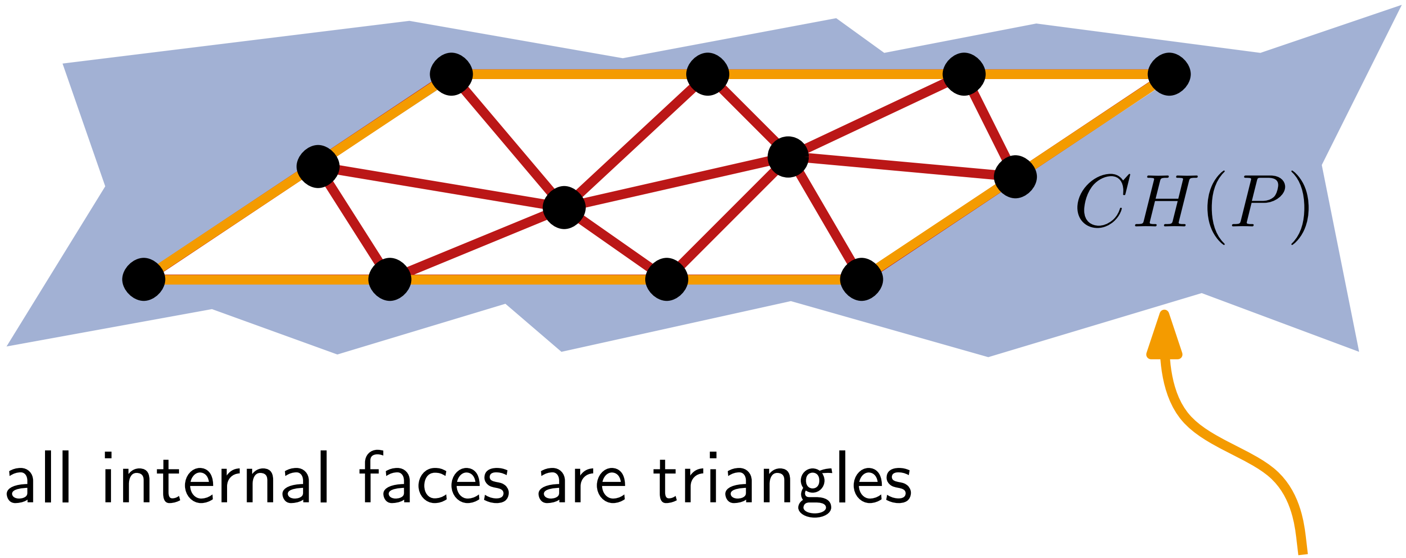

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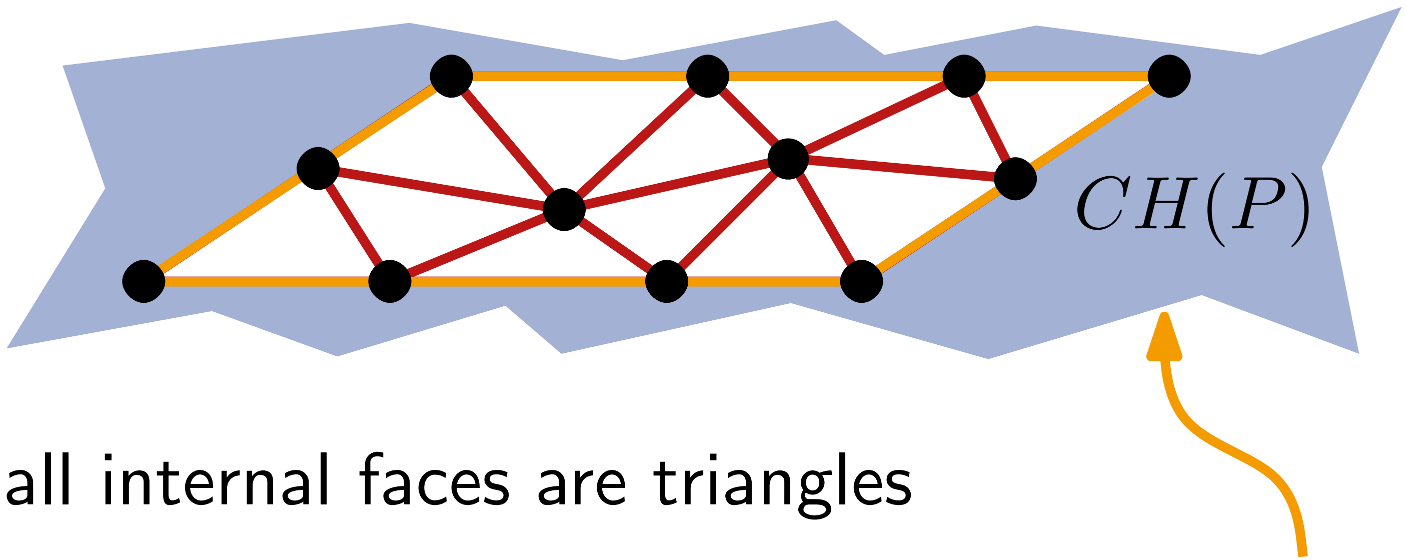

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Theorem 1: Let P be a set of n points, not all collinear. Let h be the number of points in $CH(P)$.

Then any triangulation of P has $(2n - 2 - h)$ triangles and $(3n - 3 - h)$ edges.

Exercise 5

Problem:

Let $P \subset \mathbb{R}^2$ be a set of n points.

a) There are at most $2^{\binom{n}{2}}$ triangulations of P.

b) Example: P such that for each triangulation there is at least one point having degree $n - 1$.

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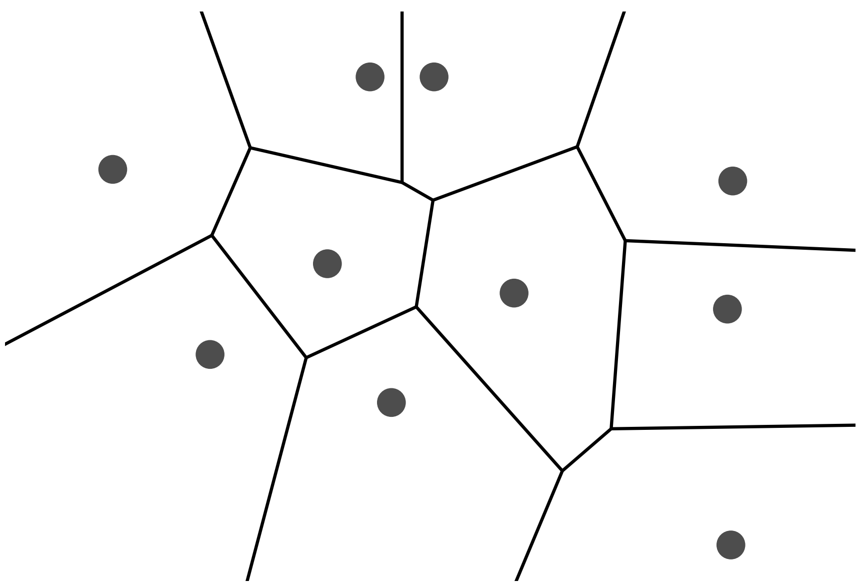
Delaunay-Triangulation

Let $\mathrm{Vor}(P)$ be the Voronoi-Diagram of a point set P.

Def.: The graph $\mathcal{G} = (P, E)$ with
$E = \{pq \mid \mathcal{V}(p)$ and $\mathcal{V}(q)$ are adjacent$\}$
is called **dual graph** of $\mathrm{Vor}(P)$.



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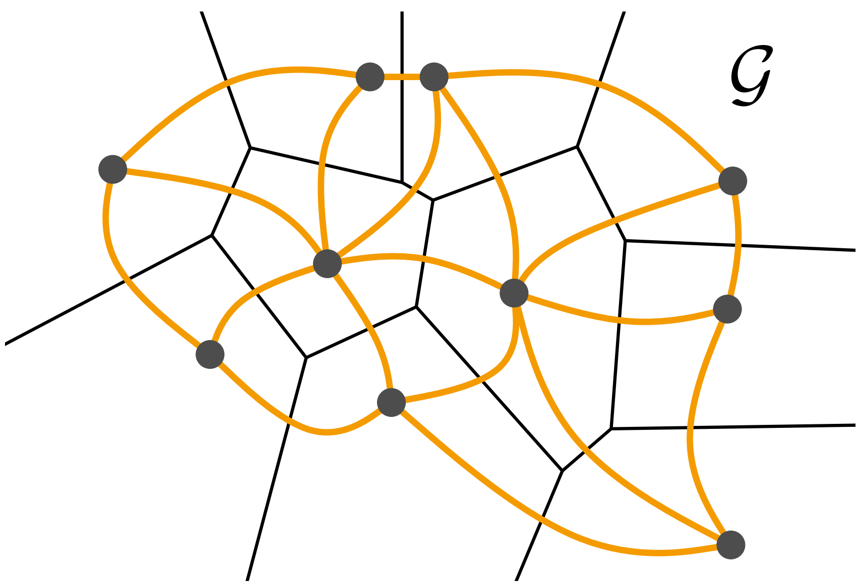

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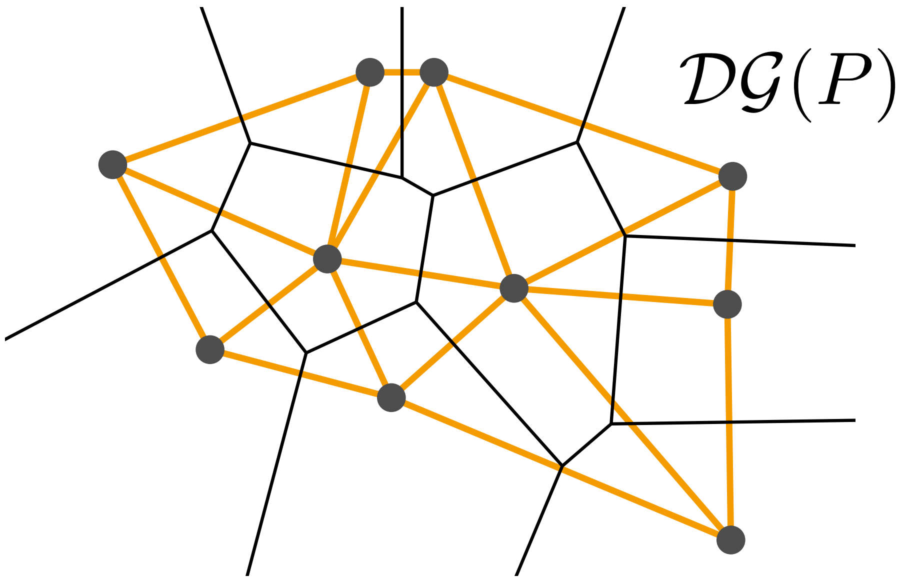
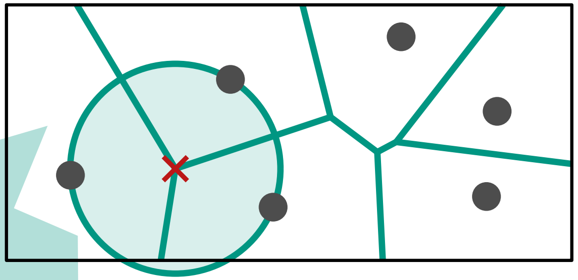
Def.: The straight-line drawing of $\mathcal{G}$ is called
Delaunay-Graph $\mathcal{DG}(P)$.

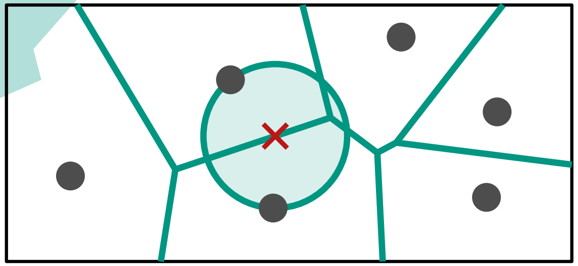


Characterization

Theorem about Voronoi-Diagram:

- point q is a Voronoy-vertex
  $\Leftrightarrow |C_P(q) \cap P| \geq 3$,
- bisector $b(p_i, p_j)$ defines a Voronoi-edge
  $\Leftrightarrow \exists q \in b(p_i, p_j)$ with $C_P(q) \cap P = \{p_i, p_j\}$.





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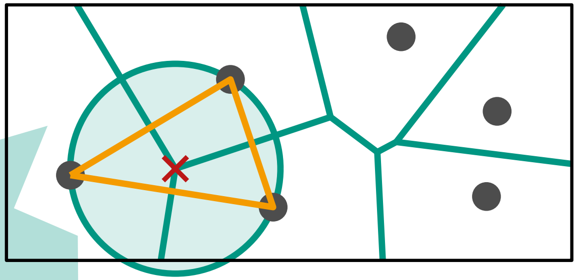

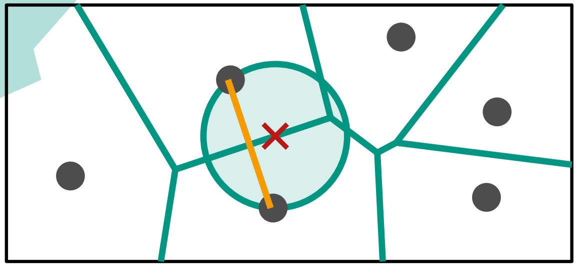

Theorem 4: Let P be a set of points.

- Points p, q, r are vertices of the same face of $\mathcal{DG}(P) \Leftrightarrow$ circle through p, q, r is empty
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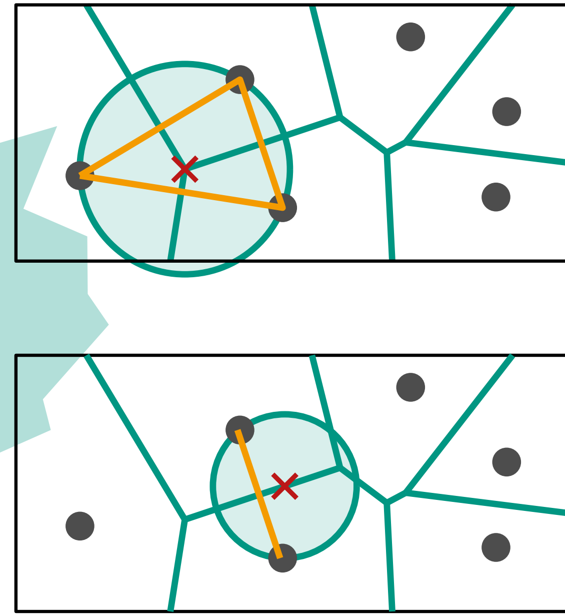

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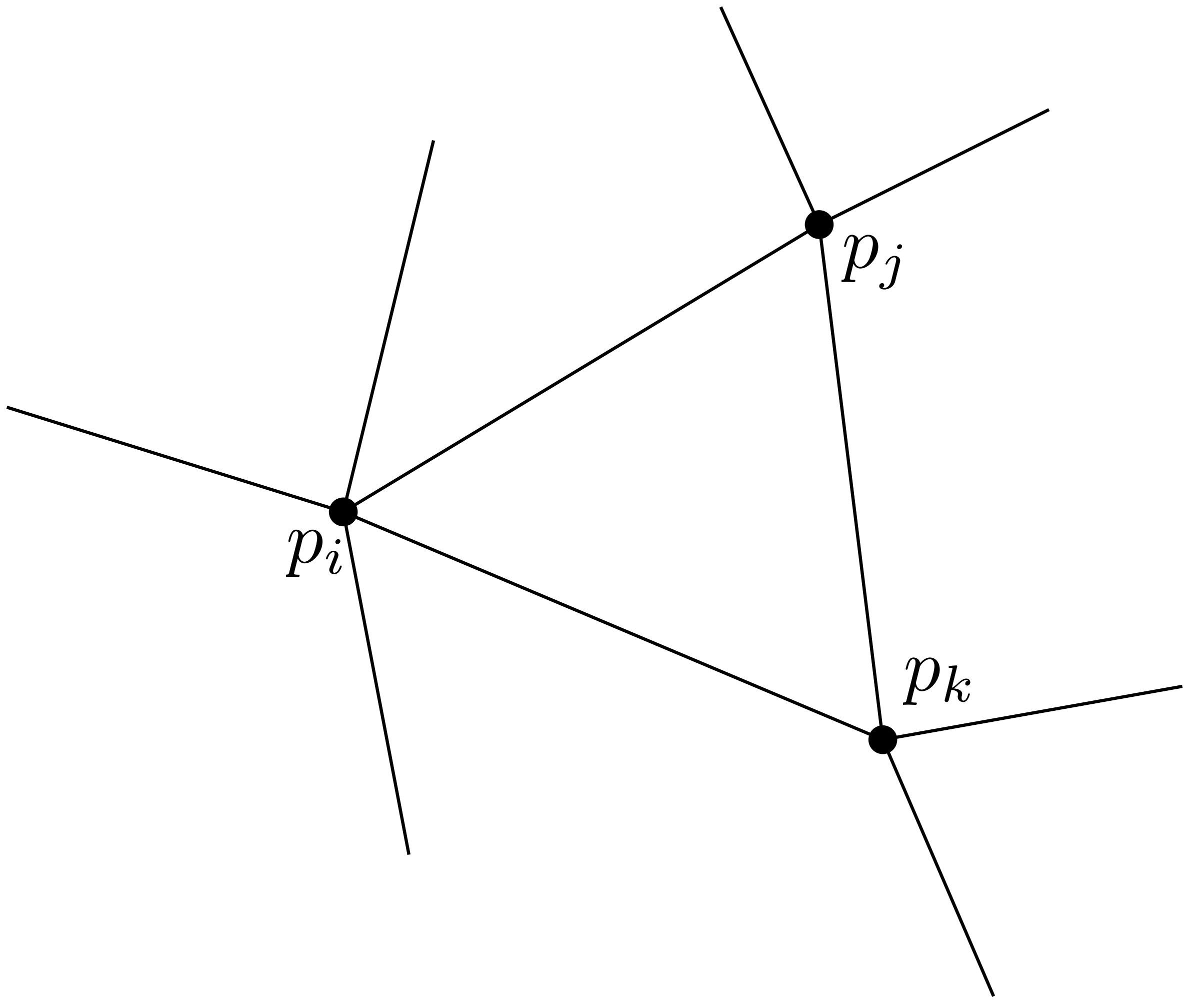

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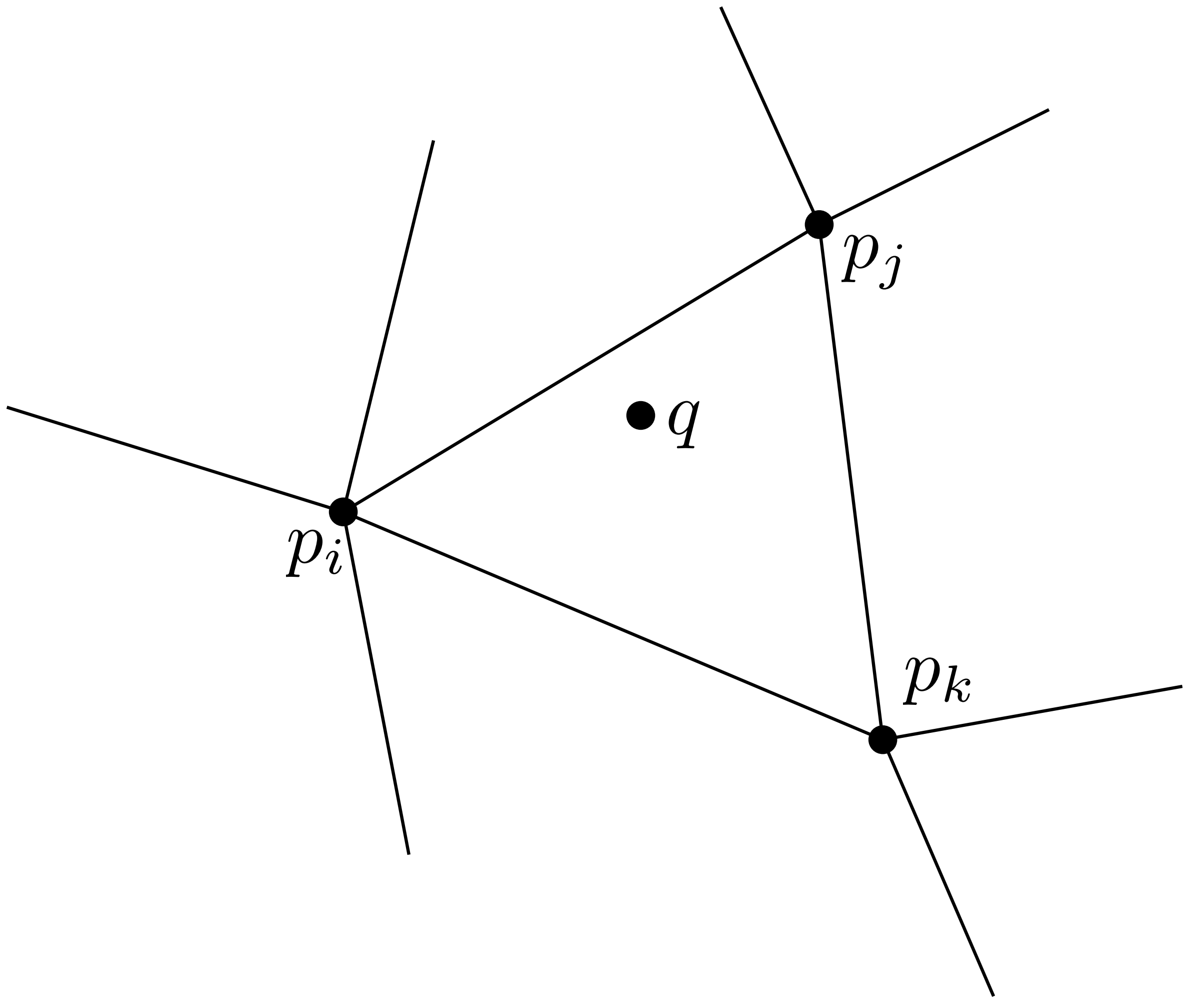

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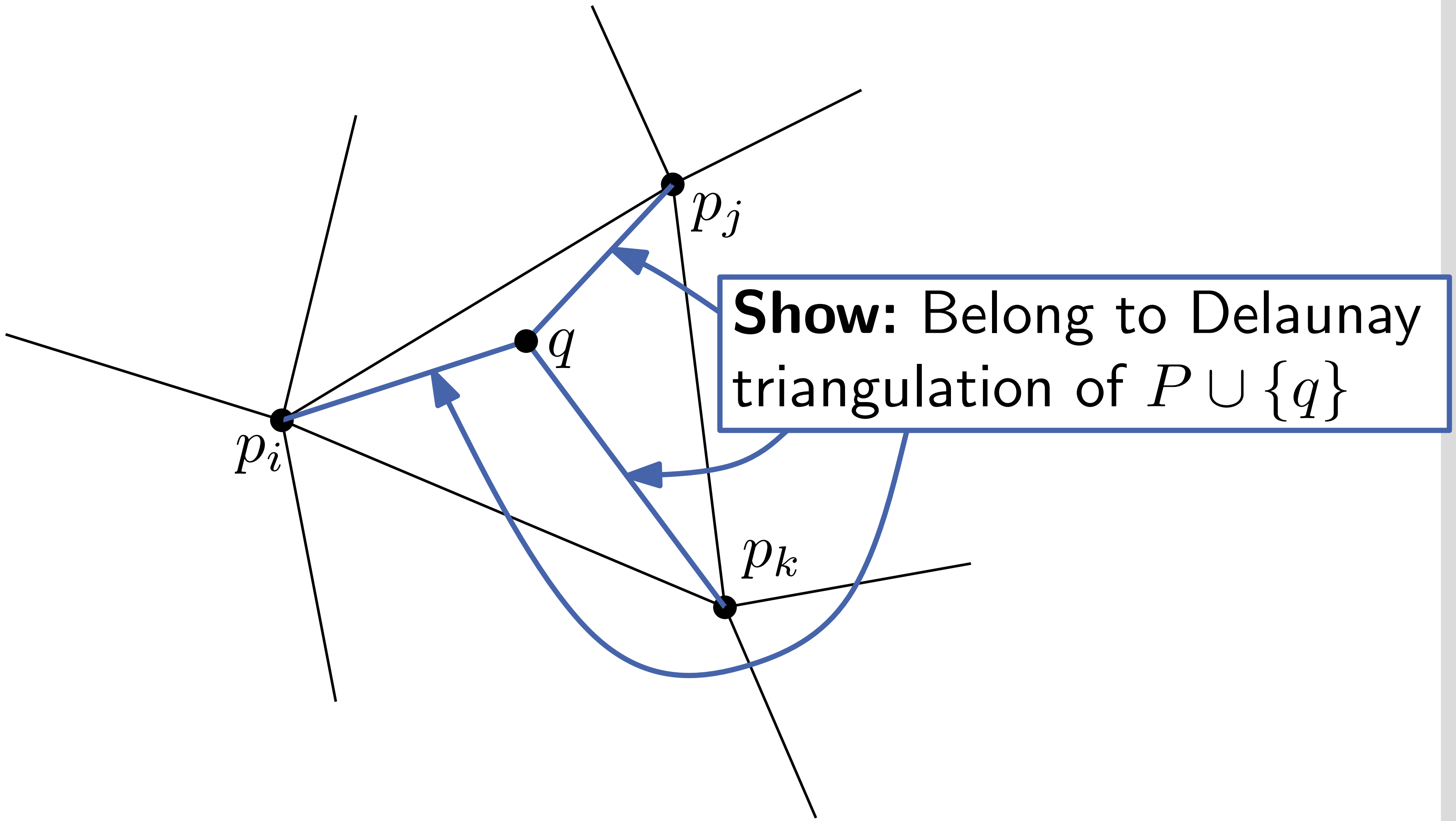

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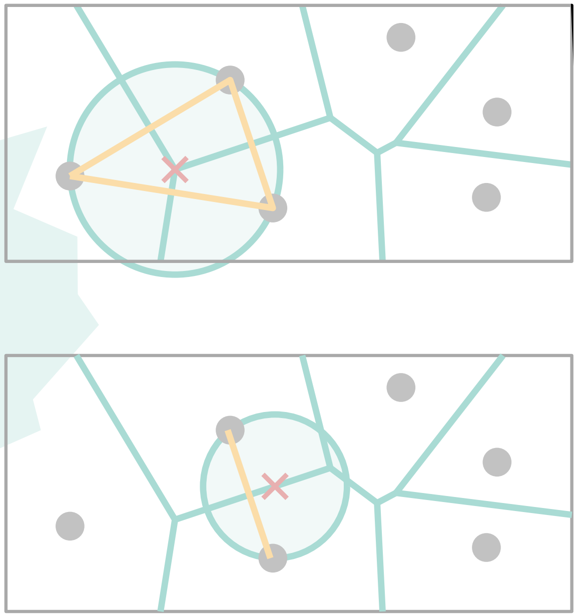

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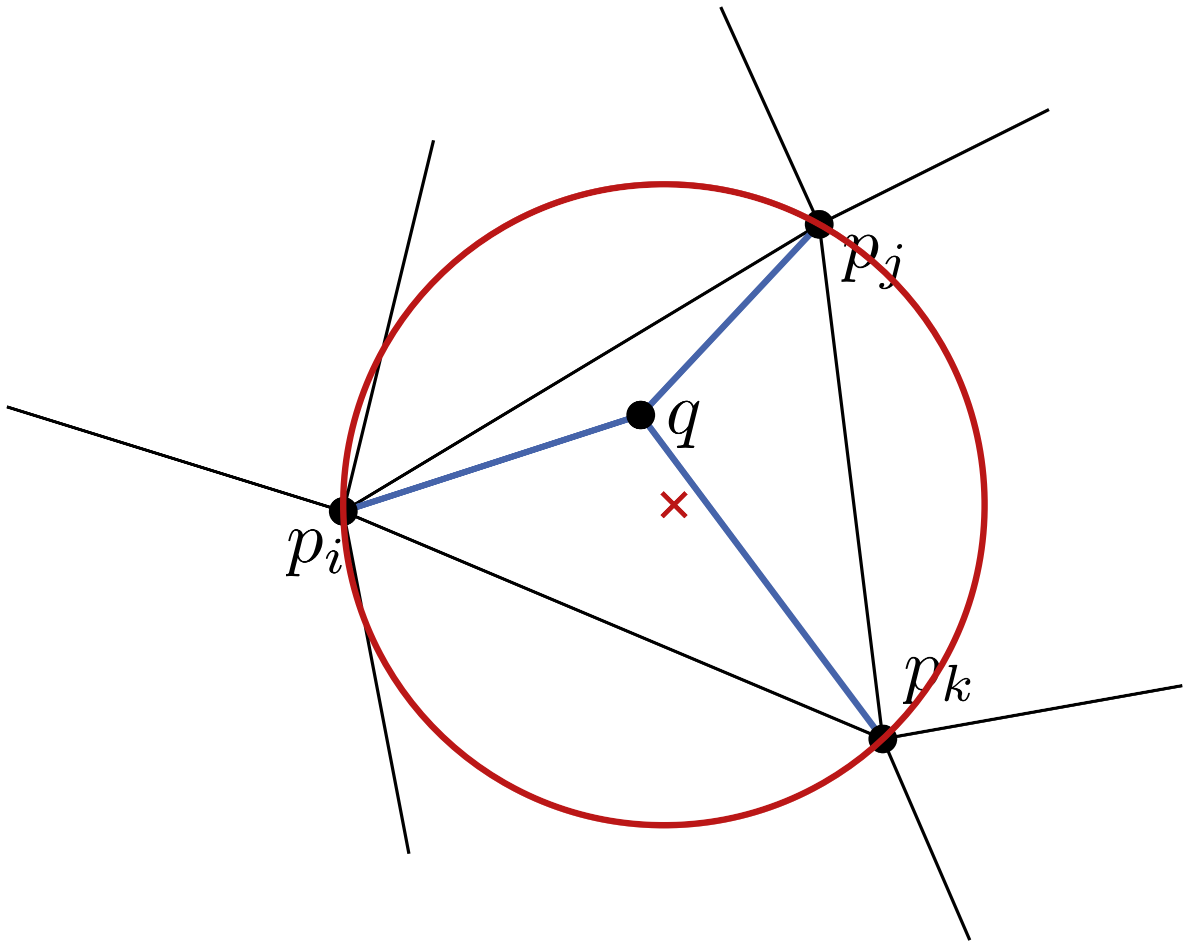

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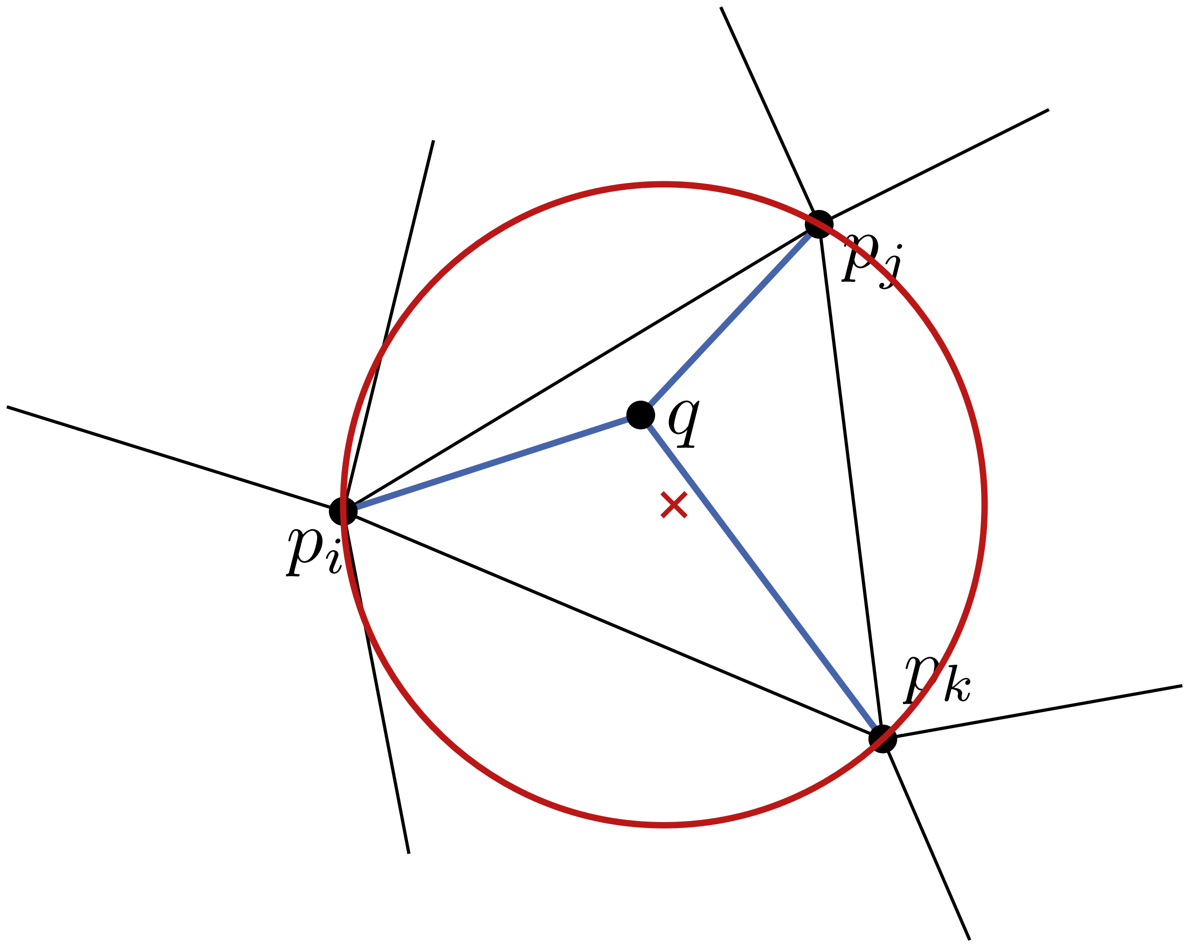

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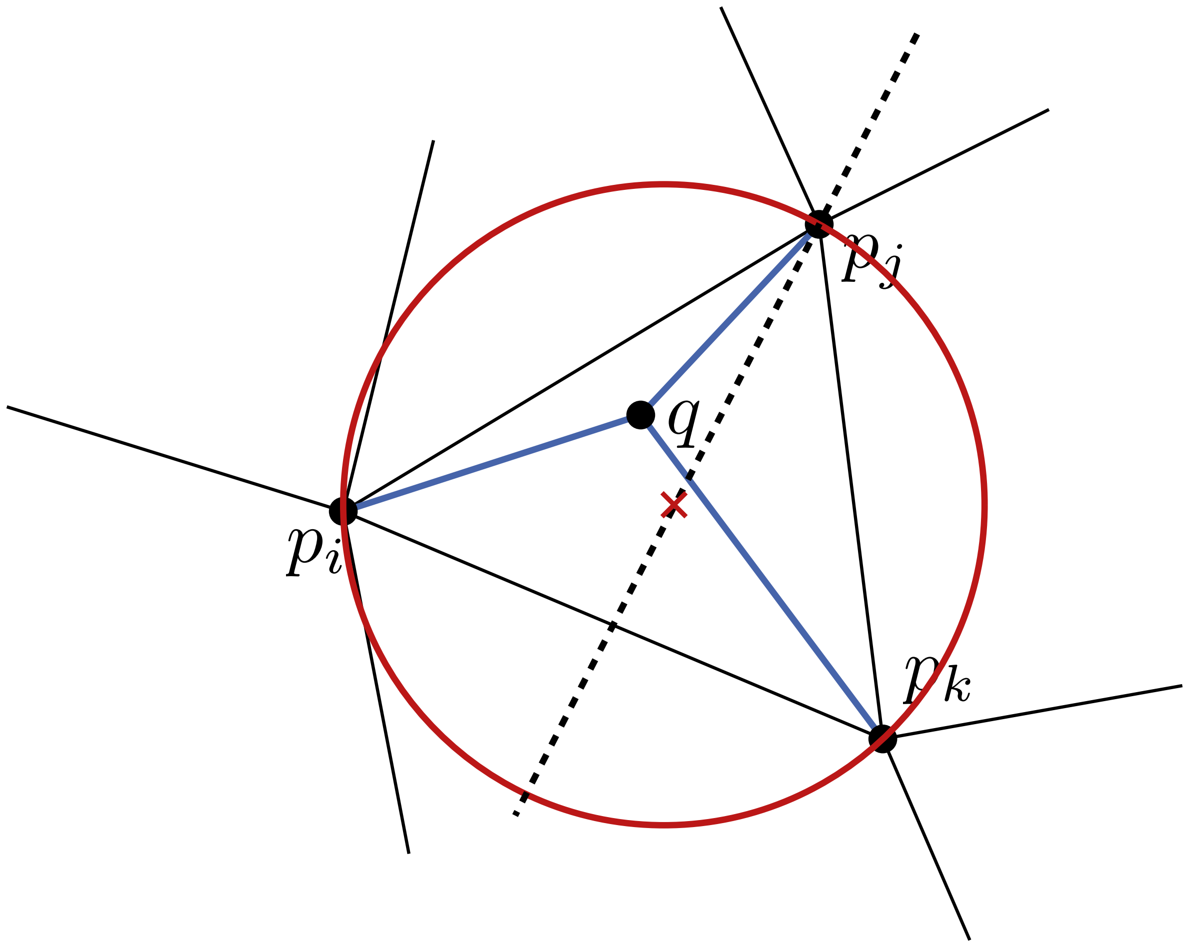

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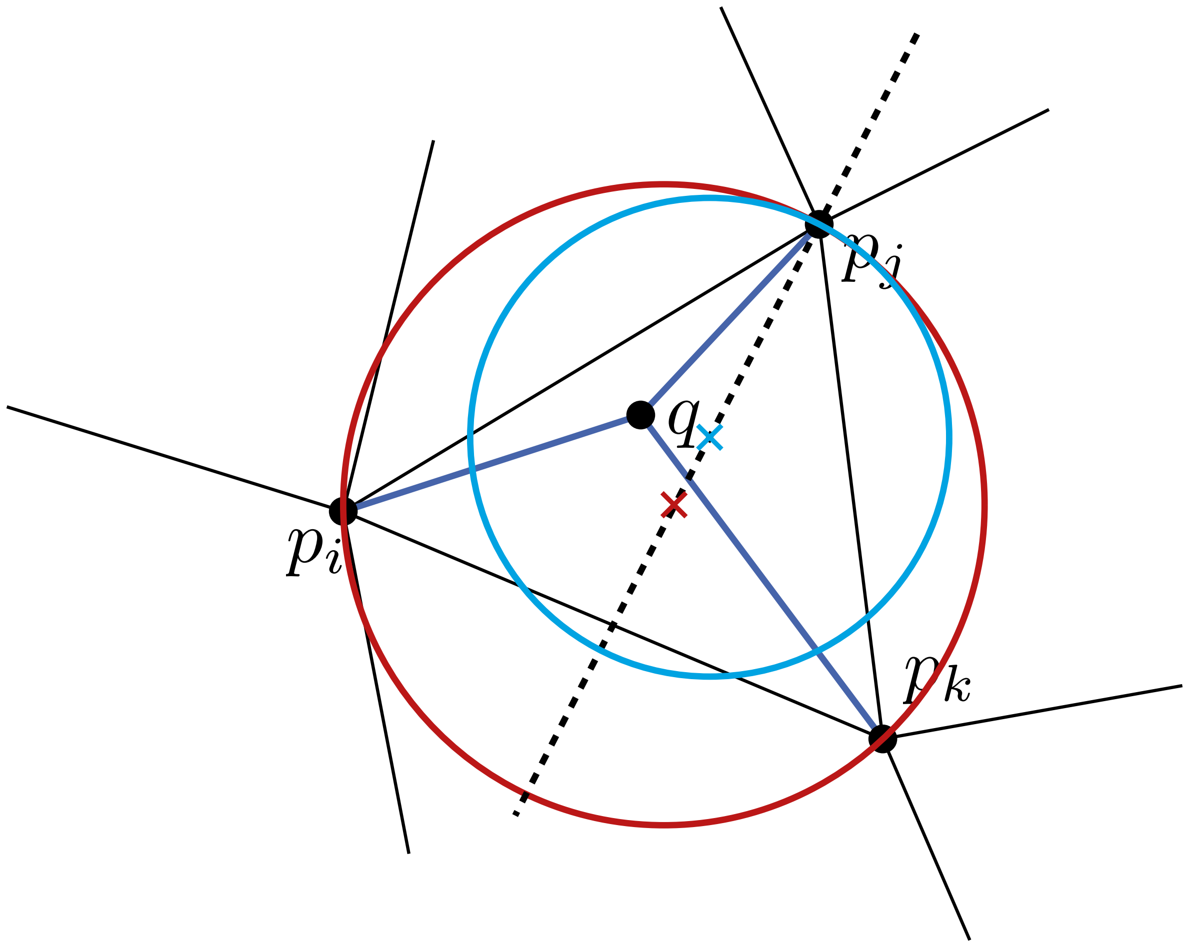

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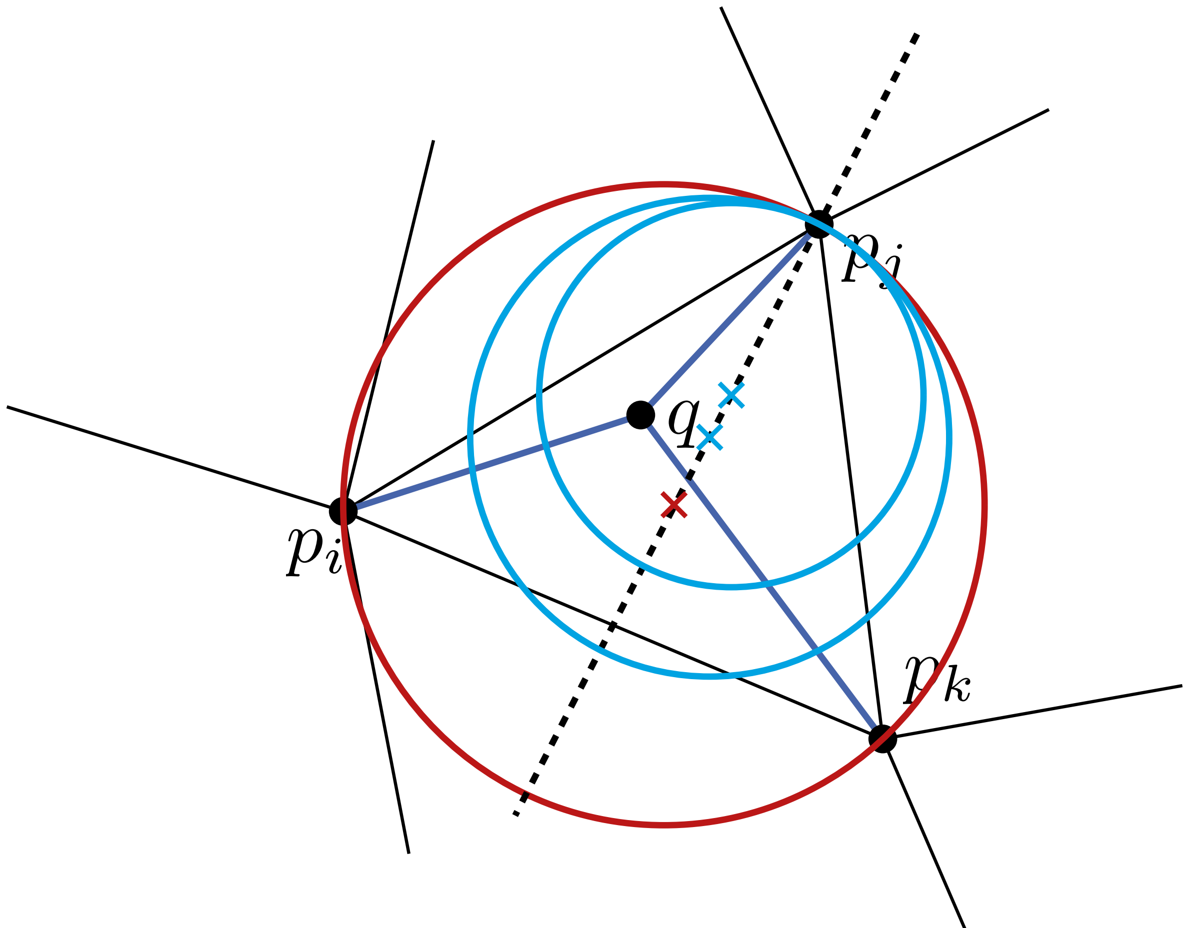

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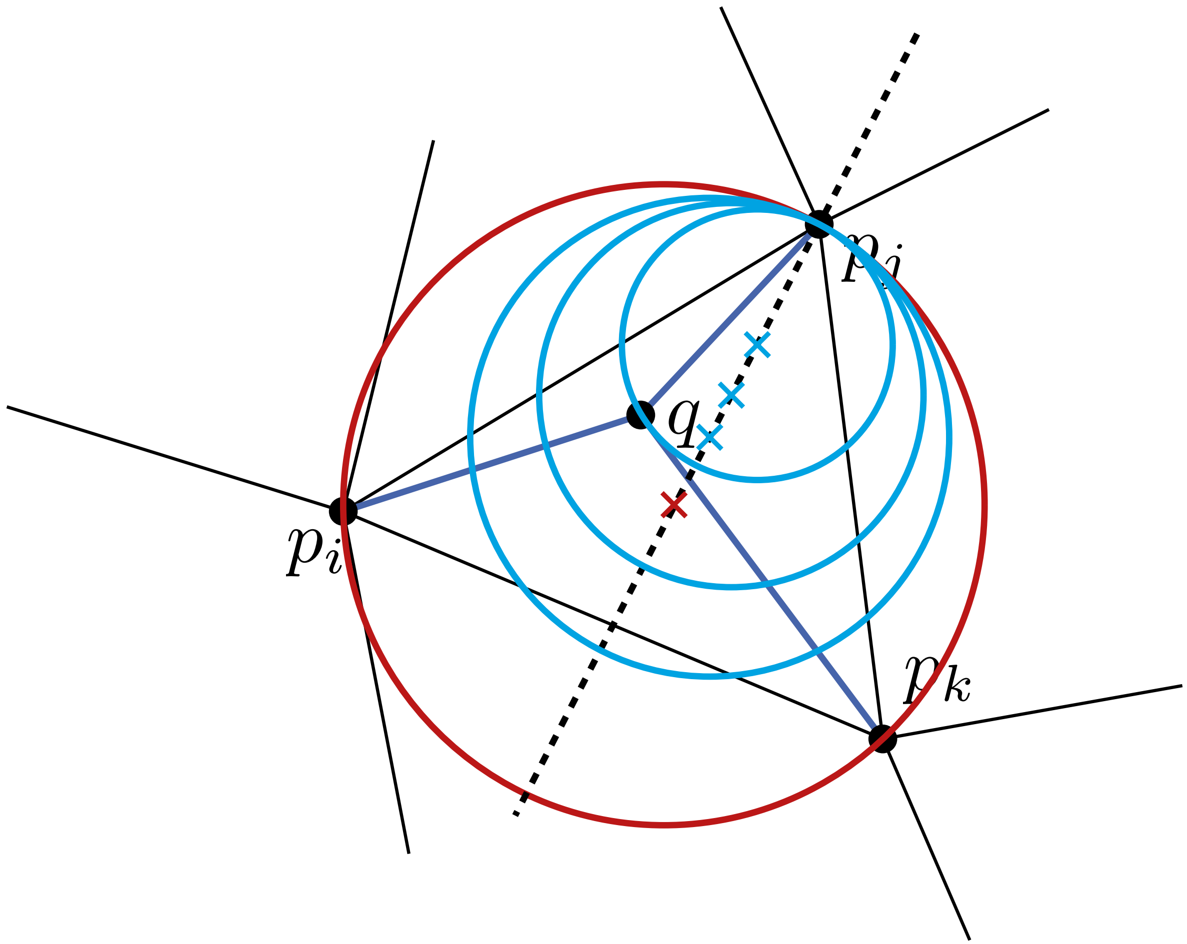

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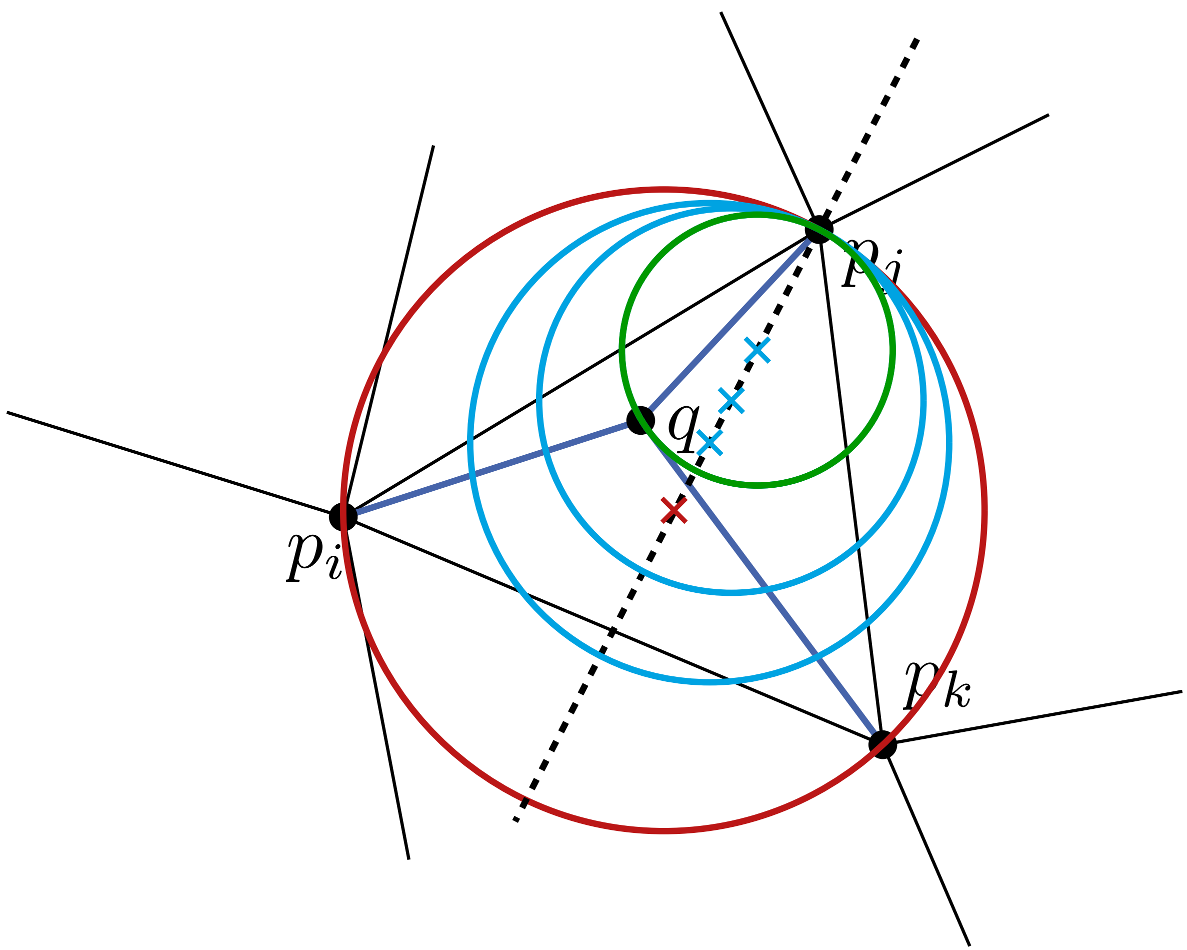

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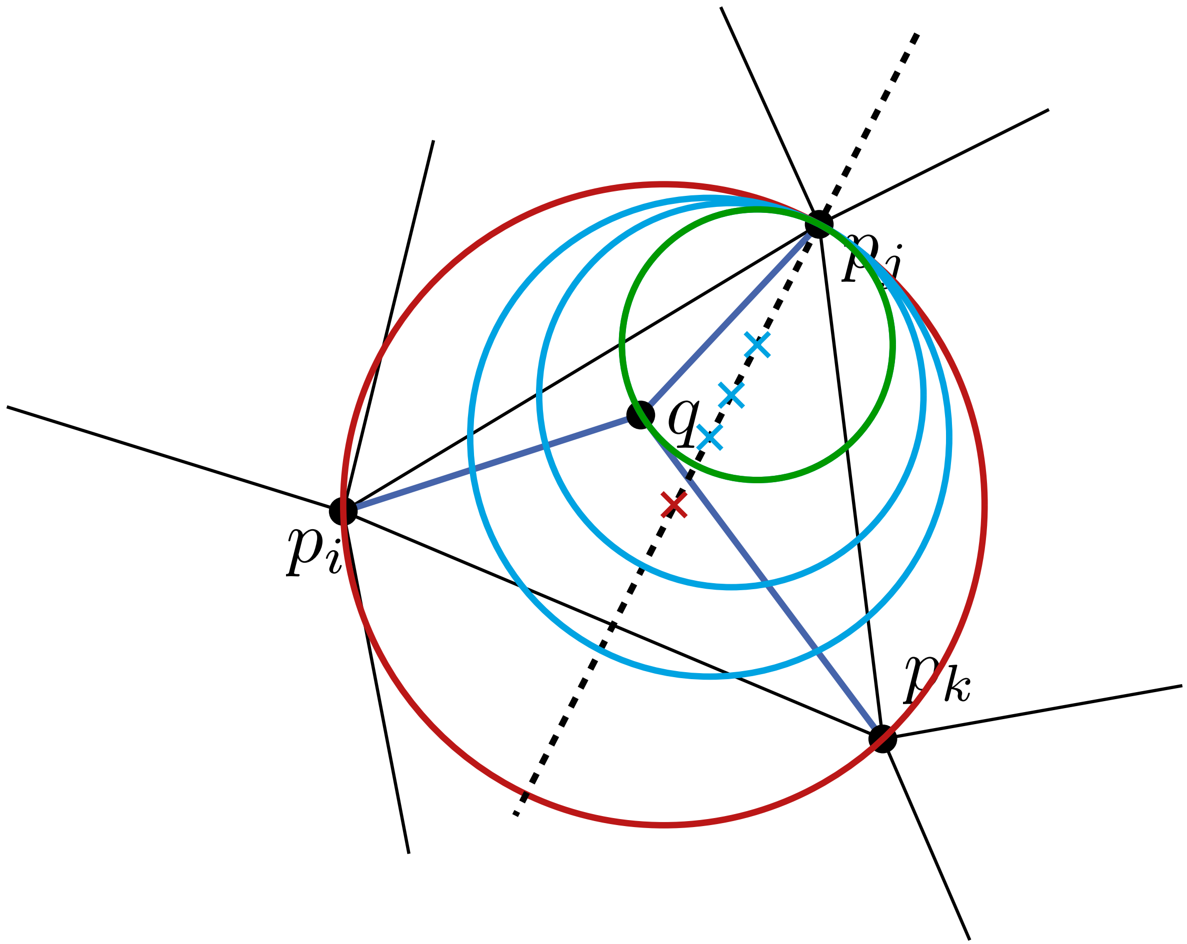

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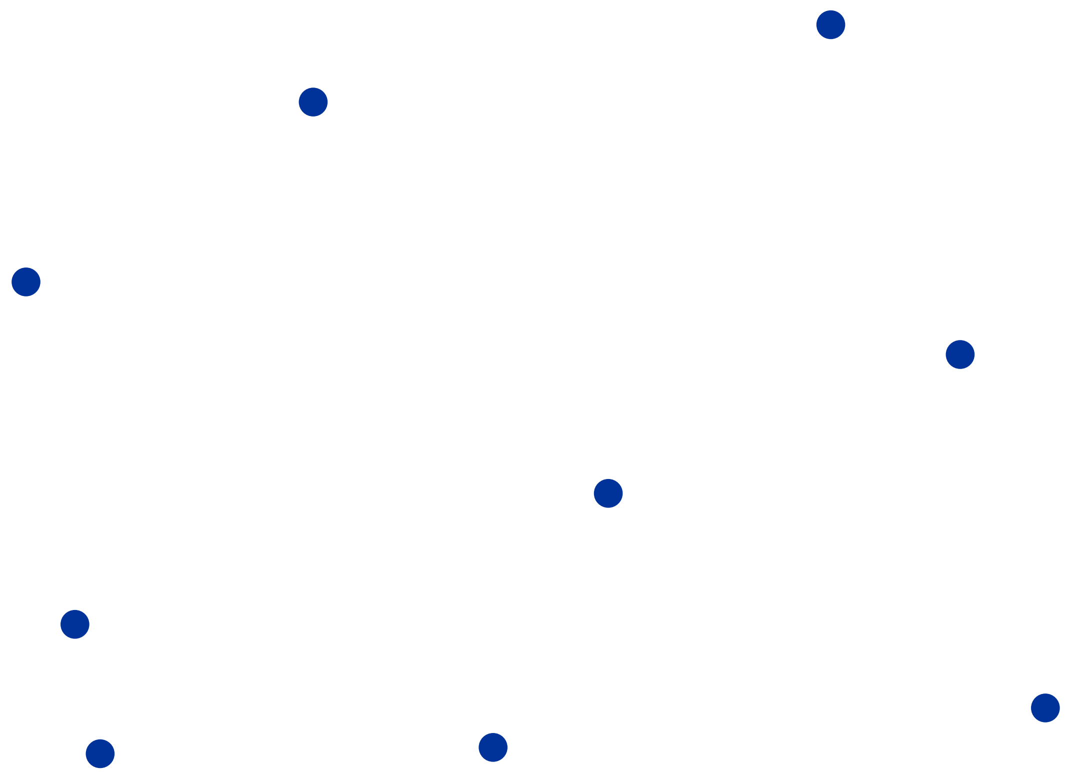

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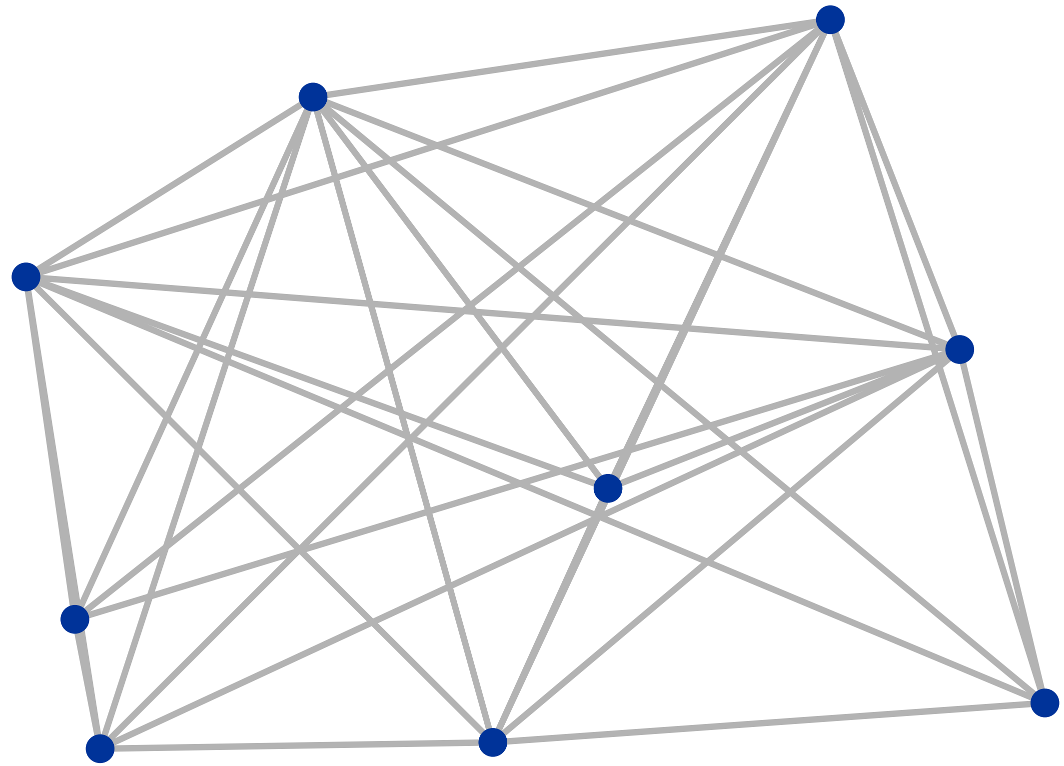

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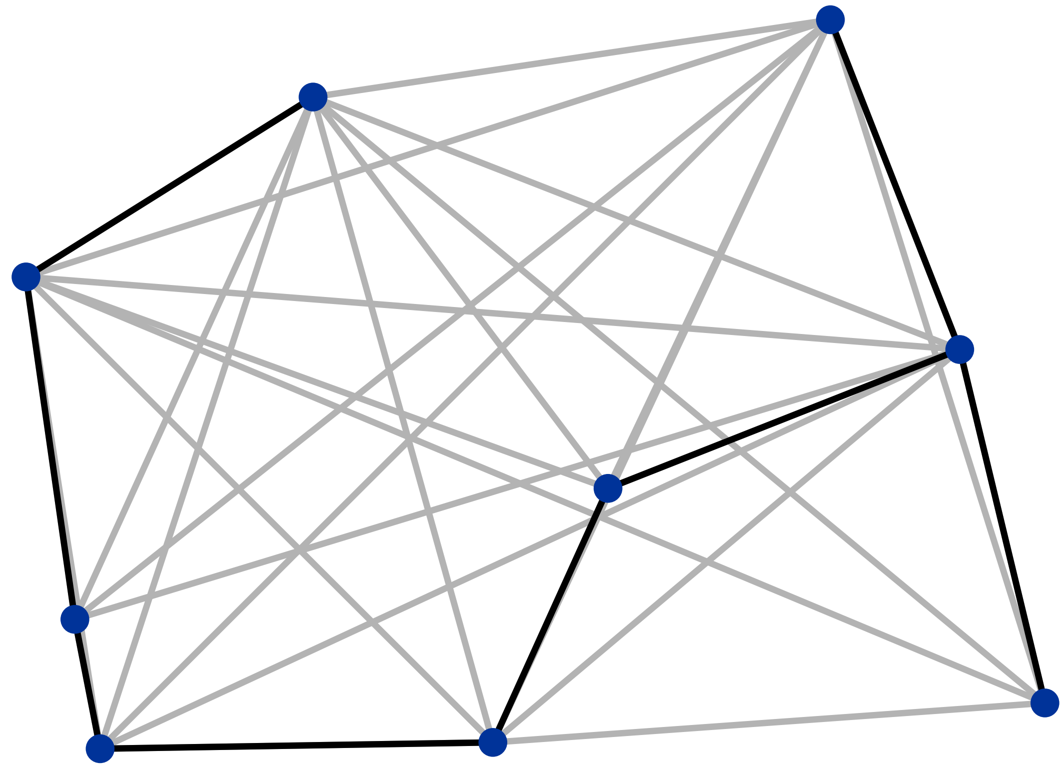

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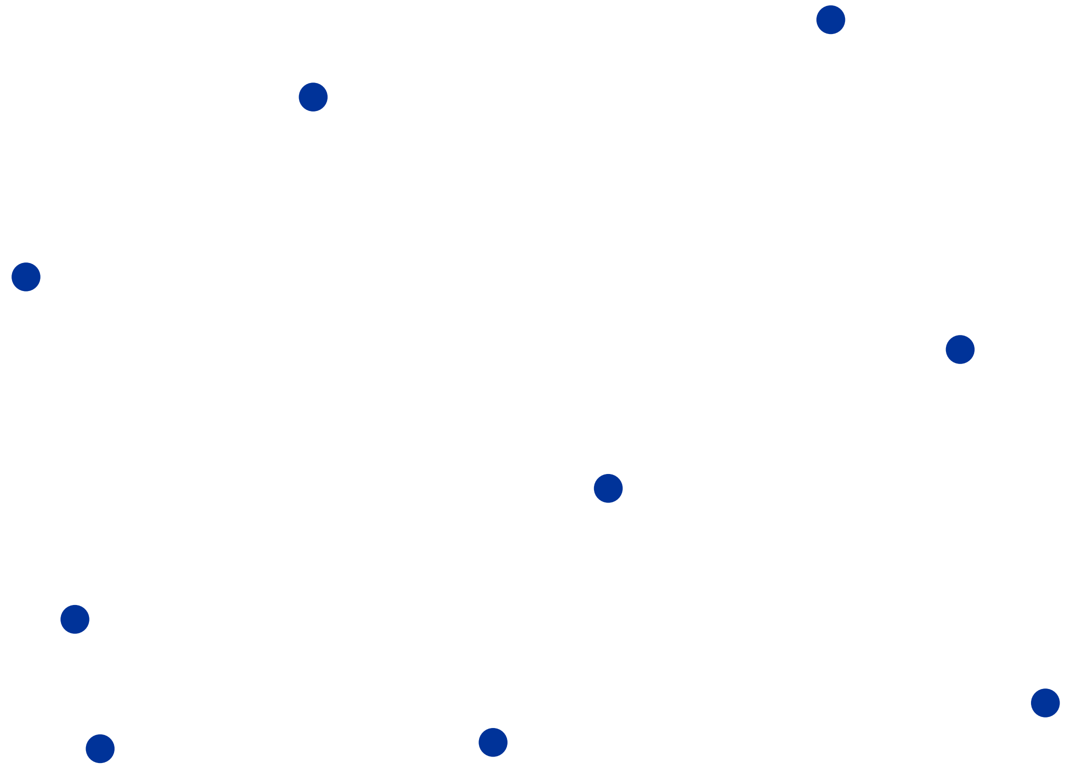

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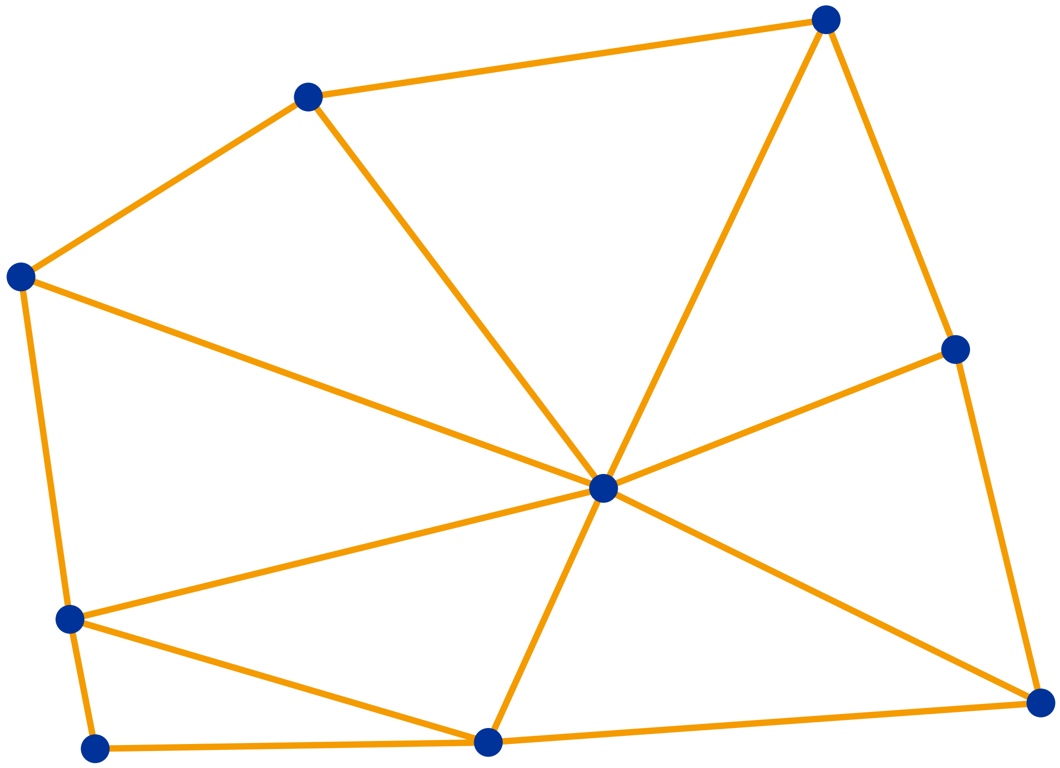

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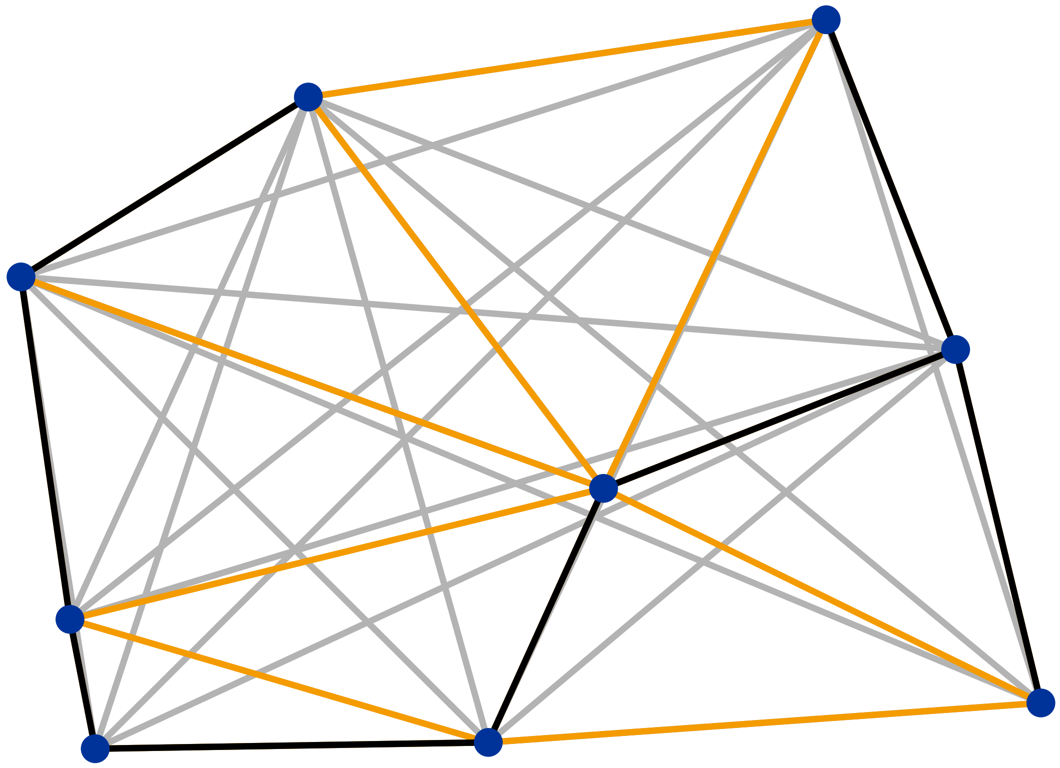

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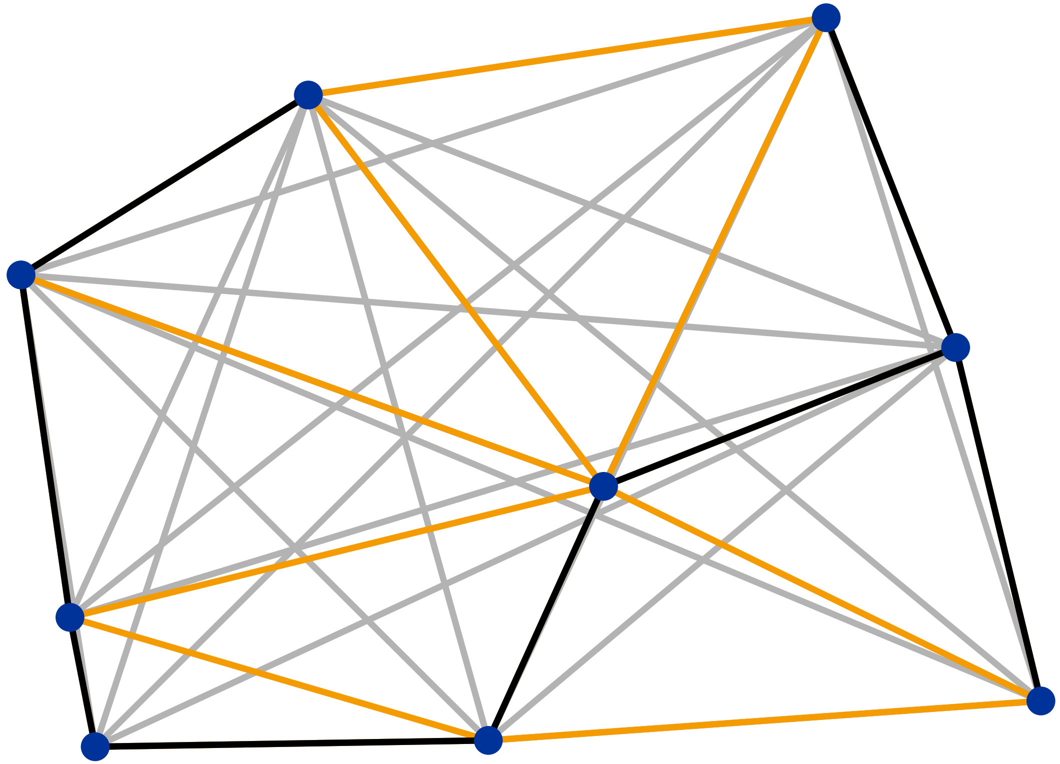

Show that edges of EMST form subset of the edges of the Delaunay-Graph

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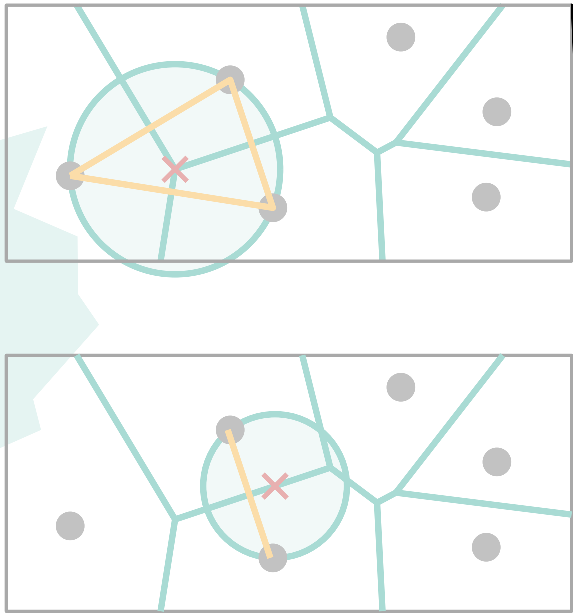

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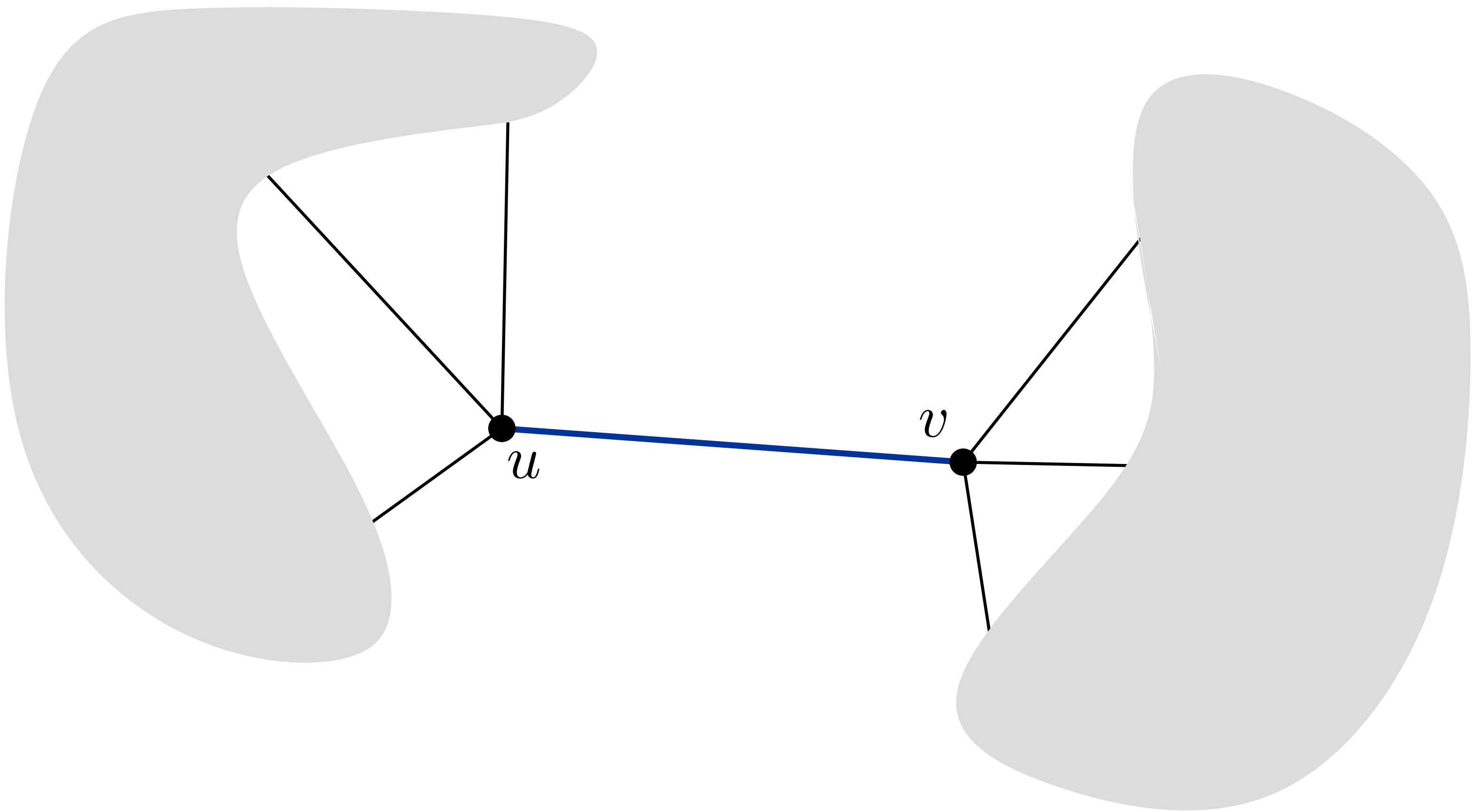

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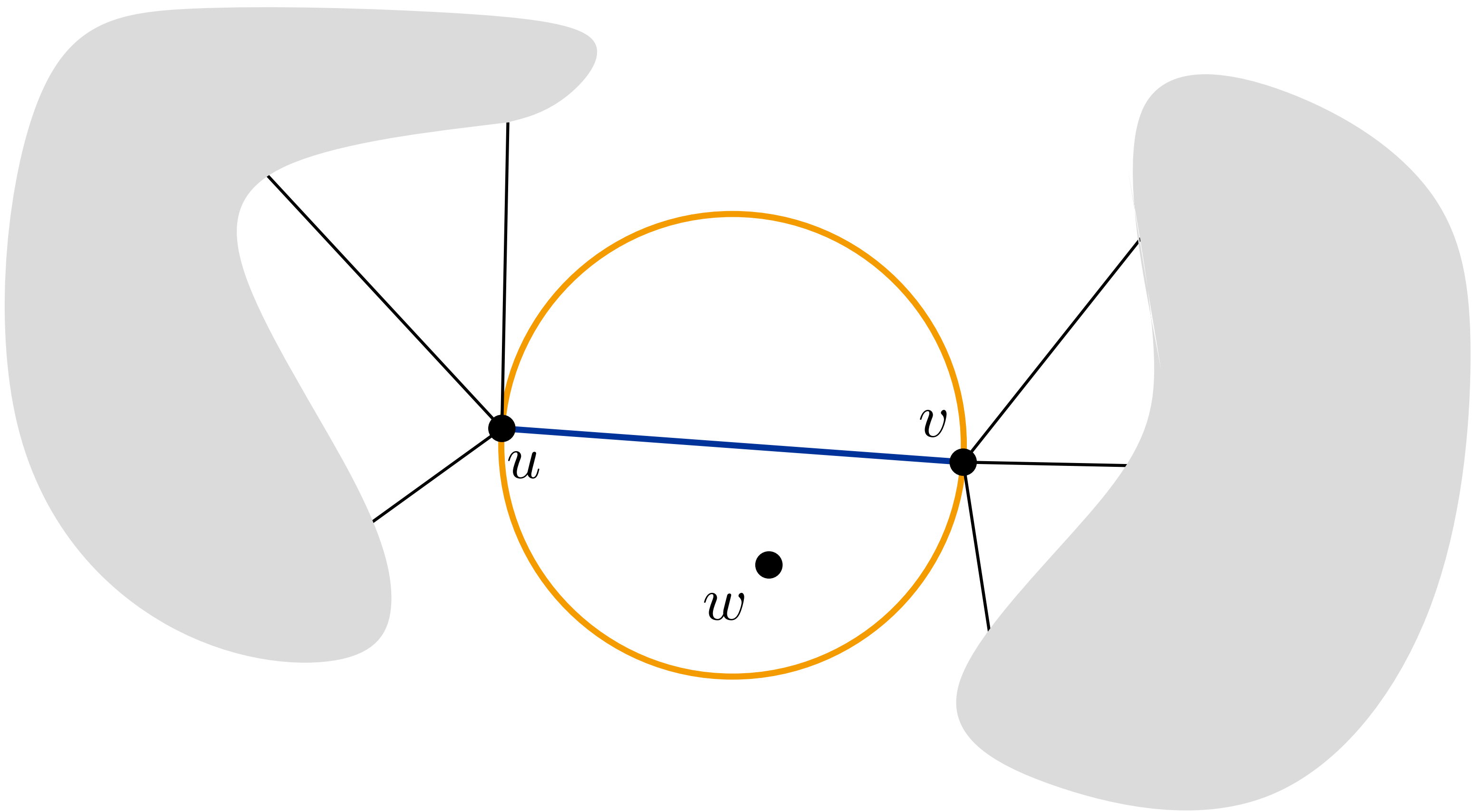

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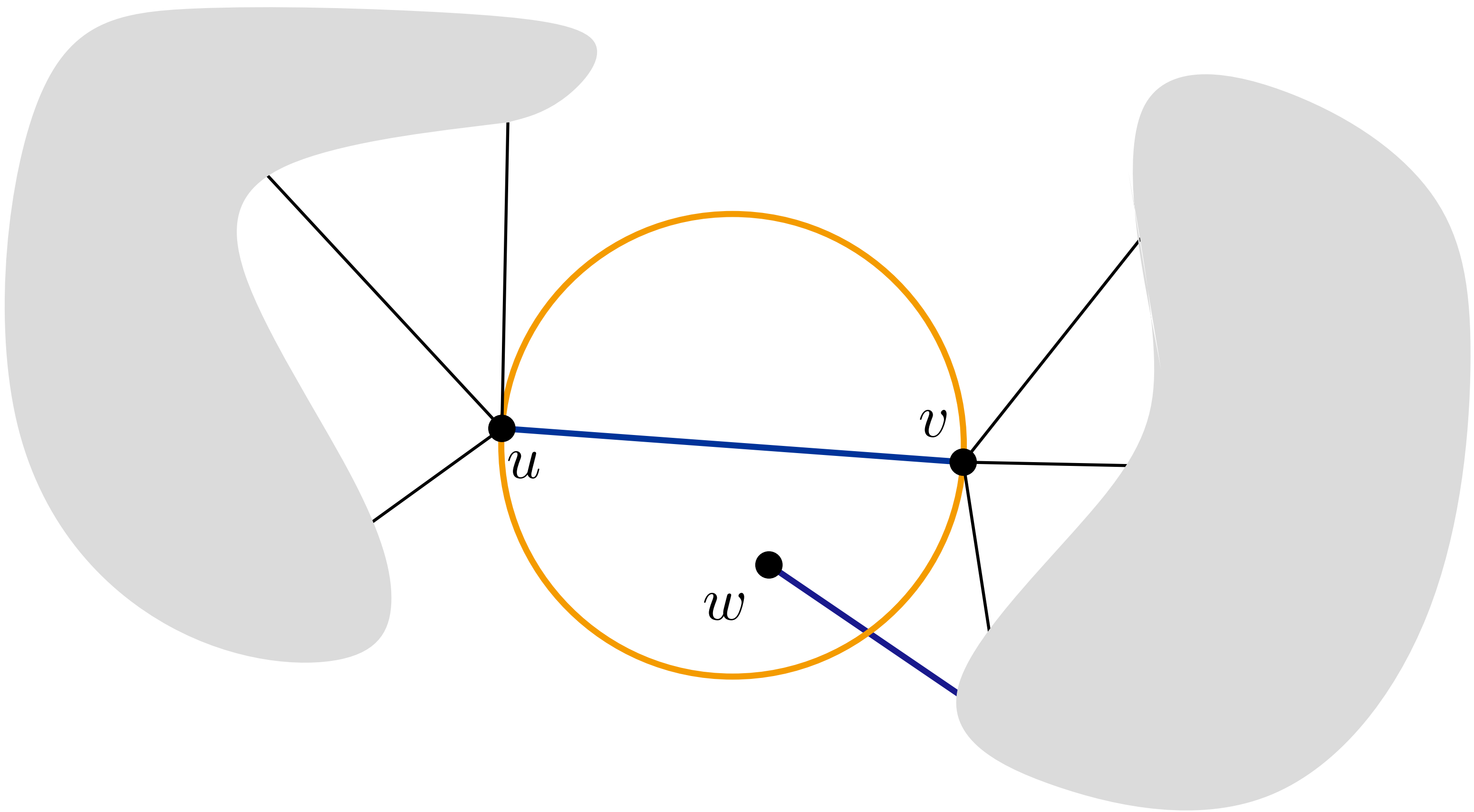

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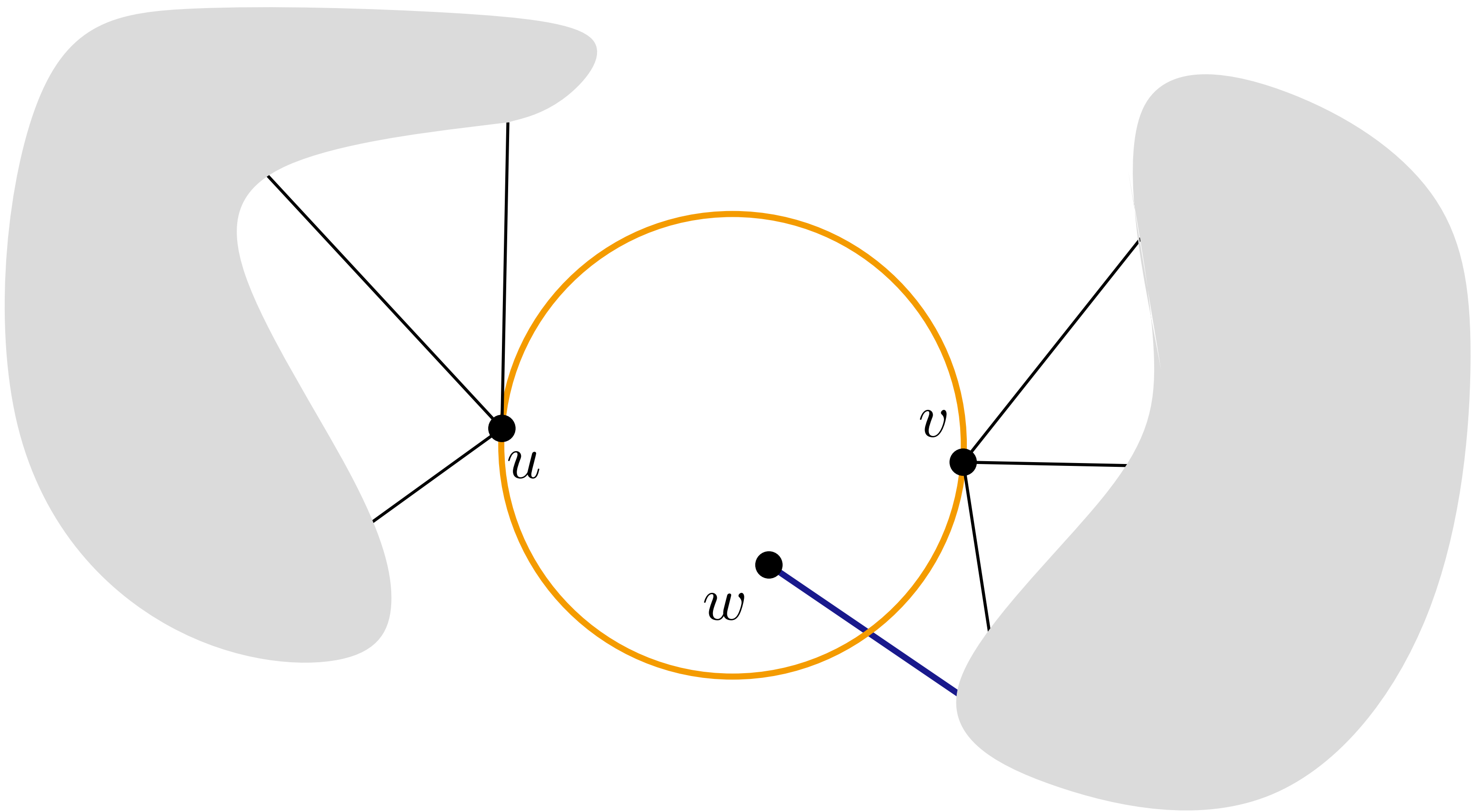

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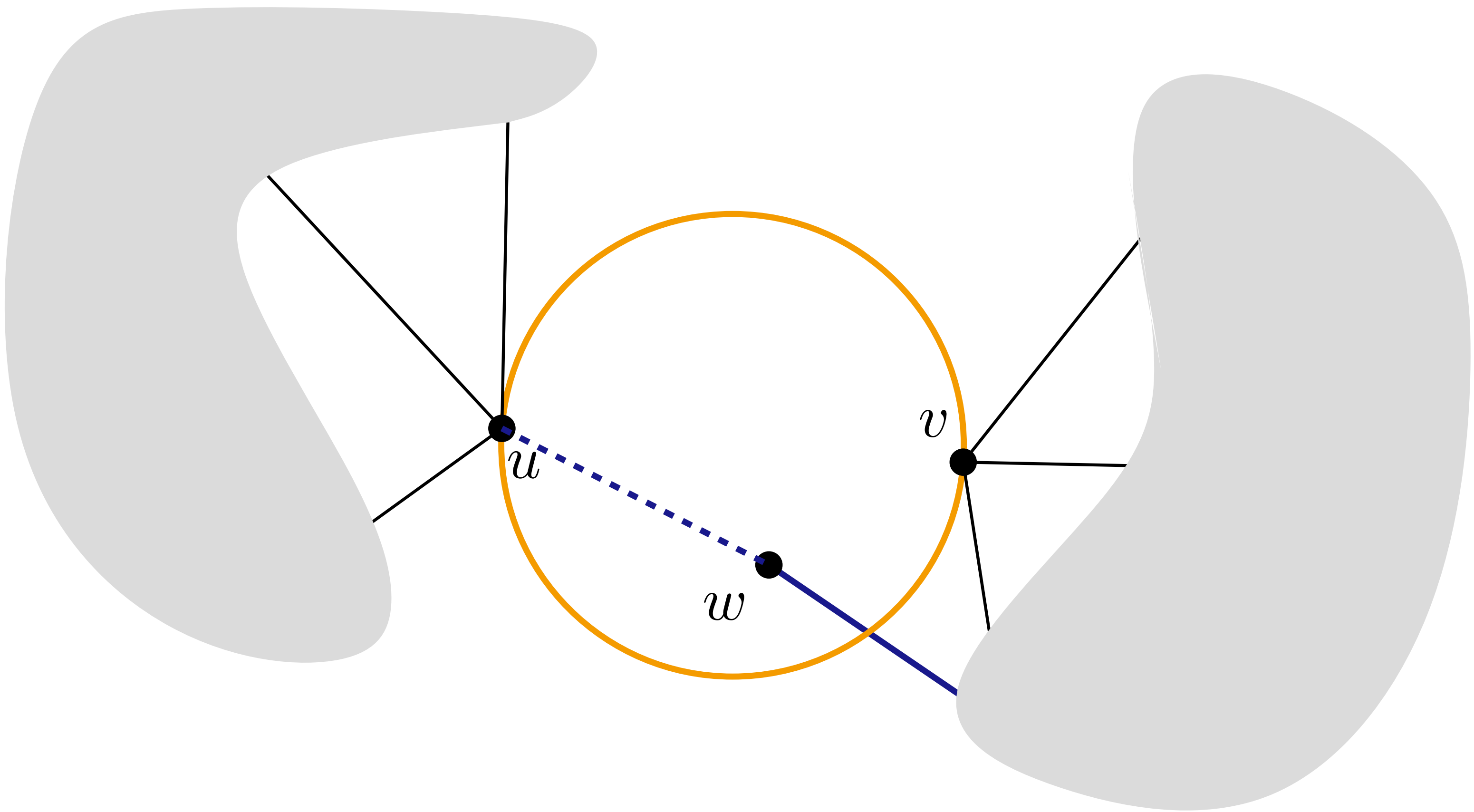

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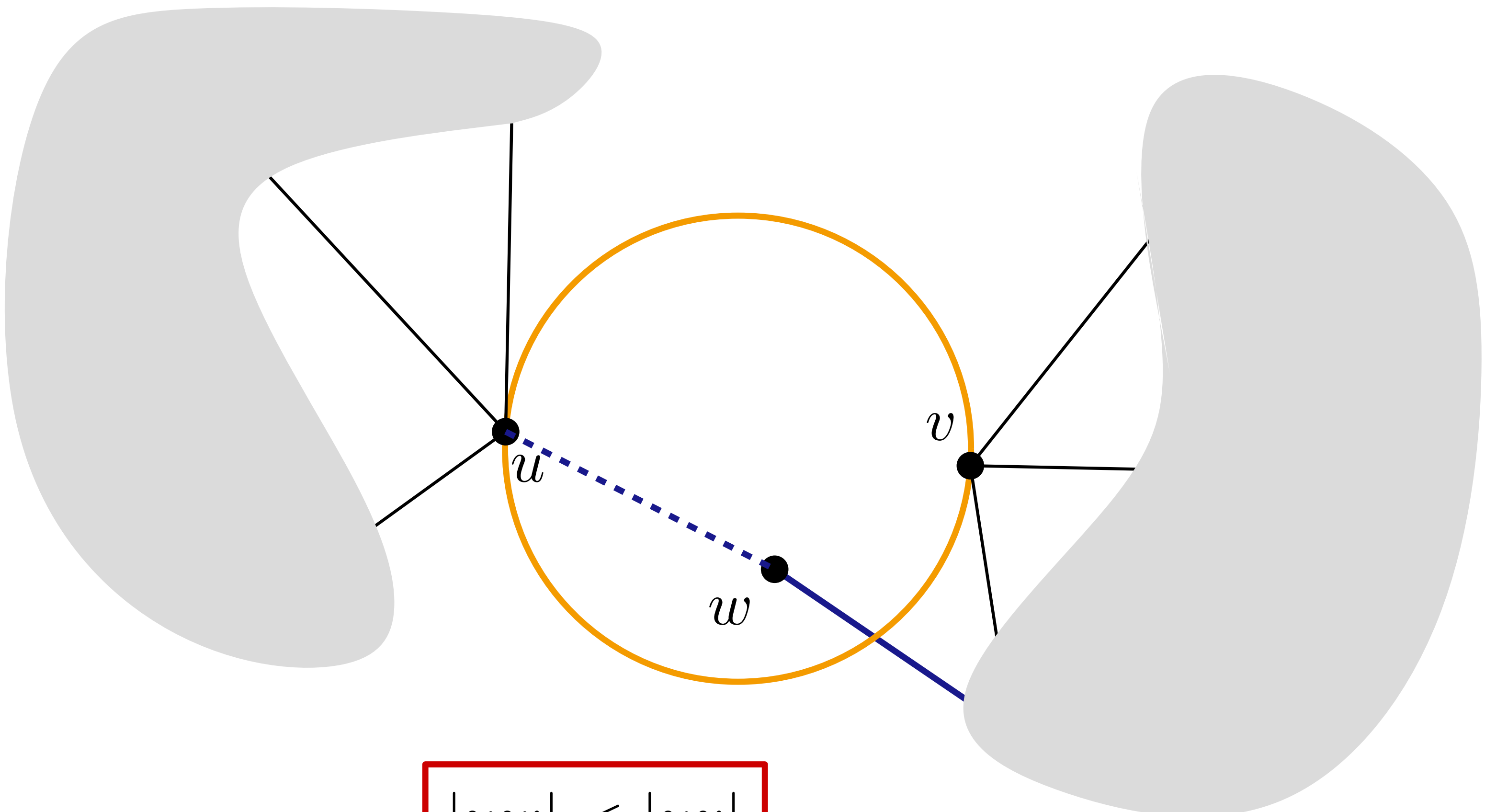

$|uw| < |uv|$

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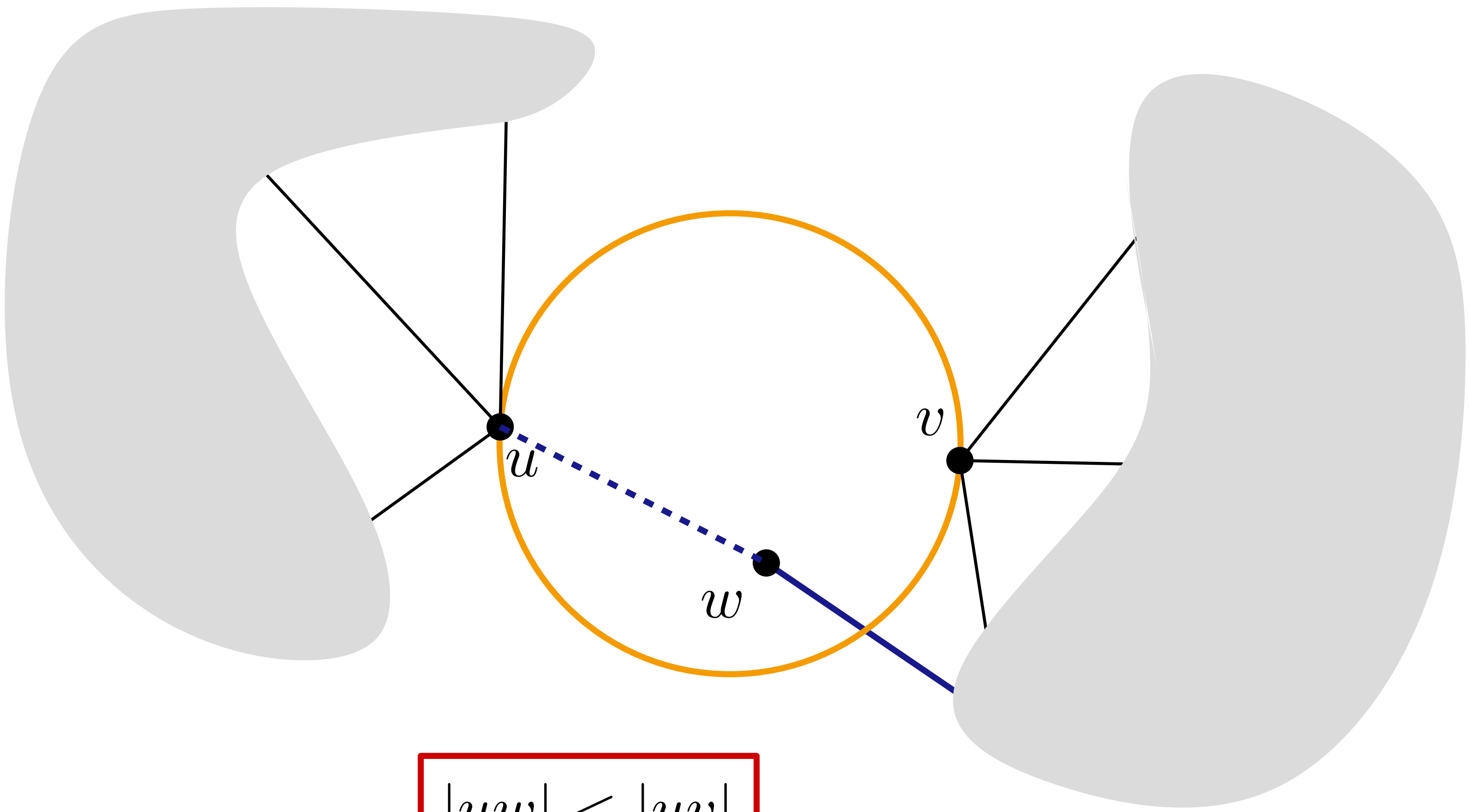

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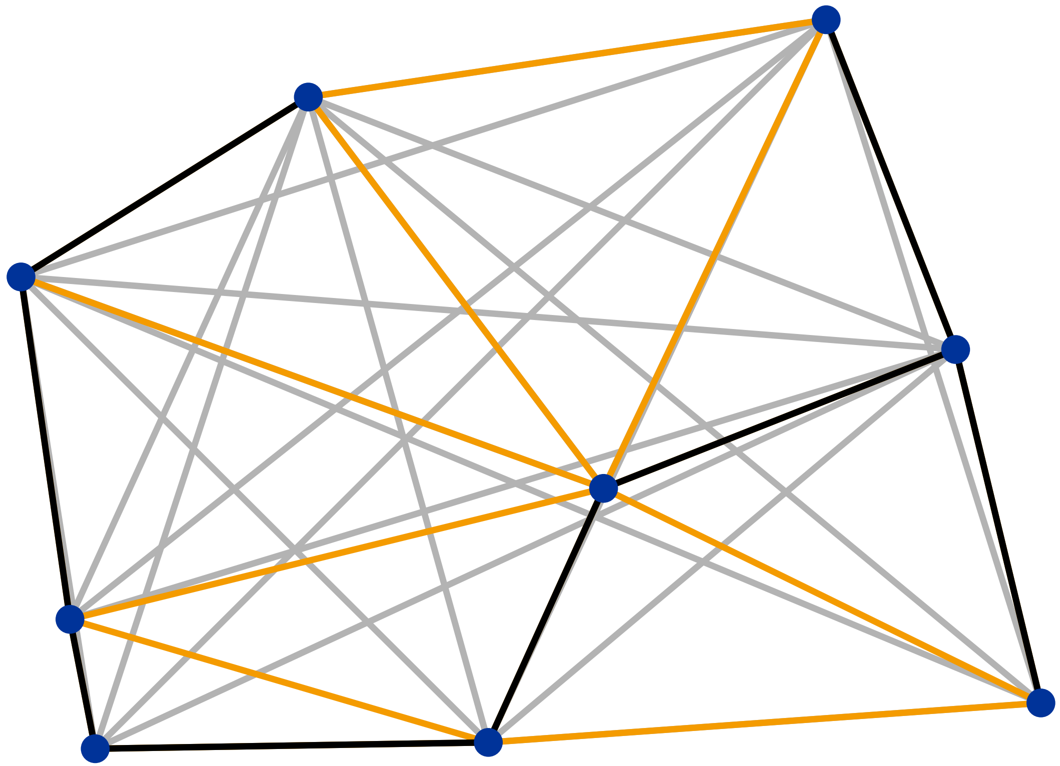

Computation of EMST in $\mathcal{O}(n \log n)$?

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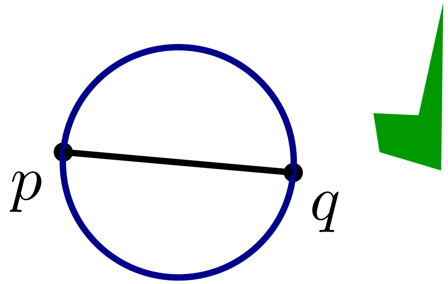

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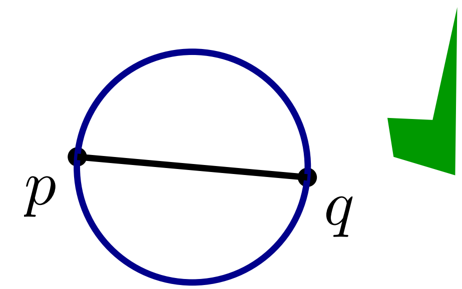

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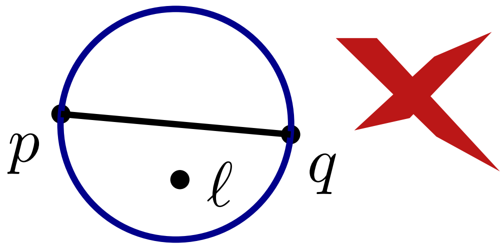

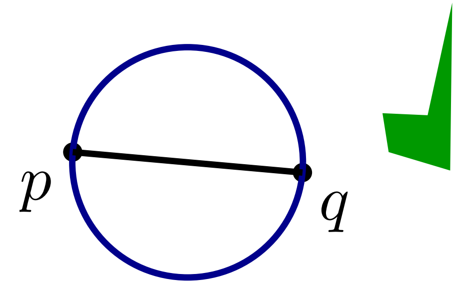

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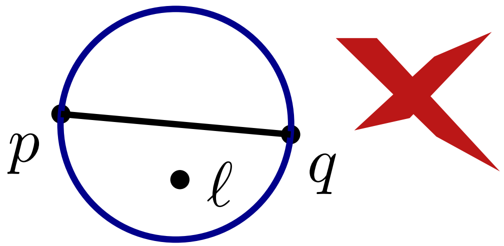
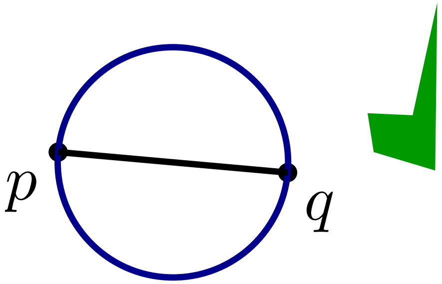


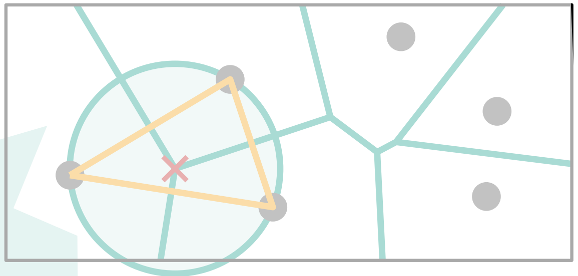


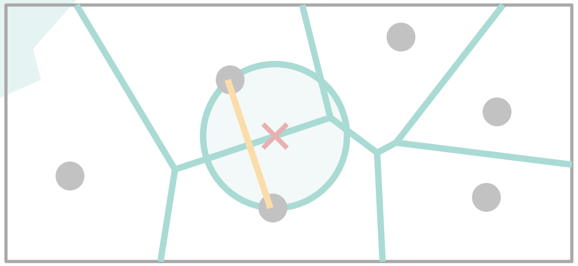
a) Show that Delaunay triangulation of P contains Gabriel graph of P.

Characterization

Theorem about Voronoi-Diagram:

- point q is a Voronoy-vertex $\Leftrightarrow |C_P(q) \cap P| \geq 3$,
- bisector $b(p_i, p_j)$ defines a Voronoi-edge $\Leftrightarrow \exists q \in b(p_i, p_j)$ with $C_P(q) \cap P = \{p_i, p_j\}$.





Theorem 4: Let P be a set of points.

- Points p, q, r are vertices of the same face of $\mathcal{DG}(P) \Leftrightarrow$ circle through p, q, r is empty
- Edge pq is in $\mathcal{DG}(P) \Leftrightarrow$ there is an empty circle $C_{p,q}$ through p and q

Theorem 5: Let P be a set of points and let $\mathcal{T}$ be a triangulation of P. $\mathcal{T}$ is Delaunay-Triangulation $\Leftrightarrow$ the circumcircle of each triangle has an empty interior.

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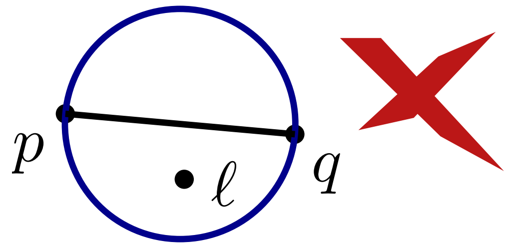

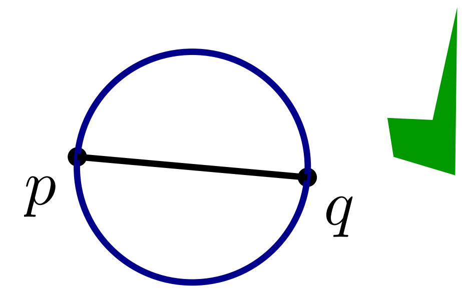

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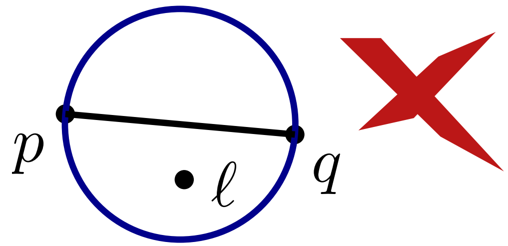

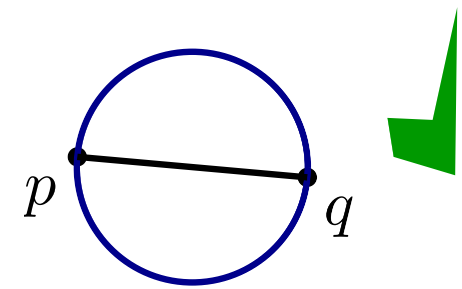

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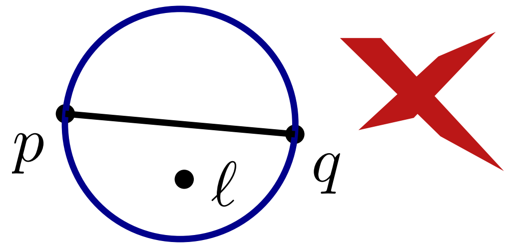

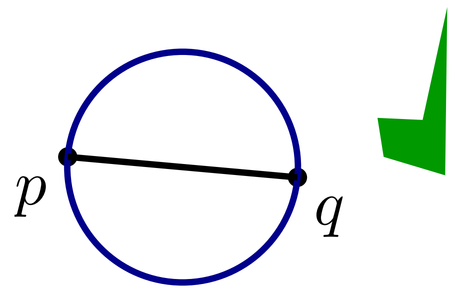

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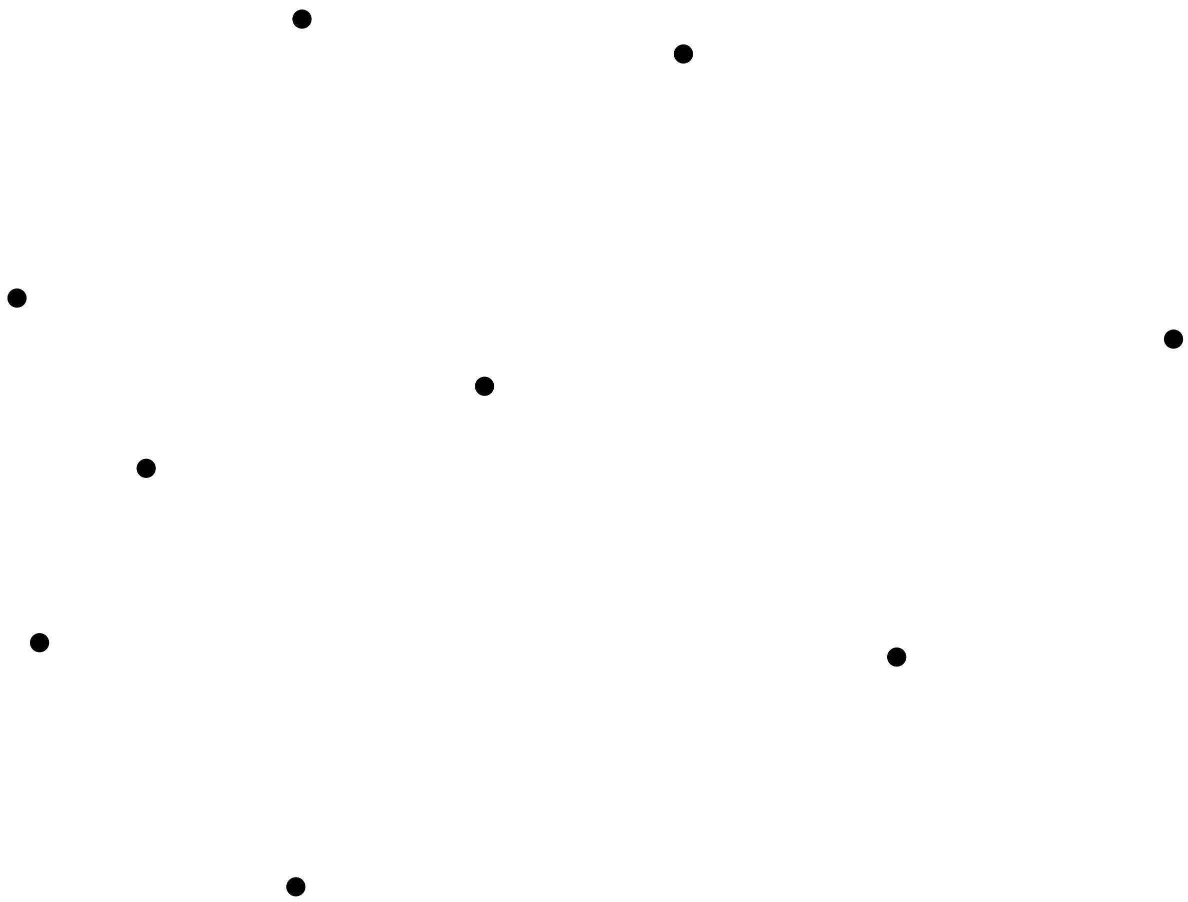

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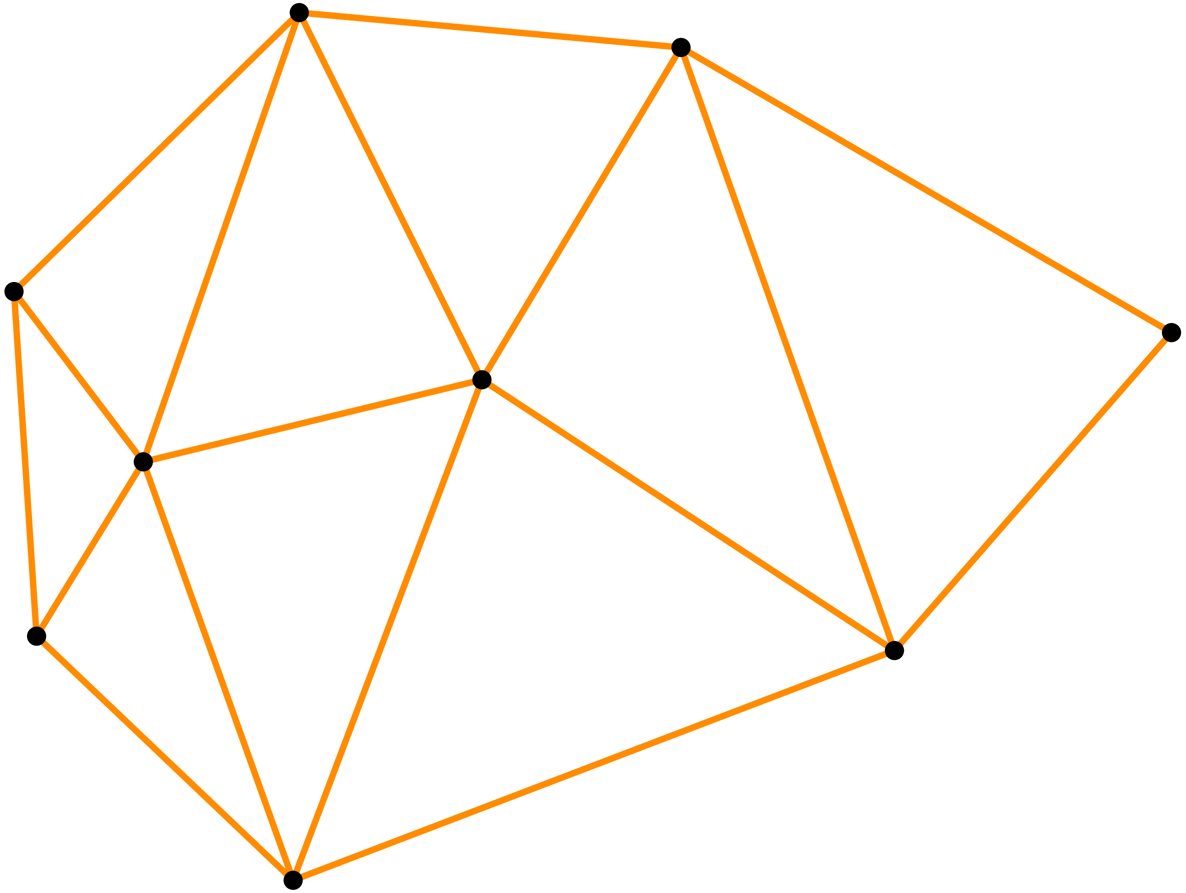

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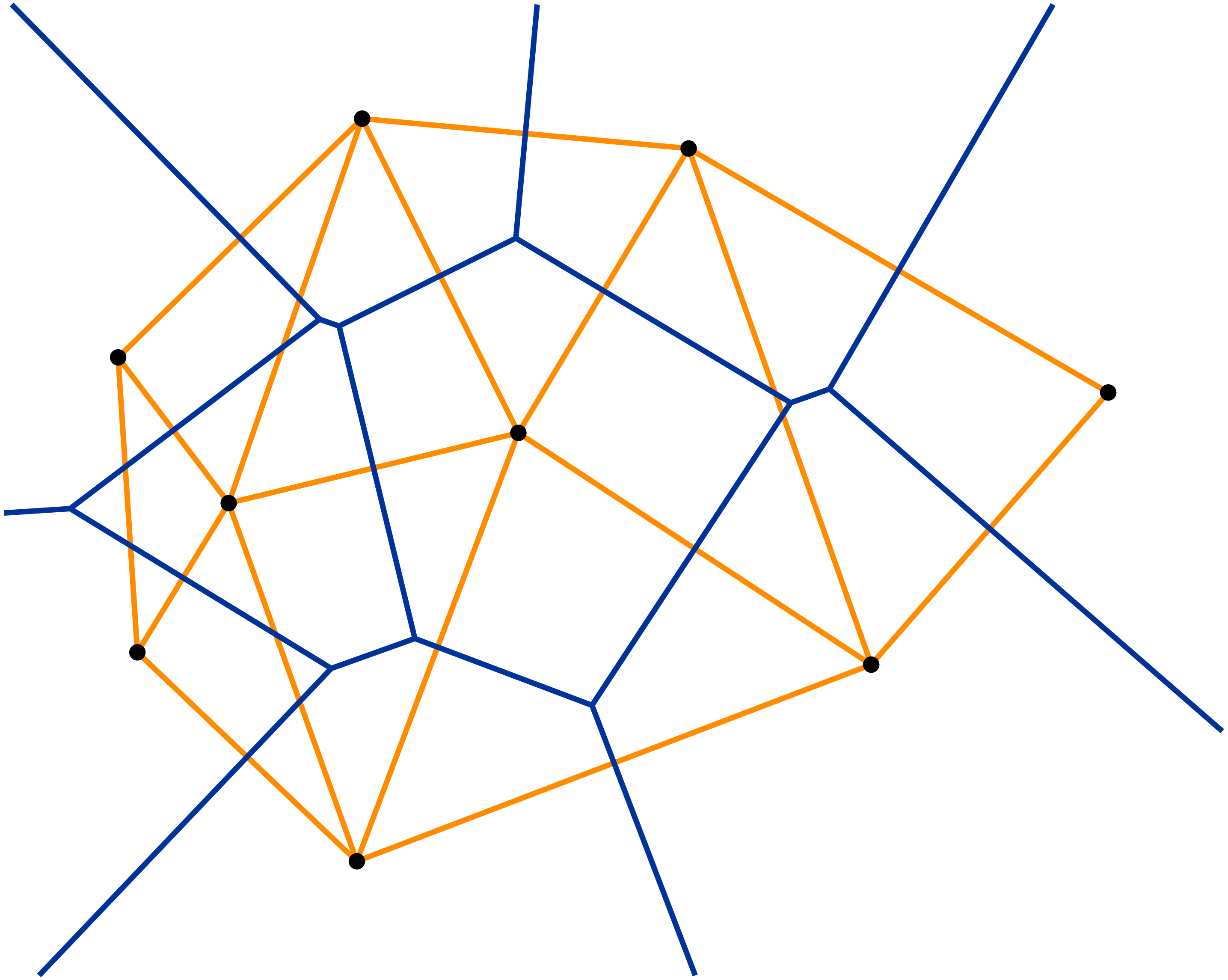

Exercise 4

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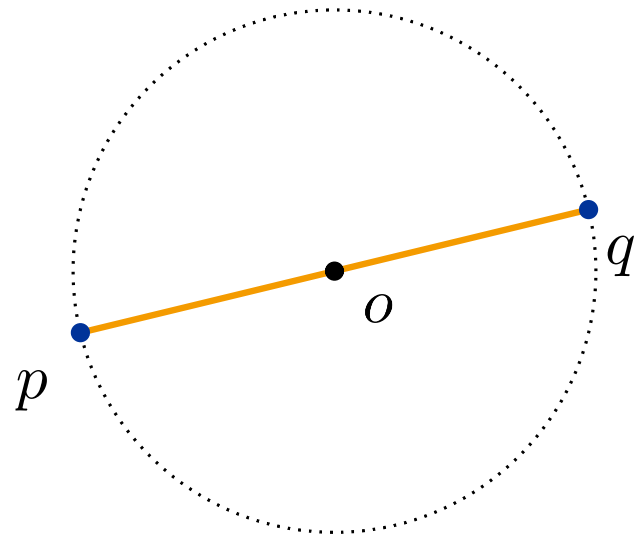

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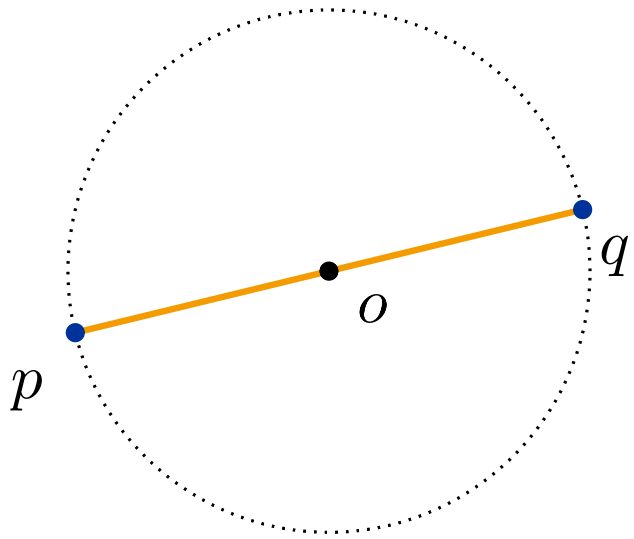

- o is closer to p, q than to all other points in P

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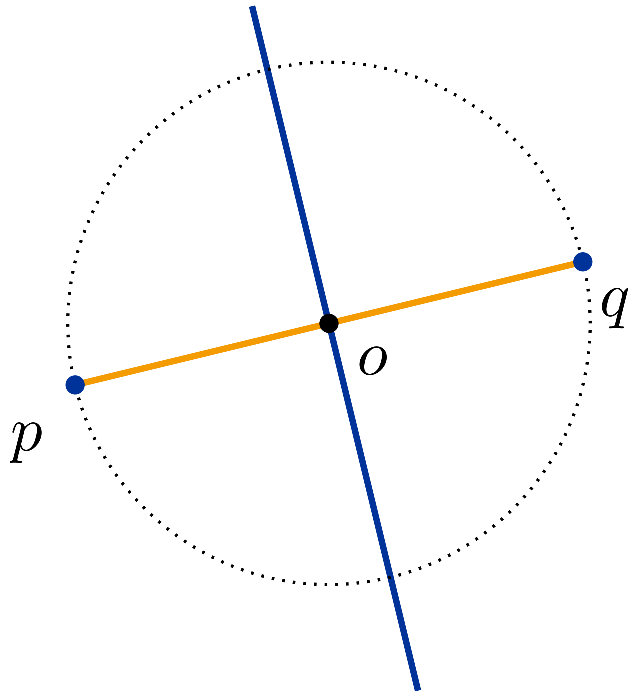
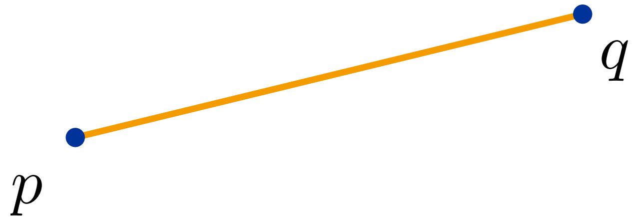


- o is closer to p, q than to all other points in P
  $\Rightarrow$ o lies on Voronoi-edge which bounds Voronoi cells of p and q.

Exercise 4

- pq intersects dual Voronoi-edge



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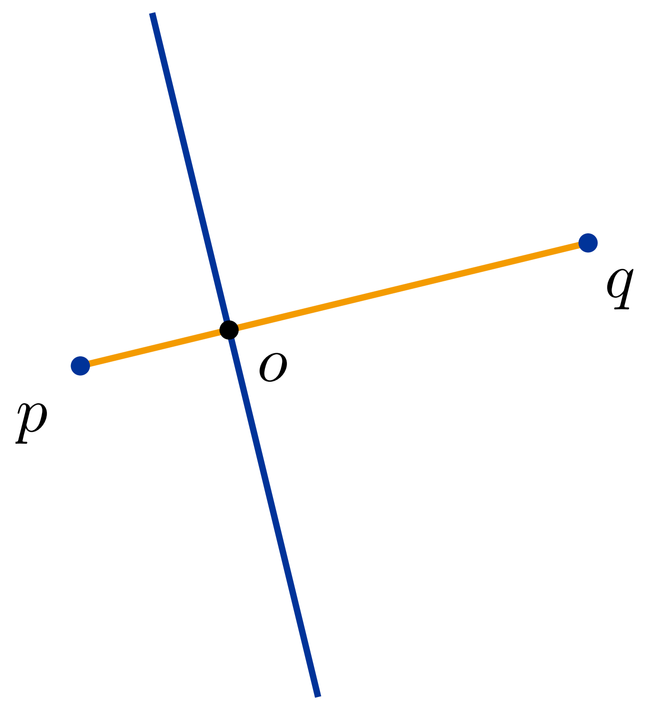

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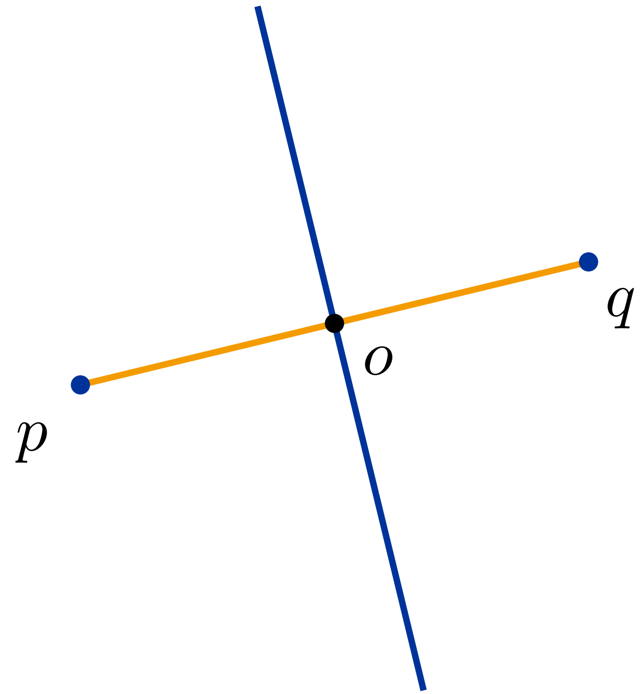

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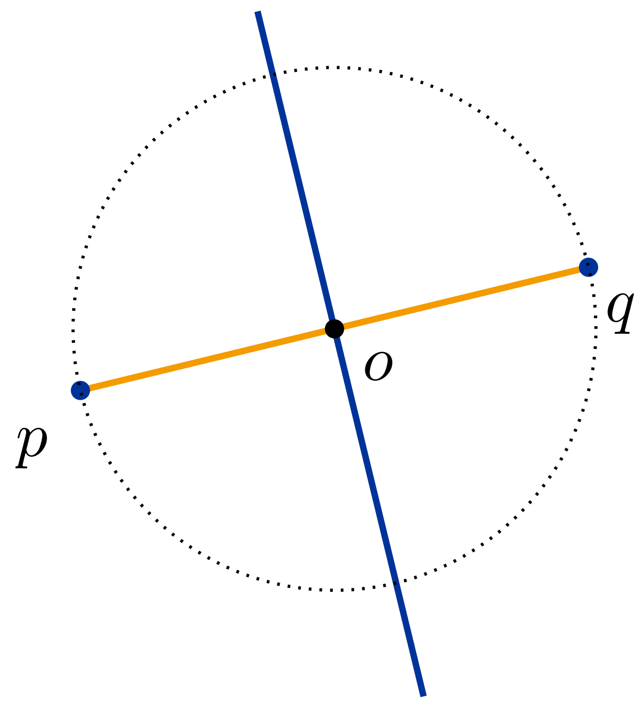

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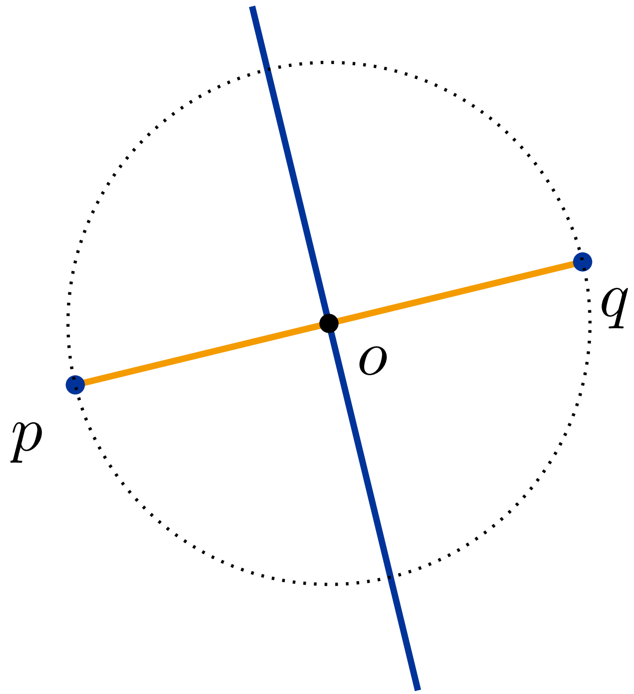

- Assume there is point r in C.

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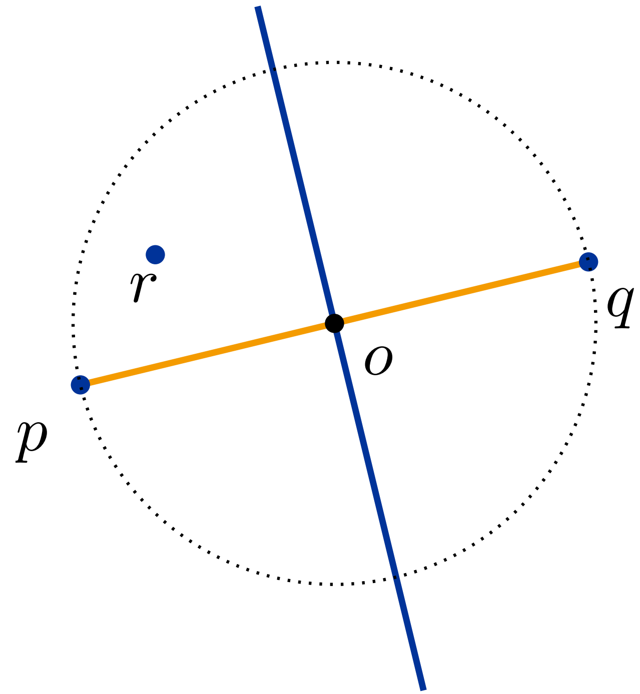

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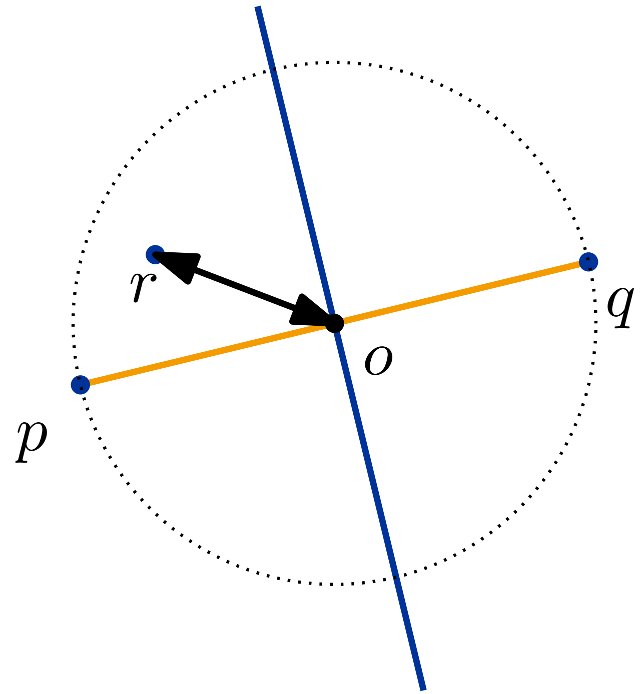

- Assume there is point r in C.
  $\Rightarrow$ Distance between o and r less than $|op|$ ($|oq|$)
    $\Rightarrow$ Contradicts assumption that o lies on Voronoi-edge.

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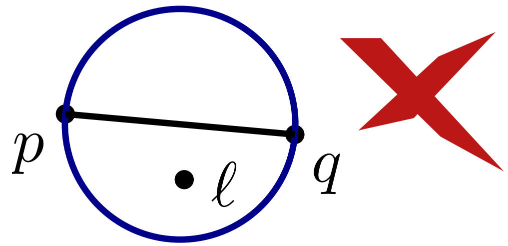

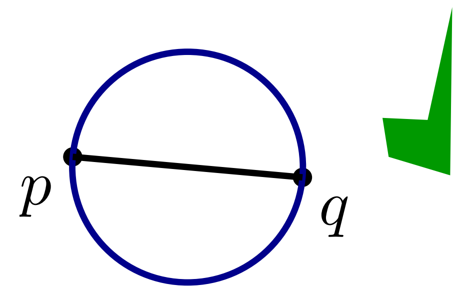

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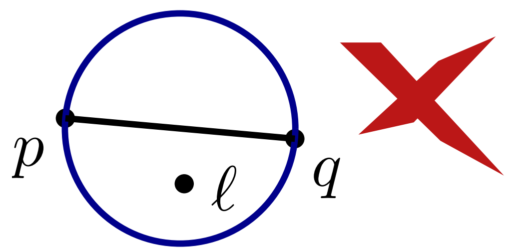

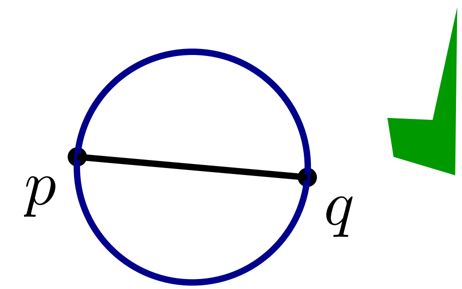

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c) $\mathcal{O}(n \log n)$ Algorithm for computing Gabriel-Graph.