Computational Geometry • Lecture

Duality of Points and Lines

INSTITUTE FOR THEORETICAL INFORMATICS · FACULTY OF INFORMATICS

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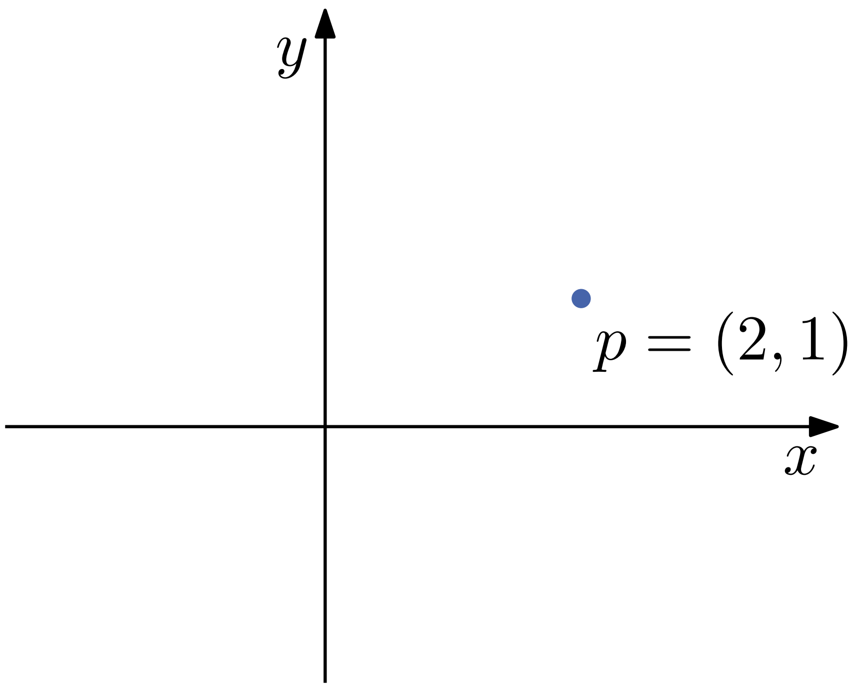
13.7.2018

Duality Transforms

We have seen duality for planar graphs and duality of Voronoi diagrams and Delaunay triangulations. Here we will see a duality of points and lines in $\mathbb{R}^2$.



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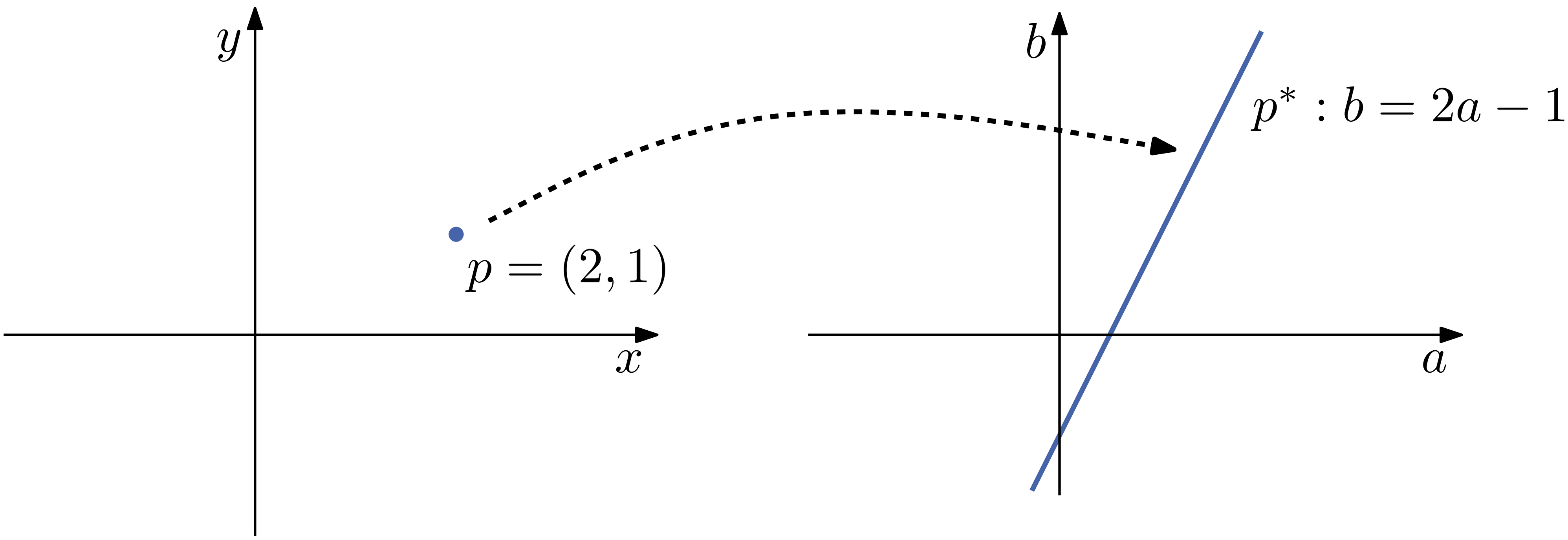

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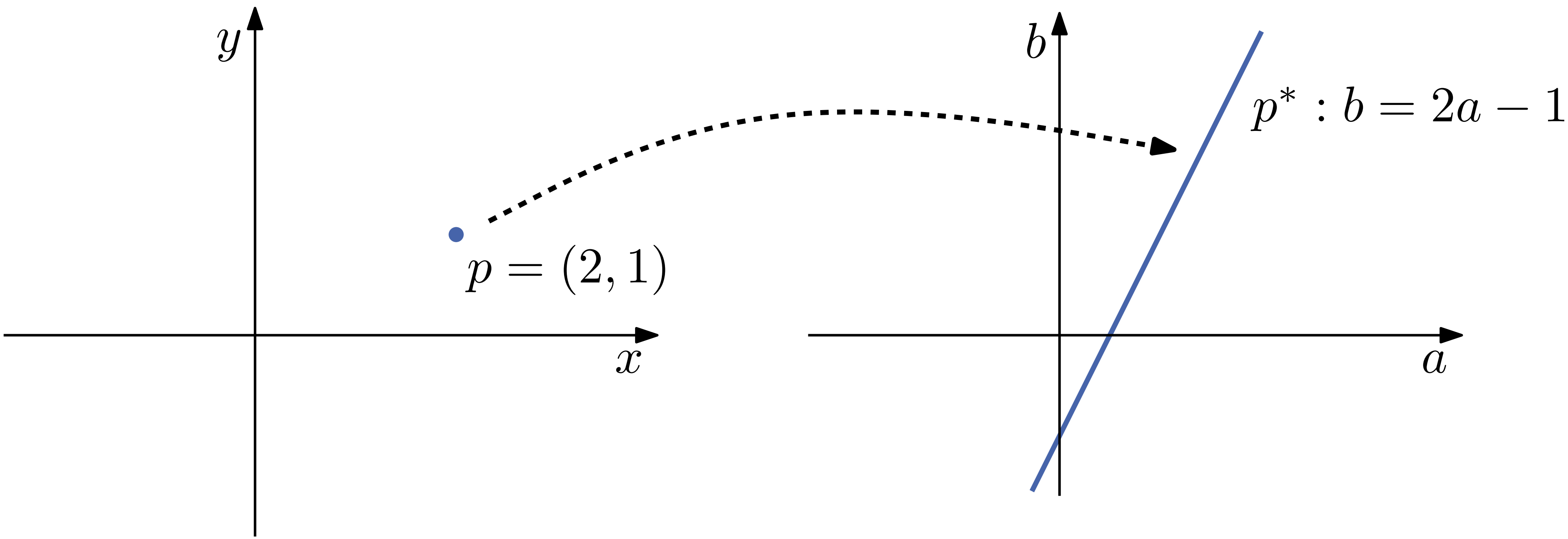

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$$p = (p_x, p_y) \quad \mapsto \quad p^* : b = p_x a - p_y$$

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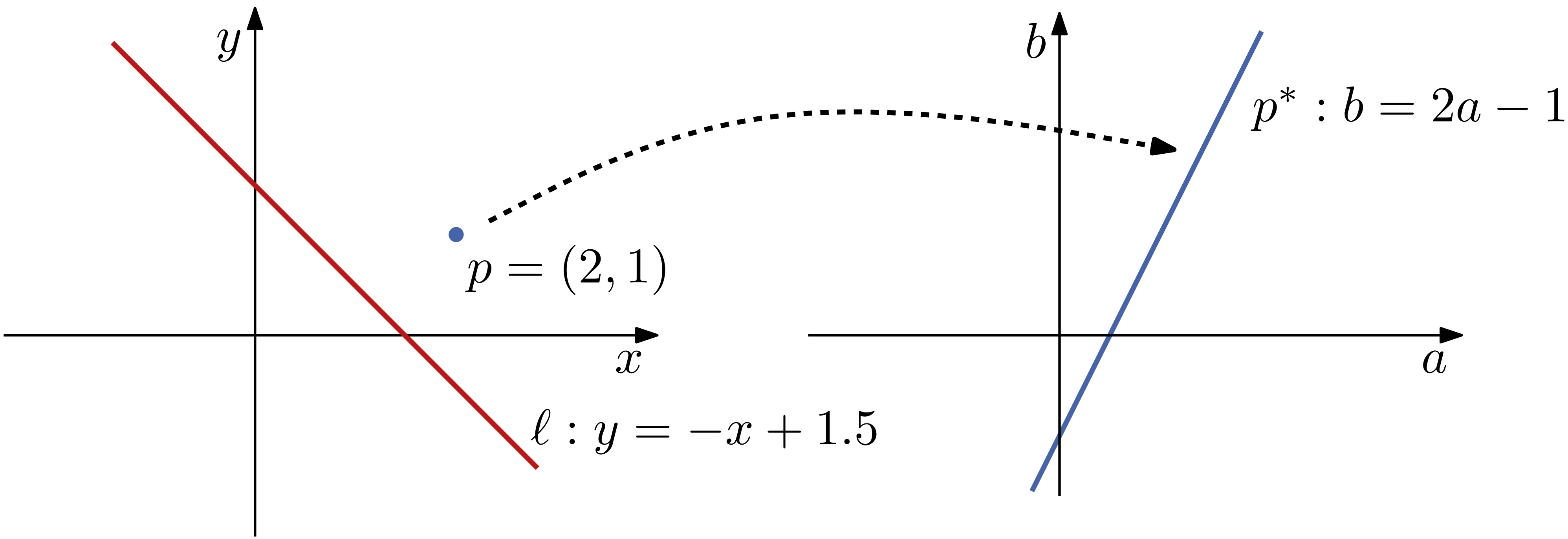

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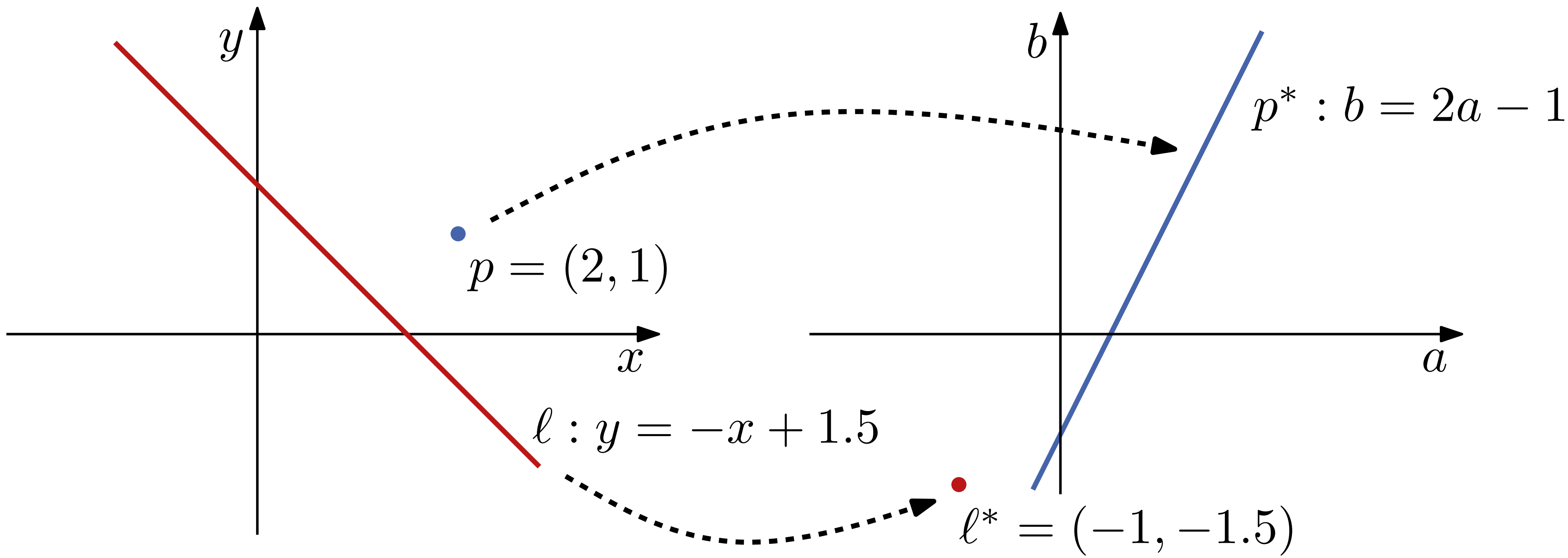

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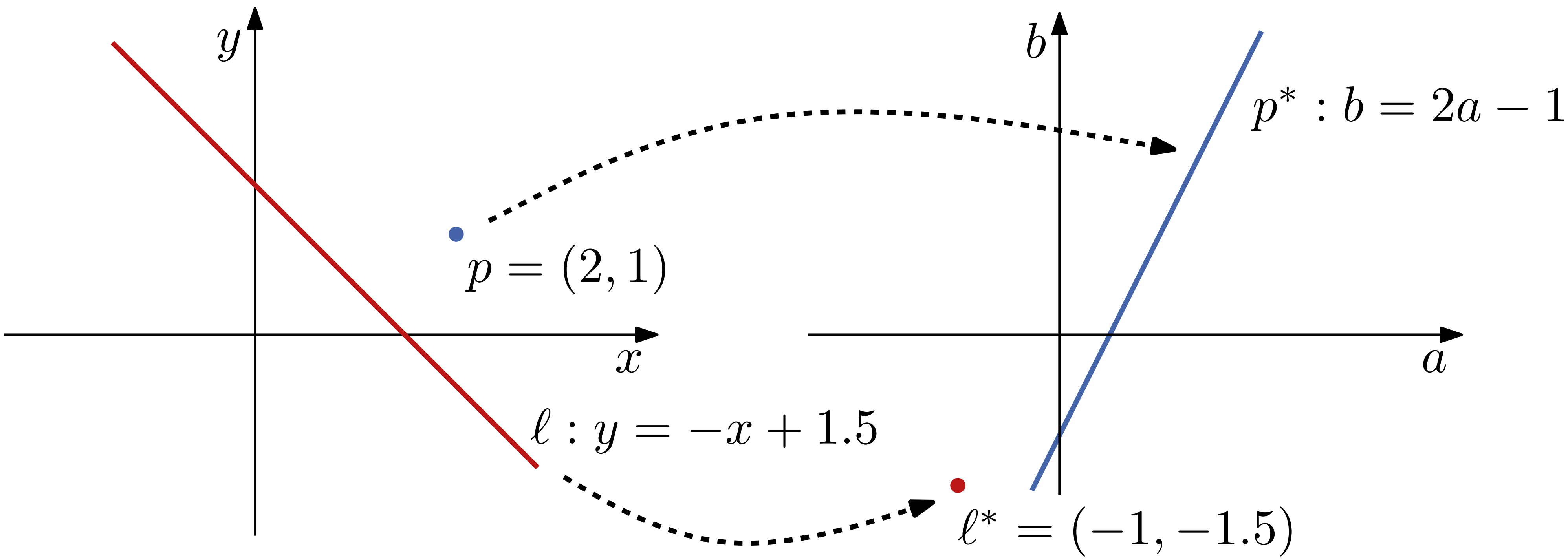

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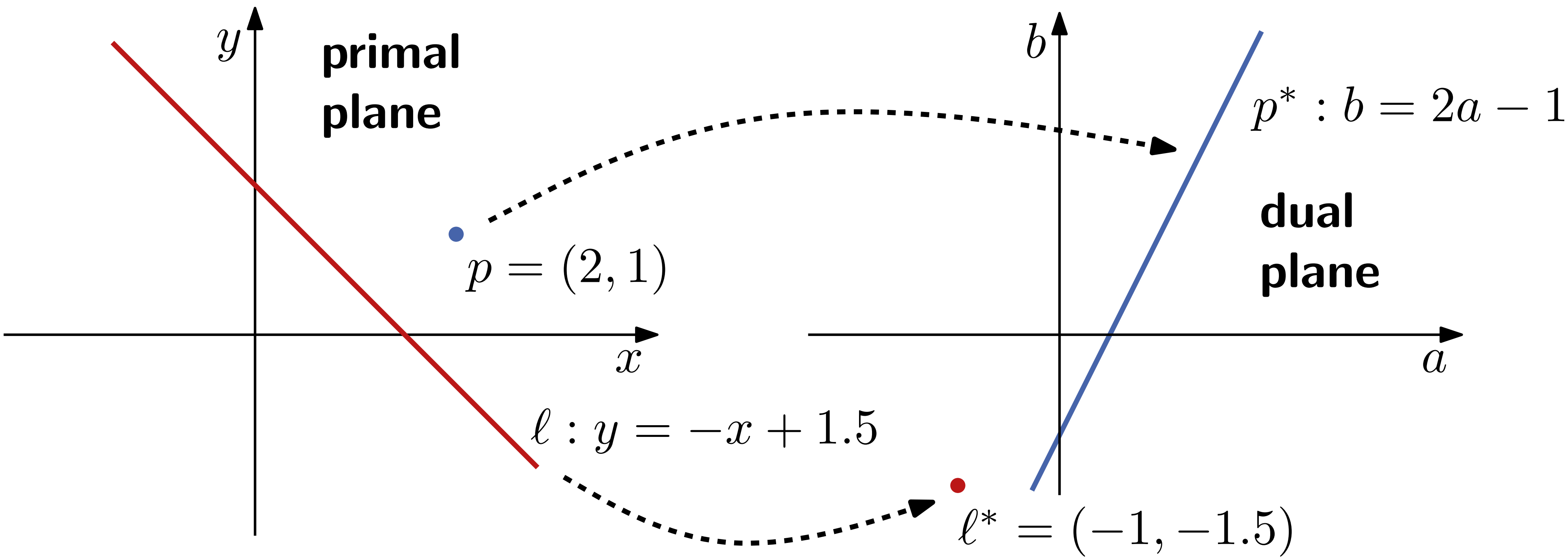

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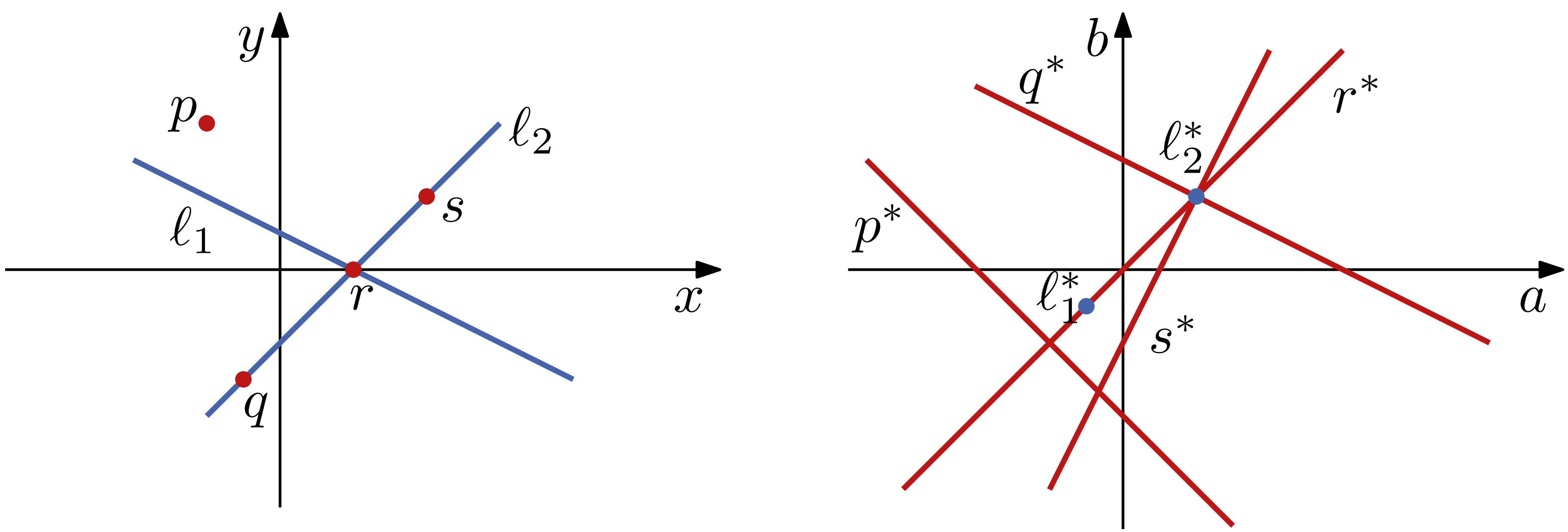
Properties

Lemma 1: The following properties hold

- $(p^*)^* = p$ and $(\ell^*)^* = \ell$
- p lies below/on/above $\ell \Leftrightarrow p^*$ passes above/through/below ℓ^*
- ℓ_1 and ℓ_2 intersect in point r
  $\Leftrightarrow r^*$ passes through ℓ_1^* and ℓ_2^*
- q, r, s collinear
  $\Leftrightarrow q^*, r^*, s^*$ intersect in a common point



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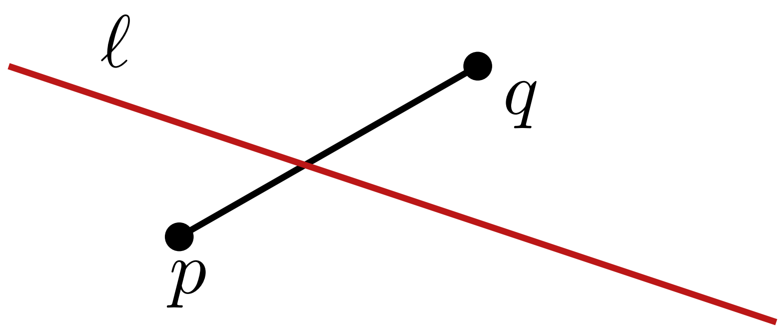

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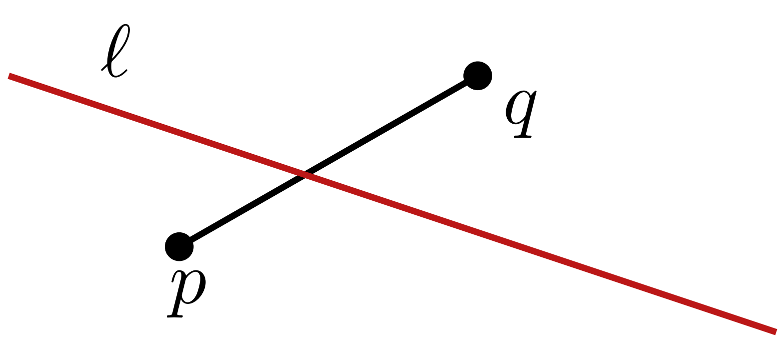
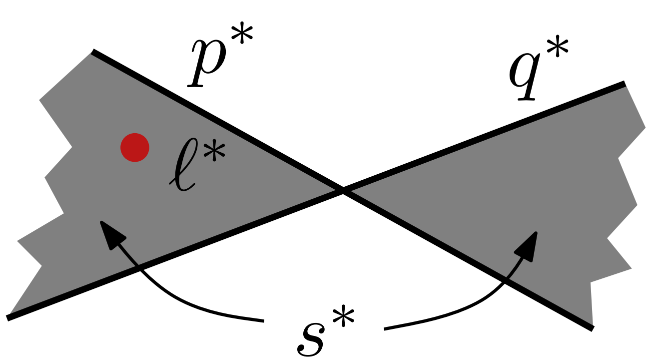




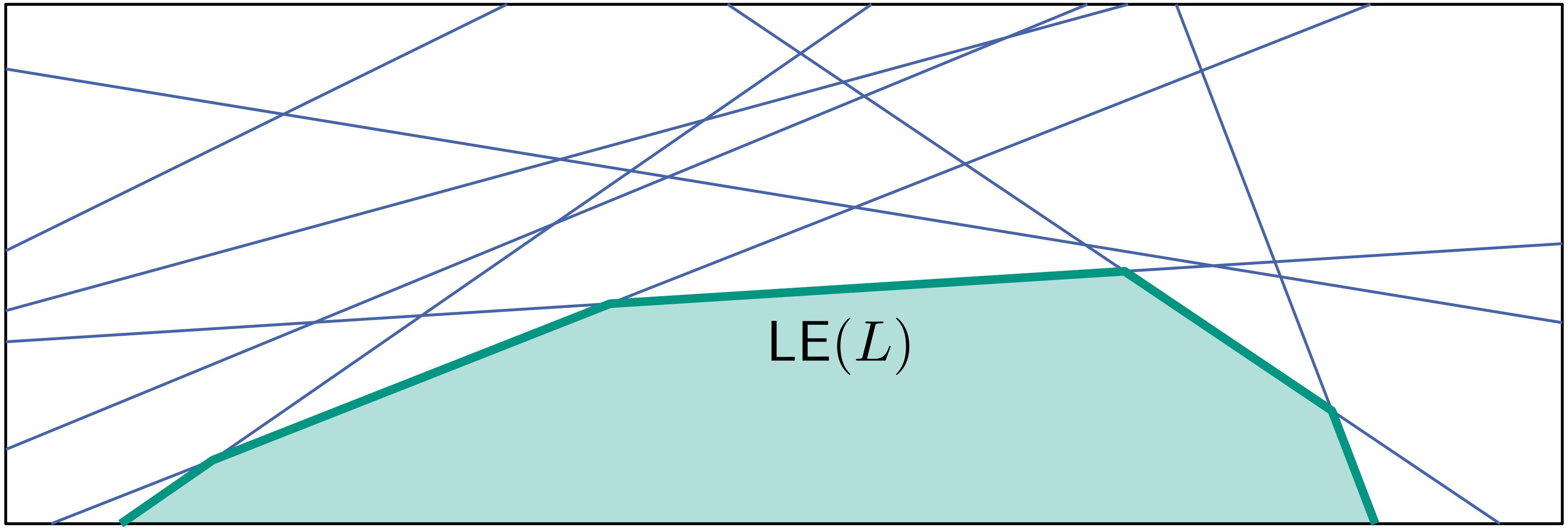
Applications of Duality

Duality does not make geometric problems easier or harder; it simply provides a different (but often helpful) perspective!

We will look at two examples in more detail:

- upper/lower envelopes of line arrangements
- minimum-area triangle in a point set

Lower Envelope

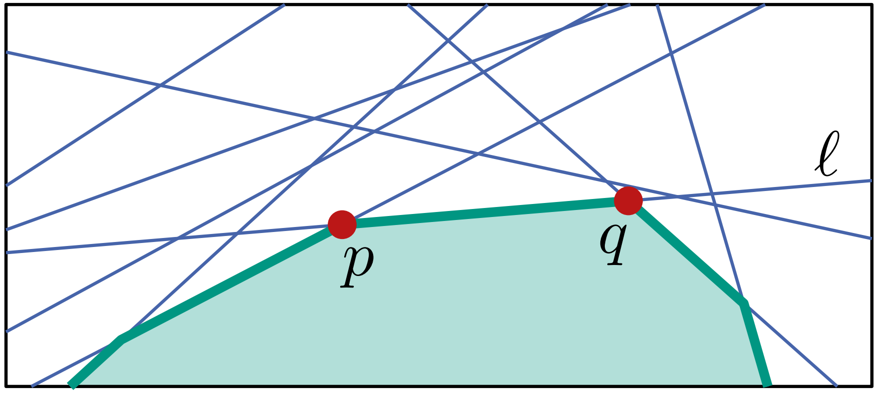


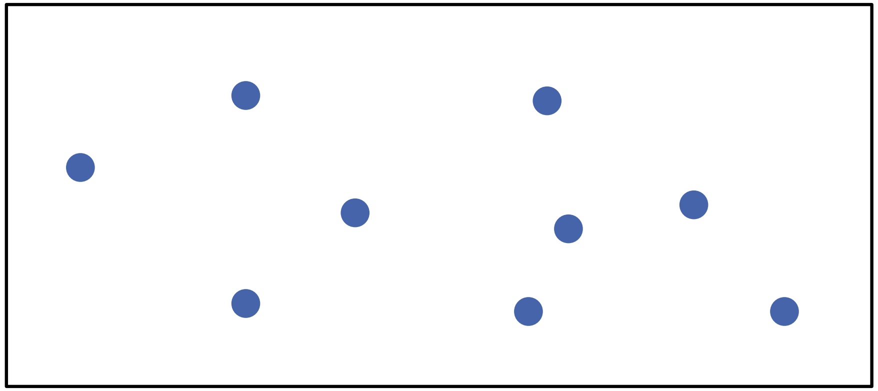
Def: For a set L of lines the **lower envelope** LE(L) of L is the set of all points in $\cup_{\ell \in L} \ell$ that are not above any line in the set L (boundary of the intersection of all lower halfplanes).

Several possibilities for computing lower envelopes

- divide&conquer or sweep-line half-plane intersection algorithms (see Chapter 4.2 in [BCKO08])
- consider the dual problem for $L^* = \{\ell^* \mid \ell \in L\}$

Envelopes and Duality





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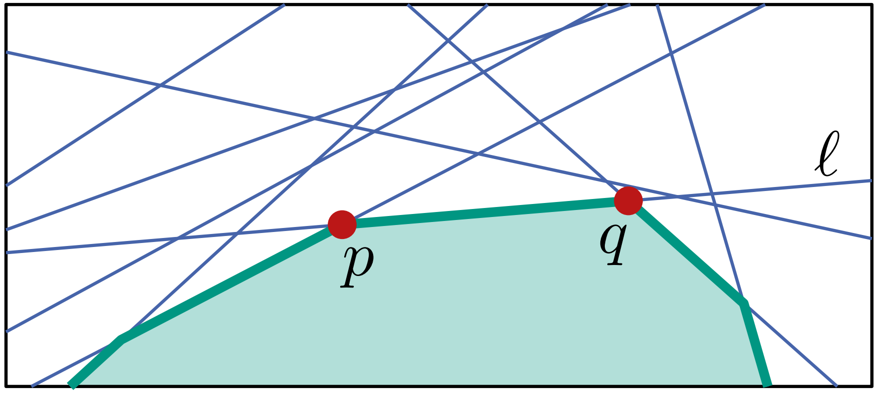

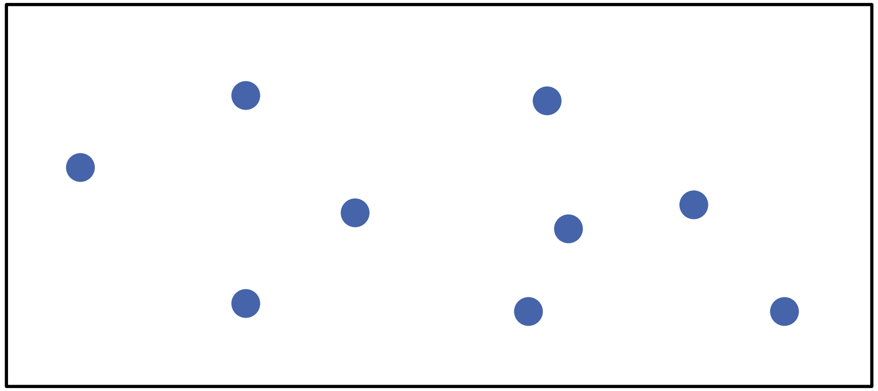

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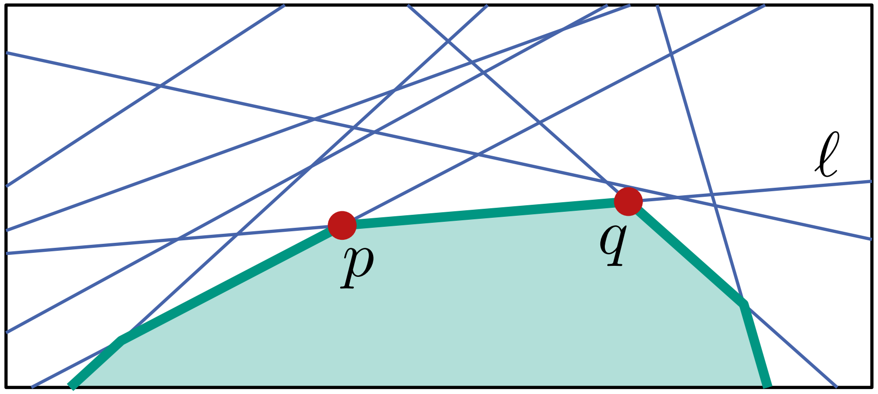

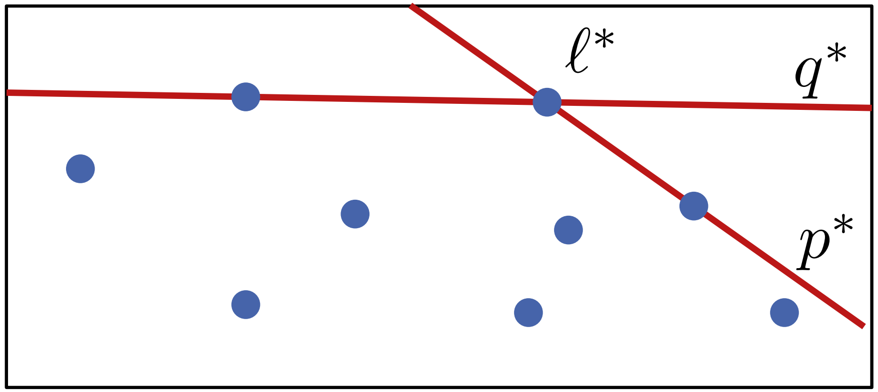

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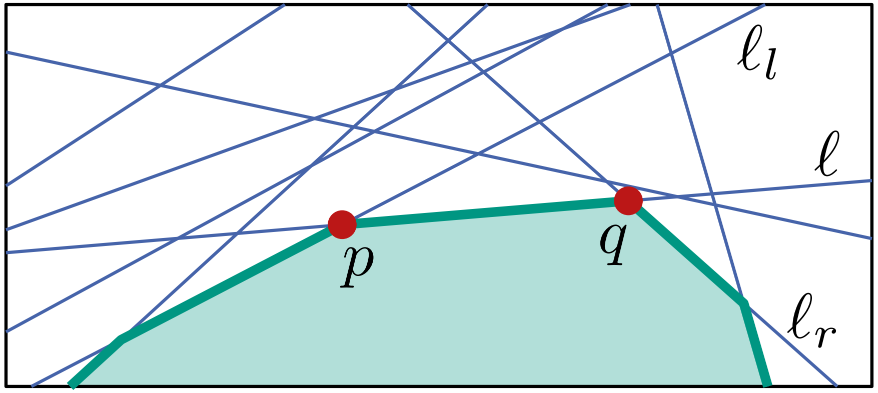

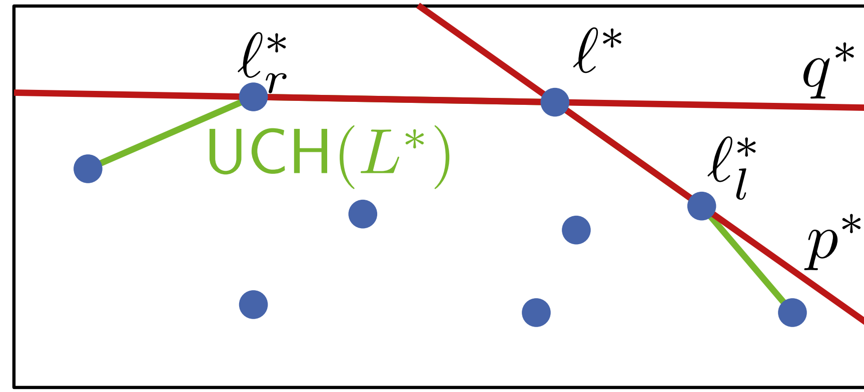

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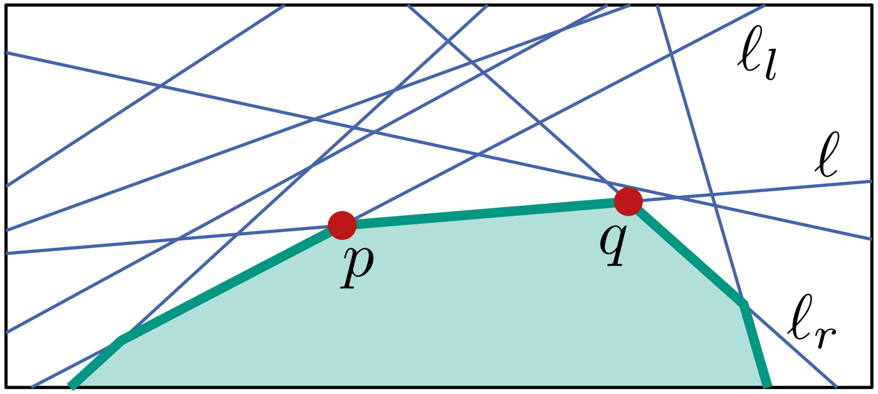

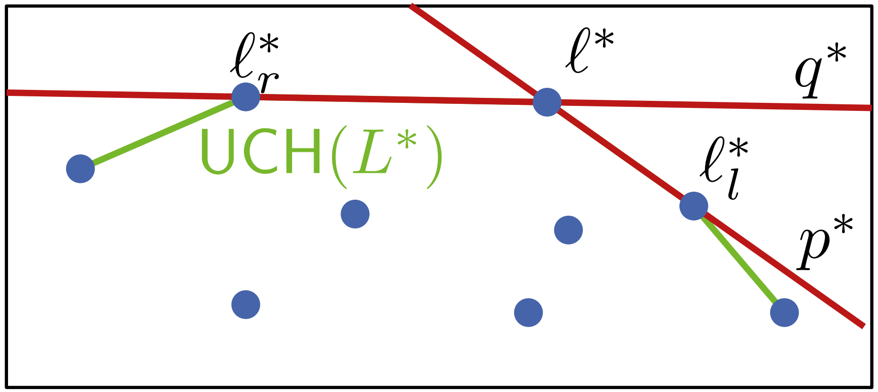

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Lemma 2: The lines on LE(L) ordered from right to left correspond to the vertices of UCH(L^*) ordered from left to right.

Computing the Envelope

- algorithm for computing upper convex hull in time $O(n \log n)$ (see Lecture 1 on convex hulls)

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When does this approach work faster?

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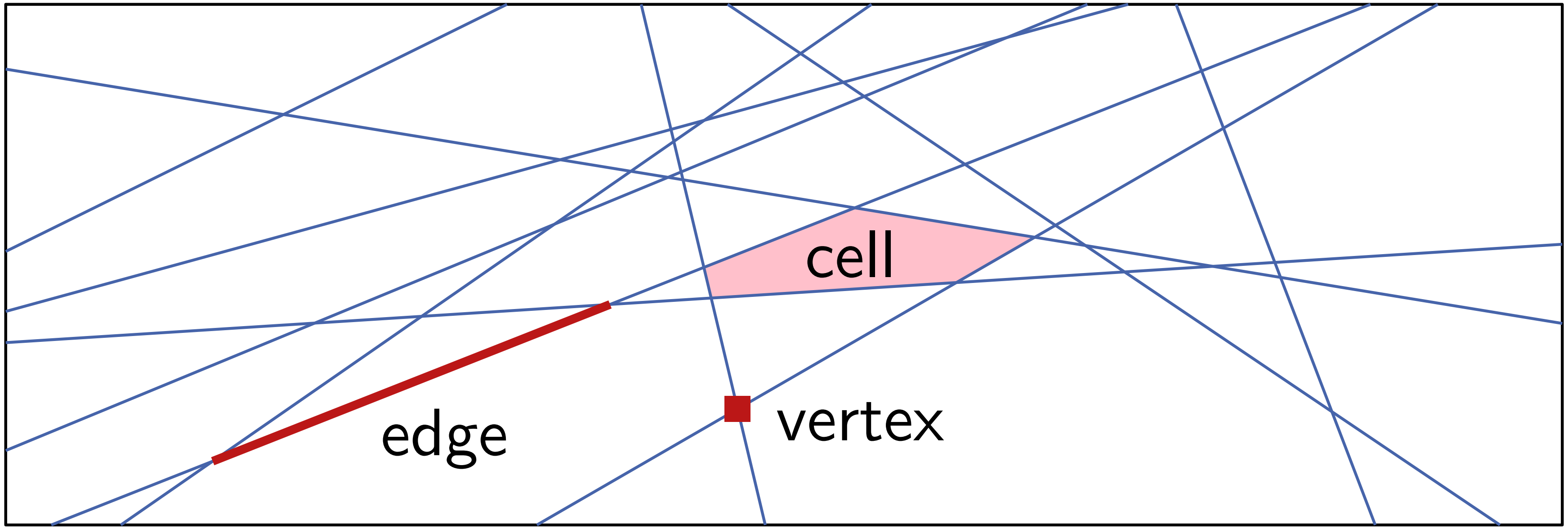
When does this approach work faster?

- output sensitive algorithm for computing convex hull with h points with time complexity $O(n \log h)$

Line Arrangements



Def: A set L of lines defines a subdivision $\mathcal{A}(L)$ of the plane (the **line arrangement**) composed of vertices, edges, and cells (poss. unbounded).
$\mathcal{A}(L)$ is called **simple** if no three lines share a point and no two lines are parallel.

Complexity of $\mathcal{A}(L)$

The combinatorial complexity of $\mathcal{A}(L)$ is the number of vertices, edges, and cells.

Theorem 1: Let $\mathcal{A}(L)$ be a simple line arrangement for n lines. Then $\mathcal{A}(L)$ has $\binom{n}{2}$ vertices, n^2 edges, and $n^2/2 + n/2 + 1$ cells.

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$\rightarrow$ could use line segment intersection plane sweep in $O(n^2 \log n)$

Incrementally Constructing $\mathcal{A}(L)$

Input: lines $L = \{\ell_1, \ldots, \ell_n\}$
Output: DCEL $\mathcal{D}$ for $\mathcal{A}(L)$
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for $i \leftarrow 1$ **to** n **do**
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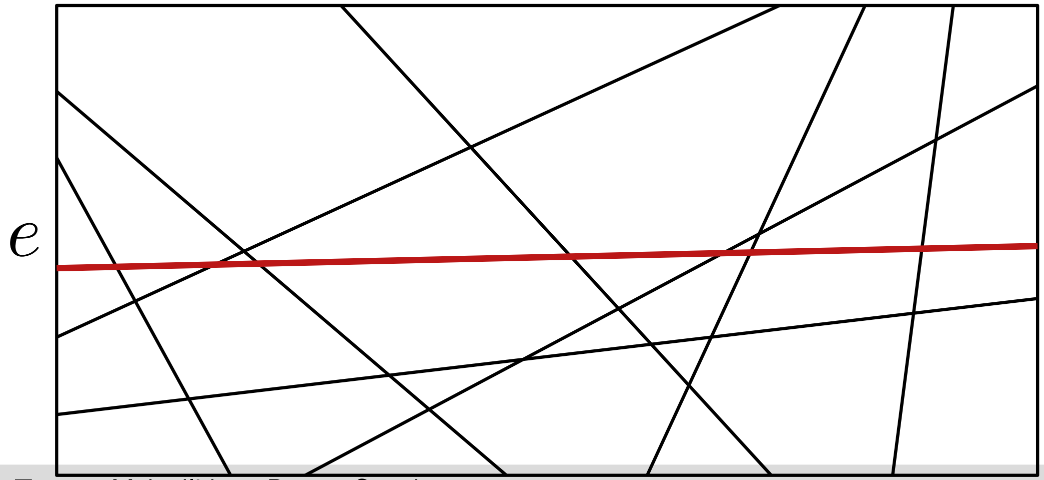

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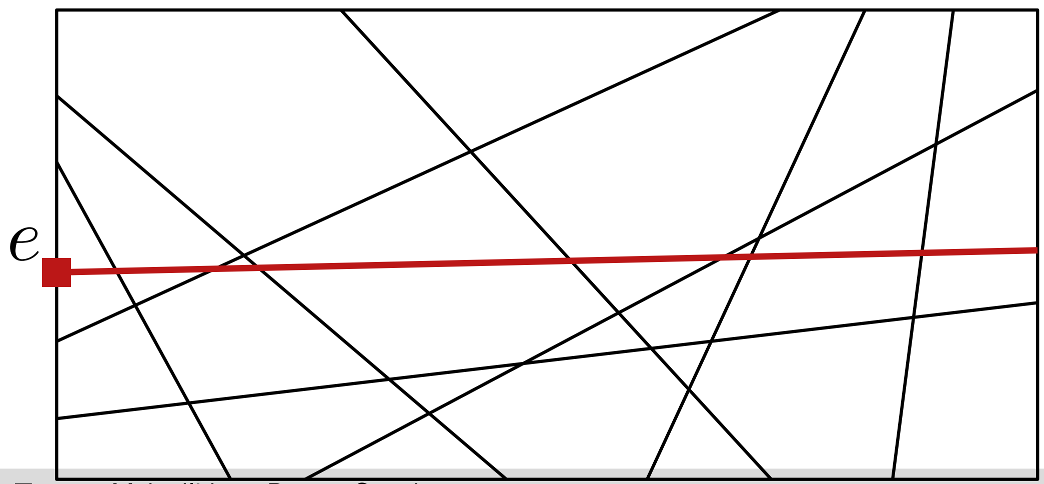

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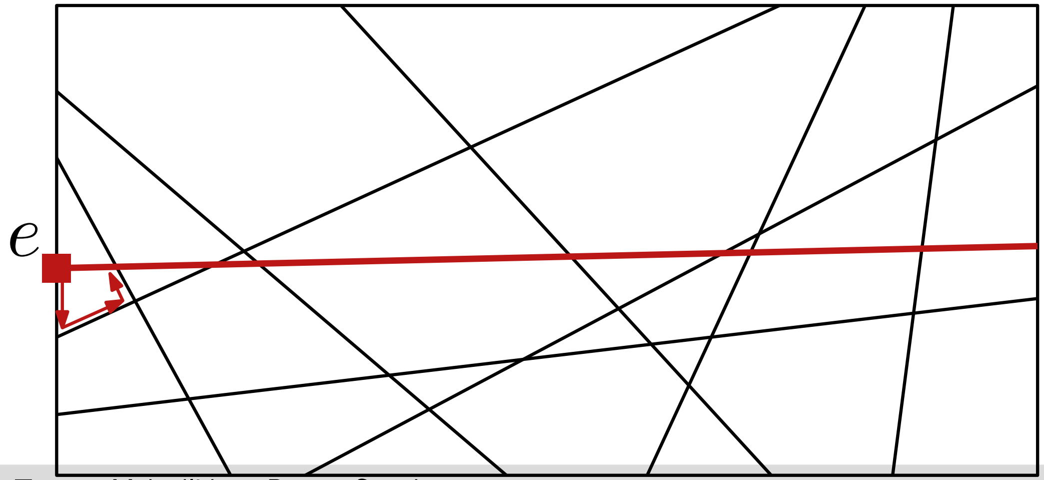

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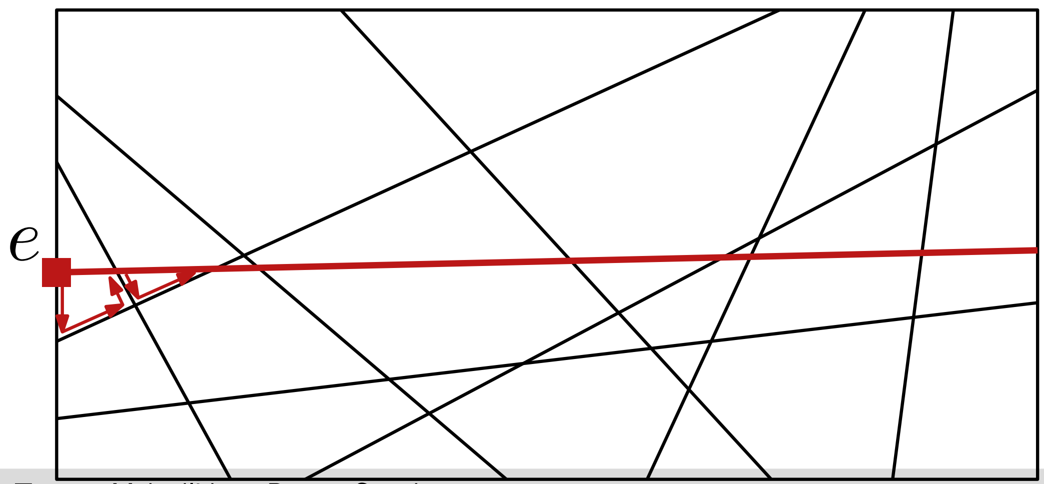

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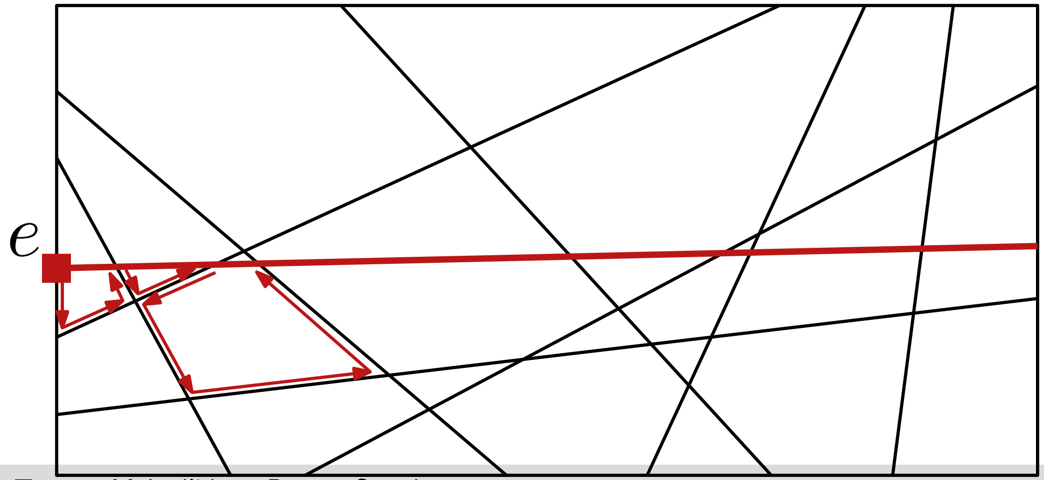

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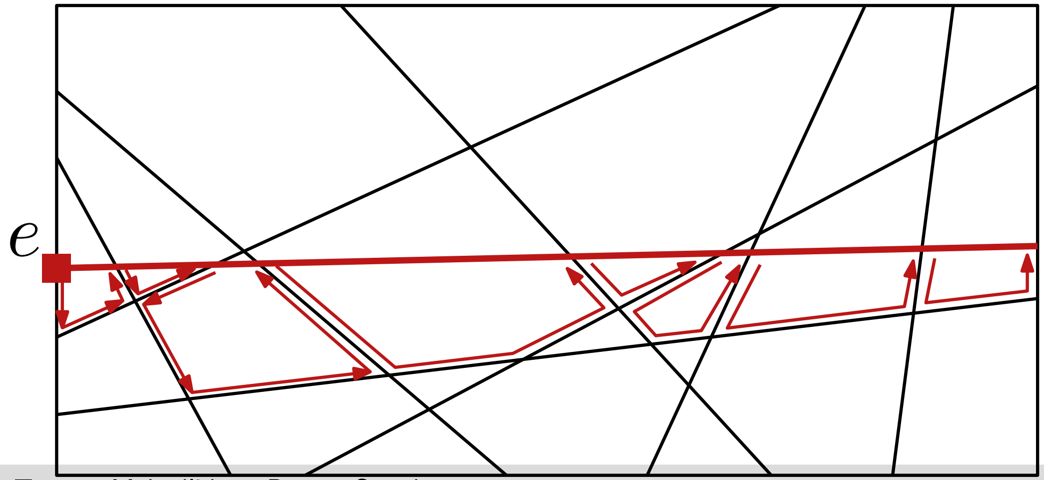

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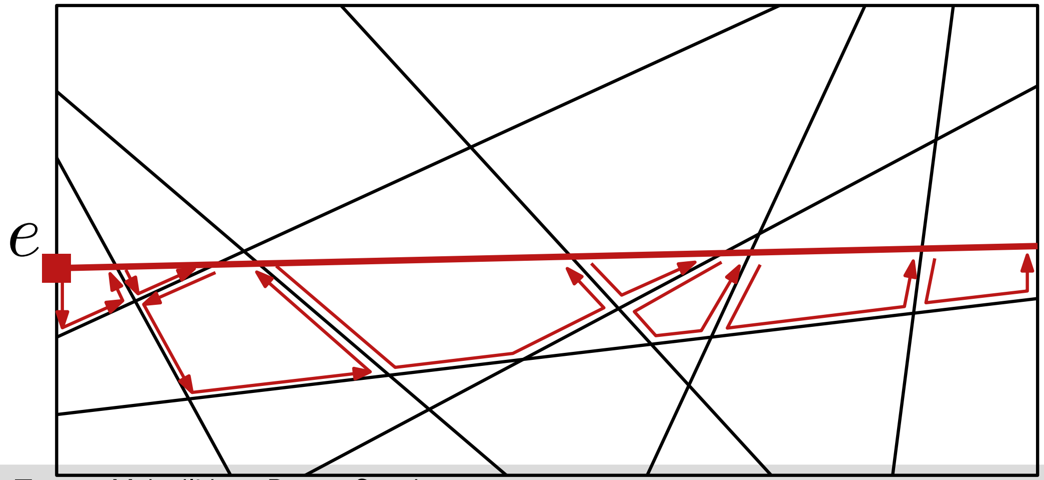

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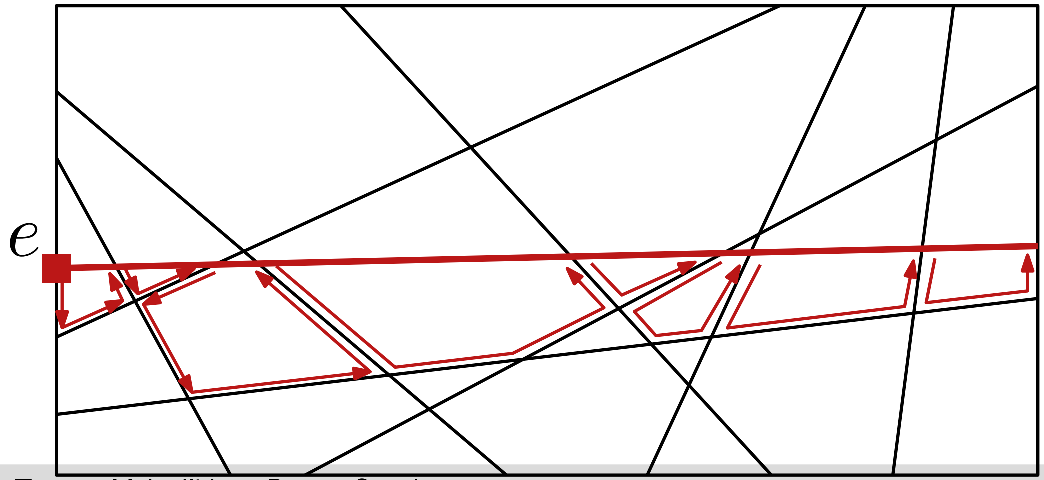

Running time?

- bounding box: $O(n^2)$
- start point of ℓ_i: $O(i)$
- **while**-loop: $O(|\text{red path}|)$

Zone Theorem

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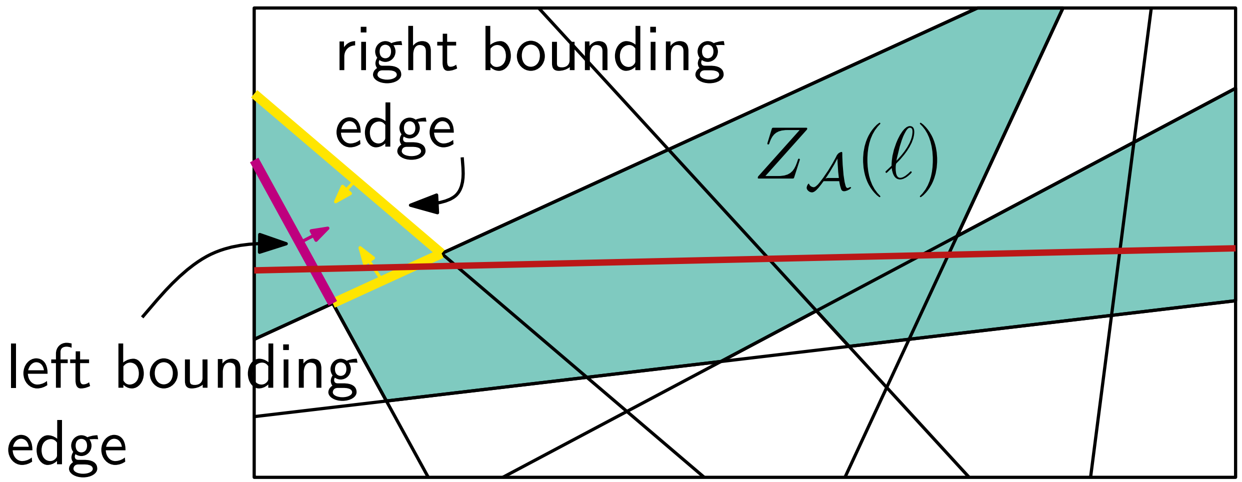

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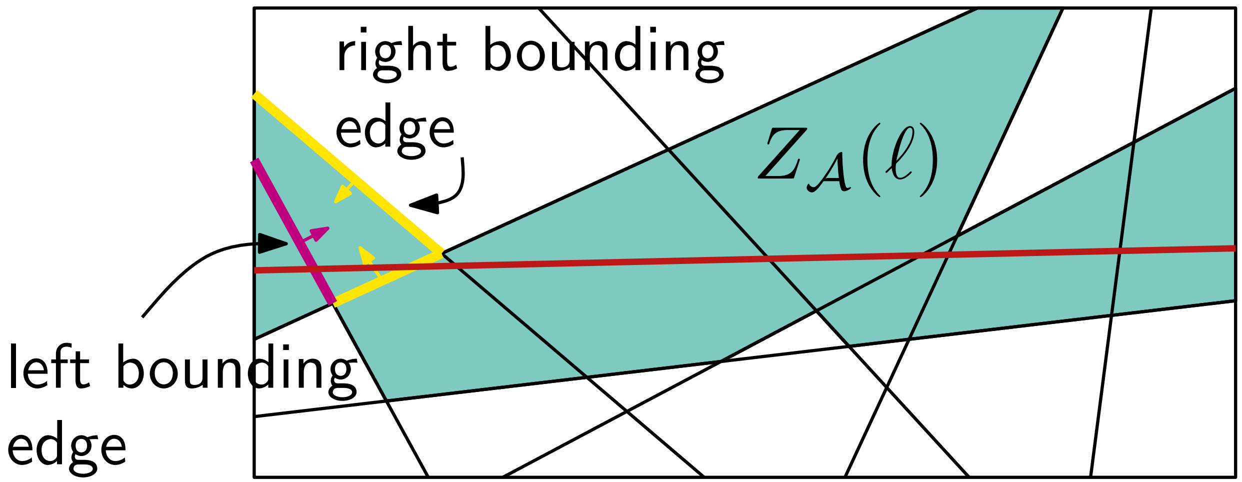

How many edges are in $Z_{\mathcal{A}}(\ell)$?

Theorem 2: For an arrangement $\mathcal{A}(L)$ of n lines in the plane and a line $\ell \notin L$ the zone $Z_{\mathcal{A}}(\ell)$ consist of at most $6n$ edges.

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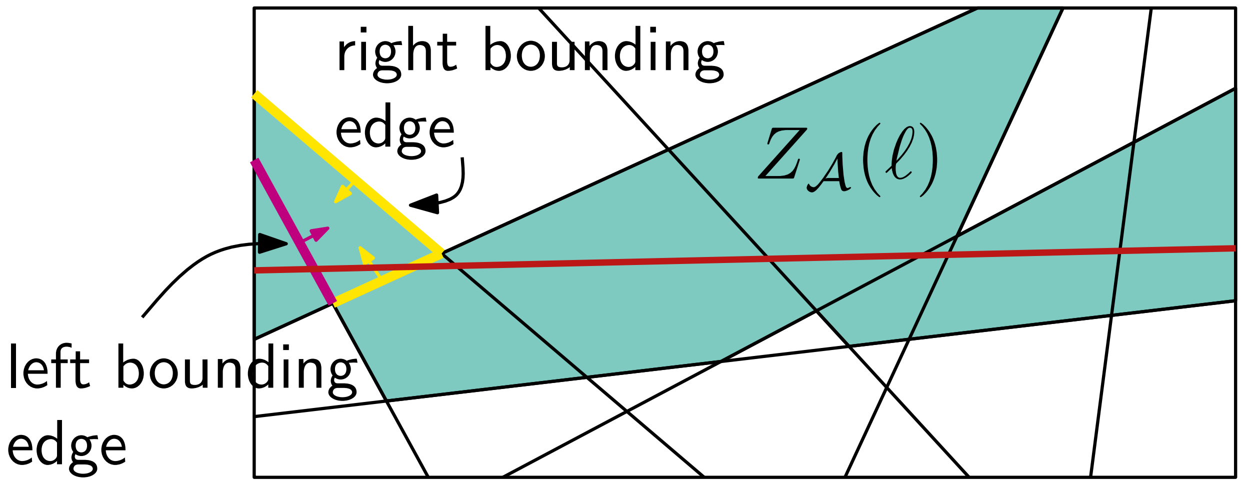


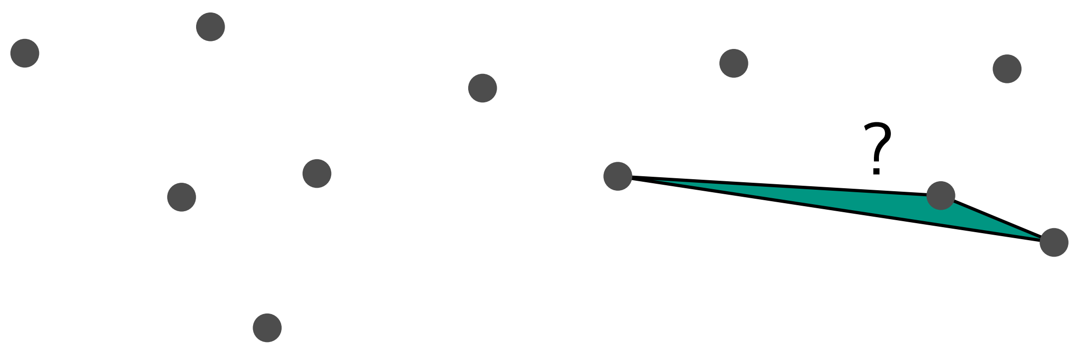
How many edges are in $Z_{\mathcal{A}}(\ell)$?

Theorem 2: For an arrangement $\mathcal{A}(L)$ of n lines in the plane and a line $\ell \notin L$ the zone $Z_{\mathcal{A}}(\ell)$ consist of at most $6n$ edges.

Theorem 3: The arrangement $\mathcal{A}(L)$ of a set of n lines can be constructed in $O(n^2)$ time and space.

Smallest Triangle

Given a set P of n points in $\mathbb{R}^2$, find a minimum-area triangle Δpqr with $p, q, r \in P$.



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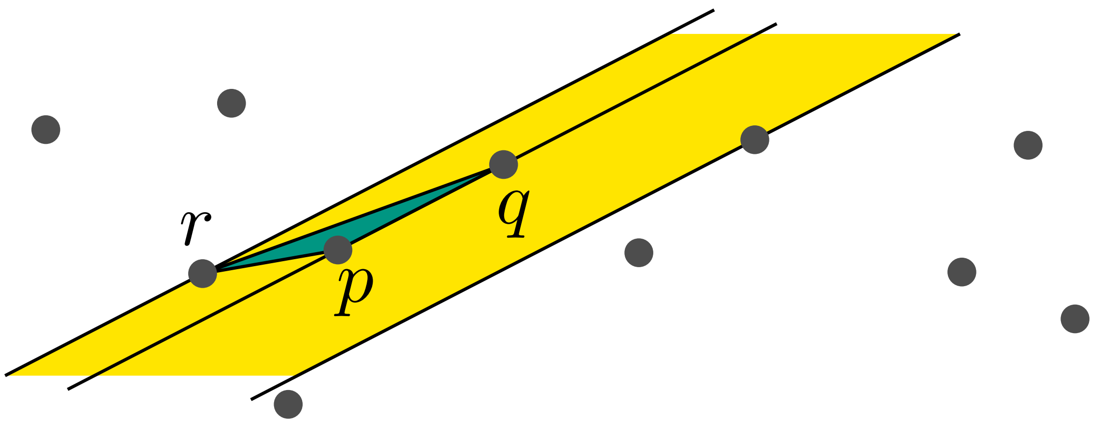

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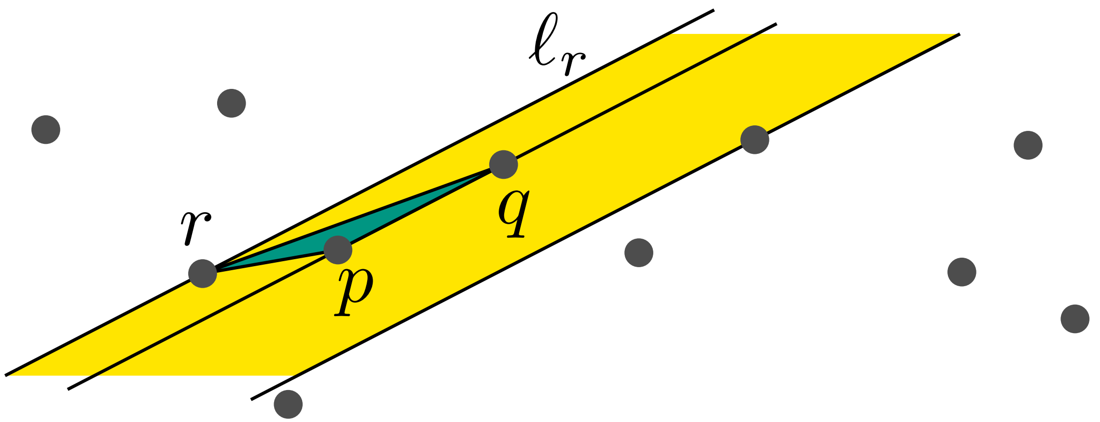

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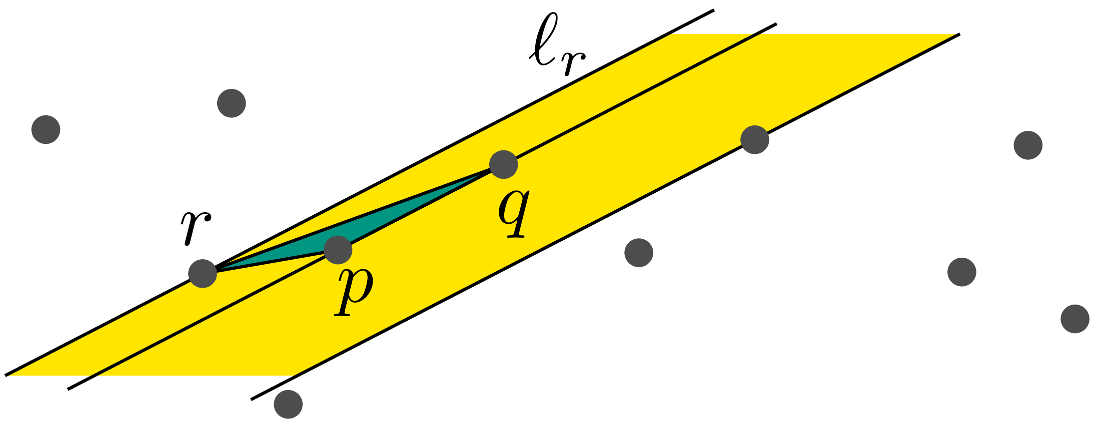

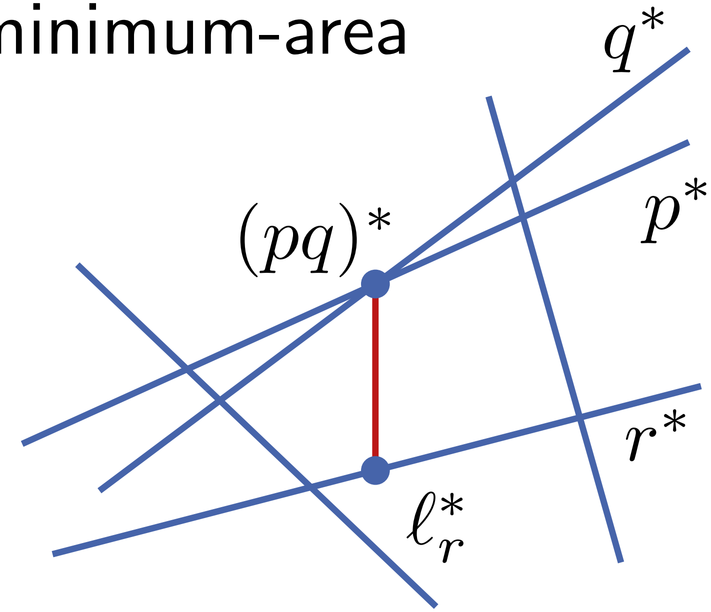

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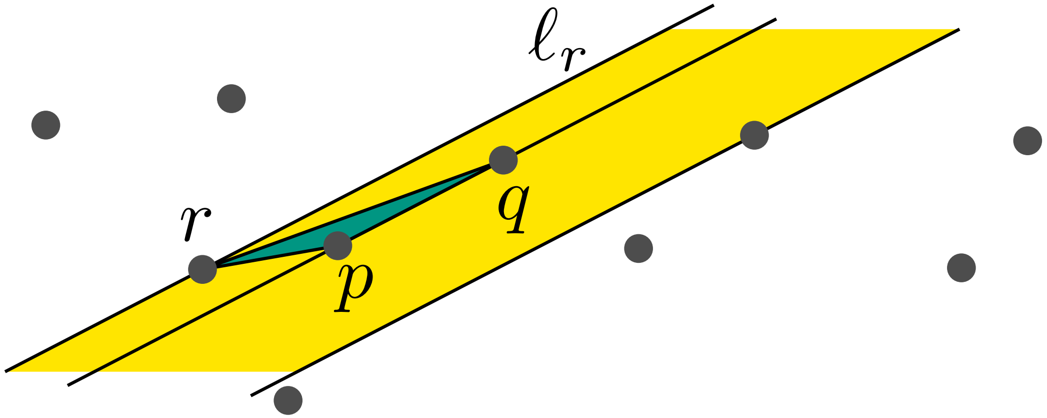
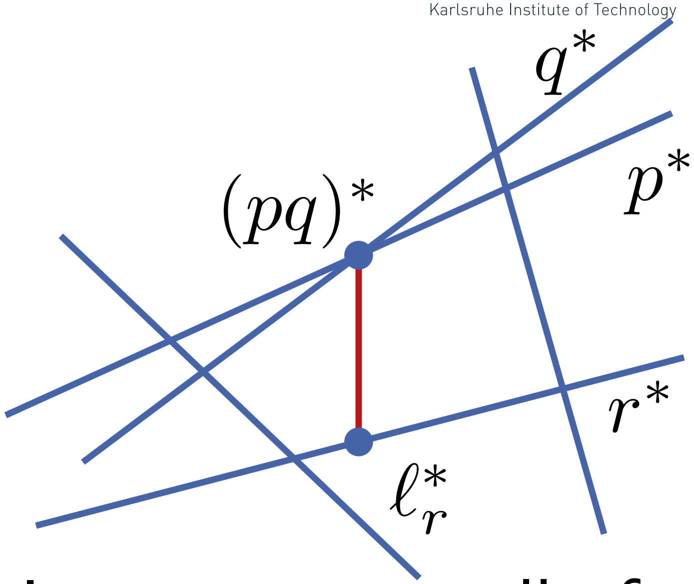
In dual plane:

- ℓ_r^* lies on r^*
- ℓ_r^* and $(pq)^*$ have identical x-coordinate
- no line $p^* \in P^*$ intersects $\overline{\ell_r^* (pq)^*}$

Computing in the Dual





- ℓ_r^* lies vertically above or below $(pq)^*$ in a common cell of $\mathcal{A}(P^*) \Rightarrow$ only two candidates

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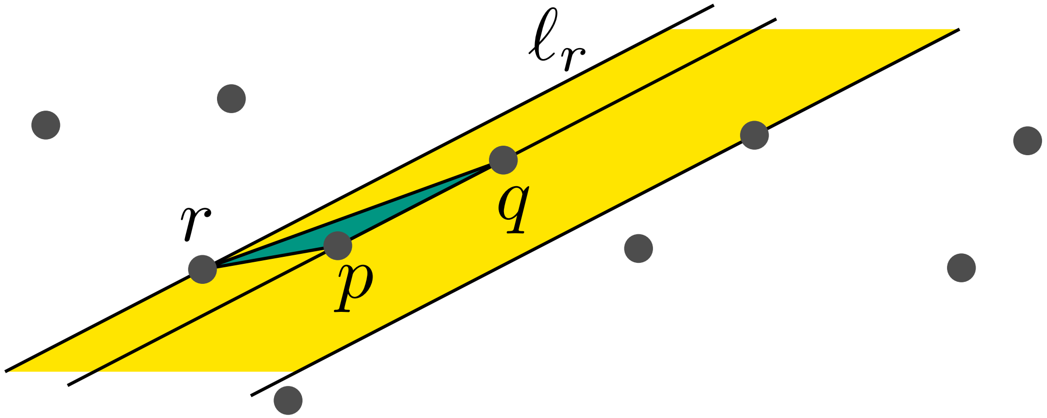

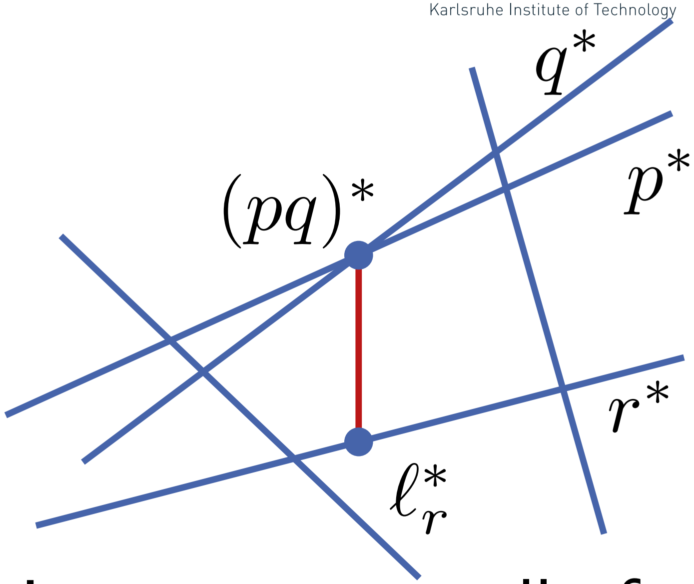

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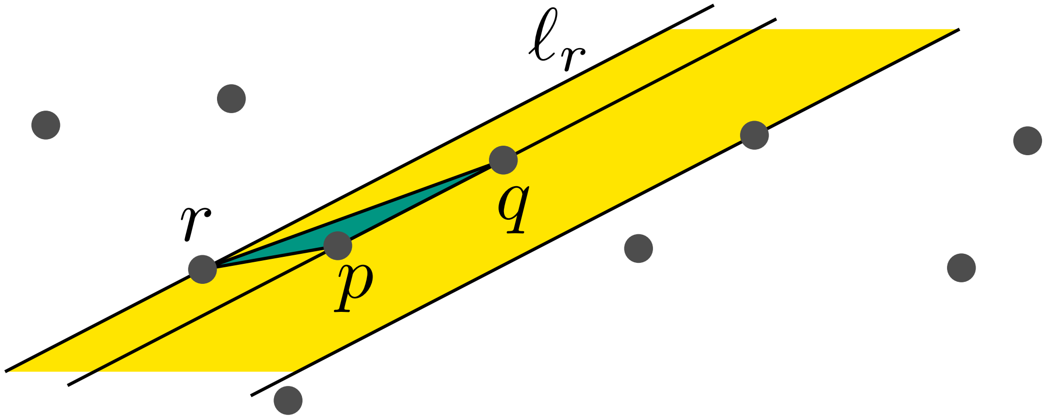

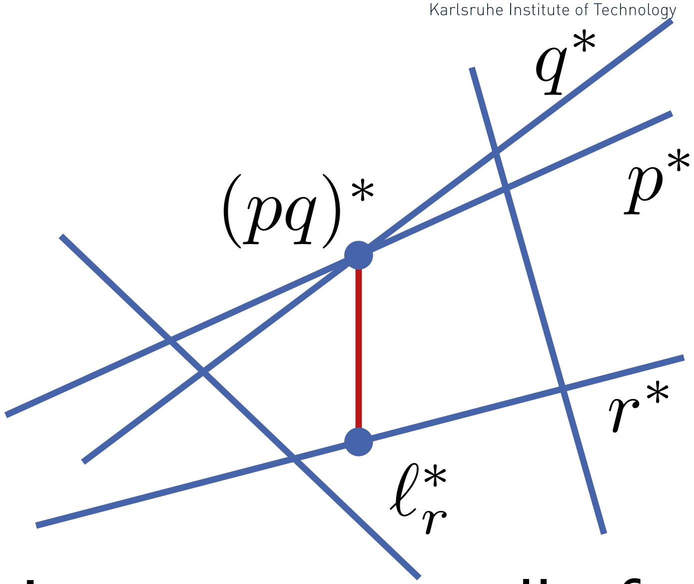

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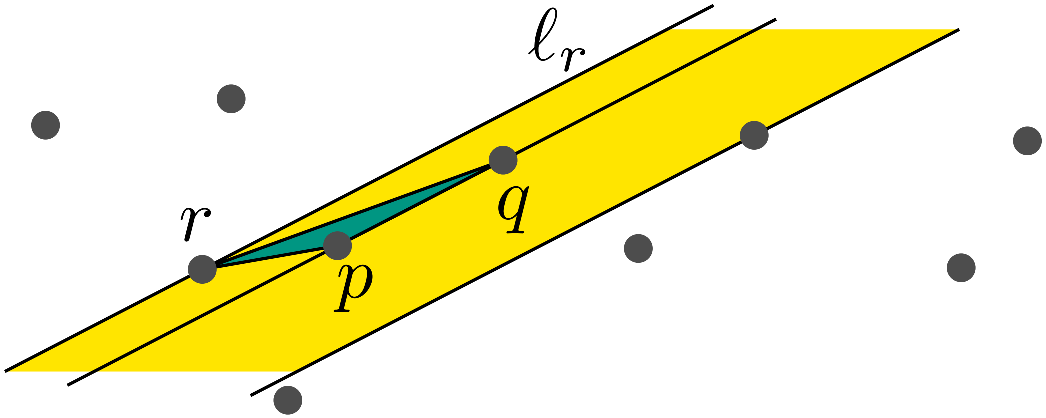

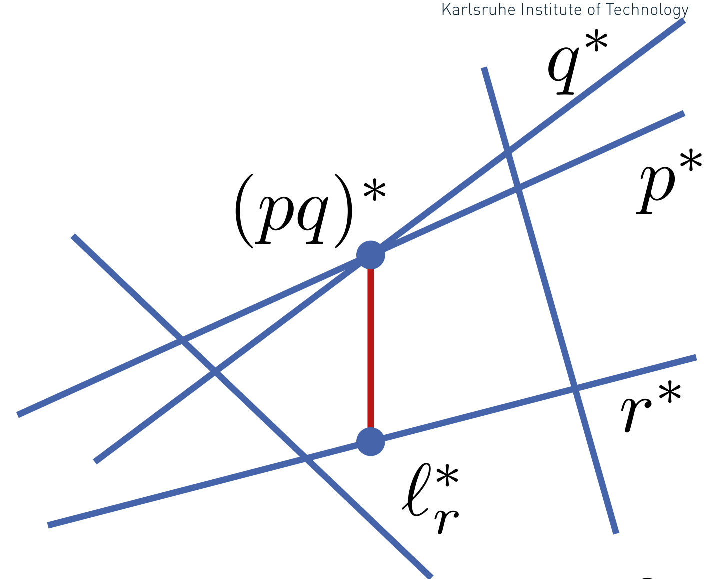

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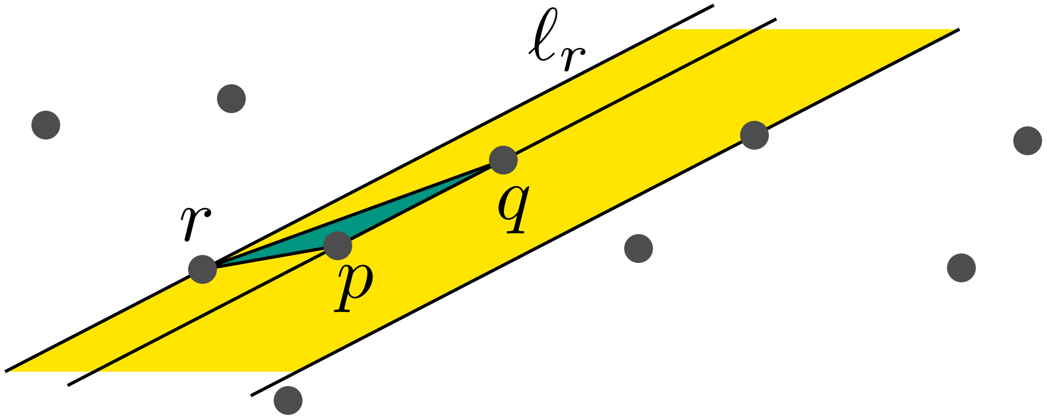

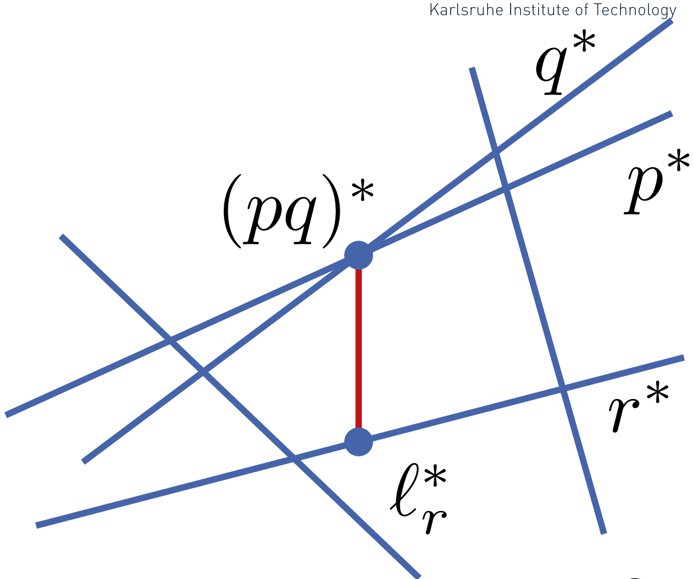

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- finds minimum in $O(n^2)$ time in total

Further Duality Applications

- Two thieves have stolen a necklace of diamonds and emeralds. They want to share fairly without destroying the necklace more than necessary. How many cuts do they need?

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Karlsruhe Institute of Technology

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Theorem 4: Let D, E be two finite sets of points in $\mathbb{R}^2$. Then there is a line ℓ that divides S and D in half simultaneously.

- Given n segments in the plane, find a maximum stabbing-line, i.e., a line intersecting as many segments as possible.

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The arrangement of n hyperplanes in $\mathbb{R}^d$ has complexity $\Theta(n^d)$. A generalization of the Zone Theorem yields an $O(n^d)$-time algorithm.