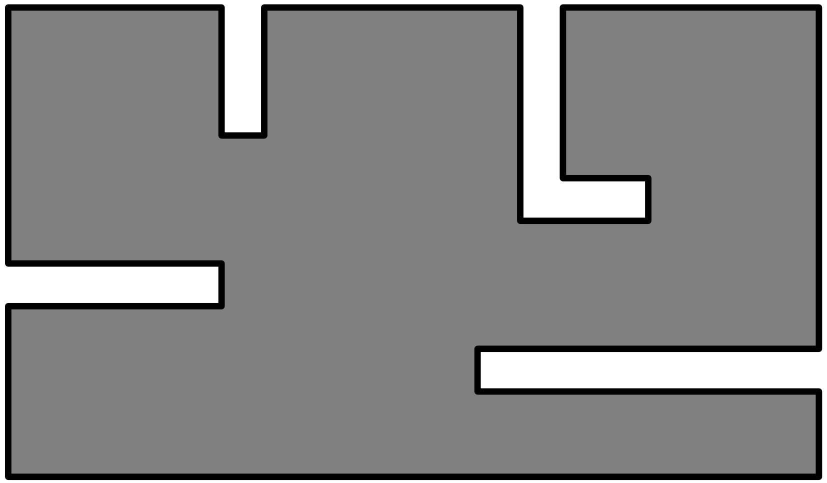
Computational Geometry – Exercise
Triangulation of Polygons & Linear Programming

LEHRSTUHL FÜR ALGORITHMIK I · INSTITUT FÜR THEORETISCHE INFORMATIK · FAKULTÄT FÜR INFORMATIK

Guido Brückner
23.05.2018

The Art-Gallery-Problem

Task: Install a number of cameras in an art gallery so that every part of the galery is visible to at least one of them.



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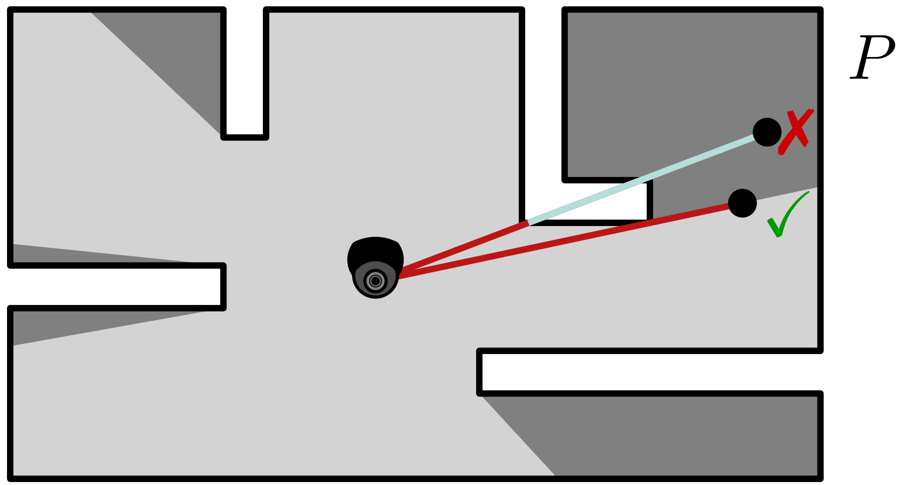

Assumption: Art gallery is a *simple* polygon P with n corners (no self-intersections, no holes)

Observation: each camera observes a star-shaped region

Definition: Point $p \in P$ is *visible* from $c \in P$ if $\overline{cp} \in P$

Goal: Use as few cameras as possible!

NP-hard!

$\rightarrow$ The number depends on the number of corners n and on the shape of P

Observation of the Border

Prove or falsify the following statement.

> *Let $\mathcal{P}$ be a simple polygon and consider a set of cameras that together observe the complete border of P, then they also observe the complete interior of P.*

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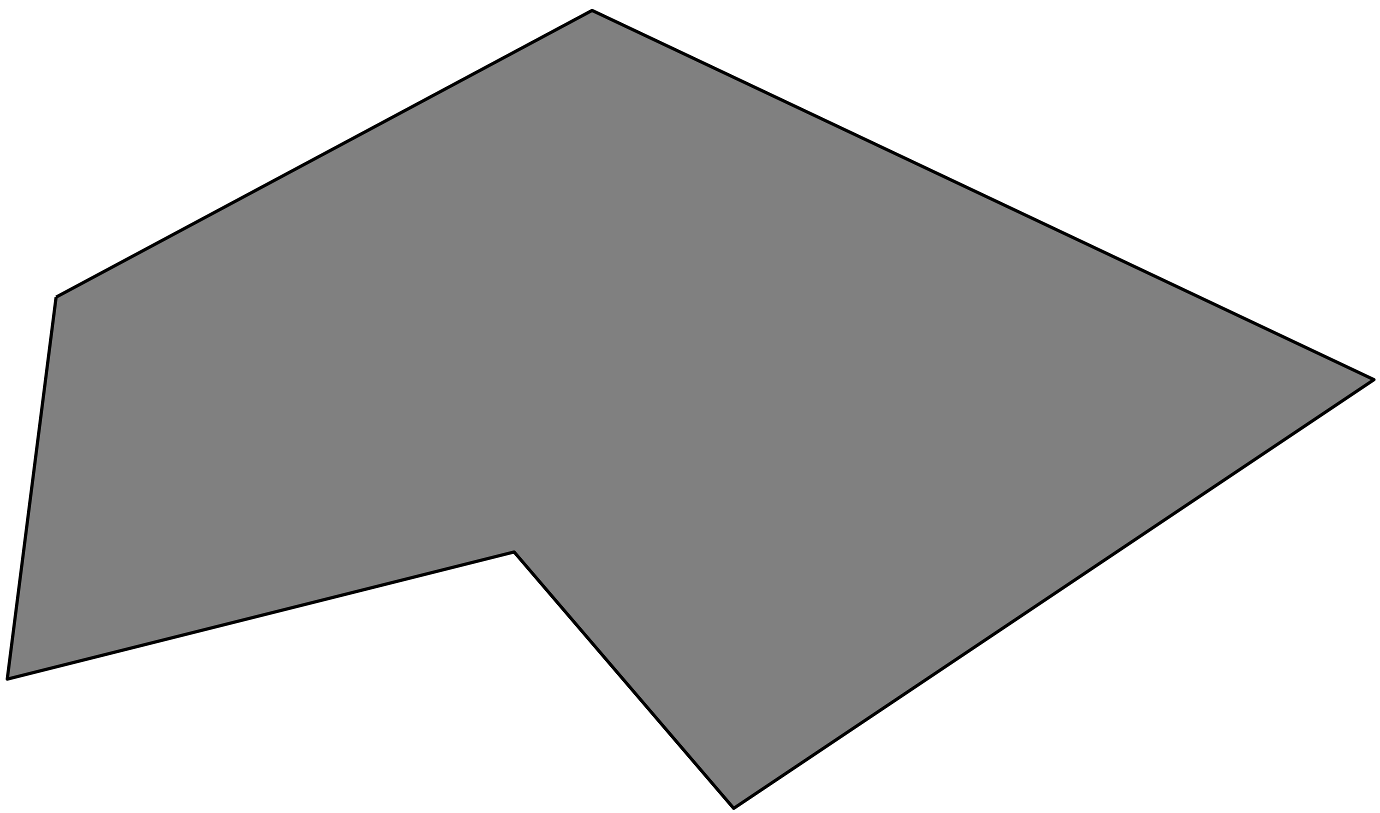

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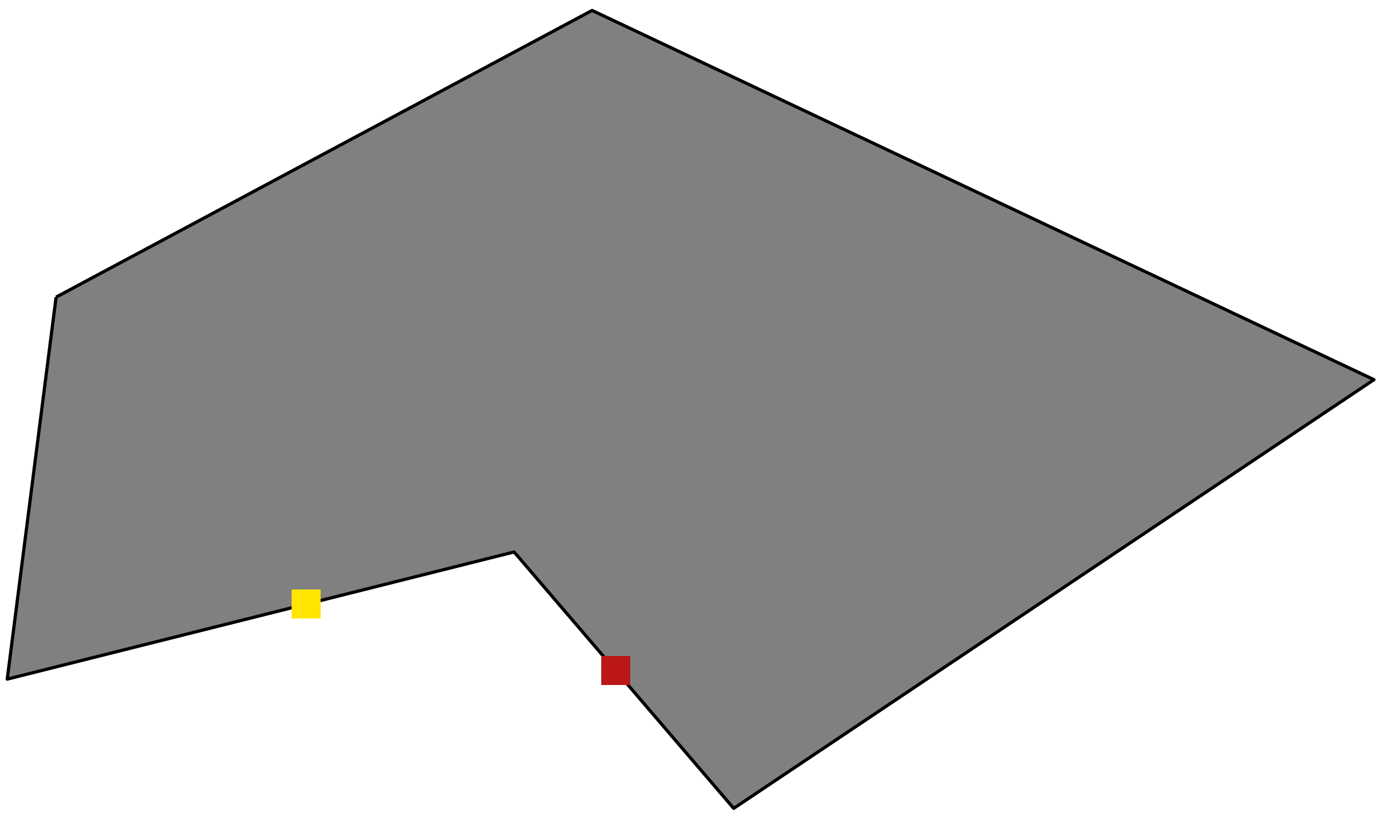

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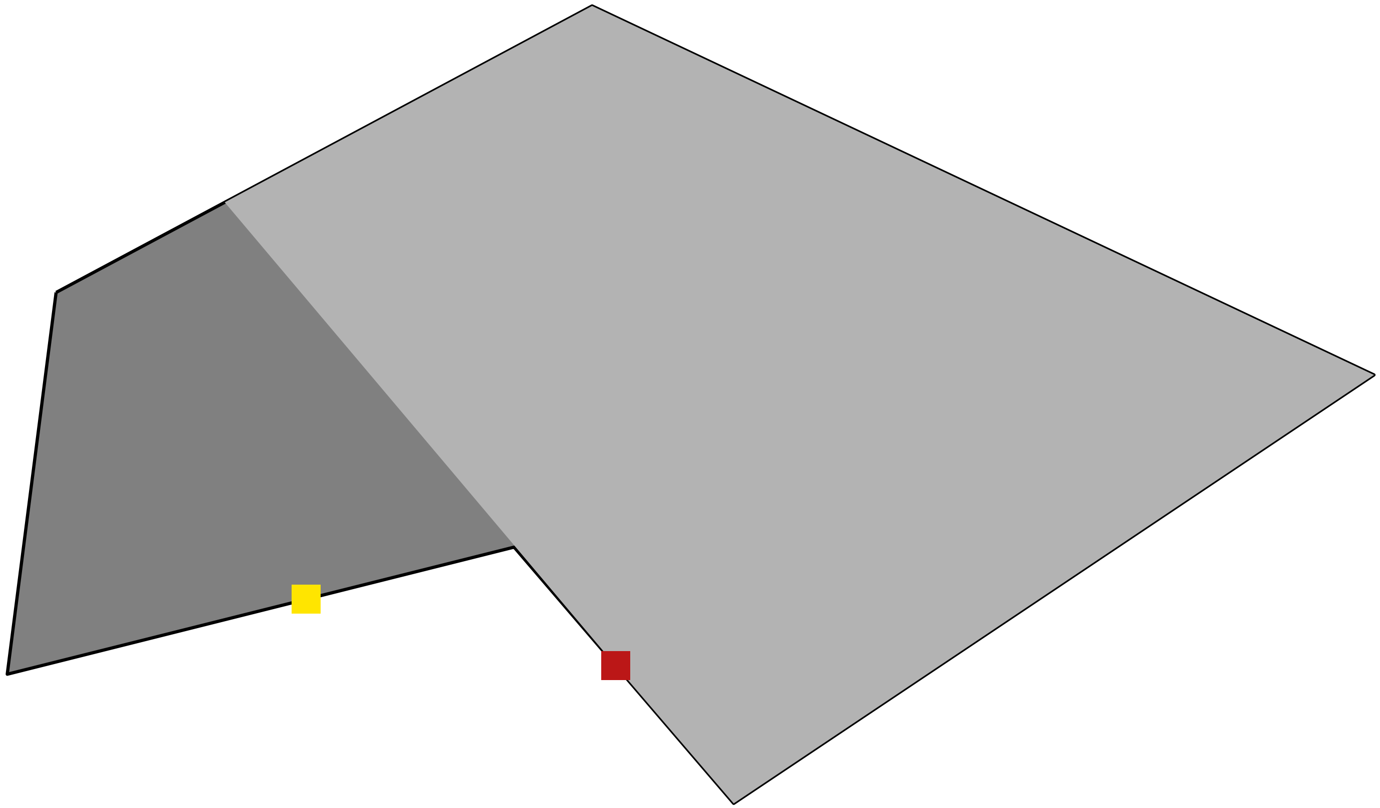

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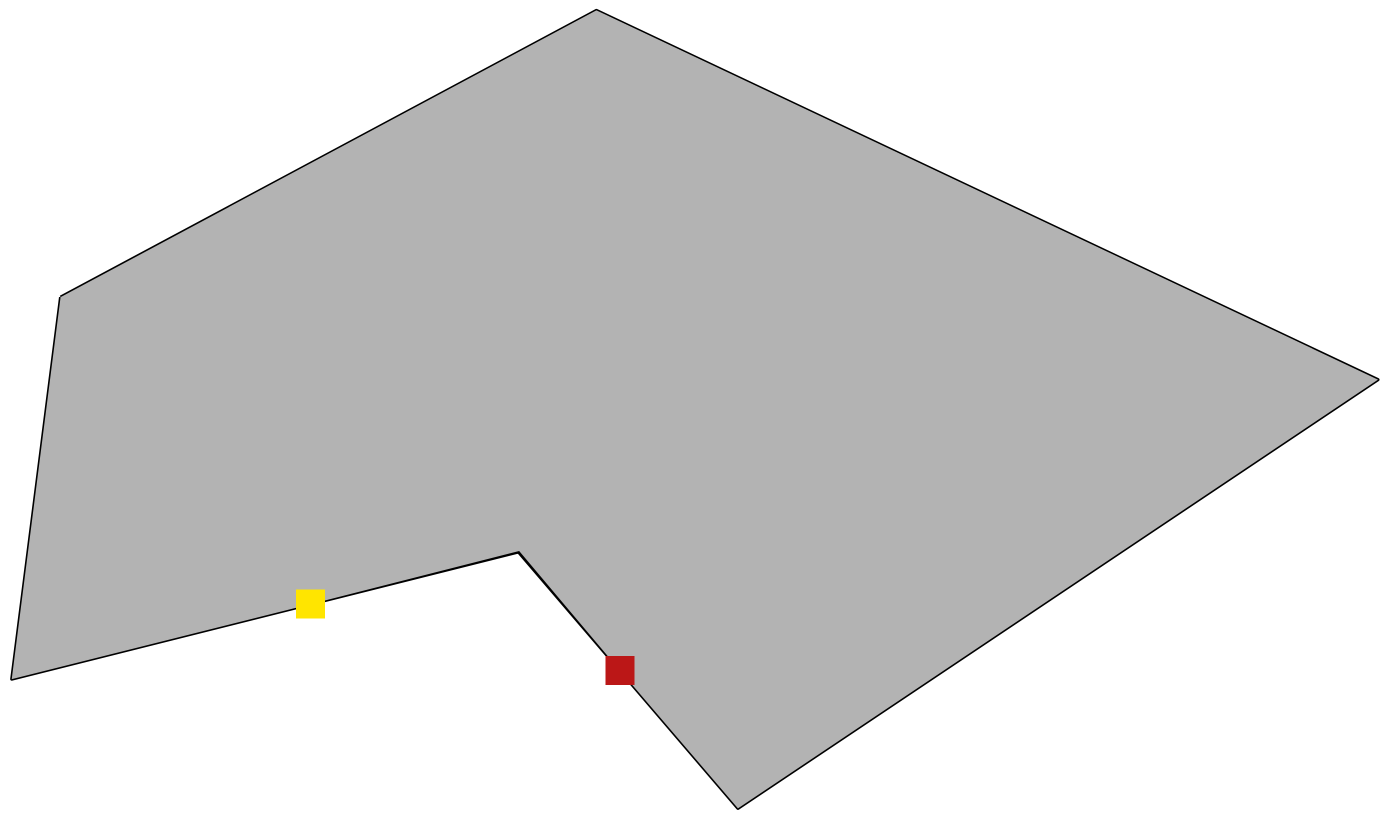

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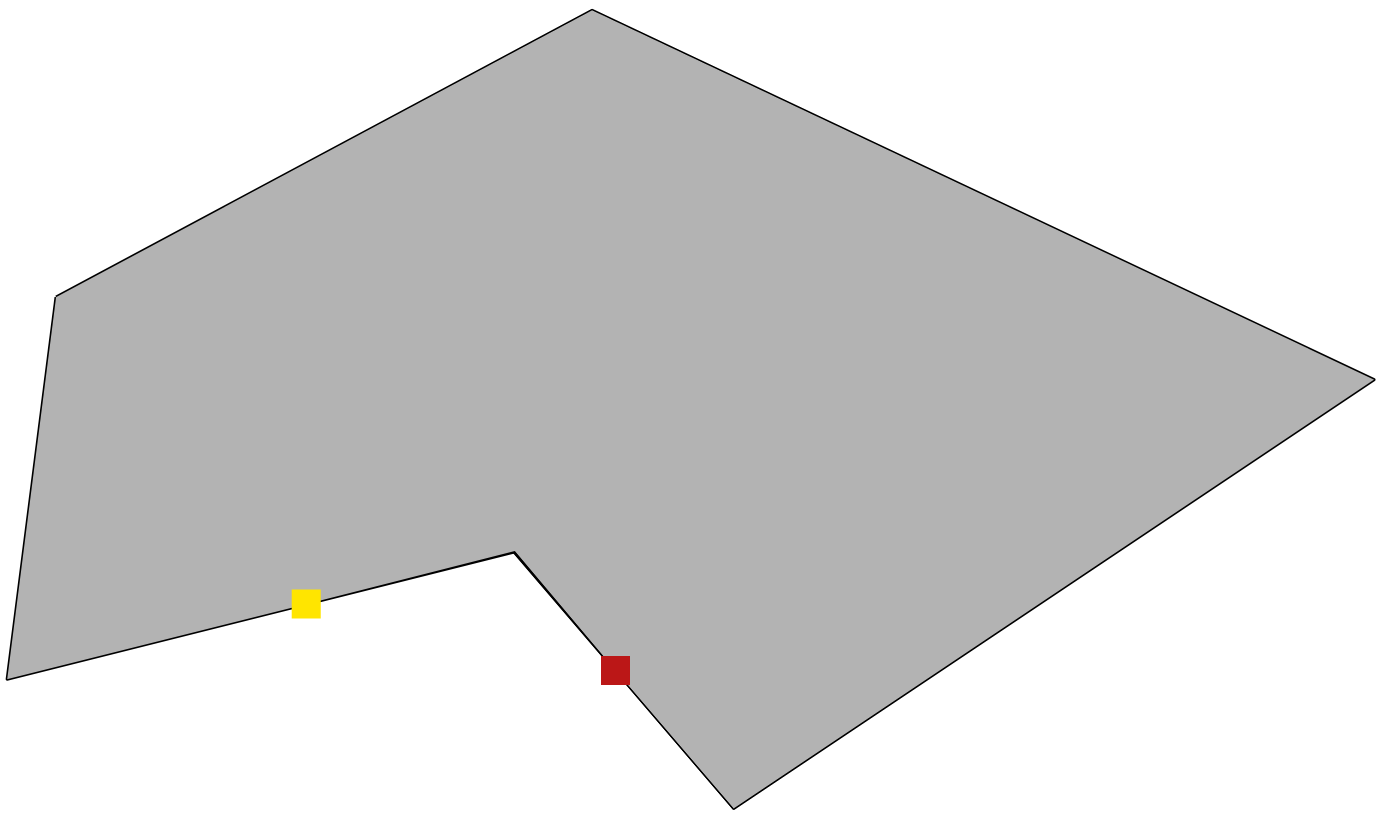

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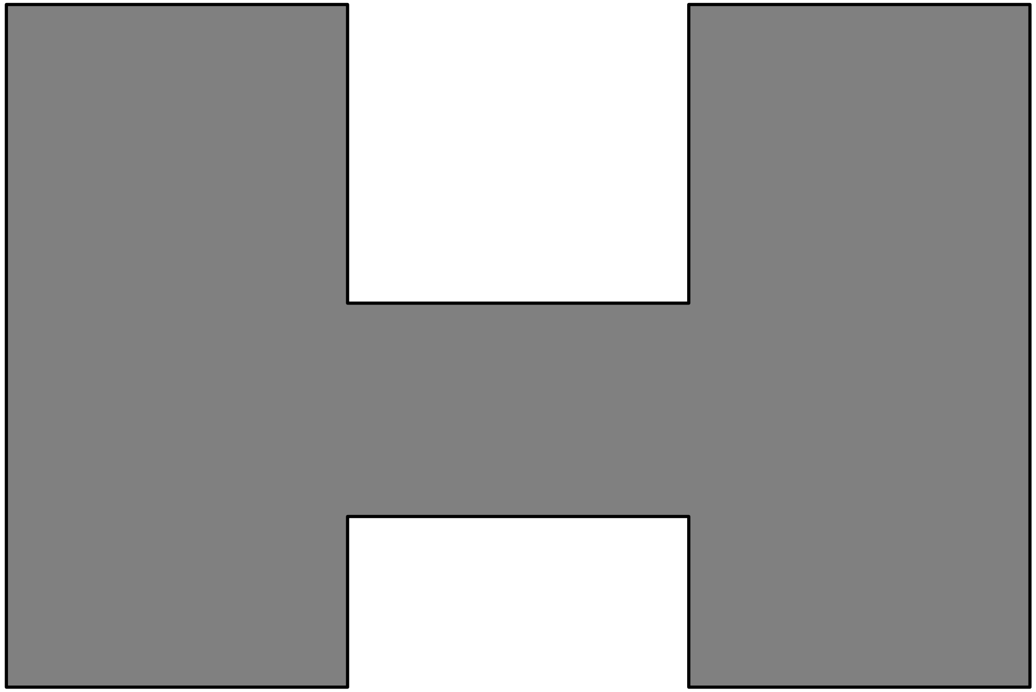

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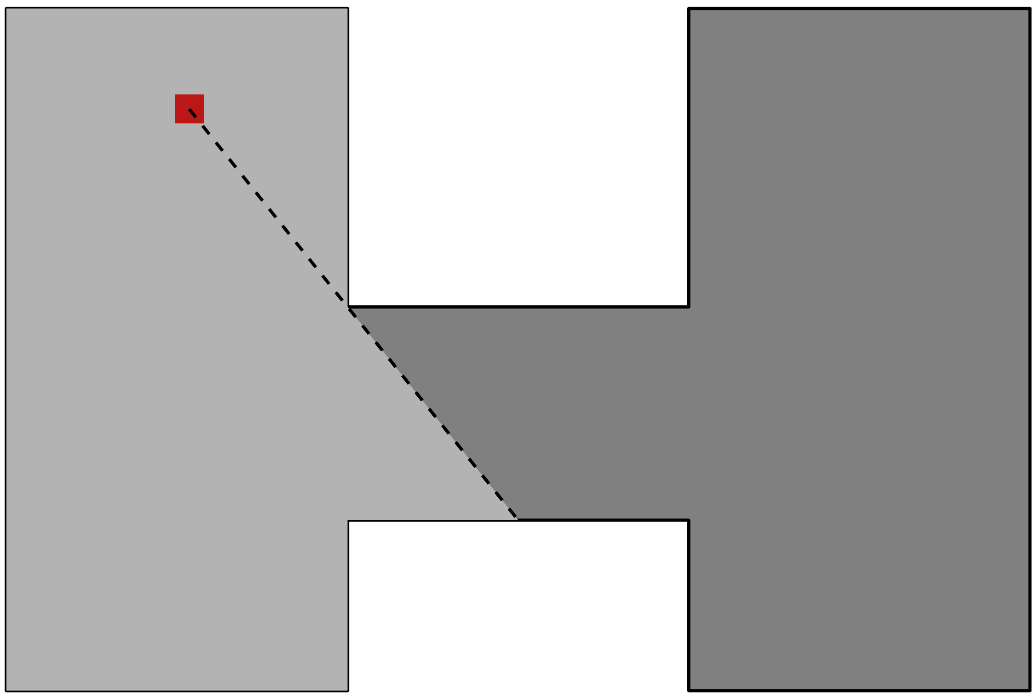

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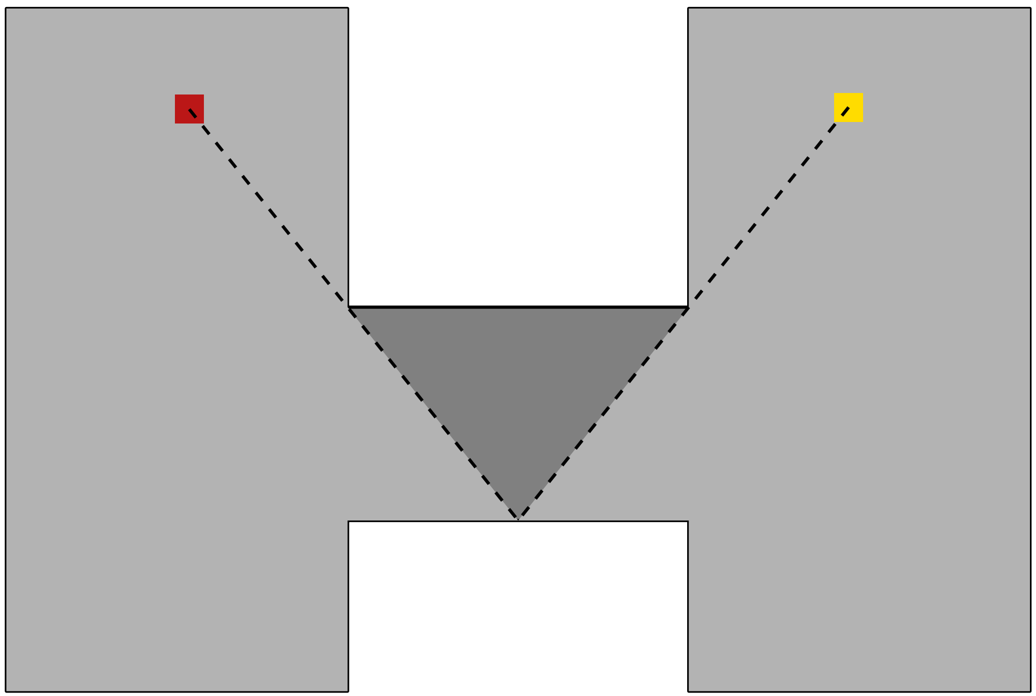

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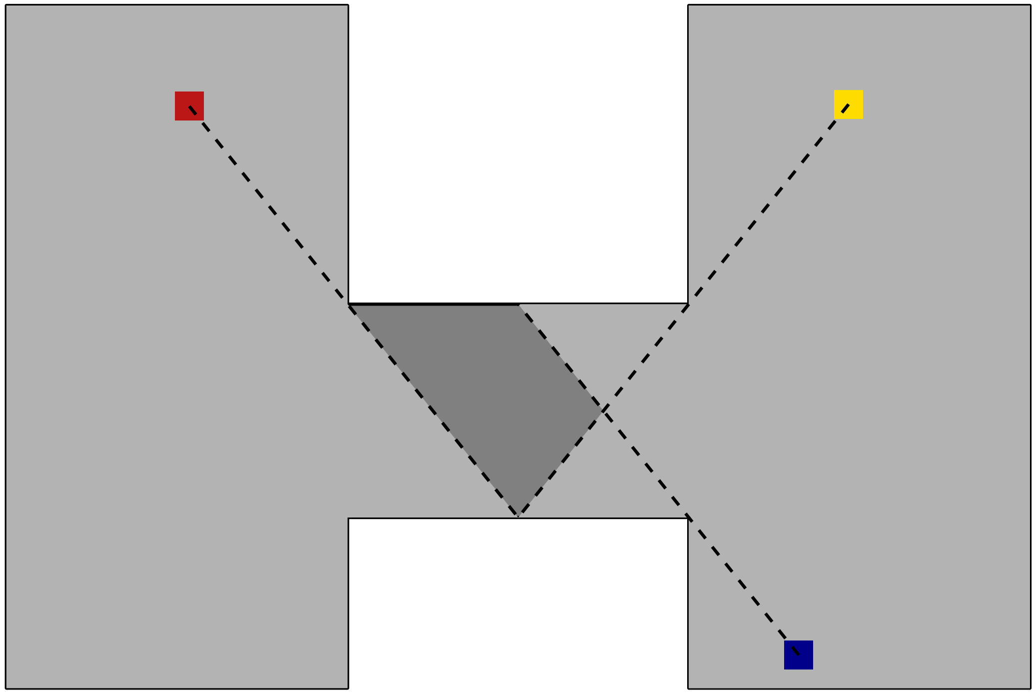

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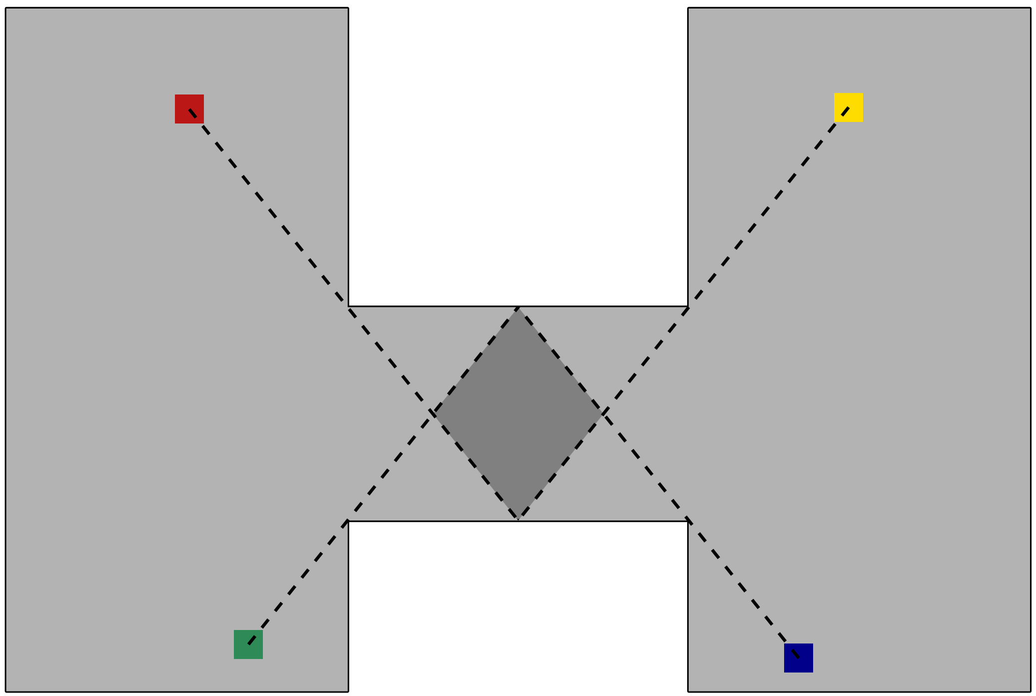
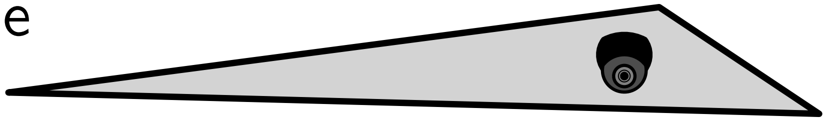


Problem Simplification

Observation: It is easy to guard a triangle



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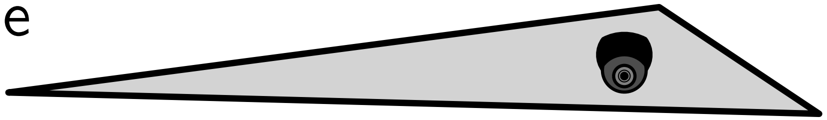
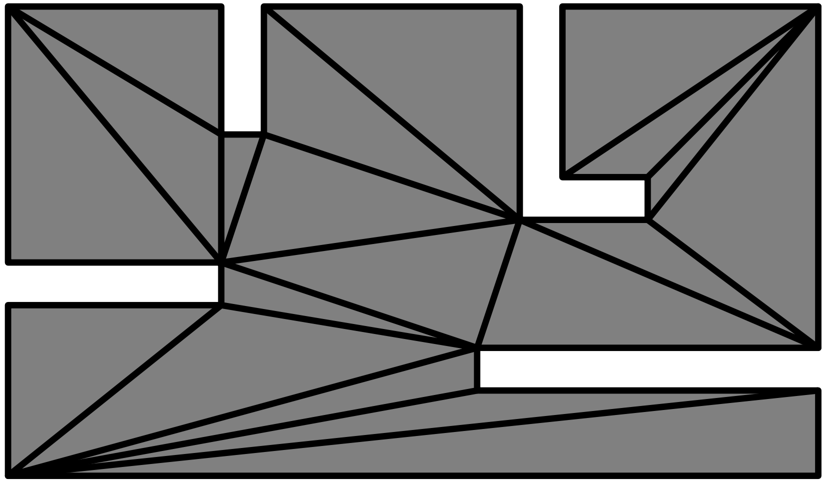


Idea: Decompose P into triangles and guard each of them



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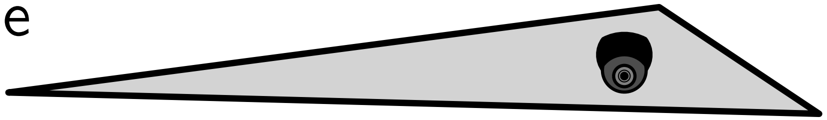

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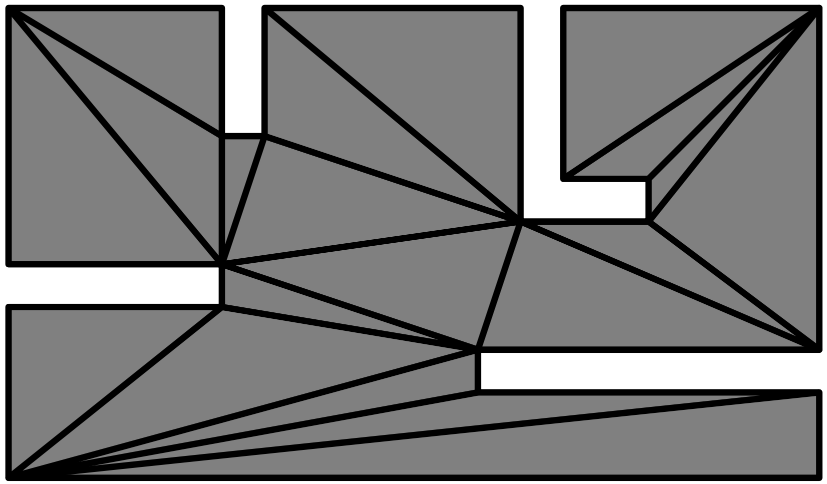

Theorem 1: Each simple polygon with n corners admits a triangulation; any such triangulation contains exactly $n-2$ triangles.

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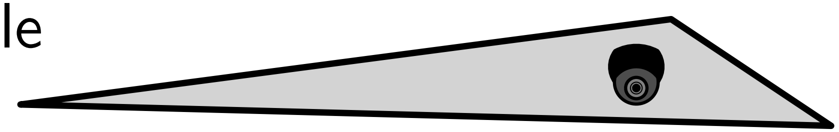

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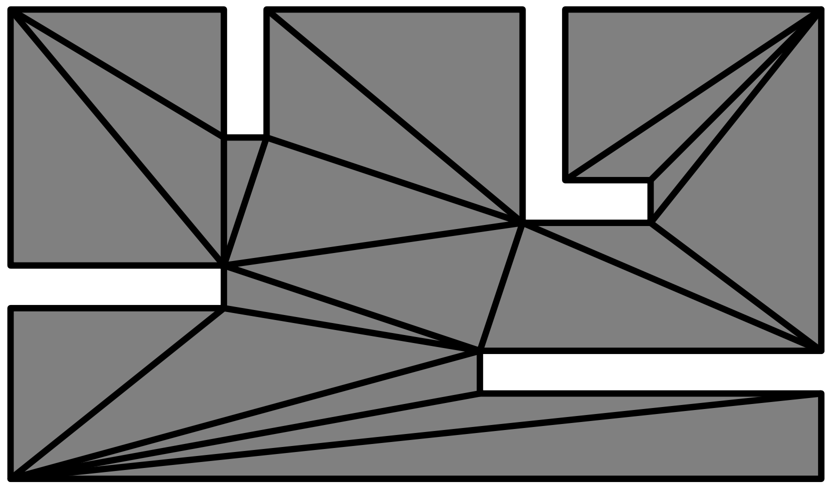

Theorem 1: Each simple polygon with n corners admits a triangulation; any such triangulation contains exactly $n-2$ triangles.

The proof implies a recursive $O(n^2)$-Algorithm!

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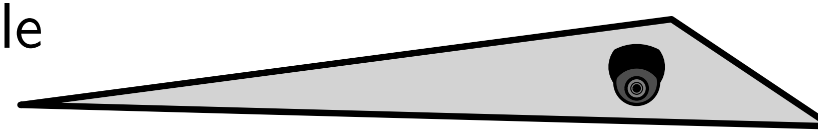

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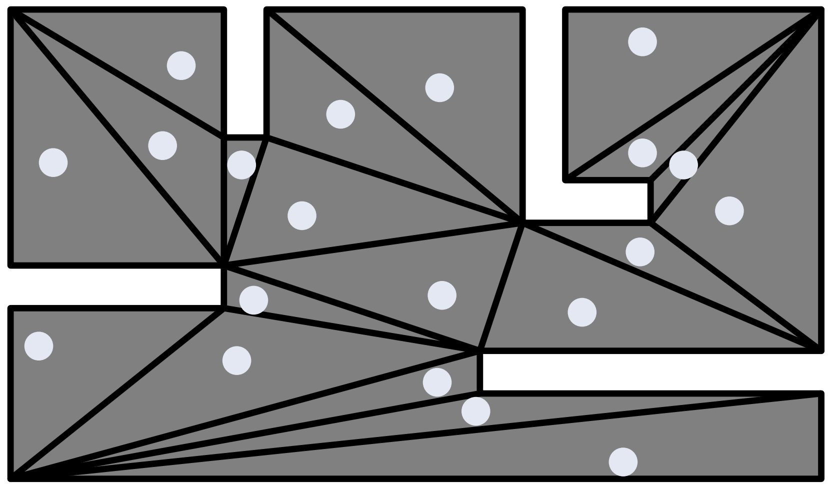

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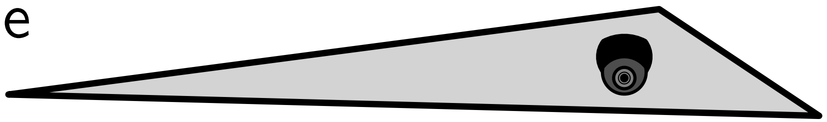

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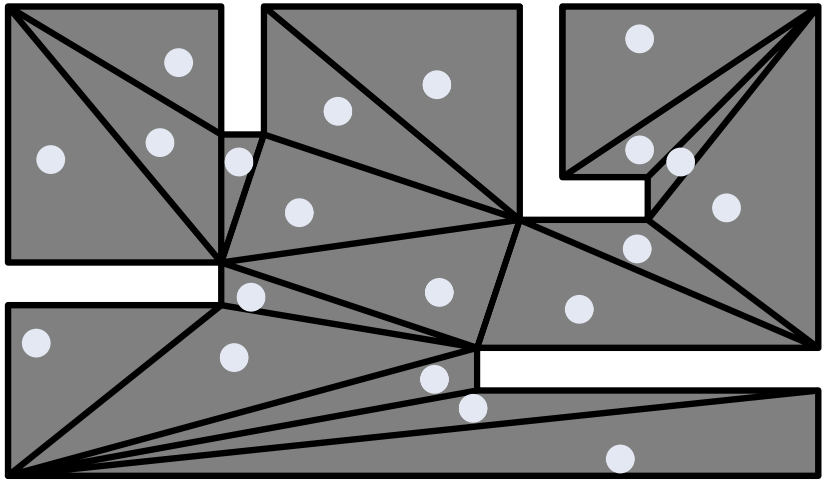

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Can we do better?

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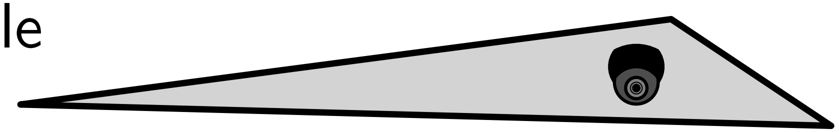

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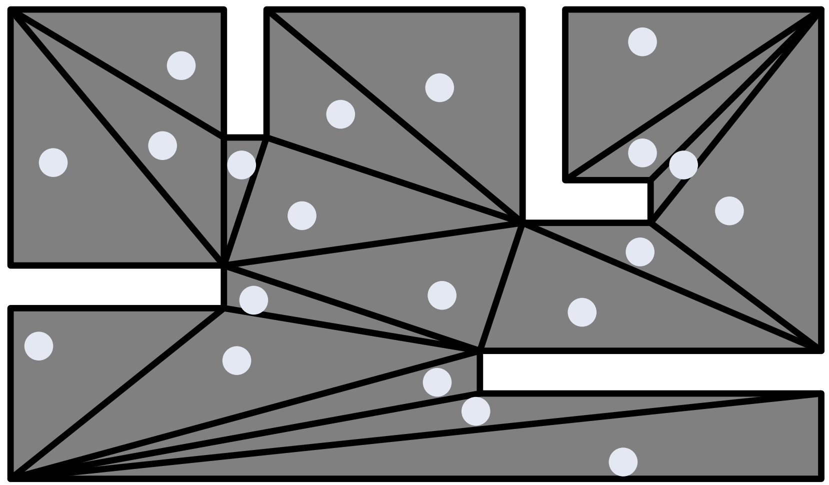

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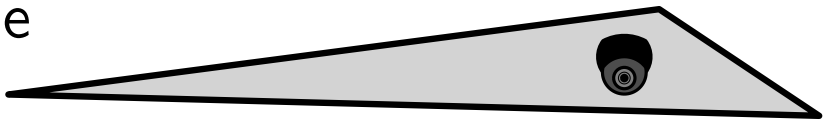

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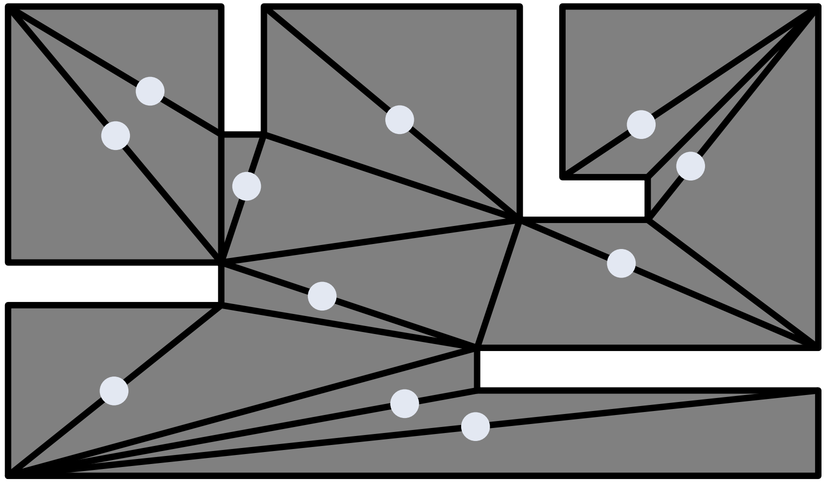

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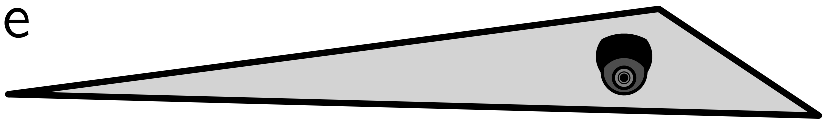

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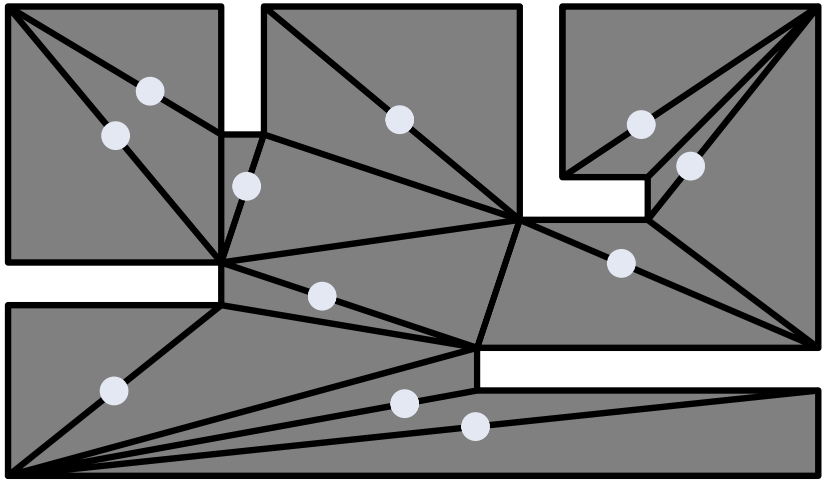

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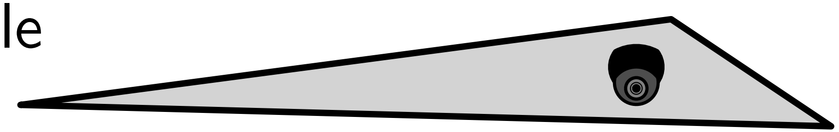

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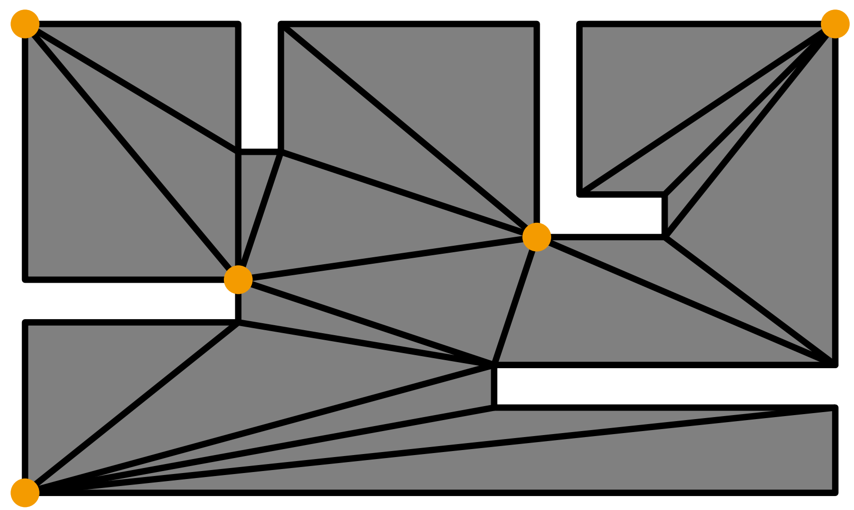

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The Art-Gallery-Theorem [Chvátal '75]

Theorem 2: For a simple polygon with n vertices, $\lfloor n/3 \rfloor$ cameras are sometimes necessary and always sufficient to guard it.

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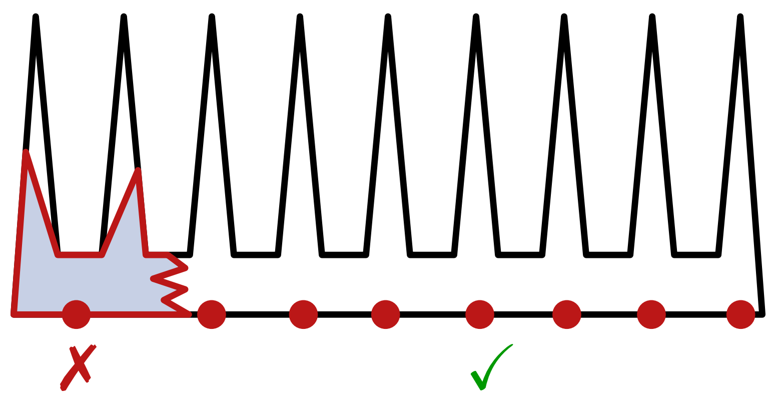
Proof:

- Find a simple polygon with n corners that requires $\approx n/3$ cameras!



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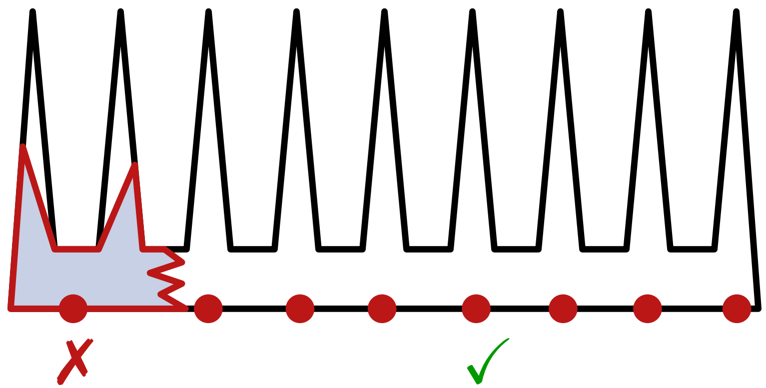

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Conclusion: Given a triangulation, the $\lfloor n/3 \rfloor$ cameras that guard the polygon can be placed in $O(n)$ time.

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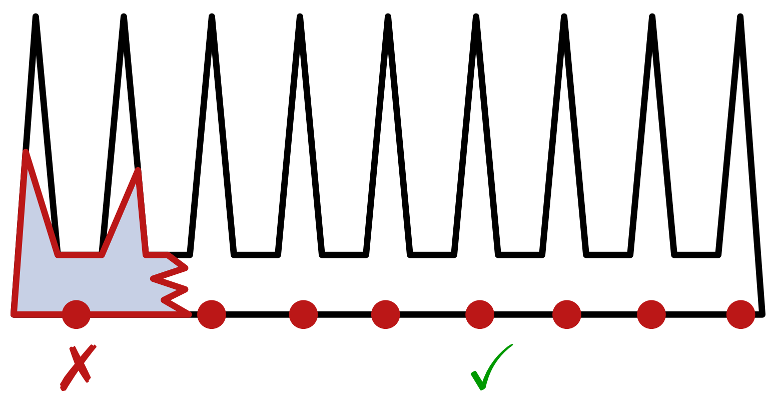

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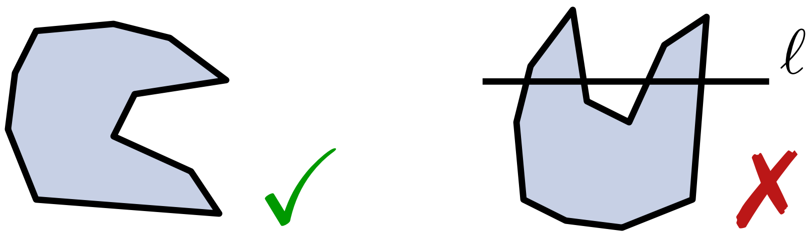
Can we do better than $O(n^2)$ described before?

Triangulation of Polygons

2-step process:

- Step 1: Decompose P into y-monotone polygons

Definition: A polygon is y-monotone, if for any horizontal line ℓ, the interection $\ell \cap P$ is connected.



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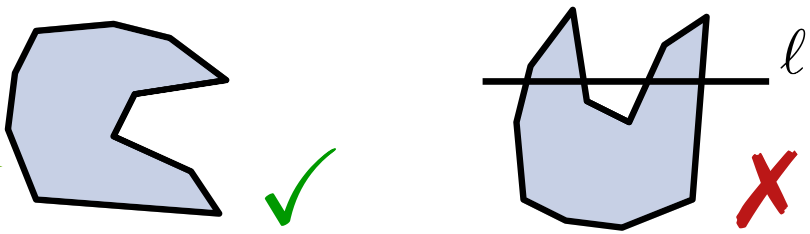
The two paths from the topmost to the bottomost point bounding the polygon, never go upward



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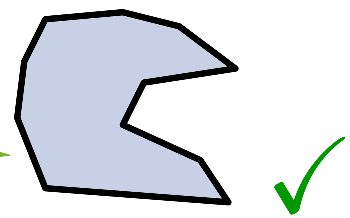

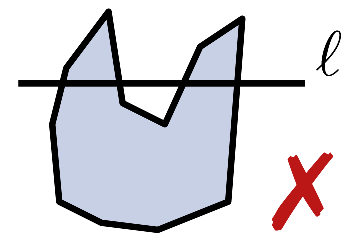

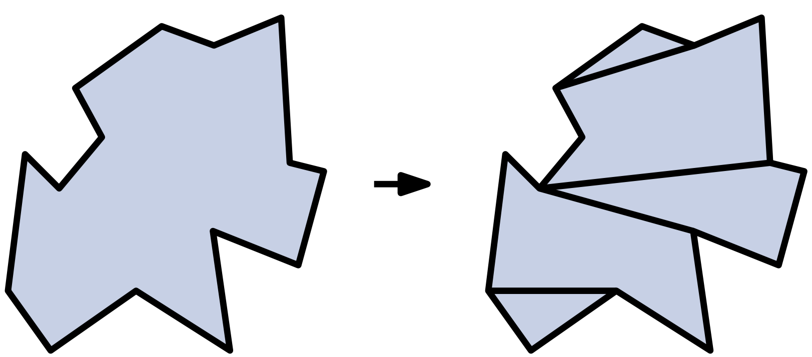

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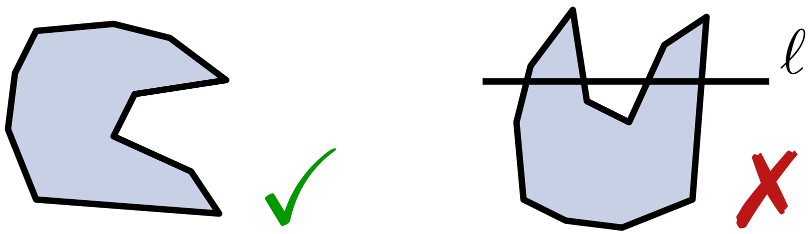
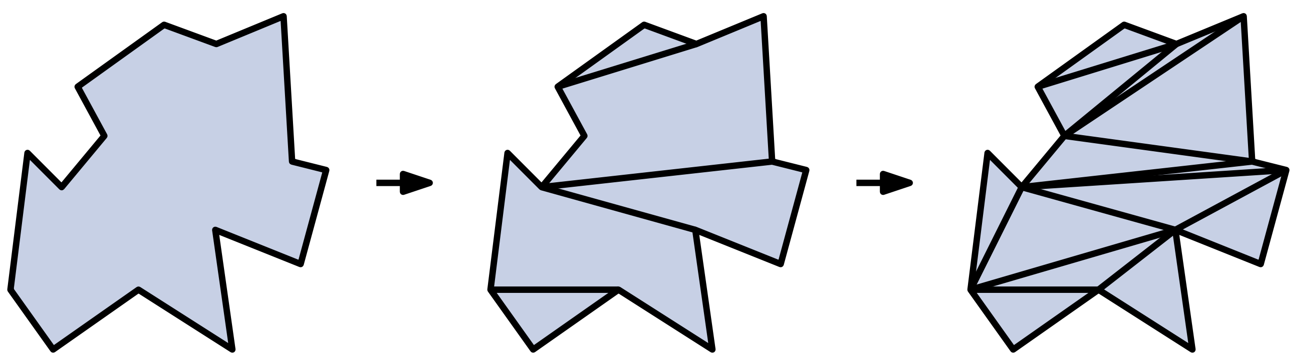
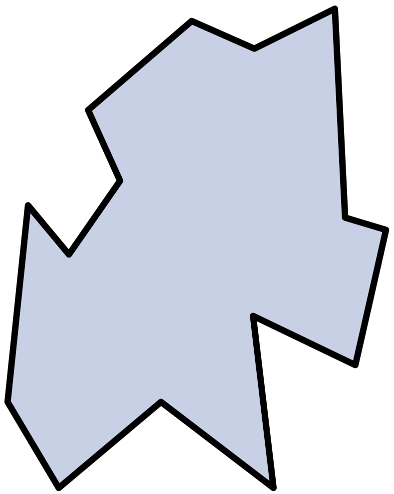


- Step 2: Triangulate the resulting y-monotone polygons



Partition into y-monotone Polygons

Idea: Five different types of vertices



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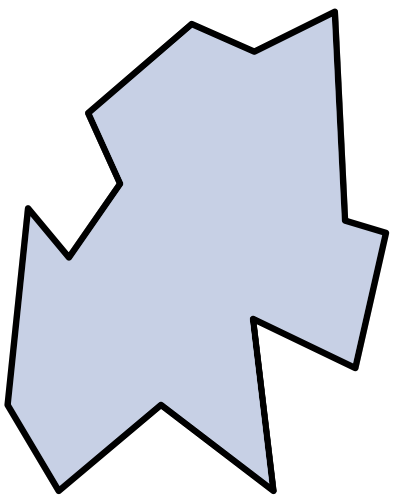
– *Turn vertices*:



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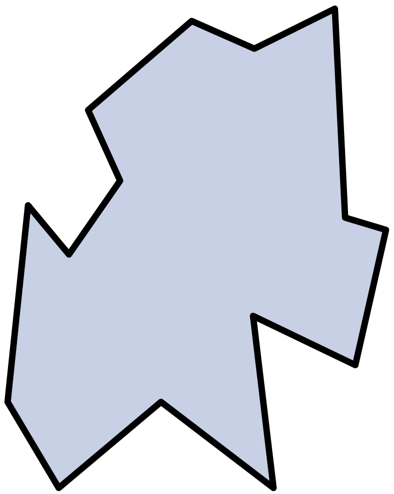

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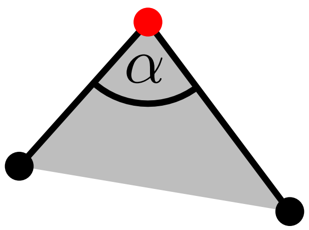
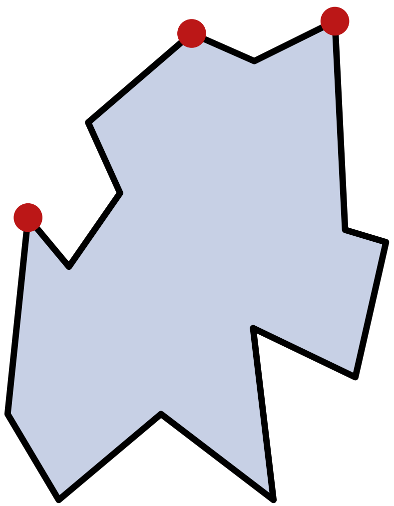
- *start* vertices



if $\alpha < 180°$



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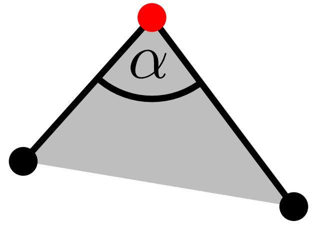

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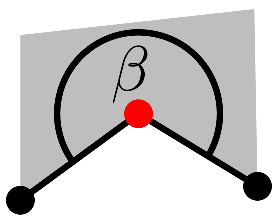
- *split* vertices



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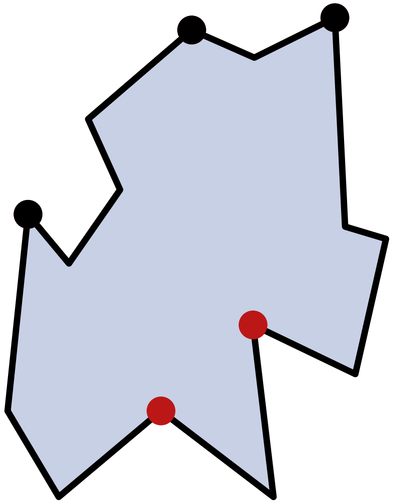

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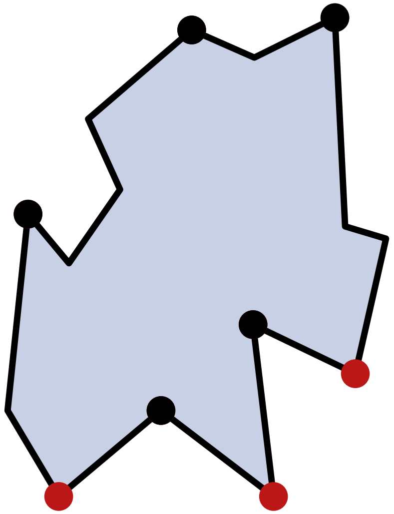

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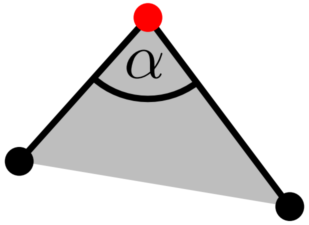

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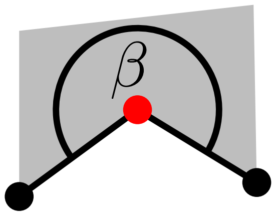
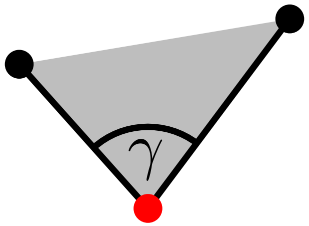


if $\beta > 180°$

- *end* vertices



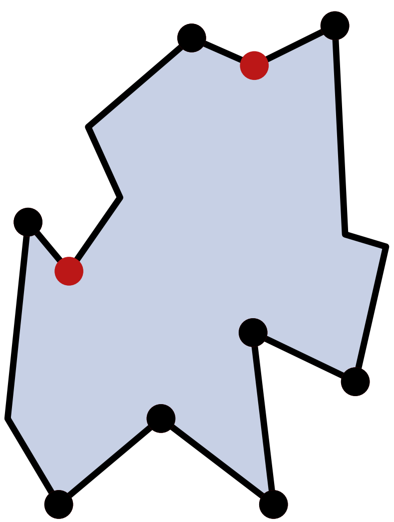
if $\gamma < 180°$

– *regular* vertices

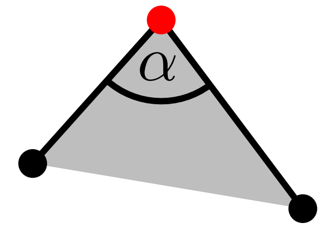
Partition into y-monotone Polygons

Idea: Five different types of vertices

– *Turn vertices*:
vertical change in direction

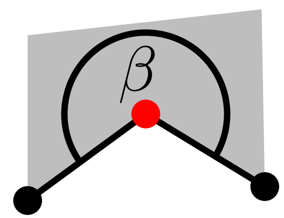


- *start* vertices



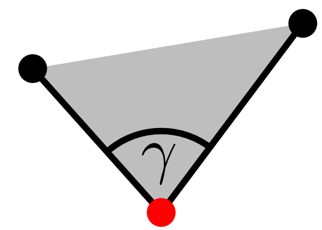
if $\alpha < 180°$

- *split* vertices



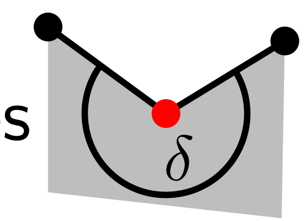
if $\beta > 180°$

- *end* vertices



if $\gamma < 180°$

- *merge* vertices



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Partition into y-monotone Polygons

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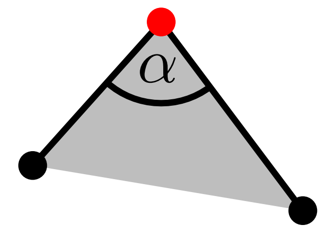

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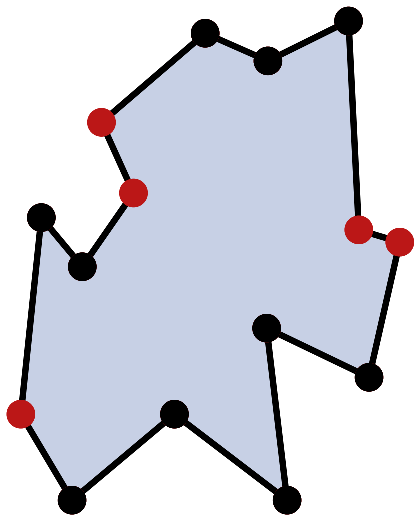

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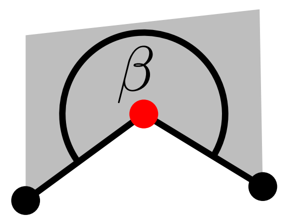

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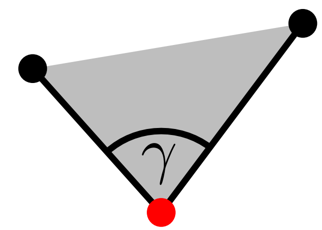

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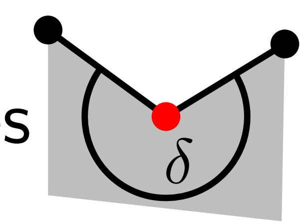

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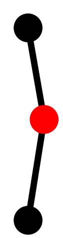
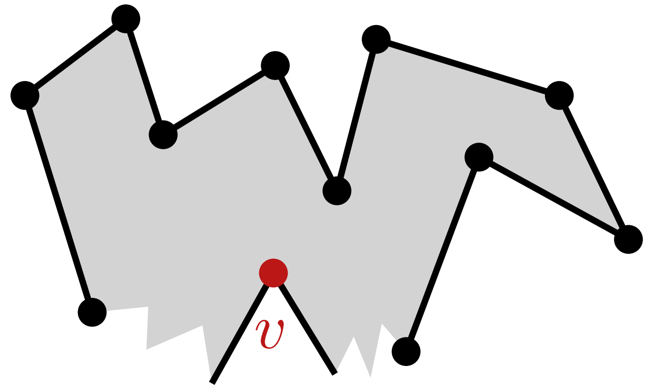


Ideas for Sweep-Line-Algorithm

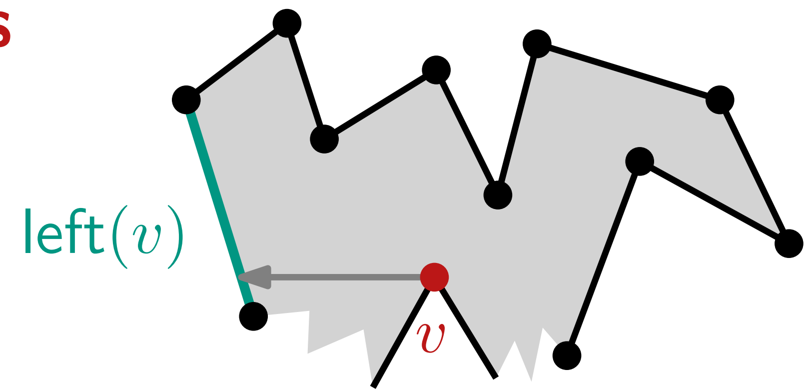
1) Diagonals for the split vertices



Ideas for Sweep-Line-Algorithm

1) Diagonals for the split vertices

- compute for each vertex v its left adjacent edge left(v) with respect to the horizontal sweep line ℓ



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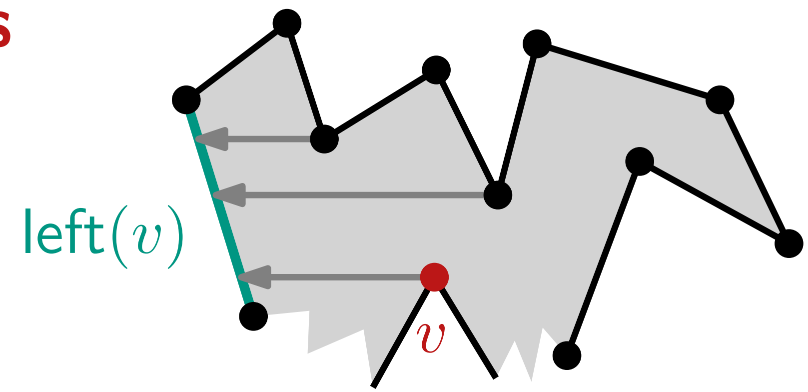

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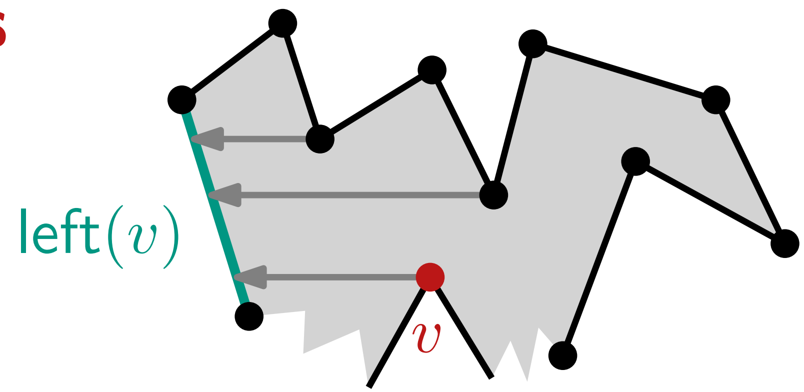

- connect split vertex v to the nearest vertex w above v, such that left$(w) =$ left(v)

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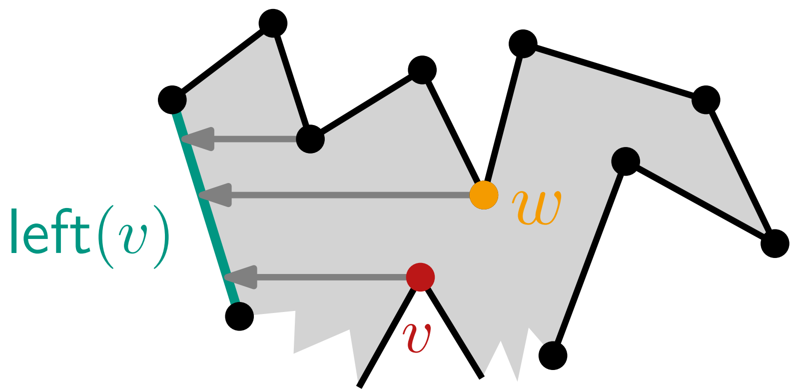

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Ideas for Sweep-Line-Algorithm

Karlsruhe Institute of Technology

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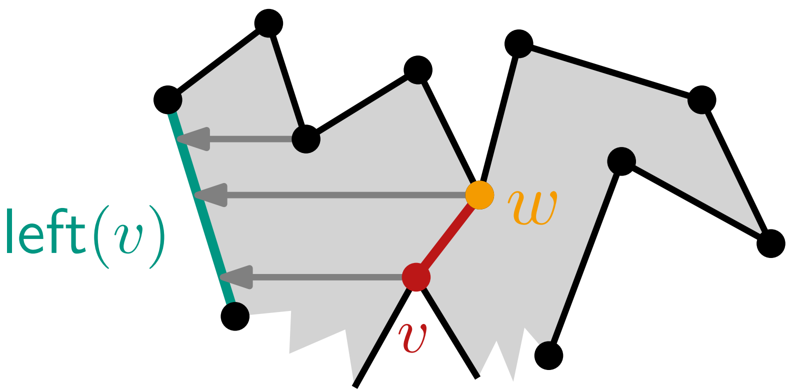

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Ideas for Sweep-Line-Algorithm

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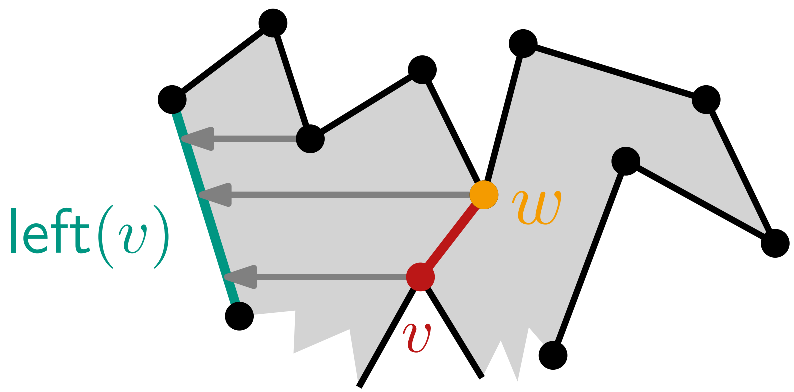
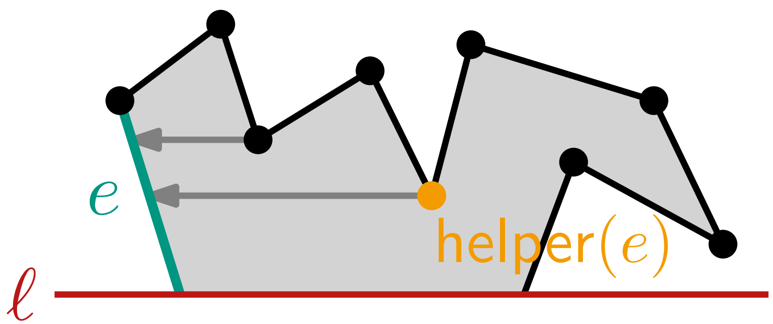


- compute for each vertex v its left adjacent edge left(v) with respect to the horizontal sweep line ℓ
- connect split vertex v to the nearest vertex w above v, such that left$(w) =$ left(v)
- for each edge e save the bottommost vertex w such that left$(w) = e$; notation helper(e) := w



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Karlsruhe Institute of Technology

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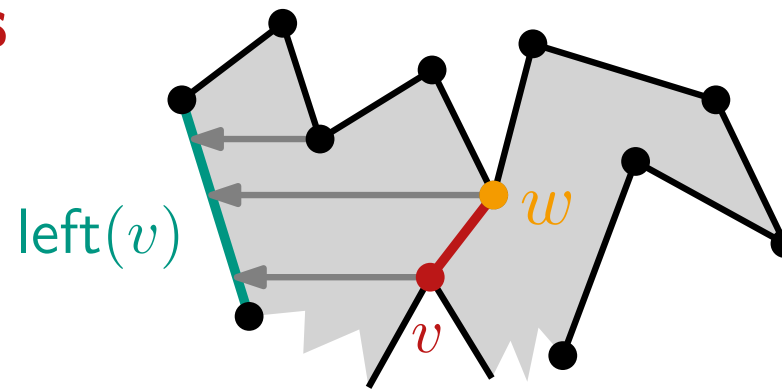
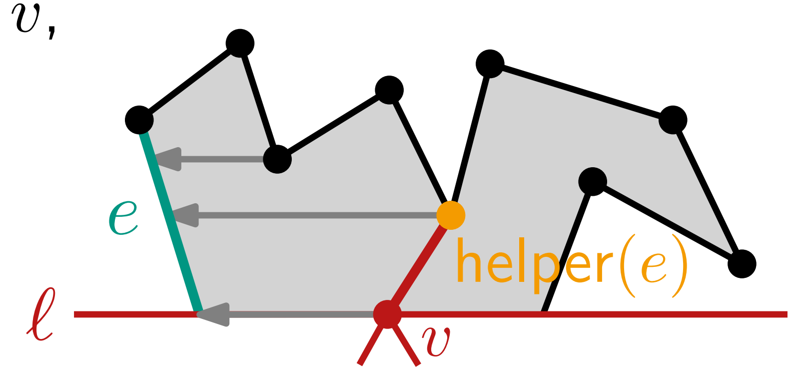


- connect split vertex v to the nearest vertex w above v, such that left(w) = left(v)

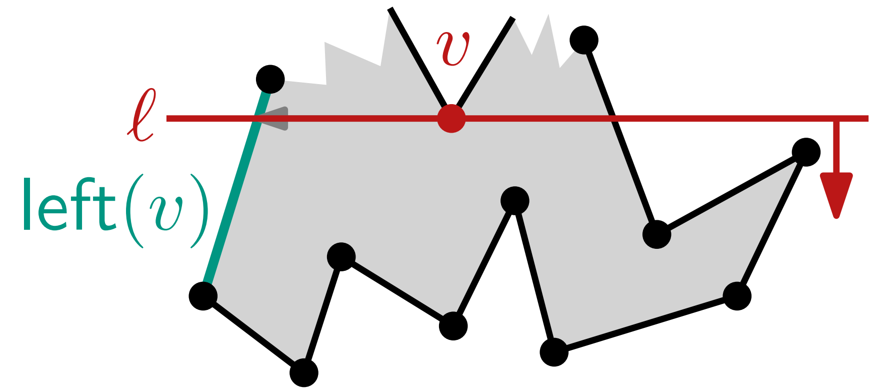
- for each edge e save the bottommost vertex w such that left$(w) = e$; notation helper(e) := w

- when ℓ passes through a split vertex v, we connect v with helper(left(v))



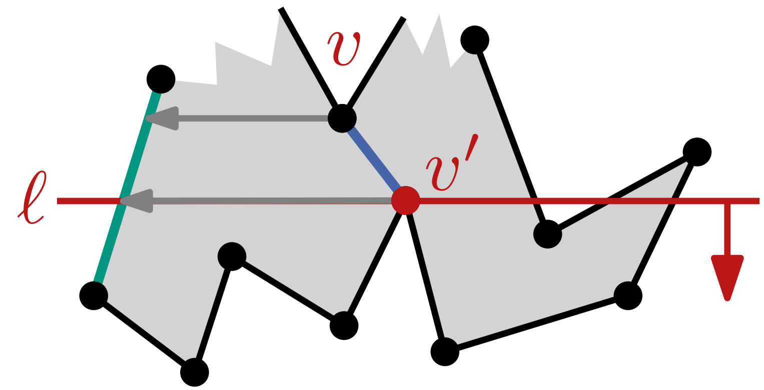
2) Diagonals for merge vertices

- when the vertex v is reached, we set helper(left(v)) $= v$



2) Diagonals for merge vertices

- when the vertex v is reached, we set helper(left$(v)) = v$
- when we reach a split vertex v' such that left(v') = left(v) the diagonal (v, v') is introduced



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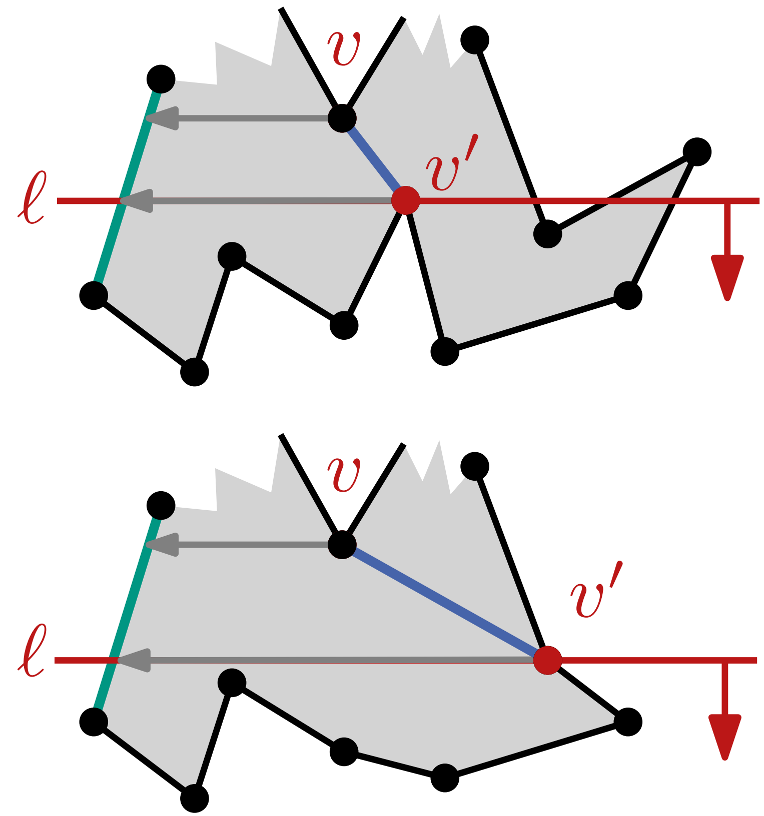
2) Diagonals for merge vertices

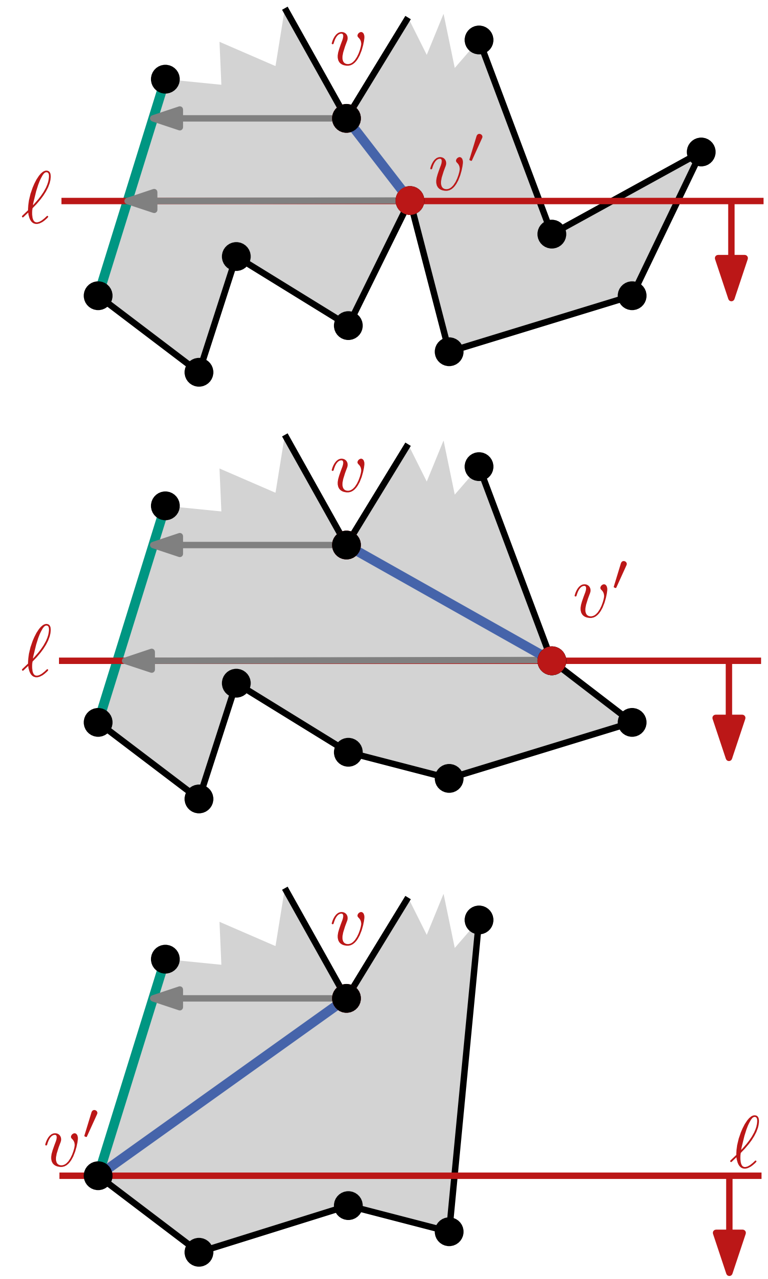
- when the vertex v is reached, we set helper(left$(v)) = v$
- when we reach a split vertex v' such that left(v') = left(v) the diagonal (v, v') is introduced
- in case we reach a regular vertex v' such that helper(left(v')) is v the diagonal (v, v') is introduced



Ideas for Sweep-Line-Algorithm

2) Diagonals for merge vertices

- when the vertex v is reached, we set helper(left(v)) $= v$
- when we reach a split vertex v' such that left(v') $=$ left(v) the diagonal (v, v') is introduced
- in case we reach a regular vertex v' such that helper(left(v')) is v the diagonal (v, v') is introduced
- if the end of v' of left(v) is reached, then the diagonal (v, v') is introduced



Algorithm MakeMonotone(P)

MakeMonotone(Polygon P)

$\mathcal{D} \leftarrow$ doubly-connected edge list for $(V(P), E(P))$
$\mathcal{Q} \leftarrow$ priority queue for $V(P)$ sorted lexicographically; $\mathcal{T} \leftarrow \emptyset$
(binary search tree for sweep-line status)
while $\mathcal{Q} \neq \emptyset$ **do**
    $v \leftarrow \mathcal{Q}$.nextVertex()
    $\mathcal{Q}$.deleteVertex(v)
    handleVertex(v)
return $\mathcal{D}$

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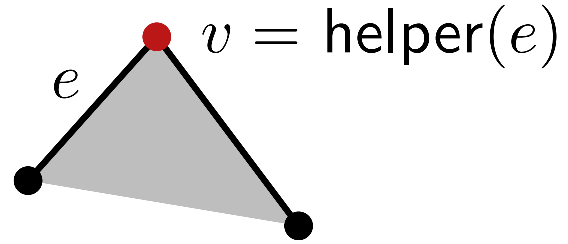
handleStartVertex(vertex v)

$\mathcal{T} \leftarrow$ add the left edge e
helper(e) $\leftarrow v$



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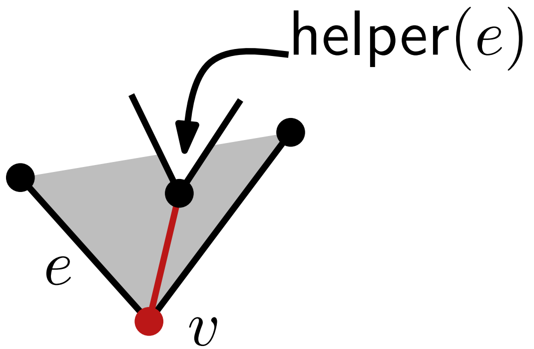

handleStartVertex(vertex v)

$\mathcal{T} \leftarrow$ add the left edge e
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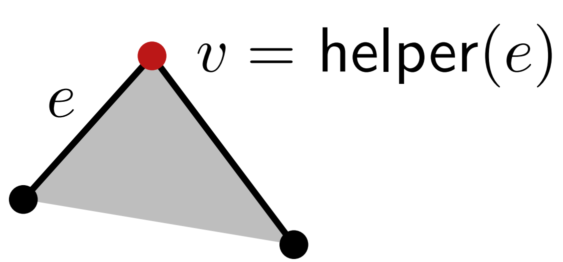

handleEndVertex(vertex v)

$e \leftarrow$ left edge
if isMergeVertex(helper(e)) **then**
  $\mathcal{D} \leftarrow$ add edge (helper(e), v)
remove e from $\mathcal{T}$

Algorithm MakeMonotone(P)

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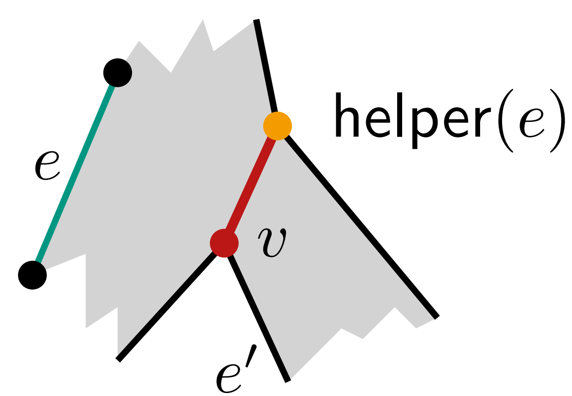
  handleVertex(v)
return $\mathcal{D}$

handleSplitVertex(vertex v)

$e \leftarrow$ Edge to the left of v in $\mathcal{T}$
$\mathcal{D} \leftarrow$ add edge (helper(e), v)
helper(e) $\leftarrow v$
$\mathcal{T} \leftarrow$ add the right edge e' of v
helper(e') $\leftarrow v$



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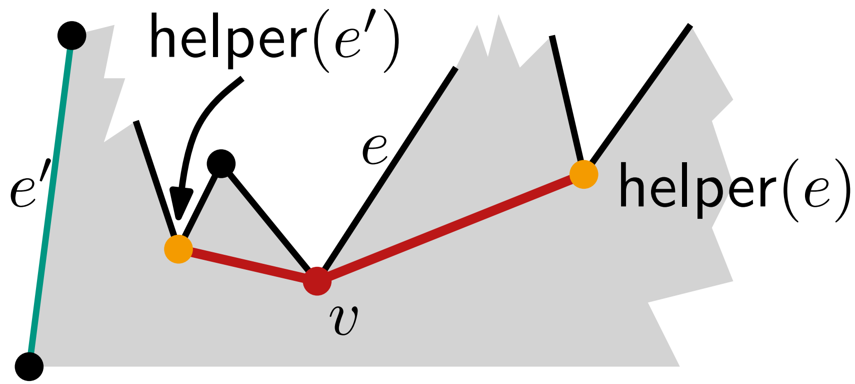

handleMergeVertex(vertex v)

$e \leftarrow$ right edge
if isMergeVertex(helper(e)) **then**
  $\mathcal{D} \leftarrow$ add edge (helper(e), v)
remove e from $\mathcal{T}$
$e' \leftarrow$ edge to the left of v in $\mathcal{T}$
if isMergeVertex(helper(e'))
 then
  $\mathcal{D} \leftarrow$ add edge (helper(e'), v)
helper(e') $\leftarrow v$

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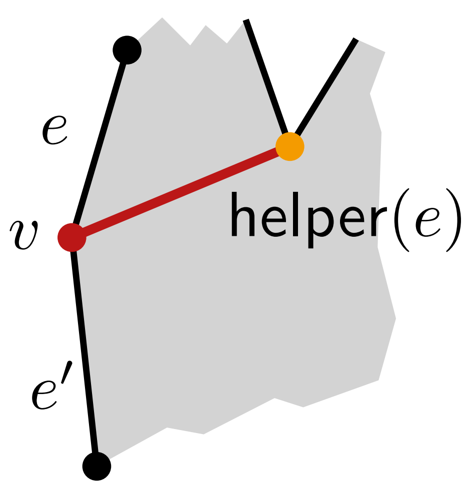
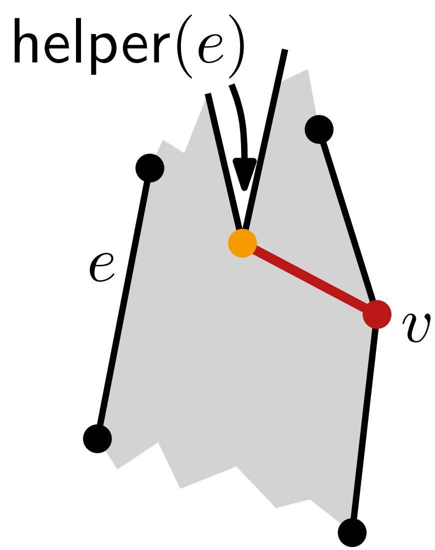
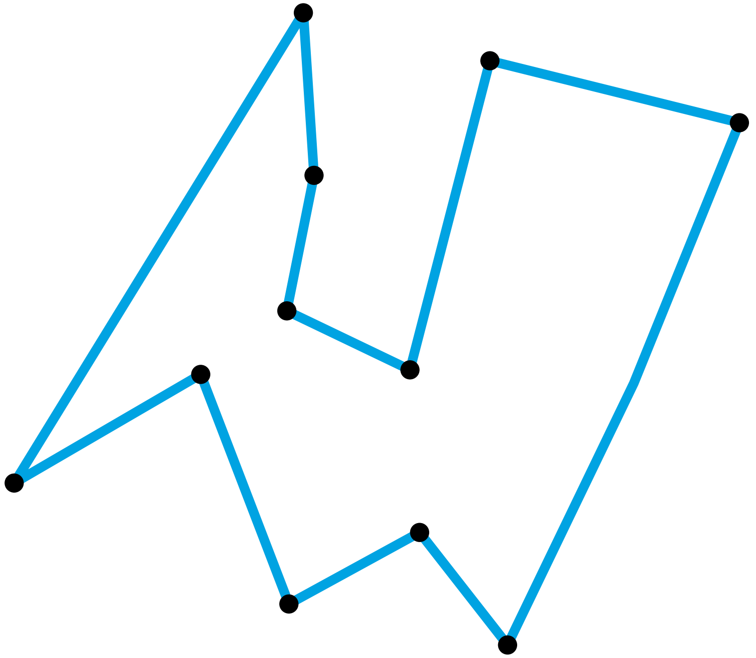


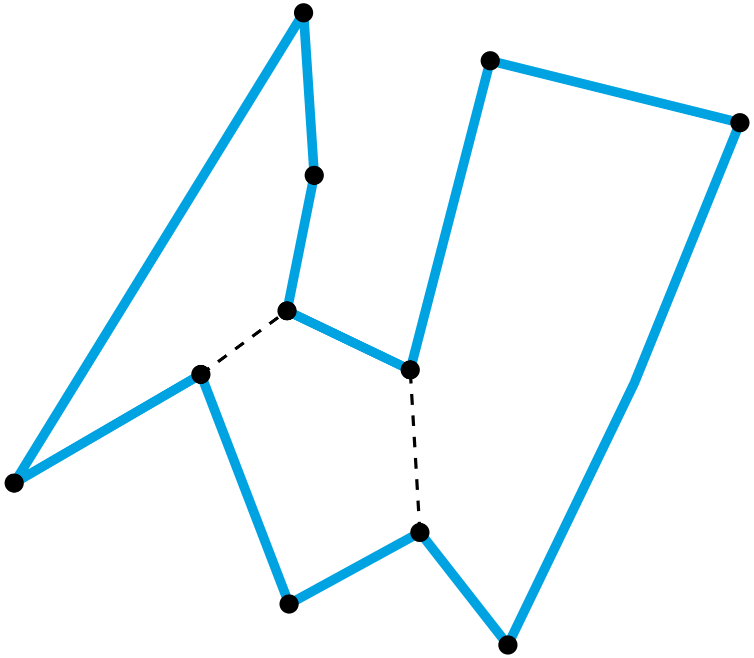


handleRegularVertex(vertex v)
if P lies locally to the right of v
then
    $e, e' \leftarrow$ above, below edge
    if isMergeVertex(helper(e))
    then
        $\mathcal{D} \leftarrow$ add edge (helper$(e), v)$
    remove e from $\mathcal{T}$
    $\mathcal{T} \leftarrow$ add e'; helper$(e') \leftarrow v$
else
    $e \leftarrow$ edge to the left of v
    ~~add e to $\mathcal{T}$~~
    if isMergeVertex(helper(e))
    then

Insertion Diagonals



Insertion Diagonals



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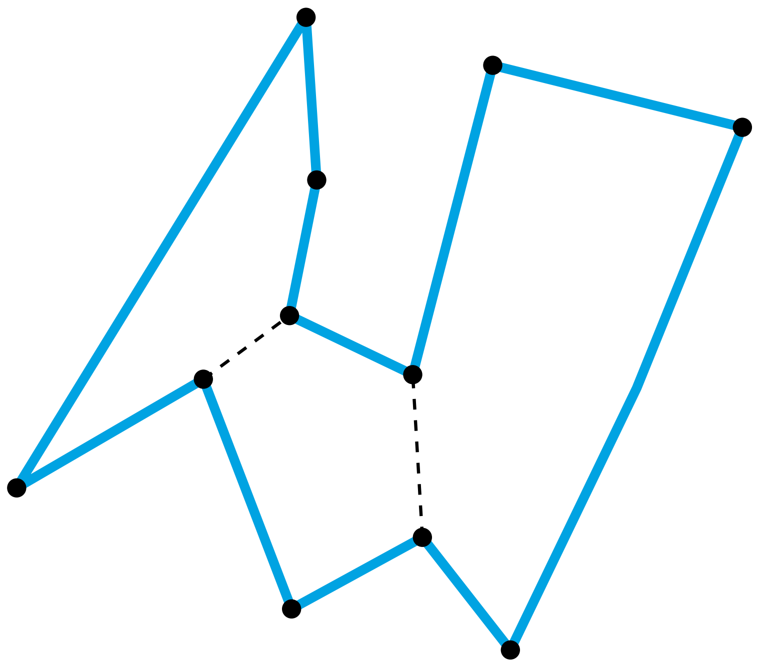
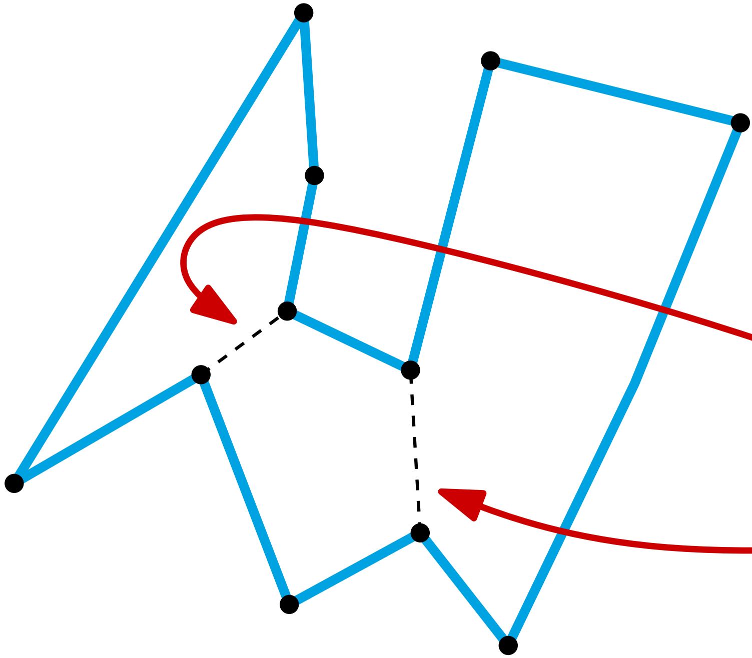


Data structure: Doubly-connected edge list (DCEL)

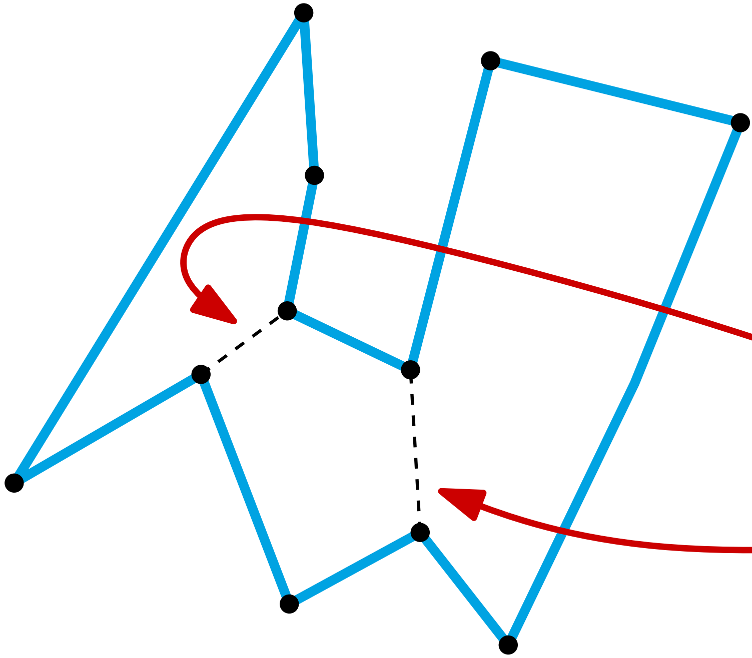
Insertion Diagonals



Claim:
Insertion of diagonals in $O(1)$ time.

Data structure: Doubly-connected edge list (DCEL)

Insertion Diagonals



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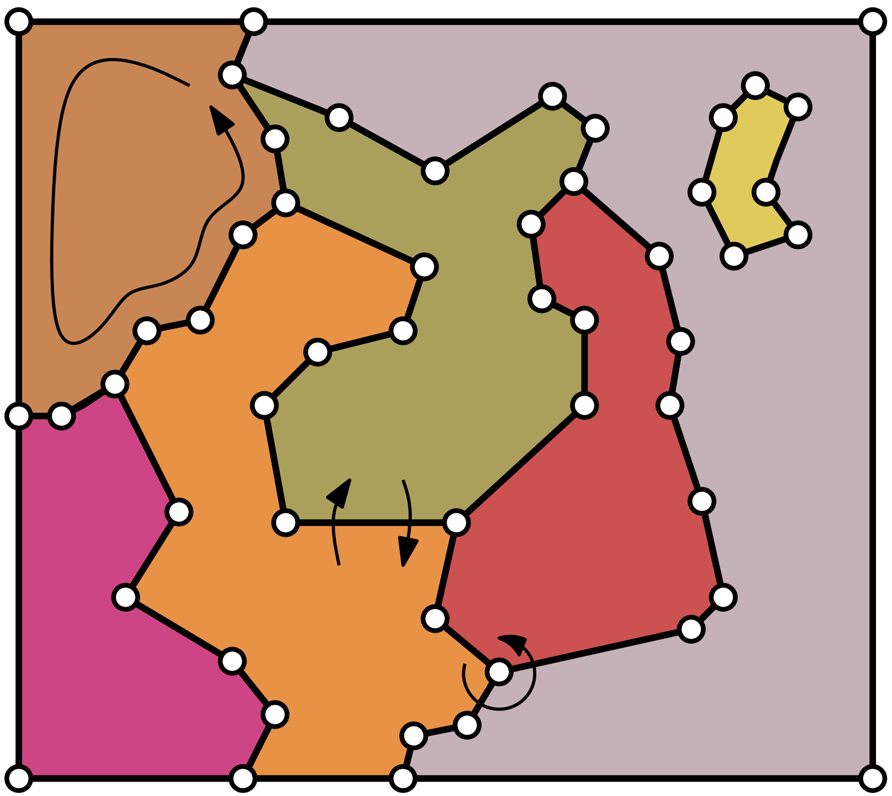
Doubly Connected Edge List



- Map corresponds with subdivision of plane into polygons.
- Subdivision corresponds with embedding of planar graph with
  - vertices
  - edges
  - faces

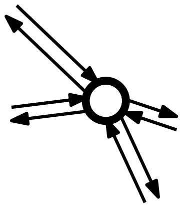
Which operations should be supported by the data structure?

- Traverse edges of face.
- Go from face to face by edges.
- Traverse neighboring vertices in cyclic order.

Doubly Connected Edge List(DCEL)

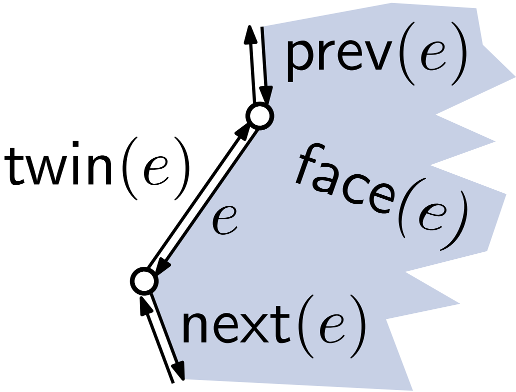
Ingredients:

- Vertices



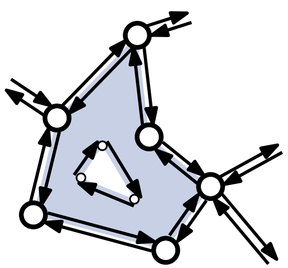
- Coordinates $(x(v), y(v))$
- (first) outgoing edge

- Edge $=$ two half-edges



- Vertex origin(v)
- Opposite edge twin(e)
- Predecessor prev(e) & Successor next(e)
- incident face

- Faces

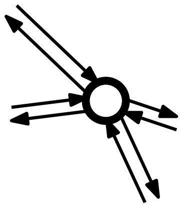


- Bounding edges for outer face.
- Edge list inner(f) for holes.

Doubly Connected Edge List (DCEL)

Ingredients:

- Vertices



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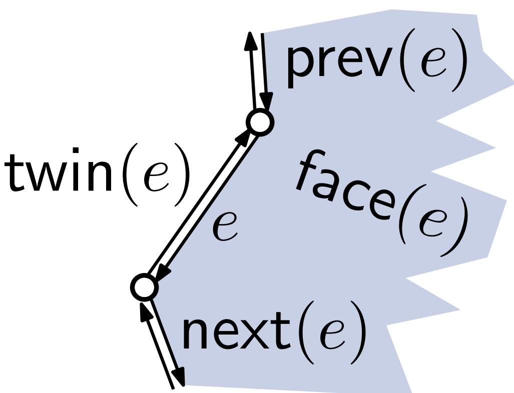

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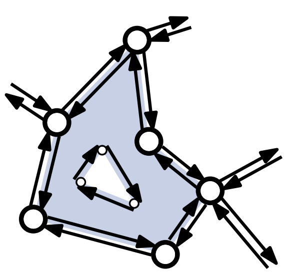

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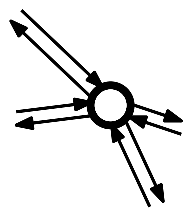
Doubly Connected Edge List (DCEL)

Ingredients:

- Vertices



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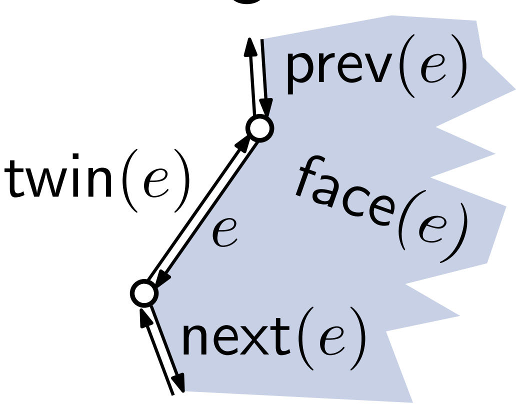
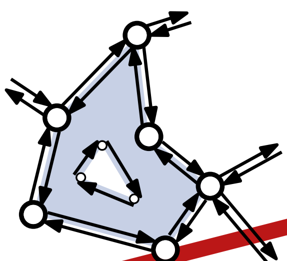


  - Vertex origin(v)
  - Opposite edge twin(e)
  - Predecessor prev(e) & Successor next(e)
  - incident face

- ~~Faces~~



  - ~~Bounding edges for outer face.~~
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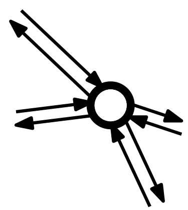
Doubly Connected Edge List (DCEL)

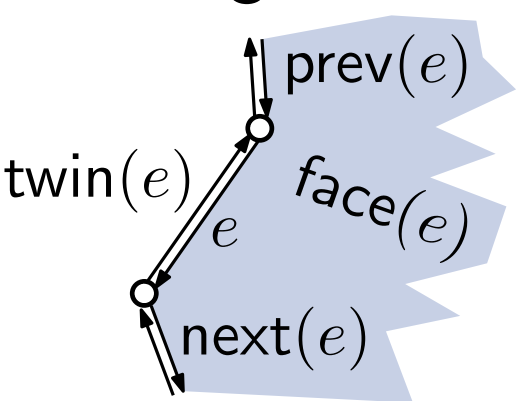
Ingredients:

- Vertices



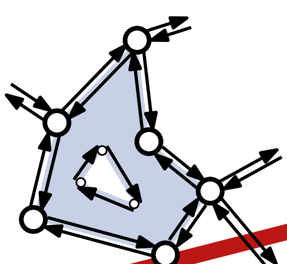
  - Coordinates $(x(v), y(v))$
  - (first) outgoing edge

- Edge $=$ two half-edges



  - Vertex origin(v)
  - Opposite edge twin(e)
  - Predecessor prev(e) & Successor next(e)
  - ~~incident face~~

- ~~Faces~~

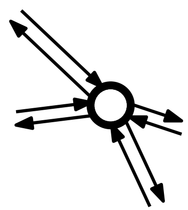


  - ~~Bounding edges for outer face.~~
  - ~~Edge list inner(f) for holes.~~

Doubly Connected Edge List (DCEL)

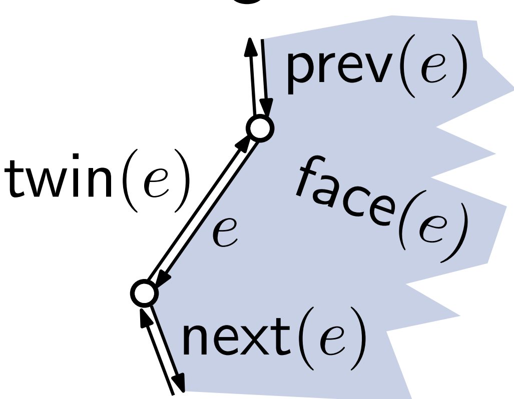
Ingredients:

- Vertices



  - Coordinates $(x(v), y(v))$ ✓
  - (first) outgoing edge

- Edge $=$ two half-edges



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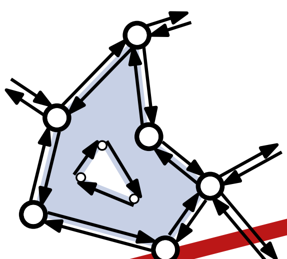
- ~~Faces~~



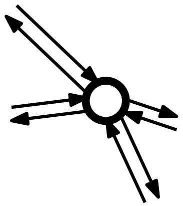
  - ~~Bounding edges for outer face.~~
  - ~~Edge list inner(f) for holes.~~

Doubly Connected Edge List (DCEL)

Karlsruhe Institute of Technology

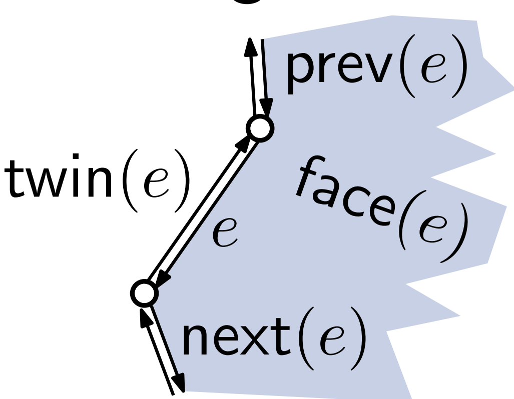
Ingredients:

- Vertices



- Coordinates $(x(v), y(v))$ ✓
- (first) outgoing edge ✓

- Edge $=$ two half-edges



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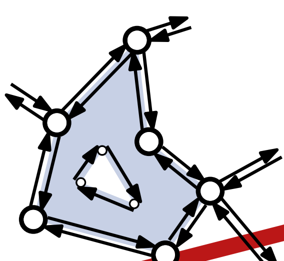
- ~~Faces~~



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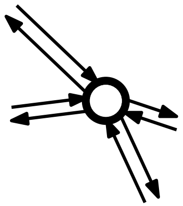
Doubly Connected Edge List (DCEL)

Ingredients:

- Vertices



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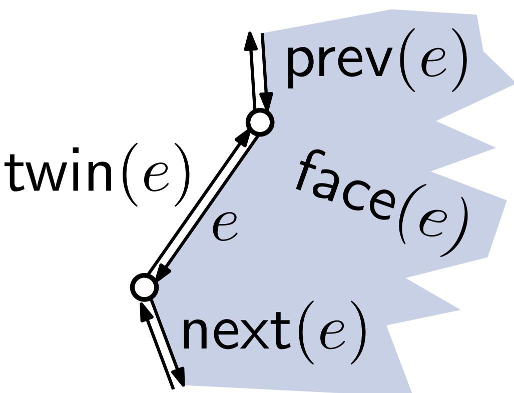

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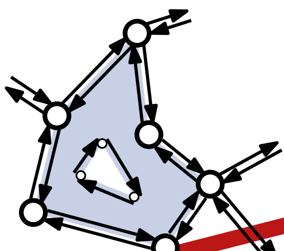

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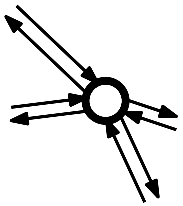
Doubly Connected Edge List (DCEL)

Ingredients:

- Vertices



- Coordinates $(x(v), y(v))$ ✓
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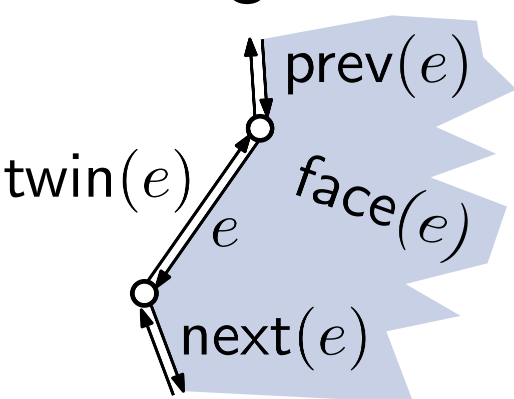

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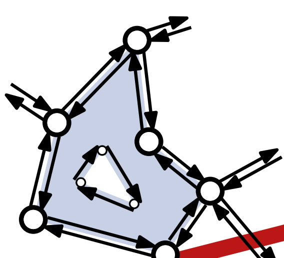

- ~~Bounding edges for outer face.~~
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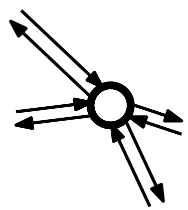
Doubly Connected Edge List (DCEL)

Ingredients:

- Vertices



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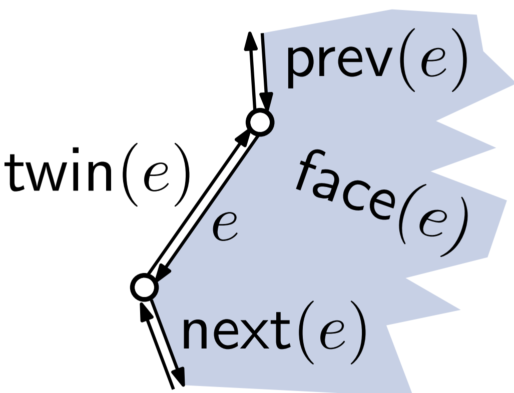
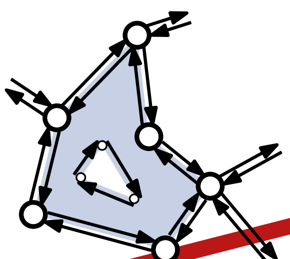


  - Vertex origin(v)
  - Opposite edge twin(e)
  - Predecessor prev(e) & Successor next(e) ?
  - ~~incident face~~

- ~~Faces~~



  - ~~Bounding edges for outer face.~~
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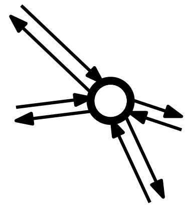
Doubly Connected Edge List (DCEL)

Ingredients:

- Vertices



  - Coordinates $(x(v), y(v))$
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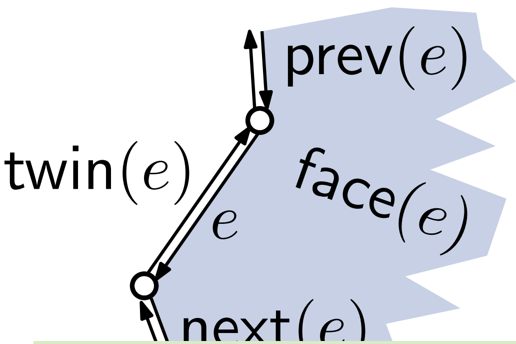
- Edge $=$ two half-edges



  - Vertex origin(v)
  - Opposite edge twin(e)
  - Predecessor prev(e) & Successor next(e) ?
  - ~~incident face~~

a) Each vertex has $O(1)$ incident edges.
- Initially each vertex has degree 2.
- Each vertex is at most once helper $+1$
- Each vertex is handled at most once : $+2$

b) Using a appropriate ordering, we can find the desired edges.

Linear Running Time

MakeMonotone(Polygon P)

$\mathcal{D} \leftarrow$ doubly-connected edge list for $(V(P), E(P))$
$\mathcal{Q} \leftarrow$ priority queue for $V(P)$ sorted lexicographically; $\mathcal{T} \leftarrow \emptyset$
(binary search tree for sweep-line status)
while $\mathcal{Q} \neq \emptyset$ **do**
  $v \leftarrow \mathcal{Q}$.nextVertex()
  $\mathcal{Q}$.deleteVertex(v)
  handleVertex(v)
return $\mathcal{D}$

Assumption: P contains $O(1)$ turn vertices.

Exercise: Adapt procedure such that it has $O(n)$ running time.

Linear Running Time

MakeMonotone(Polygon P)

$\mathcal{D} \leftarrow$ doubly-connected edge list for $(V(P), E(P))$
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while $\mathcal{Q} \neq \emptyset$ **do**
  $v \leftarrow \mathcal{Q}$.nextVertex()
  $\mathcal{Q}$.deleteVertex(v)
  handleVertex(v)
return $\mathcal{D}$

Assumption: P contains $O(1)$ turn vertices.

Exercise: Adapt procedure such that it has $O(n)$ running time.

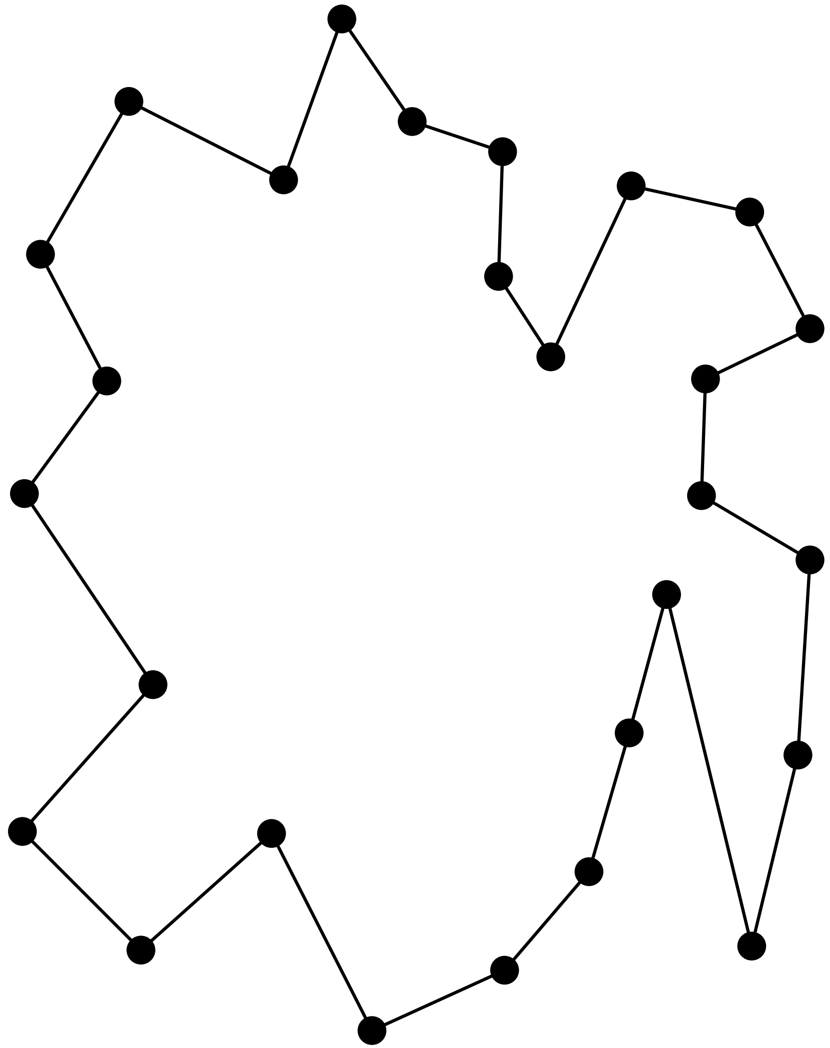
Observation:

Creation of $\mathcal{Q}$ costs $O(n \log n)$ time.

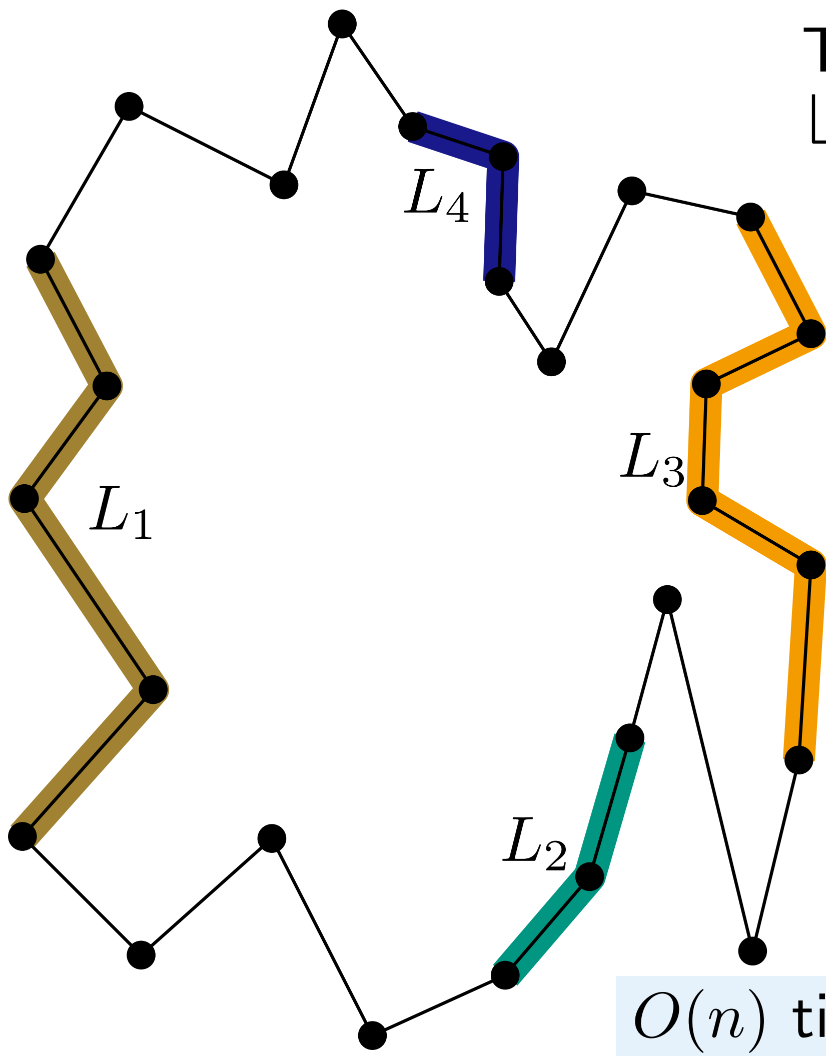
Querries in $\mathcal{T}$ cost $O(n \log n)$ time in total.

Linear Running Time

Step 1: Create queue $\mathcal{Q}$ in $O(n)$ time.



Linear Running Time



Step 1: Create queue $\mathcal{Q}$ in $O(n)$ time.

Traverse P in counter-clockwise order.
↳ Add consecutive regular vertices to a list.

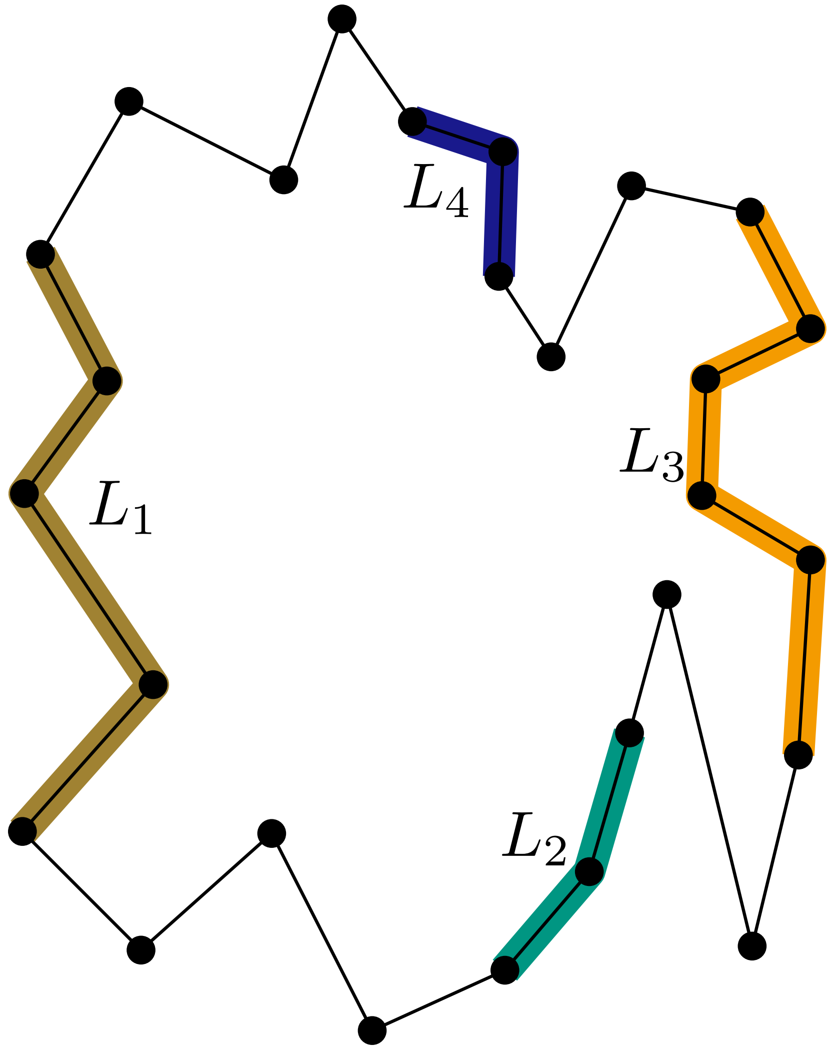
Observation:
- Lists are sorted by y-coordinate.
- $O(1)$ many lists.

1. Apply *merge*-step of Merge-Sort on lists, to obtain *one* list.
2. Insert turn vertices into list maintaining the sorting.

$O(n)$ time, since
$O(1)$ many lists and $O(1)$ many turn vertices.

⟶ Queue $\mathcal{Q}$ can be created in $O(n)$ time.

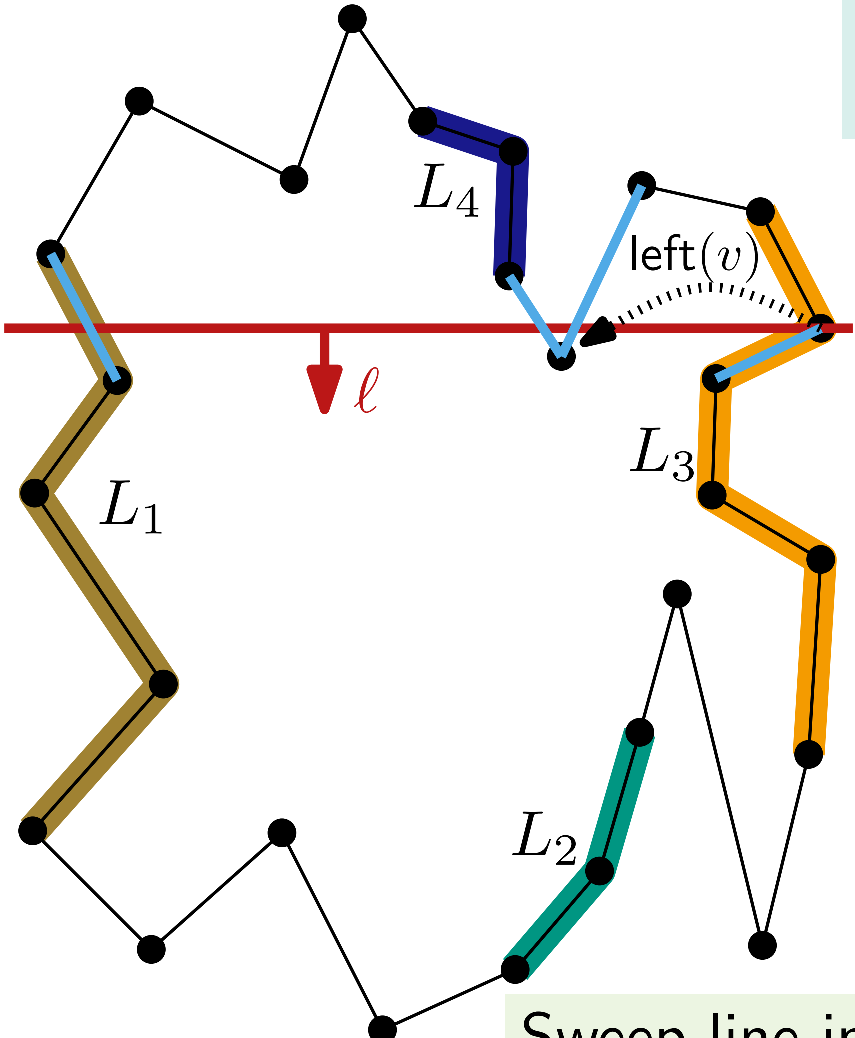
Linear Running Time



Step 2: Replace $\mathcal{T}$.

Task of $\mathcal{T}$: Determine for vertex v the edge left(v) directly left to v.

Linear Running Time



Step 2: Replace $\mathcal{T}$.

Task of $\mathcal{T}$: Determine for vertex v the edge left(v) directly left to v.

Idea: For each vertex v precompute left(v).

Sweep-Line: from top to bottom.
Sweep-State:
Edges that intersect sweep-line
Event: Vertices of polygon.
Determine edge that intersects sweep-line directly left to current node.

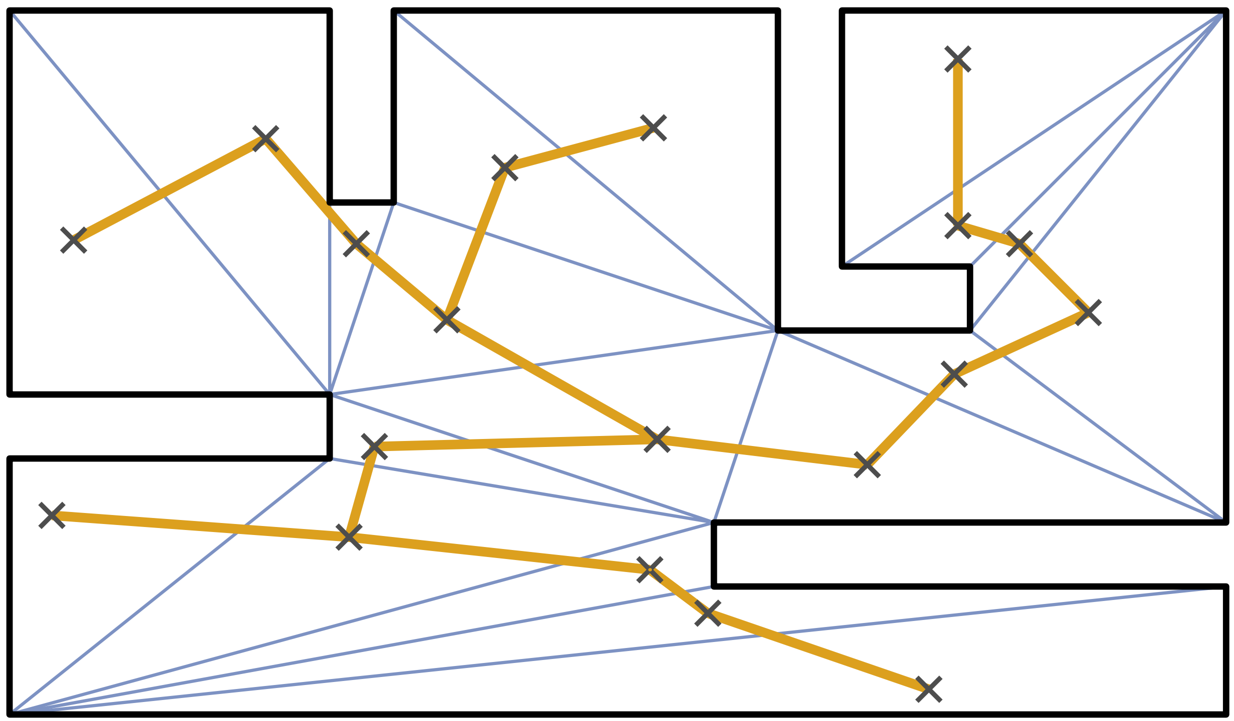
Sweep-line intersects $O(1)$ many edges,
since $O(1)$ many lists and $O(1)$ many turn vertices.

Splitting Polygons.

Given: Polygon P with n vertices.

Find: $O(n \log n)$-Algorithm, that splits P into two simple polygons such that each has at most $\lfloor 2n/3 \rfloor + 2$ vertices.

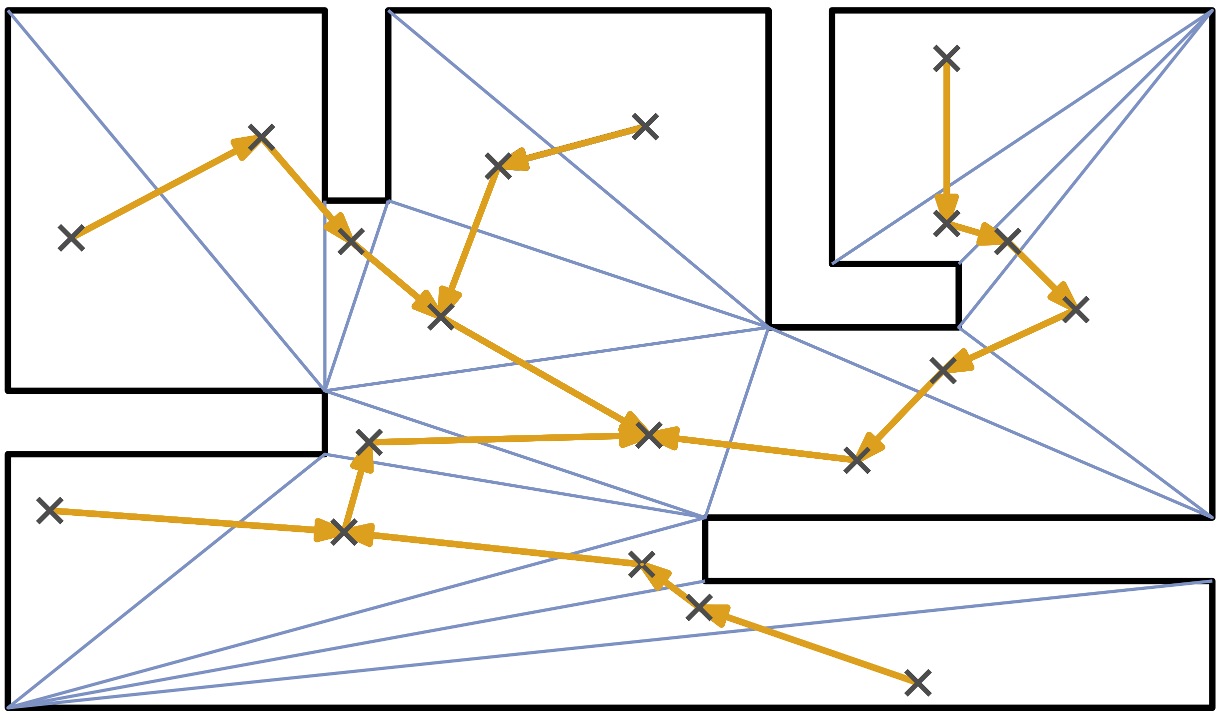
Hint: Triangulate P and make use of the dual graph of the triangulation.



Solution

Initialitation: Each vertex $u \in V$ receives weight $w(u) = 1$.

while $\textsc{True}$ **do**
  Let u be leaf of T
  while u has degree 1 **do**
    if $n - (\lfloor 2n/3 \rfloor + 2) \leq w(u) \leq \lfloor 2n/3 \rfloor + 2$ **then**
      return Sub-tree of u induces desired partition
    $w(parent(u)) \leftarrow w(parent(u)) + w(u)$
    Delete u from T
    $u \leftarrow parent(u)$



$n = 19$

$n - (\lfloor 2n/3 \rfloor + 2) = 5$

$\lfloor 2n/3 \rfloor + 2 = 14$

Annahme:
Tree has root with degree ≥ 2.
Edges are directed to the root.

Solution

Initialitation: Each vertex $u \in V$ receives weight $w(u) = 1$.

while $\textsc{True}$ **do**

- Let u be leaf of T
- **while** u has degree 1 **do**
  - **if** $n - (\lfloor 2n/3 \rfloor + 2) \leq w(u) \leq \lfloor 2n/3 \rfloor + 2$ **then**
    - **return** Sub-tree of u induces desired partition
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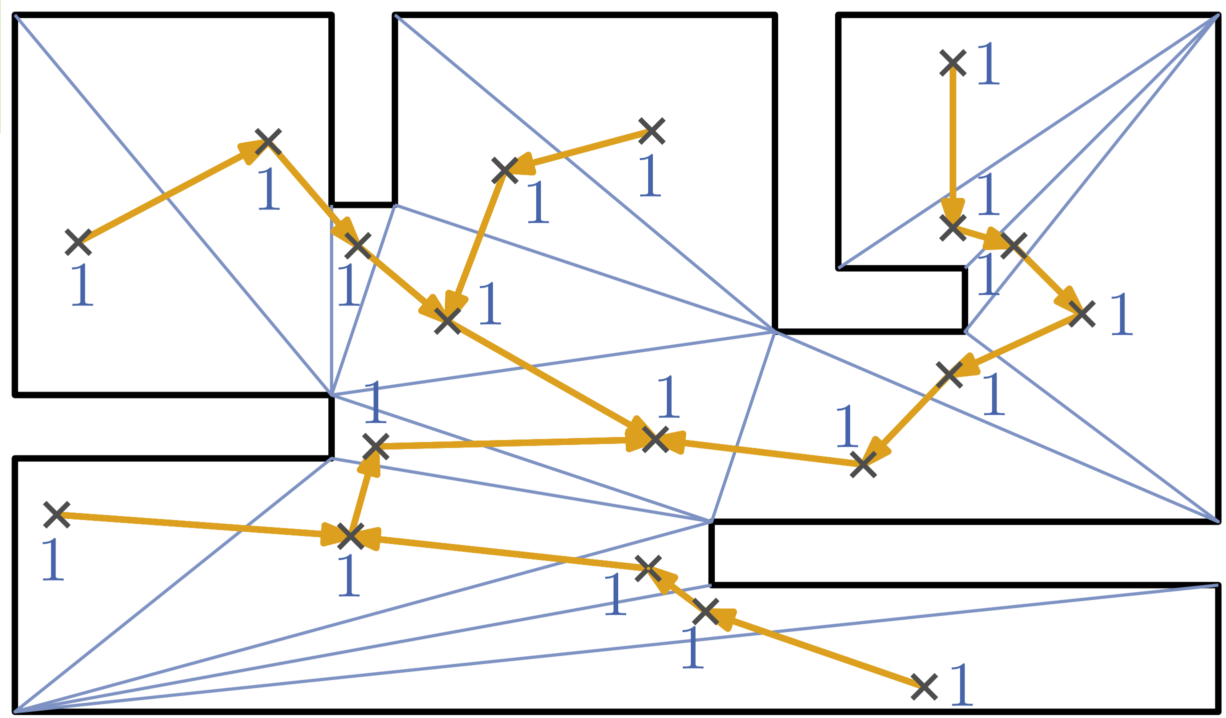

Solution

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while $\textsc{True}$ **do**
  Let u be leaf of T
  while u has degree 1 **do**
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    Delete u from T
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Tree has root with degree ≥ 2.
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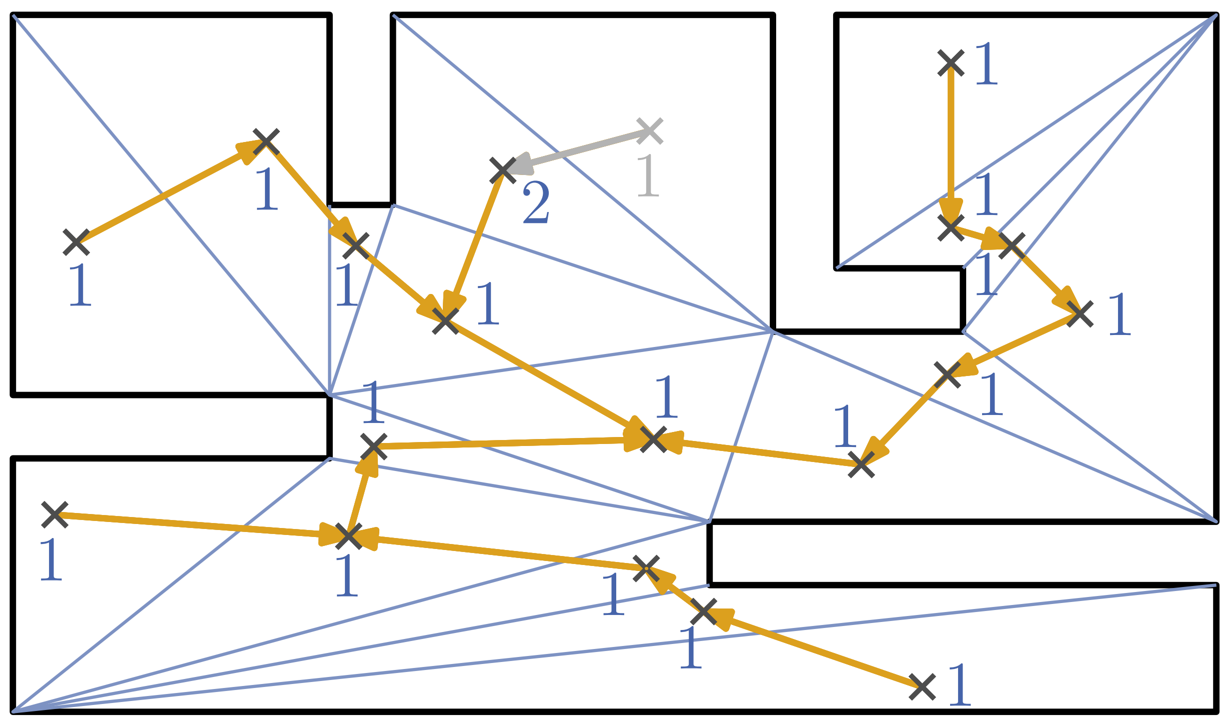

Solution

Initialitation: Each vertex $u \in V$ receives weight $w(u) = 1$.

while $\textsc{True}$ **do**
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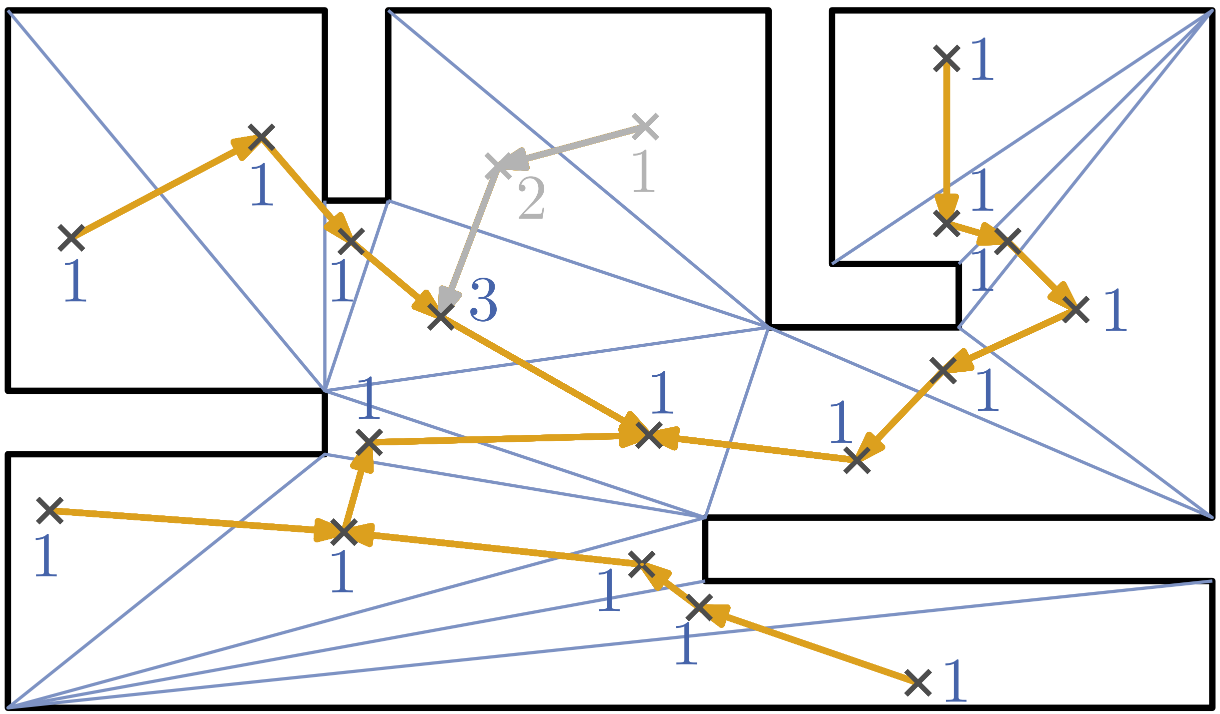
    Delete u from T
    $u \leftarrow parent(u)$

$n = 19$

$n - (\lfloor 2n/3 \rfloor + 2) = 5$

$\lfloor 2n/3 \rfloor + 2 = 14$

Annahme:
Tree has root with degree ≥ 2.
Edges are directed to the root.



Solution

Initialitation: Each vertex $u \in V$ receives weight $w(u) = 1$.

while $\textsc{True}$ **do**
- Let u be leaf of T
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  - **if** $n - (\lfloor 2n/3 \rfloor + 2) \leq w(u) \leq \lfloor 2n/3 \rfloor + 2$ **then**
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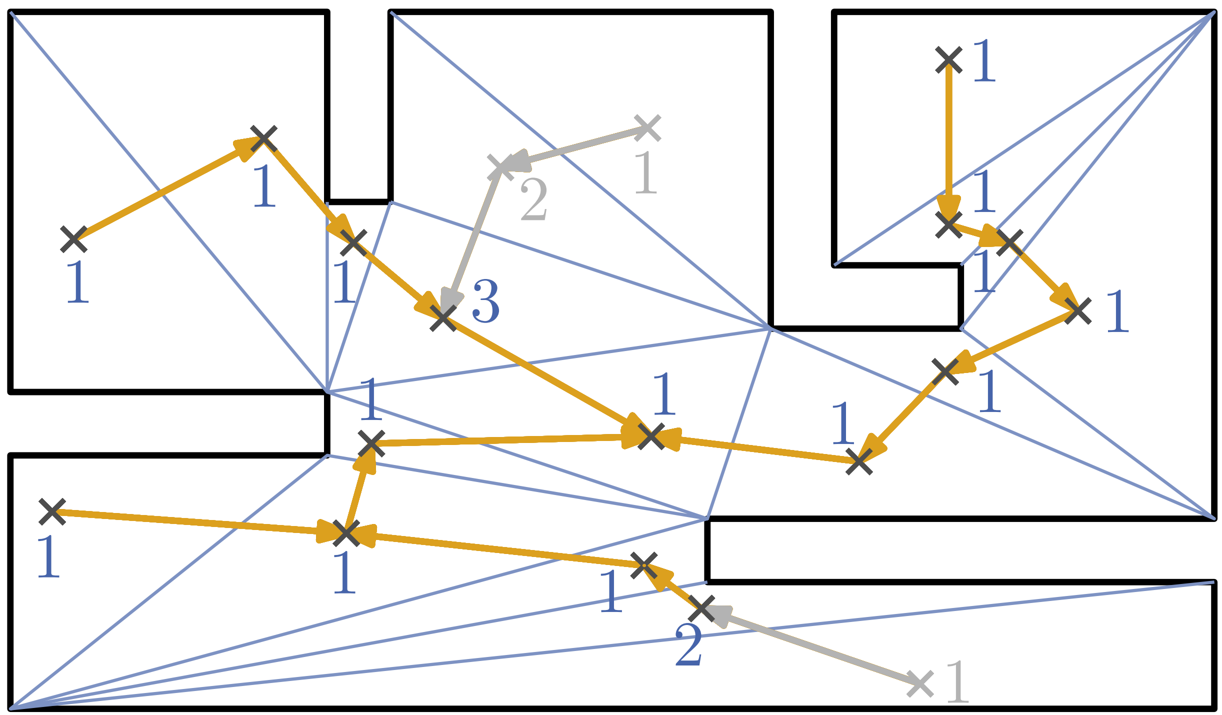
  - $w(parent(u)) \leftarrow w(parent(u)) + w(u)$
  - Delete u from T
  - $u \leftarrow parent(u)$

$n = 19$

$n - (\lfloor 2n/3 \rfloor + 2) = 5$

$\lfloor 2n/3 \rfloor + 2 = 14$

Annahme:
Tree has root with degree ≥ 2.
Edges are directed to the root.



Solution

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$n - (\lfloor 2n/3 \rfloor + 2) = 5$
$\lfloor 2n/3 \rfloor + 2 = 14$

Annahme:
Tree has root with degree ≥ 2.
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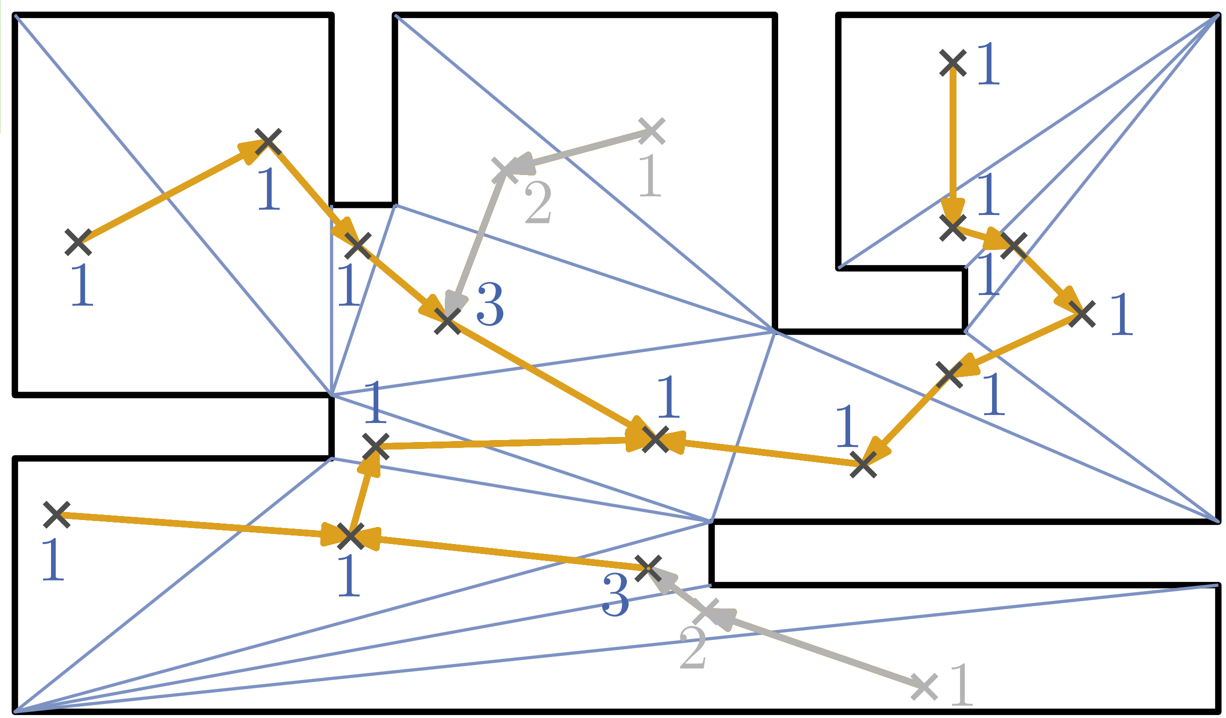

Solution

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  while u has degree 1 **do**

    if $n - (\lfloor 2n/3 \rfloor + 2) \leq w(u) \leq \lfloor 2n/3 \rfloor + 2$ **then**

      return Sub-tree of u induces desired partition

    $w(parent(u)) \leftarrow w(parent(u)) + w(u)$

    Delete u from T

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$n - (\lfloor 2n/3 \rfloor + 2) = 5$

$\lfloor 2n/3 \rfloor + 2 = 14$

Annahme:
Tree has root with degree ≥ 2.
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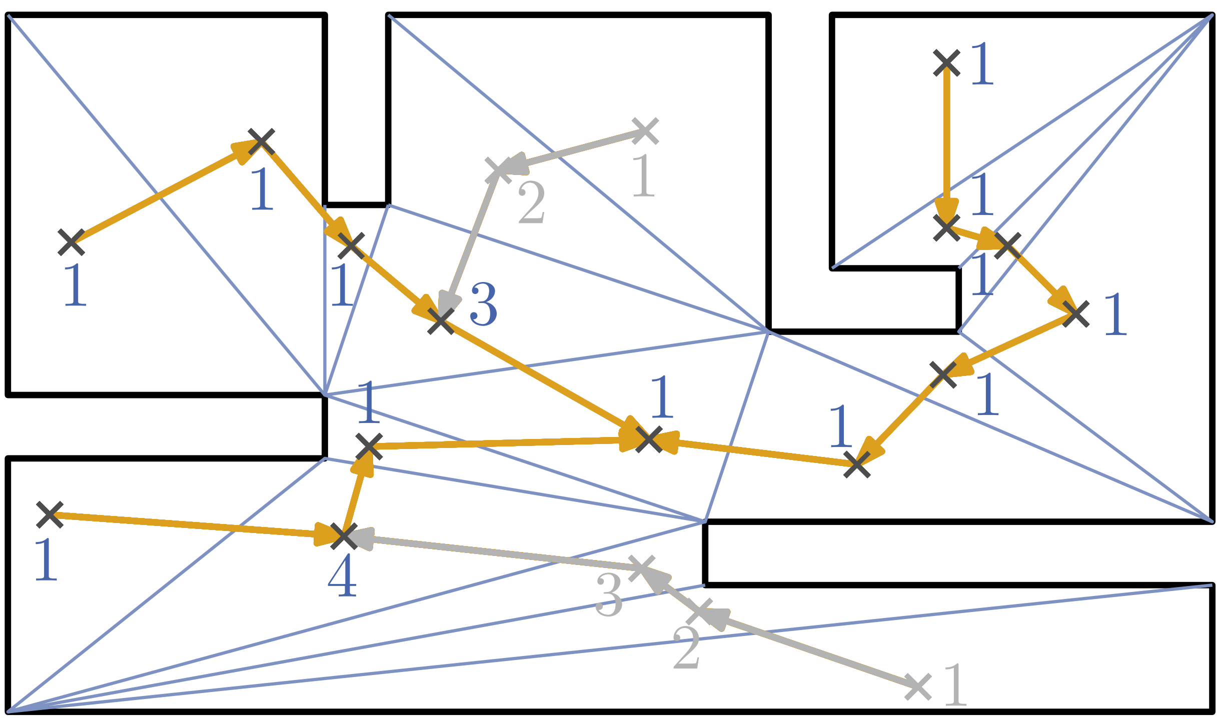

Solution

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$n - (\lfloor 2n/3 \rfloor + 2) = 5$

$\lfloor 2n/3 \rfloor + 2 = 14$

Annahme:
Tree has root with degree ≥ 2.
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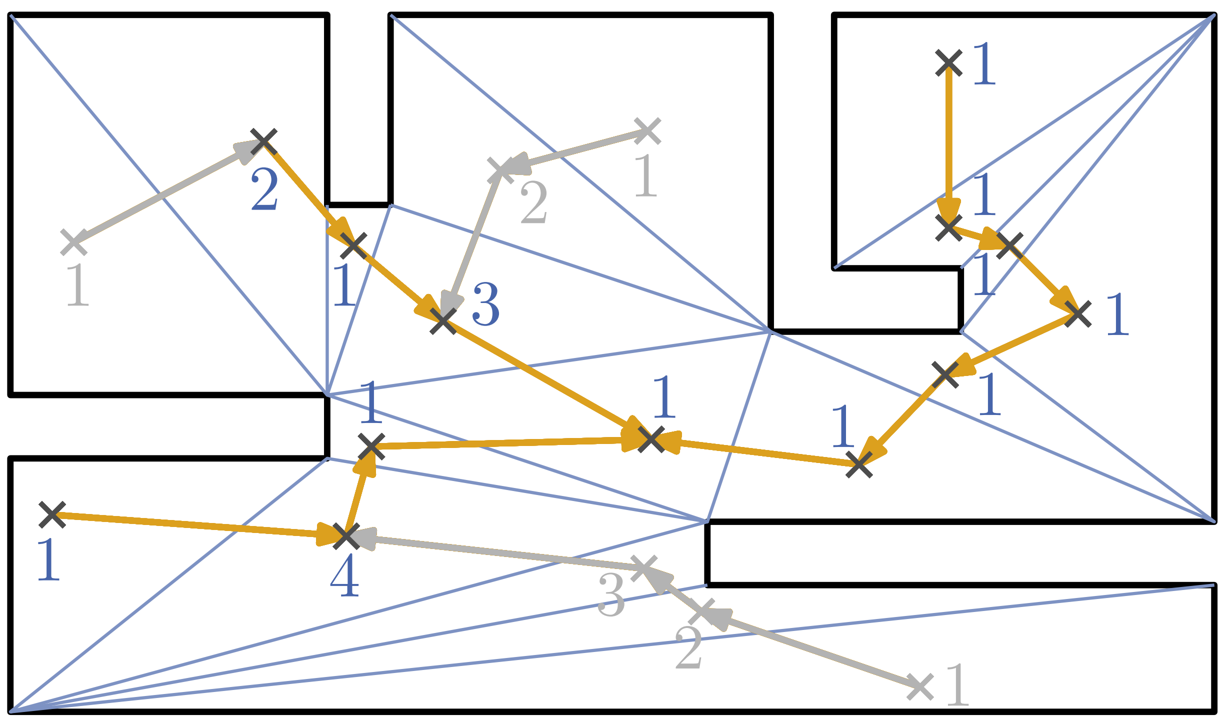

Solution

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while $\textsc{True}$ **do**

  Let u be leaf of T

  while u has degree 1 **do**

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    Delete u from T

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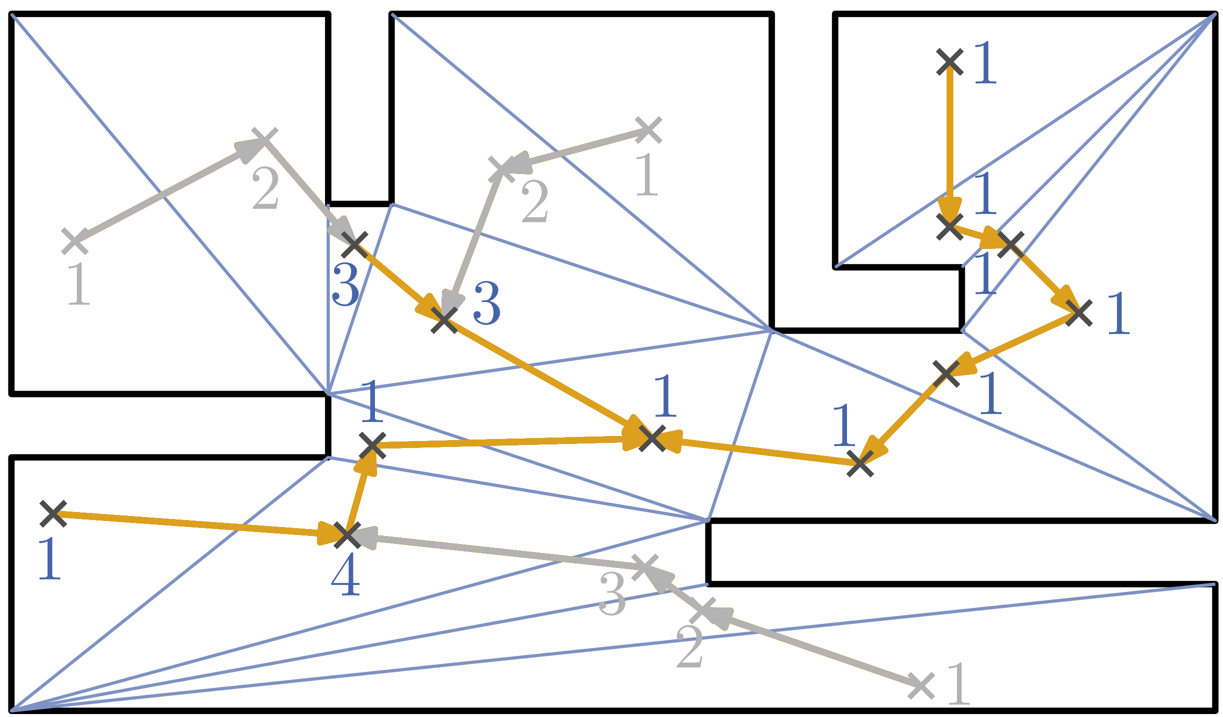

$n = 19$

$n - (\lfloor 2n/3 \rfloor + 2) = 5$

$\lfloor 2n/3 \rfloor + 2 = 14$

Annahme:
Tree has root with degree ≥ 2.
Edges are directed to the root.

Solution

Initialitation: Each vertex $u \in V$ receives weight $w(u) = 1$.

while $\textsc{True}$ **do**
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  - **if** $n - (\lfloor 2n/3 \rfloor + 2) \leq w(u) \leq \lfloor 2n/3 \rfloor + 2$ **then**
    - **return** Sub-tree of u induces desired partition
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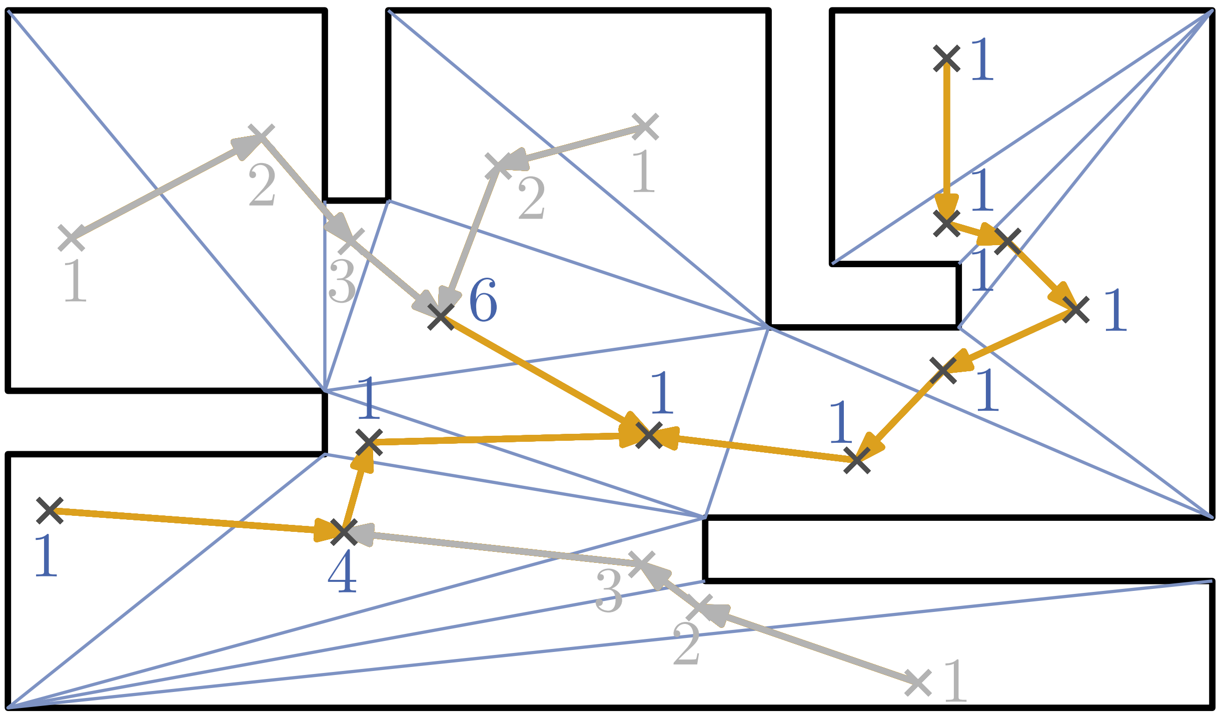
  - $u \leftarrow parent(u)$

$n = 19$

$n - (\lfloor 2n/3 \rfloor + 2) = 5$

$\lfloor 2n/3 \rfloor + 2 = 14$



Annahme:
Tree has root with degree ≥ 2.
Edges are directed to the root.

Solution

Initialitation: Each vertex $u \in V$ receives weight $w(u) = 1$.

while $\textsc{True}$ **do**
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$n - (\lfloor 2n/3 \rfloor + 2) = 5$

$\lfloor 2n/3 \rfloor + 2 = 14$

Annahme:
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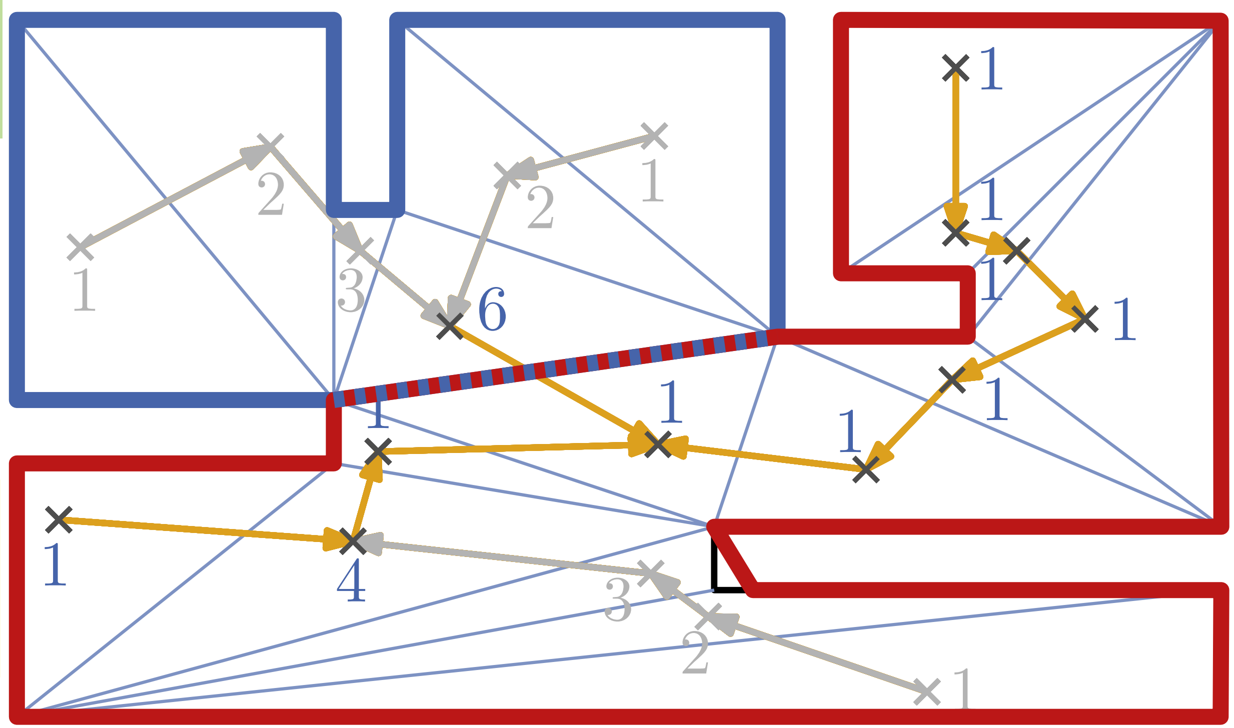

Linear programming

Definition: Given a set of linear constraints H and a linear objective function c in $\mathbb{R}^d$, a **linear program** (LP) is formulated as follows:

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$$

- H is a set of half-spaces in $\mathbb{R}^d$.
- We are searching for a point $x \in \bigcap_{h \in H} h$, that maximizes $c^T x$, i.e. $\max\{c^\mathrm{T} x \mid Ax \leq b, x \geq 0\}$.
- Linear programming is a central method in operations research.

Algorithms for LPs

There are many algorithms to solve LPs:

- Simplex-Algorithm [Dantzig, 1947]
- Ellipsoid-Method [Khatchiyan, 1979]
- Interior-Point-Method [Karmarkar, 1979]

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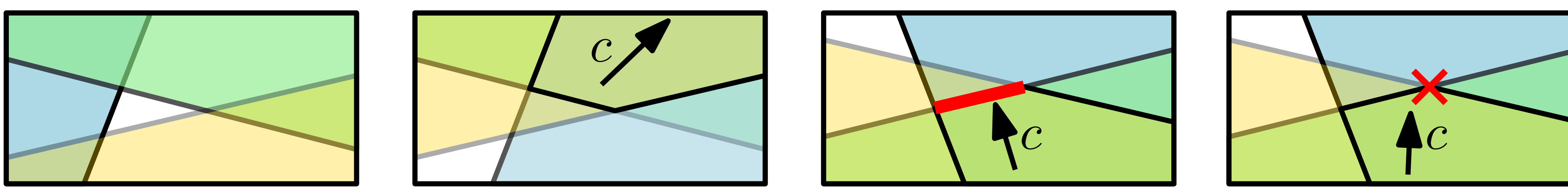
Today: Special case $d = 2$

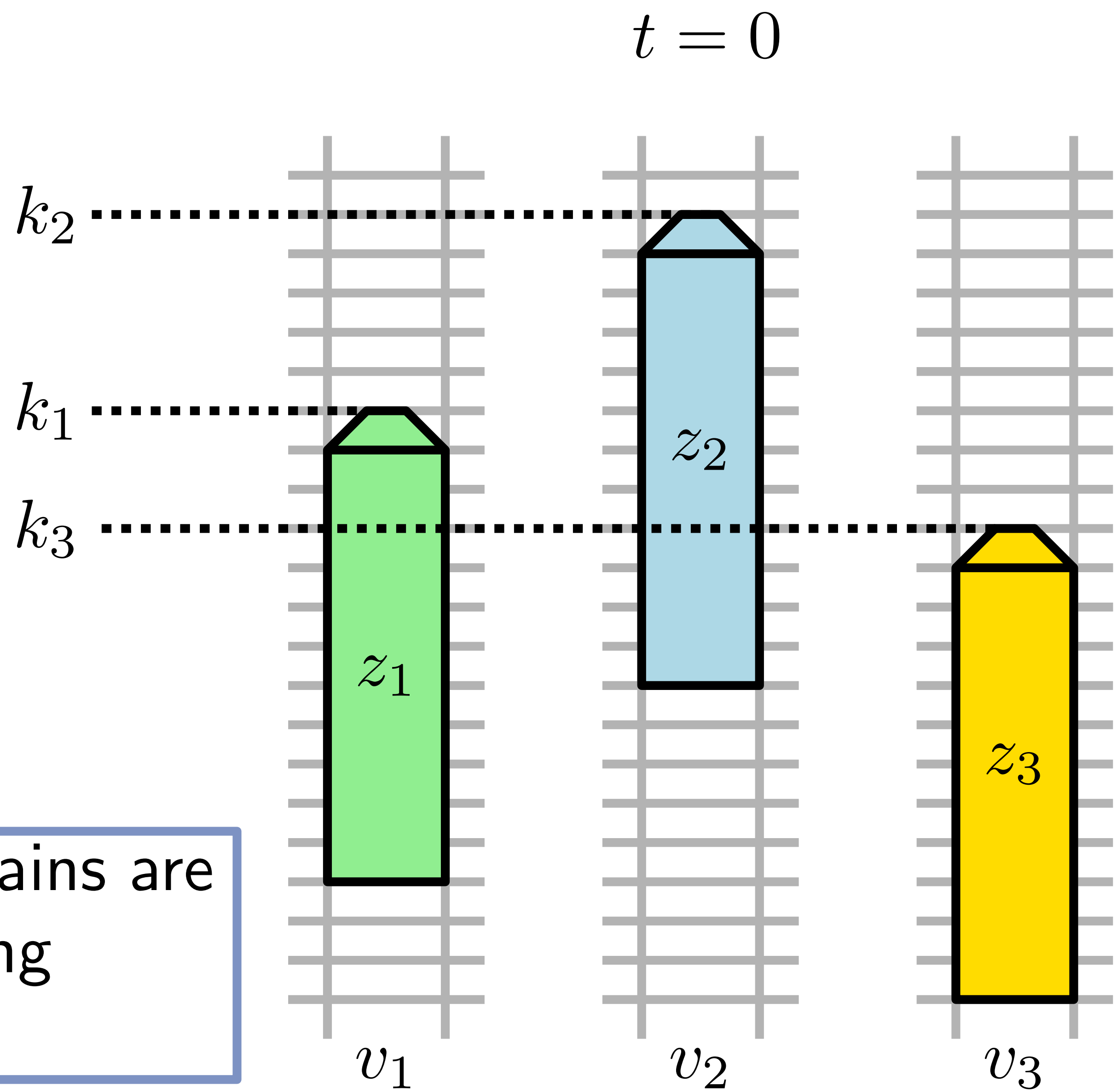
Possibilities for the solution space

feasible region $\bigcap H$ is bounded



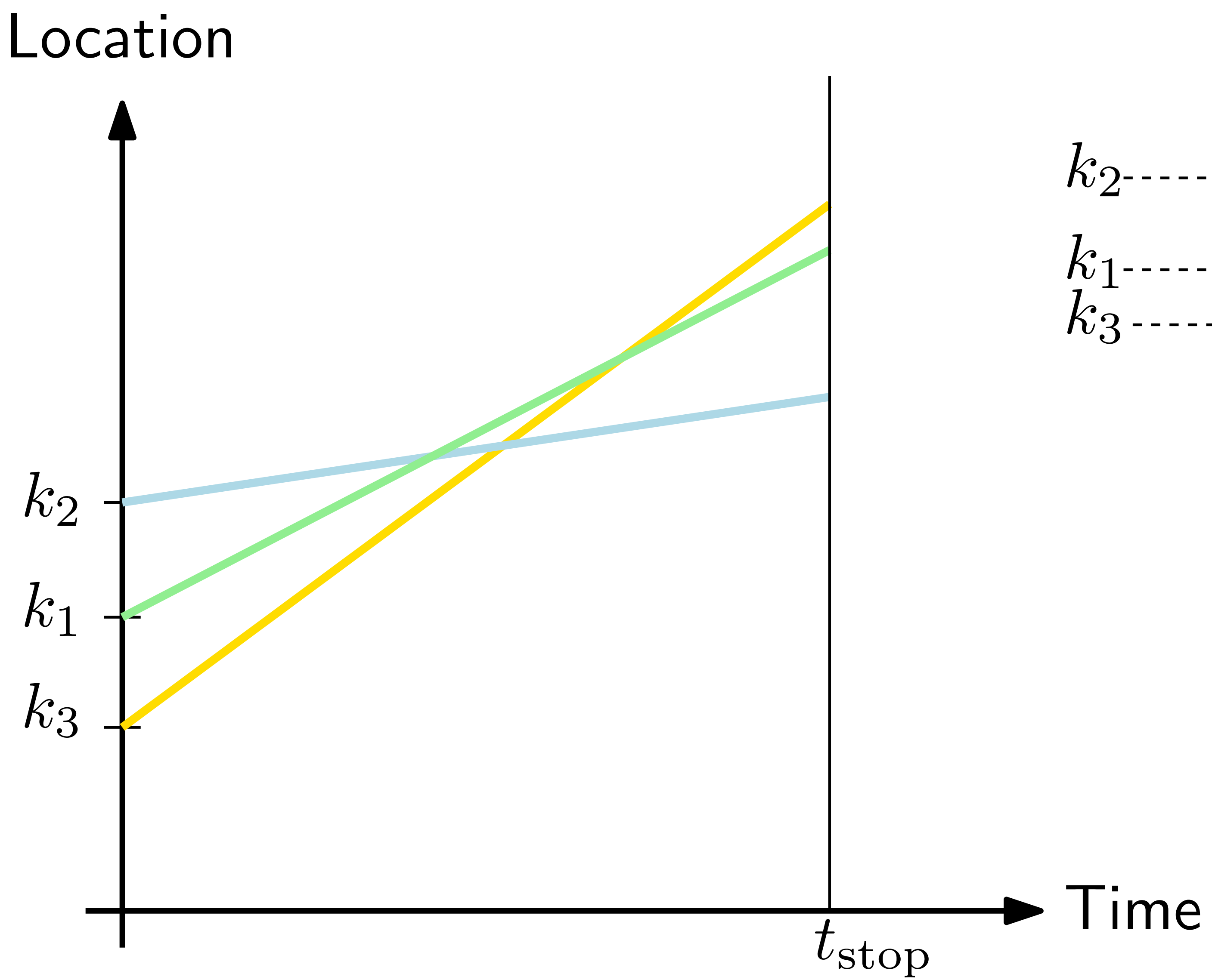
$\bigcap H = \emptyset$ infeasible | $\bigcap H$ is unbounded in the direction c | solution is not unique | unique solution

Trains



Algorithm: Which trains are at least once in leading position until time t_s

Trains



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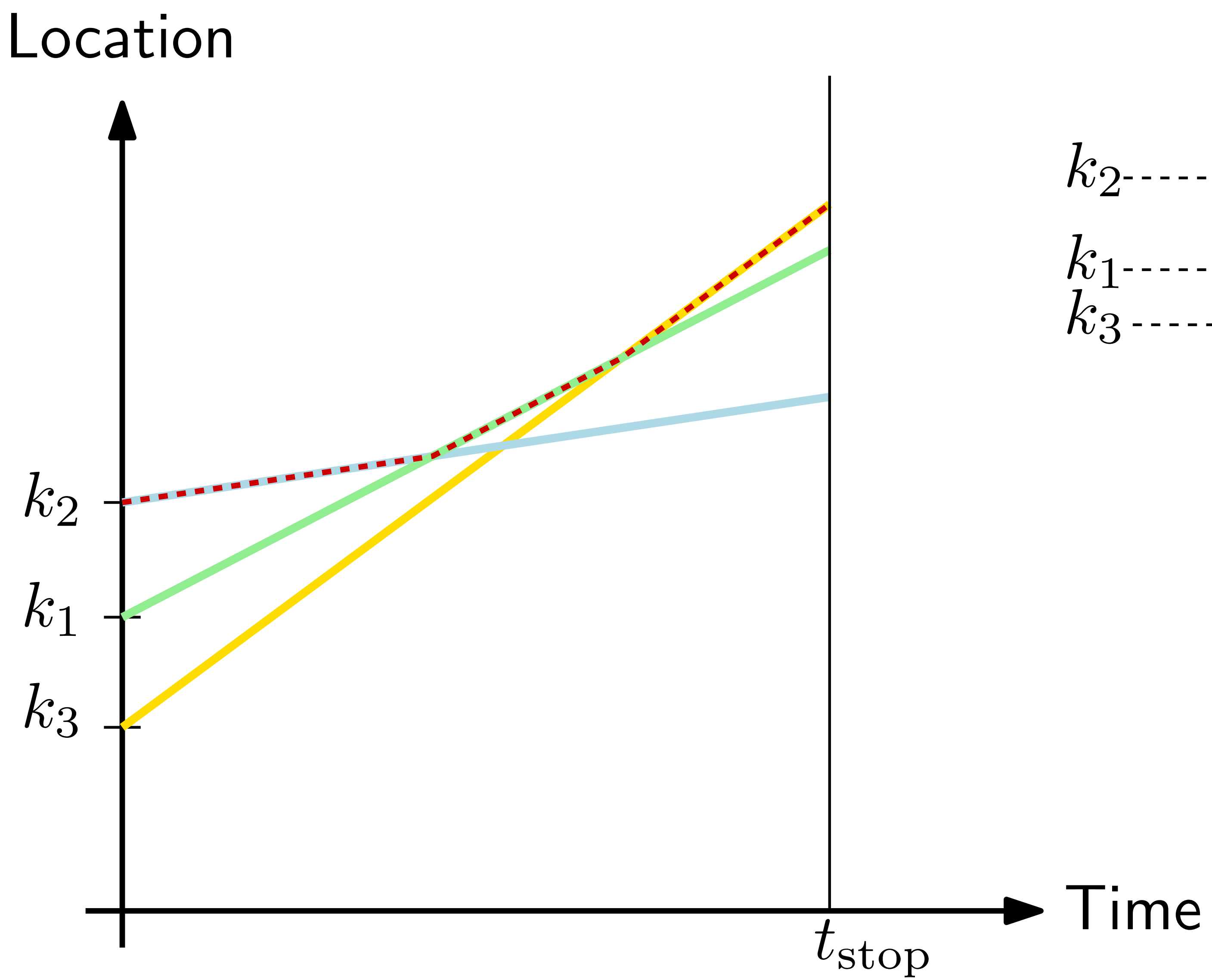

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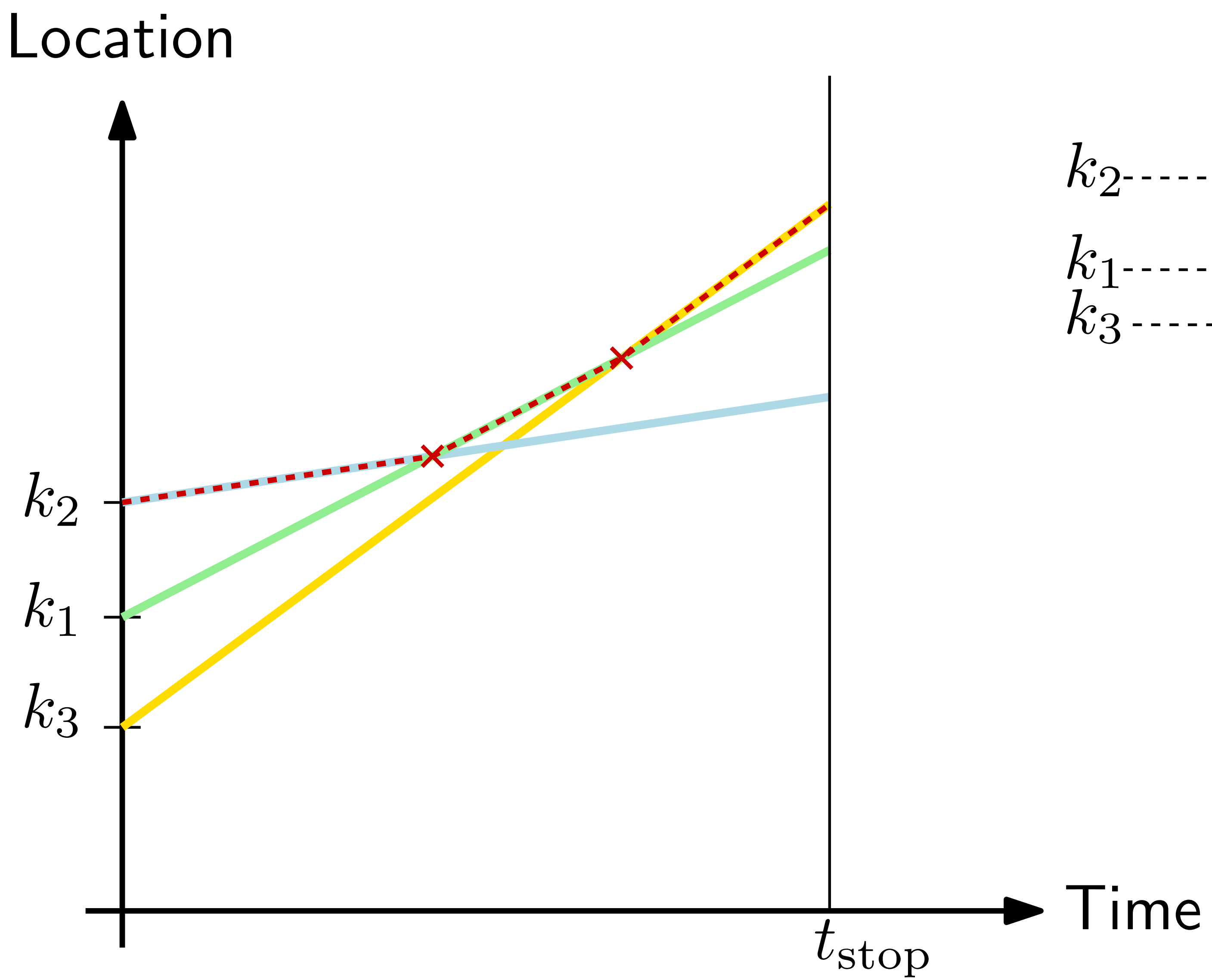

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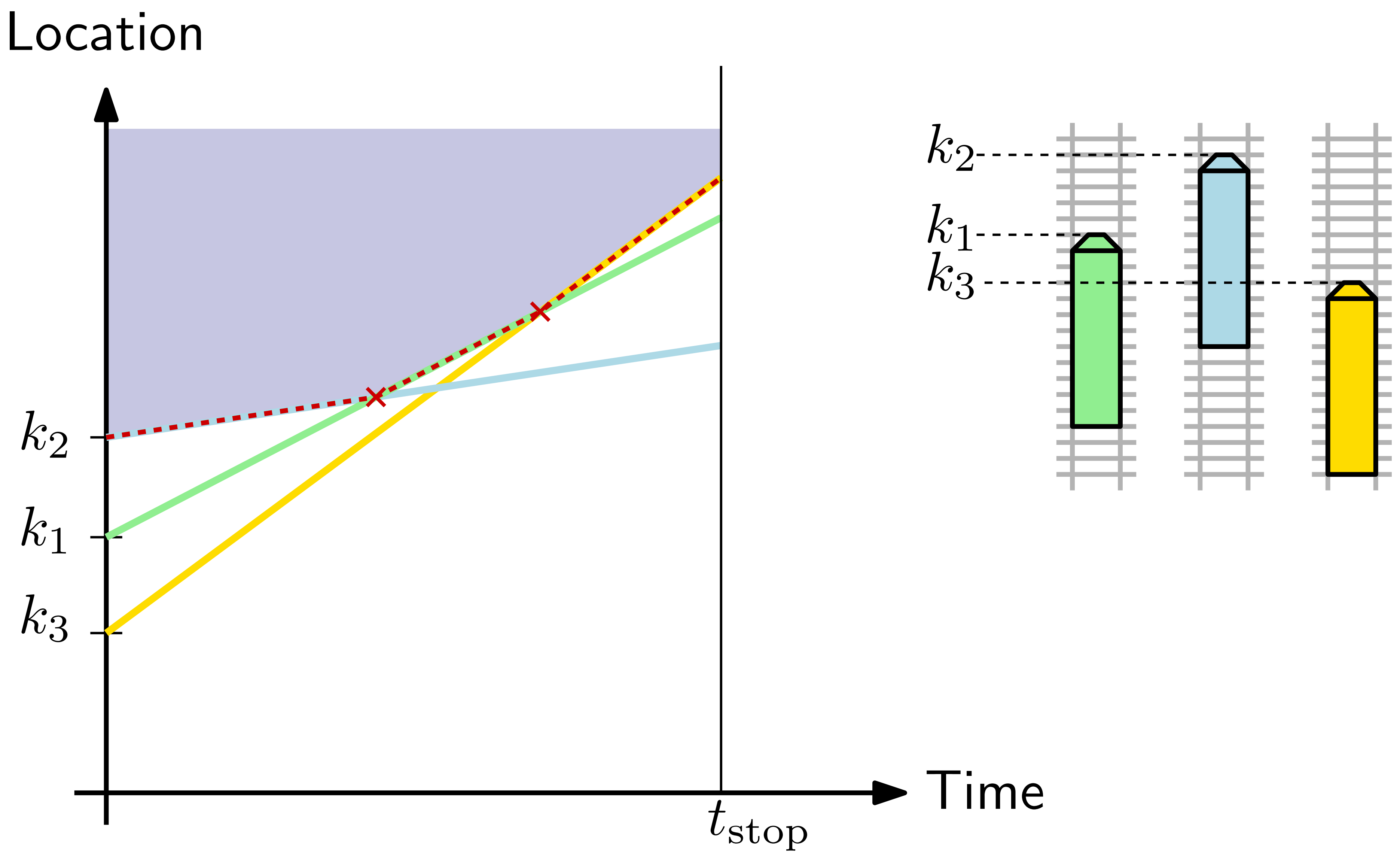

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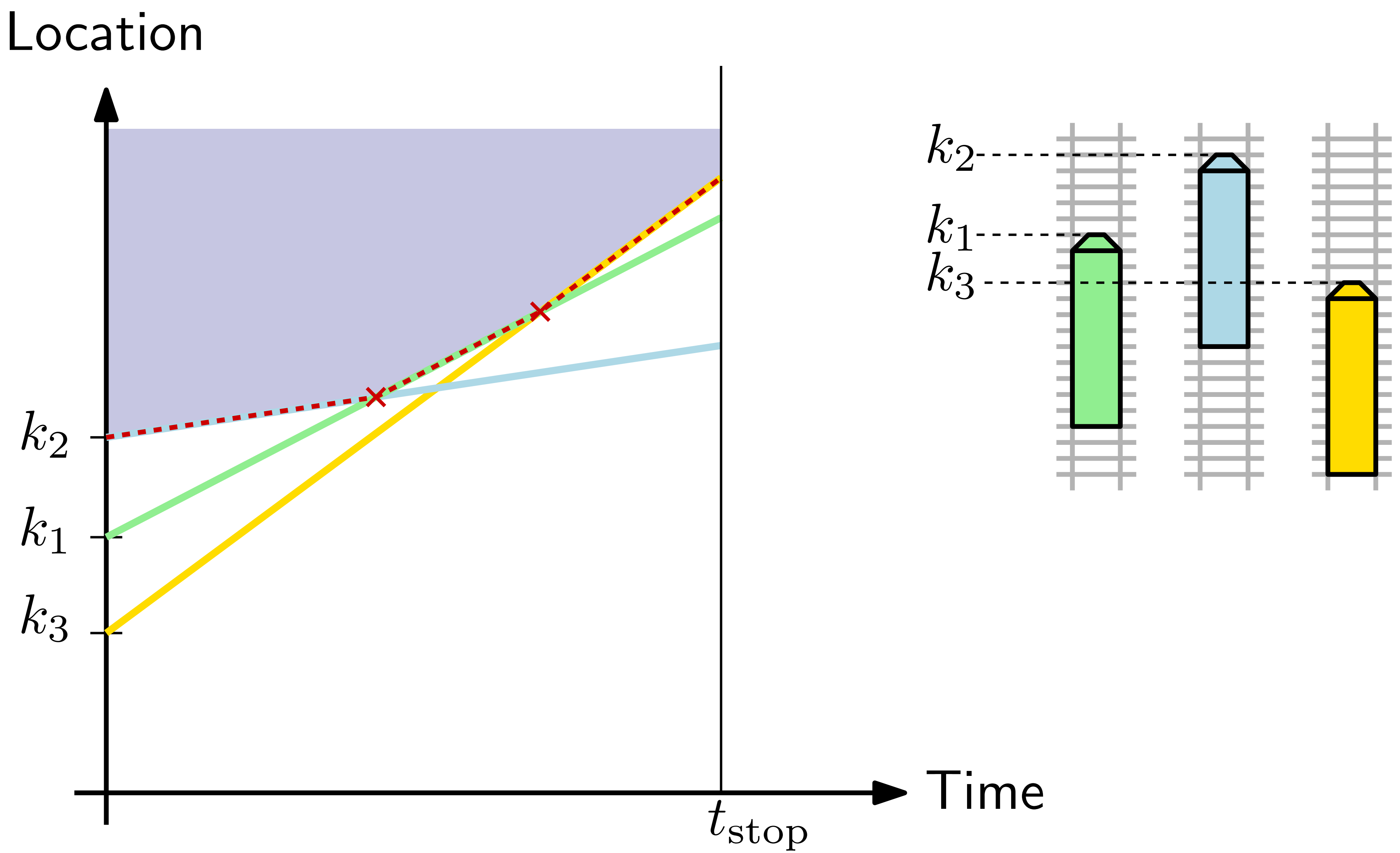

First approach

Idea: Compute the feasible region $\bigcap H$ and search for the vertex p, that maximizes $c^T p$.

- The half-planes are convex
- Let's try a simple Divide-and-Conquer Algorithm

IntersectHalfplanes(H)

if $|H| = 1$ **then**
    $C \leftarrow H$
else
    $(H_1, H_2) \leftarrow$ SplitInHalves(H)
    $C_1 \leftarrow$ IntersectHalfplanes(H_1)
    $C_2 \leftarrow$ IntersectHalfplanes(H_2)
    $C \leftarrow$ IntersectConvexRegions(C_1, C_2)
return C

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Idea: Instead of computing the feasible region and then searching for the optimal angle, do this incrementally.

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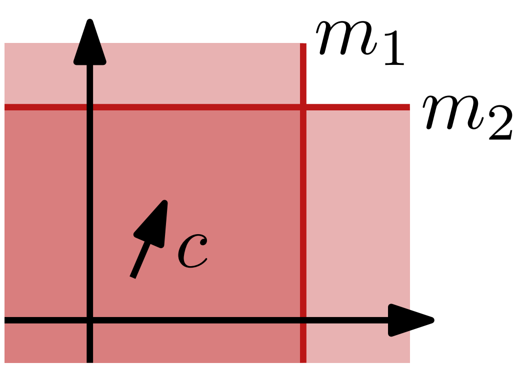
How to deal with the unbounded feasible regions?

Define two half-planes for a big enough value M

$$m_1 = \begin{cases} x \leq M & \text{if } c_x > 0 \\ -x \leq M & \text{otherwise} \end{cases} \qquad m_2 = \begin{cases} y \leq M & \text{if } c_y > 0 \\ -y \leq M & \text{otherwise} \end{cases}$$



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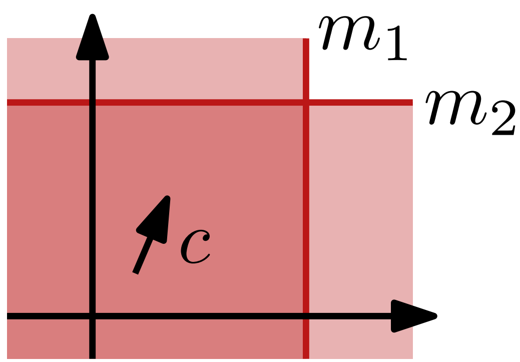

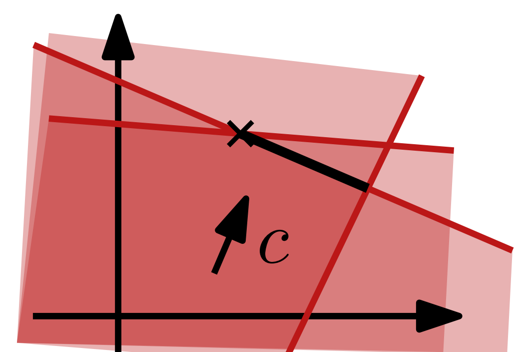

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Consider a LP (H, c) with $H = \{h_1, \ldots, h_n\}$, $c = (c_x, c_y)$. We denote the first i constraints by $H_i = \{m_1, m_2, h_1, \ldots, h_i\}$, and the feasible polygon defineed by them by $C_i = m_1 \cap m_2 \cap h_1 \cap \cdots \cap h_i$

Properties

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Lemma 1: For $1 \leq i \leq n$ and bounding line ℓ_i of h_i holds that:

1. If $v_{i-1} \in h_i$ then $v_i = v_{i-1}$,
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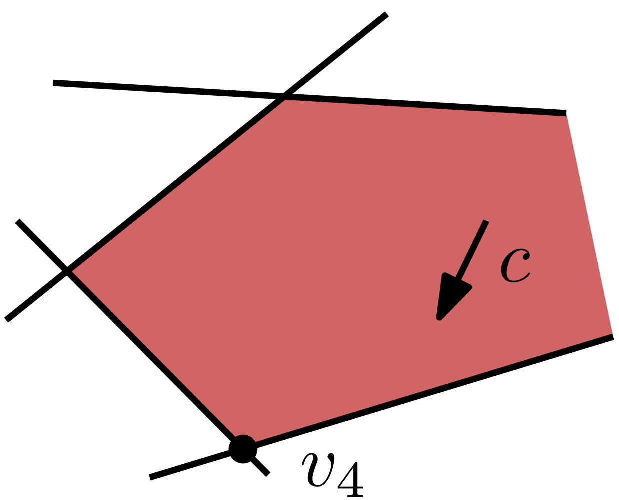

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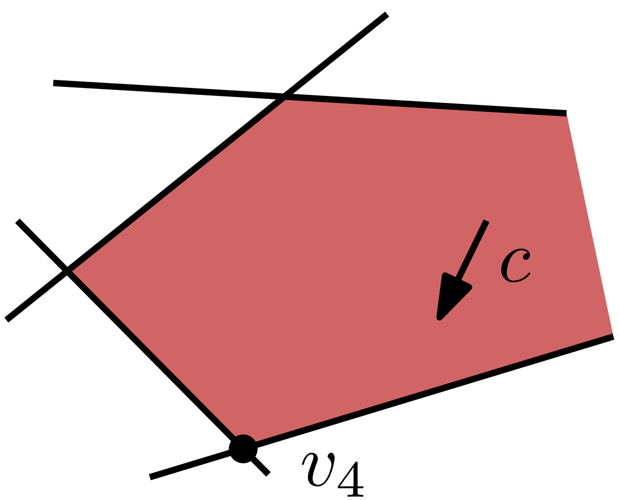

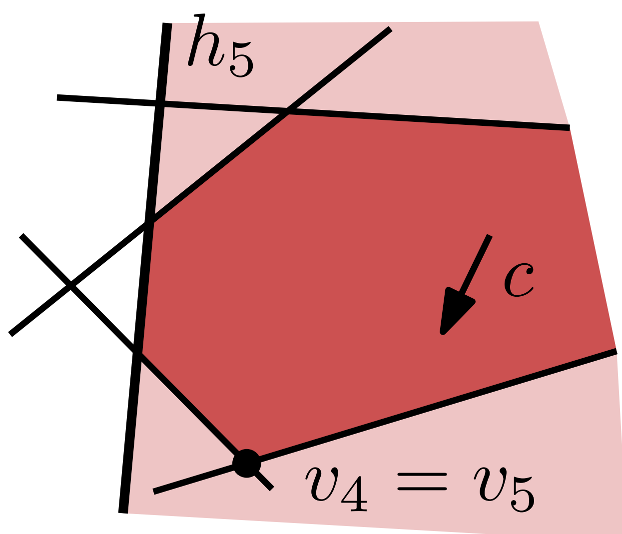

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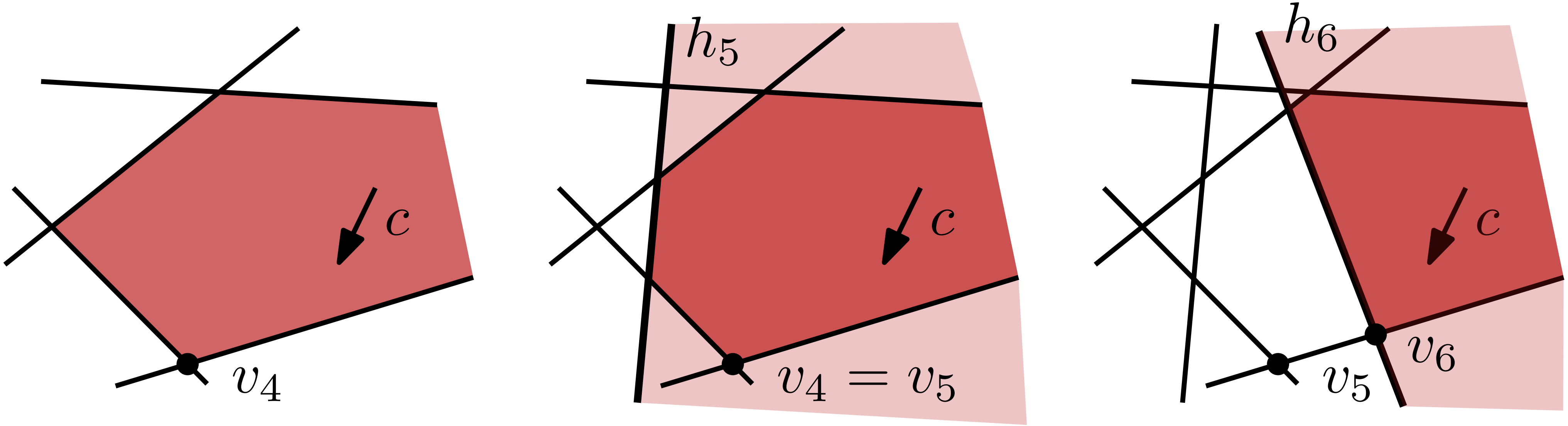

Randomized incremental algorithm

2dRandomizedBoundedLP(H, c, m_1, m_2)

$C_0 \leftarrow m_1 \cap m_2$
$v_0 \leftarrow$ unique angle of C_0
$H \leftarrow$ **RandomPermutation**(H)
for $i \leftarrow 1$ **to** n **do**
  if $v_{i-1} \in h_i$ **then**
    $v_i \leftarrow v_{i-1}$
  else
    $v_i \leftarrow$ 1dBoundedLP$(\sigma(H_{i-1}), f_c^i)$
    if $v_i =$ nil **then**
      return infeasible
return v_n

Fisher-Yates Shuffle

Proof of Correctness:

a) Prove that each permutation of A has the same probability to be chosen.

RandomPermutation(A)

Input: Array $A[1 \ldots n]$
Output: Array A, rearranged into a random permutation
for $k \leftarrow n$ **to** 2 **do**
  $r \leftarrow$ Random(k)
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b) Prove, that the stamtent of a) if not true, if we replace k by n in the second line.

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Proof by induction:

- $A[1]$ is uniformly distributed
- $A[1, \ldots, n-1]$ is uniformly distributed
- $A[n]$ is chosen uniformly at random

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The permutations of A are not chosen with the same probability.

- the algorithm uniformly generates n^n (non-distinct) permutations
- there are $n!$ distinct permutations
- since $n-1$ does not divide n, n^n is not a multiple of $n!$