Karlsruhe Institute of Technology

# Computational Geometry · Lecture

## Polygon Triangulation

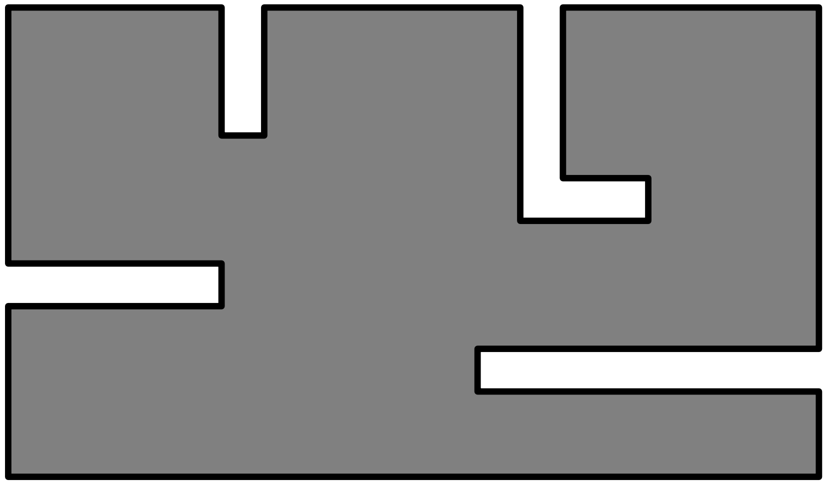
INSTITUTE FOR THEORETICAL INFORMATICS · FACULTY OF INFORMATICS

Tamara Mchedlidze

3.5.2018

# The Art-Gallery-Problem

**Task:** Install a number of cameras in an art gallery so that every part of the galery is visible to at least one of them.



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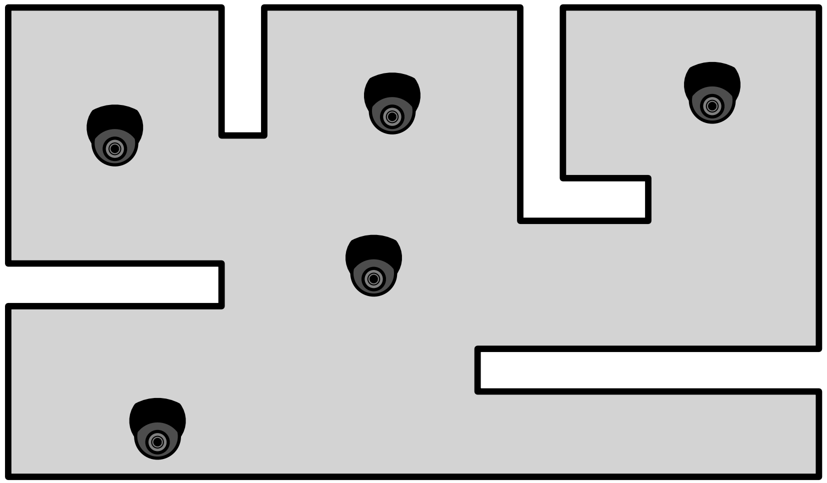

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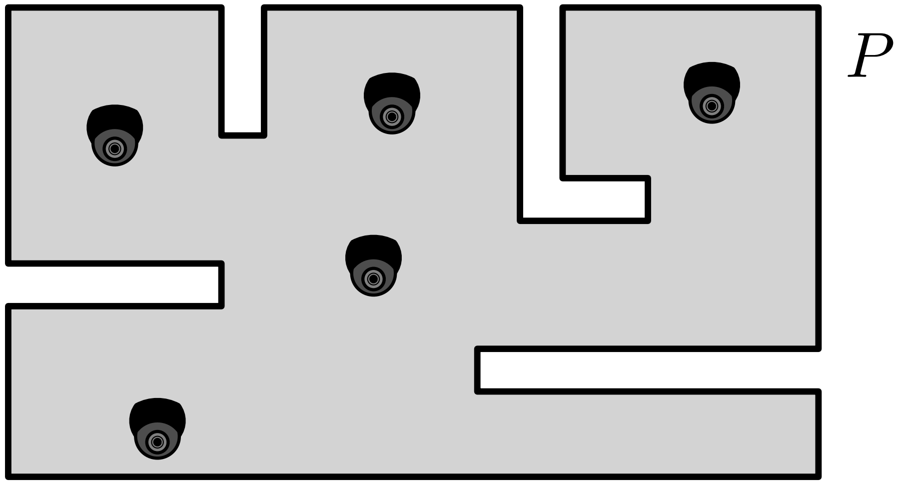


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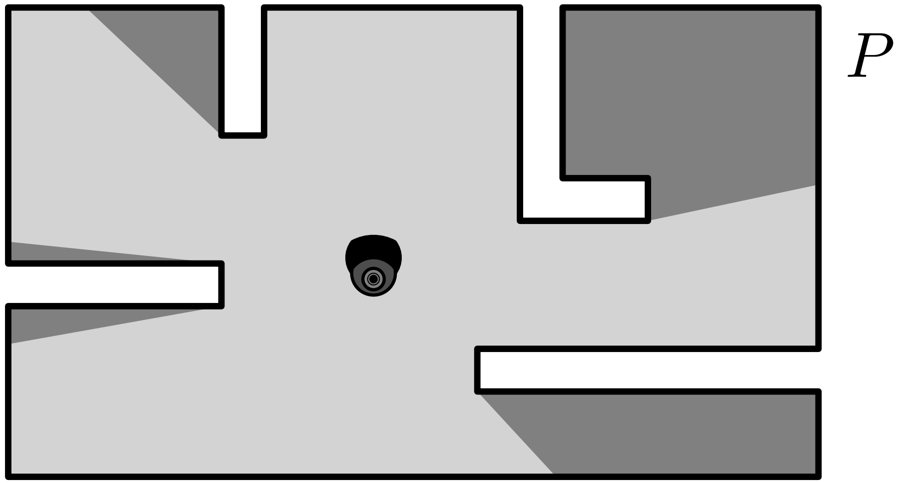


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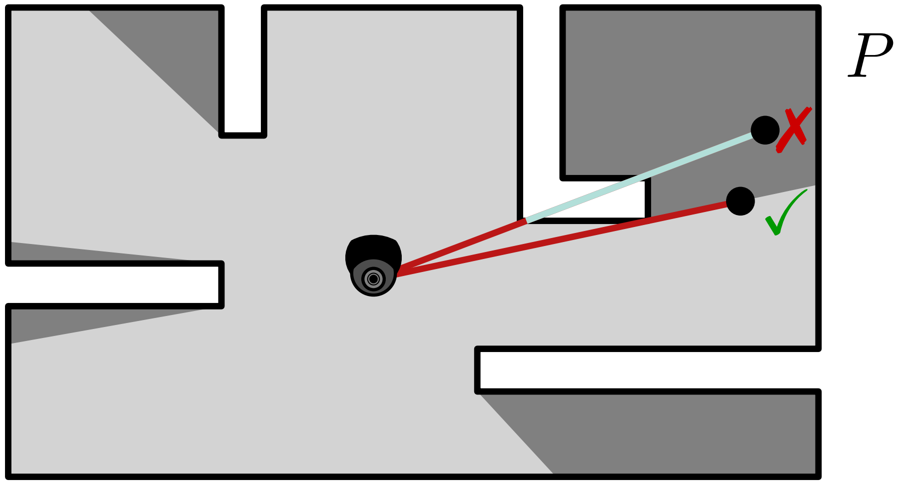


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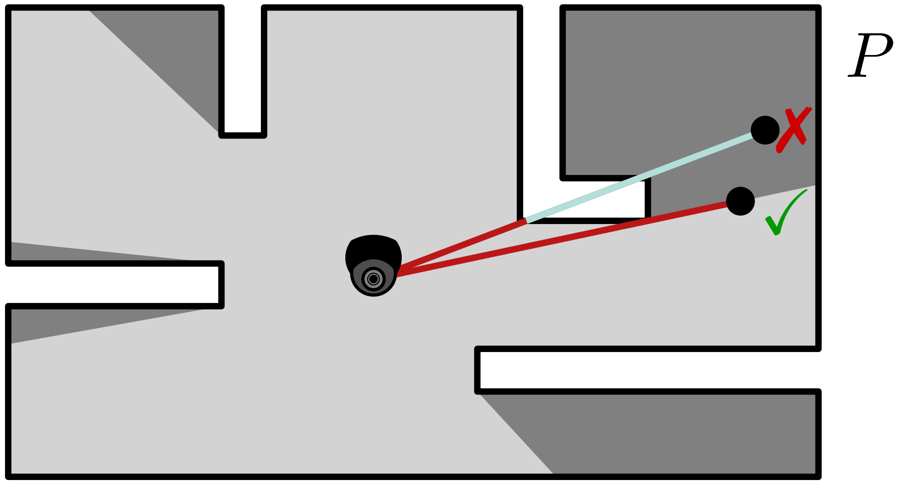


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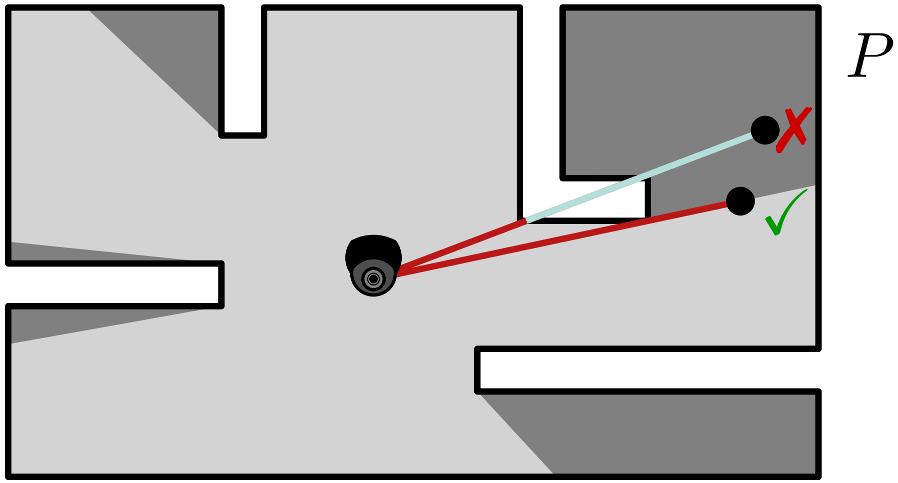


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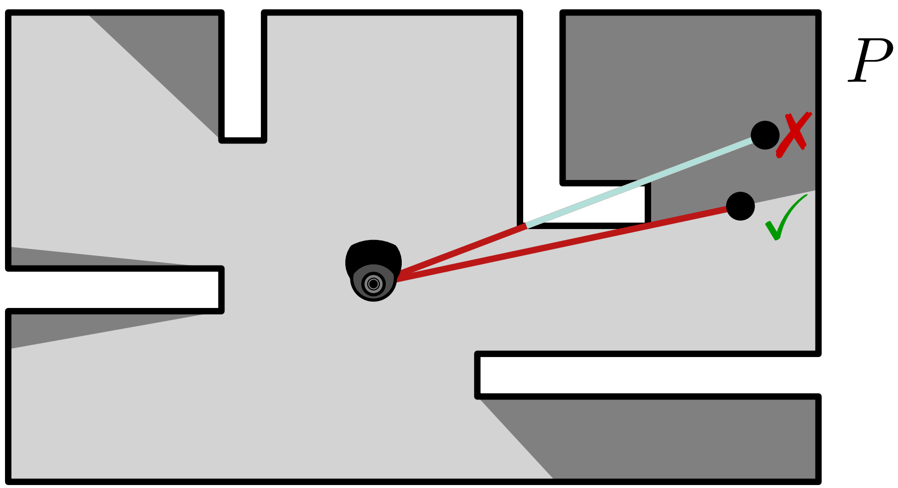


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NP-hard!

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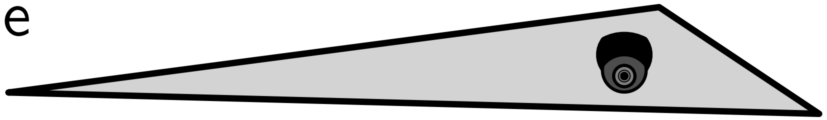
# Problem Simplification

**Observation:** It is easy to guard a triangle



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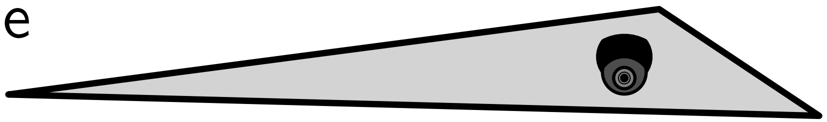
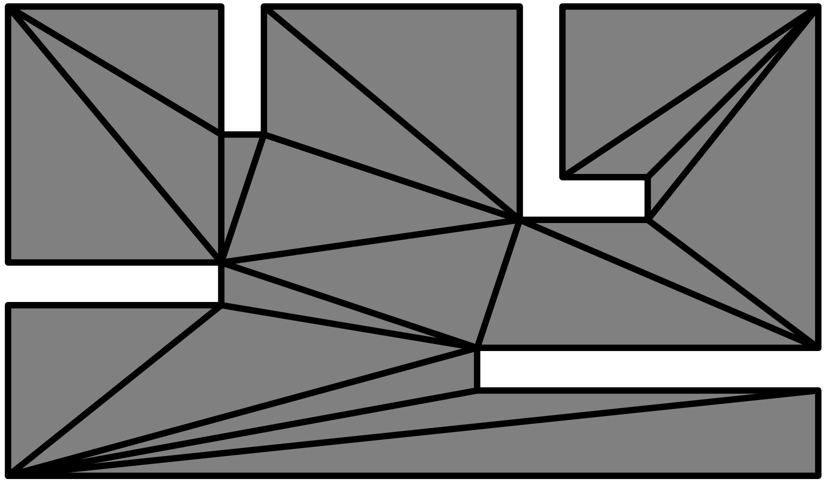


**Idea:** Decompose $P$ into triangles and guard each of them



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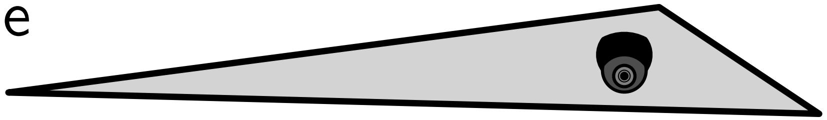

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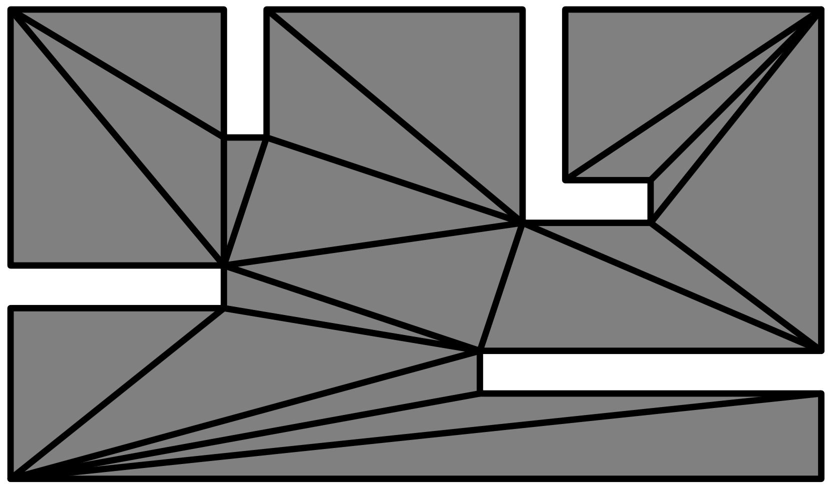

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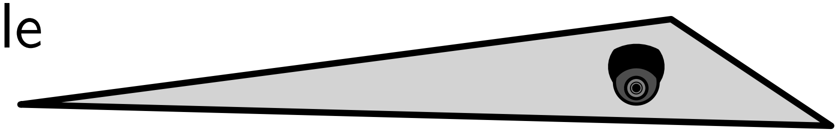

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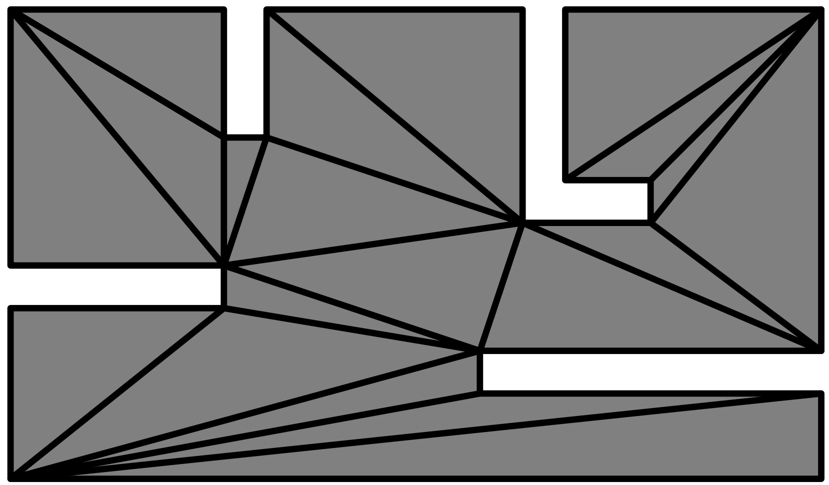

**Theorem 1:** Each simple polygon with $n$ corners admits a triangulation; any such triangulation contains exactly $n-2$ triangles.

The proof implies a recursive $O(n^2)$-Algorithm!

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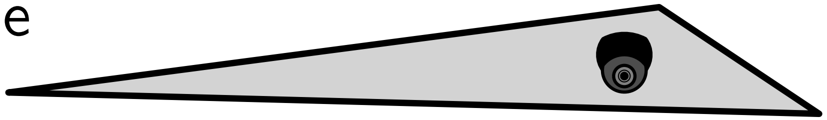

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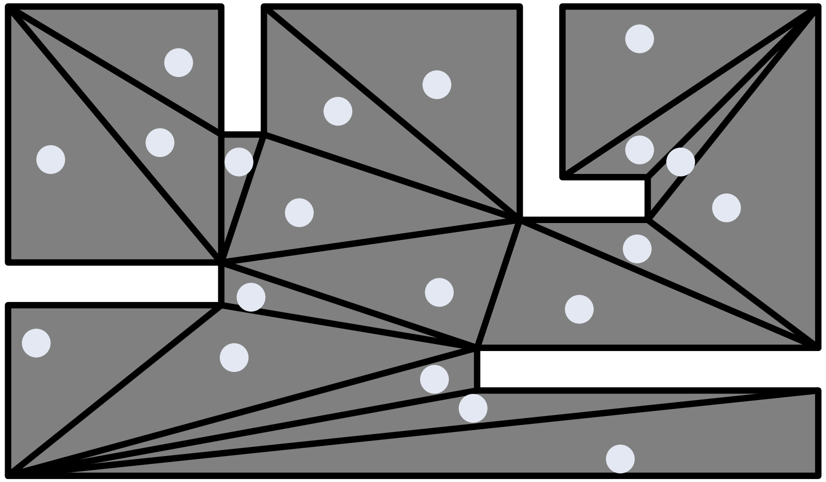

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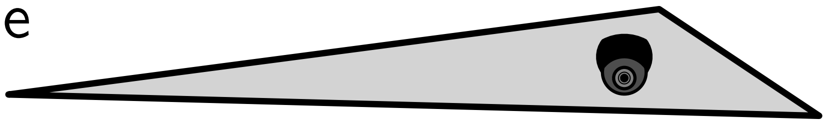

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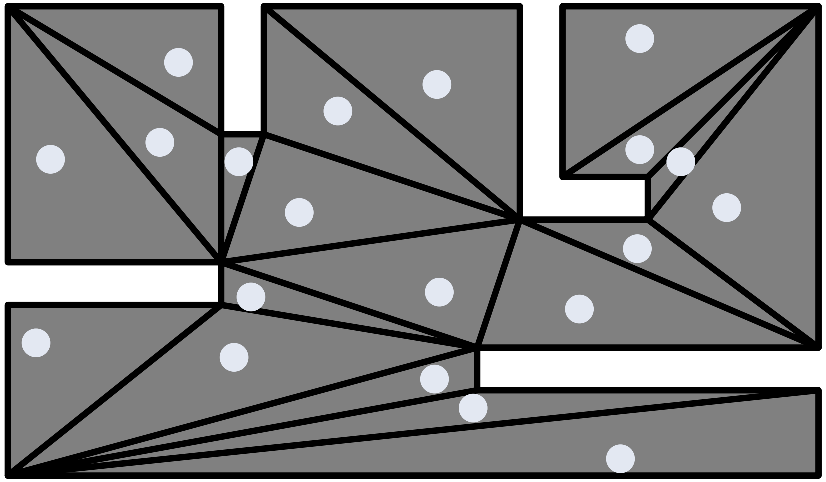

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**Can we do better?**

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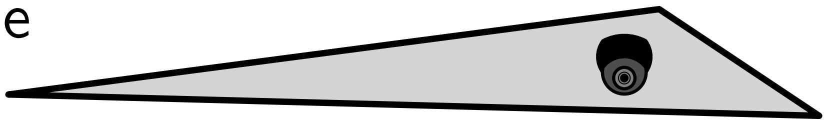

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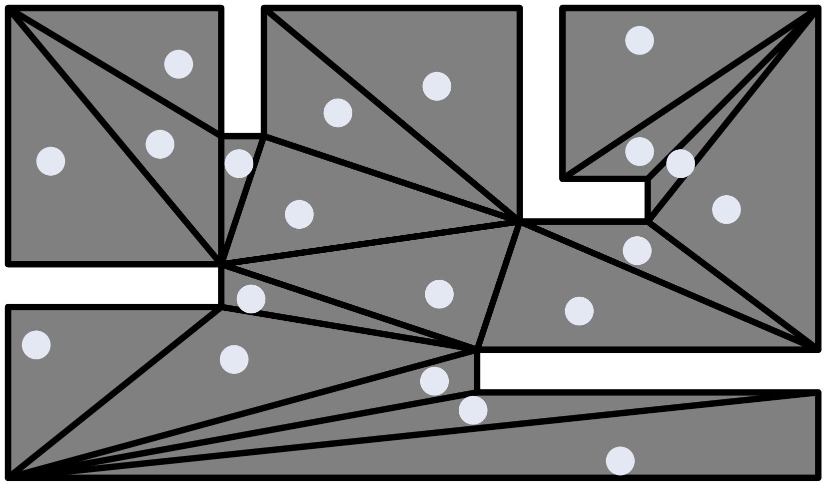

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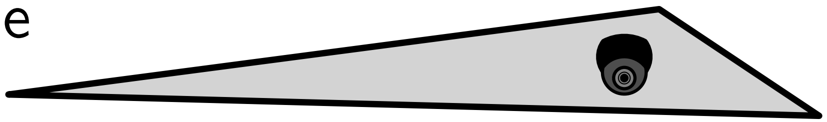

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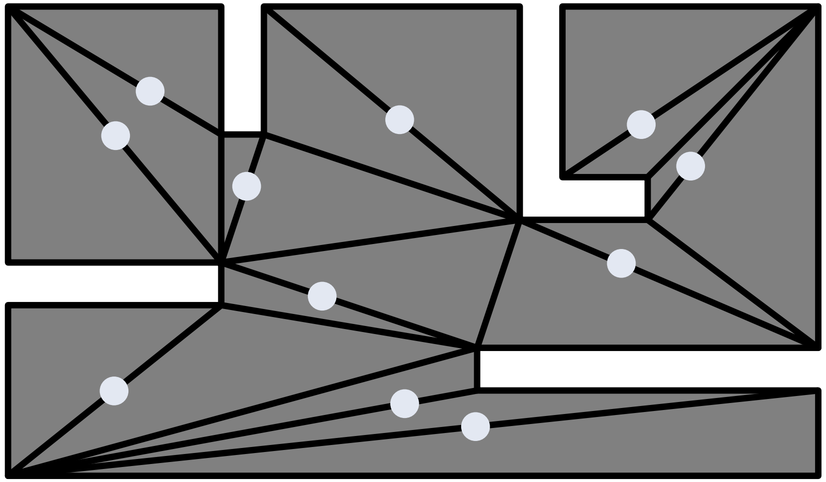

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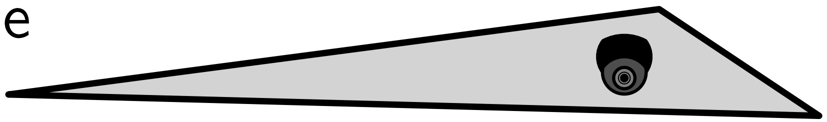

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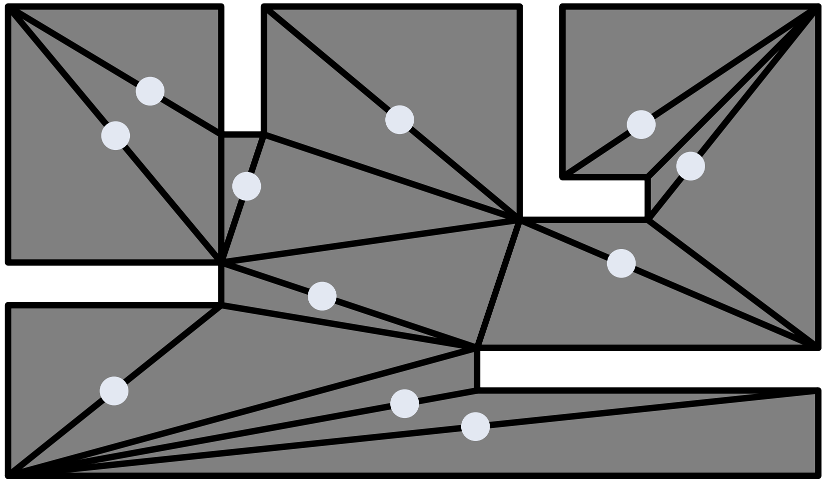

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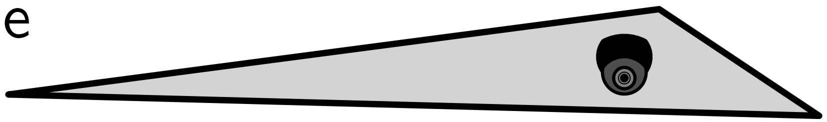

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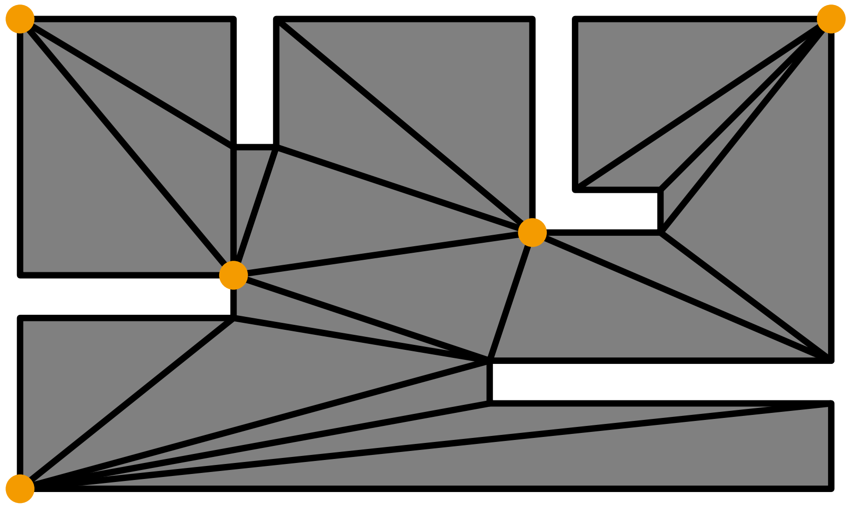

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**Theorem 2:** For a simple polygon with $n$ vertices, $\lfloor n/3 \rfloor$ cameras are sometimes necessary and always sufficient to guard it.

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**Proof:**

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**Discuss it with your neighbour for 2 minutes**

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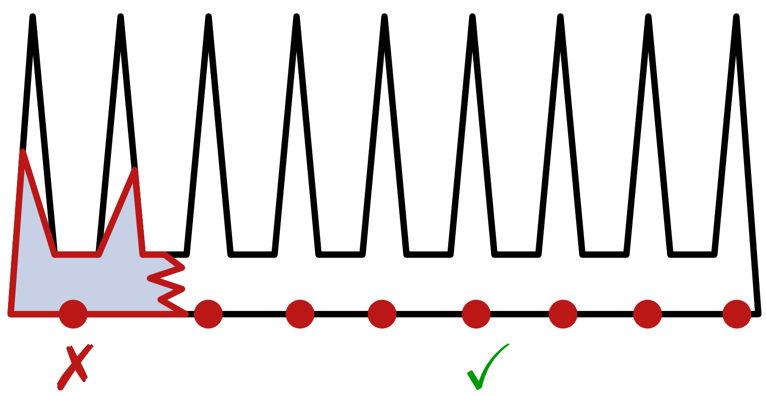

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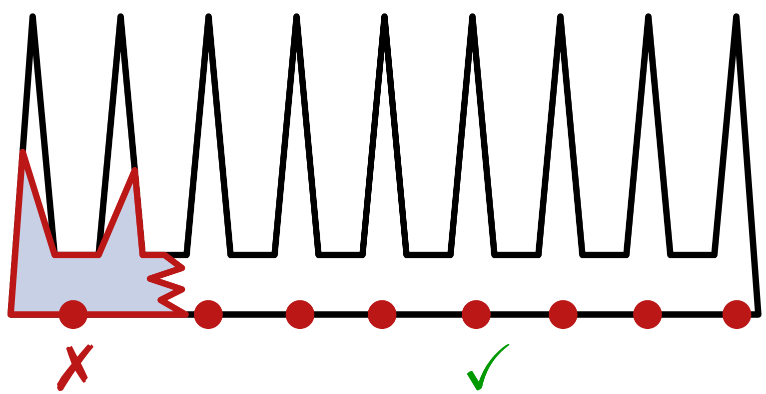


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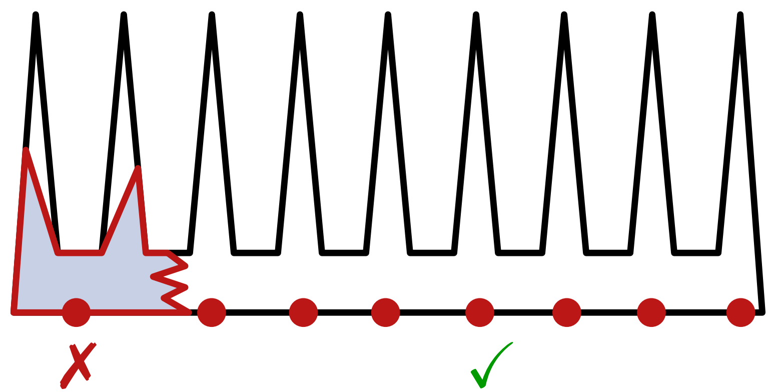


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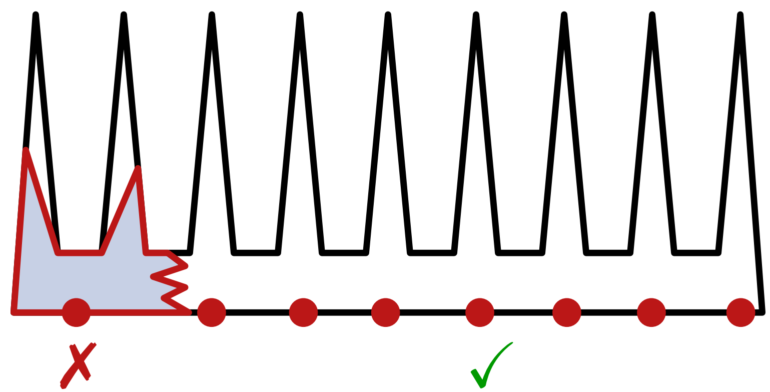


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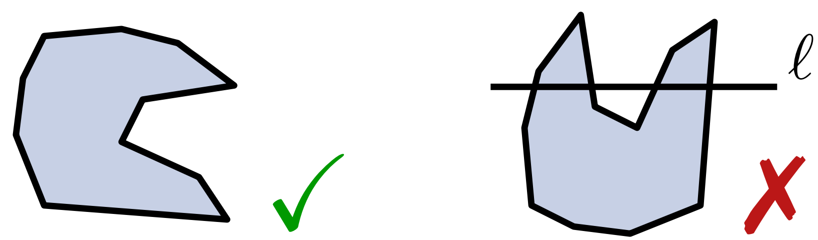
**Can we do better than $O(n^2)$ described before?**

# Proof of Art-Gallery-Theorem: Overview

3-step process:

- Step 1: Decompose $P$ into $y$-monotone polygons

**Definition:** A polygon is $y$-monotone, if for any horizontal line $\ell$, the interection $\ell \cap P$ is connected.



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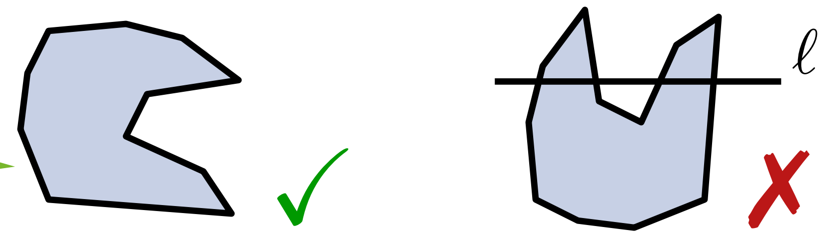

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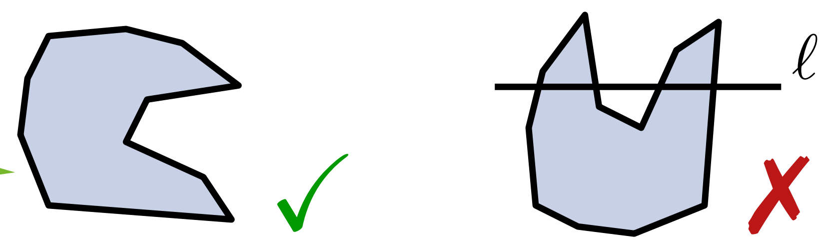


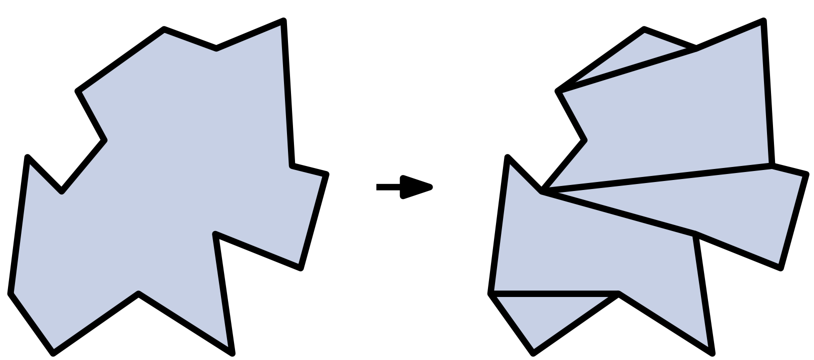

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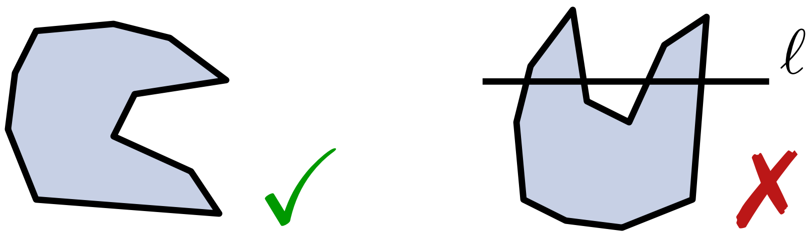


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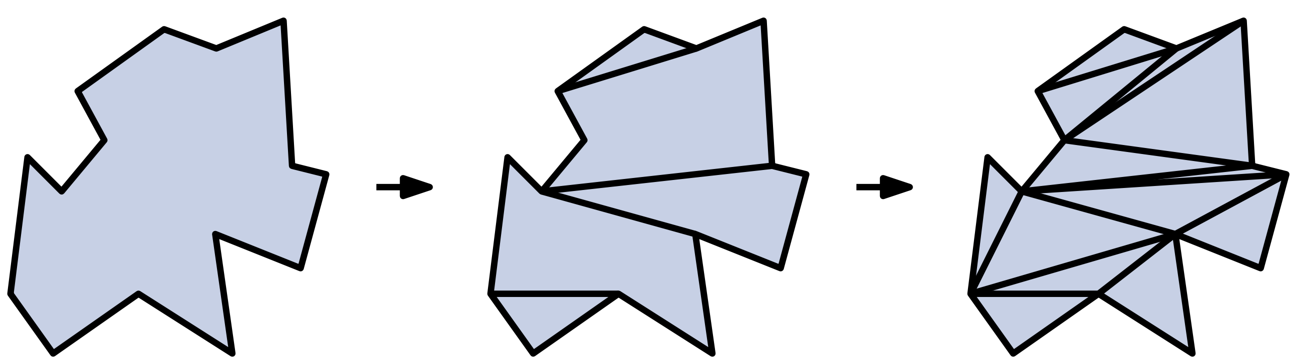

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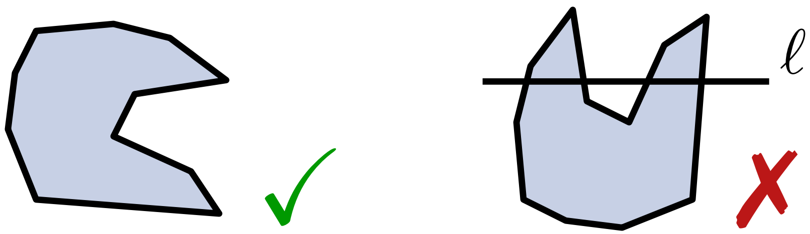
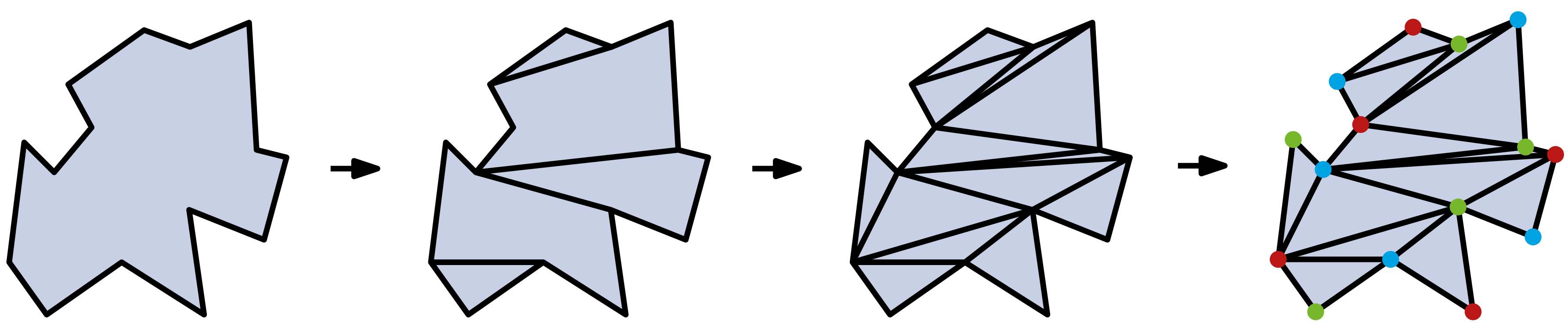
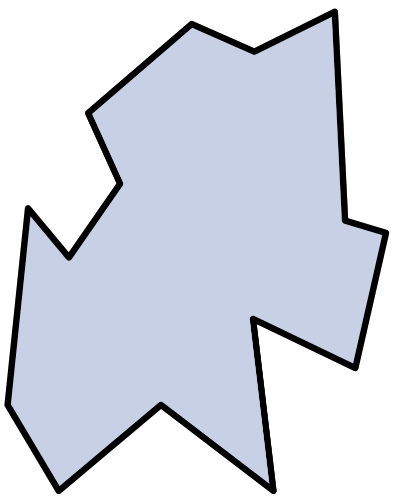



- Step 2: Triangulate the resulting $y$-monotone polygons
- Step 3: use DFS to color the vertices of the polygon



# Partition into $y$-monotone Polygons

**Idea:** Five different types of vertices



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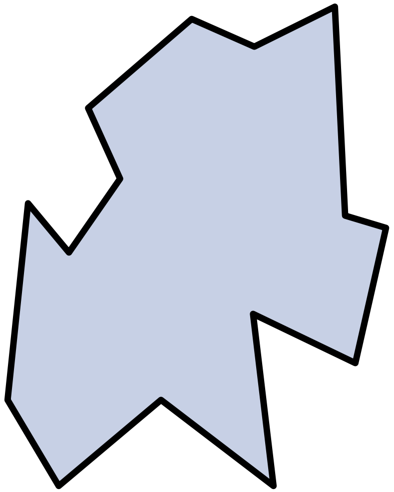

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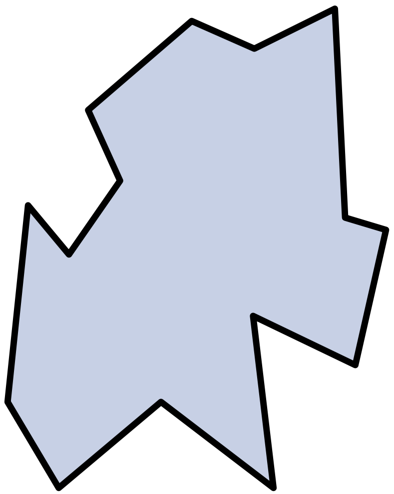

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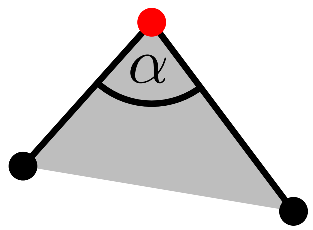
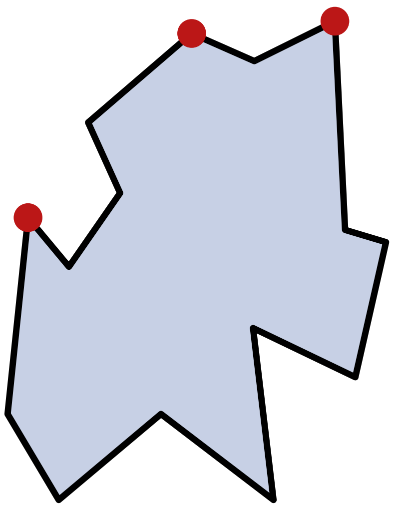



if $\alpha < 180°$



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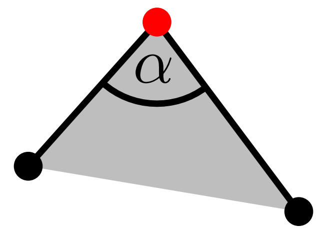
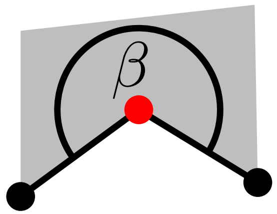
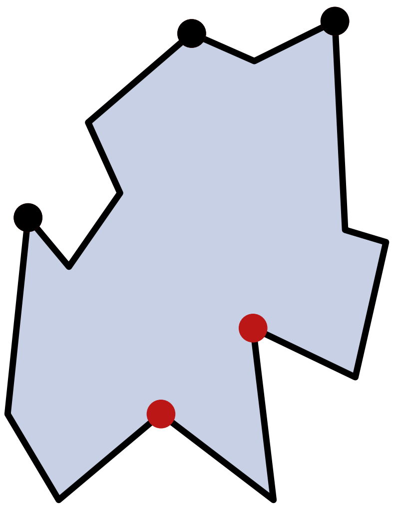
- *start* vertices  if $\alpha < 180°$



- *split* vertices  if $\beta > 180°$





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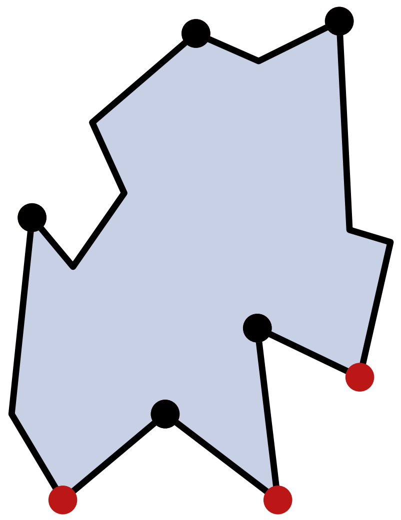
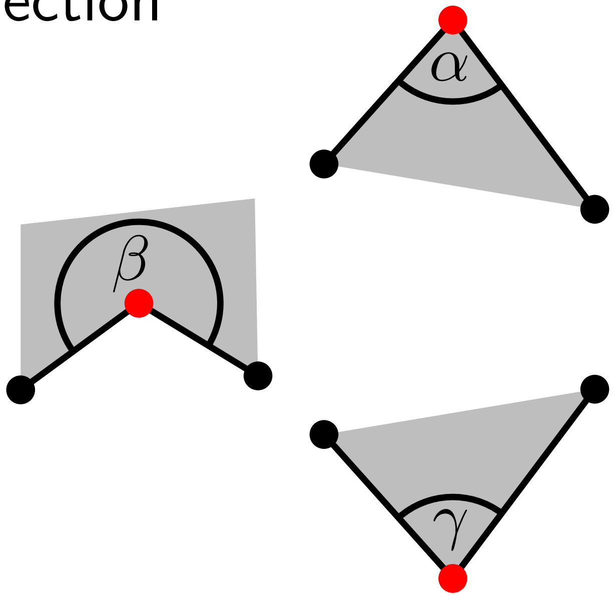
- *start* vertices
- *split* vertices
- *end* vertices






if $\alpha < 180^\circ$

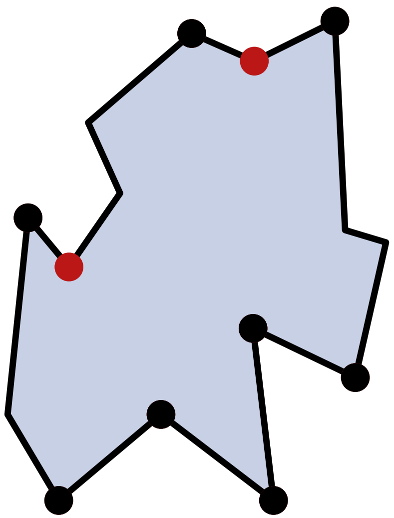
if $\beta > 180^\circ$

if $\gamma < 180^\circ$

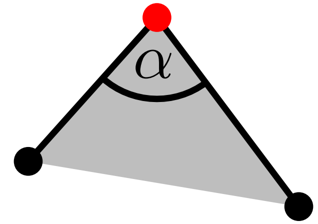
# Partition into $y$-monotone Polygons

**Idea:** Five different types of vertices

– *Turn vertices*:
vertical change in direction



- *start* vertices

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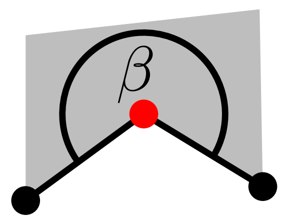
- *split* vertices

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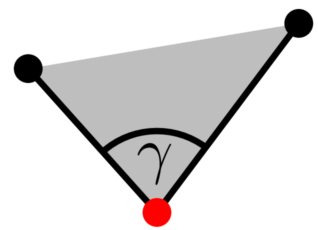
- *end* vertices

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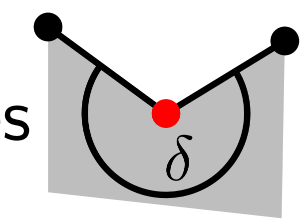
- *merge* vertices

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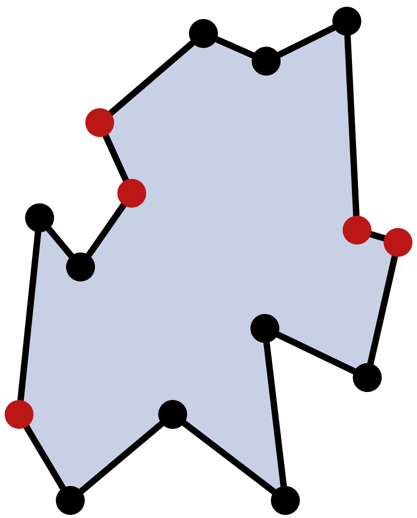
# Partition into $y$-monotone Polygons



**Idea:** Five different types of vertices



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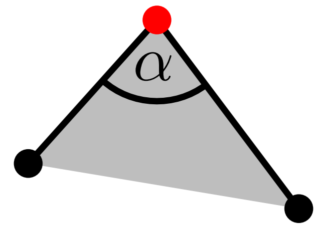


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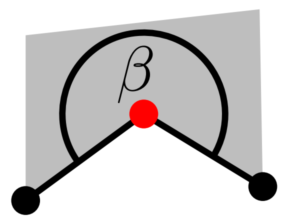


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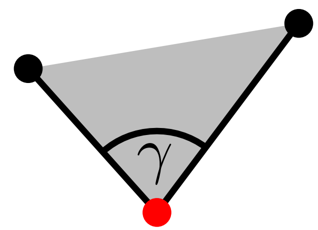


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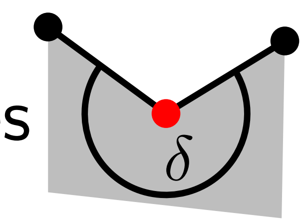


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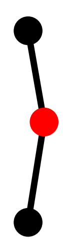
– *regular* vertices



# Characterization

**Lemma 1:** A polygon is $y$-monotone if it has no split vertices or merge vertices.

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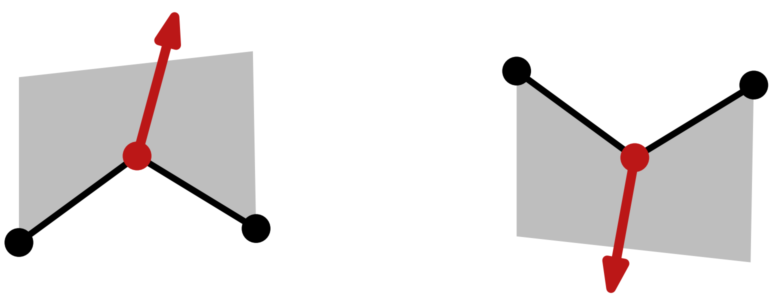

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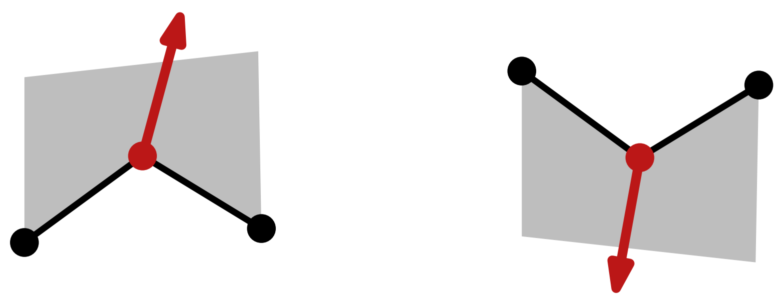
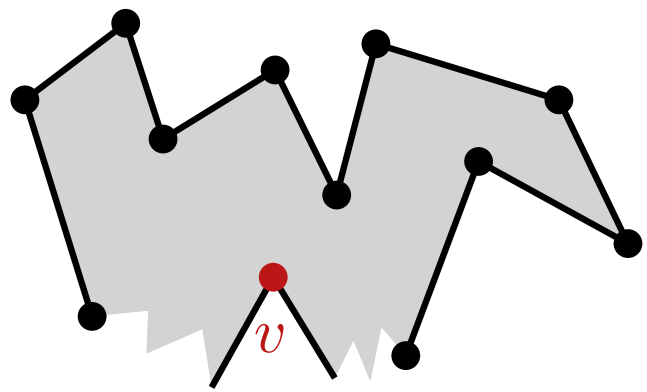


**Observation:** The diagonals should neither cross the edges of $P$ nor the other diagonals

# Ideas for Sweep-Line-Algorithm

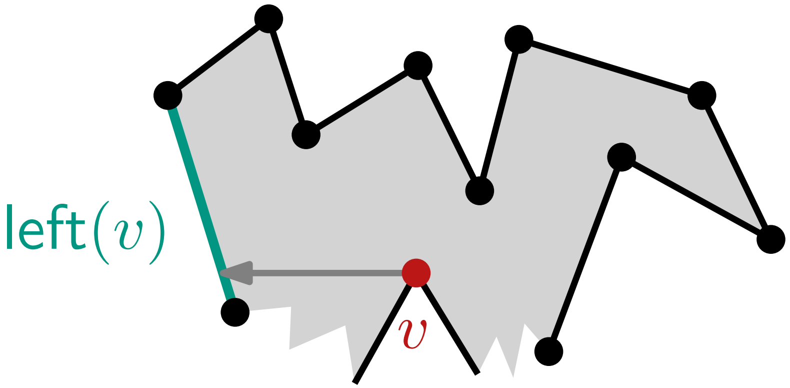
## 1) Diagonals for the split vertices



# Ideas for Sweep-Line-Algorithm

## 1) Diagonals for the split vertices

- compute for each vertex $v$ its left adjacent edge left($v$) with respect to the horizontal sweep line $\ell$



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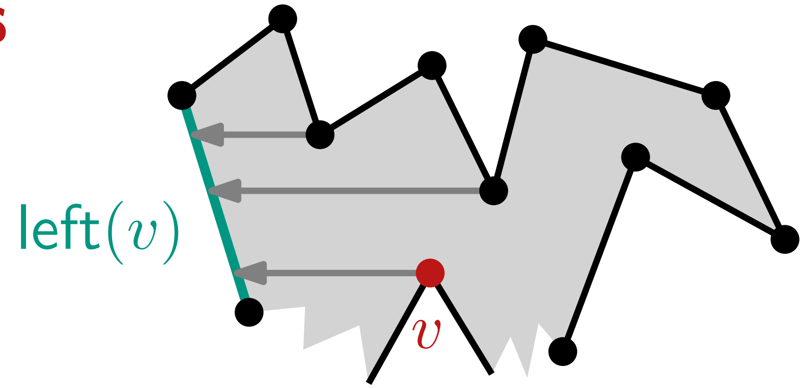

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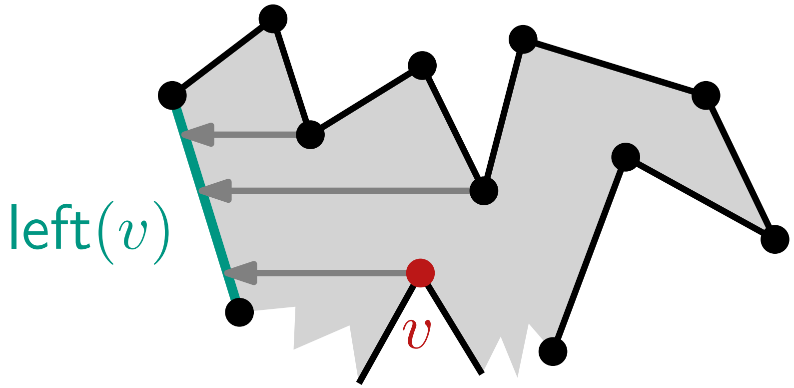


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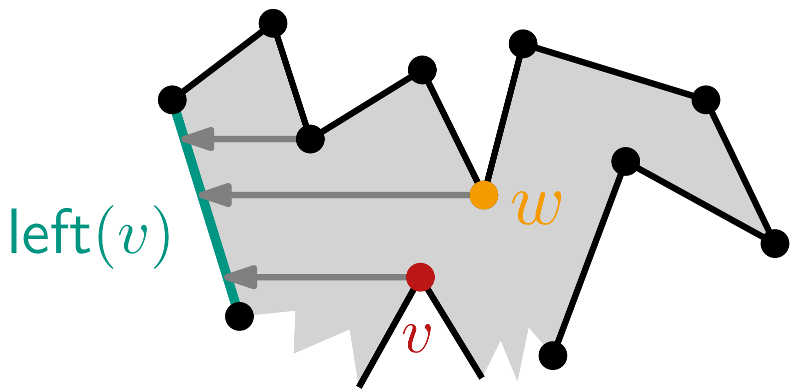


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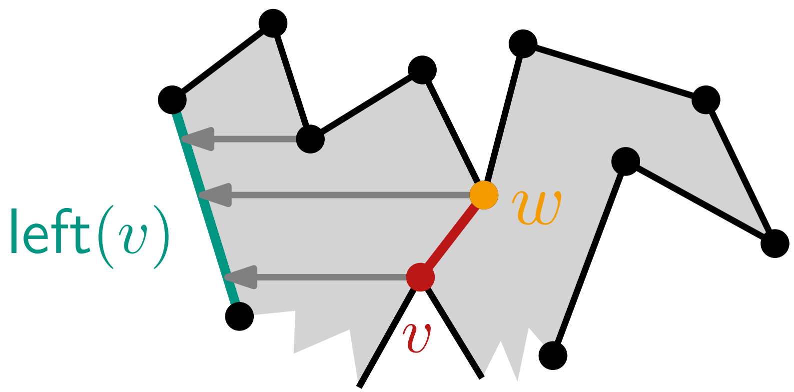


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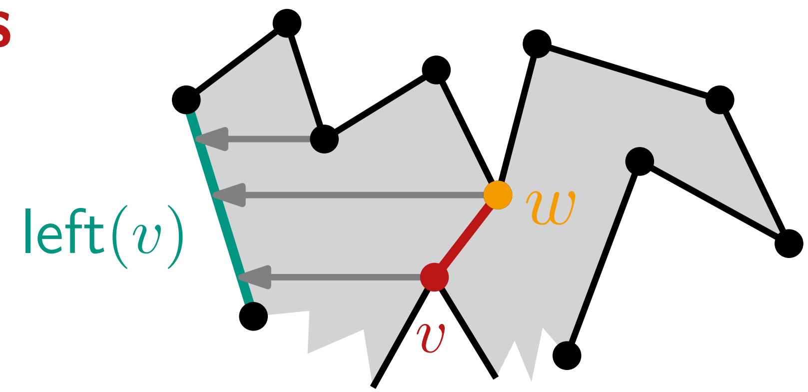


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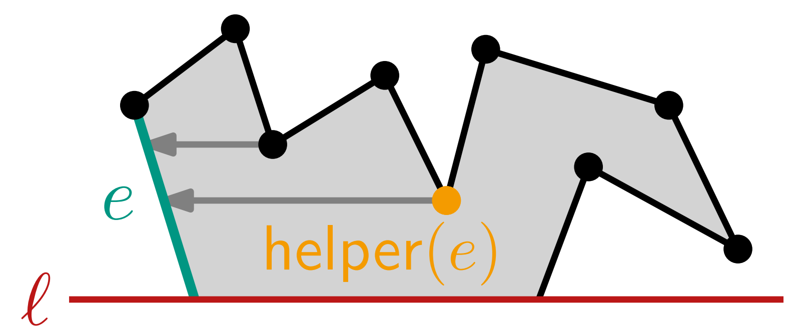
- for each edge $e$ save the botommost vertex $w$ such that left$(w) = e$; notation helper$(e) :=$ w



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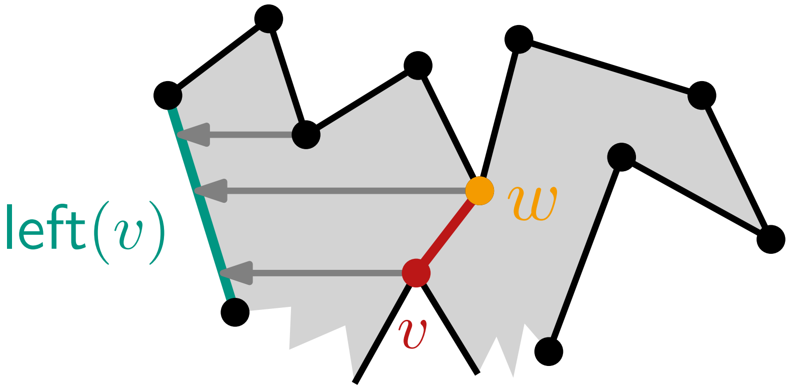


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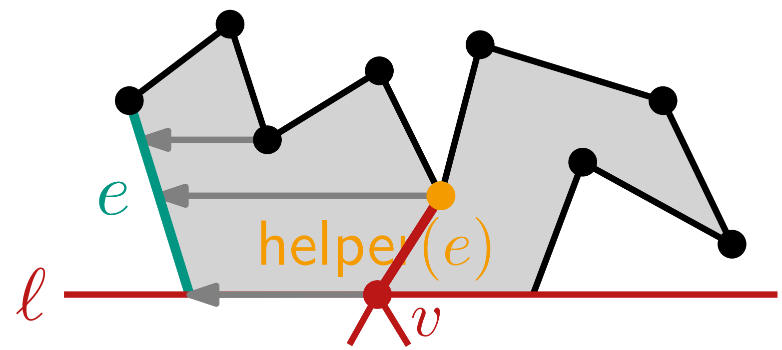
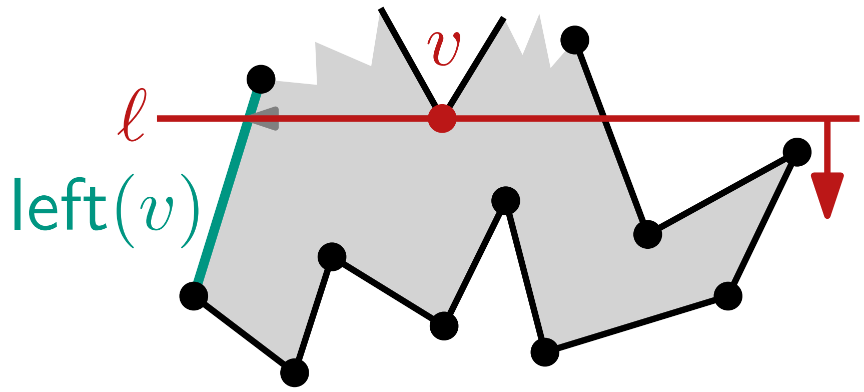
- for each edge $e$ save the botommost vertex $w$ such that left$(w) = e$; notation helper$(e)$ := w
- when $\ell$ passes through a split vertex $v$, we connect $v$ with helper(left$(v)$)



# Ideas for Sweep-Line-Algorithm

## 2) Diagonals for merge vertices

- when the vertex $v$ is reached, we set helper(left$(v)) = v$



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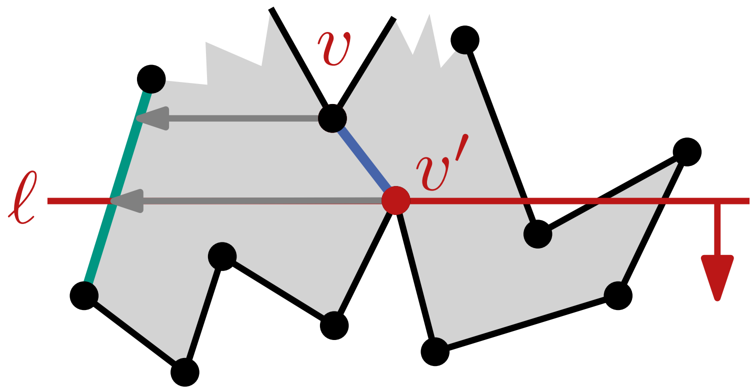
## 2) Diagonals for merge vertices

- when the vertex $v$ is reached, we set helper(left$(v)) = v$
- when we reach a split vertex $v'$ such that left$(v')$ = left$(v)$ the diagonal $(v, v')$ is introduced



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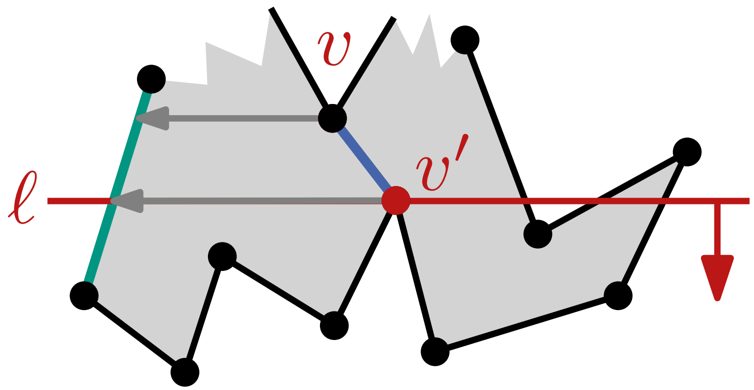
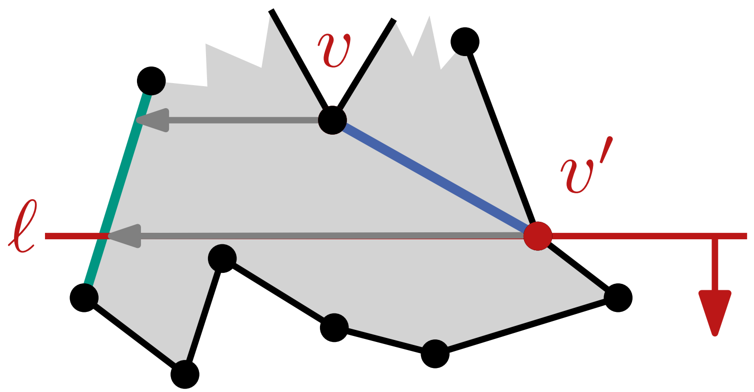



- in case we reach a regular vertex $v'$ such that helper(left$(v')$) is $v$ the diagonal $(v, v')$ is introduced



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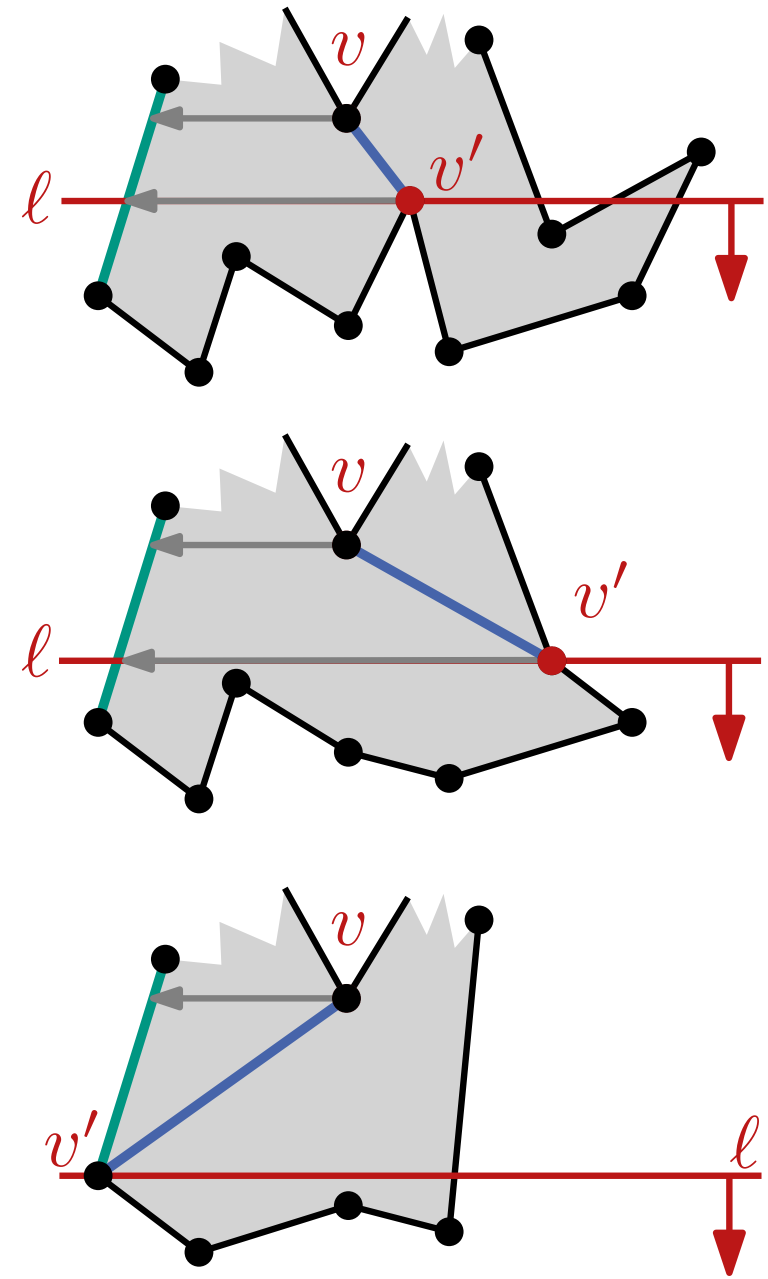
- if the end of $v'$ of left$(v)$ is reached, then the diagonal $(v, v')$ is introduced



# Algorithm MakeMonotone(P)

**MakeMonotone**(Polygon $P$)

$\mathcal{D} \leftarrow$ doubly-connected edge list for $(V(P), E(P))$
$\mathcal{Q} \leftarrow$ priority queue for $V(P)$ sorted lexicographically; $\mathcal{T} \leftarrow \emptyset$ (binary search tree for sweep-line status)
**while** $\mathcal{Q} \neq \emptyset$ **do**
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$\mathcal{T} \leftarrow$ add the left edge $e$
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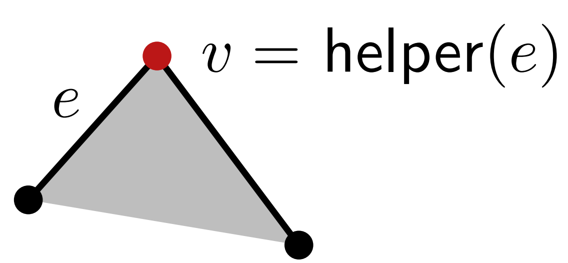

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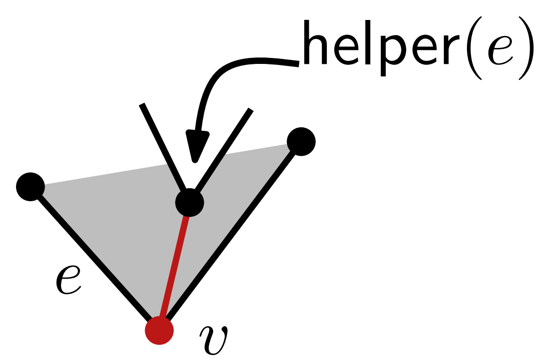


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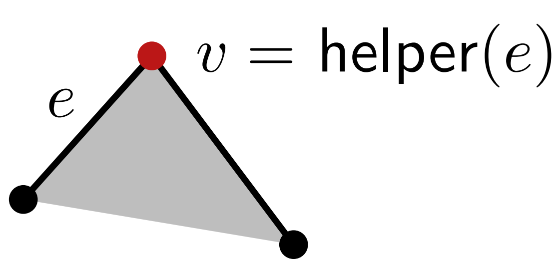


**handleEndVertex**(vertex $v$)

$e \leftarrow$ left edge
**if** isMergeVertex(helper($e$)) **then**
  $\mathcal{D} \leftarrow$ add edge (helper($e$), $v$)
remove $e$ from $\mathcal{T}$

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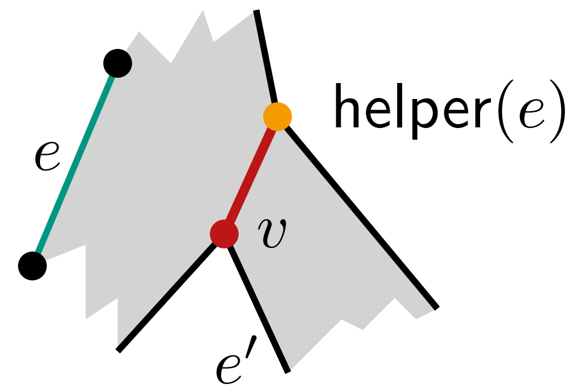
**handleSplitVertex**(vertex $v$)

$e \leftarrow$ Edge to the left of $v$ in $\mathcal{T}$
$\mathcal{D} \leftarrow$ add edge (helper($e$), $v$)
helper($e$) $\leftarrow v$
$\mathcal{T} \leftarrow$ add the right edge $e'$ of $v$
helper($e'$) $\leftarrow v$



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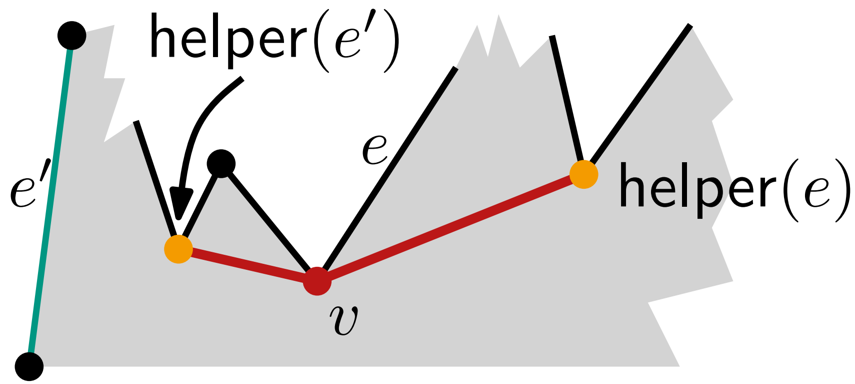


**handleMergeVertex**(vertex $v$)

$e \leftarrow$ right edge
**if** isMergeVertex(helper($e$)) **then**
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remove $e$ from $\mathcal{T}$
$e' \leftarrow$ edge to the left of $v$ in $\mathcal{T}$
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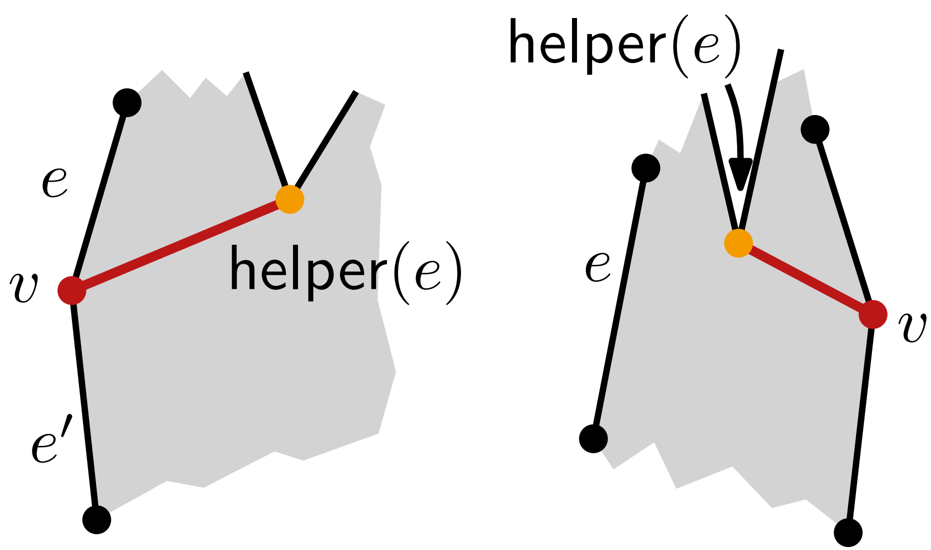


**handleRegularVertex**(vertex $v$)

**if** $P$ lies locally to the right of $v$ **then**
  $e, e' \leftarrow$ above, below edge
  **if** isMergeVertex(helper($e$)) **then**
    $\mathcal{D} \leftarrow$ add edge (helper($e$), $v$)
  remove $e$ from $\mathcal{T}$
  $\mathcal{T} \leftarrow$ add $e'$; helper($e'$) $\leftarrow v$
**else**
  $e \leftarrow$ edge to the left of $v$
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# Analysis

**Lemma 2:** The algorithm MakeMonotone computes a set of crossing-free diagonals of $P$, which partitions $P$ into $y$-monotone polygons.

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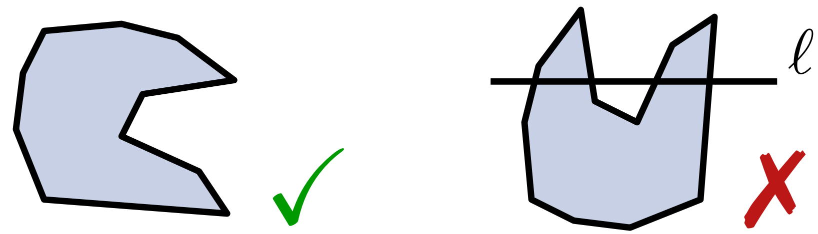
- Construct priority queue $\mathcal{Q}$: $O(n)$
- Initialize sweep-line status $\mathcal{T}$: $O(1)$
- Handle a single event: $O(\log n)$
  - $\mathcal{Q}$.deleteMax: $O(\log n)$
  - Find, remove, add element in $\mathcal{T}$: $O(\log n)$
  - Add diagonals to $\mathcal{D}$: $O(1)$
- Space: obviously $O(n)$

# Proof of Art-Gallery-Theorem: Overview

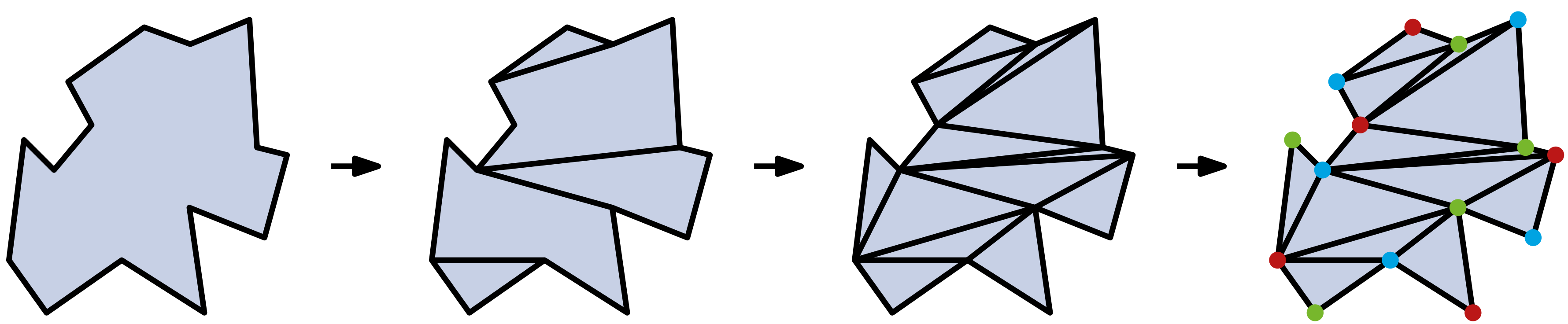
Three-step procedure:

- Step 1: Decompose $P$ in $y$-monotone polygons ✓

  **Definition:** A polygon $P$ is $y$-monotone, if for each horizontal line $\ell$ the intersection $\ell \cap P$ is connected.



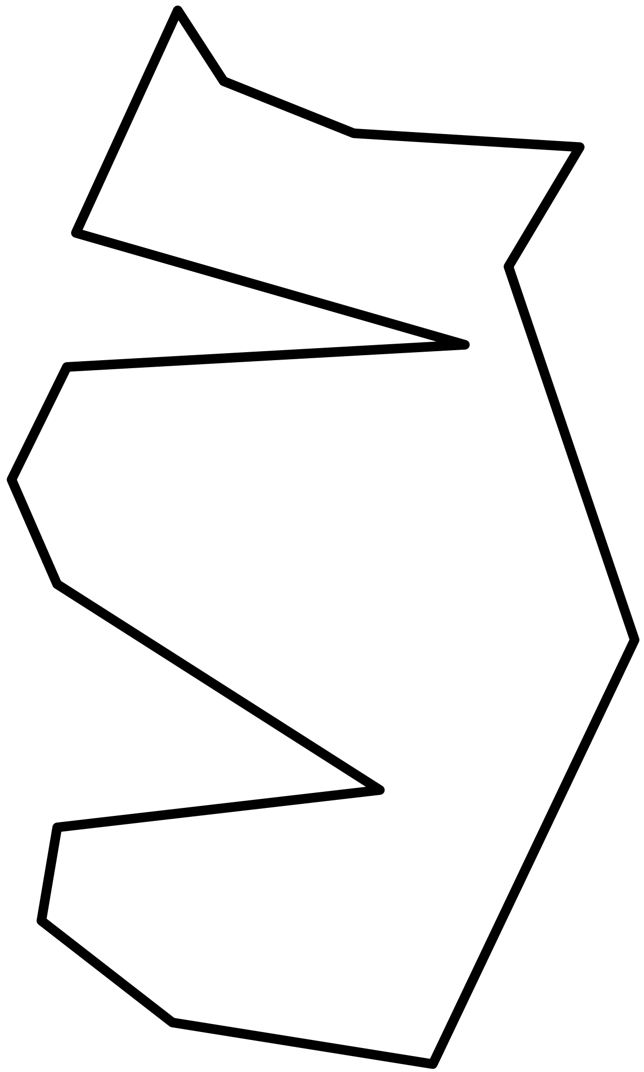

- Step 2: Triangulate $y$-monotone polygons **ToDo!**
- Step 3: use DFS to color the triangulated polygon ✓



# Triangulate $y$-monotone Polygon

**Reminder:** The left and the right boundary of the polygon have decreasing $y$-coordinates

**Approach:** Greedy, top down traversal of both sides



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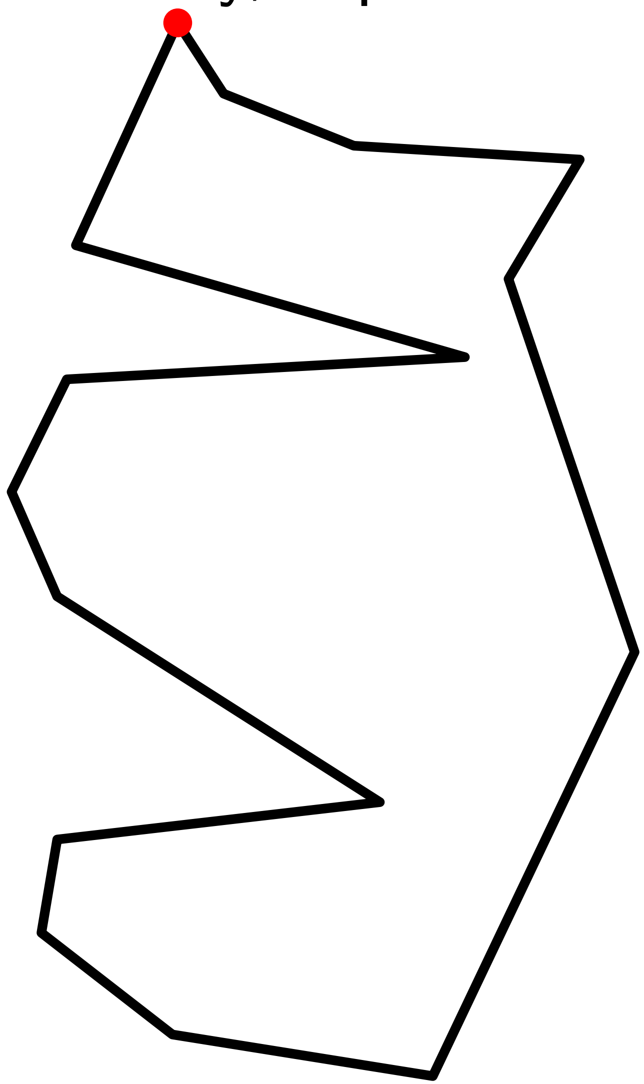

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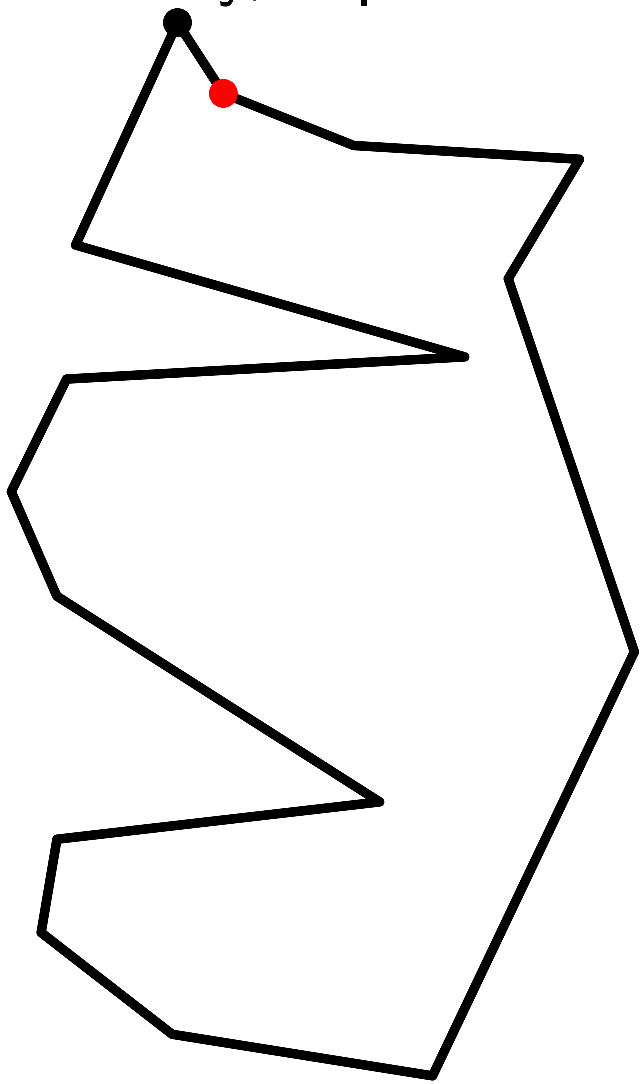

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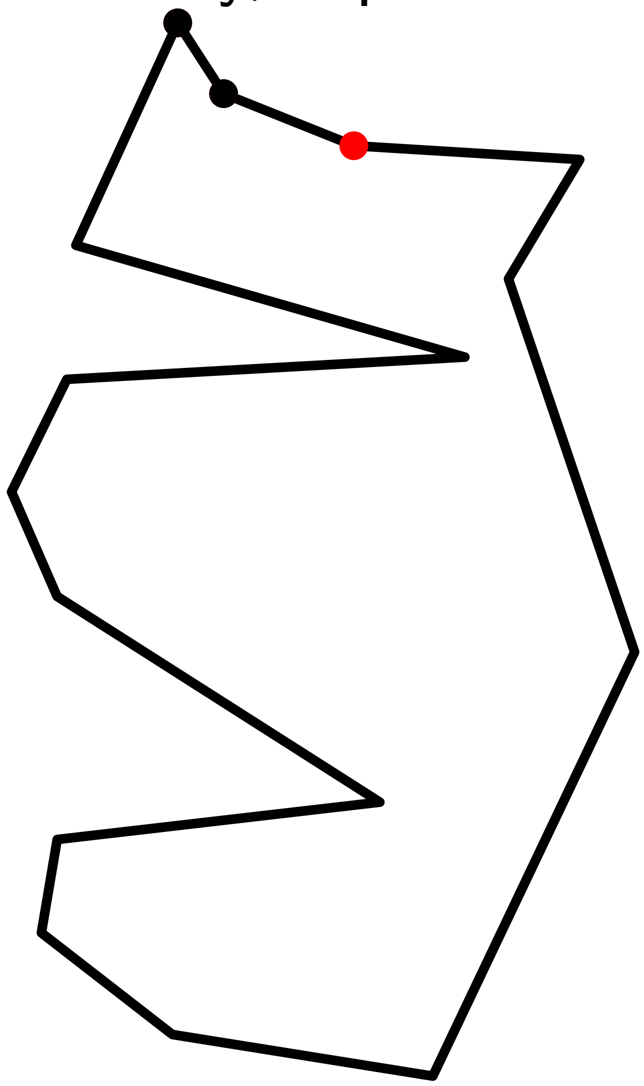

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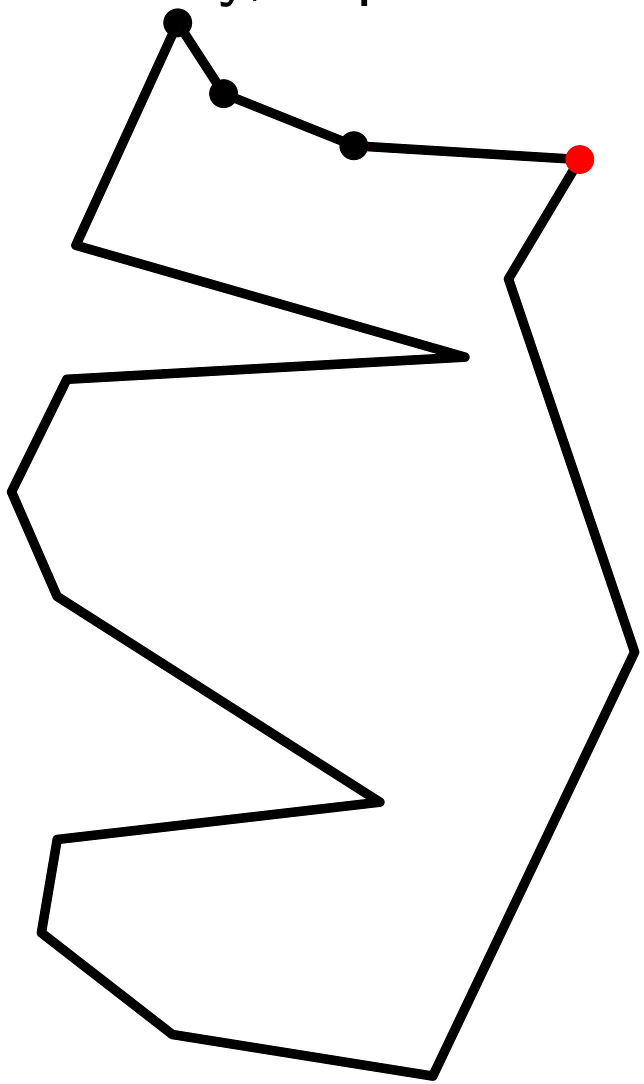

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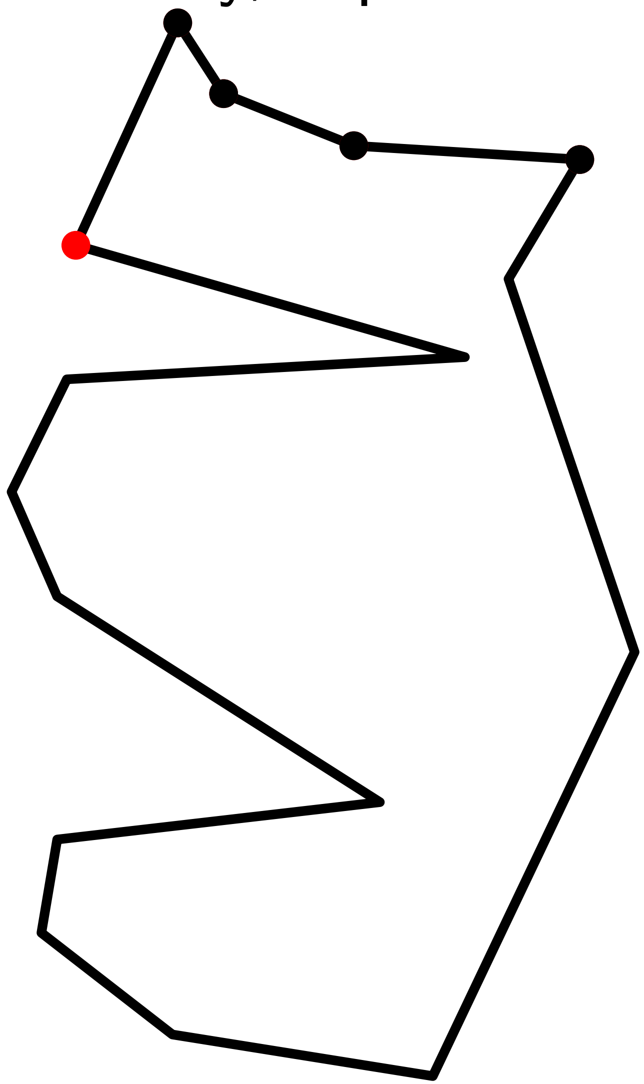

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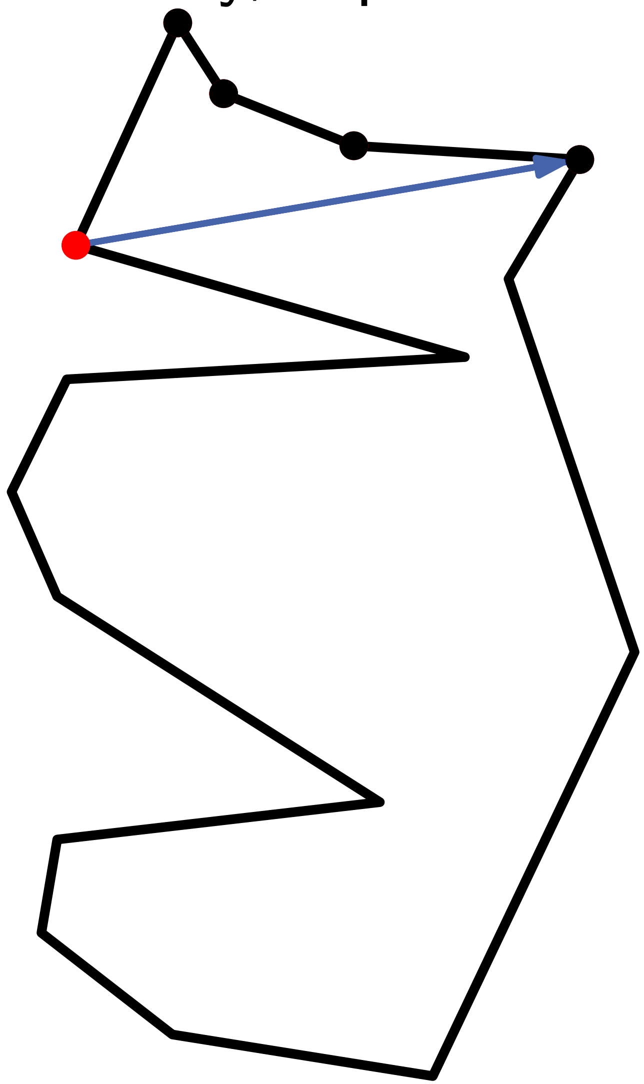

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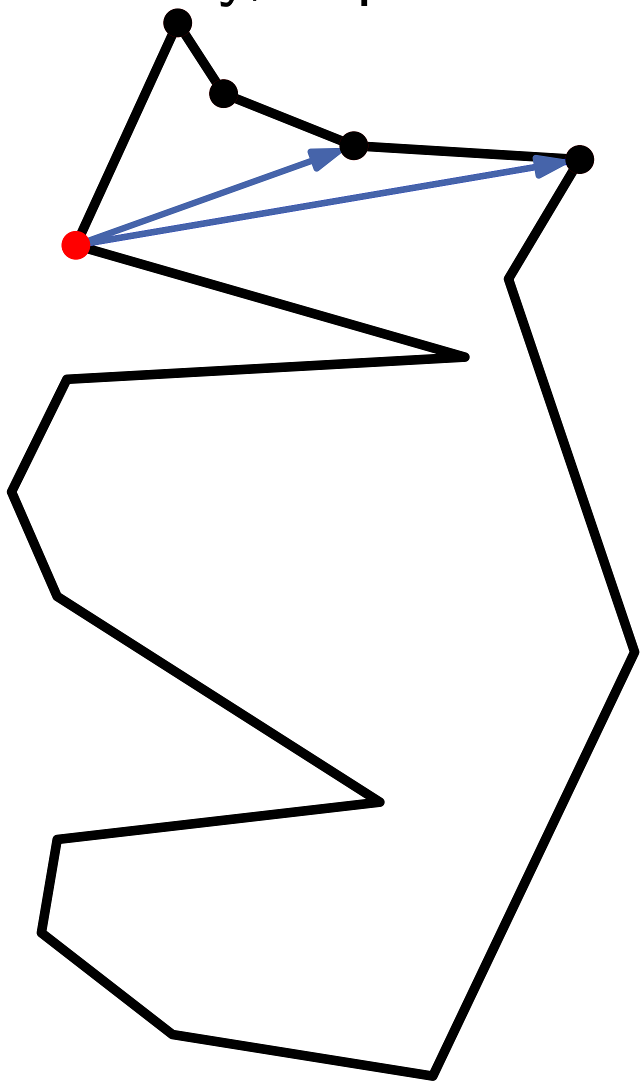

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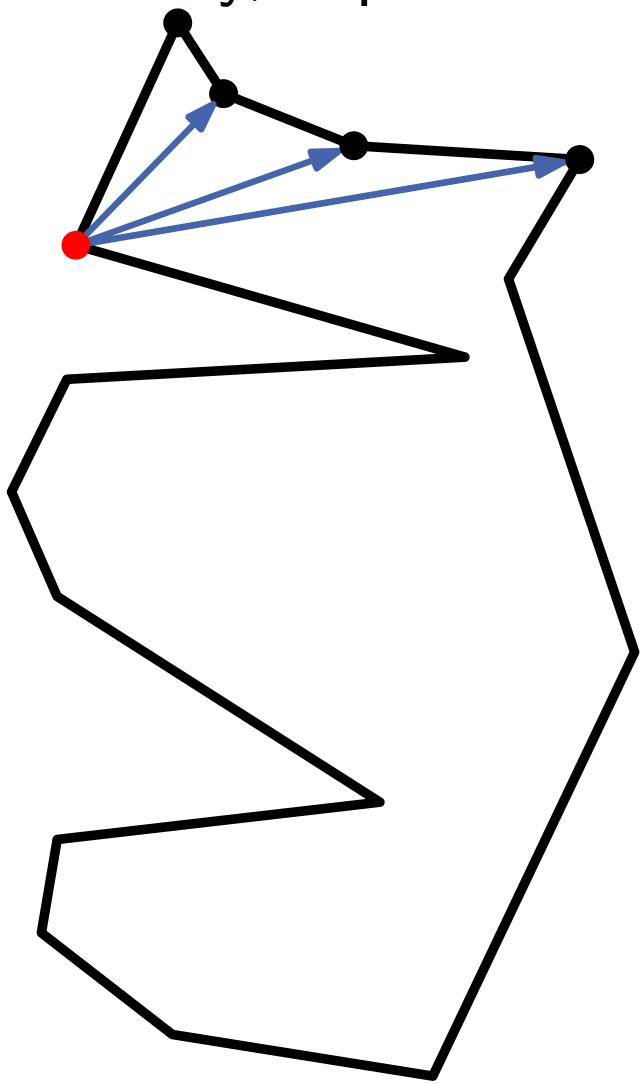

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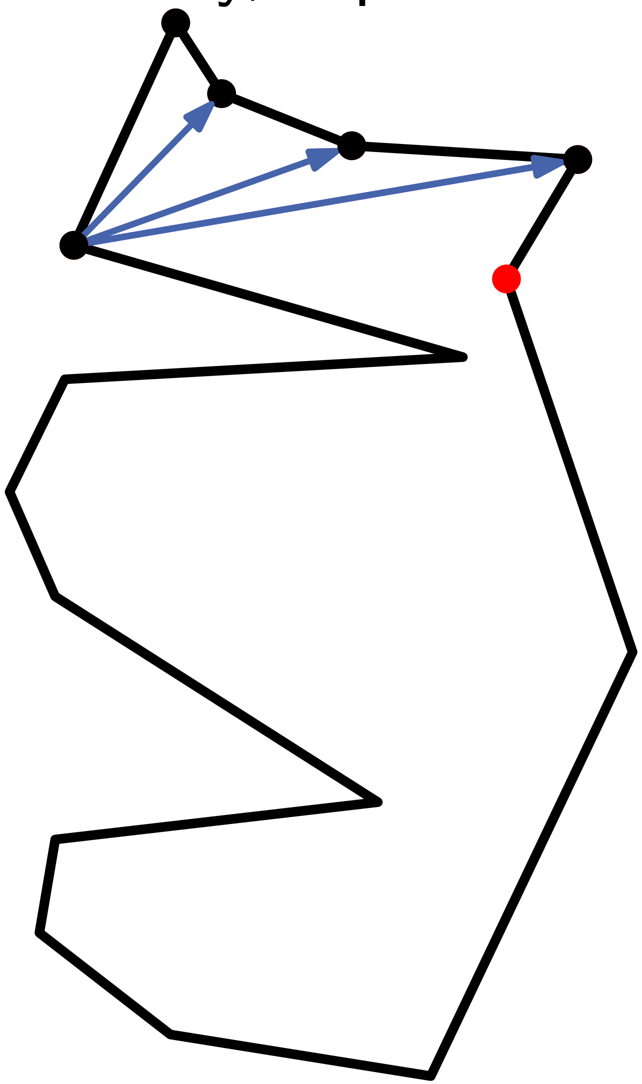

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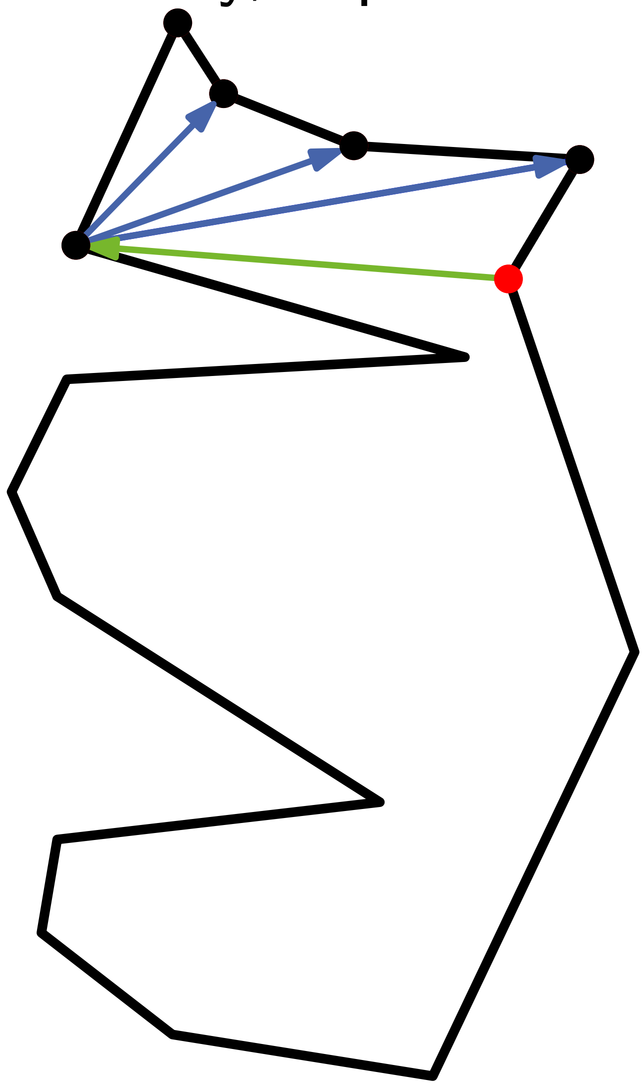

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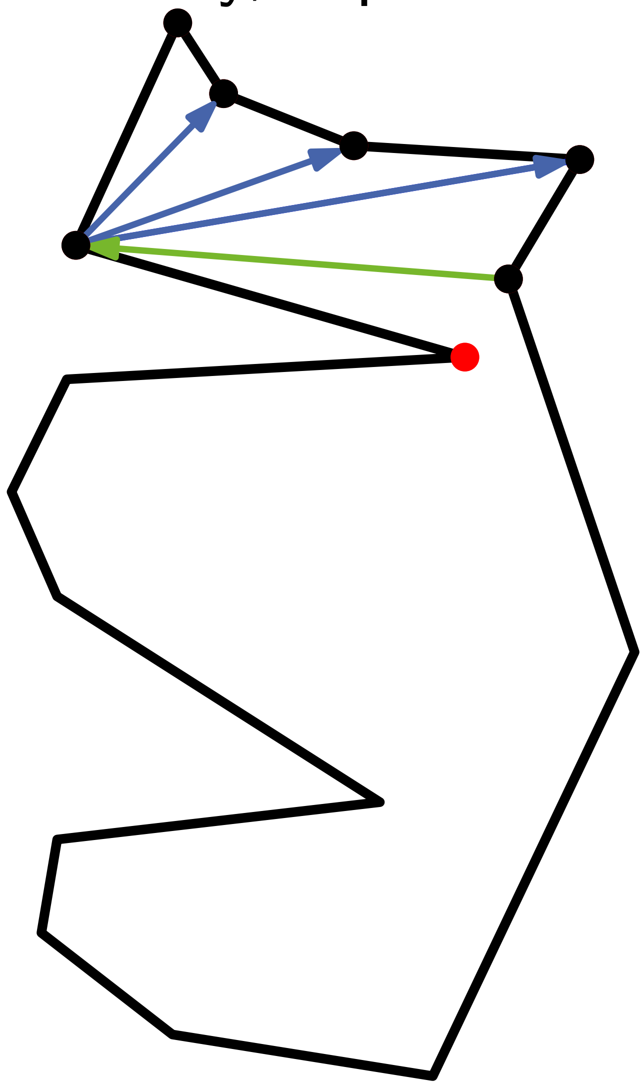

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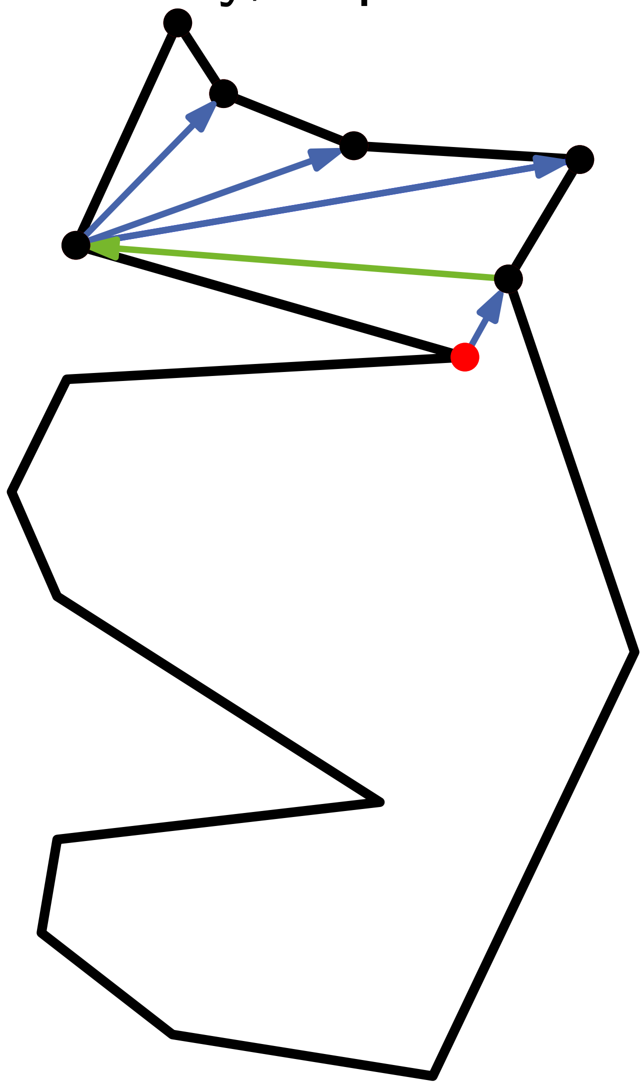

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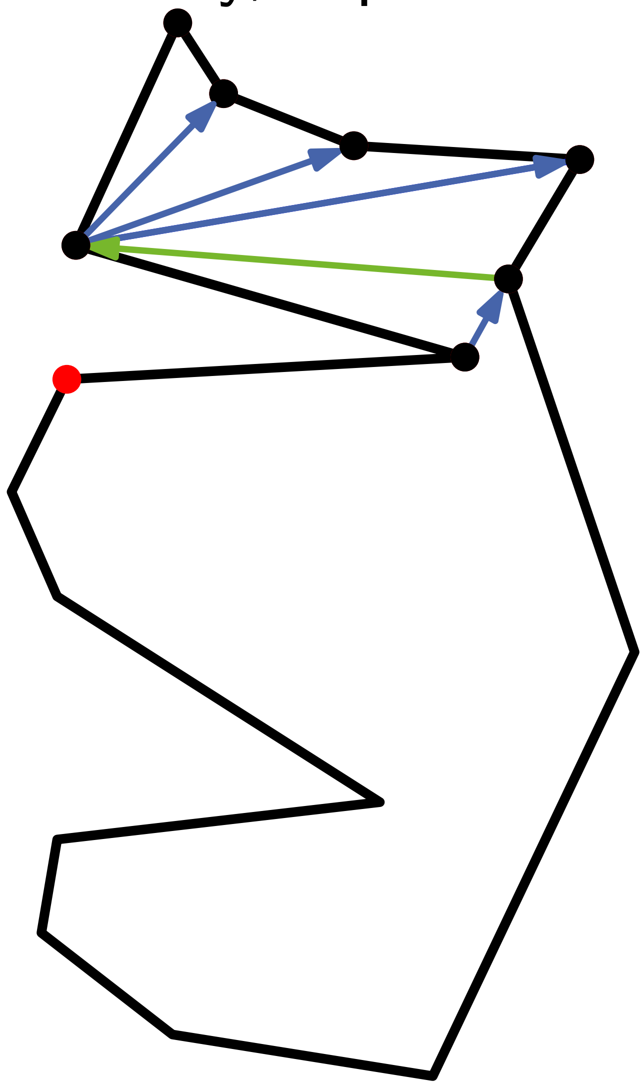

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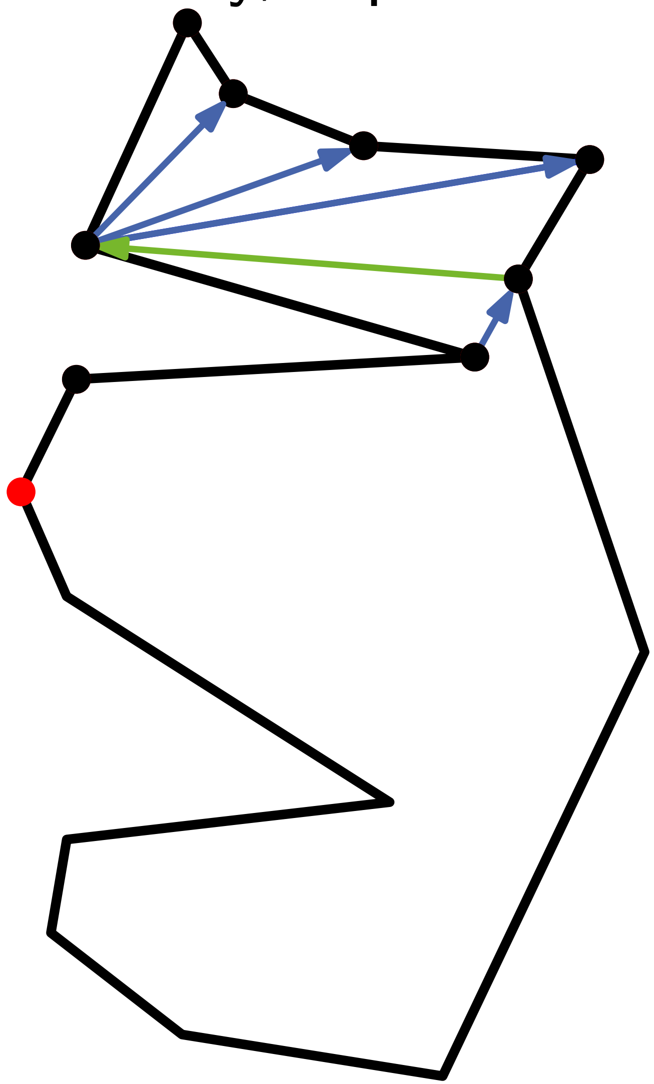

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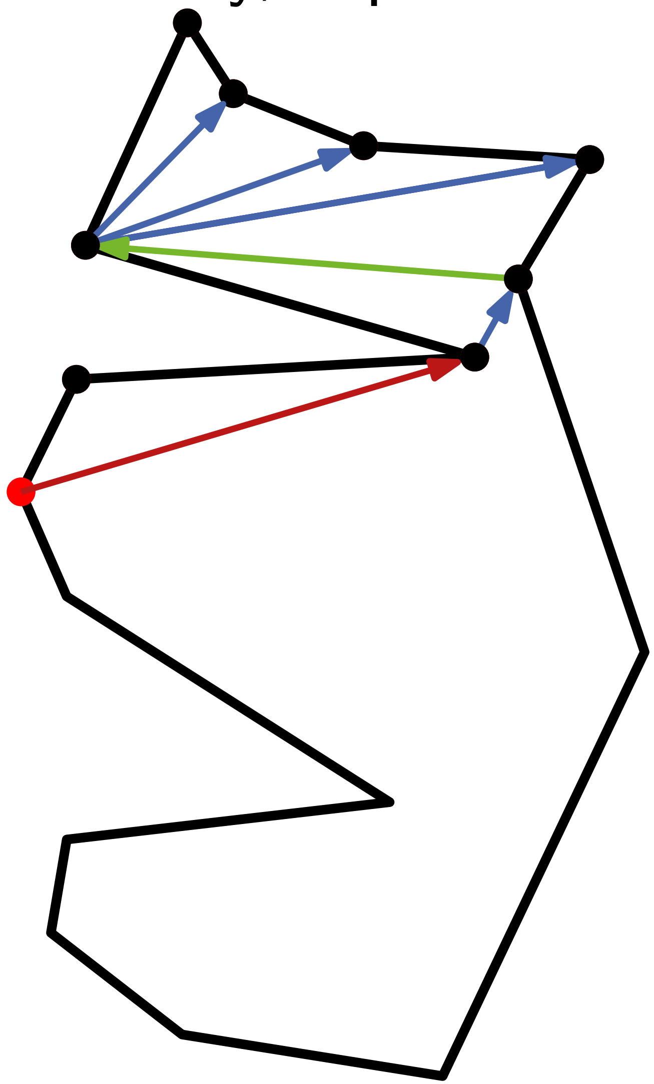

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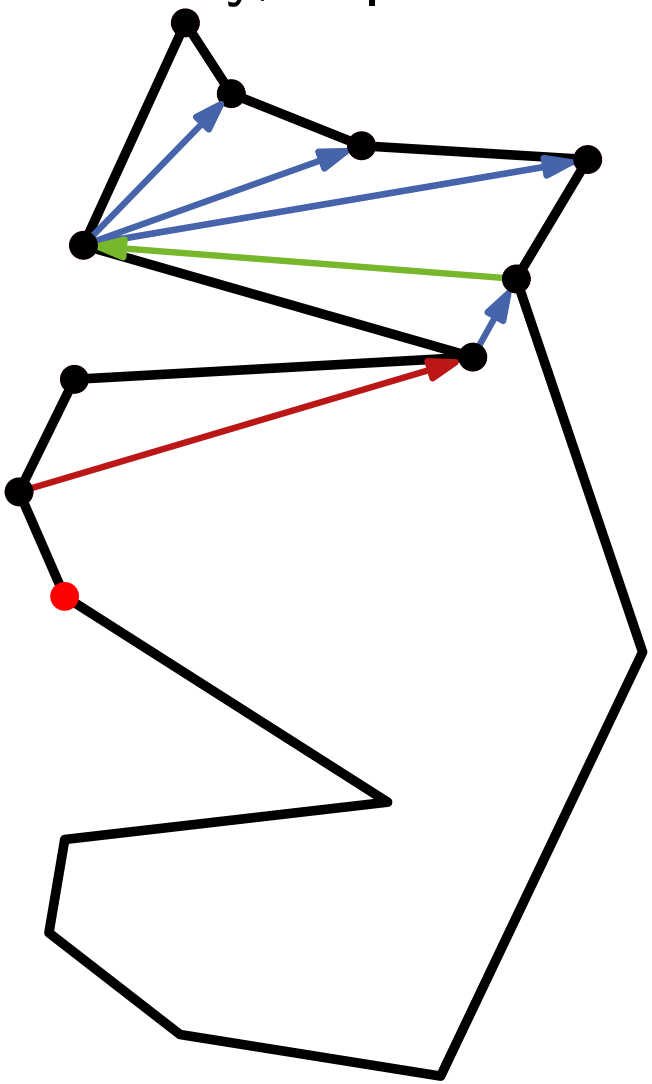

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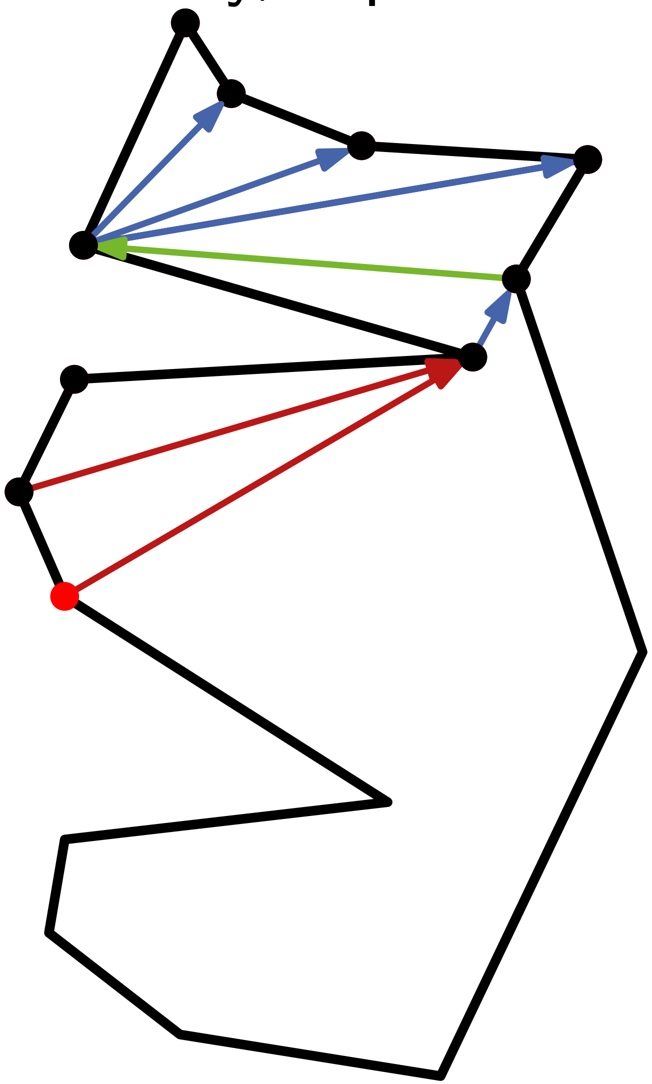

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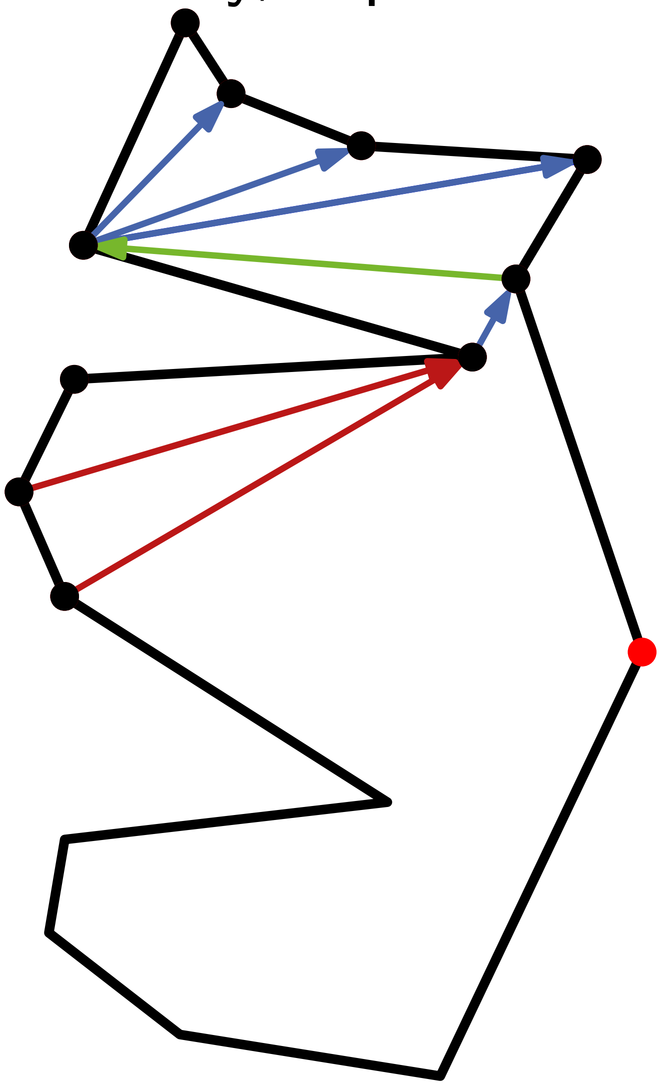

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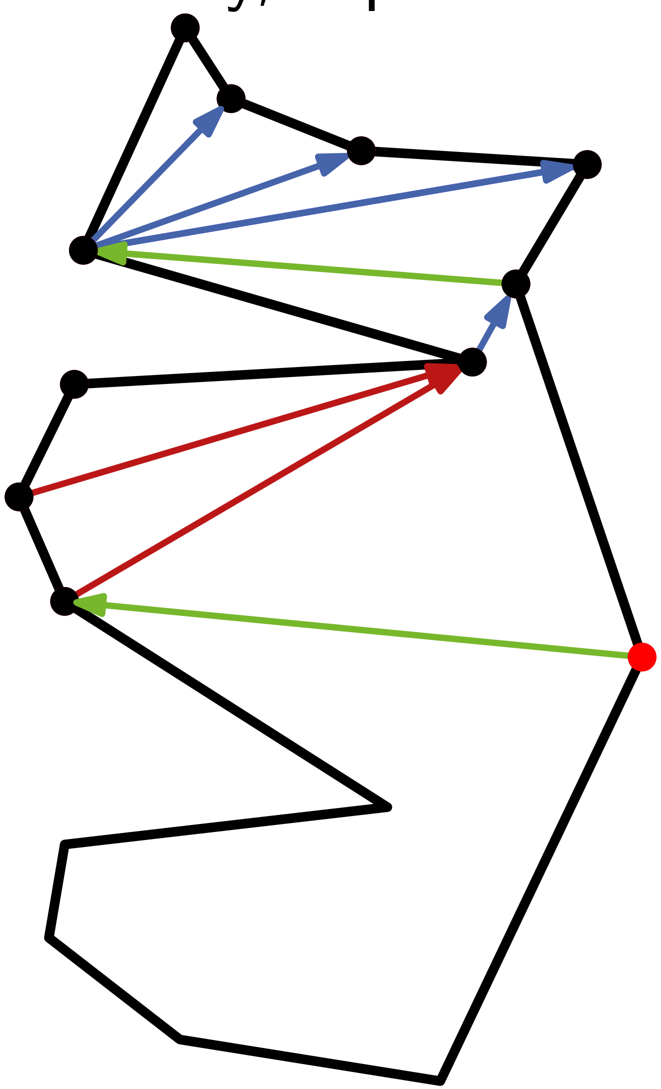

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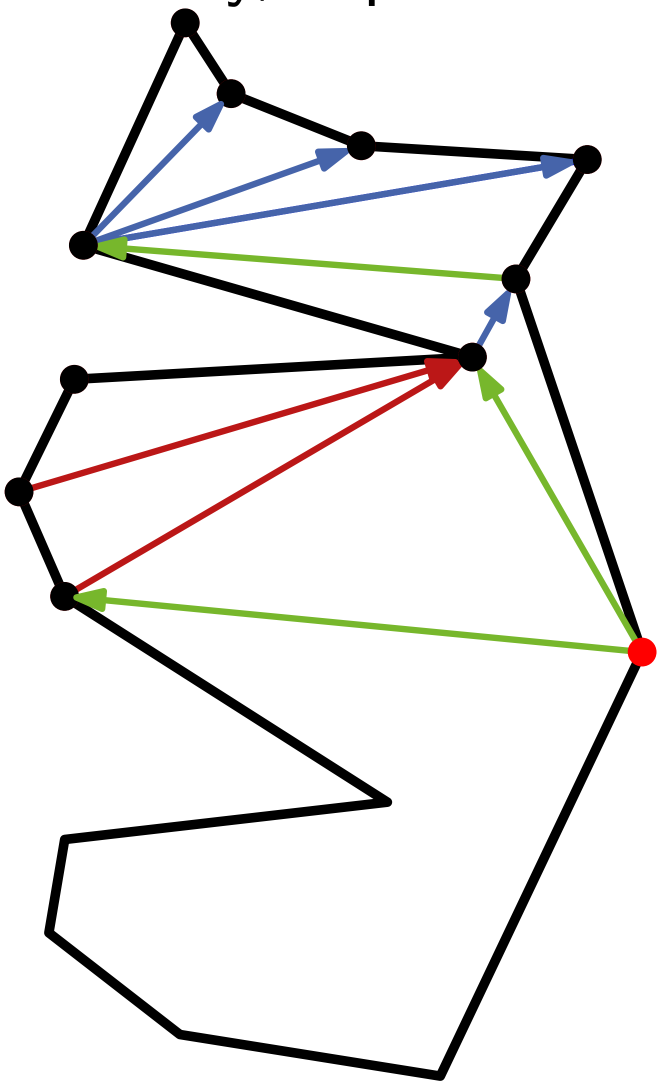

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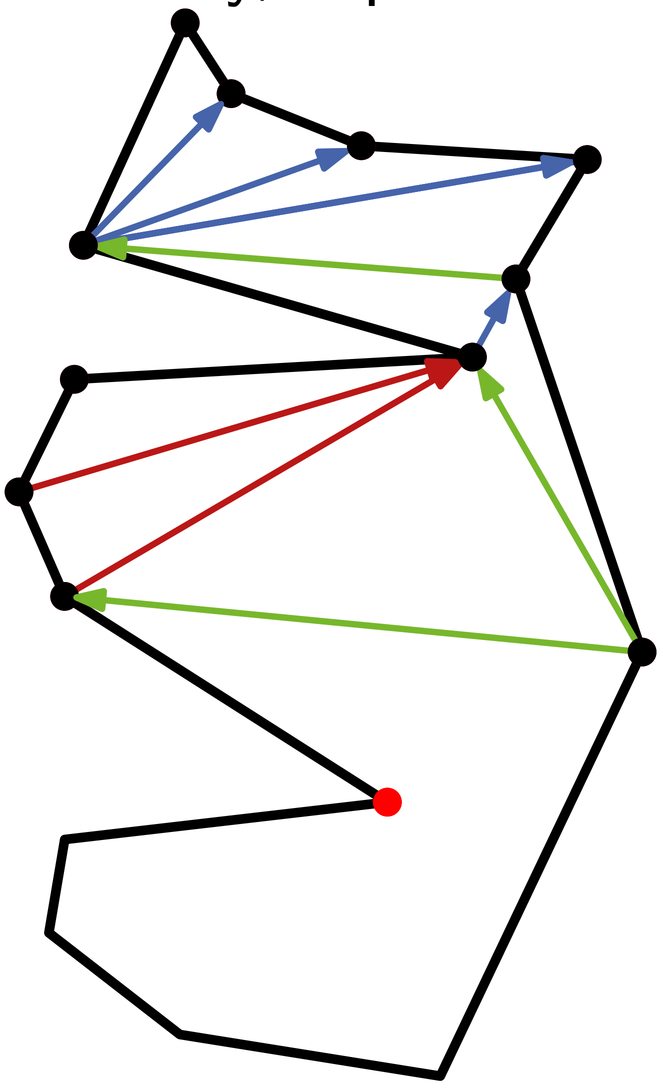

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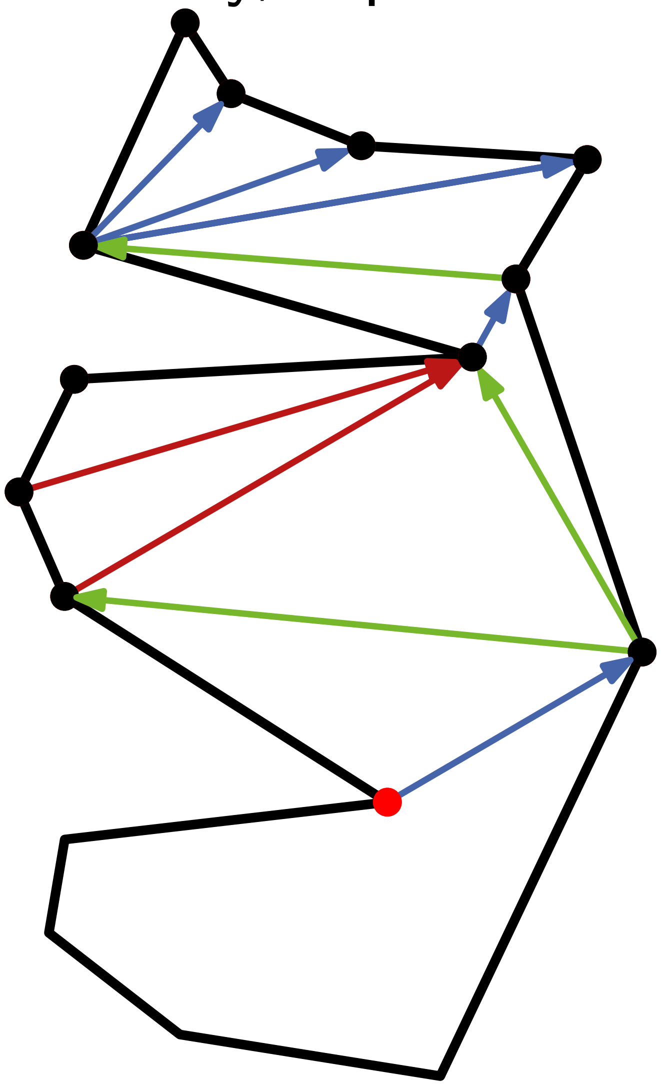

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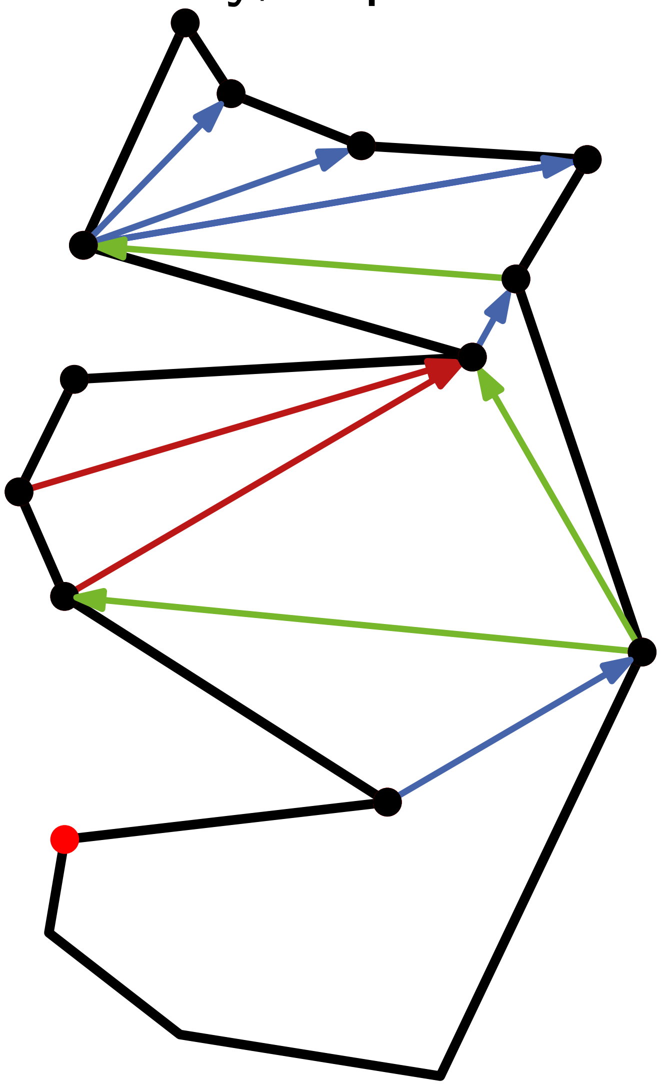

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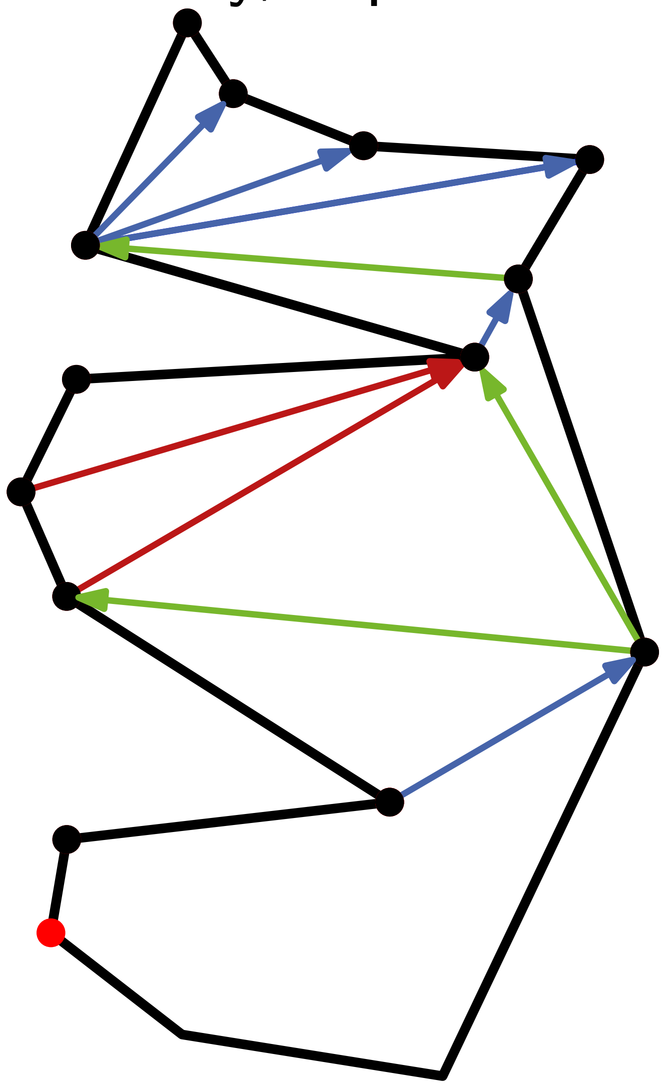

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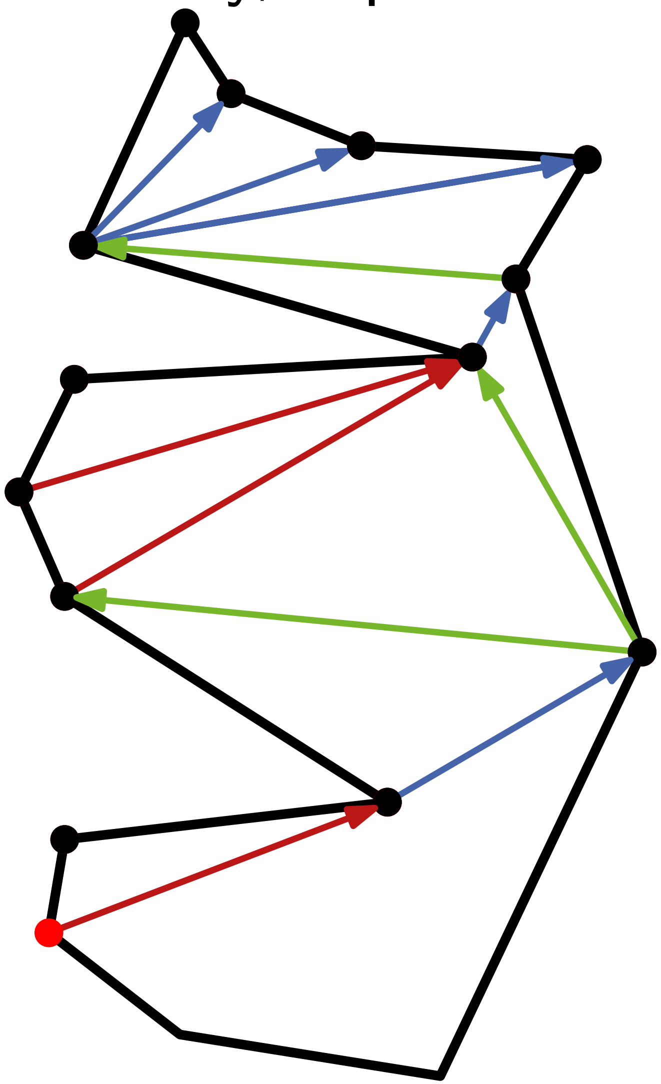

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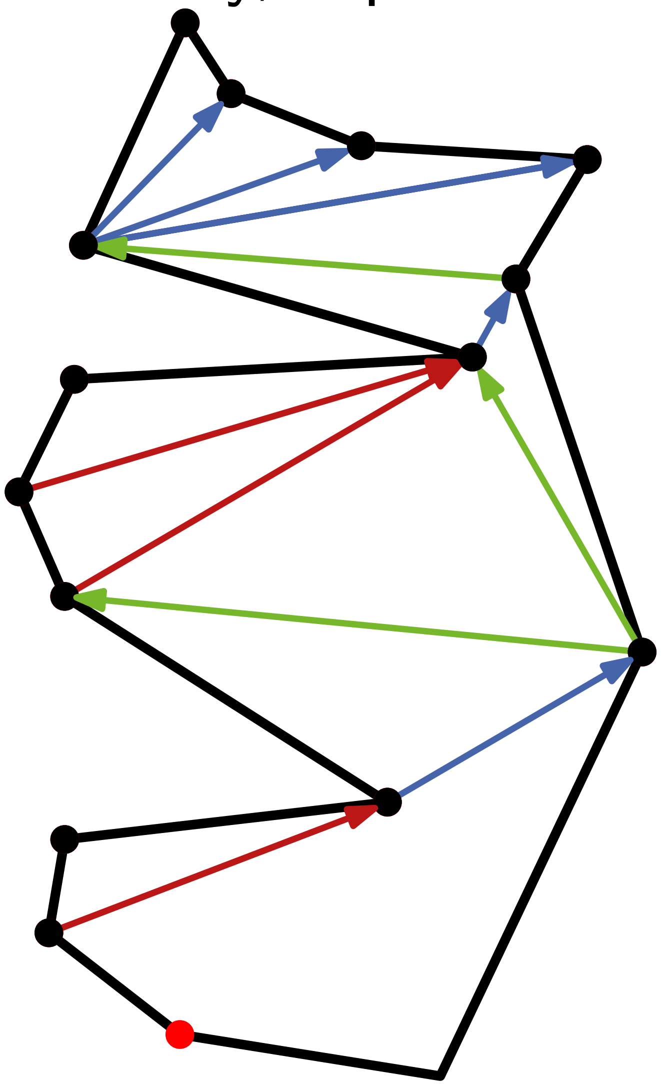

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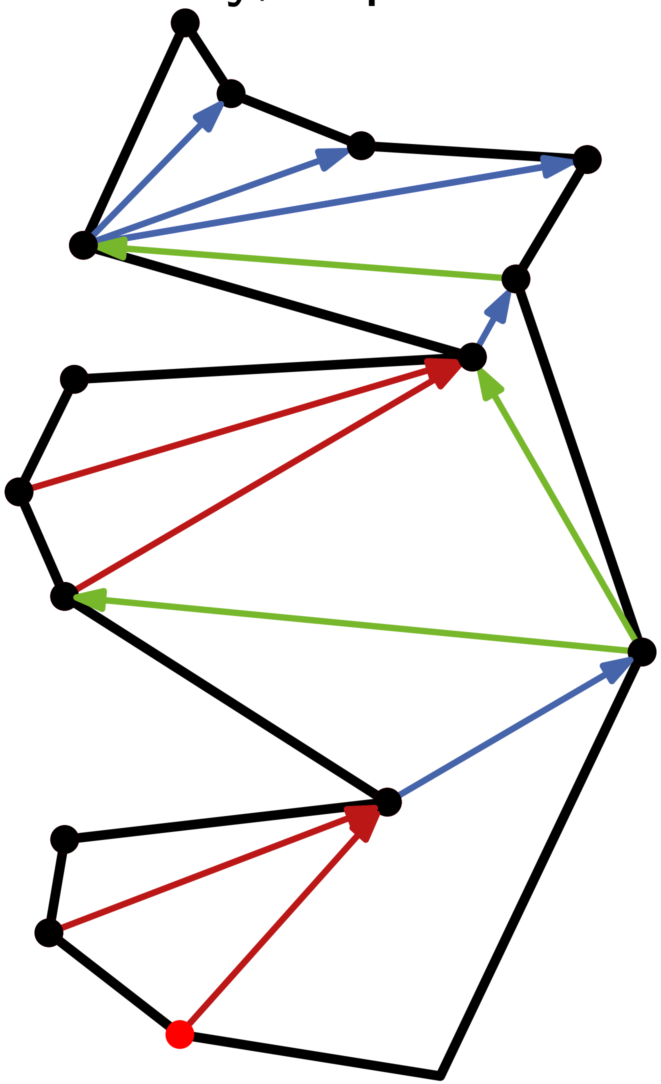

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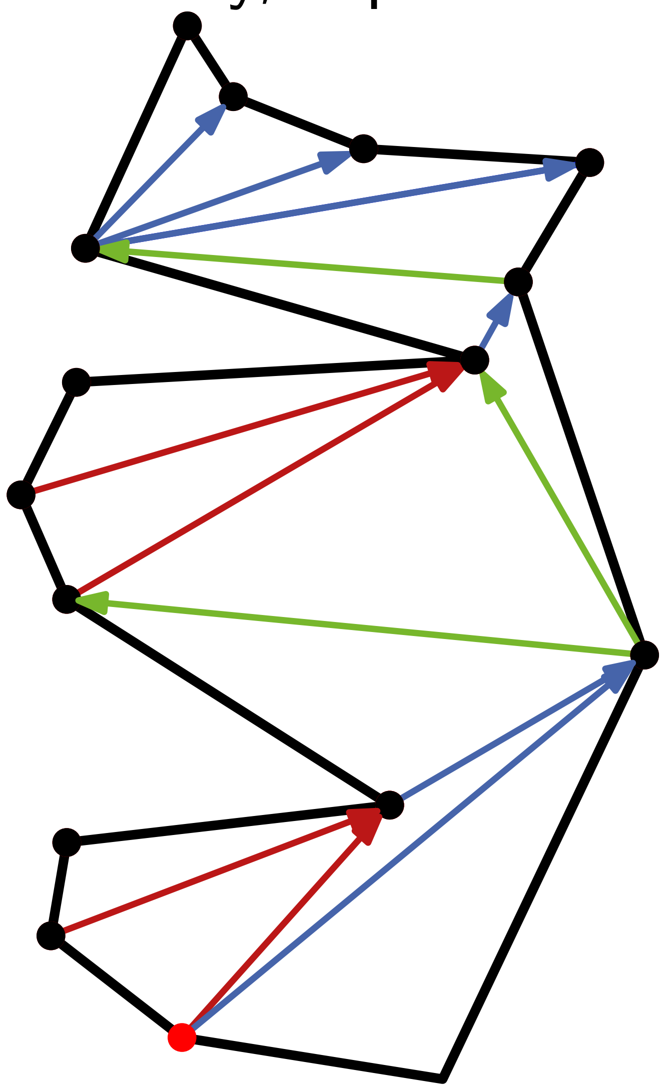

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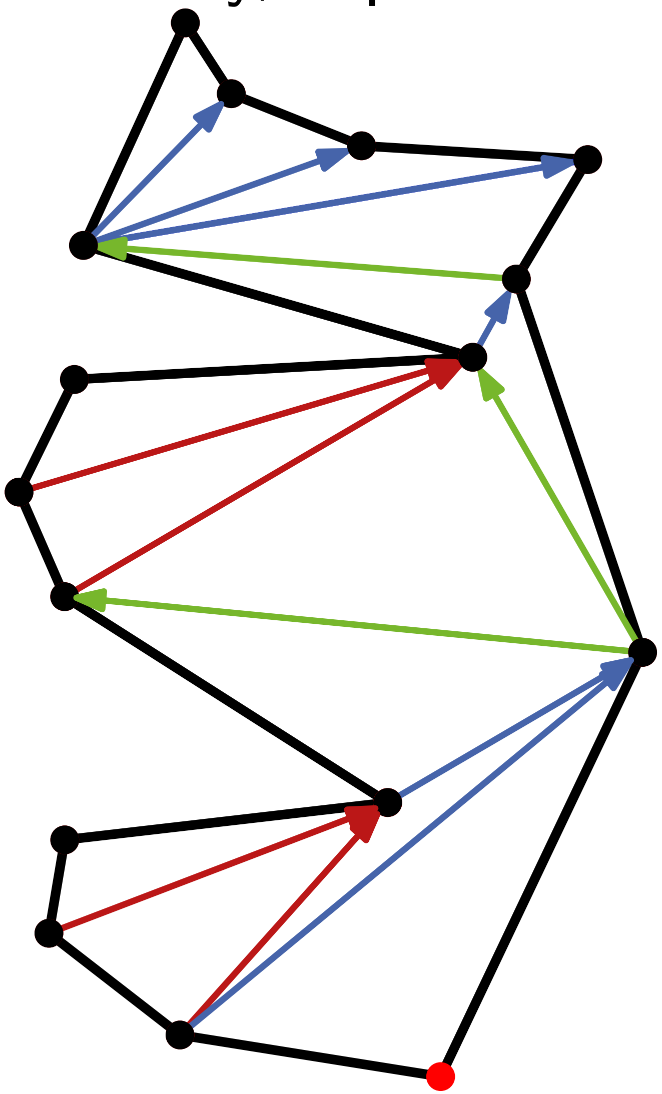

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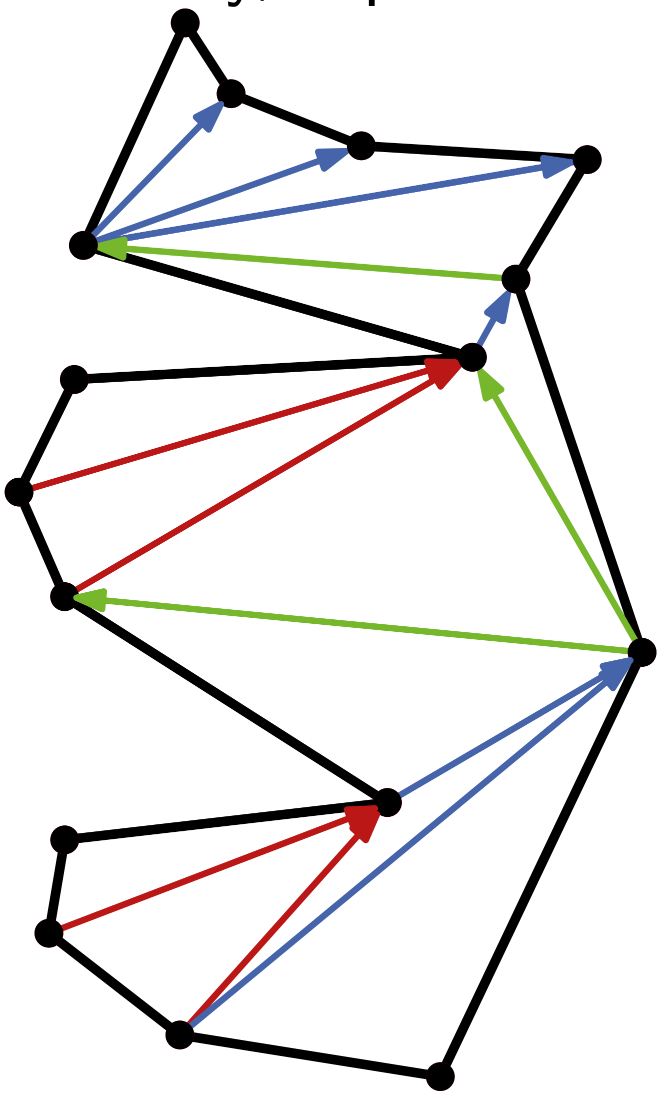

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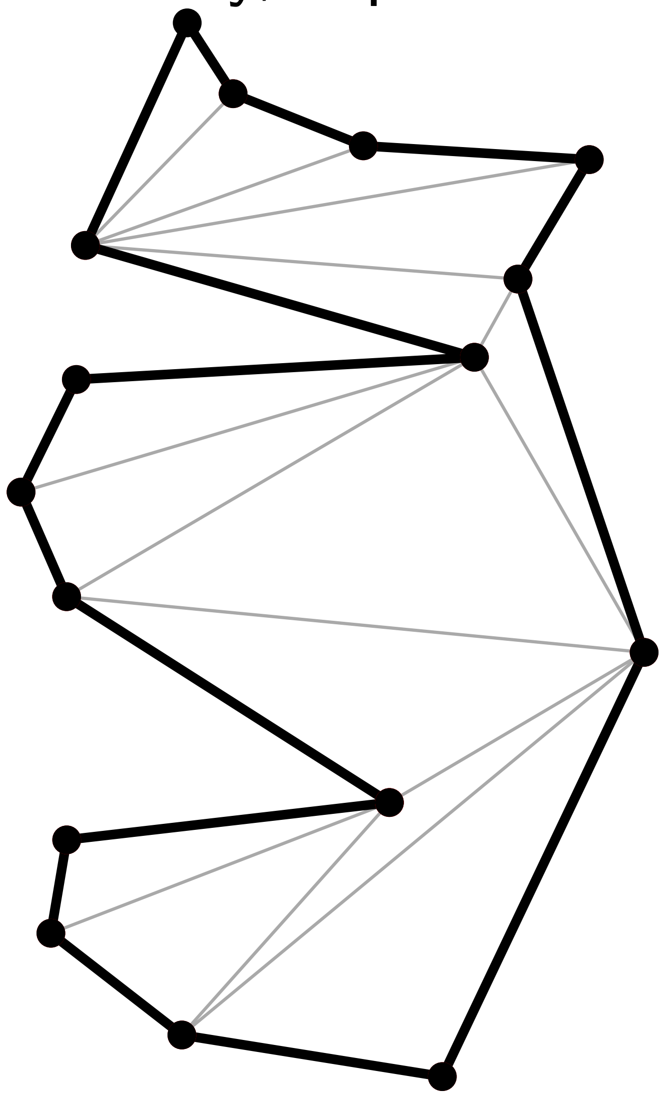

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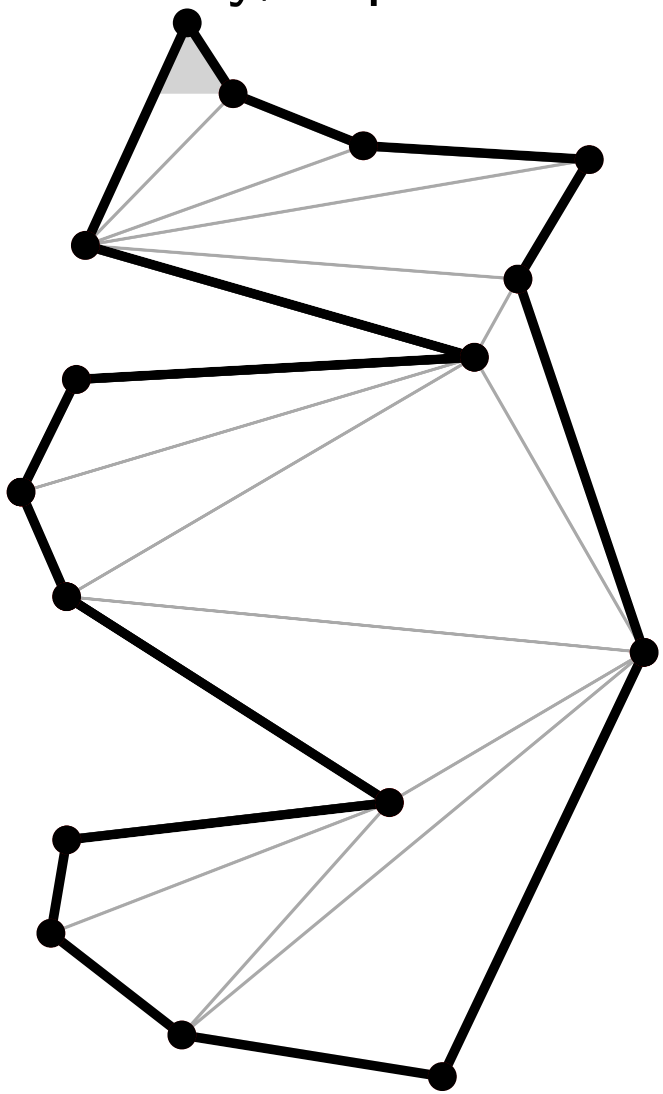

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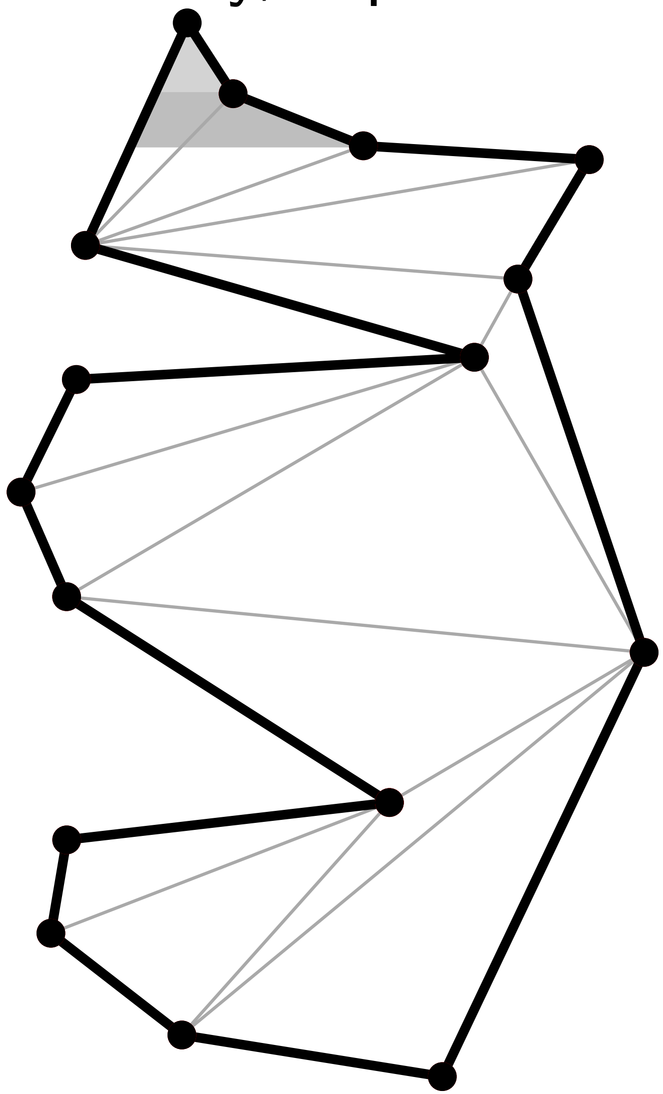

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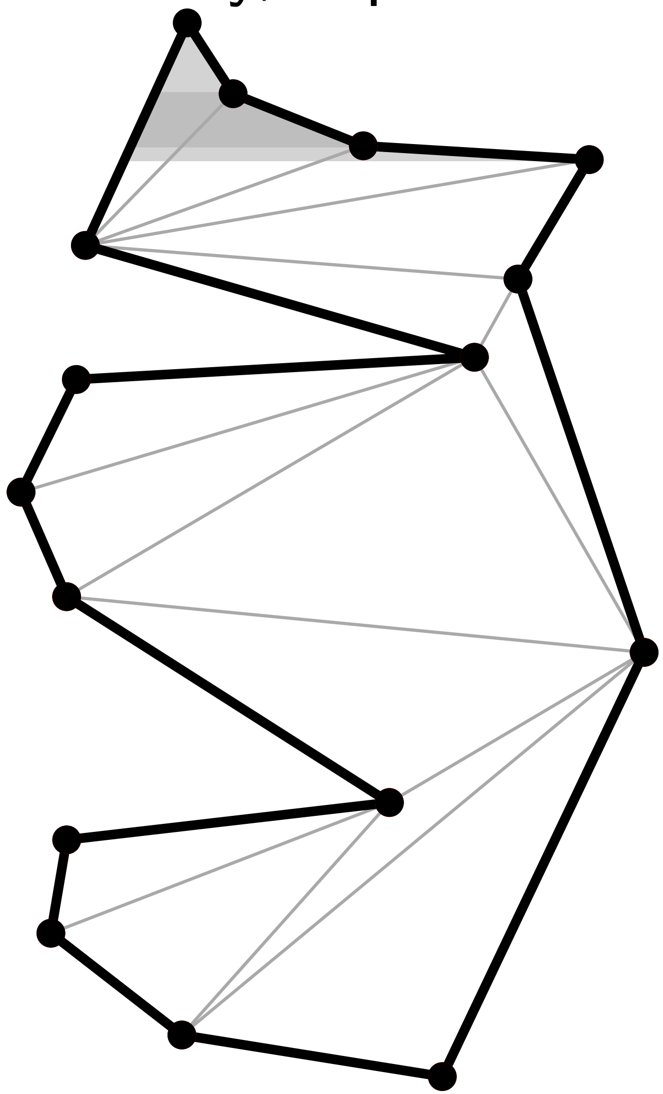

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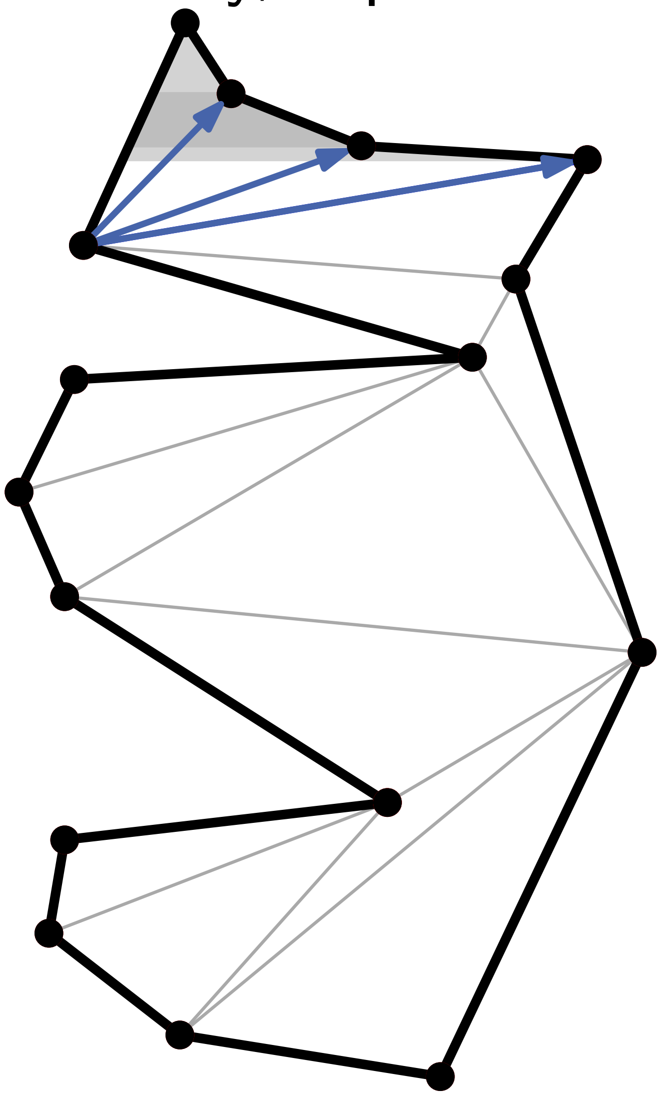

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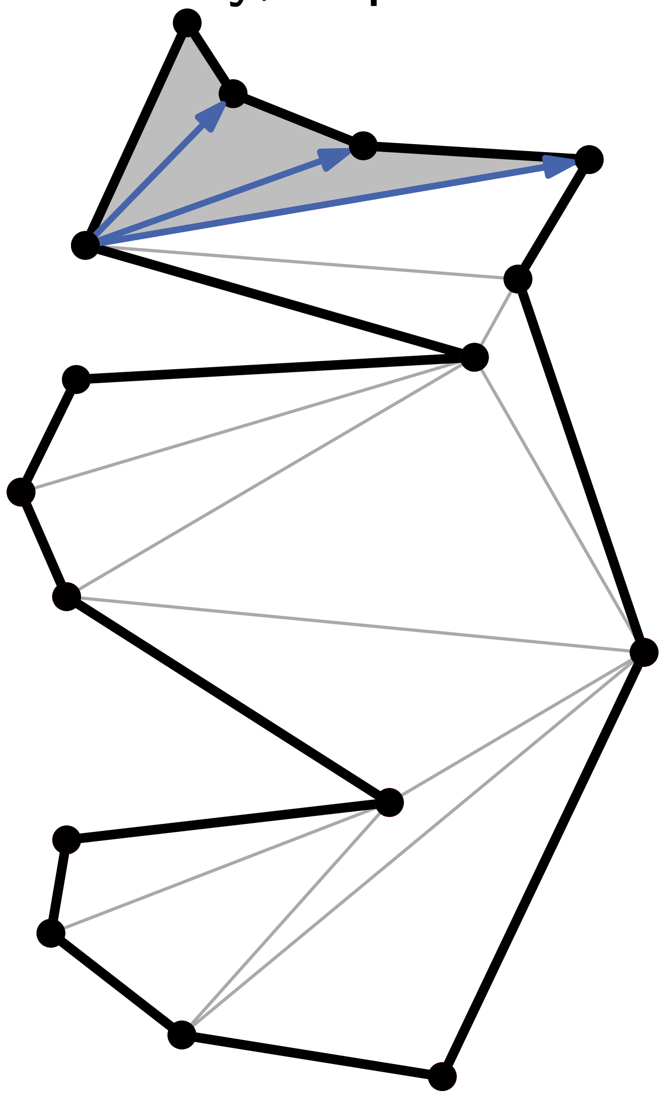

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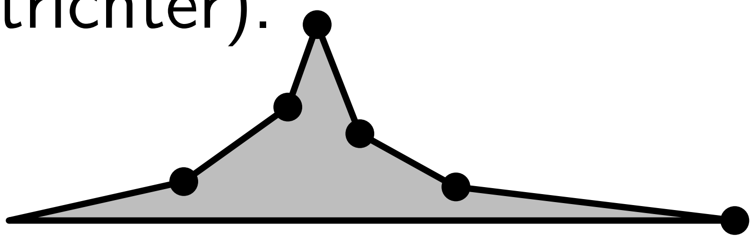

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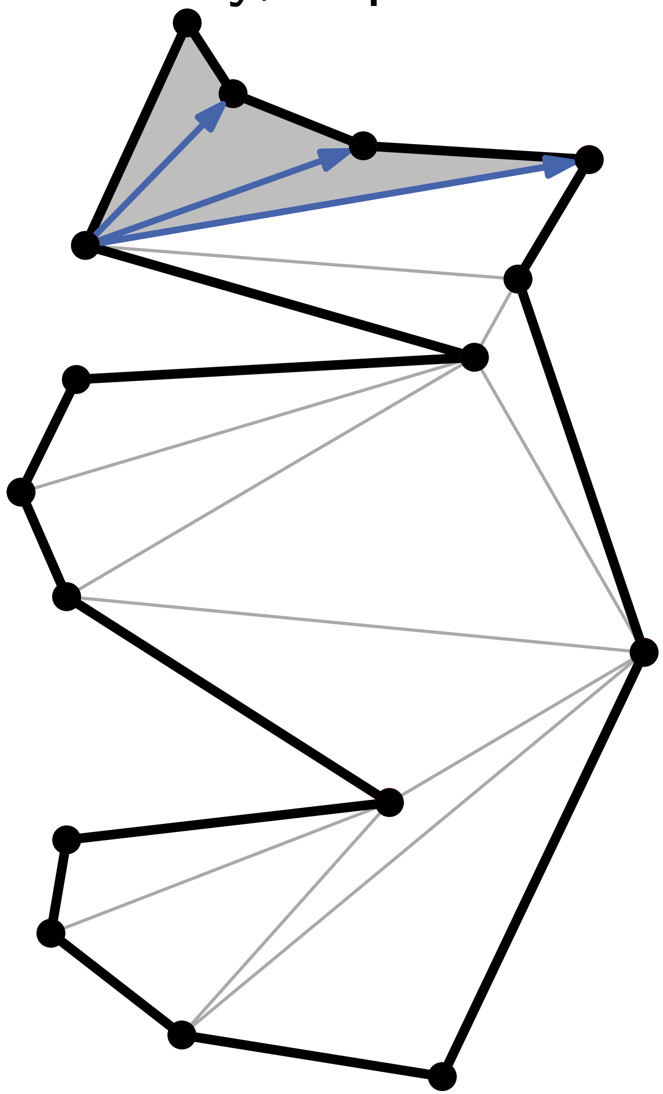

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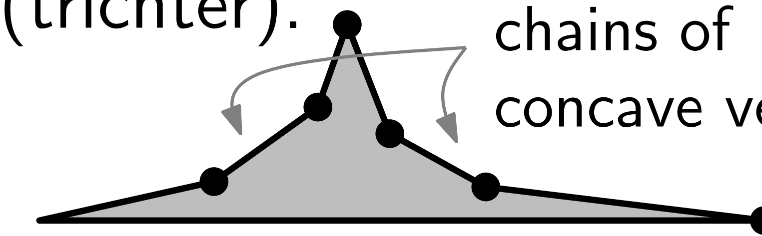

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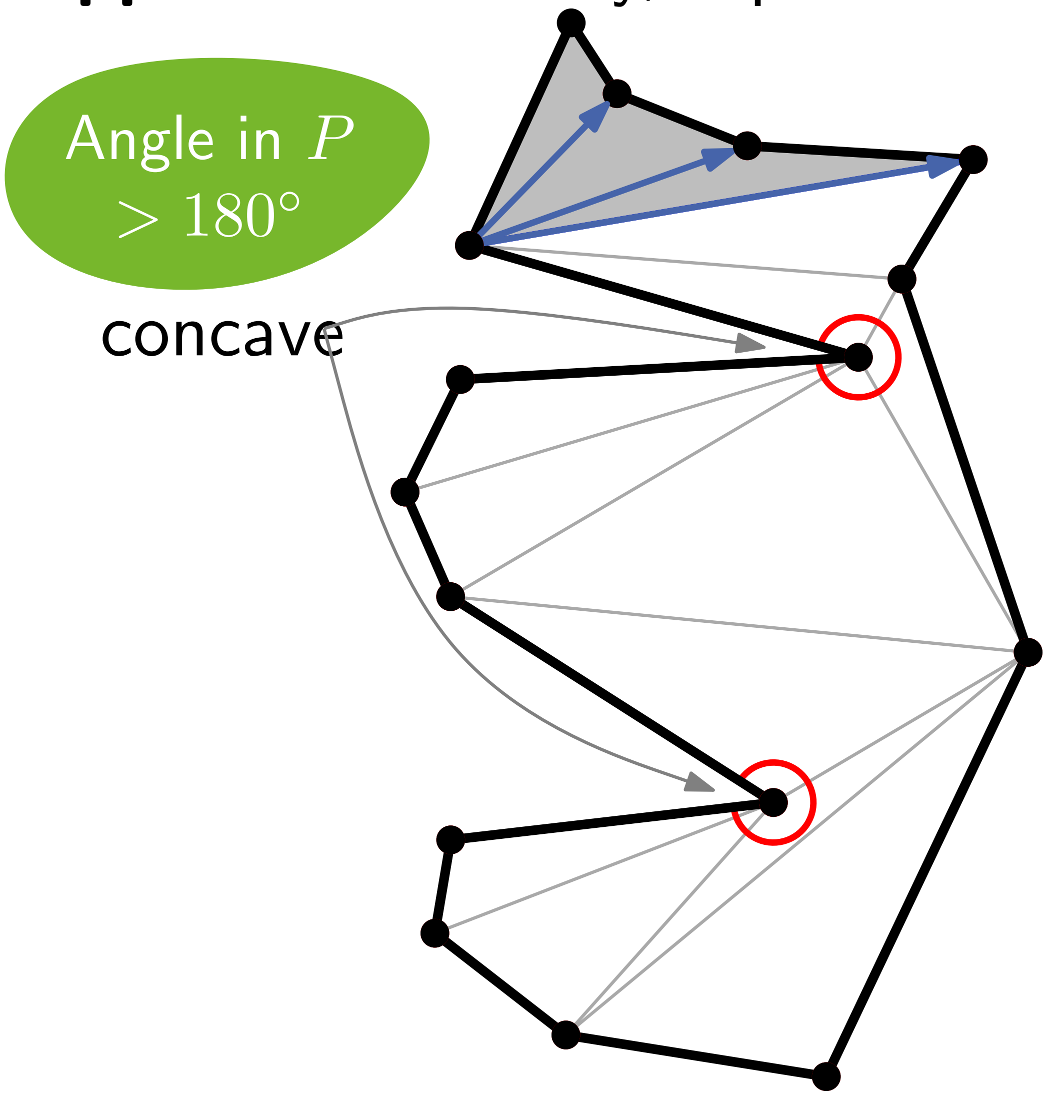


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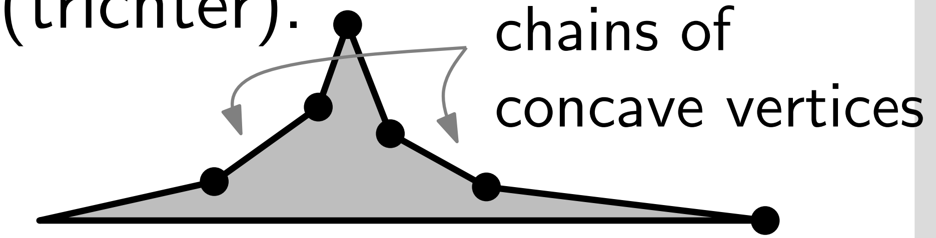

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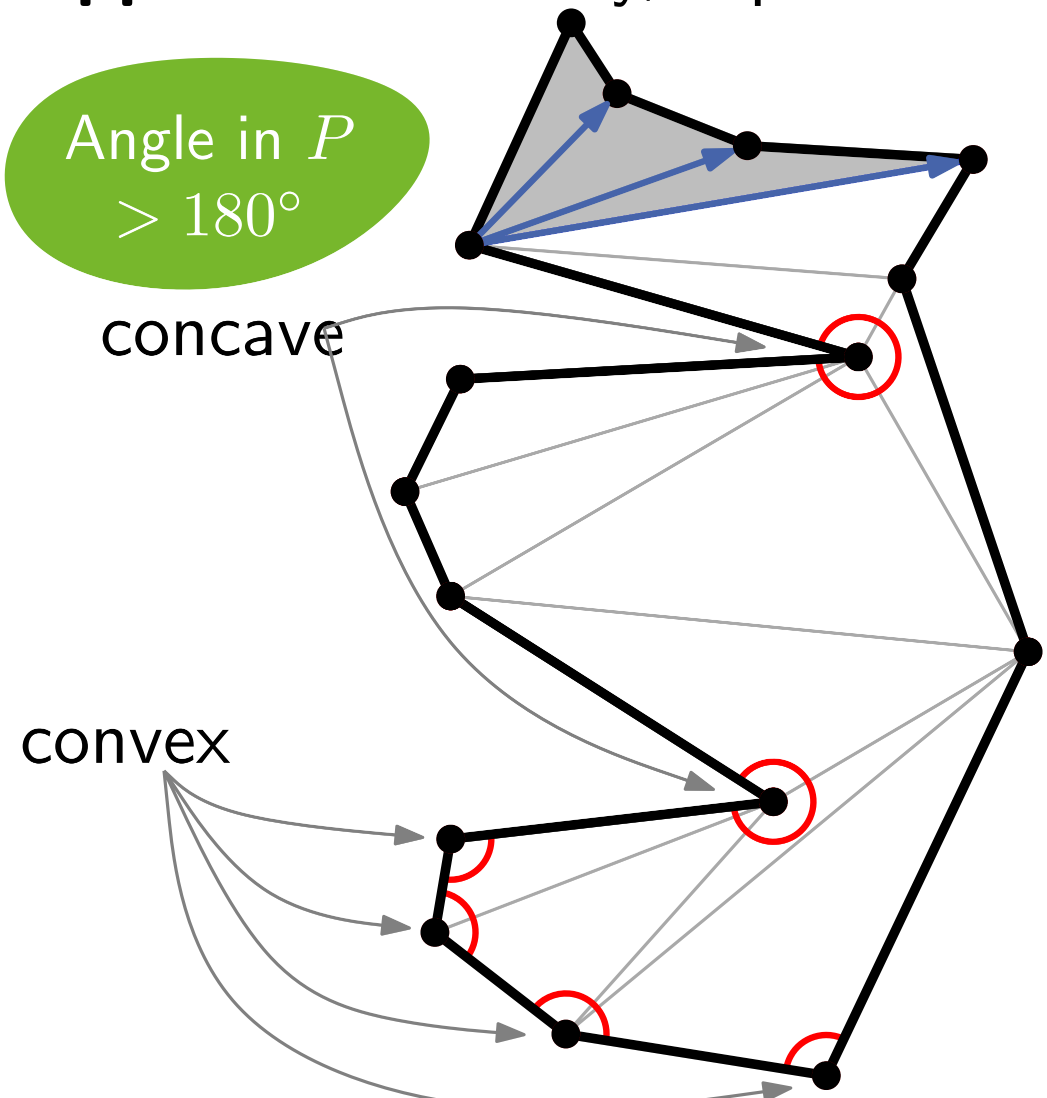


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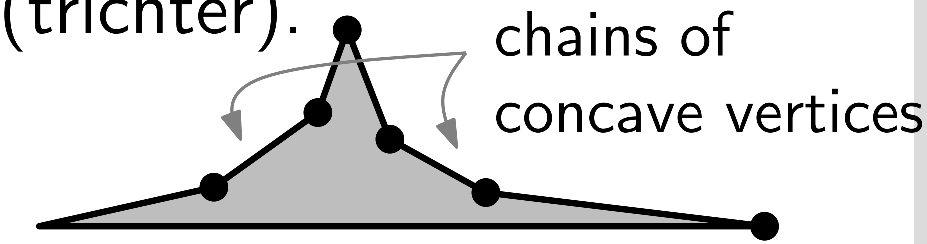

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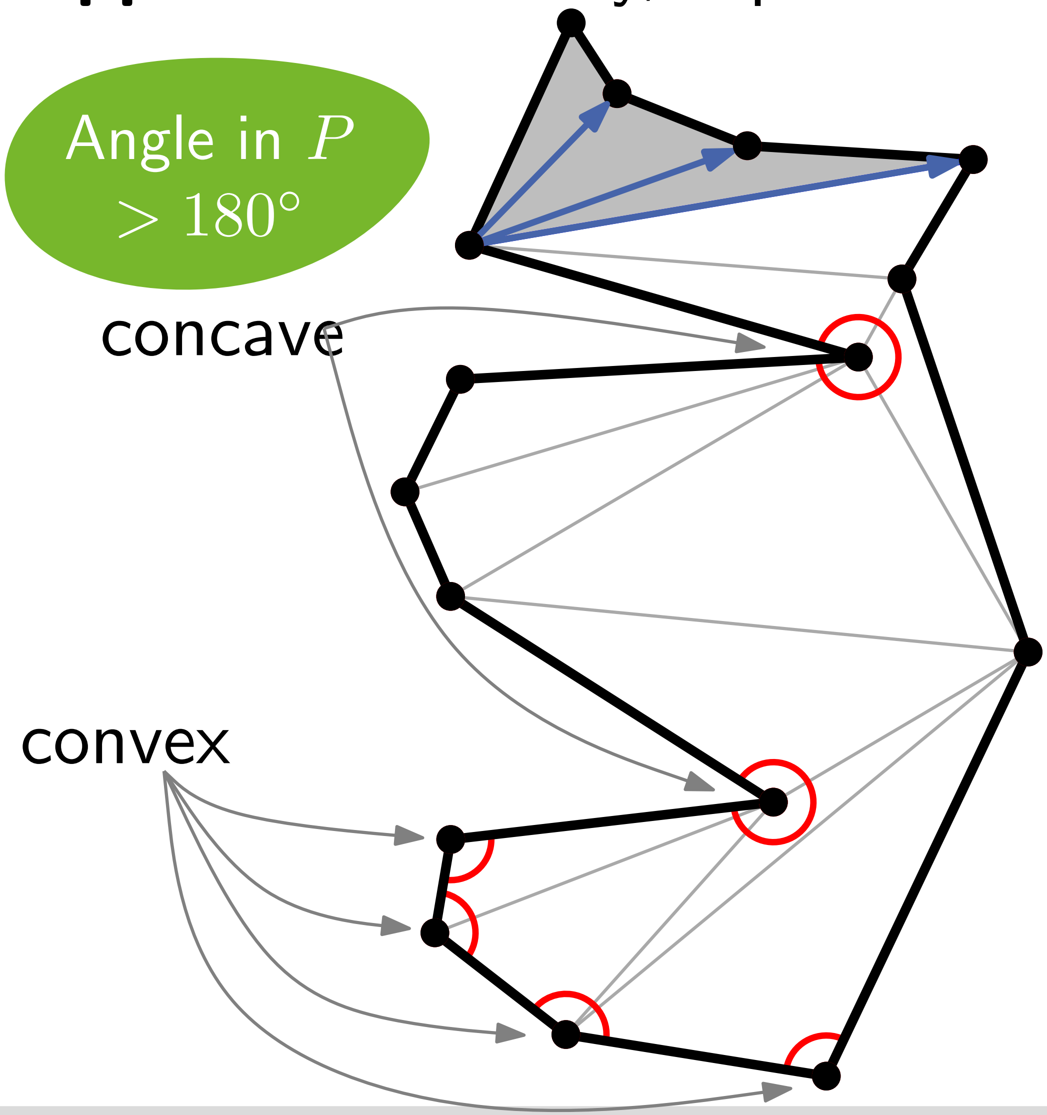


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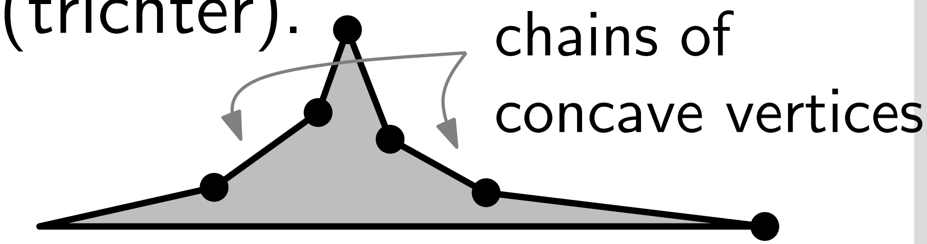


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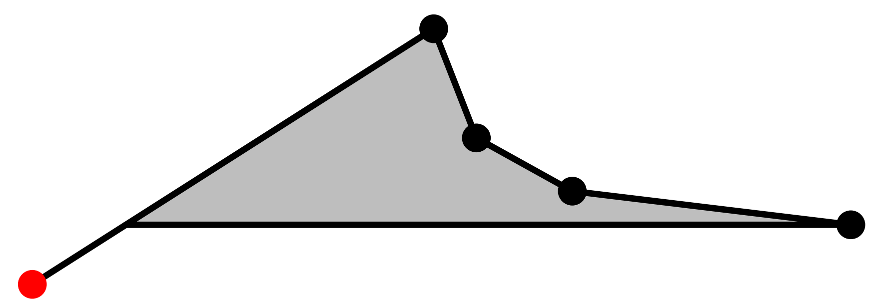

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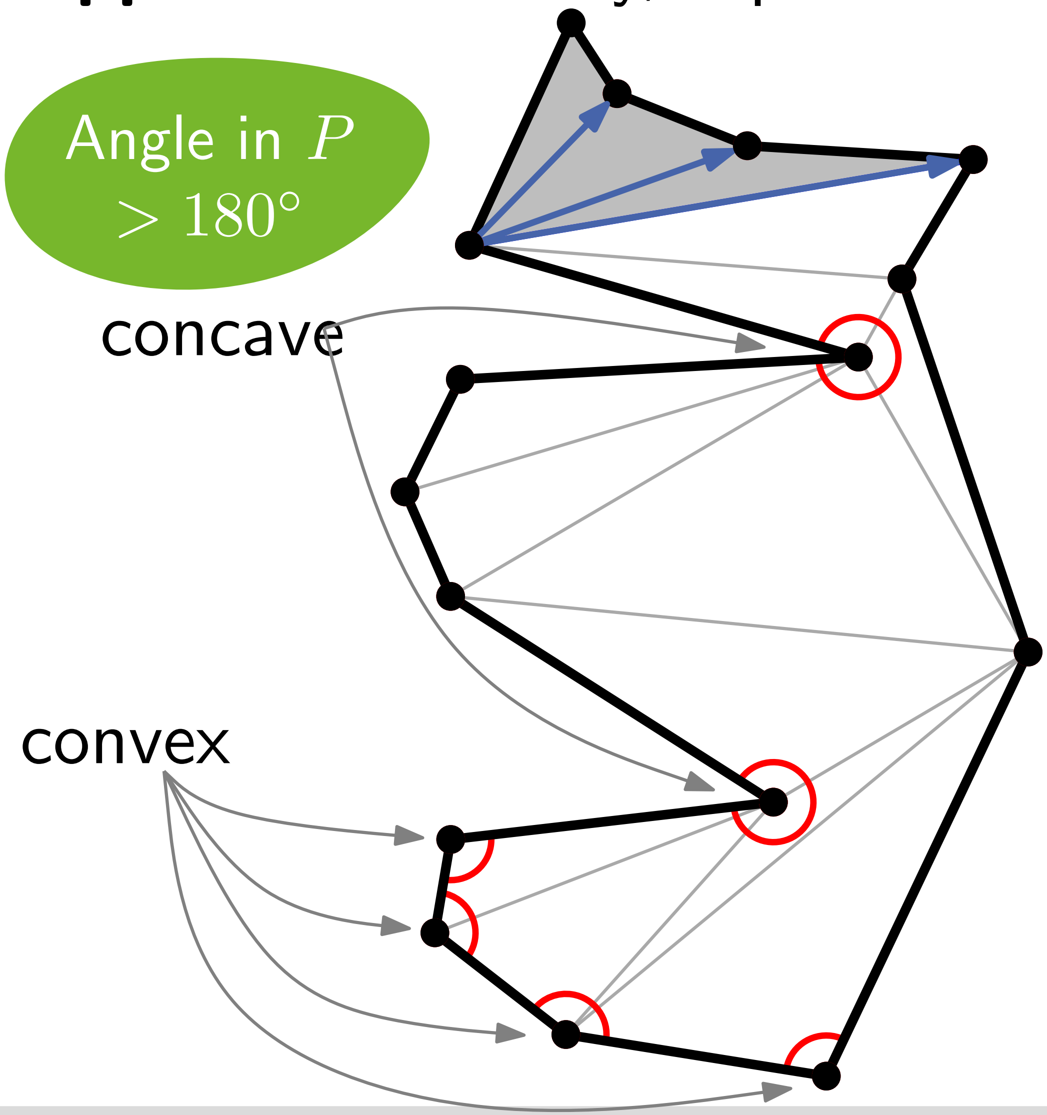


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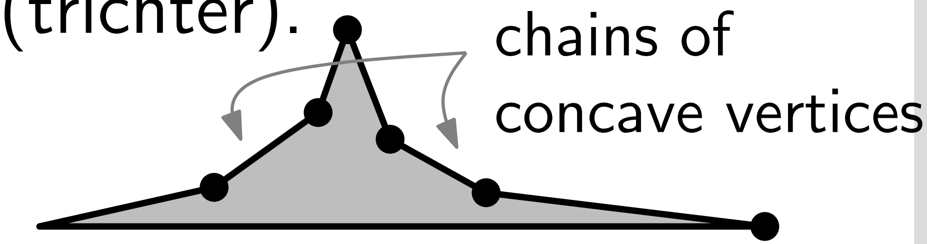


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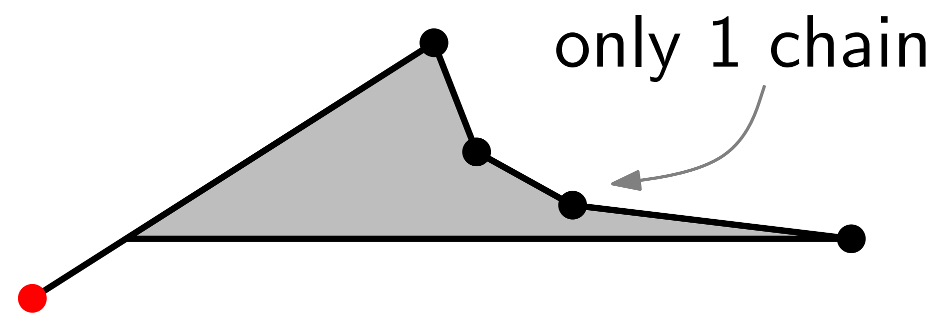

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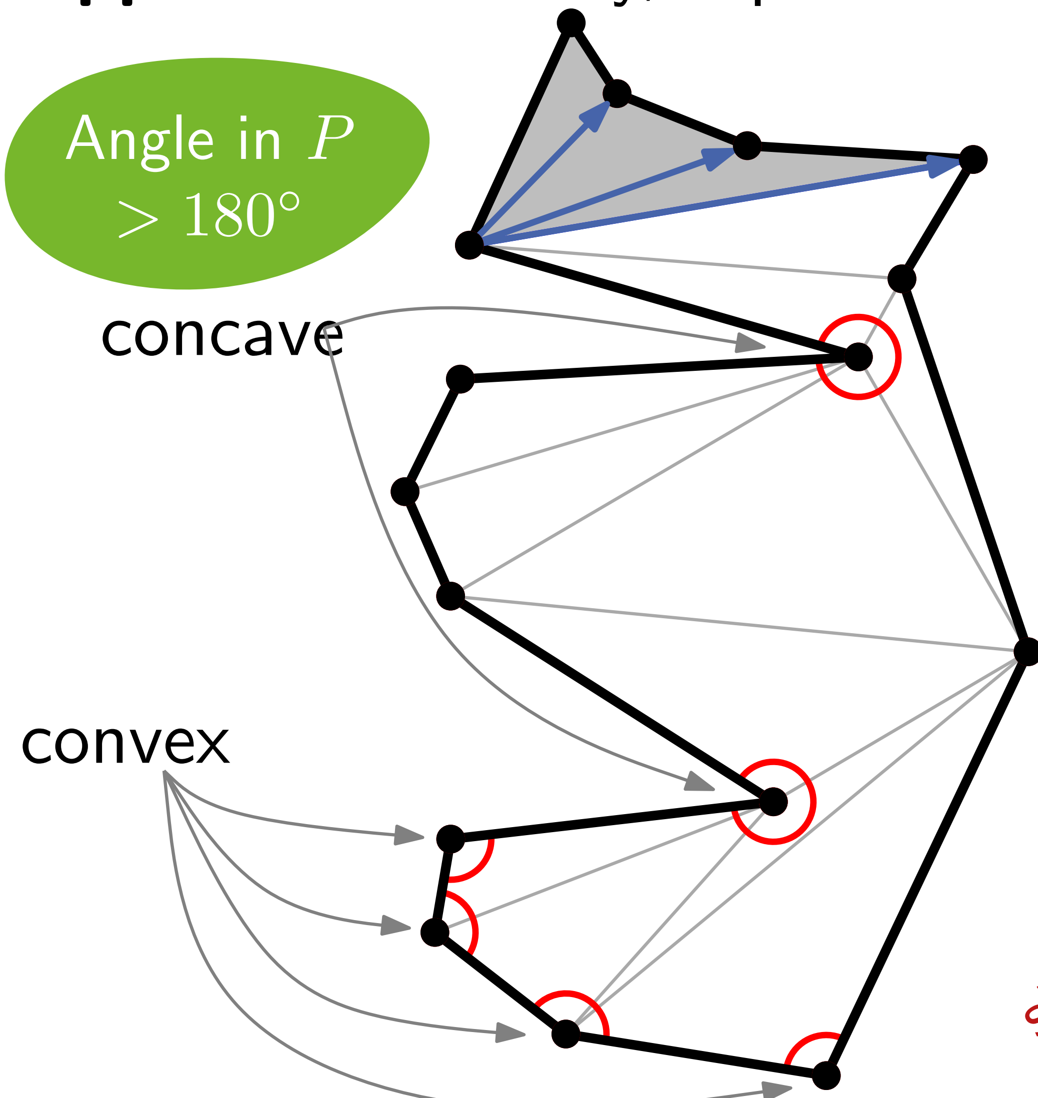


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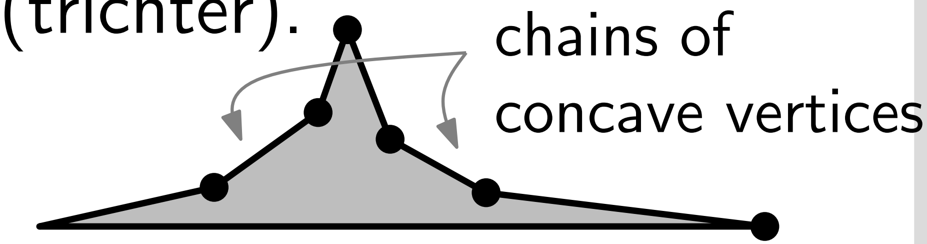


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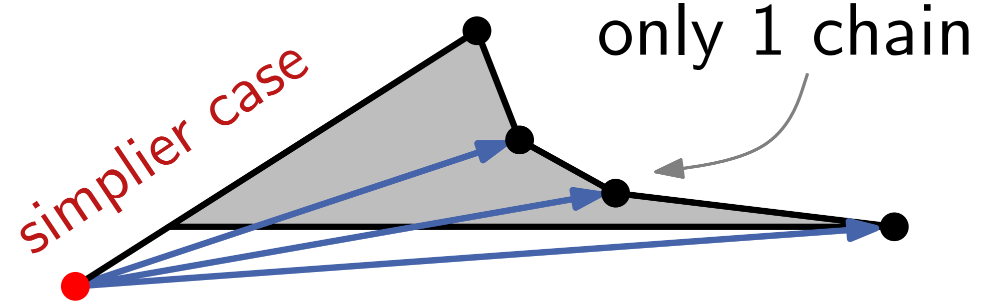

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TriangulateMonotonePolygon(Polygon $P$ as doubly-connected list of edges)

Merge vertices on left and right chains into desc. seq. $\rightarrow u_1, \ldots, u_n$

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```
Merge vertices on left and right chains into desc. seq. → u_1, ..., u_n
Stack S ← ∅; S.push(u_1); S.push(u_2)
for j ← 3 to n − 1 do
    if u_j and S.top() from different paths then
        while not S.empty() do
            v ← S.pop()
            if not S.empty() then draw (u_j, v)
        S.push(u_{j−1}); S.push(u_j)
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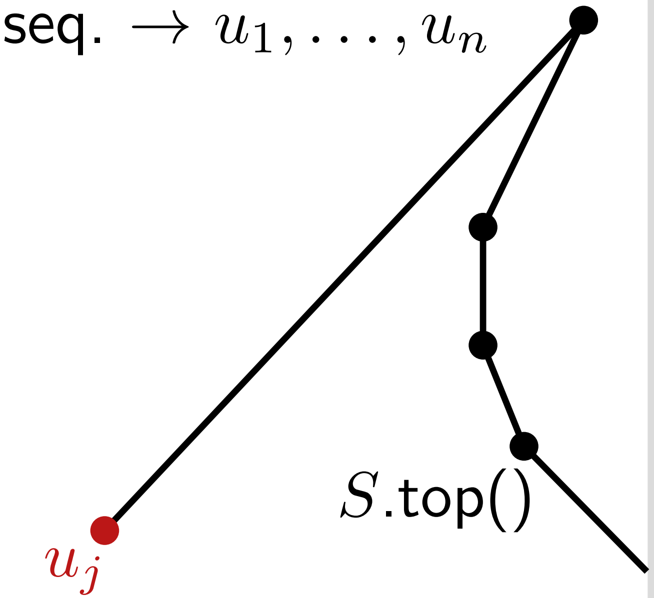

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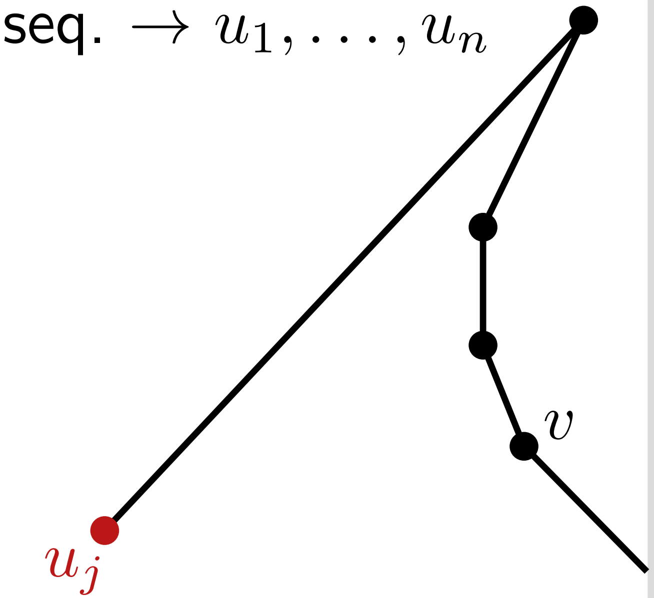

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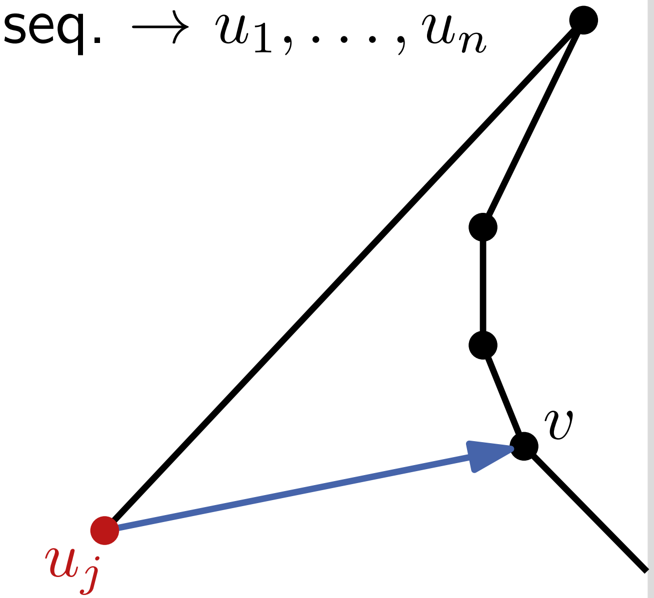

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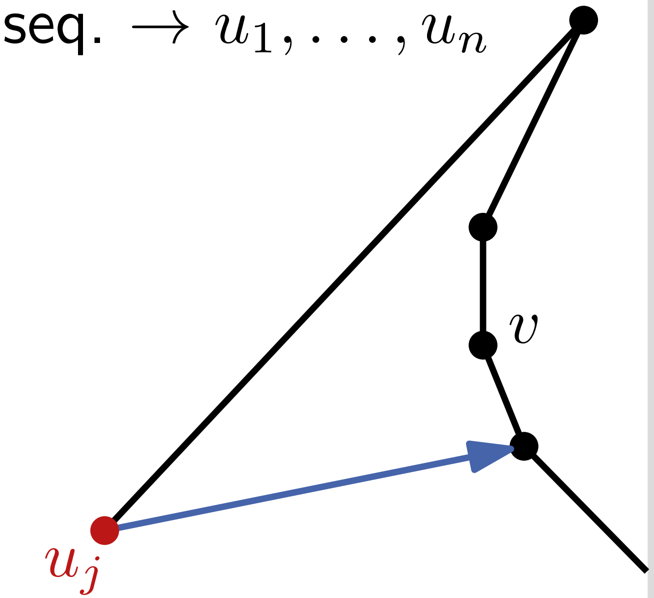

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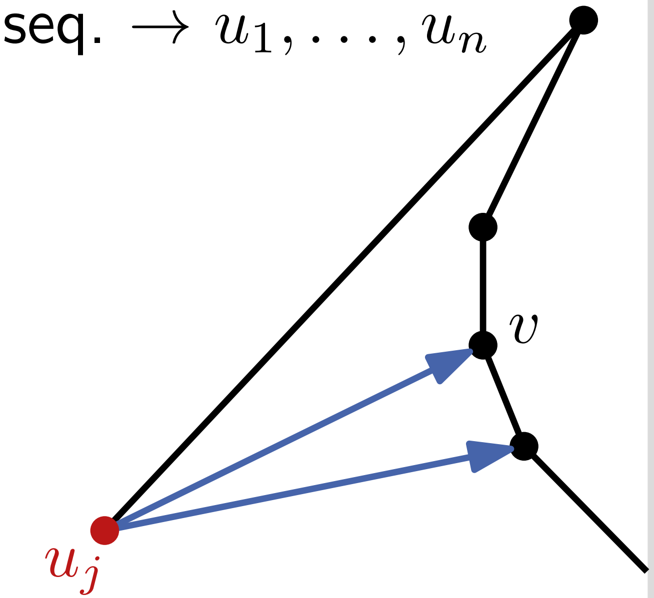

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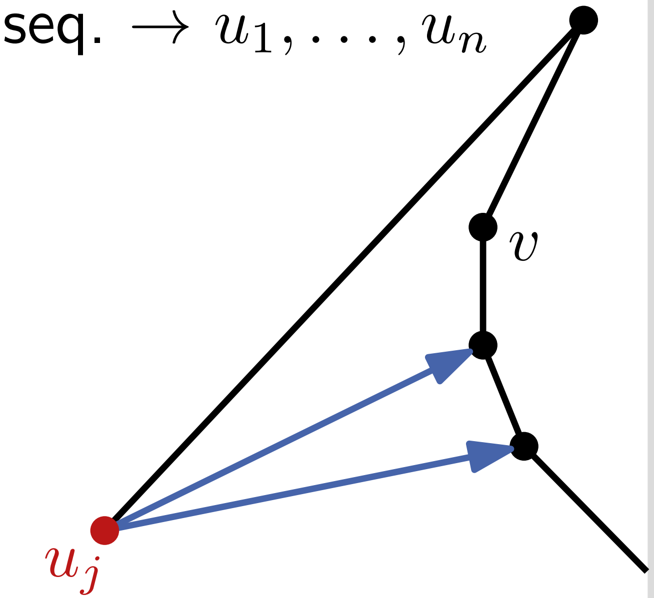

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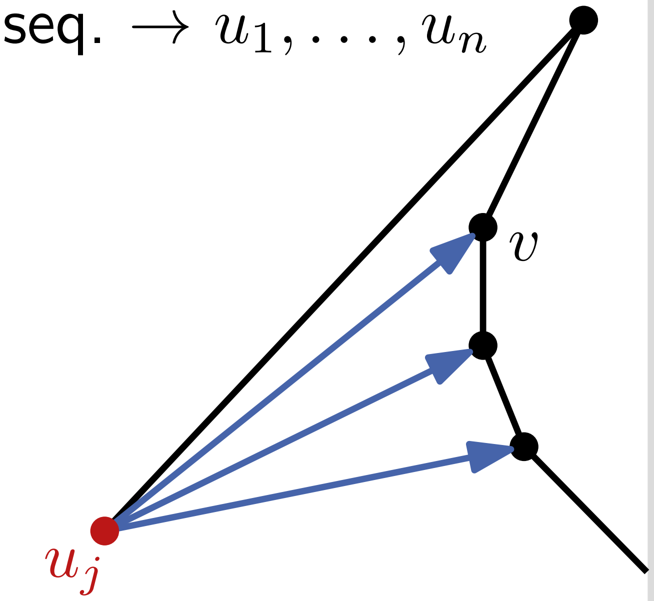

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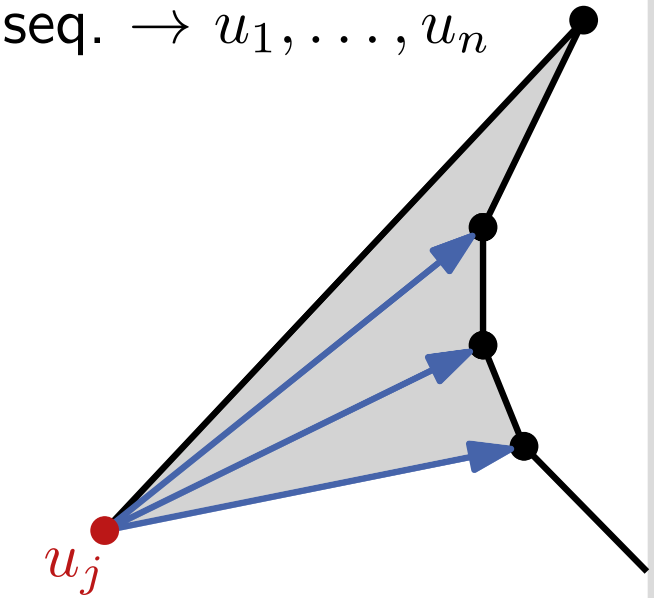

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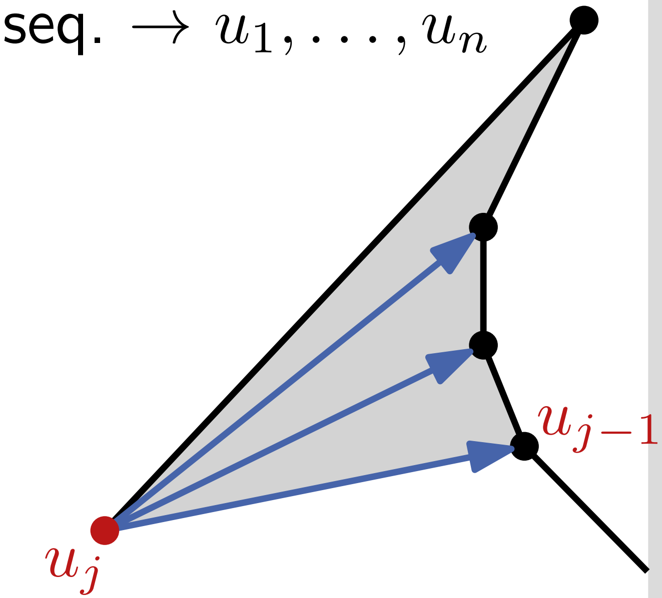

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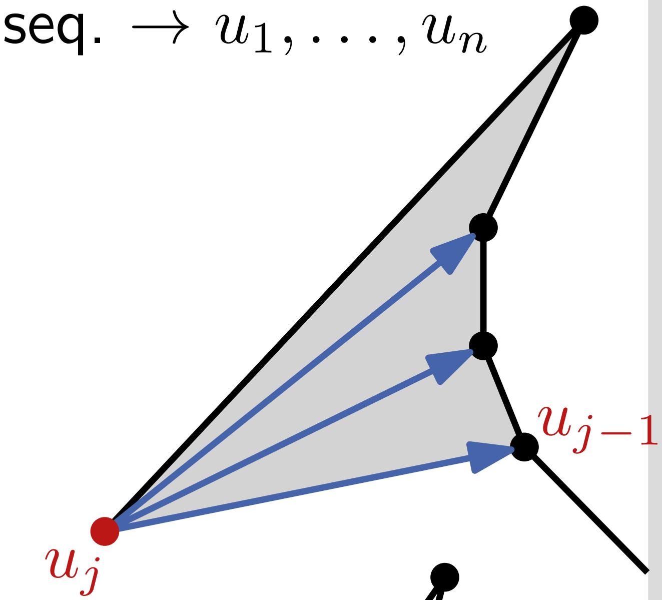


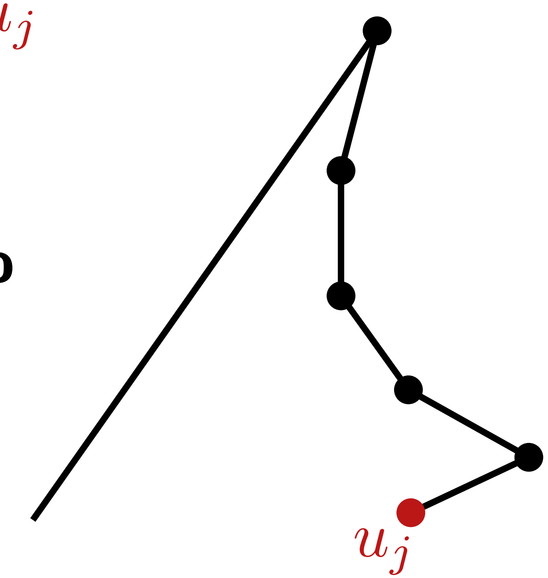

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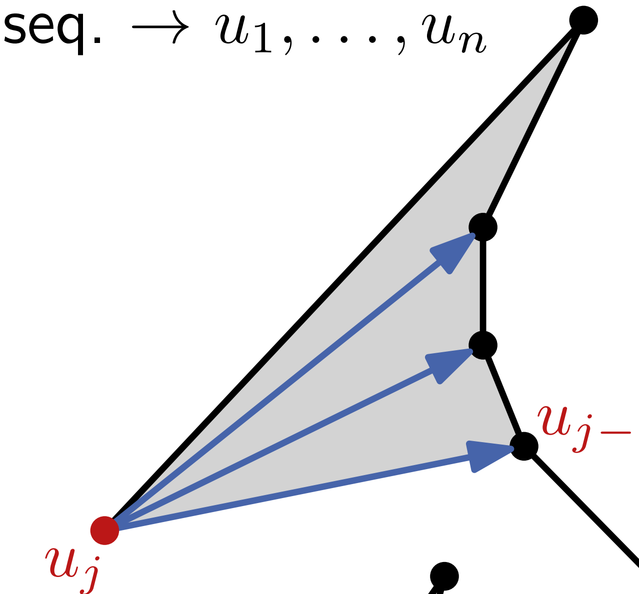


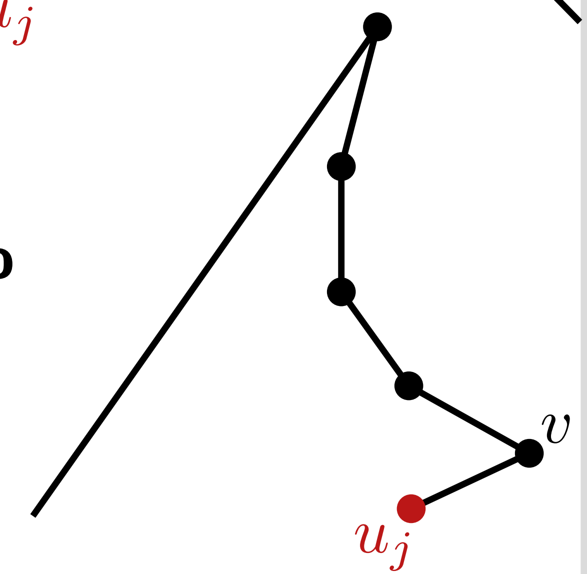

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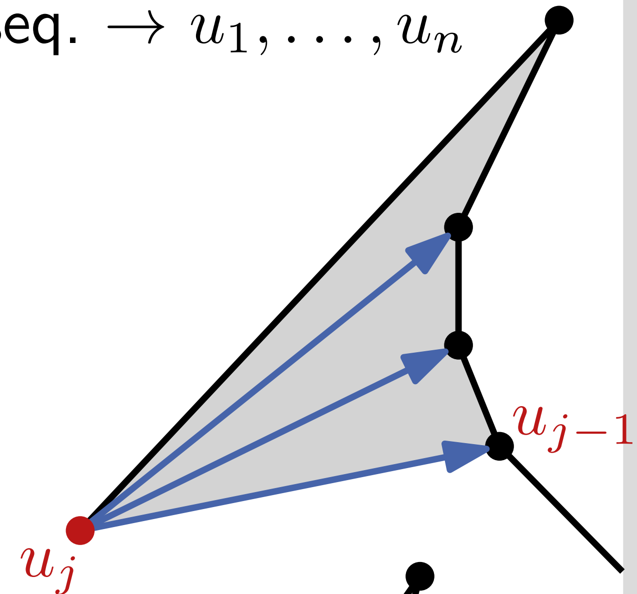


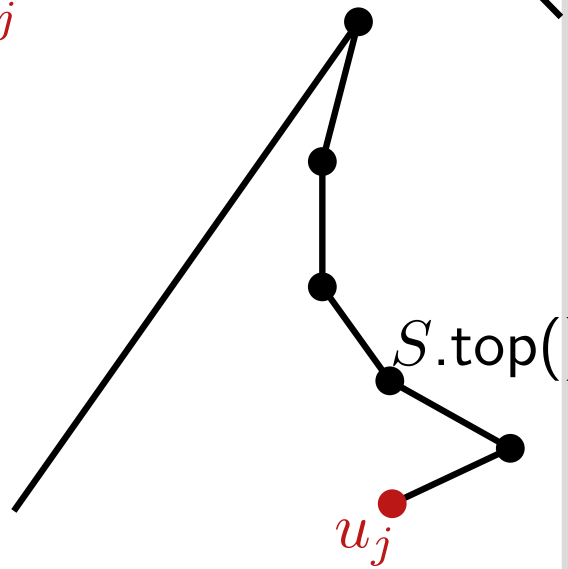

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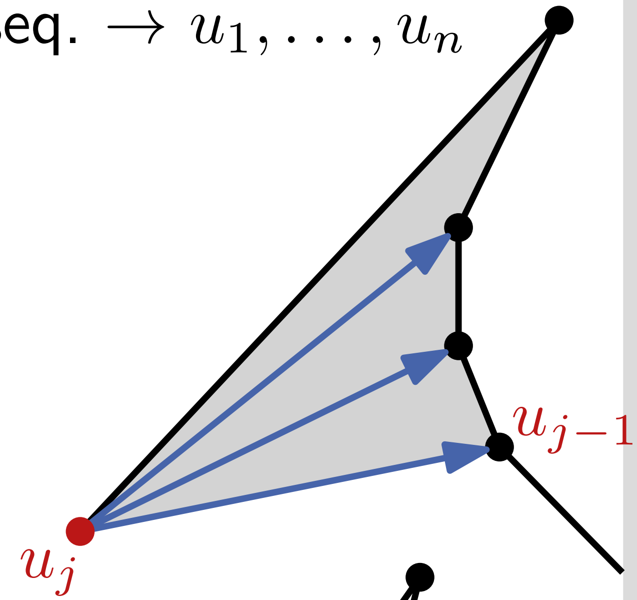


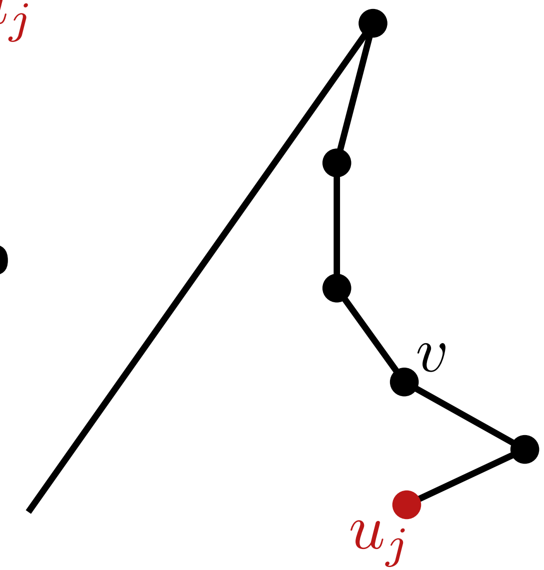

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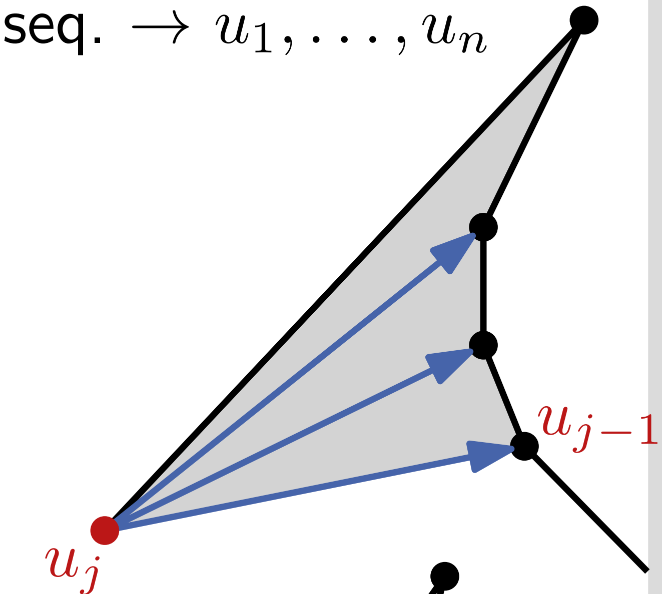


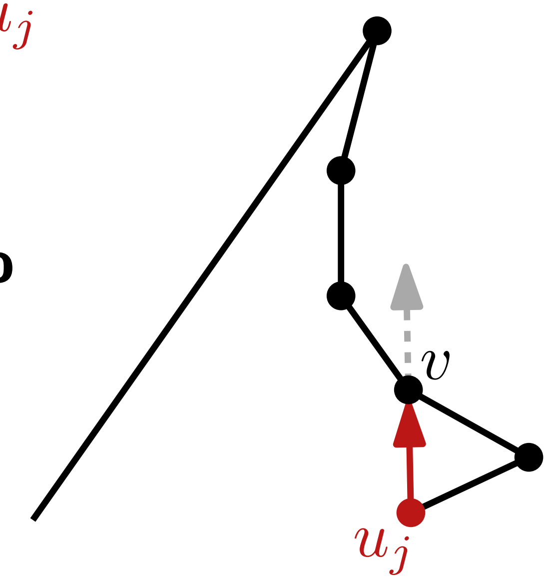

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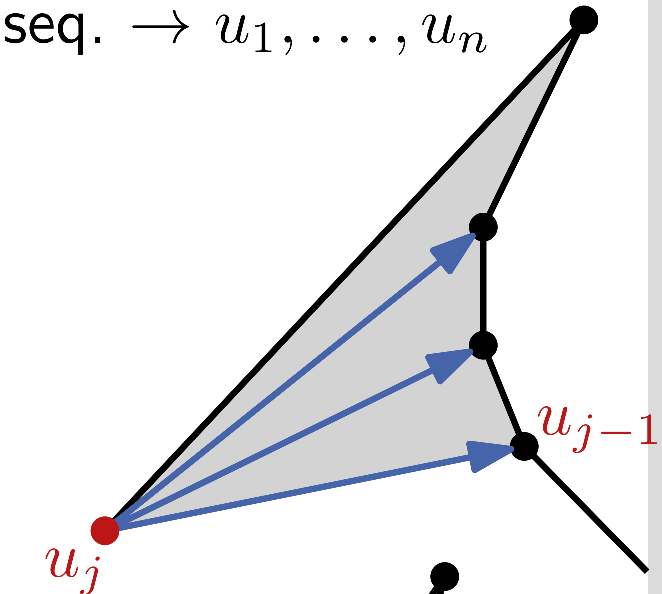


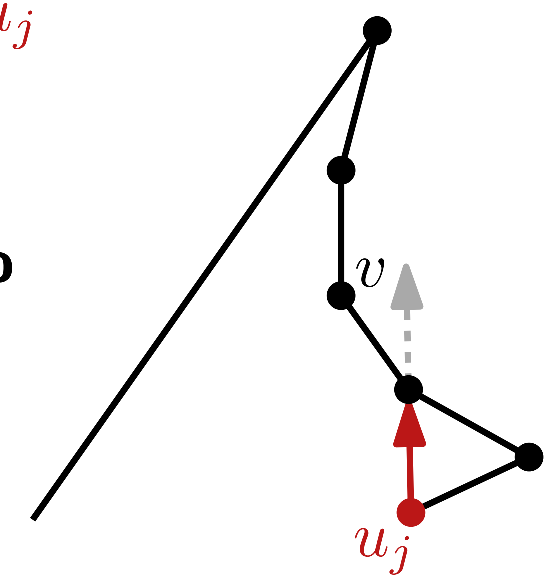

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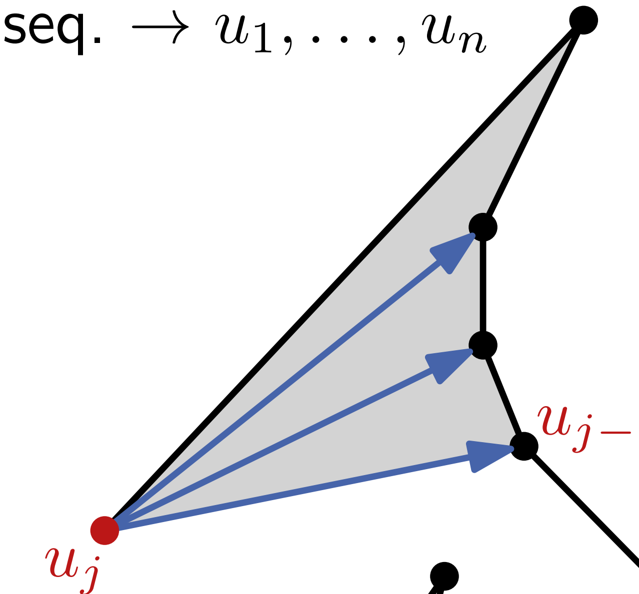


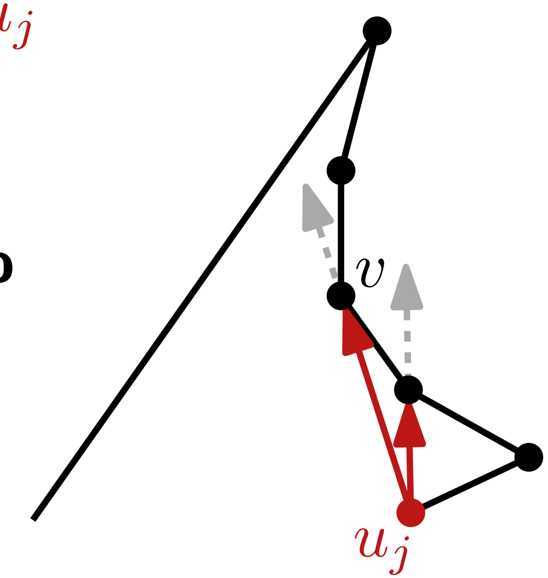

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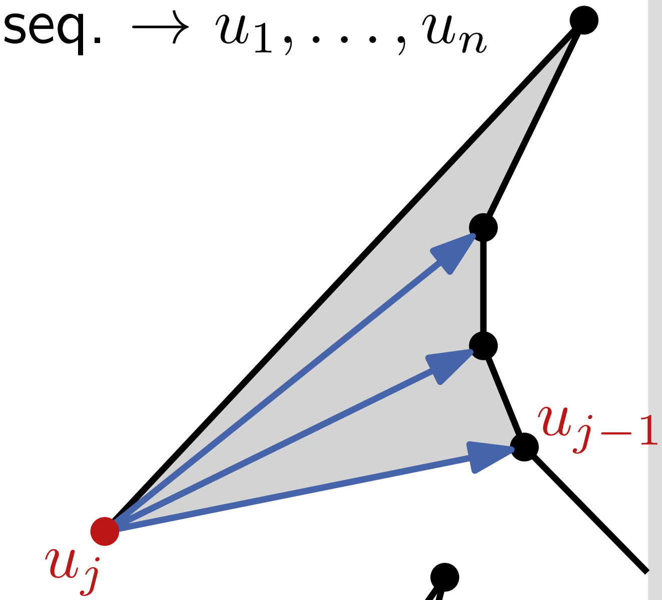


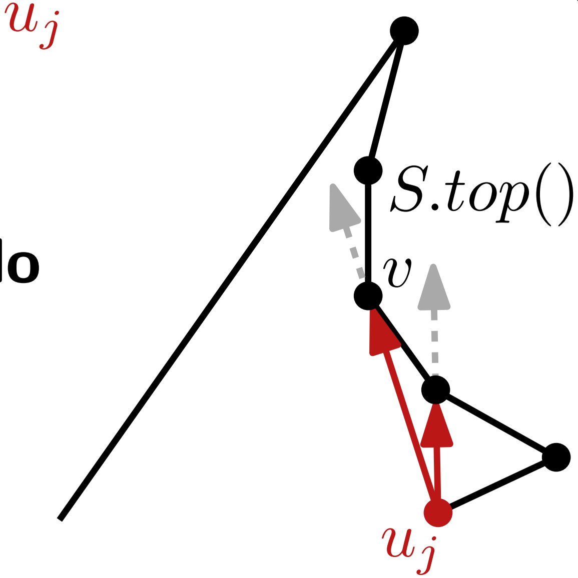

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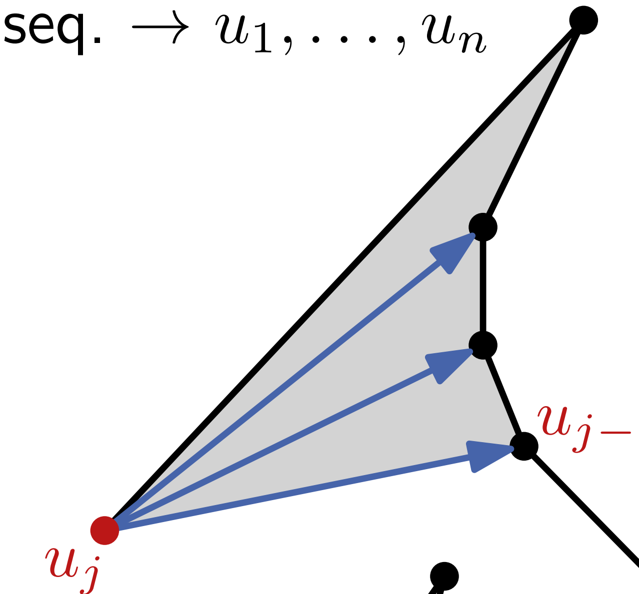


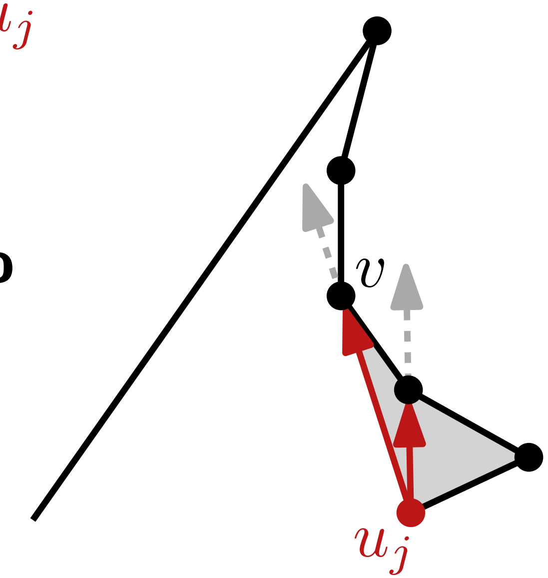

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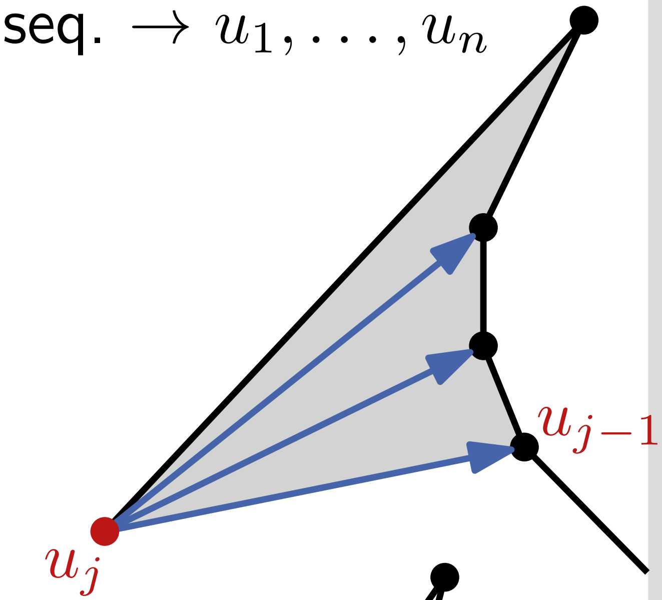


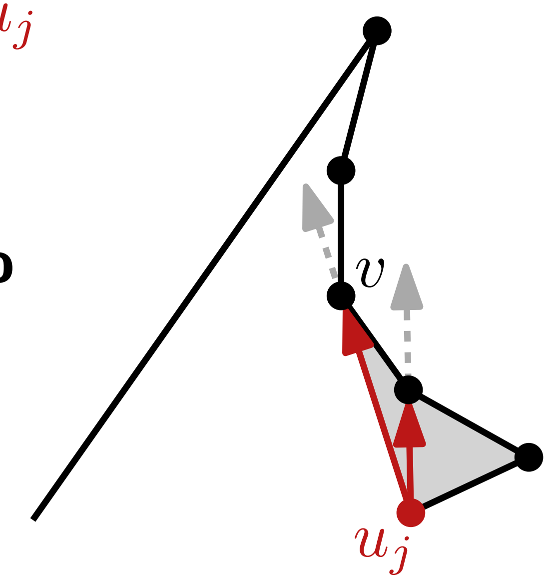

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**Task:**
What is the running time?

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$\Downarrow$

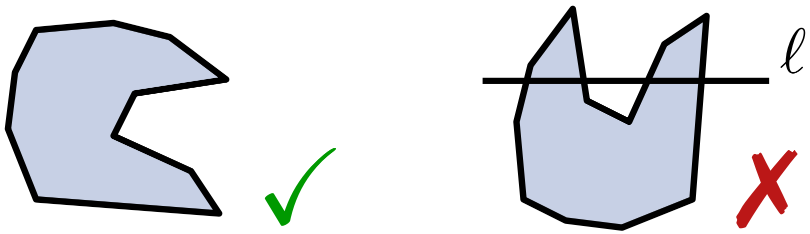
**Theorem 5:** A simple polygon with $n$ vertices can be triangulated in $O(n \log n)$ time and $O(n)$ space.

# Proof of Art-Gallery-Theorem: Overview

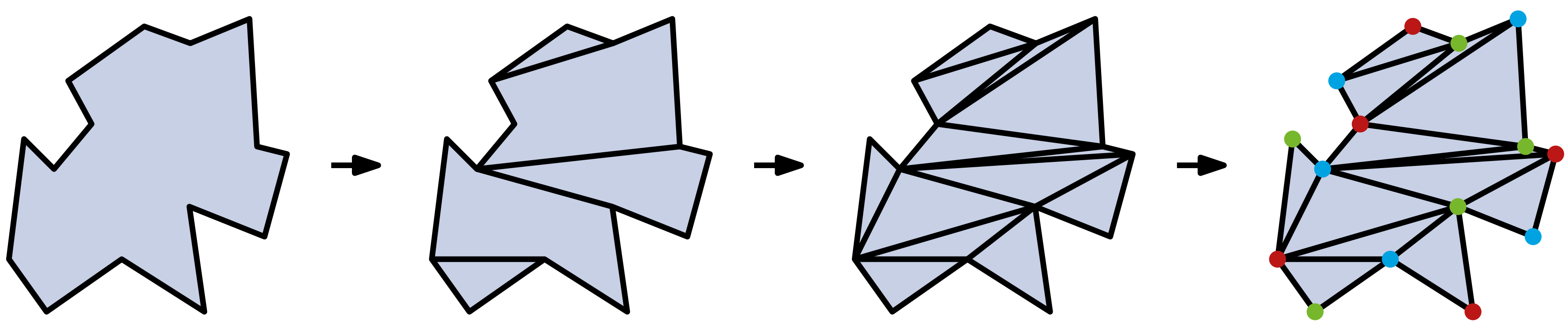
Three-step procedure:

- Step 1: Decompose $P$ in $y$-monotone polygons ✓

  **Definition:** A polygon $P$ is $y$-monotone, if for each horizontal line $\ell$ the intersection $\ell \cap P$ is connected.




- Step 2: Triangulate $y$-monotone polygons ✓
- Step 3: use DFS to color the triangulated polygon ✓



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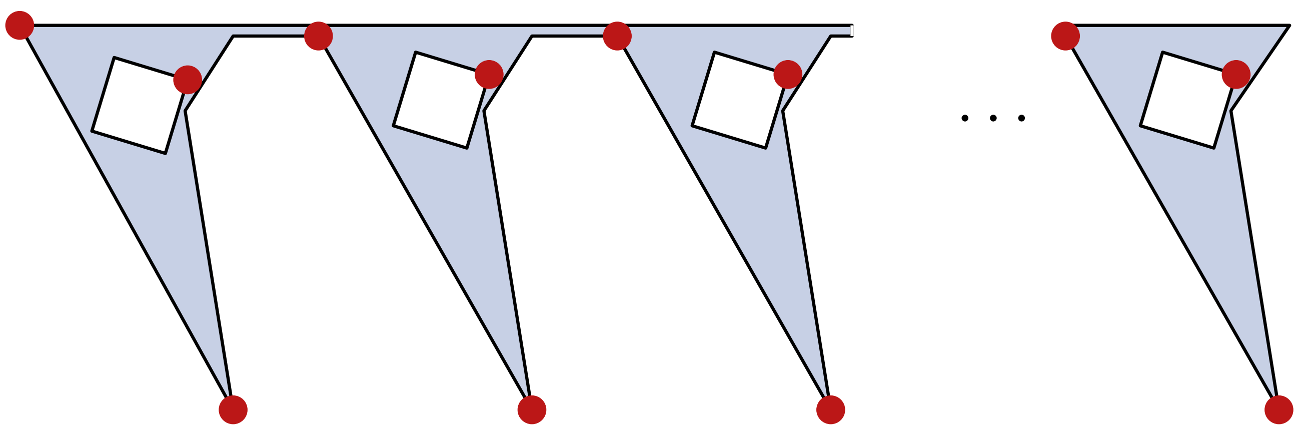

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## Can we solve the triangulation problem faster for simple polygons?

Yes. The question whether it is possible was open for more than a decade. In the end of 80's a faster randomized algorithm was given, and in 1990 Chazelle presented a deterministic linear-time algorithm (complicated).