Karlsruhe Institute of Technology

Computational Geometry · Lecture

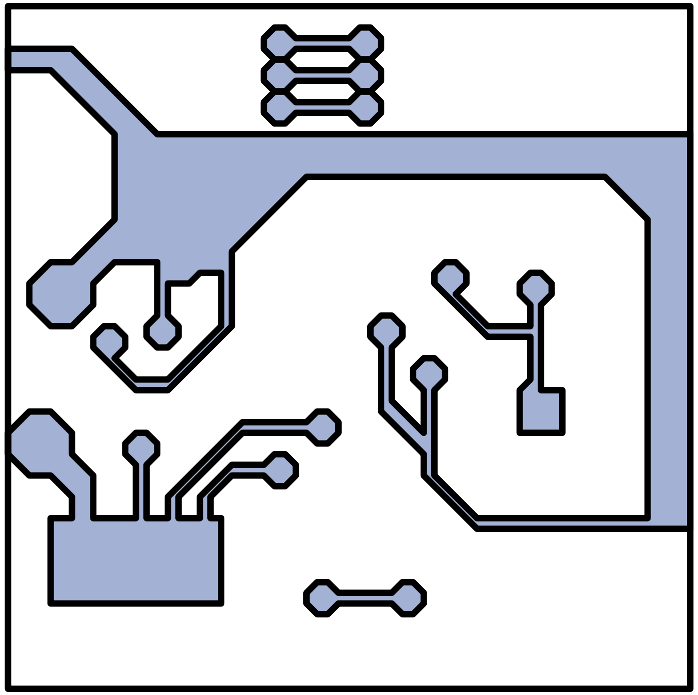
Quadtrees and Meshing

INSTITUT FÜR THEORETISCHE INFORMATIK · FAKULTÄT FÜR INFORMATIK

Tamara Mchedlidze · Chih-Hung Liu

11.07.2018

Motivation: Meshing PC Board Layouts



To simulate the heat produced on boards we can use the *finite element method* (FEM):

- decompose the board in small homogeneous elements (e.g., triangles) $\rightarrow$ mesh
- heat generation and impact on neighbors for each element known
- approximate numerically the entire heat generation of board

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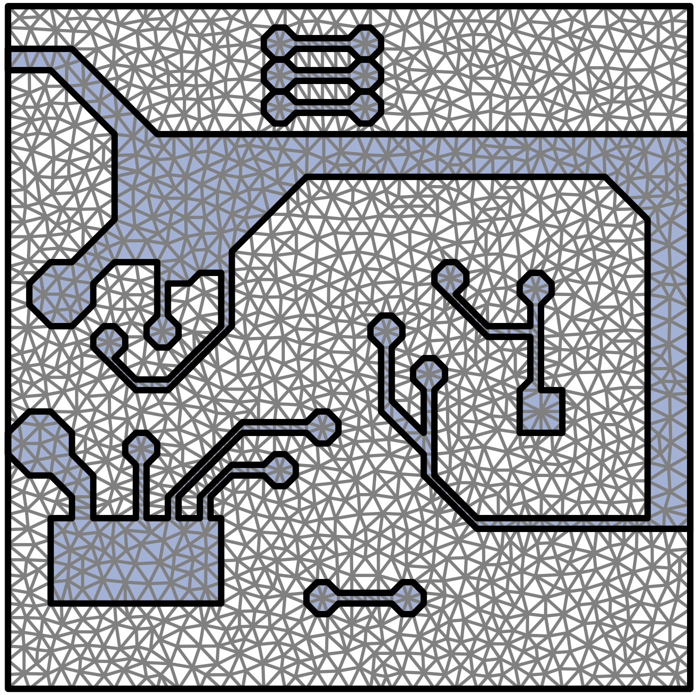

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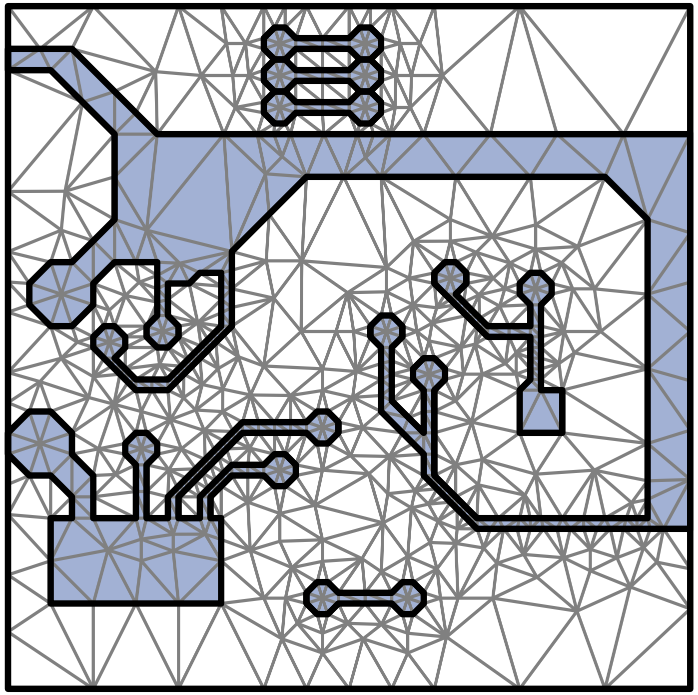

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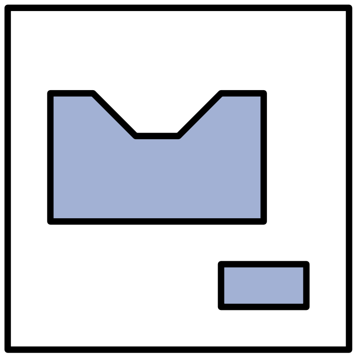
Goal:
- adaptive mesh size (small on materials, otherwise coarser)
- fat triangles (not too narrow)

Adaptive Triangular Mesh

Given: Square $Q = [0, U] \times [0, U]$ for power of two $U = 2^j$ with *octilinear*, integer-coordinate polygons inside.



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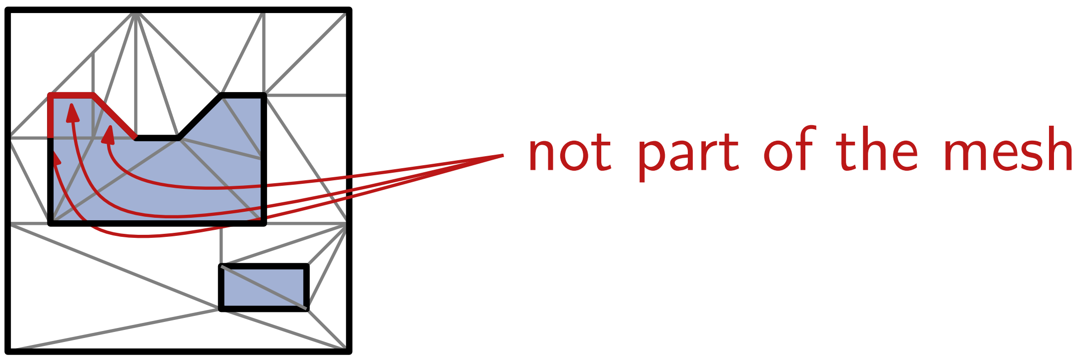

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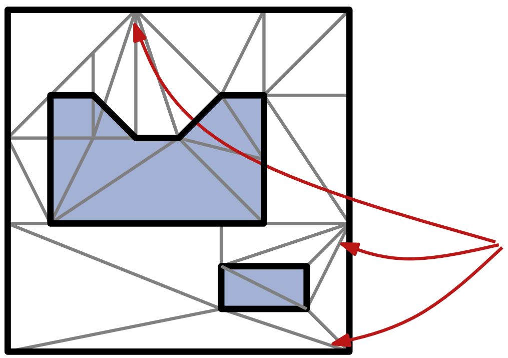

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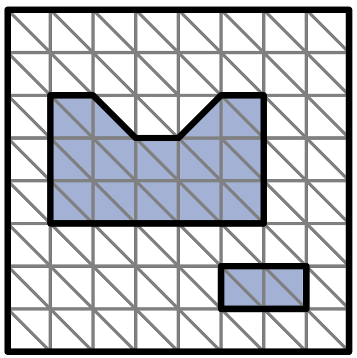

uniform mesh

Goal: Triangular mesh for Q with the following properties

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- no triangle vertex in interior of triangular mesh
- input edges must be part of the triangulation
- triangle angle between 45° and 90°
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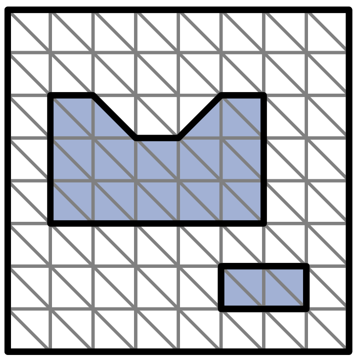

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Do we already have meaningful triangulations of Q?

Delaunay Triangulation?

- maximize smallest angle

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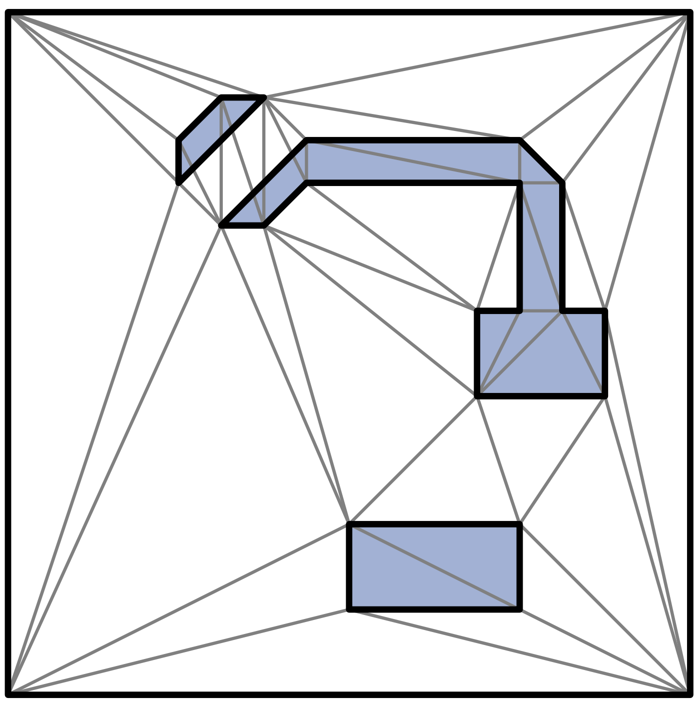

Delaunay Triangulation?

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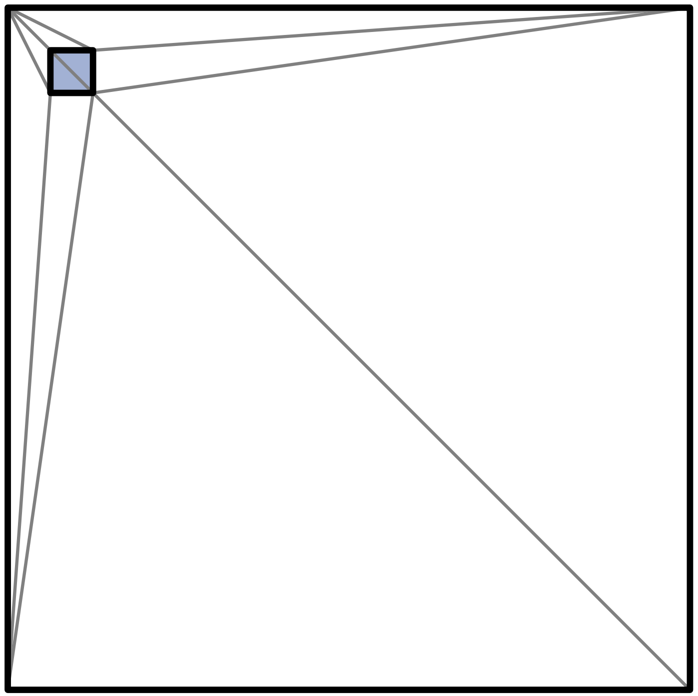

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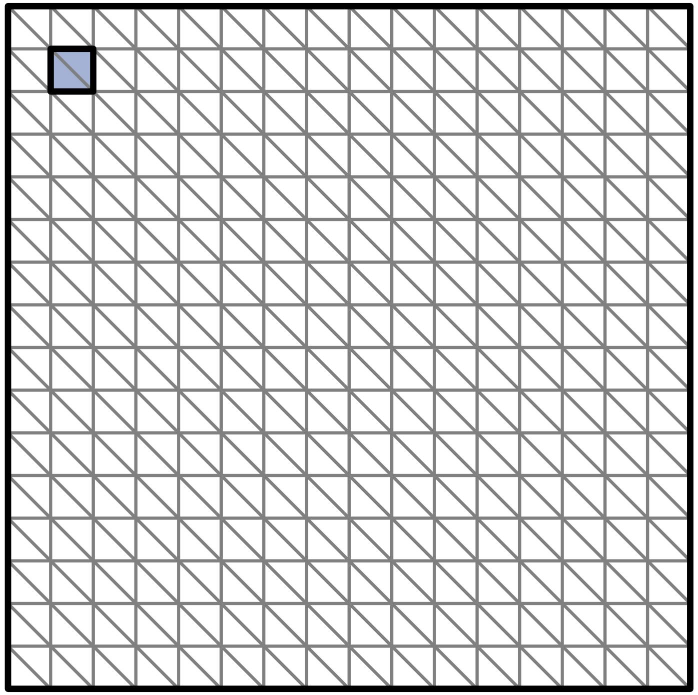
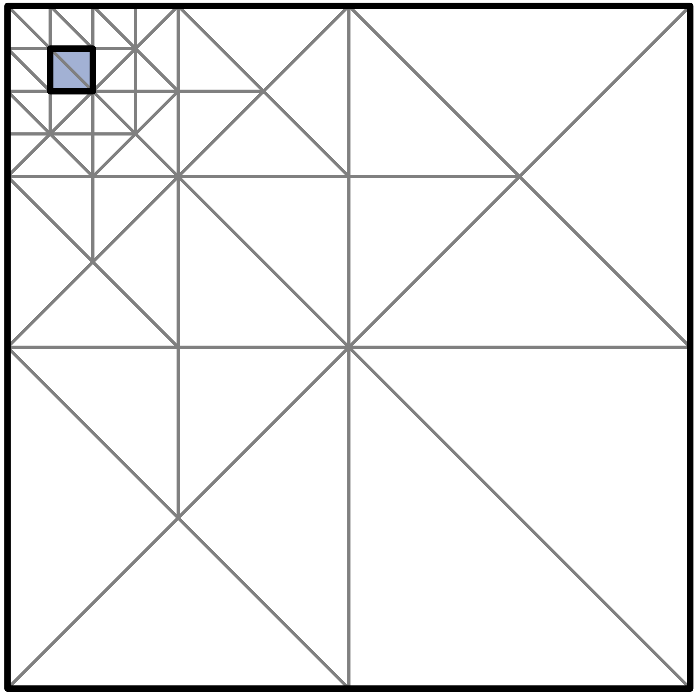
Allowed angles, but uniform



512 triangles

Allowed angles and adaptive



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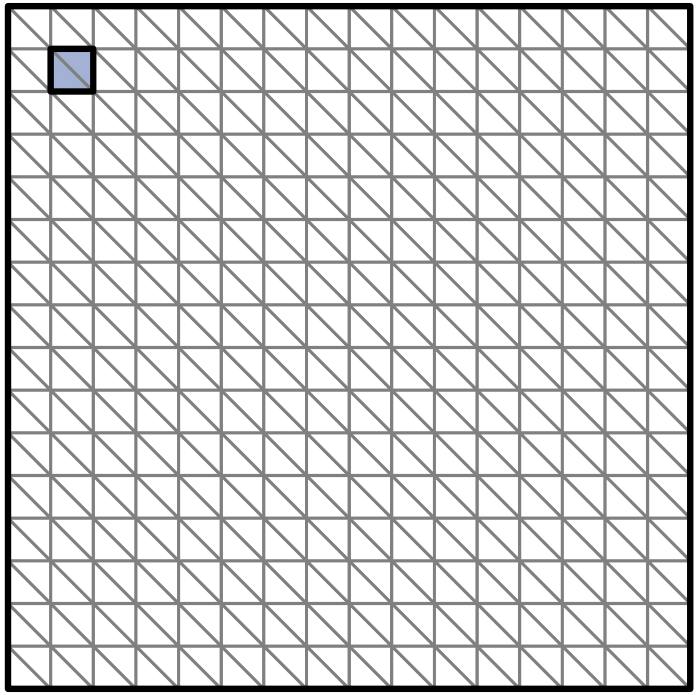

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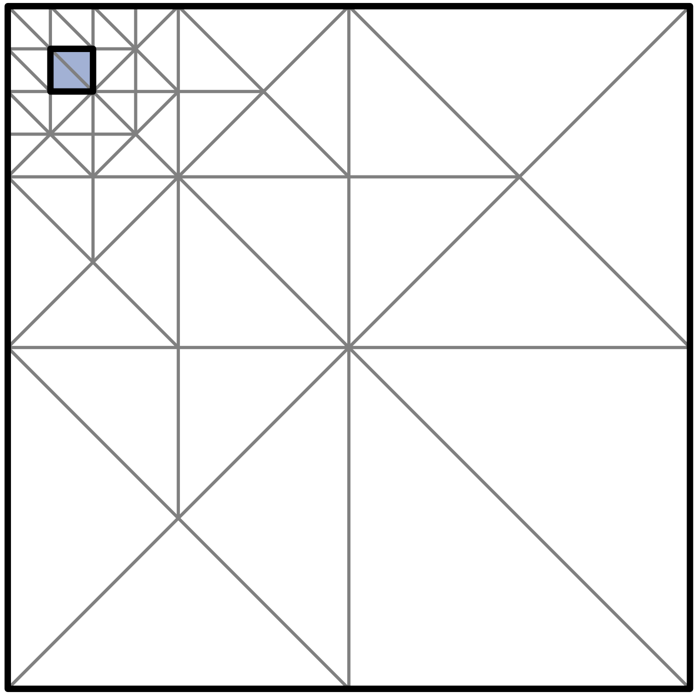

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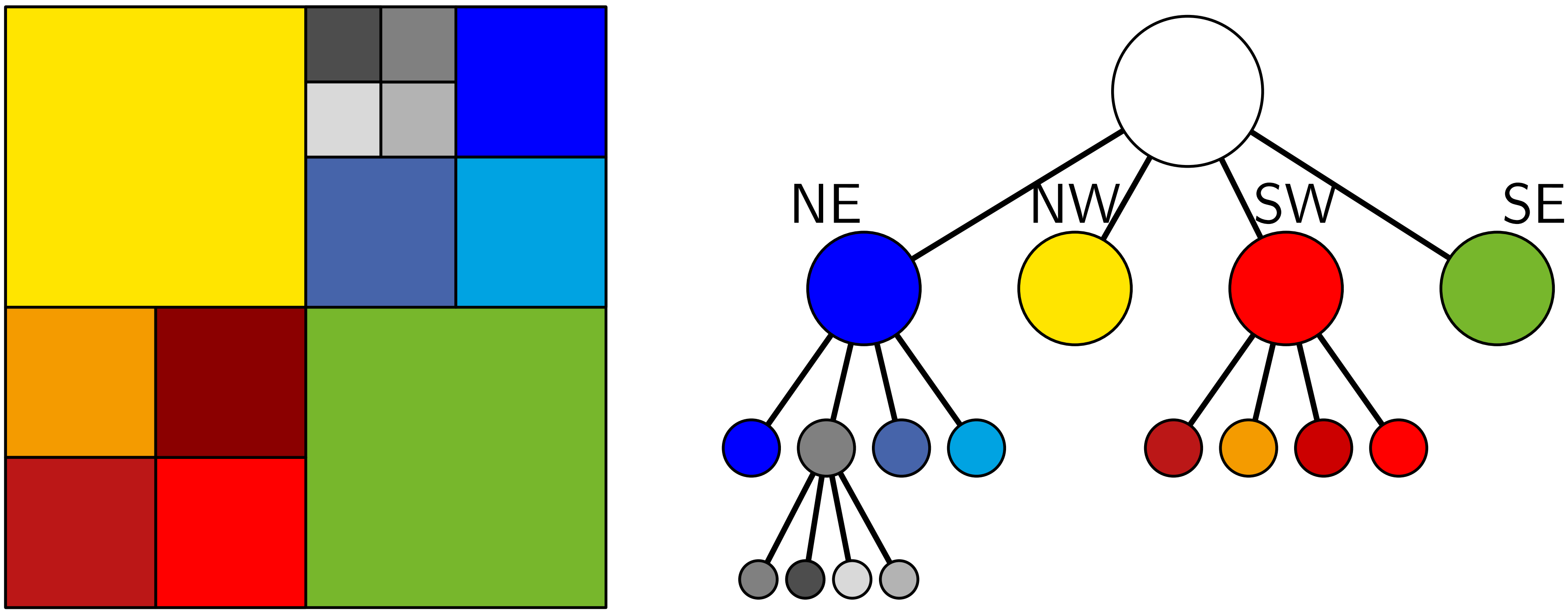
Quadtrees

Def.: A **quadtree** is a rooted tree, where each internal node has 4 children. Each node corresponds to a square, and the squares of the leaves form a partition of the root square.



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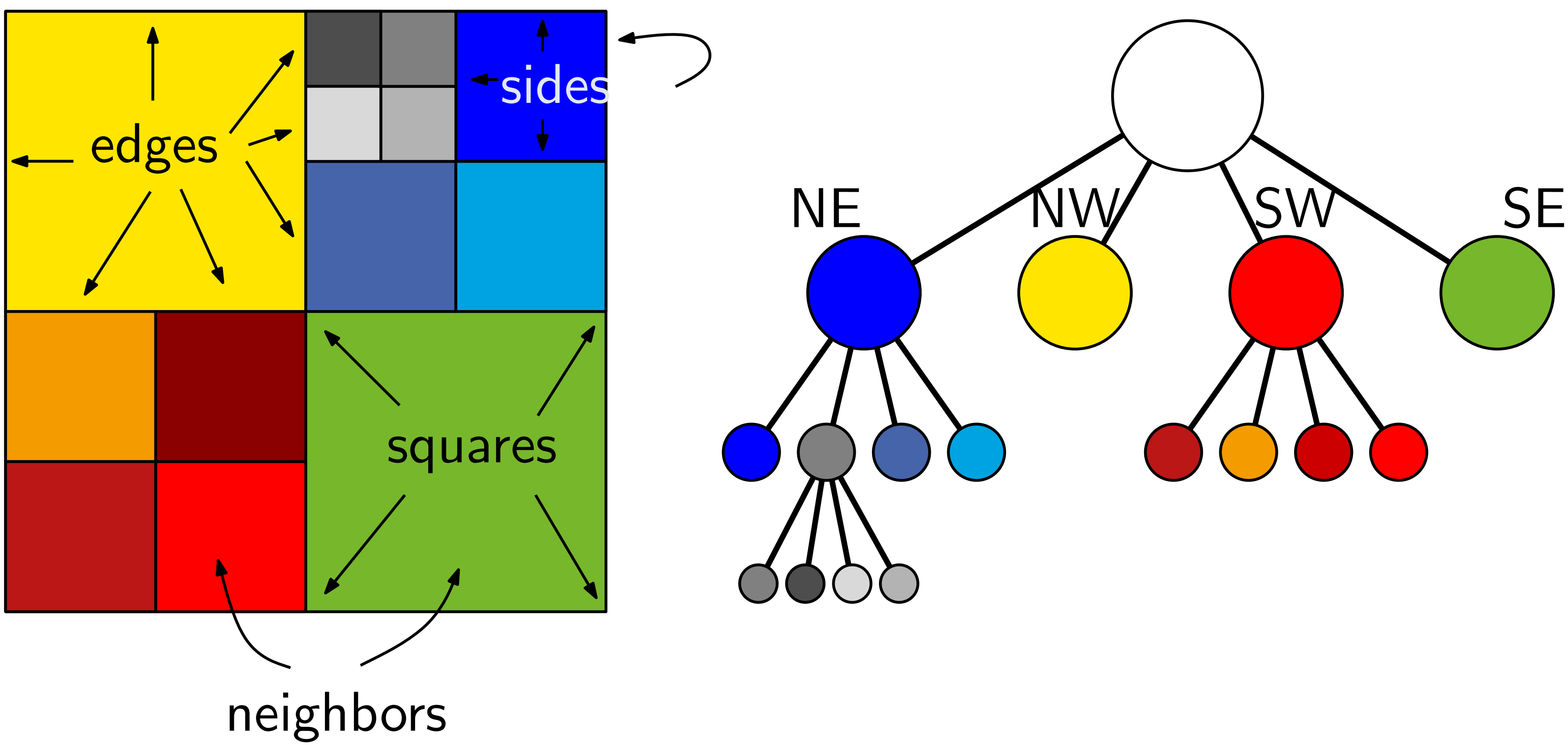

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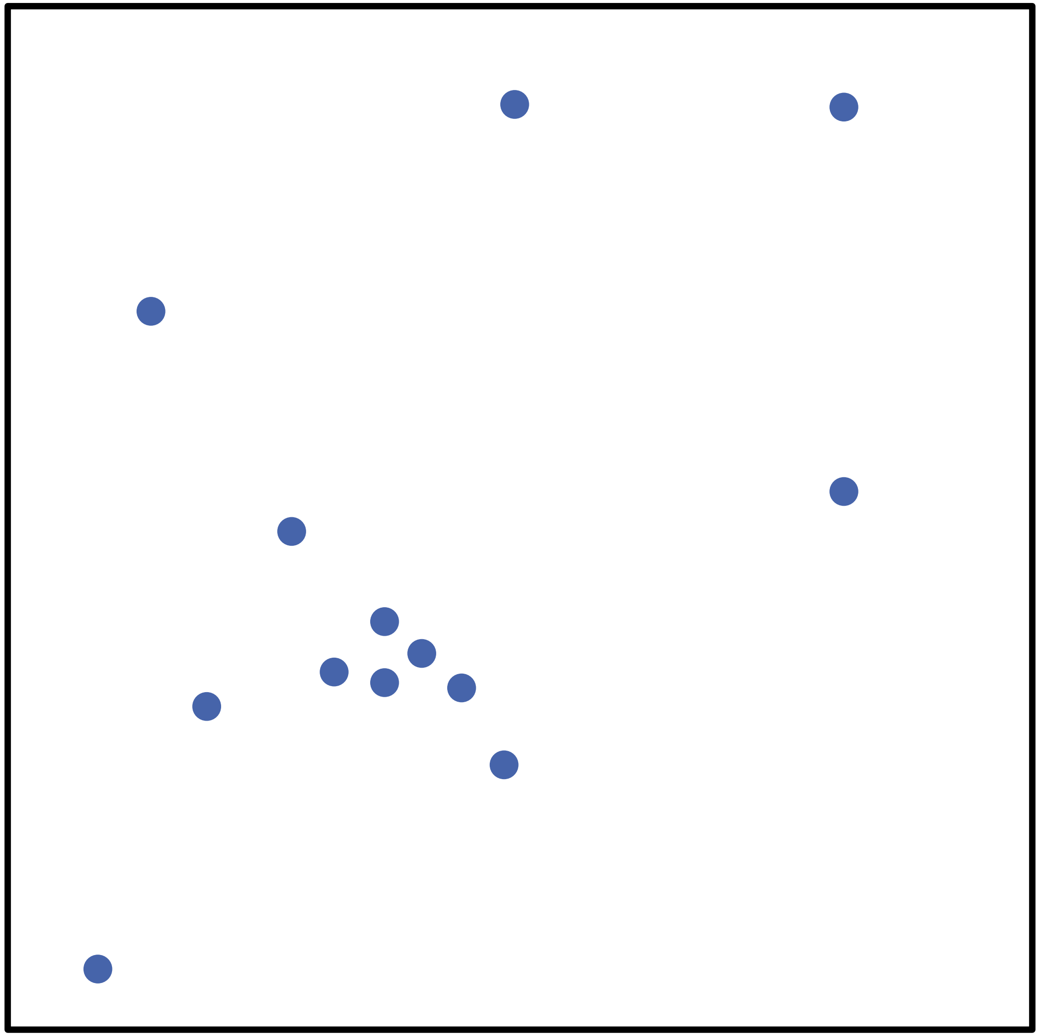
Def.: For a point set P in a square $Q = [x_Q, x'_Q] \times [y_Q, y'_Q]$ define the quadtree $\mathcal{T}(P)$

- if $|P| \leq 1$ then $\mathcal{T}(P)$ is a leaf, then Q stores P
- otherwise let $x_{\text{mid}} = \frac{x_Q + x'_Q}{2}$ and $y_{\text{mid}} = \frac{y_Q + y'_Q}{2}$ and

$$\begin{aligned}
P_{NE} &:= \{p \in P \mid p_x > x_{\text{mid}} \text{ and } p_y > y_{\text{mid}}\} \\
P_{NW} &:= \{p \in P \mid p_x \leq x_{\text{mid}} \text{ and } p_y > y_{\text{mid}}\} \\
P_{SW} &:= \{p \in P \mid p_x \leq x_{\text{mid}} \text{ and } p_y \leq y_{\text{mid}}\} \\
P_{SE} &:= \{p \in P \mid p_x > x_{\text{mid}} \text{ and } p_y \leq y_{\text{mid}}\}
\end{aligned}$$

  $\mathcal{T}(P)$ has root v, then Q has 4 children storing P_i and Q_i $(i \in \{NE, NW, SW, SE\})$.

Example



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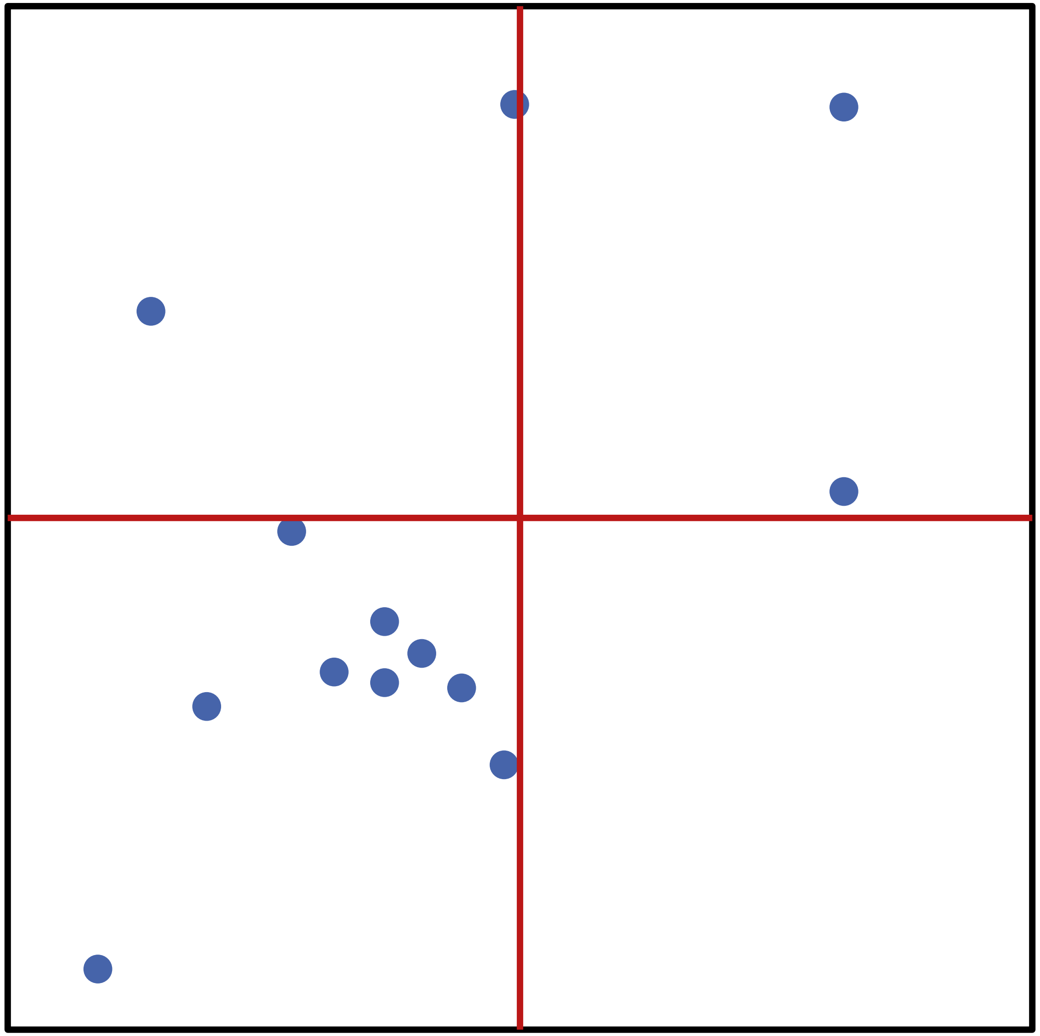

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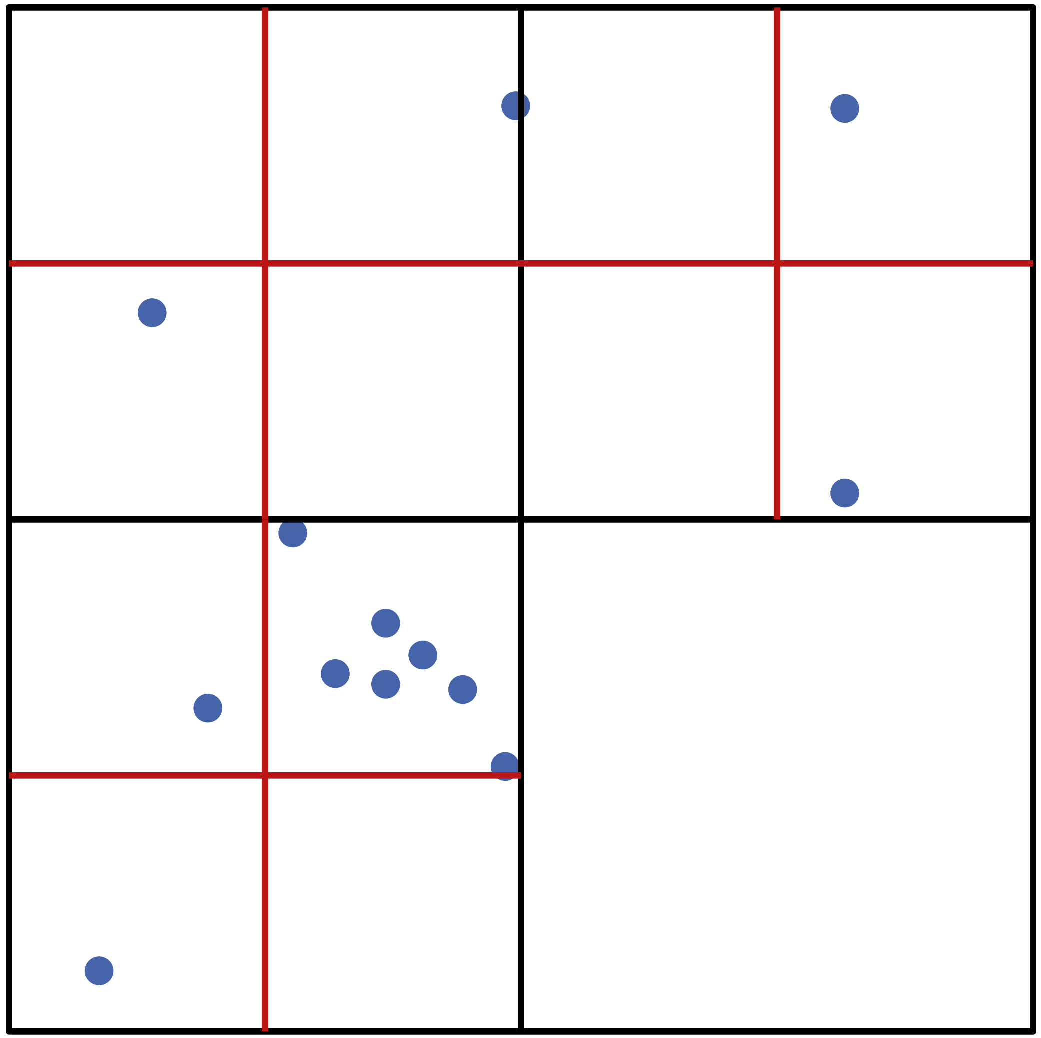

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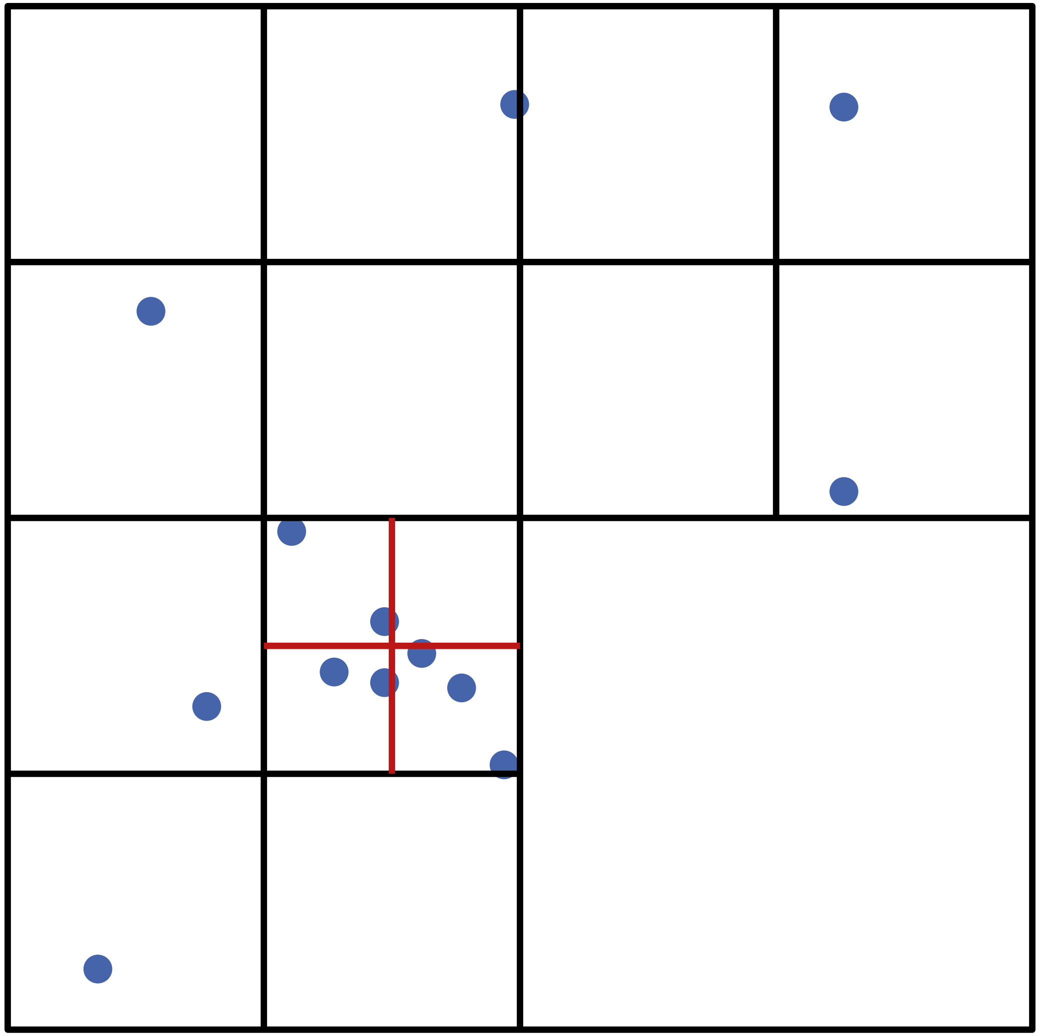

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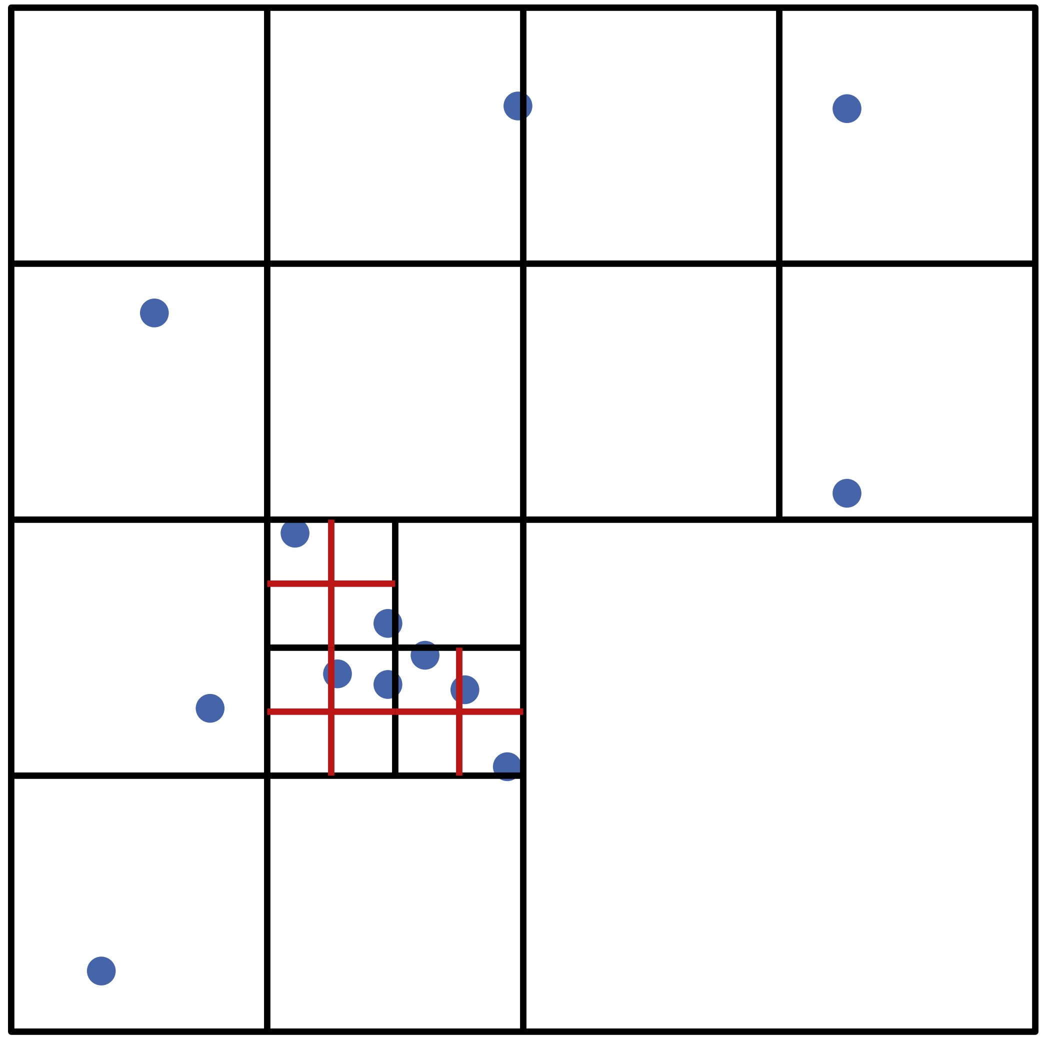

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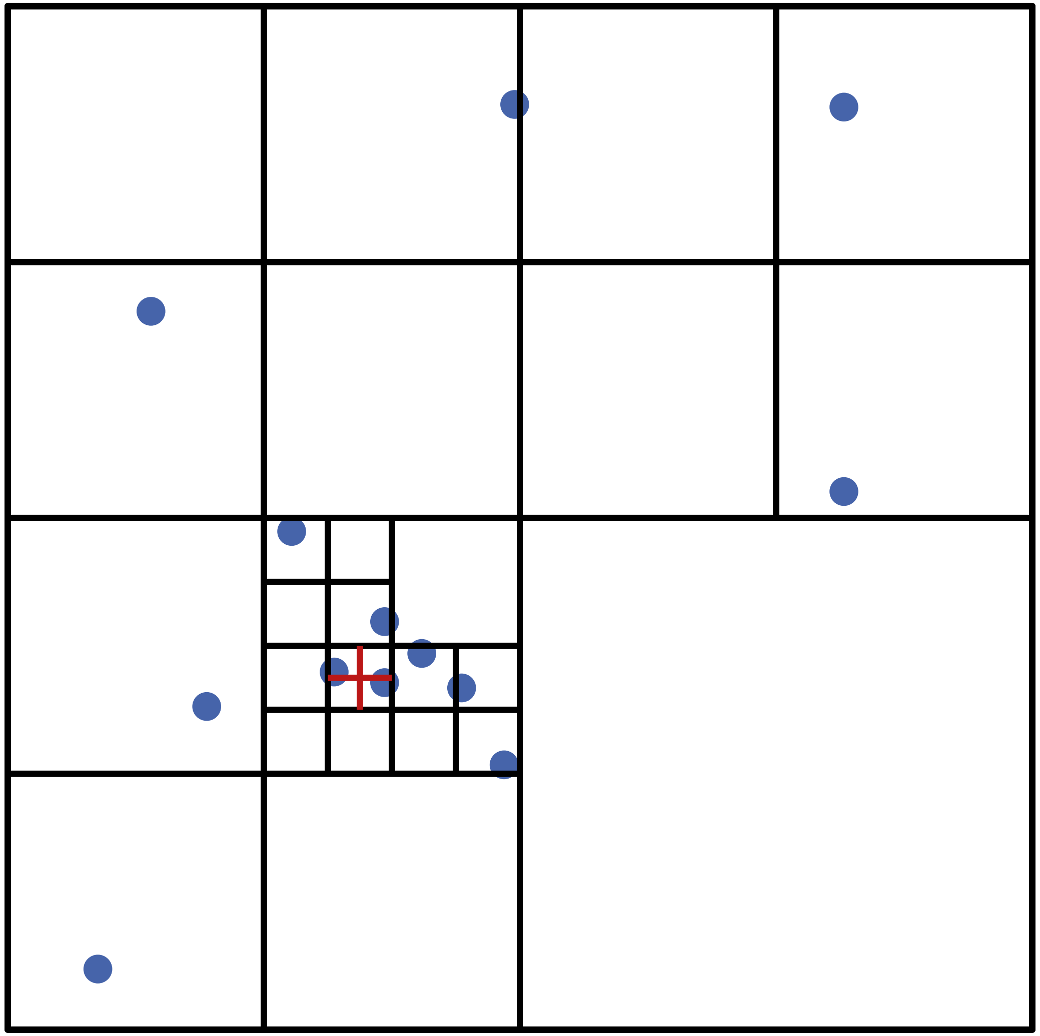

Quadtree Properties

The recursive definition of quadtrees leads directly to an algorithm for constructing them.

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Lemma 1: The depth of $\mathcal{T}(P)$ is at most $\log(s/c) + 3/2$, where c is the smallest distance in P and s is the length of a side of Q.

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What is the depth of a quadtree on n points?

Lemma 1: The depth of $\mathcal{T}(P)$ is at most $\log(s/c) + 3/2$, where c is the smallest distance in P and s is the length of a side of Q.

Theorem 1: A quadtree $\mathcal{T}(P)$ on n points with depth d has $O((d+1)n)$ nodes and can be constructed in $O((d+1)n)$ time.

Finding Neighbors

NorthNeighbor$(v, \mathcal{T})$

Input: Nodes v in quadtree $\mathcal{T}$

Output: Deepest node v' not deeper than v with $v'.Q$ to the north.
Neighbor of $v.Q$

if $v =$ root$(\mathcal{T})$ **then return** nil

$\pi \leftarrow$ parent(v)

if $v = SW$-$/SE$-child of π **then return** NW-$/NE$-child of π

$\mu \leftarrow$ NorthNeighbor$(\pi, \mathcal{T})$

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Theorem 2: Let $\mathcal{T}$ be a quadtree with depth d. The neighbor of a node v in any direction can be found in $O(d+1)$ time.

Balanced Quadtrees

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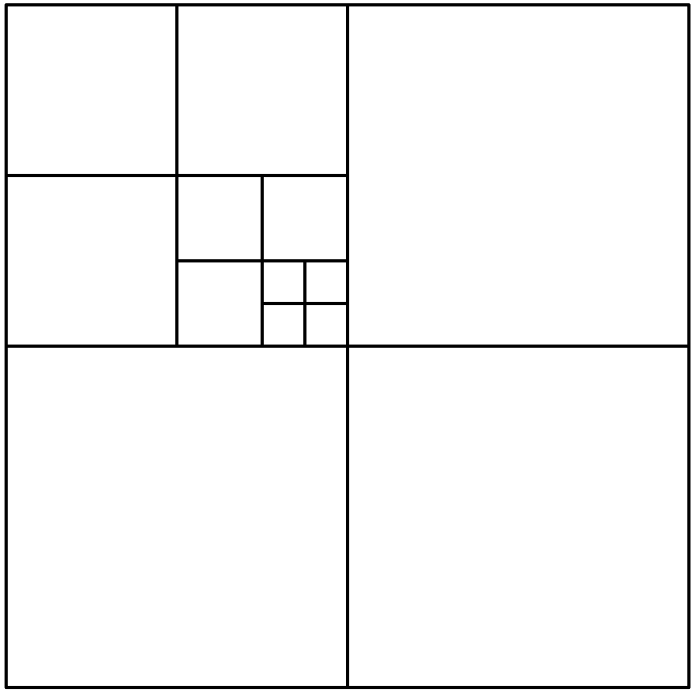

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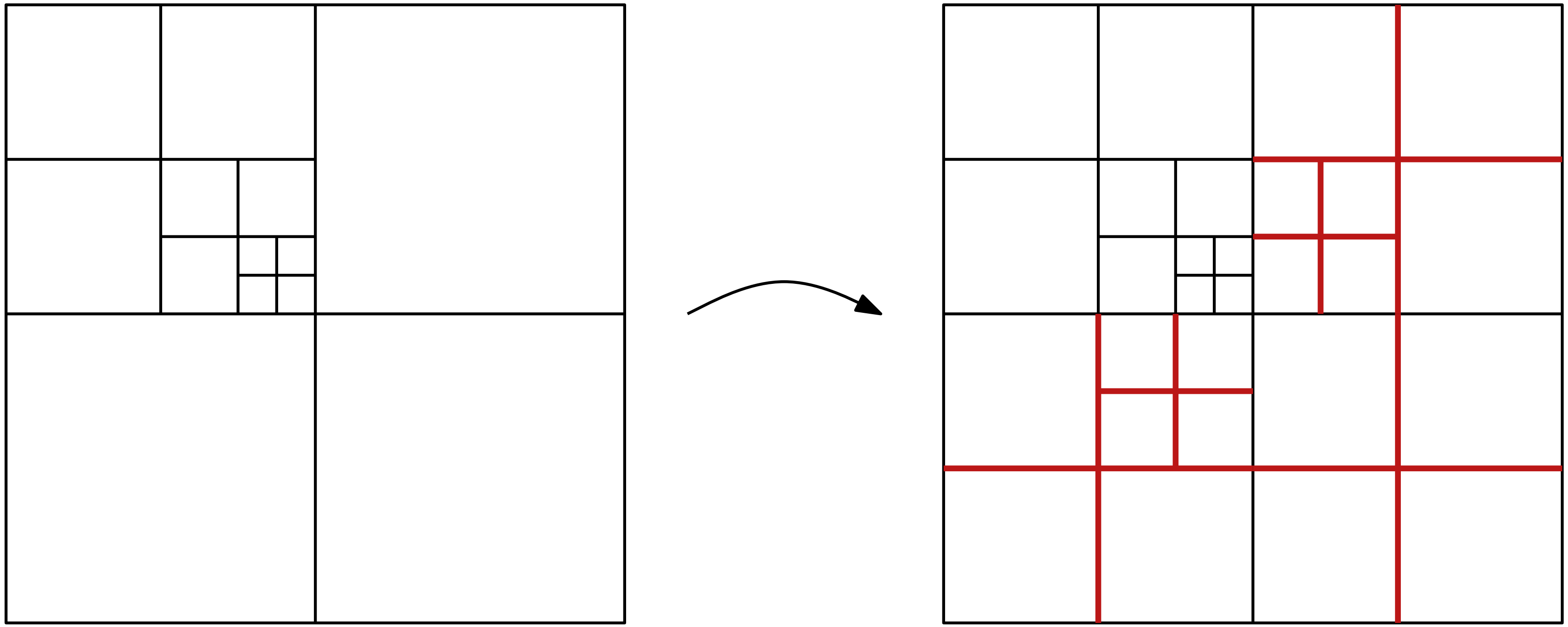

Balancing Quadtrees

BalanceQuadtree($\mathcal{T}$)

Input: Quadtree $\mathcal{T}$
Output: A balanced version of $\mathcal{T}$
$L \leftarrow$ List of all leaves of $\mathcal{T}$
while L not empty **do**
    $\mu \leftarrow$ extract leaf from L
    if $\mu.Q$ too large **then**
        Divide $\mu.Q$ into four parts and put four leaves in $\mathcal{T}$
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How large can a balanced quadtree be?

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Thm 3: Let $\mathcal{T}$ be a quadtree with m nodes and depth d. The balanced version $\mathcal{T}_B$ of $\mathcal{T}$ has $O(m)$ nodes and can be constructed in $O((d+1)m)$ time.

Quadtrees and Adaptive Triangular Meshes

Recall:

Given: Square $Q = [0, U] \times [0, U]$ for power of two $U = 2^j$
with *octilinear*, integer-coordinate polygons inside.

Goal: Triangular mesh for Q with the following properties

valid $\left\{ \begin{array}{l} \text{} \\ \text{} \\ \text{} \end{array} \right.$

- no triangle vertex in interior of triangular mesh
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What needs to be adjusted?

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Squares and edges are finished

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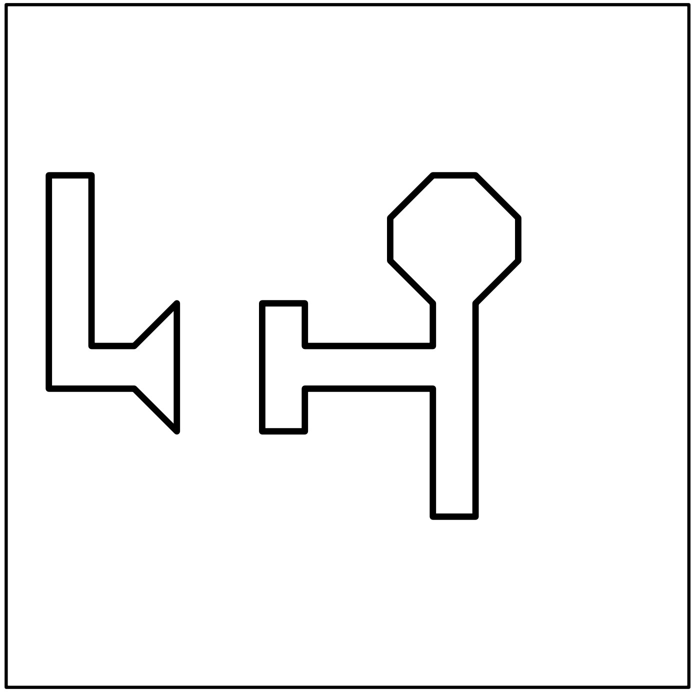
From Quadtree to Triangular Mesh



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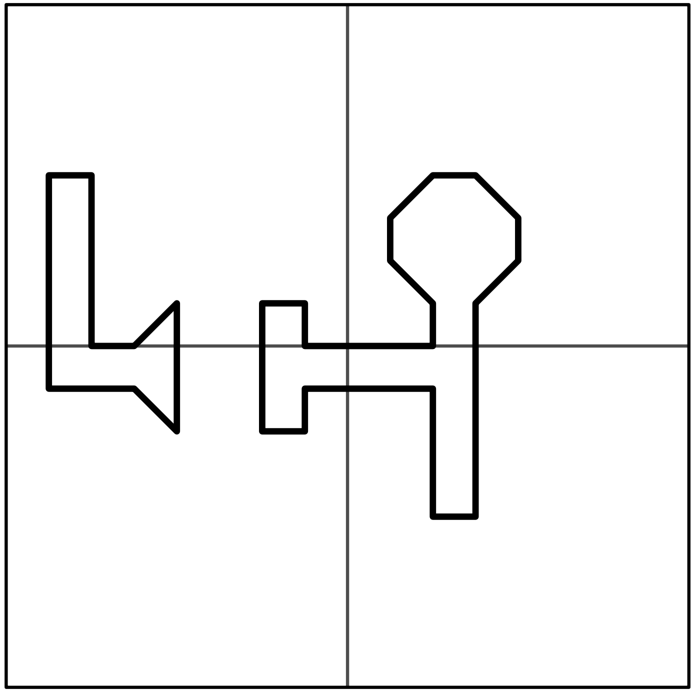

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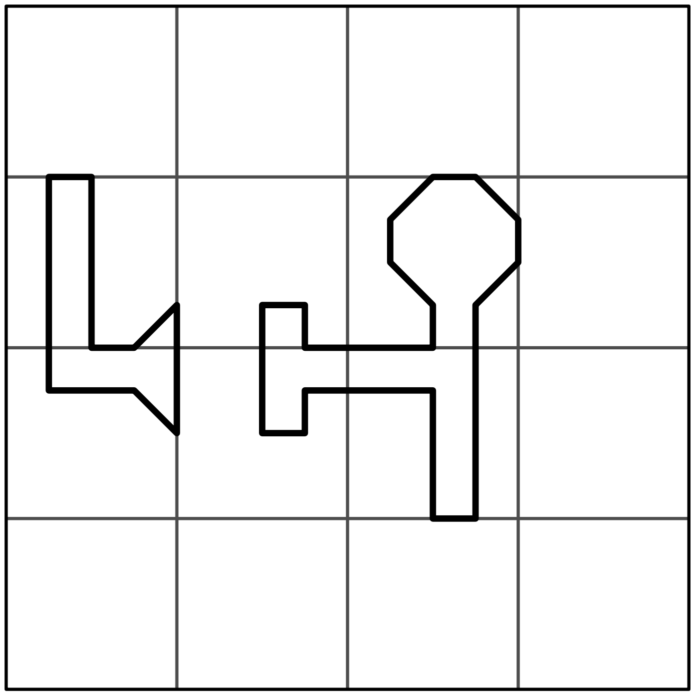

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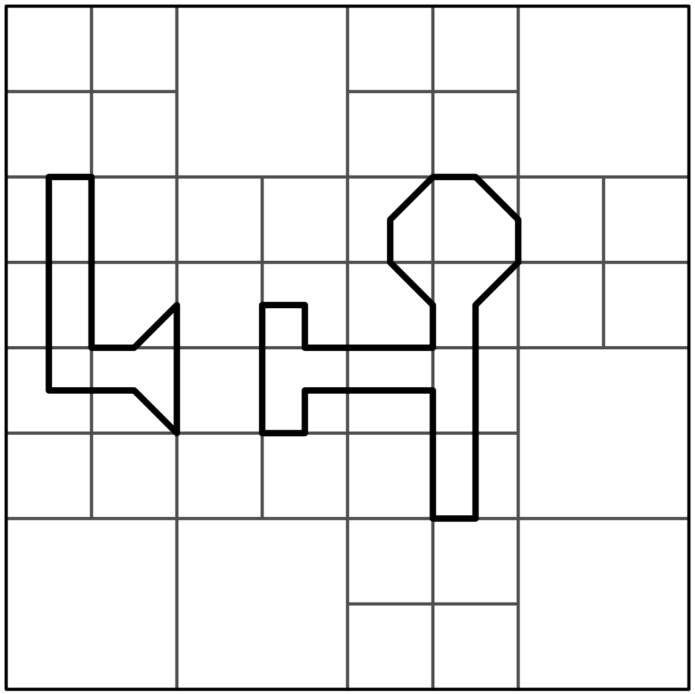

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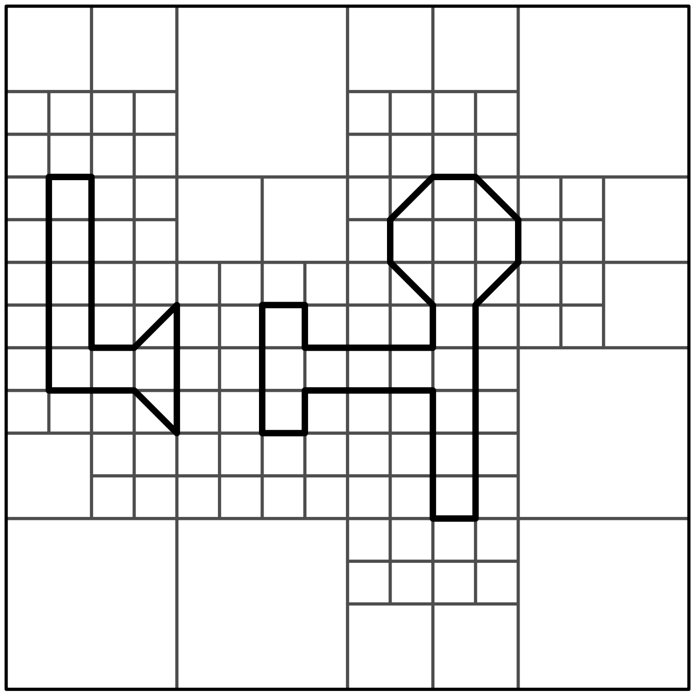

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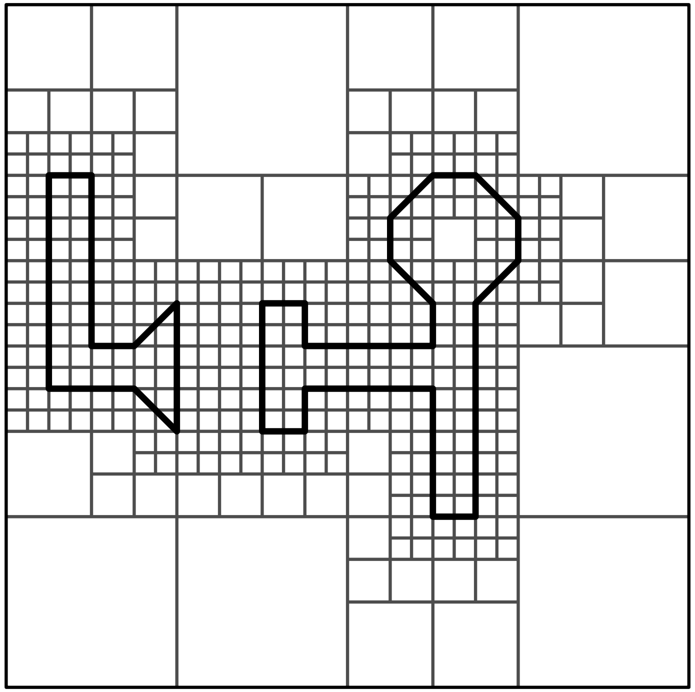

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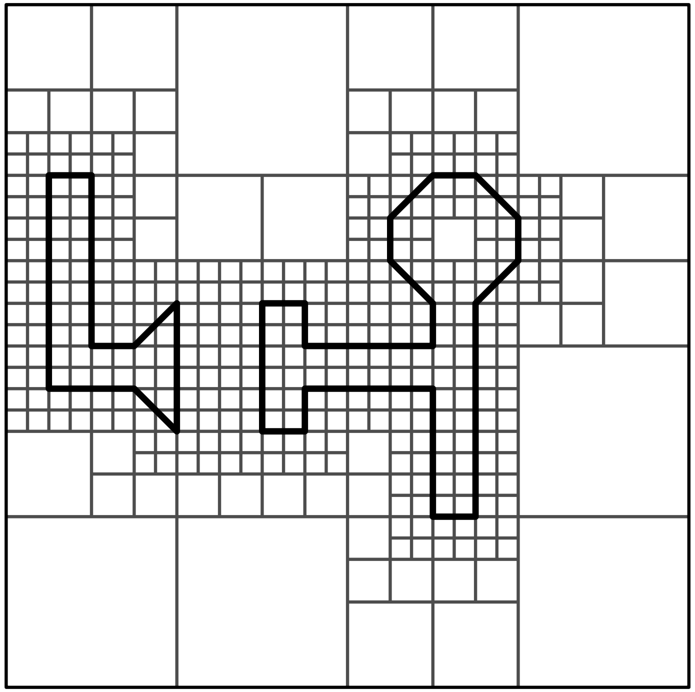

Obs.: when the interior of a square in a quadtree is intersected by an edge, then it is a diagonal.

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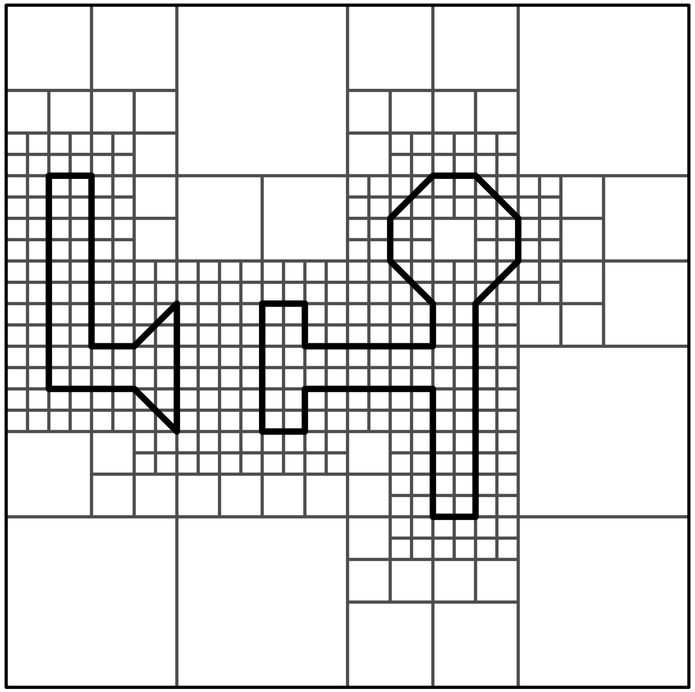

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How do you get a valid triangular mesh?

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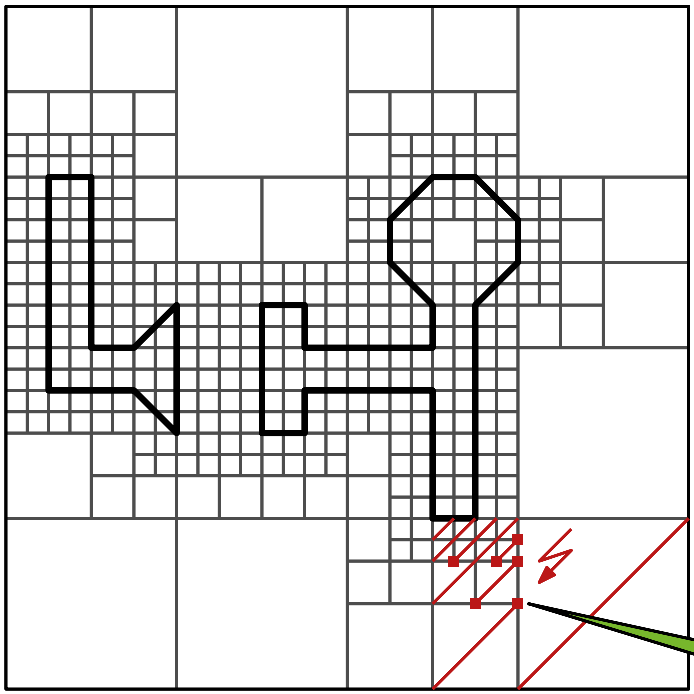

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nodes on interior edges

- Diagonals for all remaining squares?

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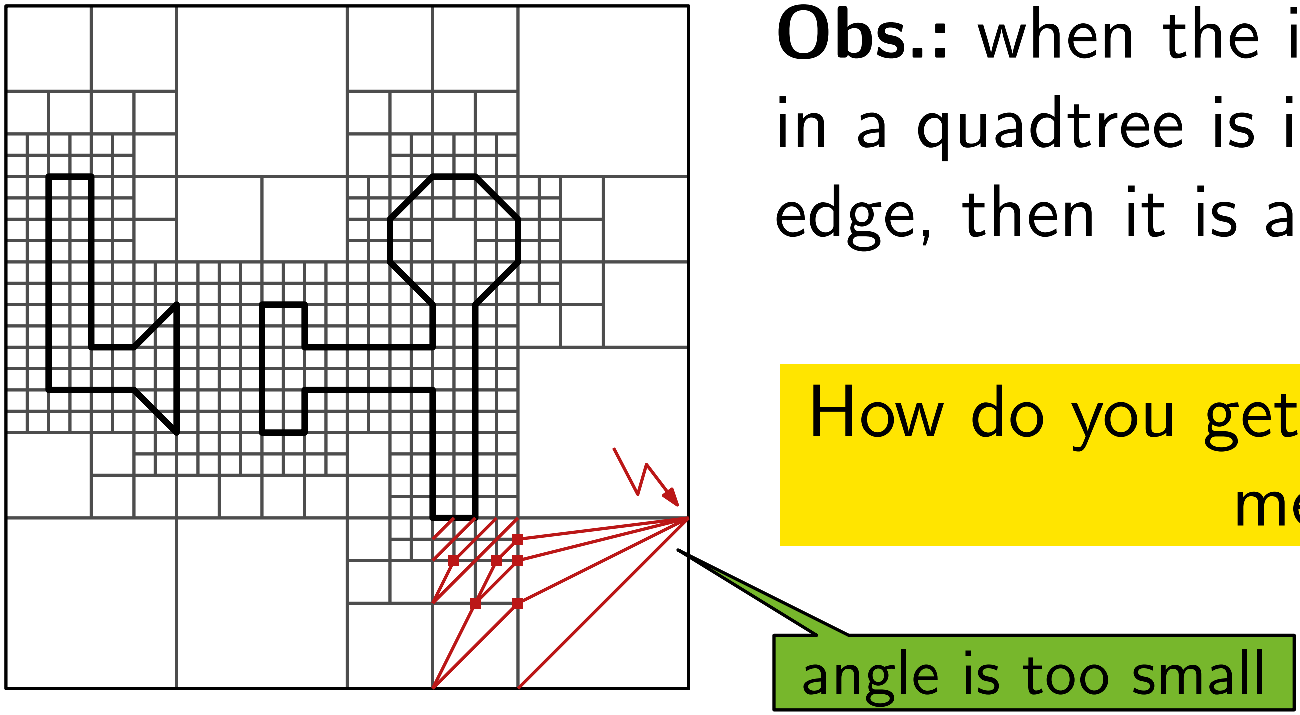

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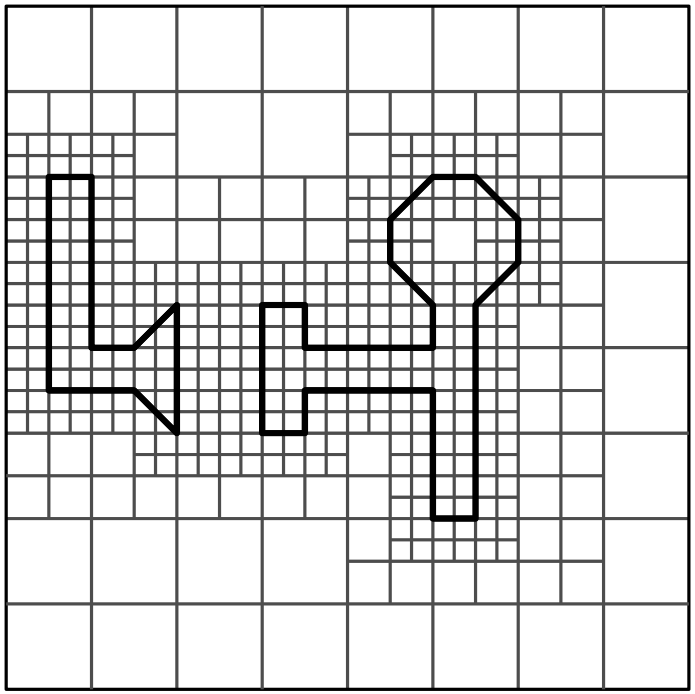
How do you get a valid triangular mesh?

- Diagonals for all remaining squares? no!
- Use subdivision nodes in triangulation?

From Quadtree to Triangular Mesh



Obs.: when the interior of a square in a quadtree is intersected by an edge, then it is a diagonal.

How do you get a valid triangular mesh?

- Diagonals for all remaining squares? no!
- Use subdivision nodes in triangulation? no!
- Balance quadtree and, if necessary, add a Steiner vertex!

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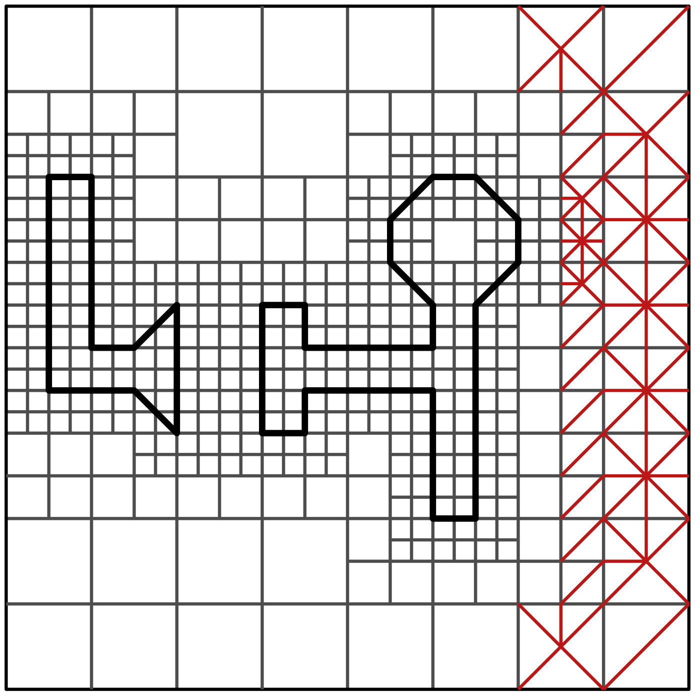

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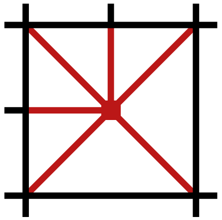

Algorithm

GenerateMesh(S)

Input: Set S octilinear, integer-coordinate polygons in
$Q = [0, 2^j] \times [0, 2^j]$

Output: valid, adaptive triangular mesh for S

$\mathcal{T} \leftarrow$ CreateQuadtree
$\mathcal{T} \leftarrow$ BalanceQuadtree($\mathcal{T}$)
$\mathcal{D} \leftarrow$ DCEL for subdivisions of Q by $\mathcal{T}$
foreach Face f in $\mathcal{D}$ **do**
    if int(f) $\cap S \neq \emptyset$ **then**
        add appropriate diagonals in f to $\mathcal{D}$
    else
        if Nodes only on the corners of f **then**
            add a diagonal in f to $\mathcal{D}$
        else
            generate Steiner point in the middle of f and connect it to all nodes in ∂f of $\mathcal{D}$
return $\mathcal{D}$

Summary

Theorem 4: For a set S of disjoint octilinear, integer-coordinate polygons with total perimeter $p(S)$ in a square $Q = [0, U] \times [0, U]$ we can compute in $O(p(S) \log^2 U)$ time a valid adaptive triangular mesh with $O(p(S) \log U)$ triangles.

Discussion

Karlsruhe Institute of Technology

Are there quadtree variants with space linear in n?

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Yes, if you contract internal nodes with only one non-empty child you get a so-called *compressed quadtree* (see exercise); a more advanced data structure is the *skip quadtree* with $O(n)$ space and insert, remove, and search in $O(\log n)$ time. [Eppstein et al., '05]

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As always: higher dimensions?

Quadtrees can be easily generalized to higher dimensions. Then they are also called octrees.