Computational Geometry • Lecture

Linear Programming

INSTITUTE FOR THEORETICAL INFORMATICS · FACULTY OF INFORMATICS

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Profit optimization

- You are the boss of a company, that produces two products P_1 und P_2 from three raw materials R_1, R_2 und R_3.
- Let's assume you produce x_1 items of the product P_1 and x_2 items of product P_2.
- Assume that items $P_1,\ P_2$ get profit of $300€$ and $500€$, respectively. Then the total profit is:

$$G(x_1, x_2) = 300x_1 + 500x_2$$

- Assume that the amout of raw material you need for P_1 and P_2 is:

$$\begin{aligned} P_1: &\quad 4R_1 + R_2 \\ P_2: &\quad 11R_1 + R_2 + R_3 \end{aligned}$$

- And in your warehouse there are $880R_1$, $150R_2$ and $60R_3$. So:

$$\begin{aligned} R_1: &\quad 4x_1 + 11x_2 \leq 880 \\ R_2: &\quad x_1 + x_2 \leq 150 \\ R_3: &\quad x_2 \leq 60 \end{aligned}$$

- Which choice for (x_1, x_2) maximizes your profit?

Solution

Linear constraints:

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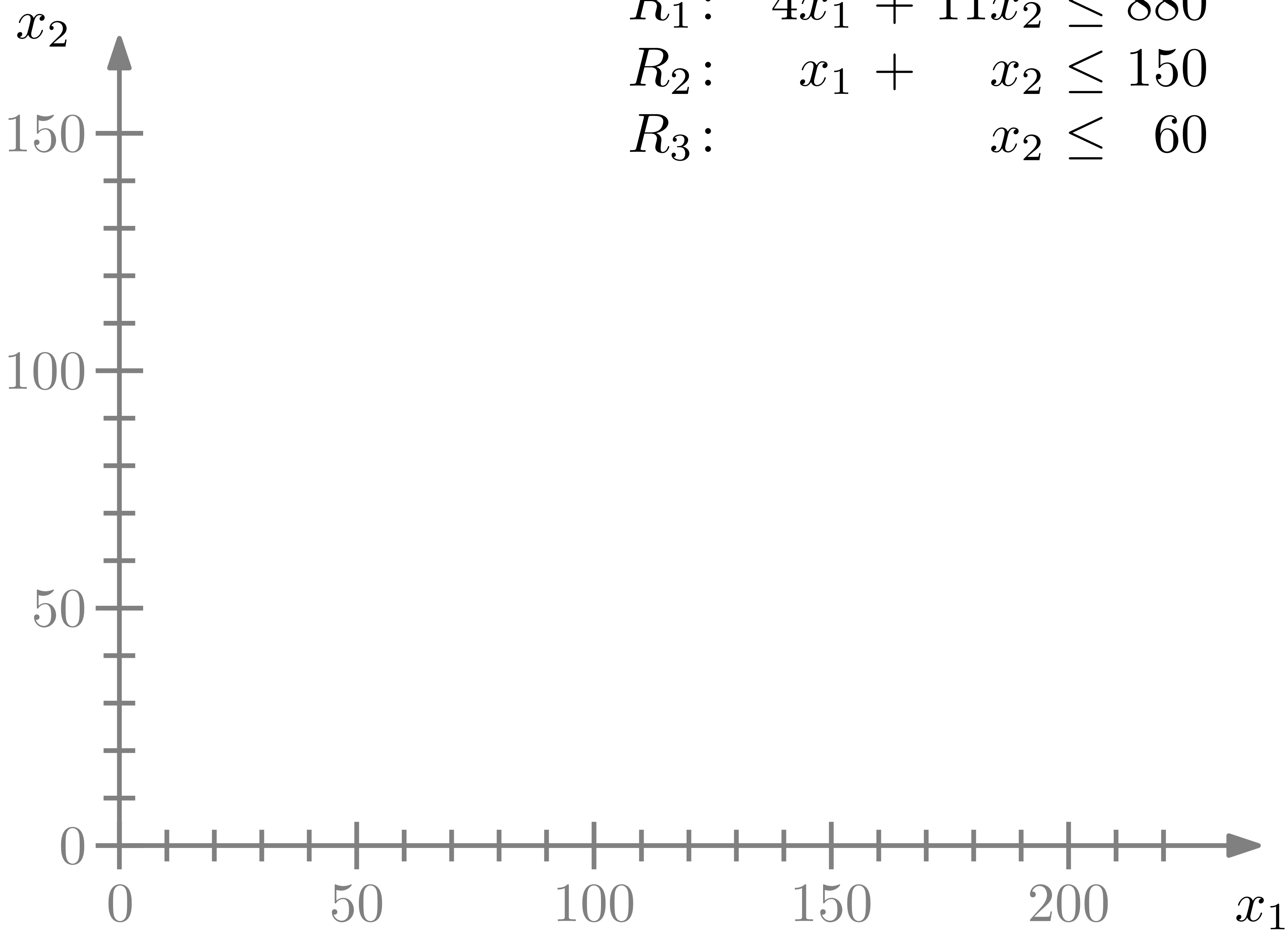

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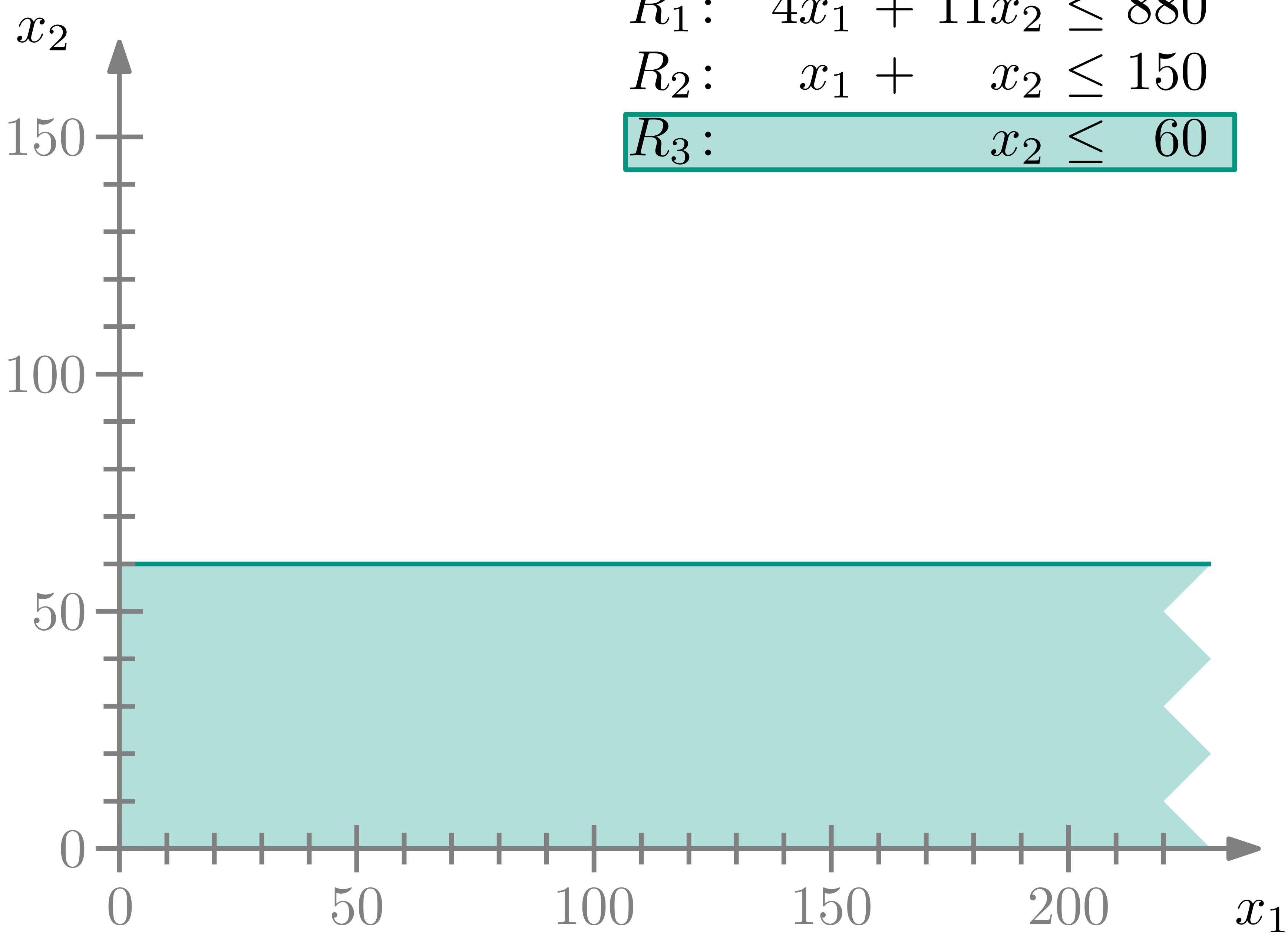

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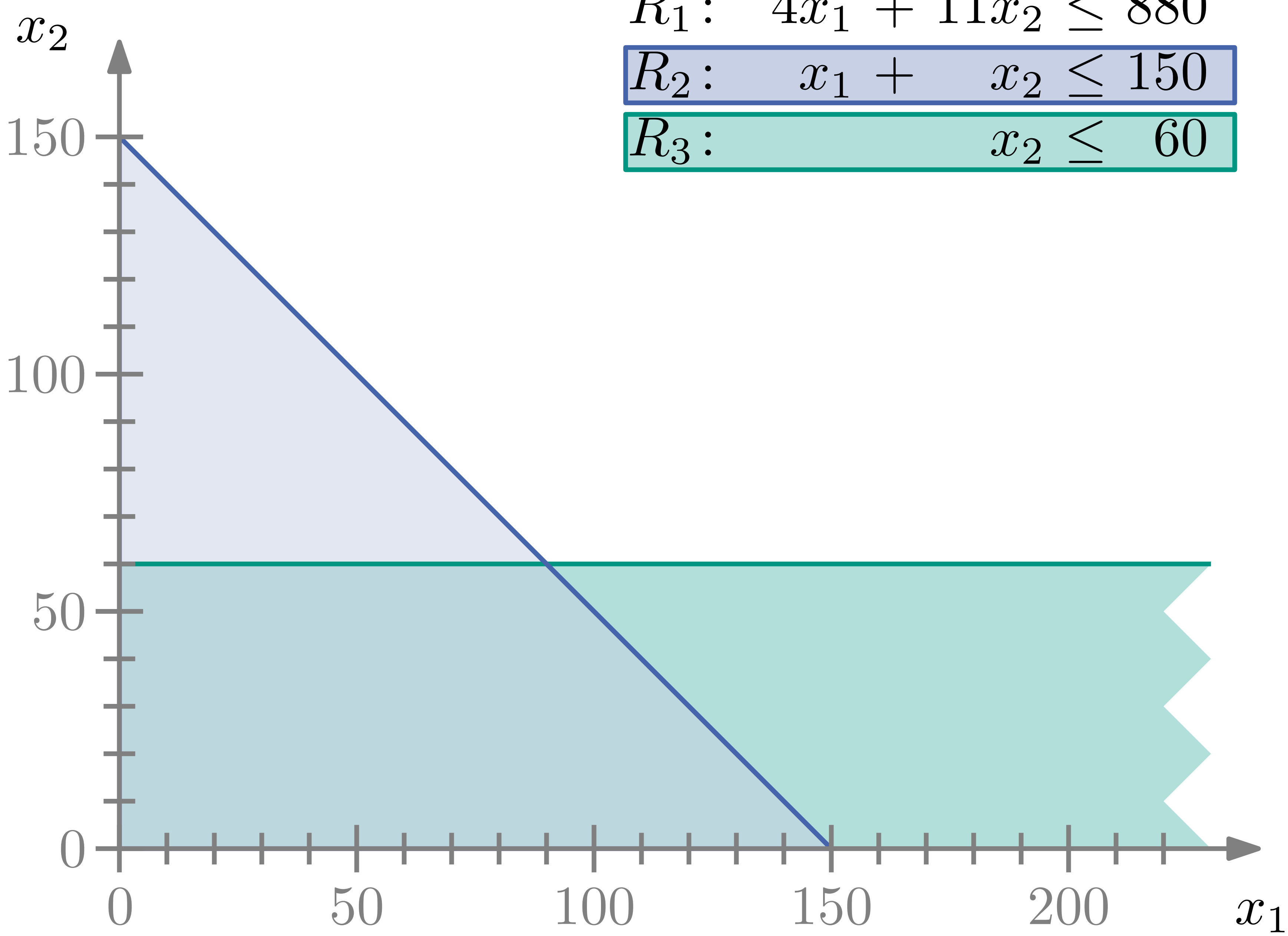

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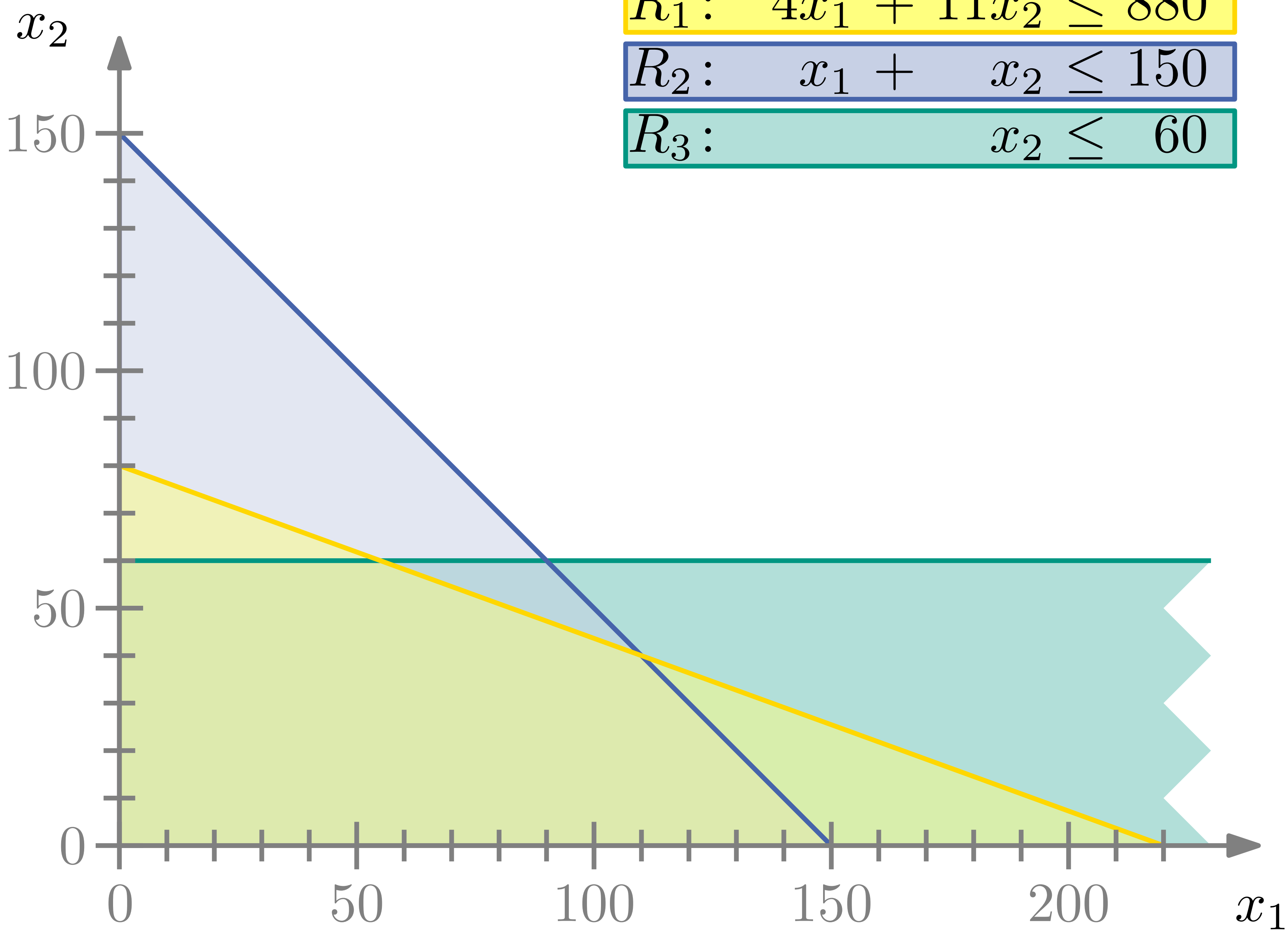

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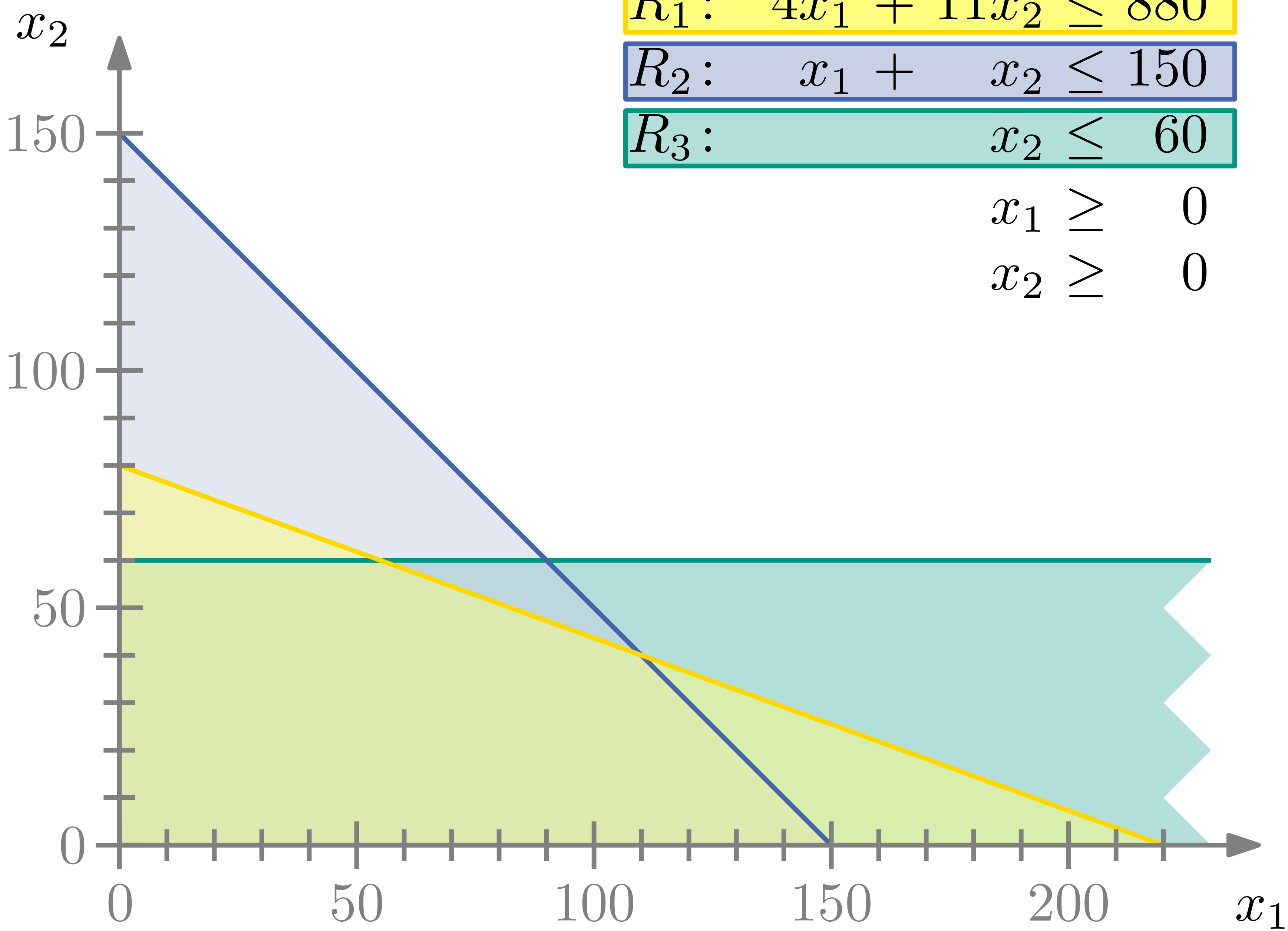

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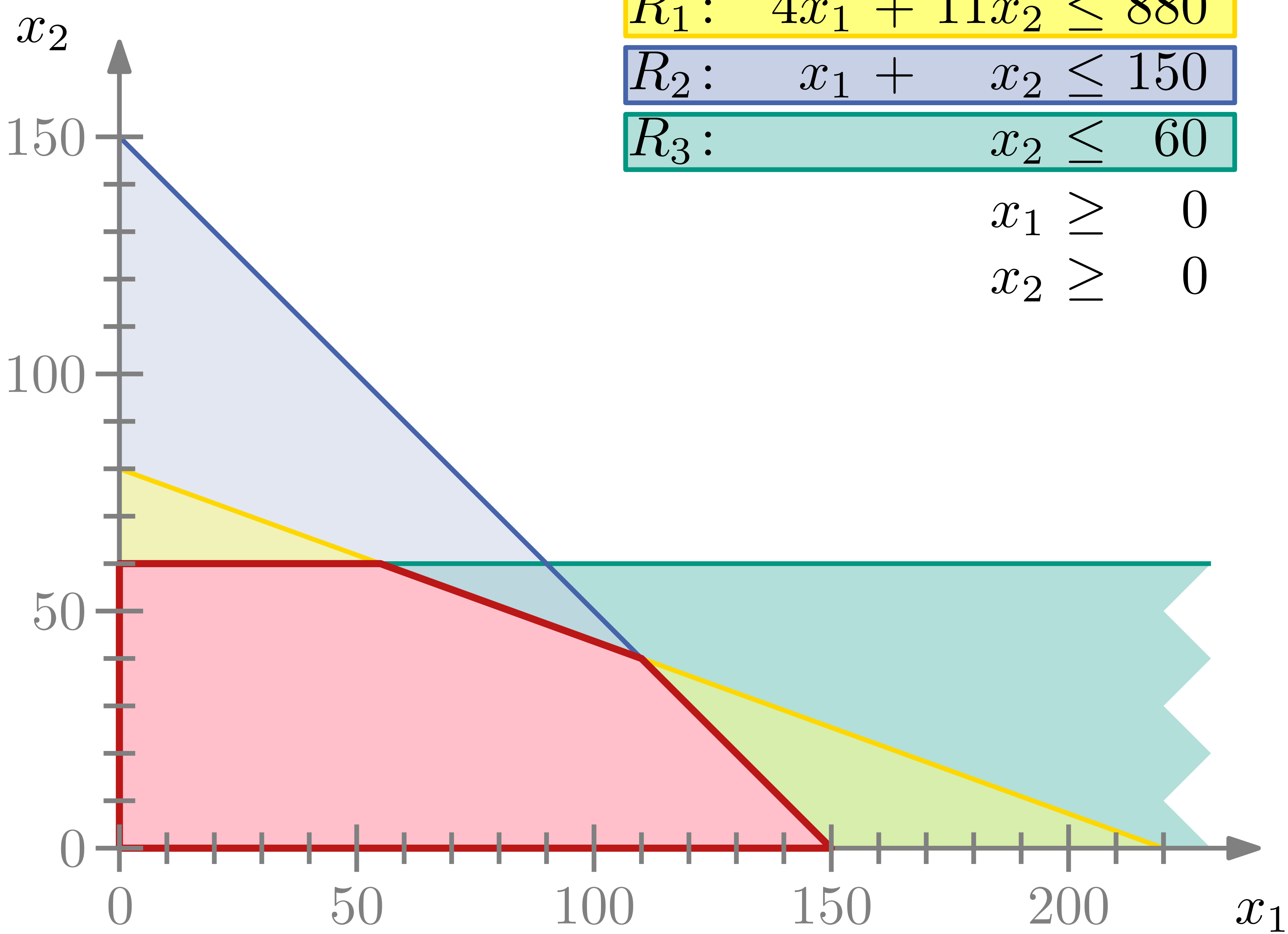

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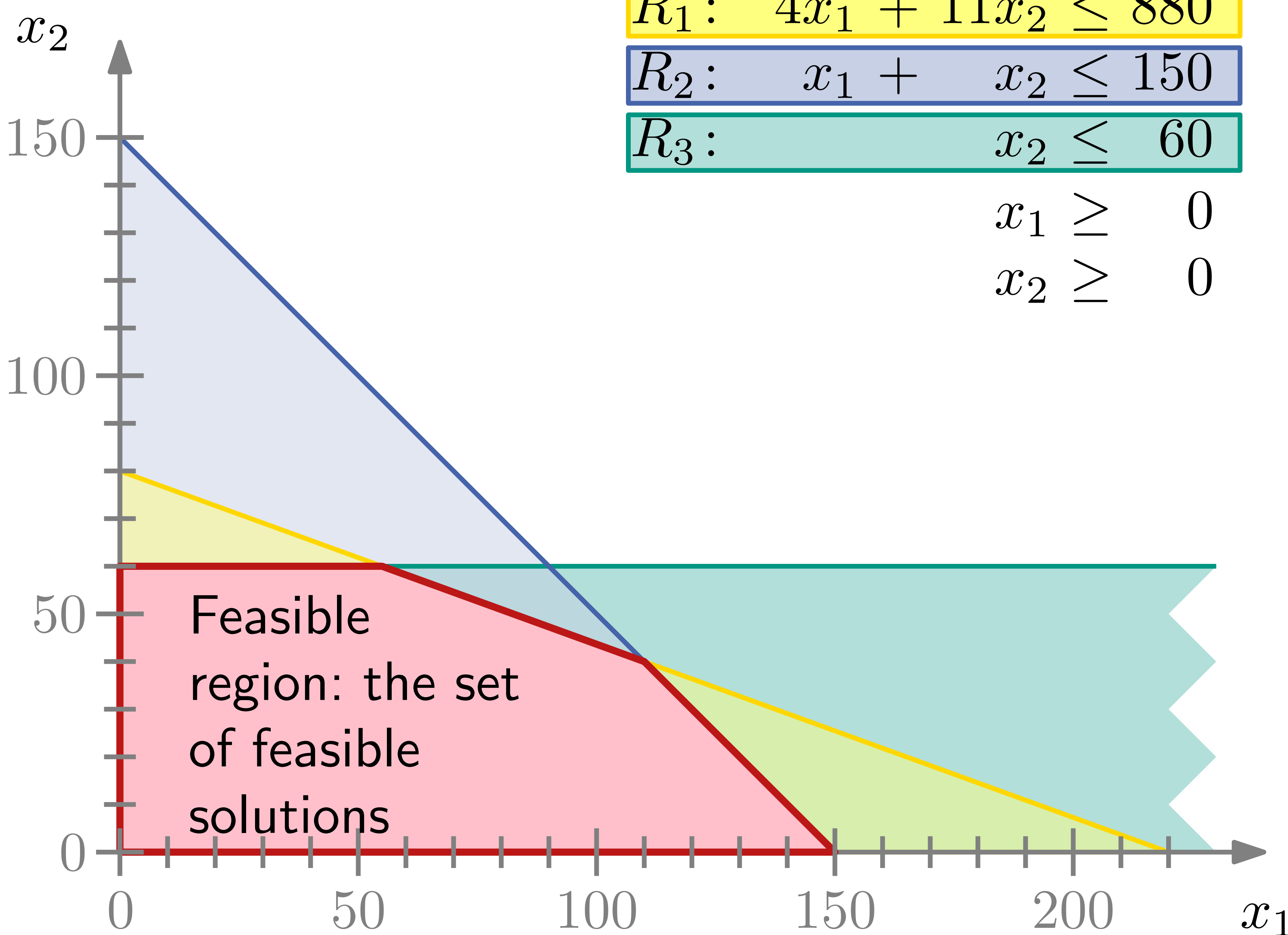

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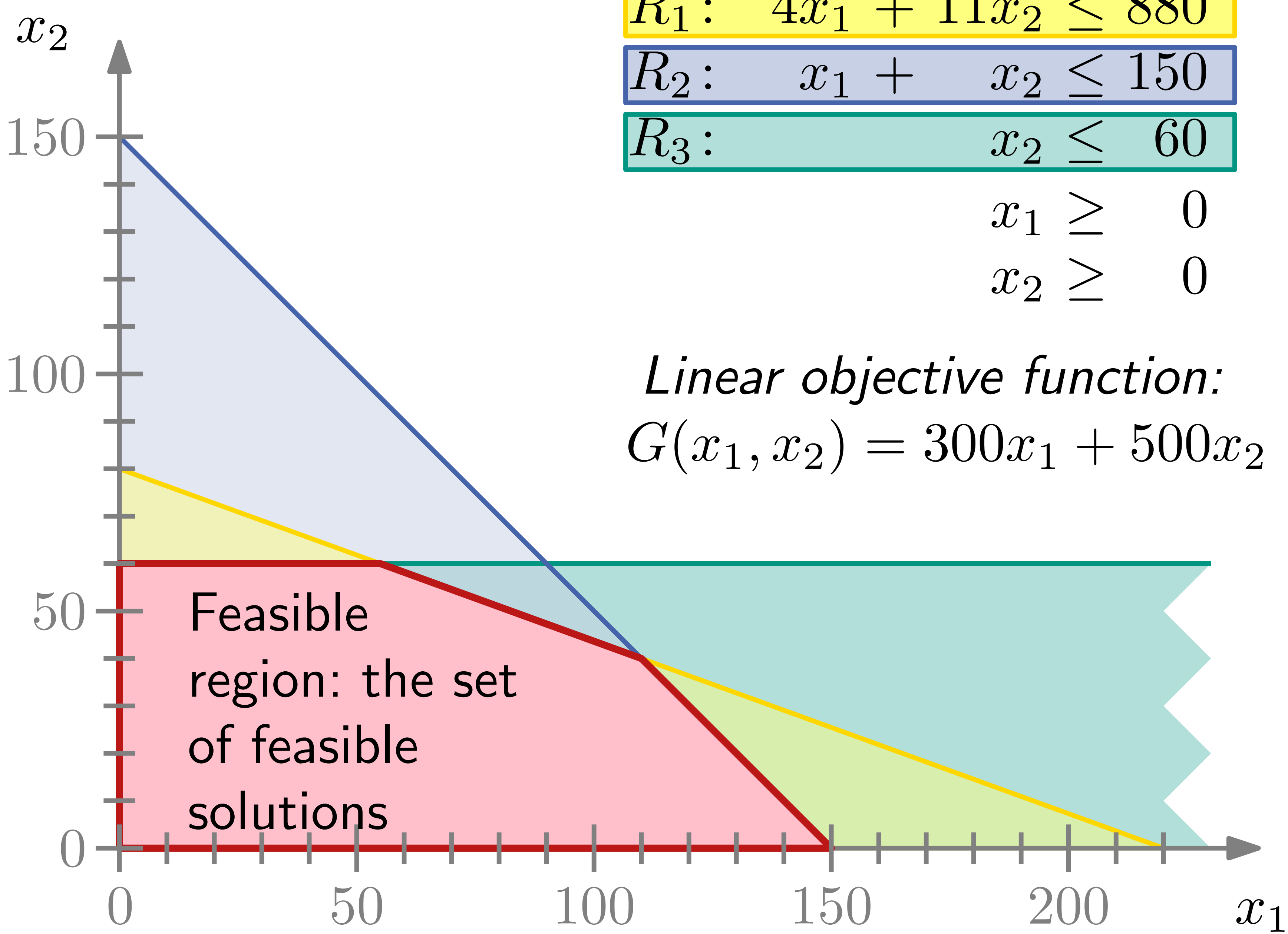

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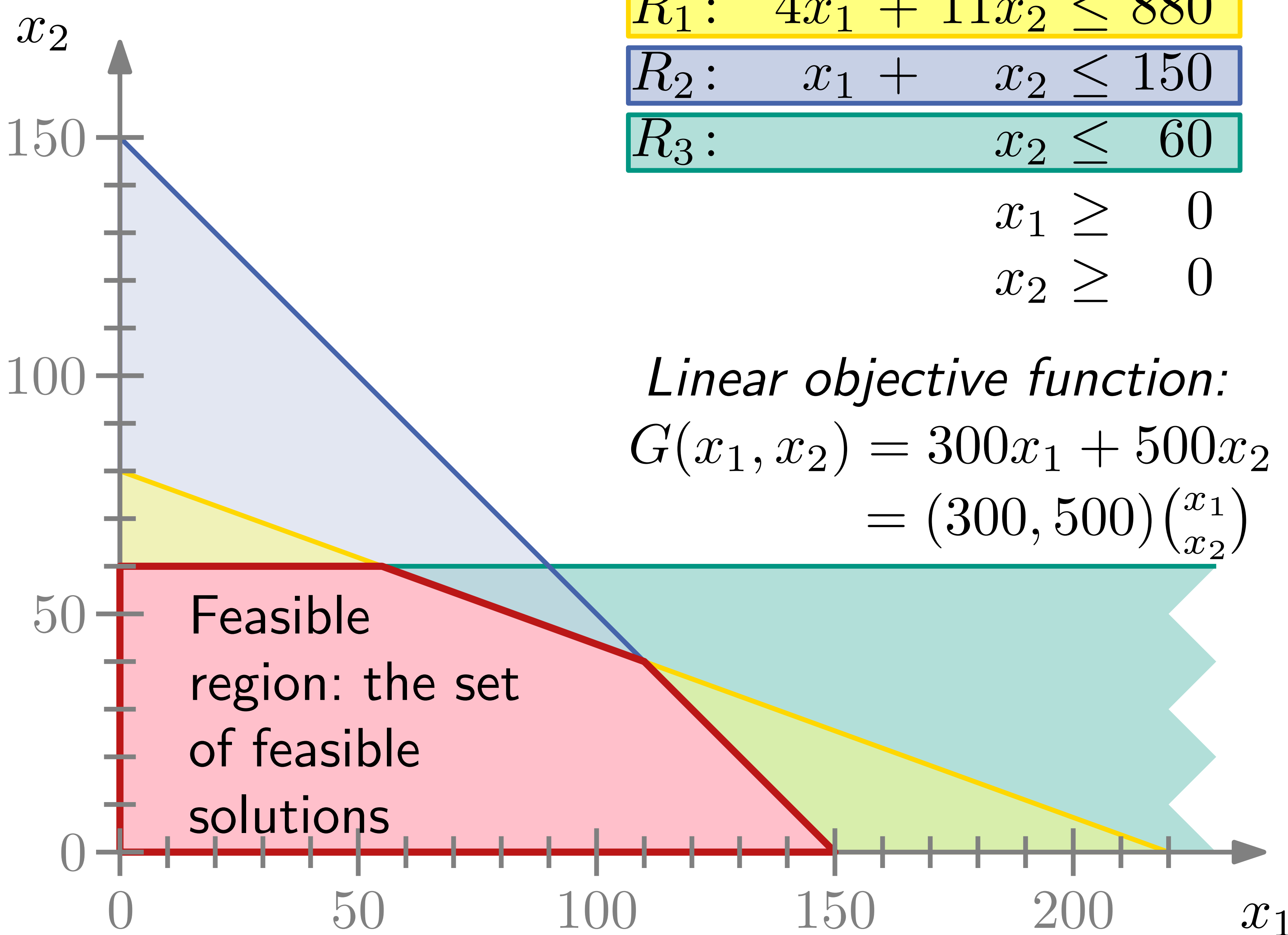

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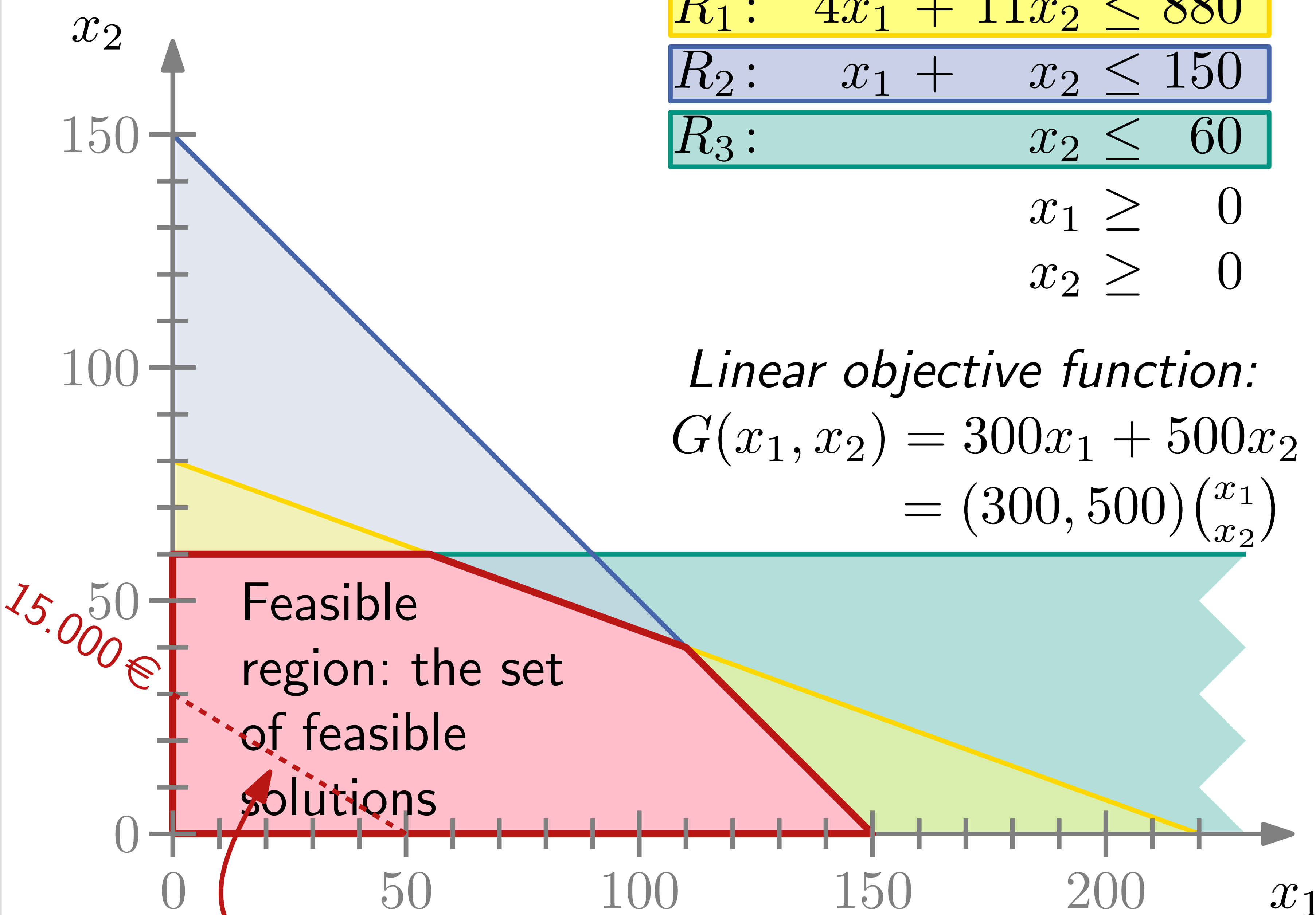

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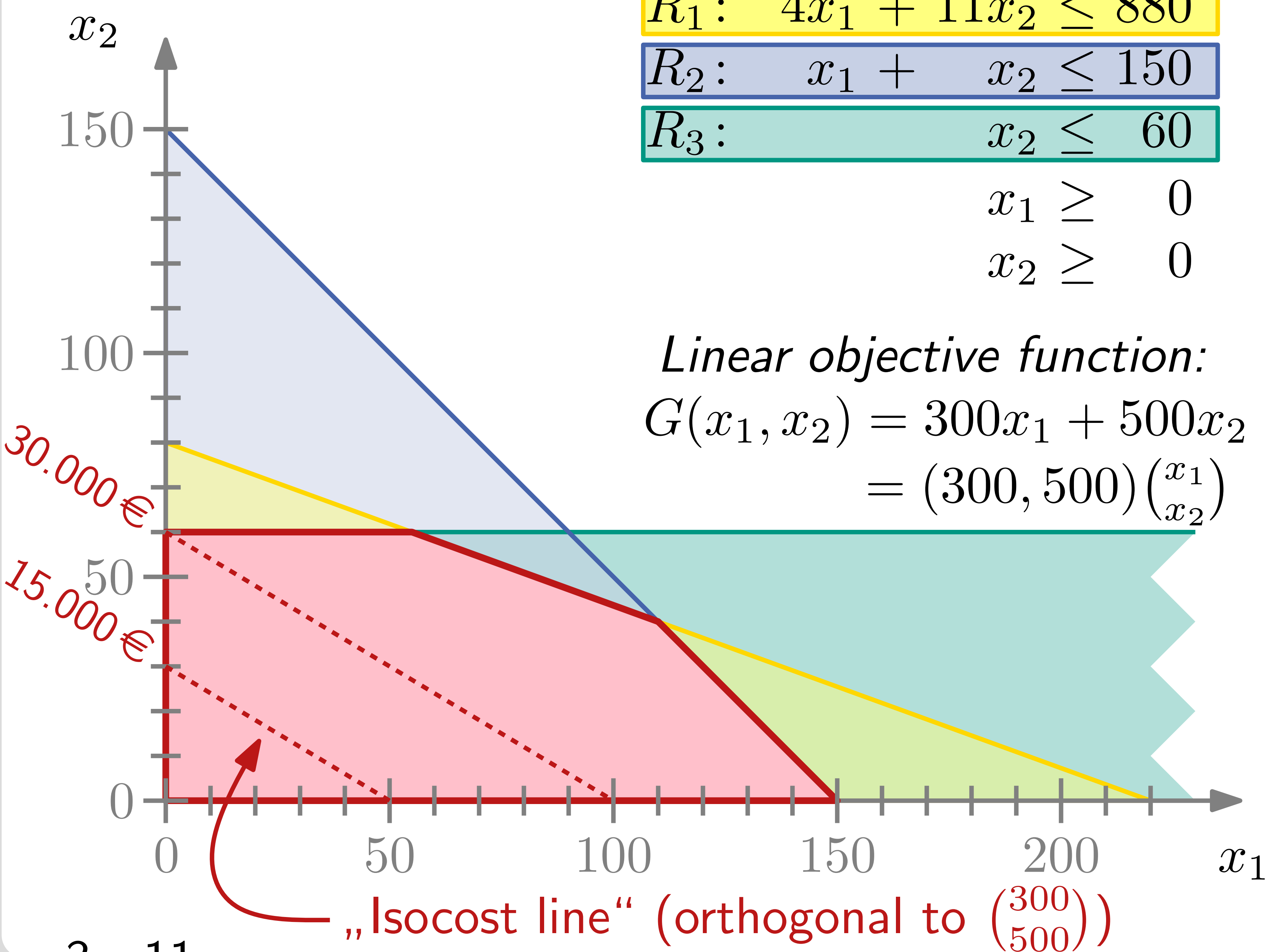

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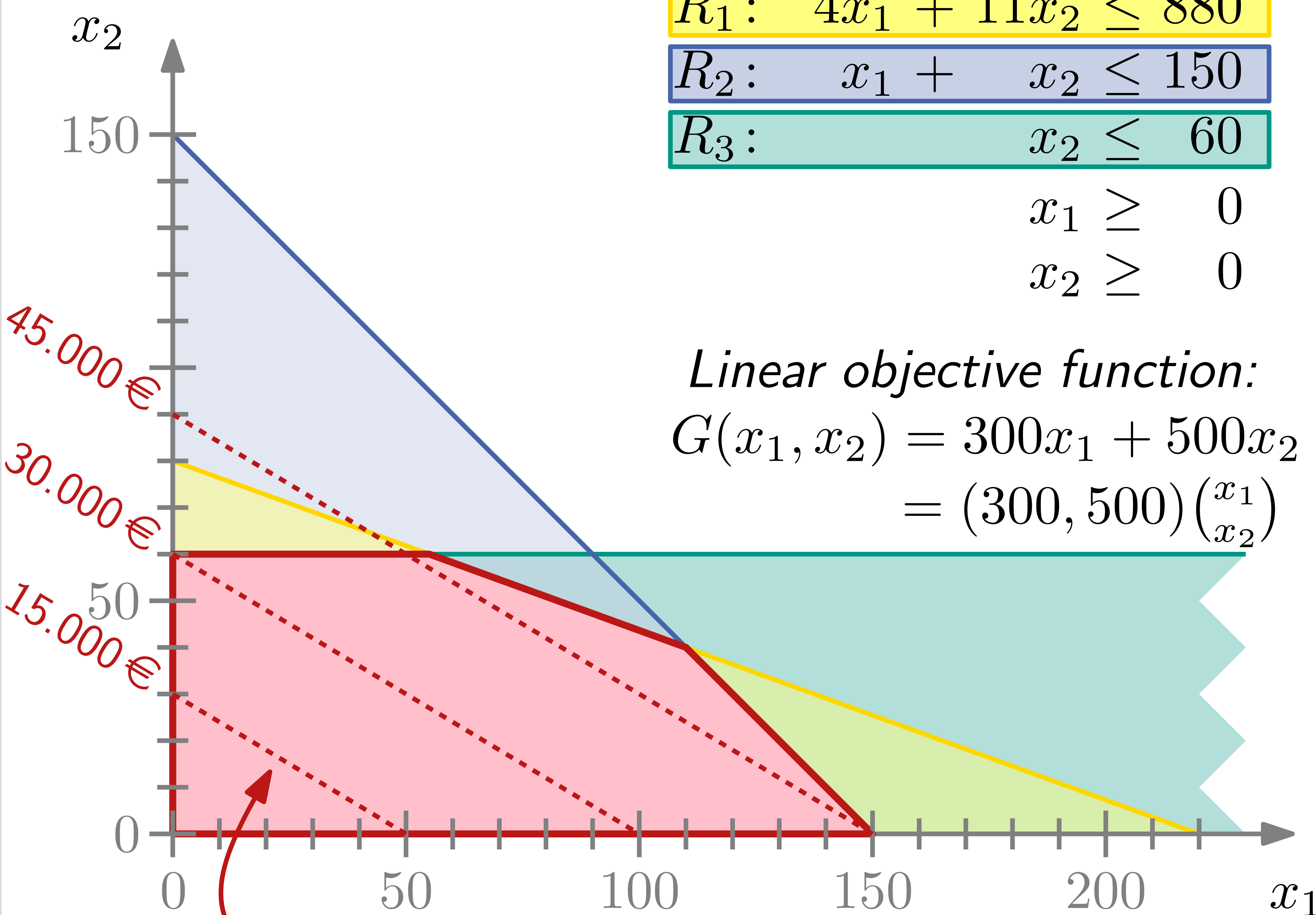

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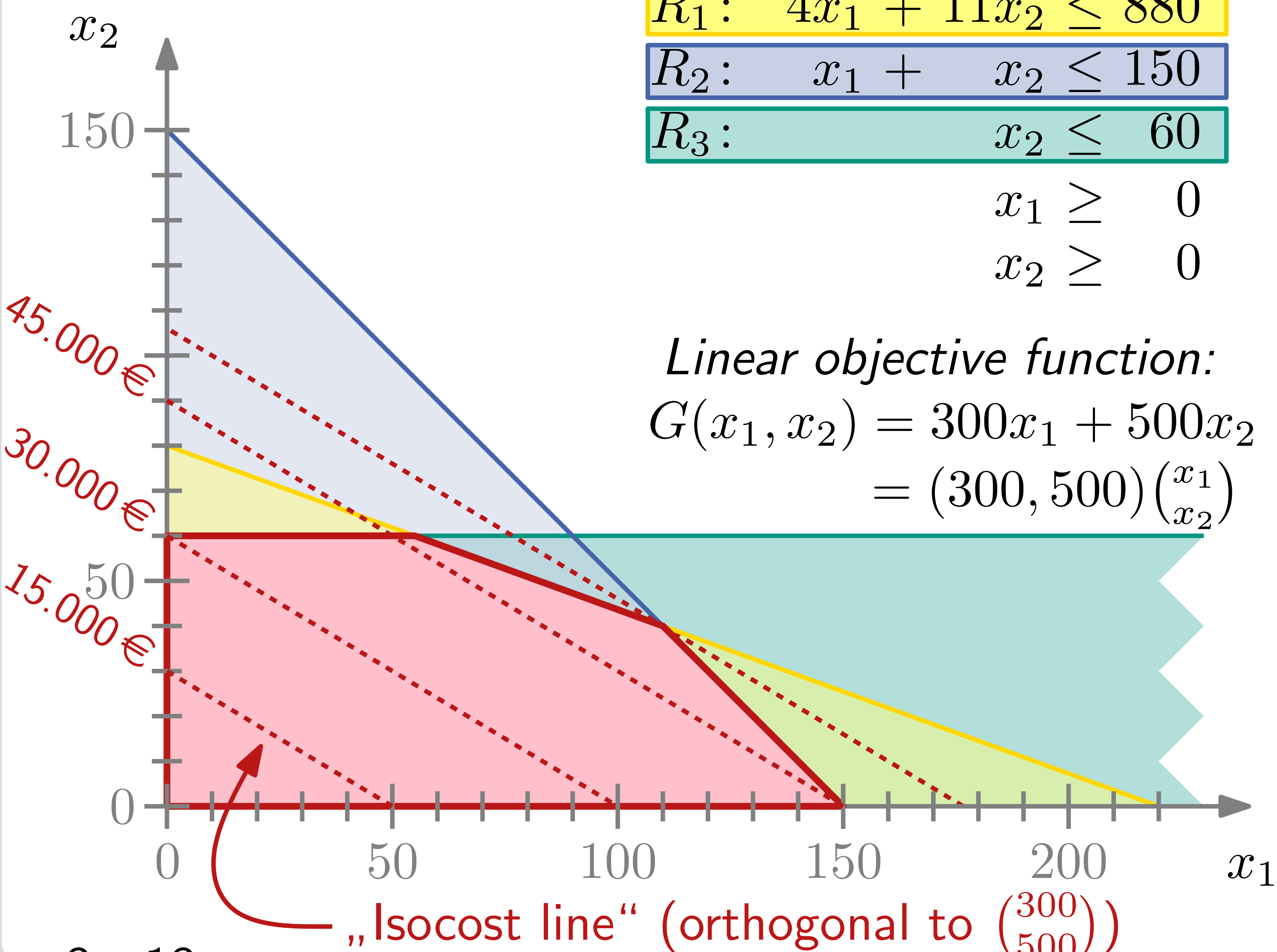

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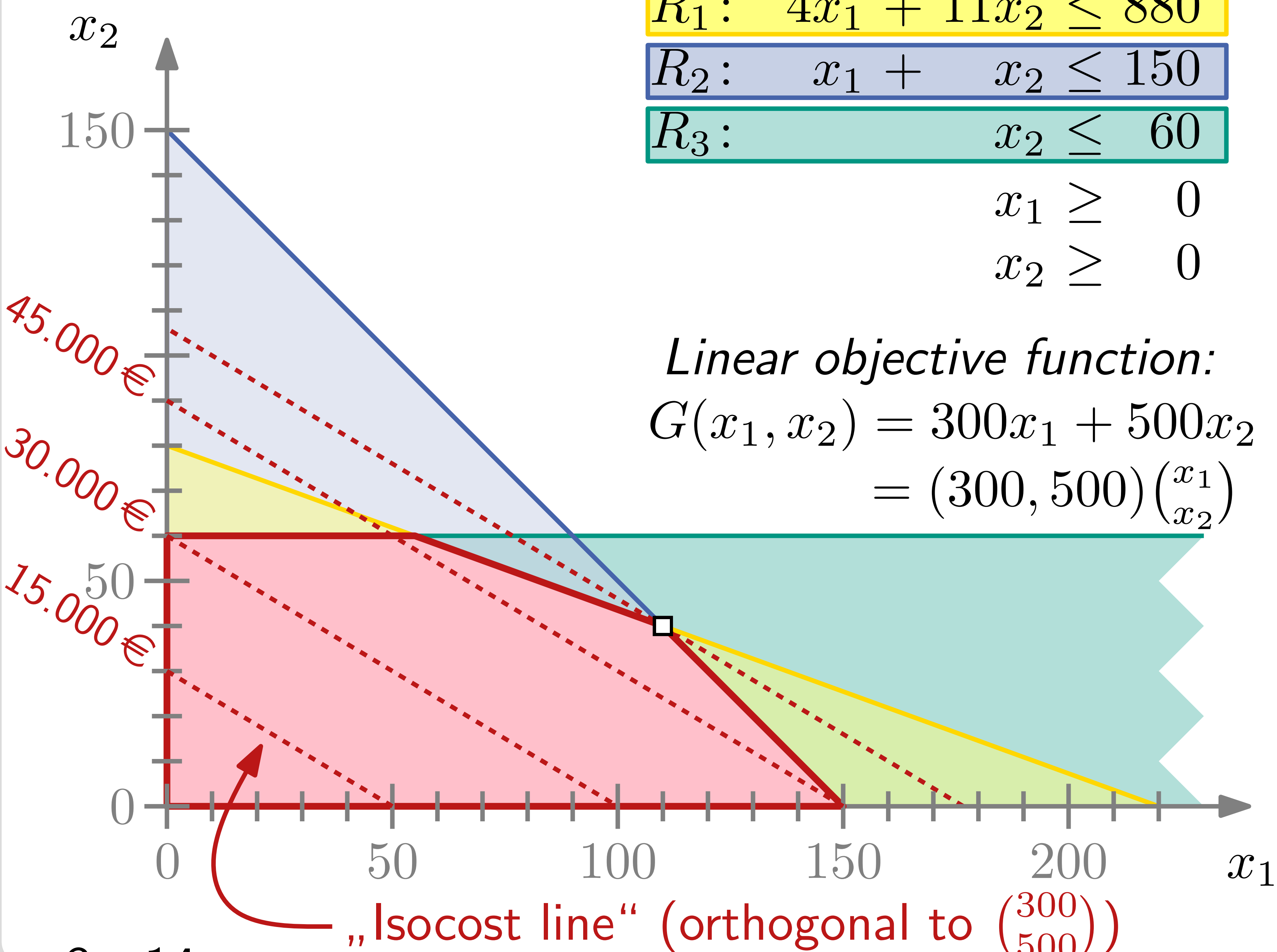

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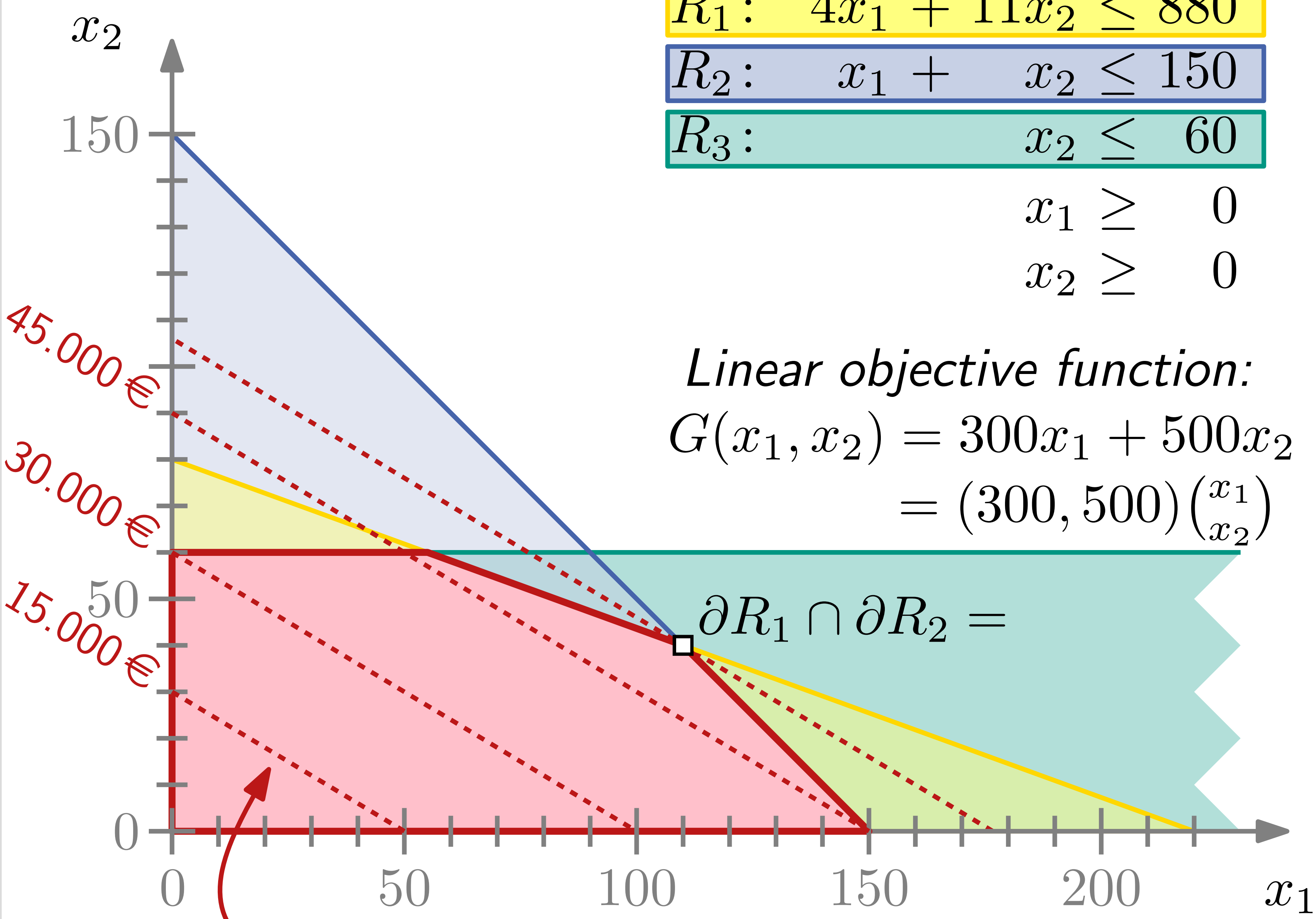

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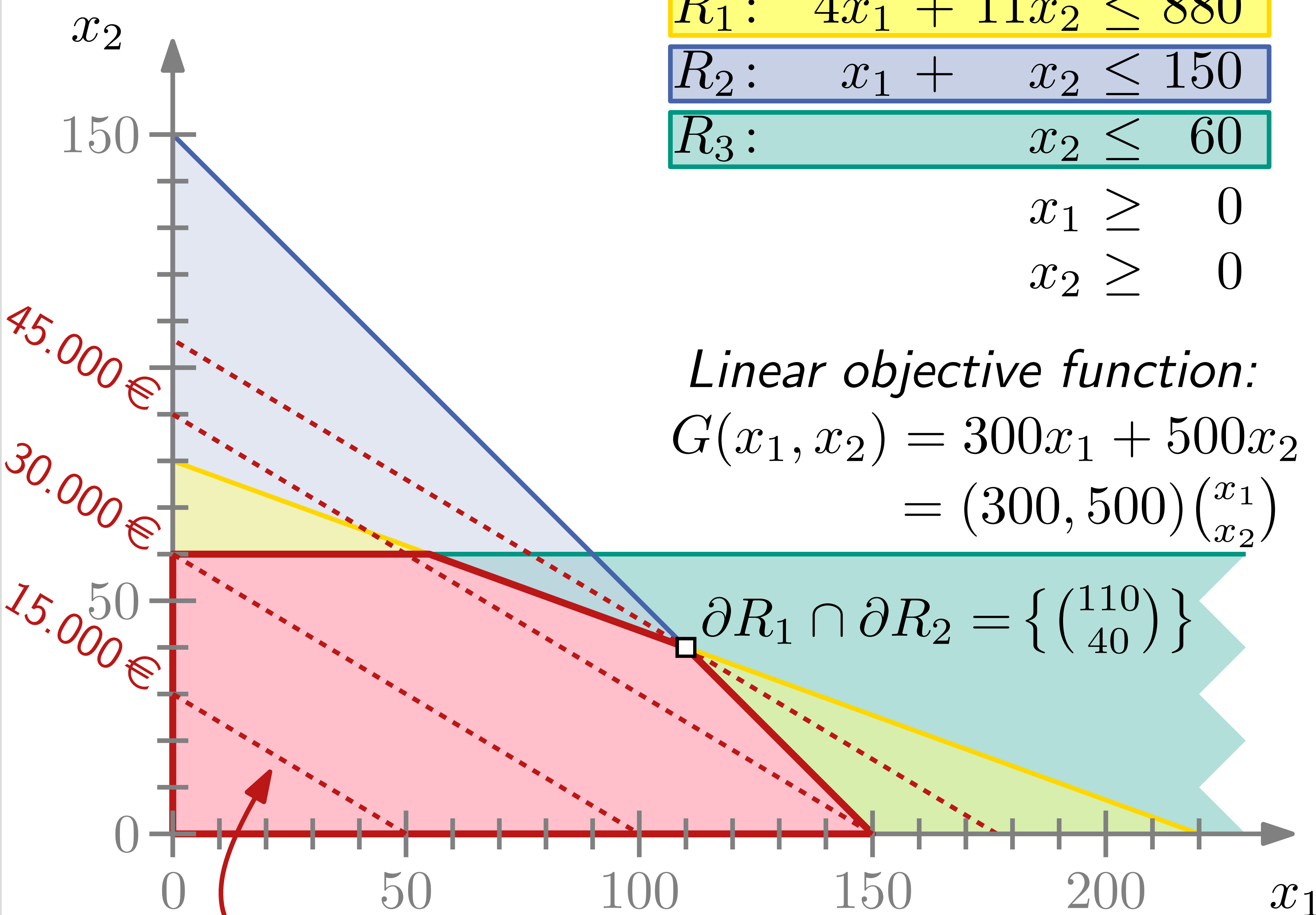

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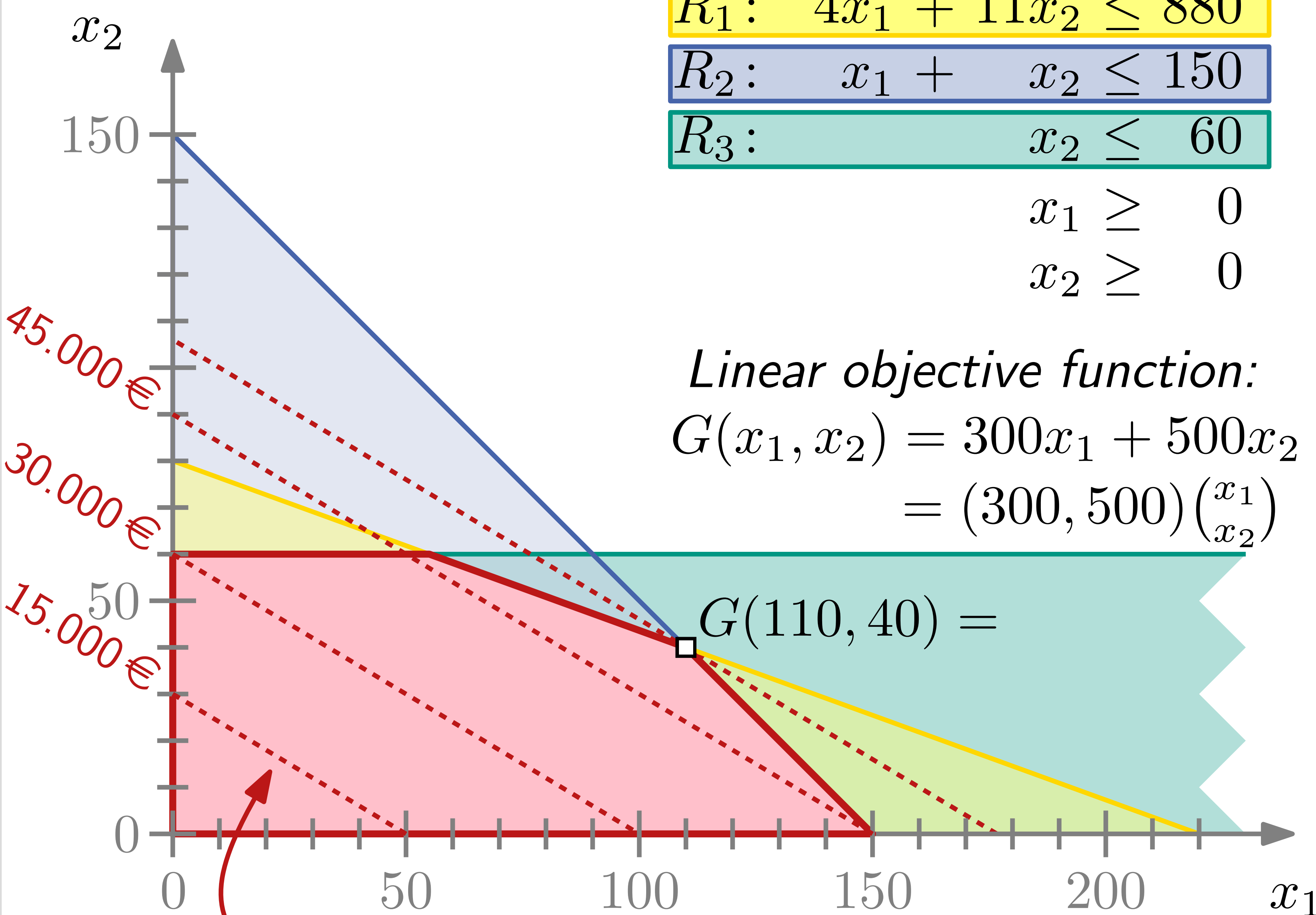

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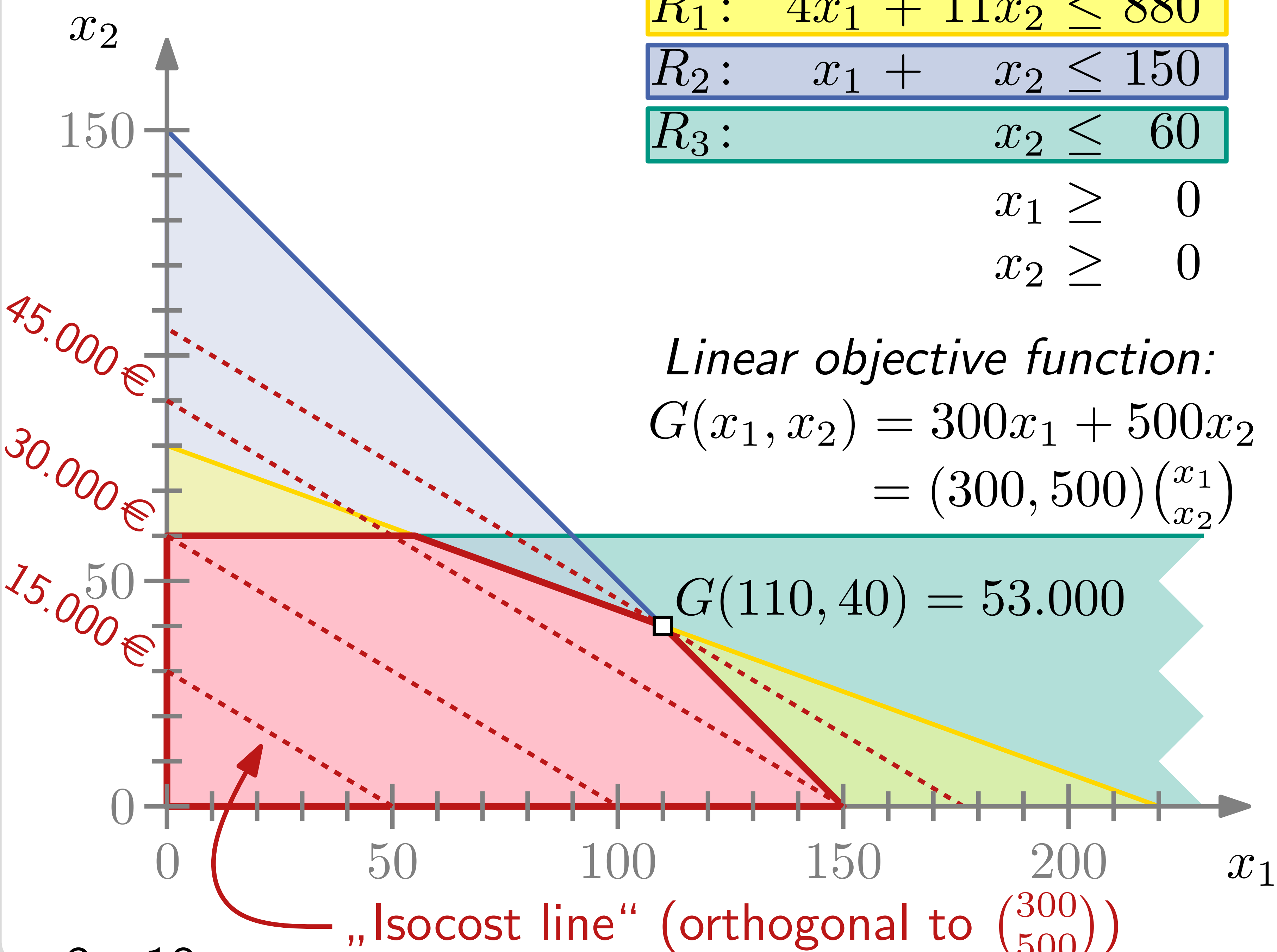

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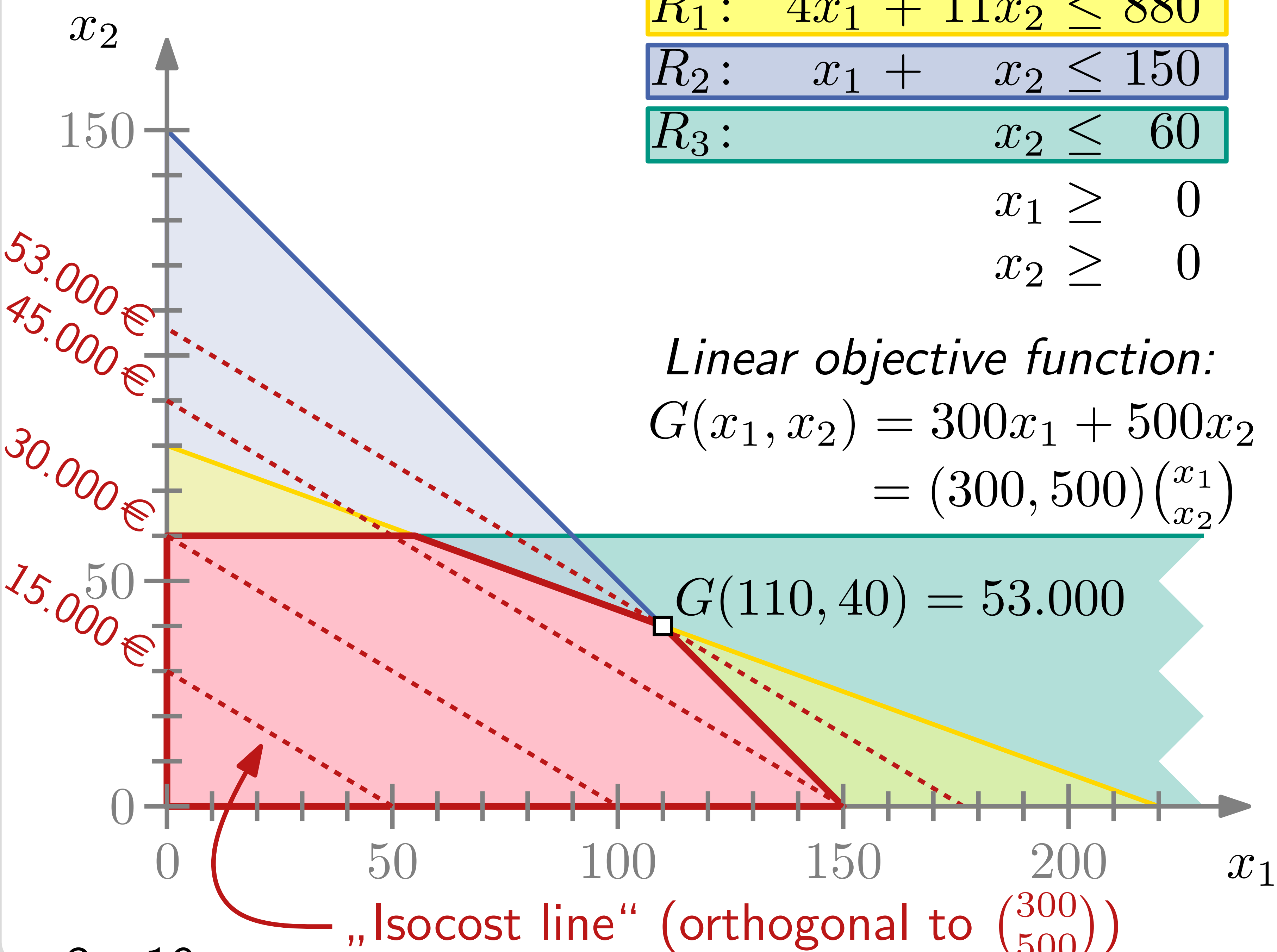

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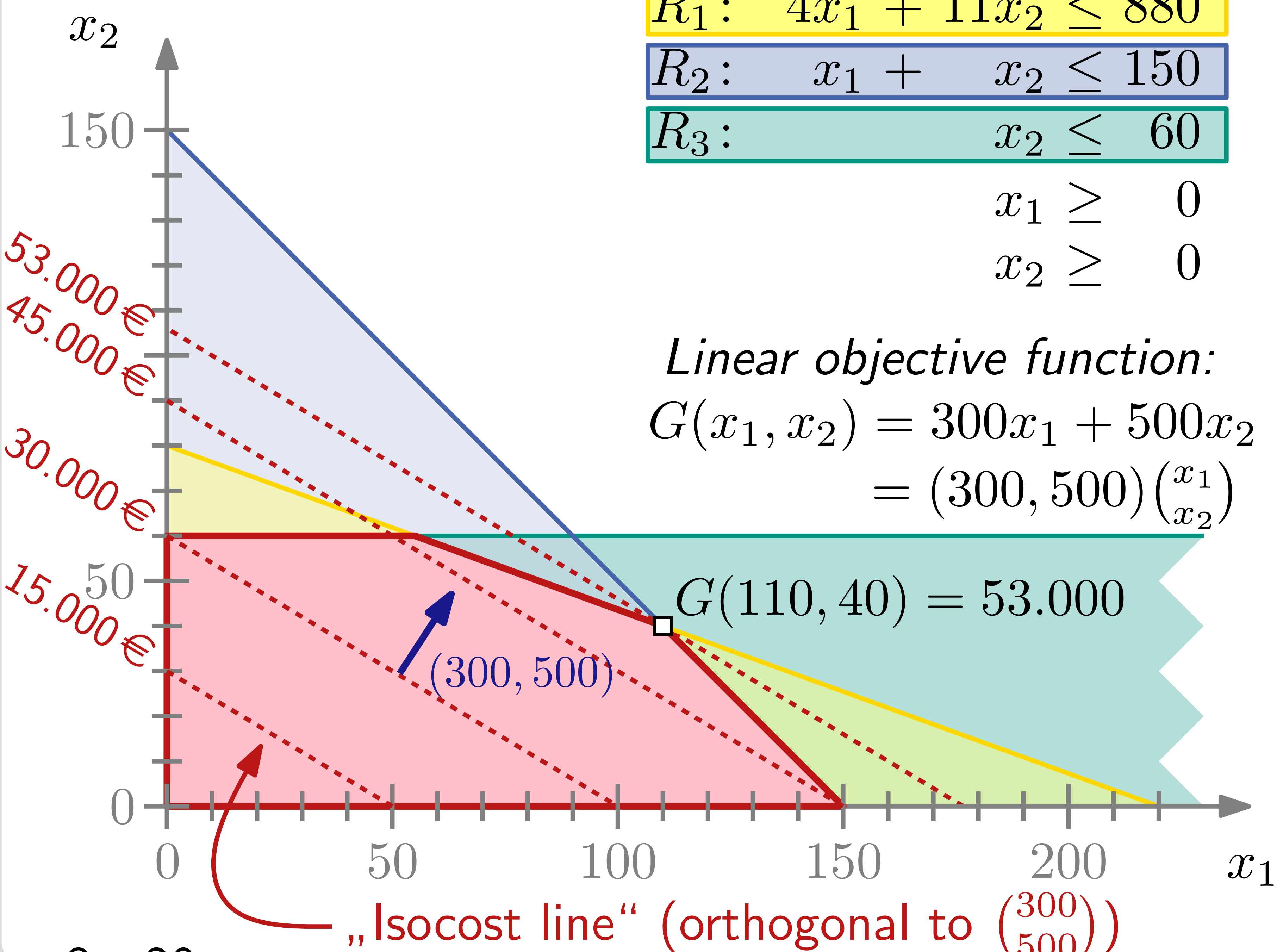

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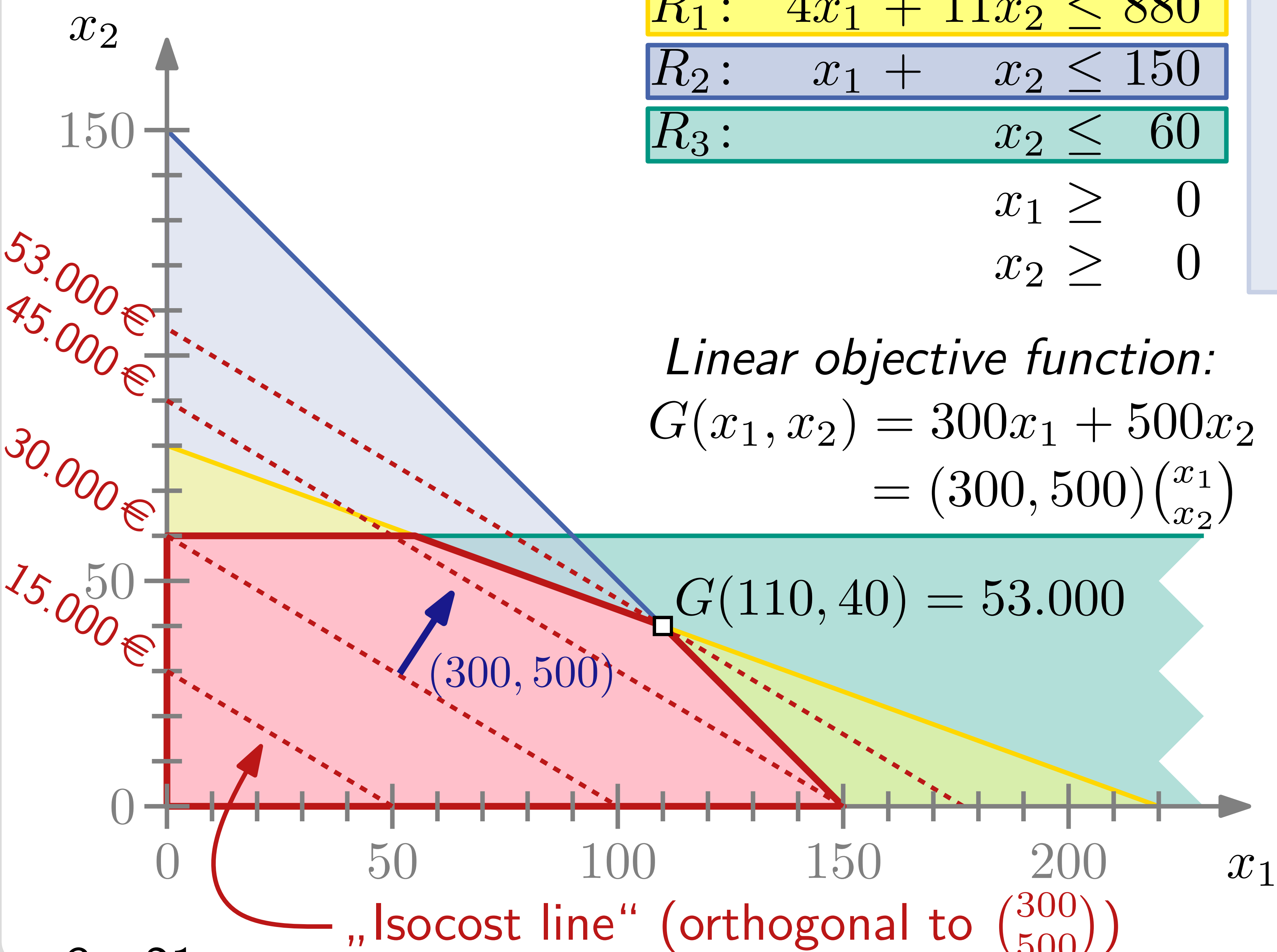

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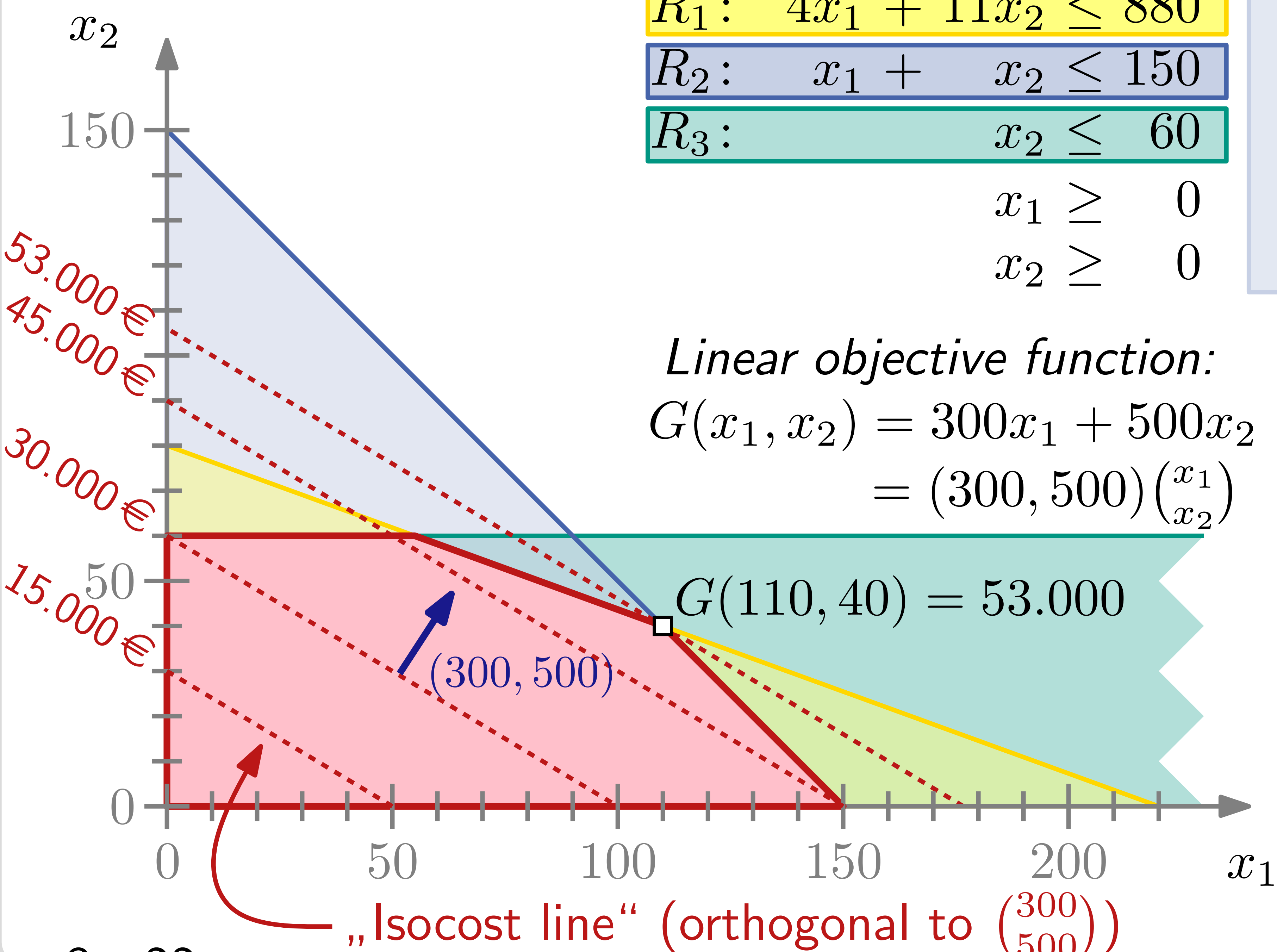

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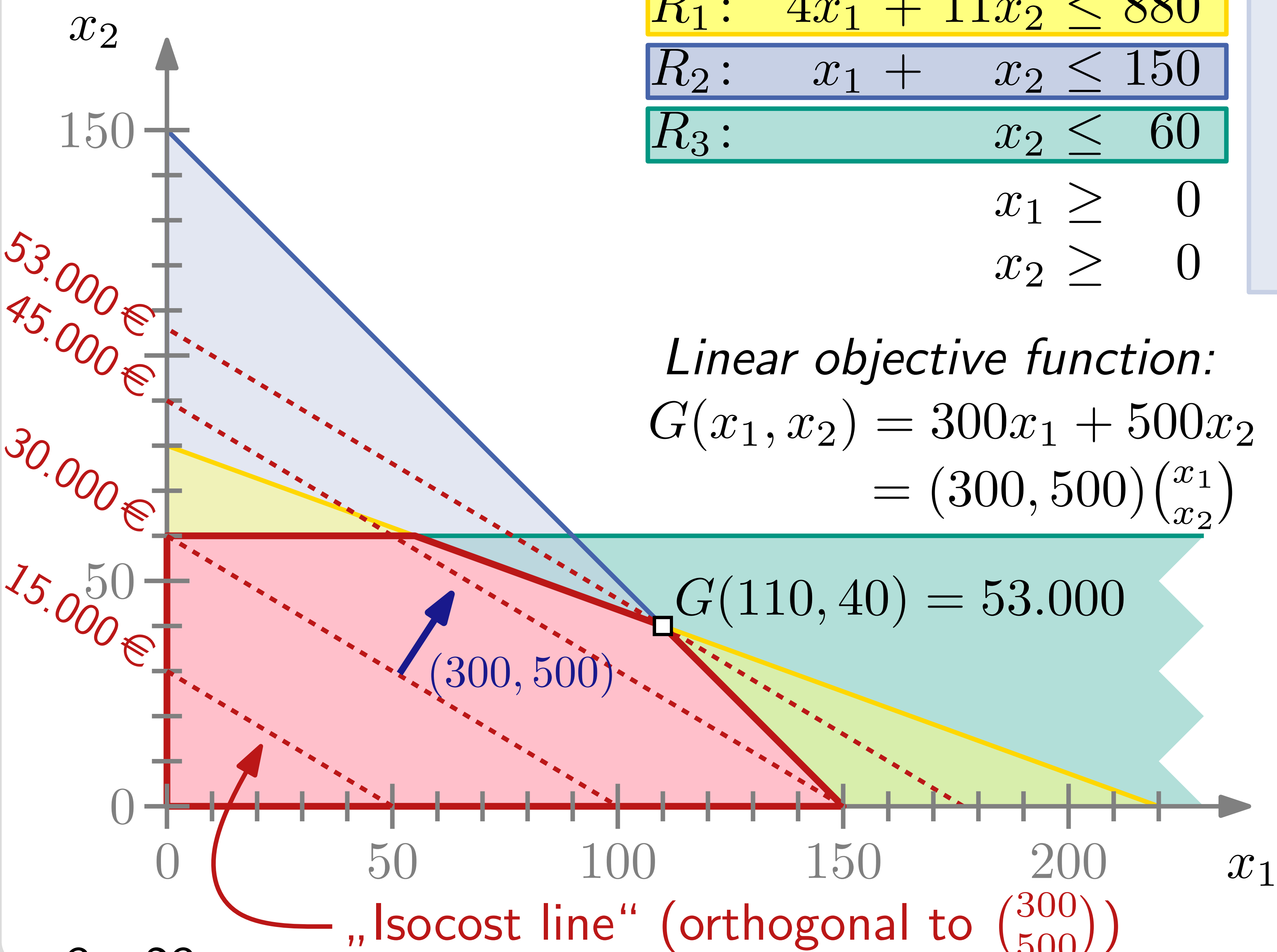

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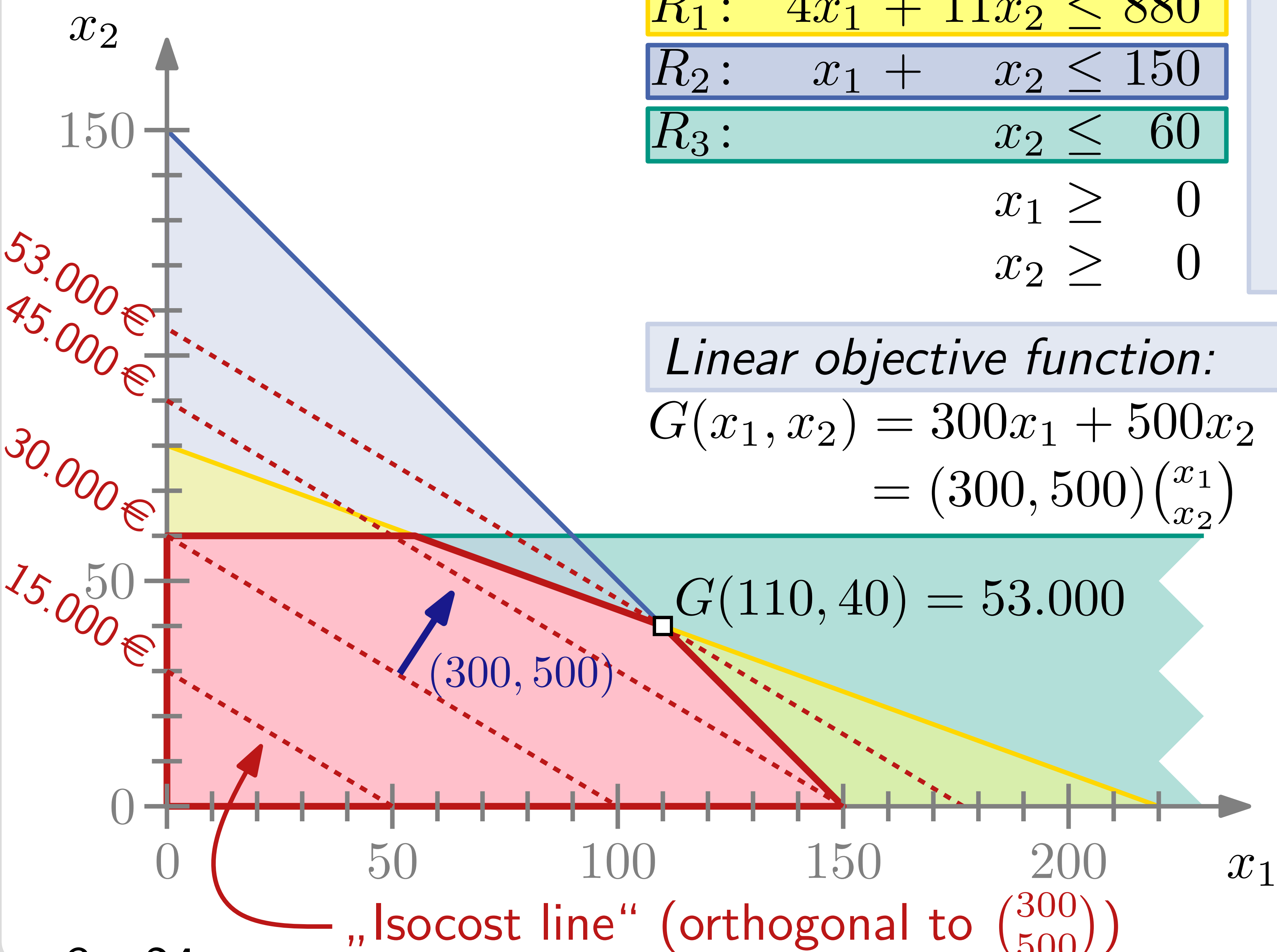

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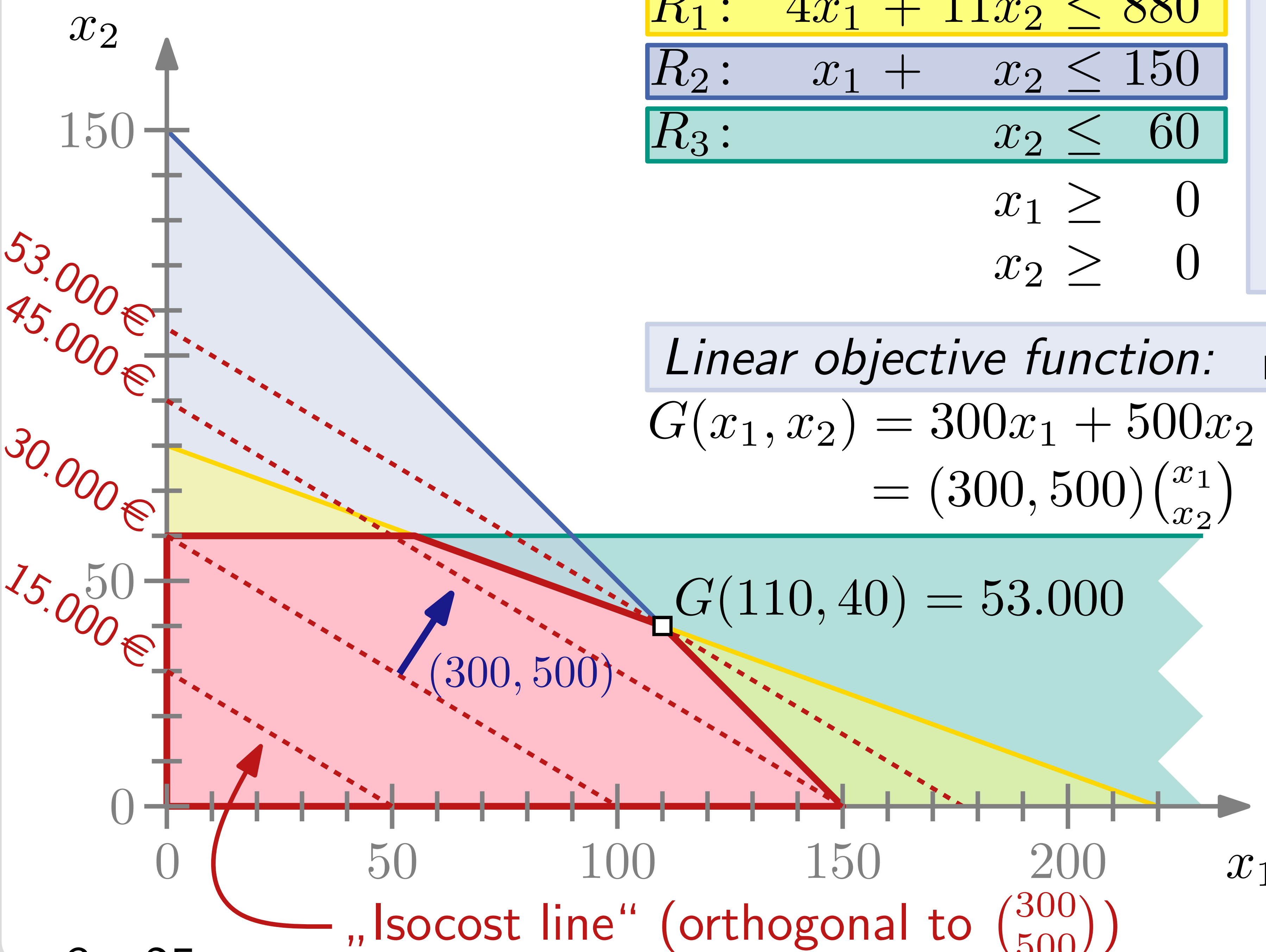

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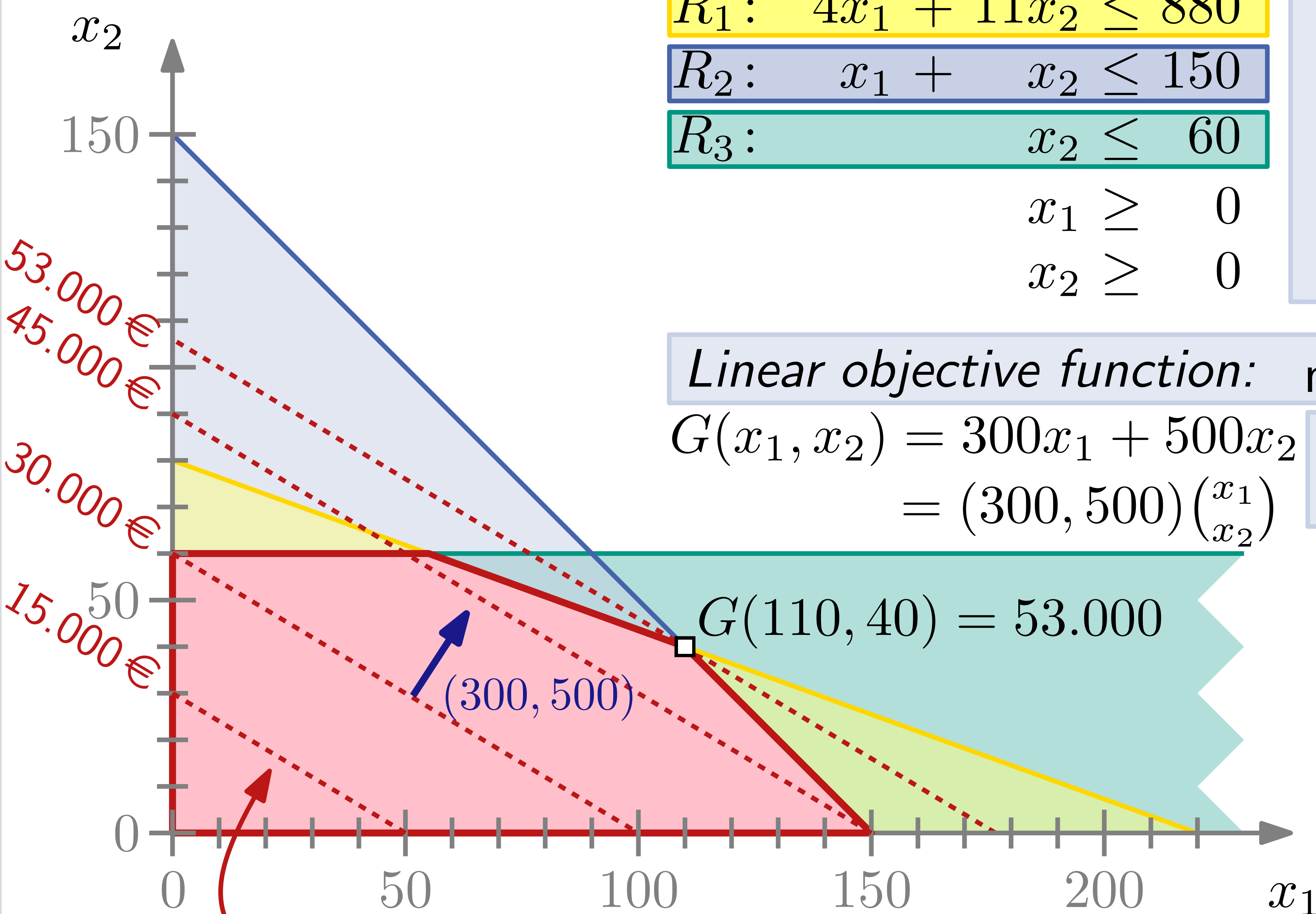

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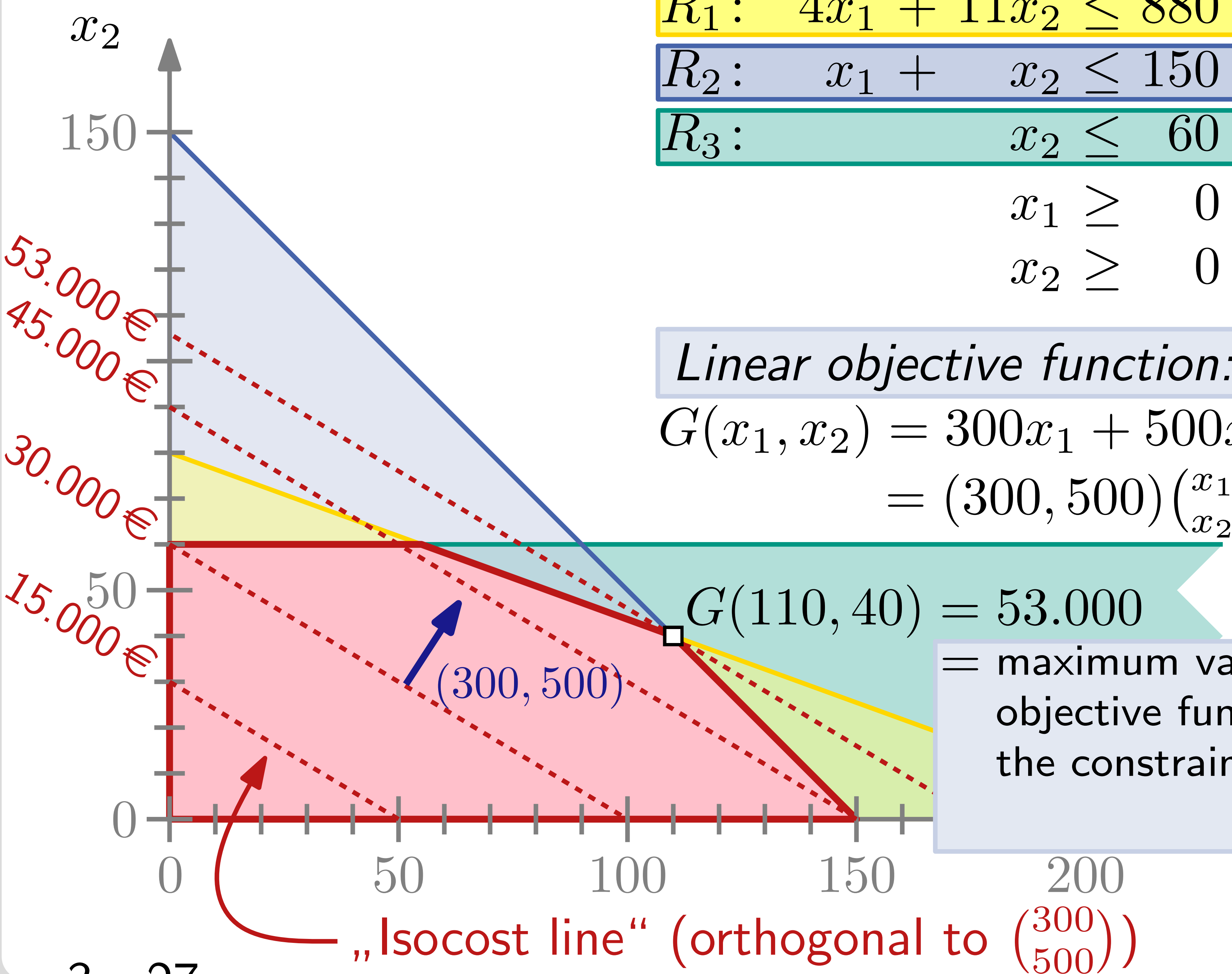

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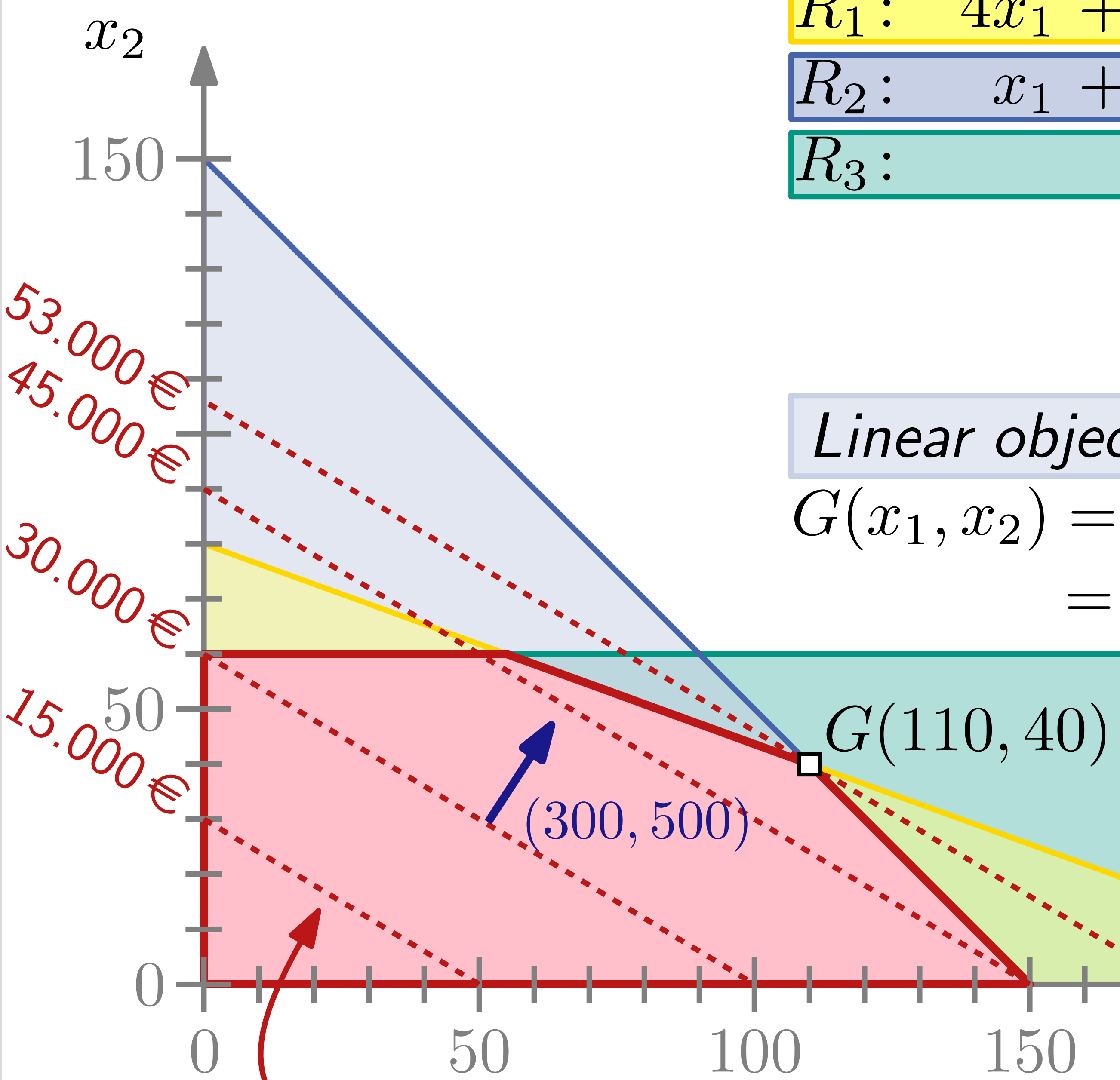

$G(110, 40) = 53.000$

= maximum value of the objective function under the constraints.

$\max\{c^{\mathrm{T}}x \mid Ax \leq b, x \geq 0\}$

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Linear programming

Definition: Given a set of linear constraints H and a linear objective function c in $\mathbb{R}^d$, a **linear program** (LP) is formulated as follows:

$$\begin{array}{lrcl} \text{maximize} & c_1x_1 + c_2x_2 + \cdots + c_dx_d & & \\ \text{under constr.} & a_{1,1}x_1 + \cdots + a_{1,d}x_d & \leq & b_1 \\ & a_{2,1}x_1 + \cdots + a_{2,d}x_d & \leq & b_2 \\ & & \vdots & \\ & a_{n,1}x_1 + \cdots + a_{n,d}x_d & \leq & b_n \end{array} \left.\vphantom{\begin{array}{c}1\\1\\1\\1\end{array}}\right\} H$$

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- H is a set of half-spaces in $\mathbb{R}^d$.
- We are searching for a point $x \in \bigcap_{h \in H} h$, that maximizes $c^T x$, i.e. $\max\{c^{\mathrm{T}}x \mid Ax \leq b, x \geq 0\}$.
- Linear programming is a central method in operations research.

Algorithms for LPs

There are many algorithms to solve LPs:

- Simplex-Algorithm [Dantzig, 1947]
- Ellipsoid-Method [Khatchiyan, 1979]
- Interior-Point-Method [Karmarkar, 1979]

They work well in practice, especially for large values of n (number of constraints) and d (number of variables).

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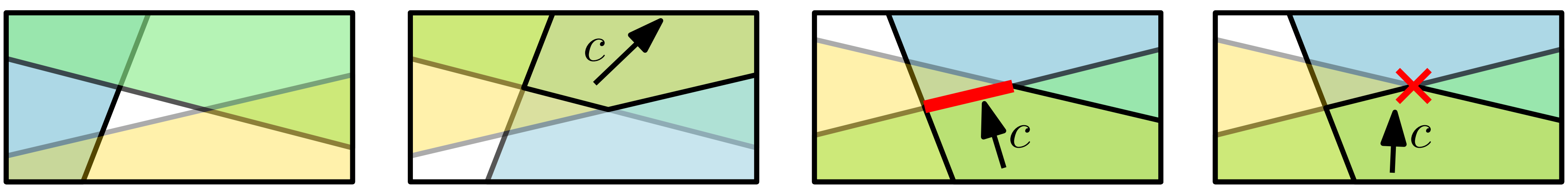
Possibilities for the solution space

feasible region $\bigcap H$ is bounded



$\bigcap H = \emptyset$ infeasible | $\bigcap H$ is unbounded in the direction c | solution is not unique | unique solution

First approach

Idea: Compute the feasible region $\bigcap H$ and search for the angle p, that maximizes $c^T p$.

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How can we proceed?

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- Let's try a simple Divide-and-Conquer Algorithm

```
IntersectHalfplanes(H)
  if |H| = 1 then
  |  C ← H
  else
  |  (H_1, H_2) ← SplitInHalves(H)
  |  C_1 ← IntersectHalfplanes(H_1)
  |  C_2 ← IntersectHalfplanes(H_2)
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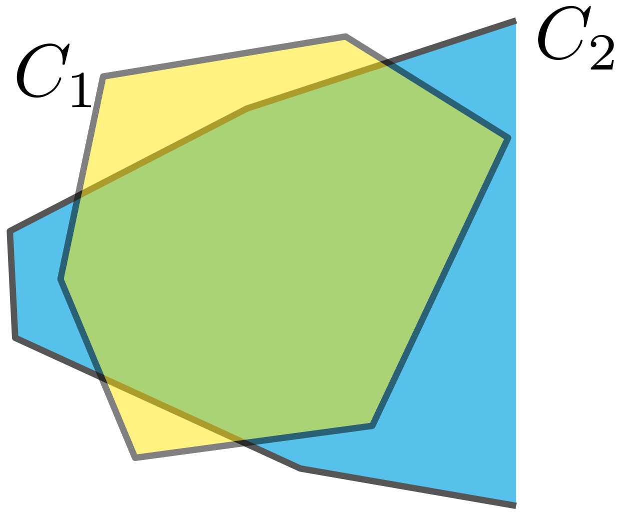

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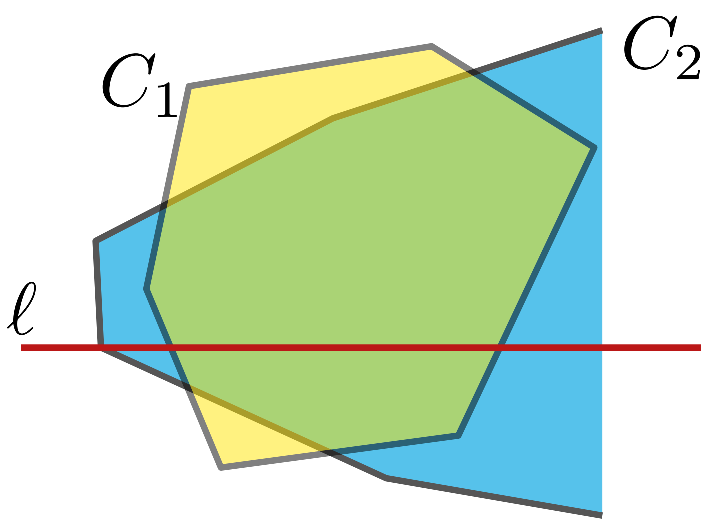

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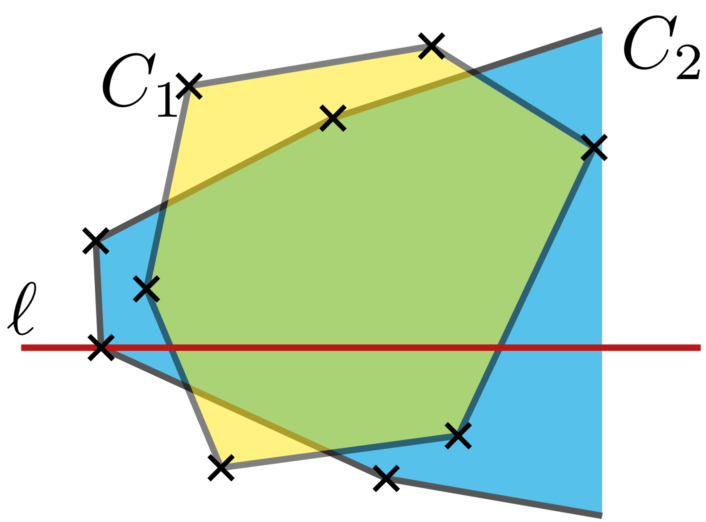

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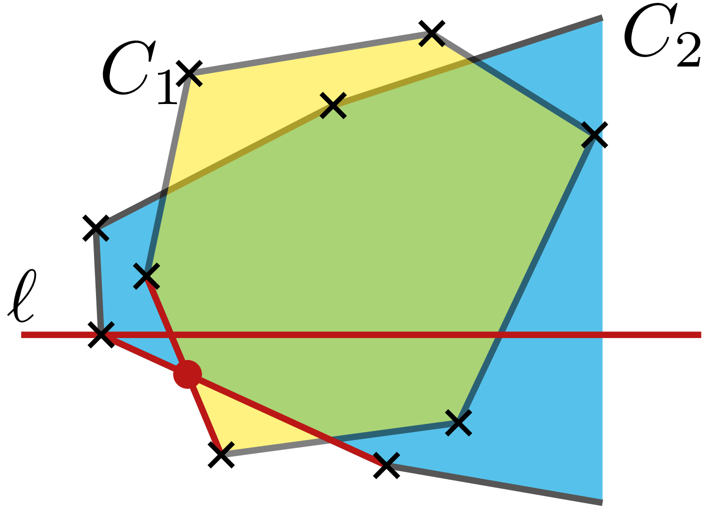

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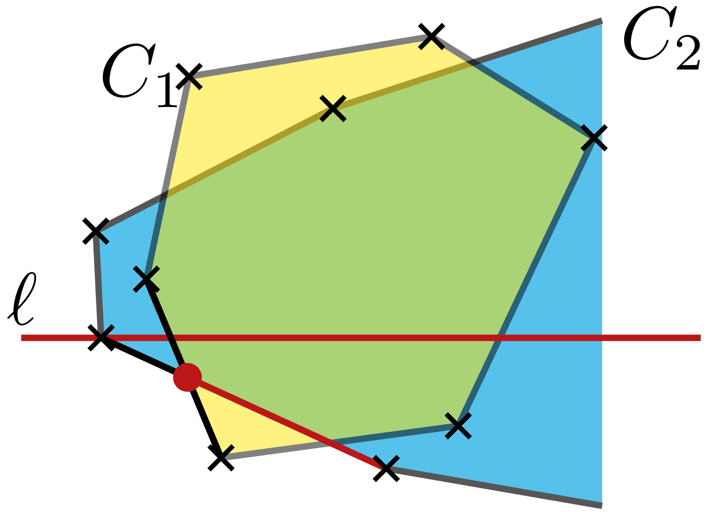

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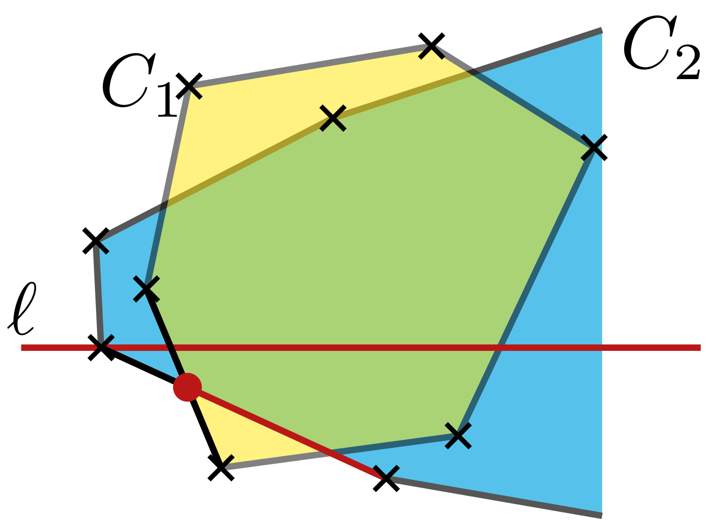

Theorem 1:
The intersection of two convex polygonal regions in the plane with $n_1 + n_2 = n$ nodes can be computed in $O(n)$ time.

Running time of IntersectHalfplanes(H)

IntersectHalfplanes(H)

```
if |H| = 1 then
    C ← H
else
    (H_1, H_2) ← SplitInHalves(H)
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$$T(n) = \begin{cases} O(1) & \text{when } n = 1 \\ O(n) + 2T(n/2) & \text{when } n > 1 \end{cases}$$

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KIT Karlsruhe Institute of Technology

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- feasible region $\bigcap H$ can be found in $O(n \log n)$ time
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Can we do better?

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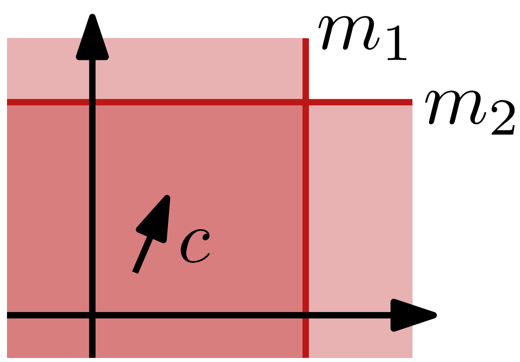

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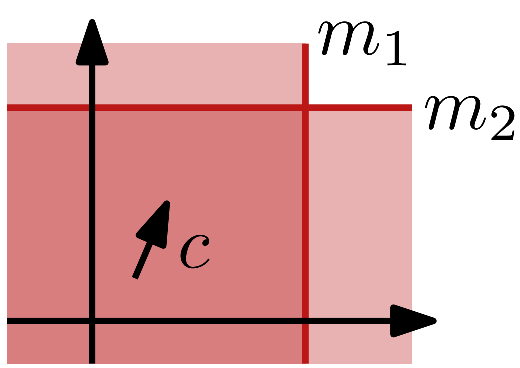

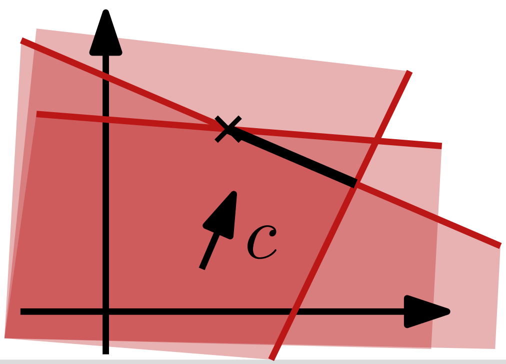

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Consider a LP (H, c) with $H = \{h_1, \ldots, h_n\}$, $c = (c_x, c_y)$. We denote the first i constraints by $H_i = \{m_1, m_2, h_1, \ldots, h_i\}$, and the feasible polygon defineed by them by $C_i = m_1 \cap m_2 \cap h_1 \cap \cdots \cap h_i$

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Lemma 1: For $1 \leq i \leq n$ and bounding line ℓ_i of h_i holds that:

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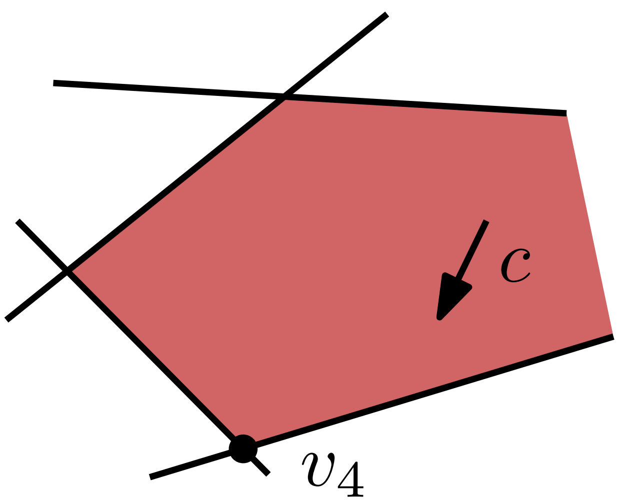

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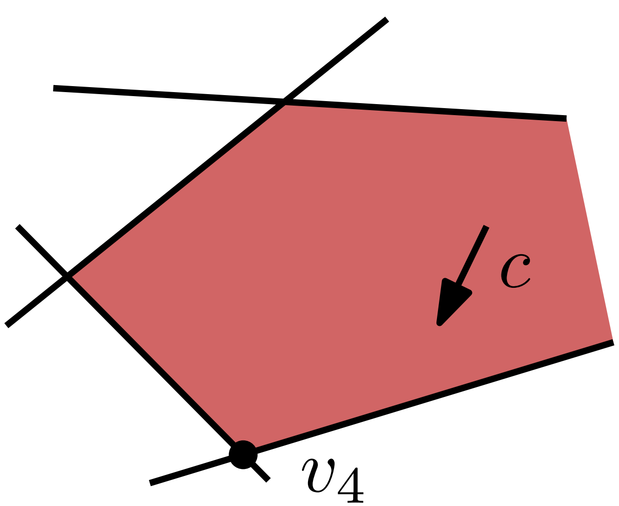

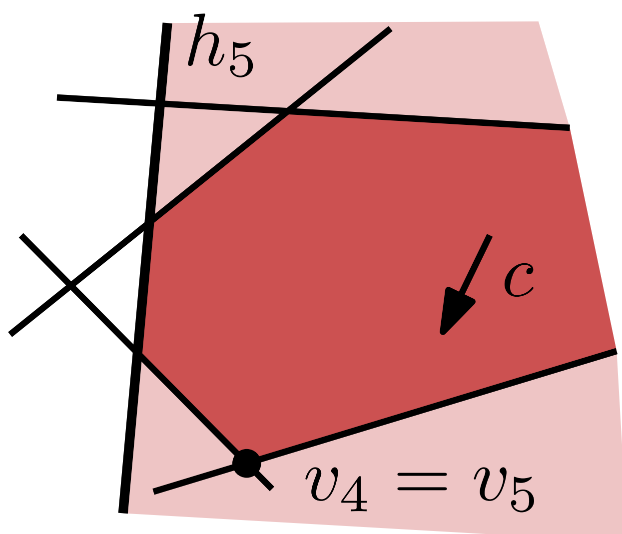

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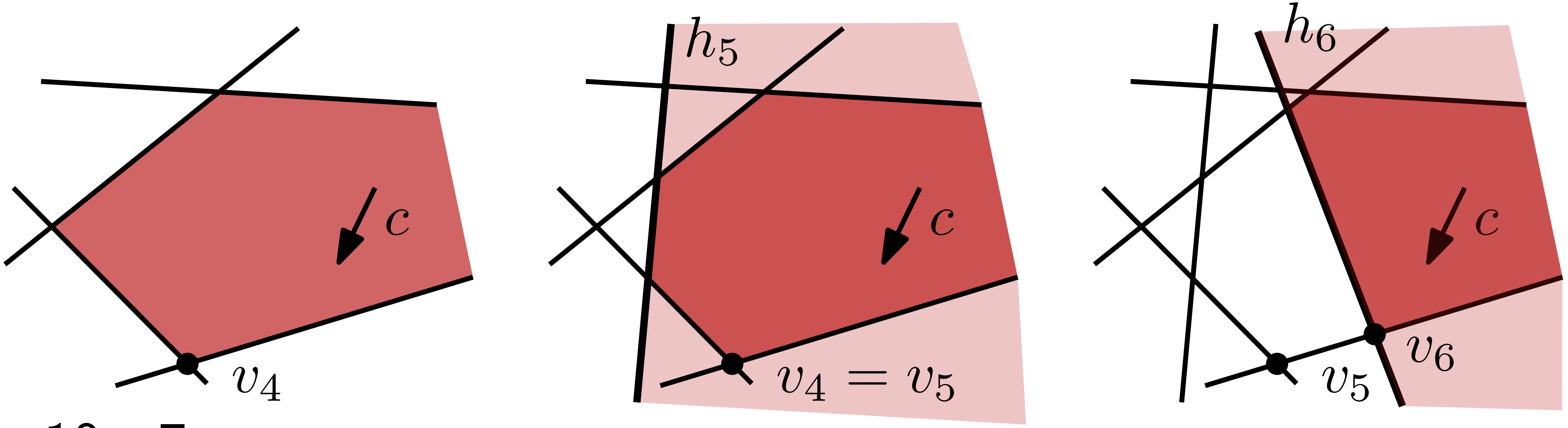

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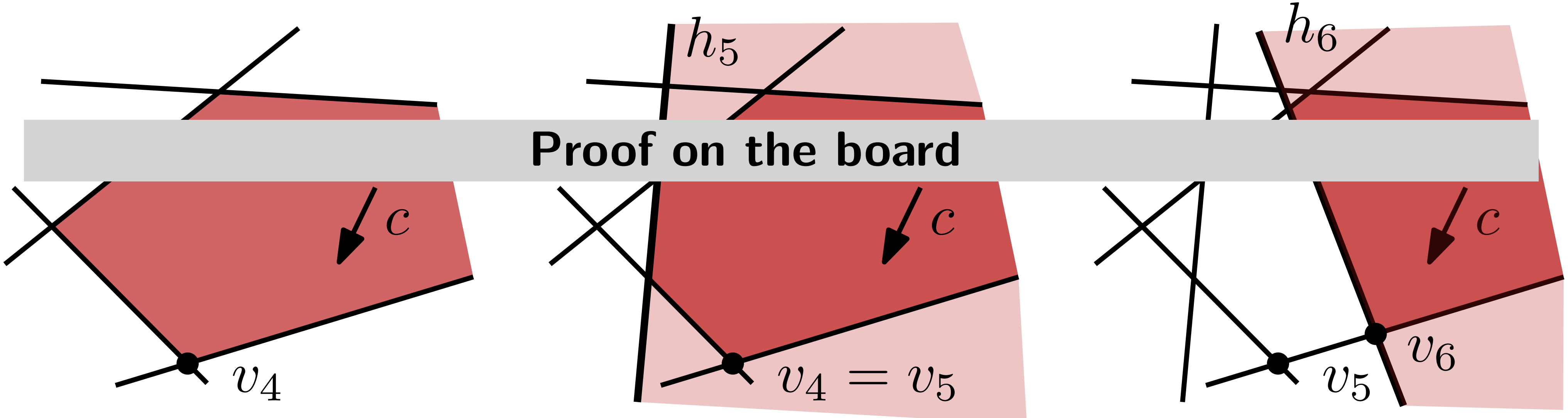
Proof on the board



One-dimentional LP

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This gives us the following one-dimentional LP:

maximize $\quad f_c^i(x) = c_x x + c_y(ax+b)$

with constr. $x \leq \sigma_x(\ell_j, \ell_i)$ if $\ell_i \cap h_j$ is limited to the righ
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How to solve this LP? Running time?

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Lemma 2: A one-dimentional LP can be solved in linear time. In particular, in case (ii), one can compute the new angle v_i or decide whether $C_i = \emptyset$ in $O(i)$ time.

Incremental Algorithm

2dBoundedLP(H, c, m_1, m_2)

```
C_0 ← m_1 ∩ m_2
v_0 ← unique angle of C_0
for i ← 1 to n do
    if v_{i-1} ∈ h_i then
        v_i ← v_{i-1}
    else
        v_i ← 1dBoundedLP(σ(H_{i-1}), f_c^i)
        if v_i = nil then
            return infeasible
return v_n
```

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for $i \leftarrow 1$ **to** n **do**
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worst-case running time:
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Question: Can in reality the case(ii) happen n times?

Incremental Algorithm

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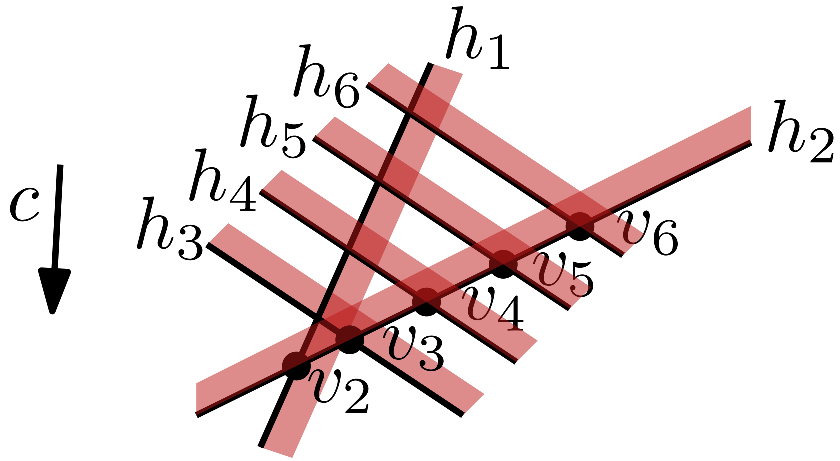
Lemma 3: Algorithmus 2dBoundedLP needs $\Theta(n^2)$ time to solve an LP with n contraints and 2 variables.

What else can we do?

Obs.: It is not the half-planes H that force the high running time, but the order in which we consider them.



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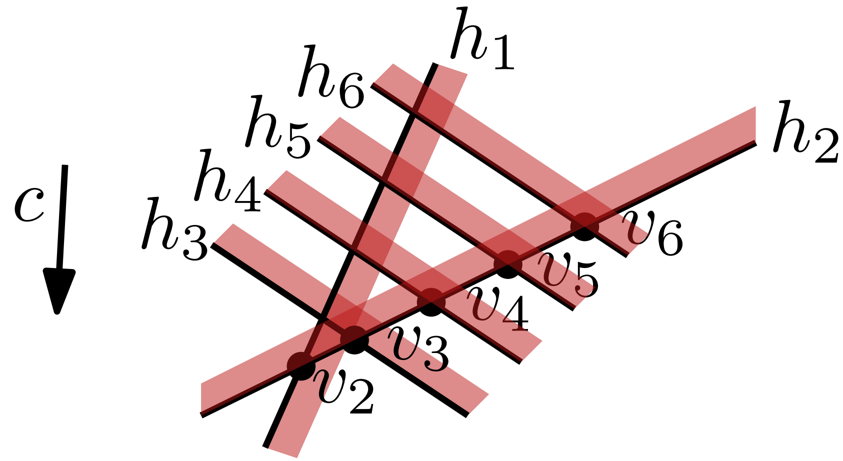

How to find (quickly) a good ordering?

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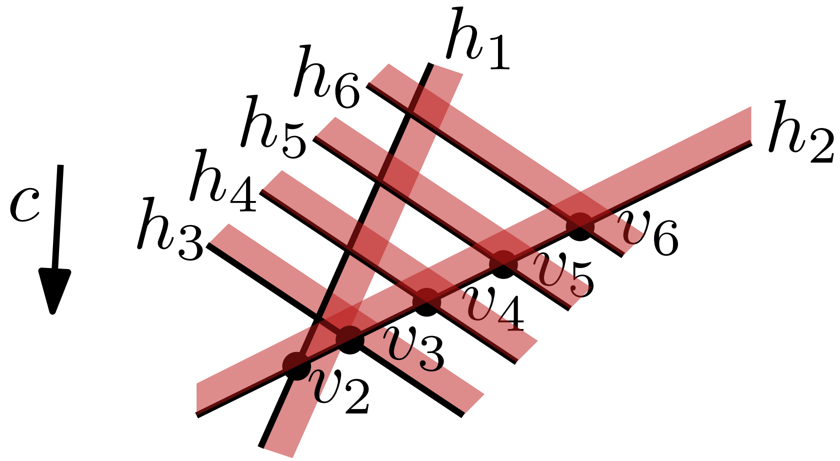

How to find (quickly) a good ordering?

Randomized incremental algorithm

2dRandomizedBoundedLP(H, c, m_1, m_2)

$C_0 \leftarrow m_1 \cap m_2$
$v_0 \leftarrow$ unique angle of C_0
$H \leftarrow$ **RandomPermutation**(H)
for $i \leftarrow 1$ **to** n **do**
  if $v_{i-1} \in h_i$ **then**
    $v_i \leftarrow v_{i-1}$
  else
    $v_i \leftarrow$ 1dBoundedLP$(\sigma(H_{i-1}), f_c^i)$
    if $v_i =$ nil **then**
      return infeasible
return v_n

Random permutation

RandomPermutation(A)

Input: Array $A[1 \ldots n]$

Output: Array A, rearranged into a random permutation

for $k \leftarrow n$ **to** 2 **do**

  $r \leftarrow$ Random(k)

  exchange $A[r]$ and $A[k]$

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Obs.: The running time of 2dRandomizedBoundedLP depends on the random permutation computed by the procedure RandomPermutation. In the following we compute the **expected running time**.

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Obs.: The running time of 2dRandomizedBoundedLP depends on the random permutation computed by the procedure RandomPermutation. In the following we compute the **expected running time**.

Theorem 2: A two-dimentional LP with n constraints can be solved in randomized expected time.

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for $k \leftarrow n$ **to** 2 **do**
  $r \leftarrow$ Random(k)
  exchange $A[r]$ and $A[k]$

Random num. between 1 and k

Obs.: The running time of 2dRandomizedBoundedLP depends on the random permutation computed by the procedure RandomPermutation. In the following we compute the **expected running time**.

Theorem 2: A two-dimentional LP with n constraints can be solved in $O(n)$ randomized expected time.

Random permutation

RandomPermutation(A)

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Proof on the board

Unbounded LPs

Till now: Artificial contraints to bound C by m_1 and m_2

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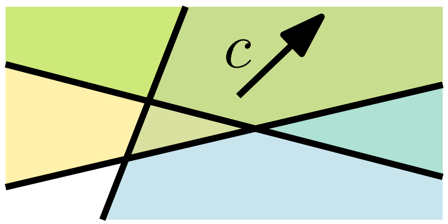
Next: recongnize and deal with an unbonded LP



$\bigcap H$ unbounded in the direction c

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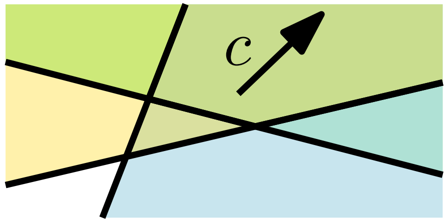

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Def.: A LP (H, c) is called **unbounded**, if there exists a ray $\rho = \{p + \lambda d \mid \lambda > 0\}$ in $C = \bigcap H$, such that the value of the objective function f_c becomes arbitrarily large along ρ.

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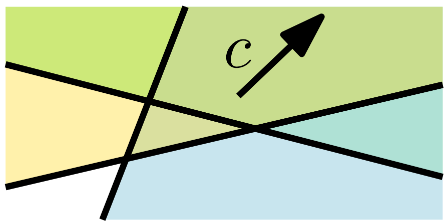

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It must be that:

- $\langle d, c \rangle > 0$
- $\langle d, \eta(h) \rangle \geq 0$ for all $h \in H$ where $\eta(h)$ is the **normal** vector of h oriented towards the feasible side of h

Characterization

Lemma 4: A LP (H, c) is unbounded iff there is a vector $d \in \mathbb{R}^2$ such that

- $\langle d, c \rangle > 0$
- $\langle d, \eta(h) \rangle \geq 0$ for all $h \in H$
- LP (H', c) with $H' = \{h \in H \mid \langle d, \eta(h) \rangle = 0\}$ is feasible.

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Test whether (H, c) is unbounded with a one-dimentional LP:

Step 1:

- rotate coordinate system till $c = (0, 1)$
- normalize vector d with $\langle d, c \rangle > 0$ as $d = (d_x, 1)$
- For normal vector $\eta(h) = (\eta_x, \eta_y)$ it should hold that $\langle d, \eta(h) \rangle = d_x \eta_x + \eta_y \geq 0$
- Let $\bar{H} = \{d_x \eta_x + \eta_y \geq 0 | h \in H\}$
- Check whether this one-dim. LP $\bar{H}$ is feasible

Test auf Unbeschränktheit

Step 2: If Step 1 returns a feasible solution $d_x^\star$

- compute $H' = \{h \in H \mid d_x^\star \eta_x(h) + \eta_y(h) = 0\}$
- Normals to H' are orthogonal to $d = (d_x, 1) \Rightarrow$ lines bounding half-planes of H' are parallel to d
- intersect the bounding lines of H' with x-axis $\rightarrow$ 1d-LP

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If the two steps result in a feasible solution, the LP (H, c) is unbounded and we can construct the ray ρ.

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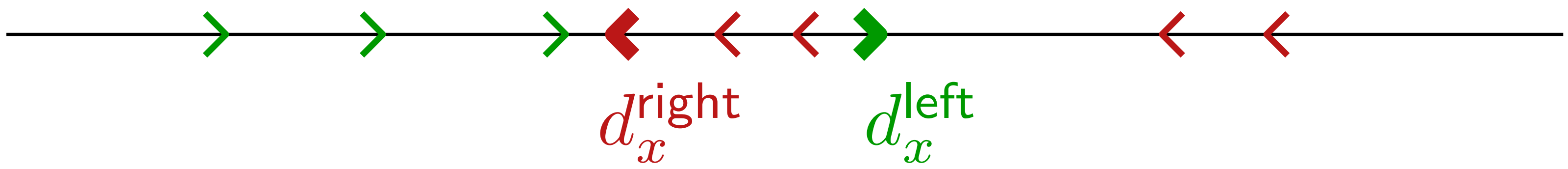
If the LP $\bar{H}$ of the Step 1 is infeasible, then by Lemma 4, (H, c) is bounded.

Certificates of boundness

Obs.: When the LP $\bar{H} = \{d_x \eta_x + \eta_y \geq 0 | h \in H\}$ of the Step 1 is infeasible, we can use this information further!



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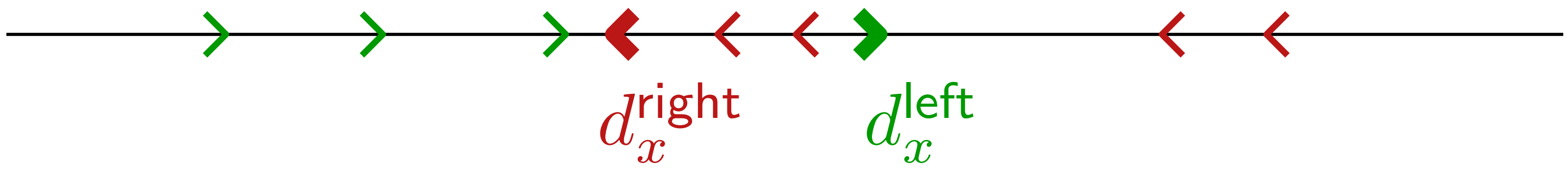

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- let h_1 and h_2 be the half planes corresponding to d_x^{left} and d_x^{right}
- There is no vector d that would "satisfy" h_1 and h_2, thus
- the LP $(\{h_1, h_2\}, c)$ is already bounded
- h_1 and h_2 are **certificates** of the boundness
- use h_1 and h_2 in 2dRandomizedBoundedLP as m_1 and m_2

Algorithms

2dRandomizedLP(H, c)

$\exists$? Vector d with $\langle d, c\rangle > 0$ and $\langle d, \eta(h)\rangle \geq 0$ for all $h \in H$

if d exists **then**

  $H' \leftarrow \{h \in H \mid \langle d, \eta(h)\rangle = 0\}$

  if H' feasible **then**

    return (ray ρ, unbounded)

  else

    return infeasible

else

  $(h_1, h_2) \leftarrow$ Certificates for the boundness of (H, c)

  $\overline{H} \leftarrow H \setminus \{h_1, h_2\}$

  return 2dRandomizedBoundedLP$(\overline{H}, c, h_1, h_2)$

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Theorem 3: A two-dimentional LP with n constraints can be solved in $O(n)$ randomized expected time.

Discussion

Can the two-dimentional algorithms be generalized to more dimentions?

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Yes! The same way as the two-dimentional LP is solved incrementally with reduction to a one-dimentional LP, a d-dimentional LP can be solved by a randomized incremental algorithm with a reduction to $(d-1)$-dimentional LP. The expected running time is then $O(c^d d!\, n)$ for a constant c. The algorithm is therefore usefull only for small values on d.

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The simple randomized incremental algorithm for two and more dimentions given in this lecture is due to Seidel (1991).