Algorithms for Route Planning

KIT/ITI, Lectures 14-15: Introduction to TDSP & TD-Oracles

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Time Dependent Shortest Path

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- Graph elements **added/removed** in real-time. /* Dynamic Shortest Path */
- Metric demonstrates **stochastic behavior**. /* Sthochastic Shortest Path */
- Graph is **fixed**, metric **changes with the value of a parameter** $\gamma \in [0, 1]$ in a predetermined fashion. /* Parametric Shortest Path */
- Graph is **fixed**, metric **changes over time** in a predetermined fashion. /* Time-Dependent Shortest Path */

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- Graph is **fixed**, metric **changes over time** in a predetermined fashion. /* Time-Dependent Shortest Path */
  - Arcs are allowed to become **occasionally unavailable** (e.g., due to periodic maintenance, saving consumption of resources, etc), for predetermined unavailability time-intervals (discrete domain).
  - Arc lengths (e.g., traversal-time / consumption) **change with departure-time from tail** which is treated as a real-valued variable (functions with continuous domain, but not necessarily continuous range).

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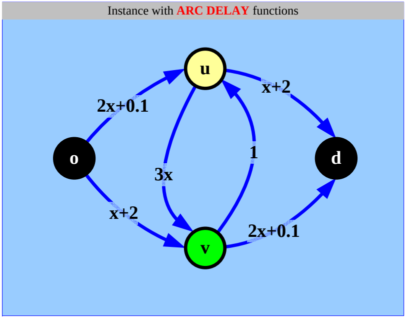

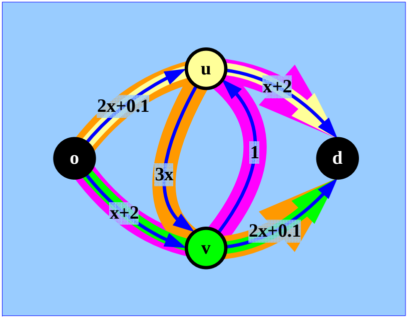

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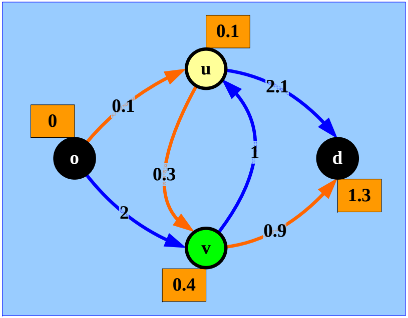

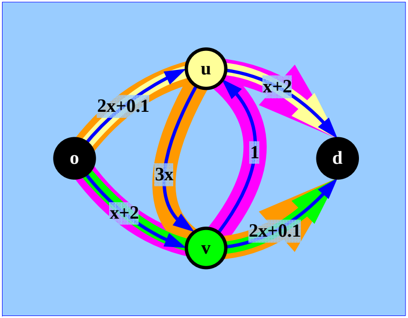

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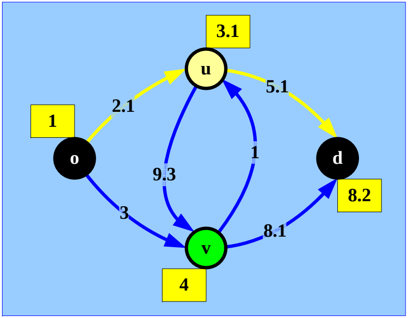

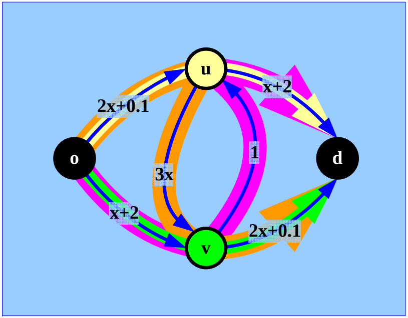

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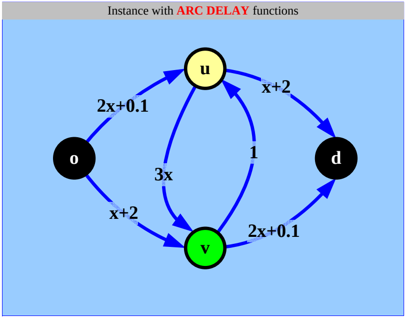

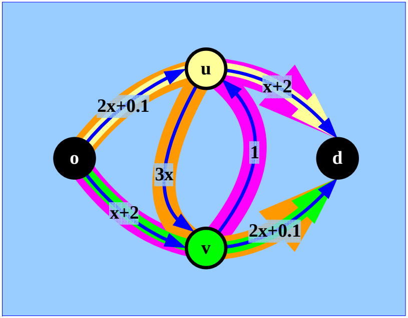

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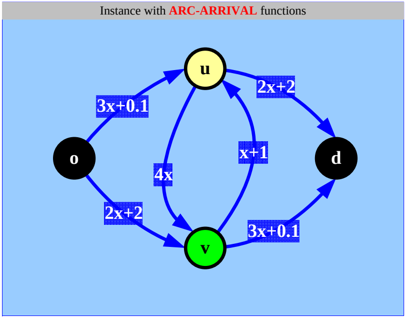

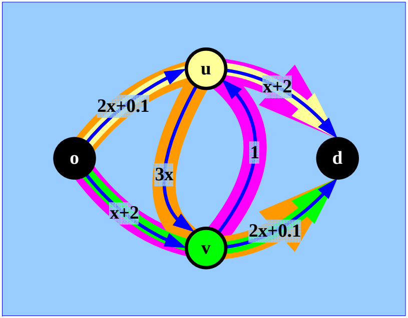

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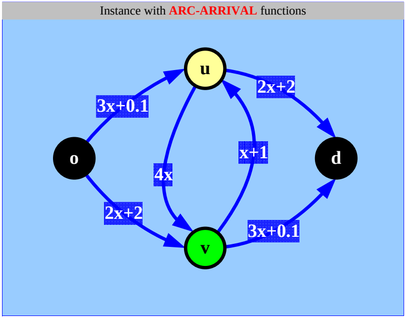

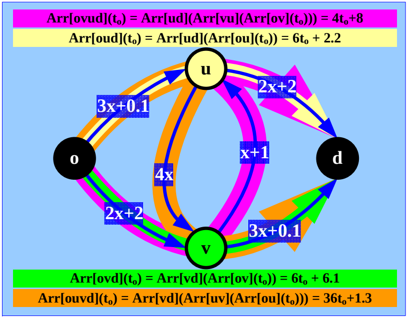

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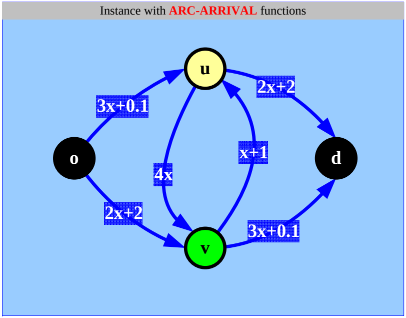

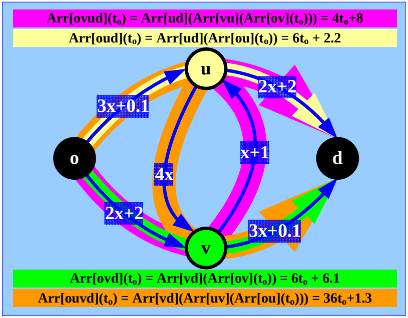

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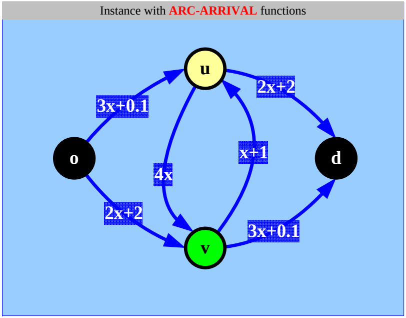
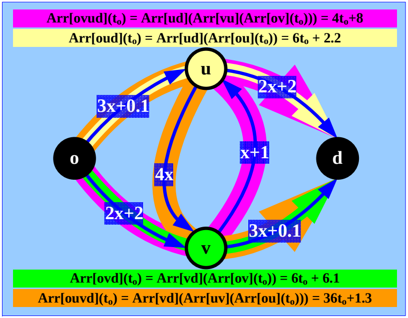
$$\text{shortest } od\text{–path} = \begin{cases} \textbf{orange path}, & \text{if } t_o \in [0, 0.03] \\ \textbf{yellow path}, & \text{if } t_o \in [0.03, 2.9] \\ \textbf{purple path}, & \text{if } t_o \in [2.9, +\infty) \end{cases}$$

TDSP :: EXAMPLE 2 (Waiting Times)





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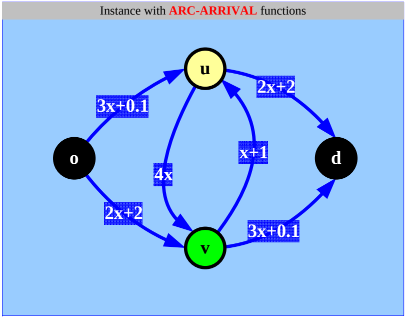

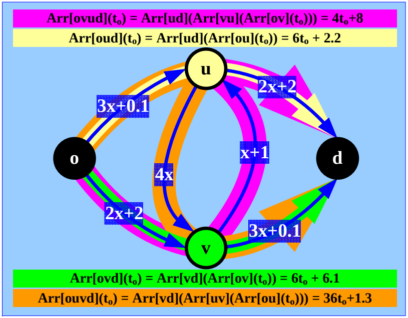

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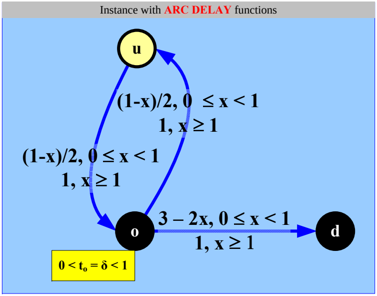

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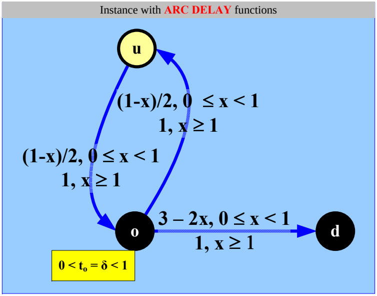

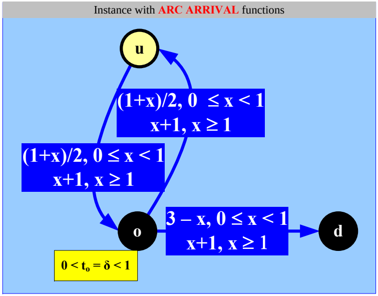

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Q2 Would **waiting-at-nodes** be worth it in this case?

A2 YES, wait until time 1 and then traverse *od*, if already present at o at time $t_o < 1$. Otherwise, traverse *od* immediately.

Waiting Policies

Unrestricted Waiting (UW) Unlimited waiting is allowed at every node along an *od*-path.

Origin Waiting (OW) Unlimited waiting is only allowed at the origin node of each *od*-path.

Forbidden Waiting (FW) No waiting is allowed at any node of each *od*-path.

> Depending on the *waiting policy*, the scheduler has to decide not only for an optimal connecting path (that assures the earliest arrival at the destination), but also for the appropriate optimal waiting times at the nodes along this path.

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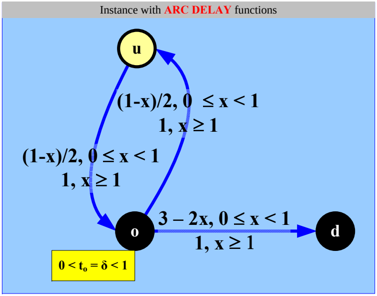

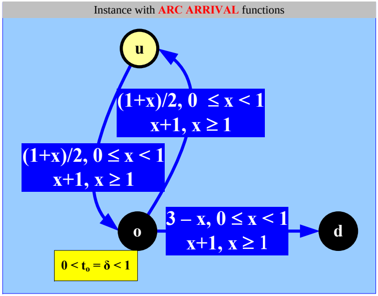

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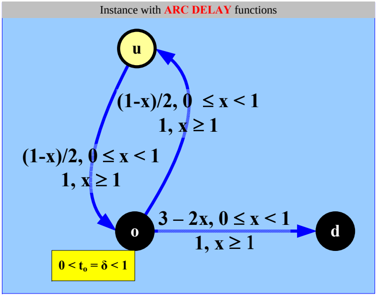

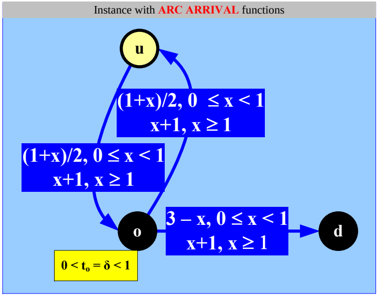

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A3 An infinite, non-simple TD shortest *od*-path with finite delay.

o					d
δ					$3-\delta$

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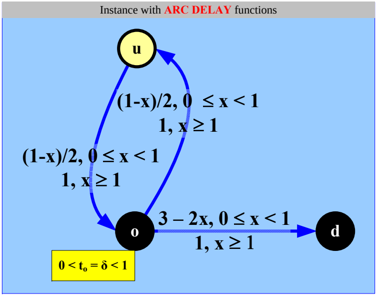

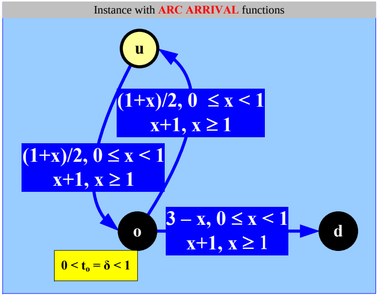

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o	u	o			d
δ	$\frac{1+\delta}{2}$	$\frac{1}{2} + \frac{1+\delta}{4}$			$3 - \frac{1}{2} - \frac{1+\delta}{4} > 2$

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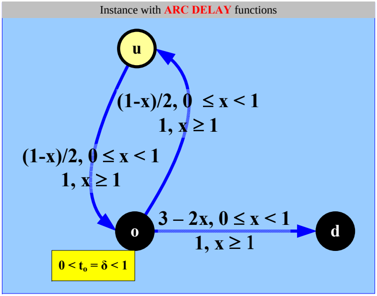

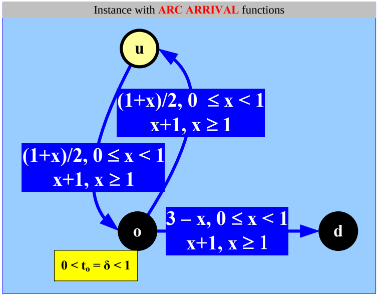

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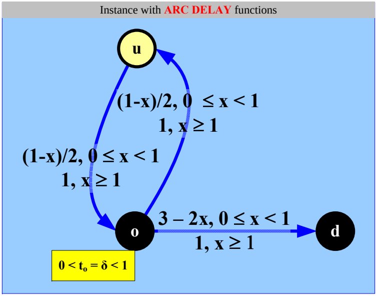

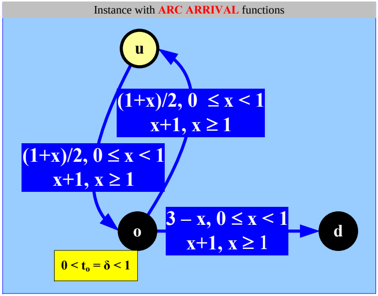

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o	u	o	$<u,o,\ldots,o,u,o>$	d
δ	$\frac{1+\delta}{2}$	$\frac{1}{2}+\frac{1+\delta}{4}$	$\uparrow \sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k = 1$	$t_d \downarrow 2$

Subpath optimality and *shortest path simplicity* **not guaranteed** for TDSP, if waiting-at-nodes is forbidden.

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\begin{aligned}
D[od](t) &= \begin{cases} 100, & t \leq 10, \\ 1, & t > 10 \end{cases} \\
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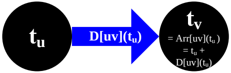
> From now on we assume that optimal waiting times at nodes exist (and are polynomial-time computable).

FIFO vs non-FIFO Arc Delays

- **(Strict) FIFO Arc-Delays:** The slopes of all the *arc-delay* functions are at least equal to (greater than) -1.

Equivalently: *Arc-arrival* functions are **non-decreasing** (aka **no-overtaking** property).



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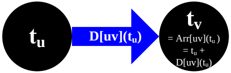

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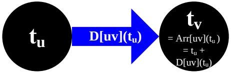

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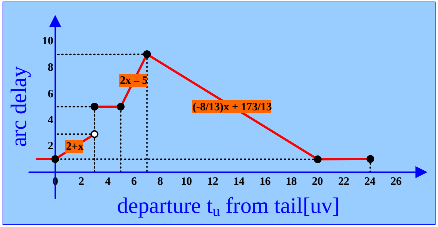

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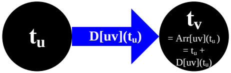

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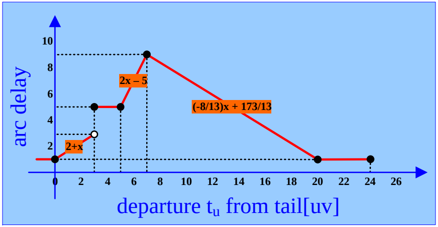

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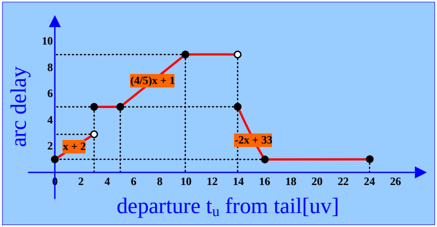
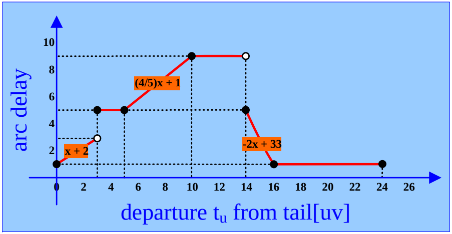
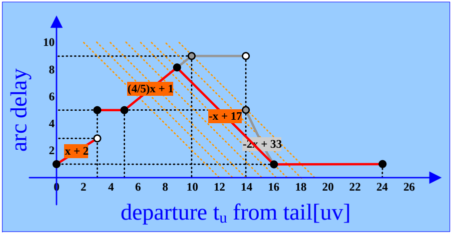


Non-FIFO arc delay example

Non-FIFO+UW Network ⇔ FIFO Network



Non-FIFO+UW arc delay function



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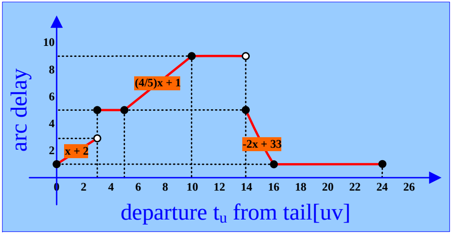

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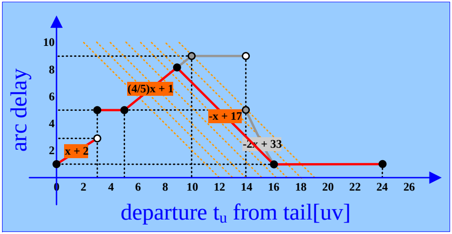

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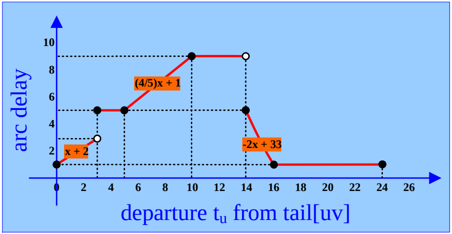

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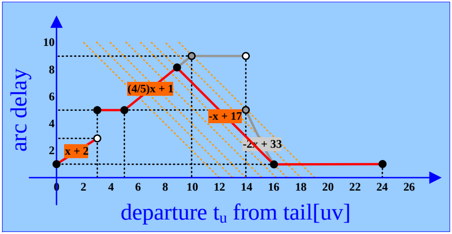

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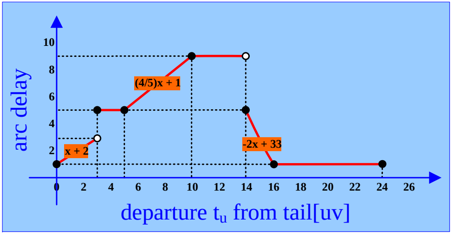

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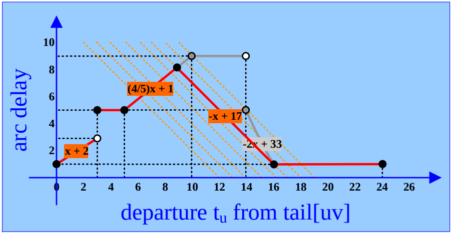

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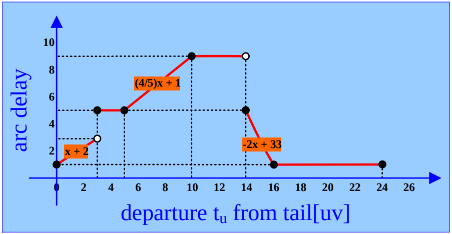

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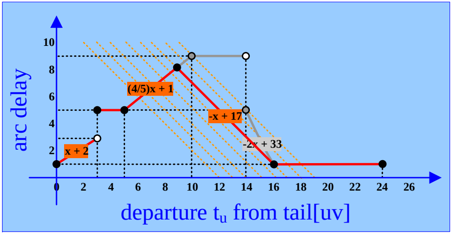

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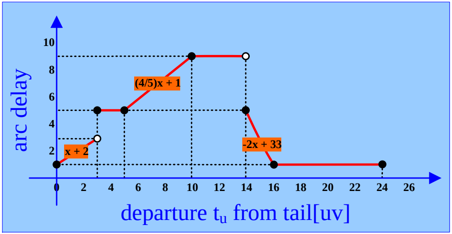

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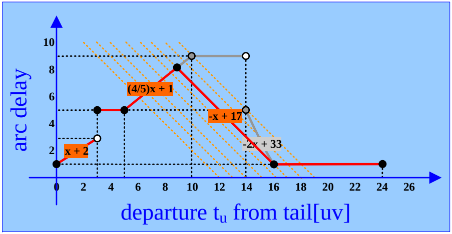

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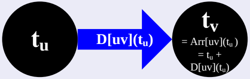
Interested in programming the transformation? Let me know!

Variants of Time-Dependent Shortest Path

DEFINITION: Time-Dependent Shortest Paths

INPUT:

- *Directed* graph $G = (V, A)$ with **succinctly represented** **arc-travel-time** functions $(D[\alpha])_{\alpha\in A}$. $(Arr[\alpha] = ID + D[\alpha])_{\alpha\in A}$.



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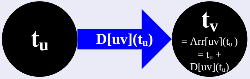

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- **Earliest-arrival / Shortest-travel-time** functions: $Arr[o,d] = \min_{p \in P_{o,d}} \{ \; Arr[p] \; \}$, $D[o,d] = Arr[o,d] - ID$.

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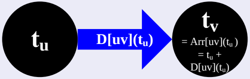

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GOAL1: For departure-time t_o from o, determine $t_d = Arr[o,d](t_o)$.

GOAL2: Provide a **succinct representation** of $Arr[o,d]$ (or $D[o,d]$).

Why Care for Both Goals?

1. Not always sure when to depart (still think about it)! Possessing the entire *distance function* $D[o,d]$ allows for easy answers (e.g., via look-ups) in several queries for varying departure times, or even finding the *minimum travel / ealriest-arrival time* within a window of possible departure times.

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- **Preprocessing** of distance summaries (as in static case) requires to precompute functions instead of scalars.

Consequencies of Different Network Models

- [Dreyfus (1969)] Prefix-subpath optimality holds in **Non-FIFO+UW** networks (given that optimal waiting times *exist*). The same applies for the **FIFO** networks.

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- For arbitrary (o, d) queries (**GOAL2**):
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    - ☹ Complexity is **polynomial** on number of "elementary" *functional operations*. (EVAL, LINEAR COMBINATION, MIN, COMPOSITION)
    - ☹ Not so "elementary" operations after all (see next slides)!!!

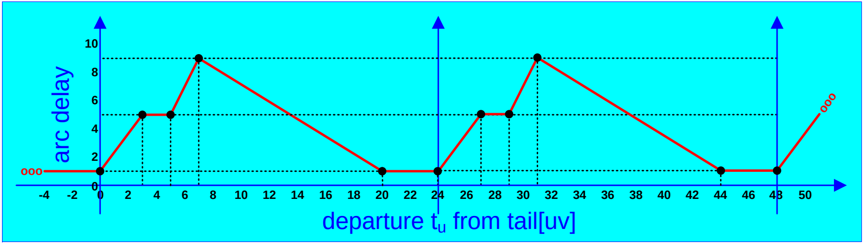
Algorithms for TDSP

in FIFO, Continuous, Pwl Instances

Input/Output Data

PWL Arc Delays

Forward Description (as function of departure times from origin)



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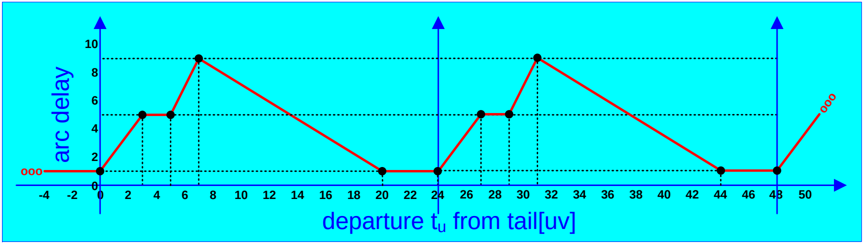
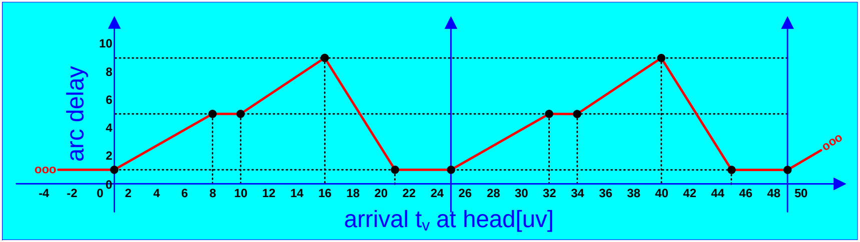
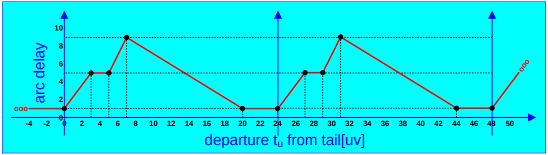


Reverse Description (as function of arrival times at destination)



How to Store/Access PWL Arc Delays



- Exploit *periodicity* and *piecewise-linearity*:

$$\forall t_u \in \mathbb{R},\quad \overrightarrow{D}[uv](t_u) = \begin{cases} \frac{4}{3}t_u + 1, & 0 \leq t_u \mod T \leq 3 \\ 5, & 3 \leq t_u \mod T \leq 5 \\ 2t_u - 5, & 5 \leq t_u \mod T \leq 7 \\ -\frac{8}{13}t_u + \frac{173}{13}, & 7 \leq t_u \mod T \leq 20 \\ 1, & 20 \leq t_u \mod T \leq 24 \end{cases}$$

- Representation: Array of **(slope-constant) triples** equipped with advanced (eg, binary / predecessor) *search capabilities*.

$\left(\frac{4}{3}, 1, 3\right)$	$(0, 5, 5)$	$(2, -5, 7)$	$\left(-\frac{8}{13}, \frac{173}{13}, 20\right)$	$(0, 1, 24)$

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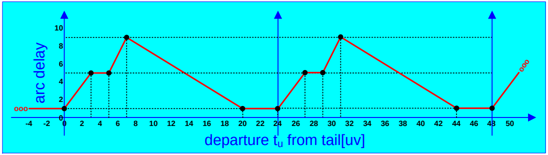

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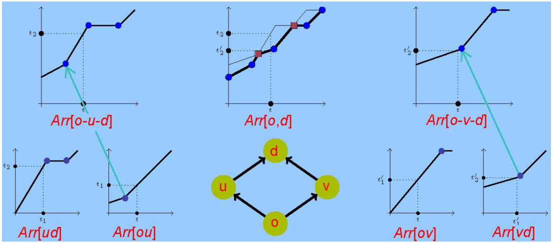
$(0,1)$	$(3,5)$	$(5,5)$	$(7,9)$	$(20,1)$

Piecewise Linearity of Path / Earliest Arrivals



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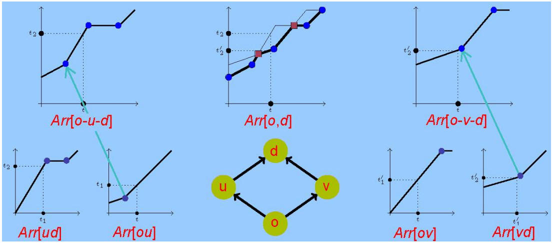

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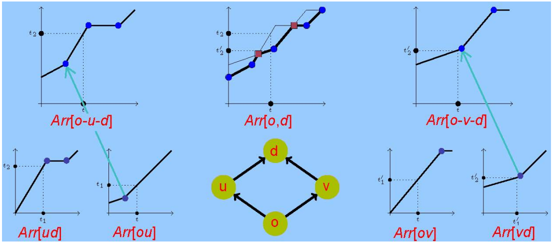

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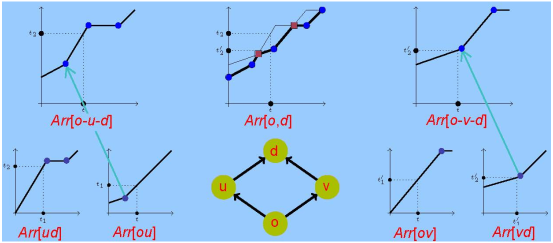

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- Periodicity of arc-delays implies periodicity of earliest-arrival function $Arr[o, d]$.

Known Issues wrt Representations

- **Same representation** both for arc-arrival (or delay) functions and earliest-arrival (or shortest-travel-time) functions.
  - Convenient for handling artificial arcs (representing shortest-travel-time functions) in *overlay abstractions* of the road network.

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- We need only $O\left(\frac{1}{\varepsilon} \cdot \log\left(\frac{D_{\max}[o,d]}{D_{\min}[o,d]}\right)\right)$ breakpoints for a $(1+\varepsilon)$ **upper approximation** $\overline{D}[o,d]$ of $D[o,d]$, for the case of linear arc-delays.

Complexity of TDSP

Lower Bound: $|BP(Arr_{\text{pwl}}[o,d])| = n^{\Omega(\log n)}$ (I)

A Useful Observation (L2.1-2.2 in FHS11)

For any pair of **monotone, pwl** functions f and g, both their composition $f \circ g$ and their minimum $\min\{f, g\}$ are **monotone, pwl** functions as well.

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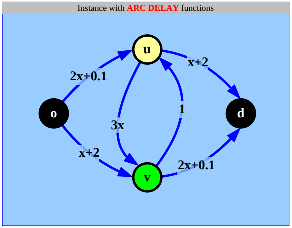
Parametric Shortest Path (PSP): A Similar Problem

- INPUT: $G = (V,A)$, $o,d \in V$. A *linear length function* $\ell[\alpha](\gamma) = \lambda[\alpha] \cdot \gamma + \mu[\alpha]$ per edge $\alpha \in A$ (negative lengths are **allowed**).
- DEFINITIONS:
  - **Path-length:** $\forall p \in G, L[p](\gamma) = \sum_{\alpha \in p} \ell[\alpha](\gamma)$.
  - **Min-length:** $\forall x,y \in V, L[x,y](\gamma) = \min_{p \in P_{x,y}}\{L[p](\gamma)\}$.
- GOAL1: Compute $L[o,d]$ for a **given value** of γ.
- GOAL2: Compute $L[o,d]$ for **all (real) values** of γ.

TDSP vs PSP?



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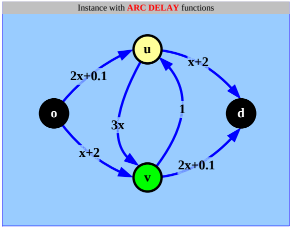

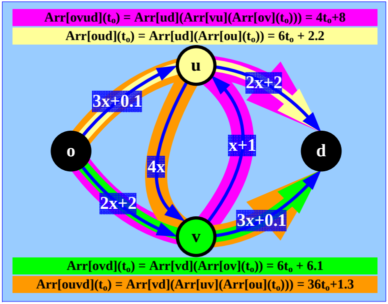

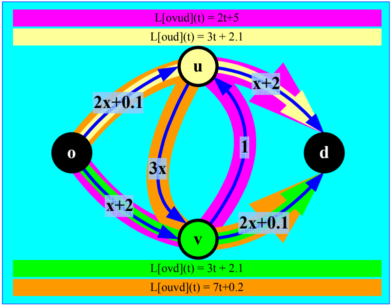

TDSP: Delay composition along paths

PSP: Delay addition along paths

Lower Bound: $|BP(Arr_{\mathrm{pwl}}[o,d])| = n^{\Omega(\log n)}$ (II)

Known Fact [Carstensen (1984), Mulmuley-Shah (2000)]

There exists (linear) PSP-instance with $n^{\Omega(\log n)}$ BPs in $L[o,d]$.

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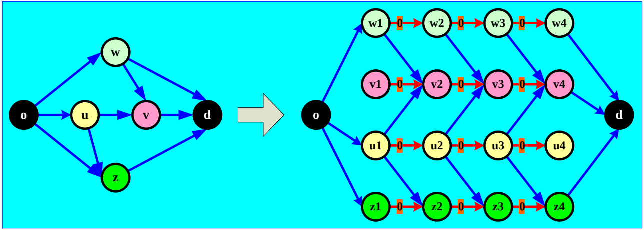
There exists (linear) PSP-instance with $n^{\Omega(\log n)}$ BPs in $L[o,d]$.

Main Steps for TDSP Lower Bound:

1. Assure *non-negativity of lengths* in the PSP instance, in the **departure-time interval of interest**.
2. Scale properly the PSP instance.
3. Consider the corresponding TDSP instance, with parameter γ handled as time.
4. Prove that $L[o,d]$ (for PSP instance) and $D[o,d]$ (for TDSP instance) have (almost) the **same number** of BPs.

Lower Bound: $|BP(Arr_{\text{pwl}}[o,d])| = n^{\Omega(\log n)}$ (III)

1. Construct a **layered-graph**, in a **path-length-preserving** manner:



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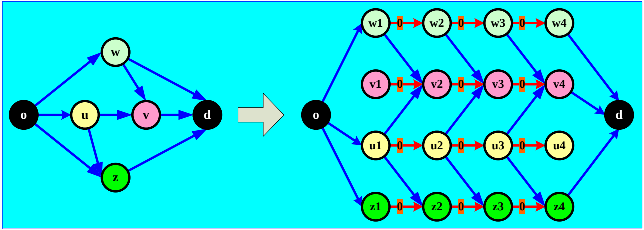

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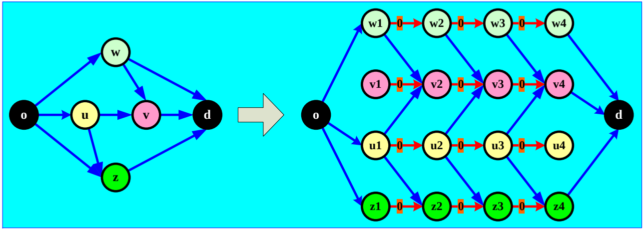

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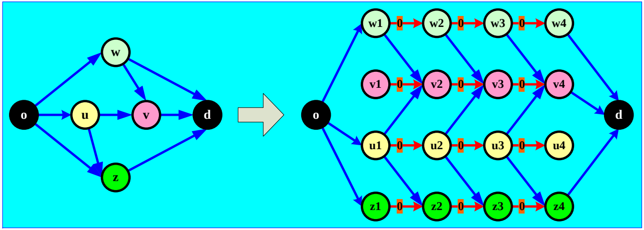

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3. For the TDSP resulting from the scaled PSP when considering γ as departure time, prove that $\forall j \in \{1, \ldots, N-1\}$, at **"time"** $\bar{\gamma}_j \equiv \frac{\gamma_j + \gamma_{j+1}}{2}$ both instances return *the same* shortest $od-$path p_j.

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How it works: At given $j \in \{1,\ldots,N-1\}$:

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- **Departure-time perturbations:** Small-enough so as to cause *not too large arc-delay perturbations*: $\forall\alpha\in A$, $\forall\delta_\alpha\in(0,\delta^*]$,

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$\therefore$ Small time-perturbations guarantee sufficiently small **arc-delay perturbations**, and thus, optimality of p_j:

$$\boxed{\begin{aligned} D[p_j](\bar{\gamma}_j) &\leq \mu\cdot\bar{L}_j + \mu\cdot\frac{(n-1)\Delta_{\min}}{2n} \\ &< \mu\cdot L'_j - \mu\frac{(n-1)\Delta_{\min}}{2n} \leq D[q](\bar{\gamma}_j),\ \forall q\neq p_j \end{aligned}}$$

QED

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Observation: (L4.1 in FHS11)

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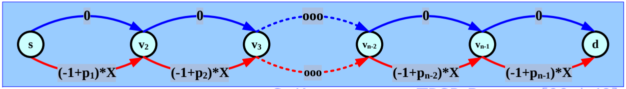

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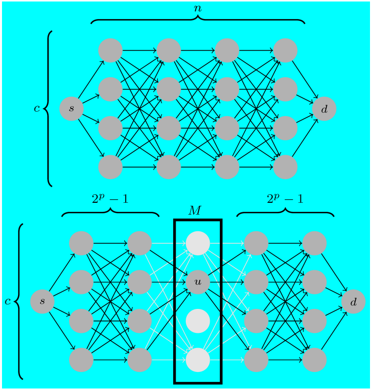

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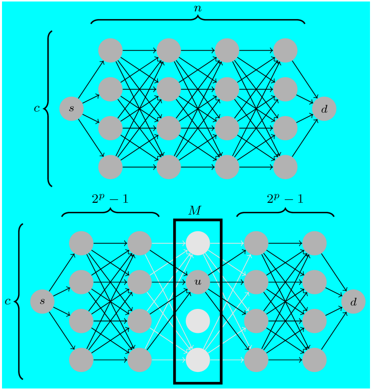

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THM4.4 (FHS11)

$|BP(Arr_{\mathrm{lin}}[o,d])| = n^{O(\log n)}$ in any graph G and pair of nodes $o, d \in V(G)$.

(Exact) Output Sensitive Algorithm for Earliest-Arrival Functions

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3. Interesting to discover whether the complexity of the earliest-arrival functions is indeed so bad **in real (e.g., road) networks**.

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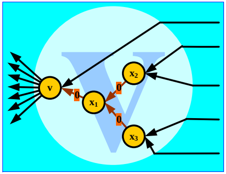

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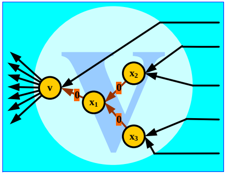

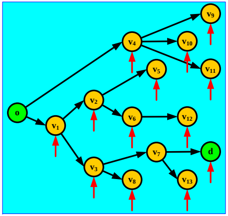

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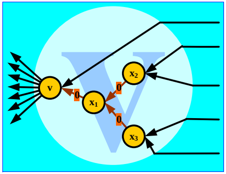

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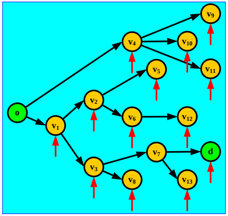

  - $\forall v \in V$, two short alternatives when departing from o at time t_0: Earliest-arrival to each parent, plus delay of corresponding incoming arc.
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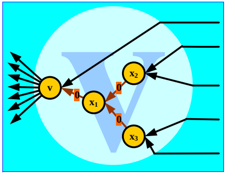

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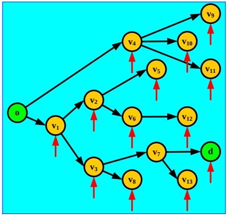

- All $(m+n)$ certificates temporarily stored in a *priority queue*.

The Output-Sensitive Algorithm (II)

When current time $t_1 > t_0$ matches the earliest failure-time of a certificate in the queue:

if minimization-certificate failure, at node $v \in V$:

then (1) Update shortest $ov-$path

/* **ONE-BIT** change in combinatorial structure */

(2) Update $Arr[o, x]$ and $t_{fail}[x]$, $\forall x \in T_v$.

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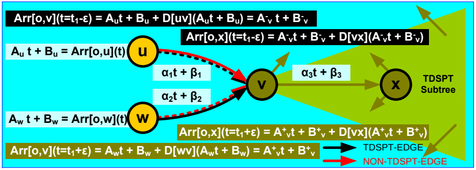

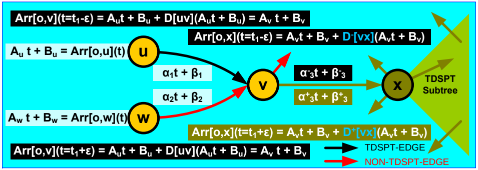

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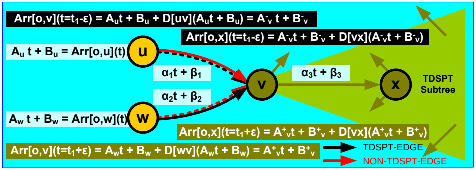

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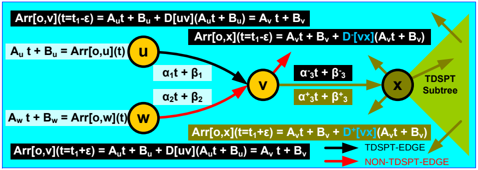

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- What to keep in memory:
  - Breakpoint triples for earliest-arrival functions, plus ONE bit (indicating the parent).
  - Advanced search structures, if number of BPs is large.
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  - In the *original graph* (in worst-case): $\mathrm{O}\left(m \times \log^2 n\right)$. Second logarithmic factor is due to **updates of tournament trees** implementing the MIN operator at a particular node, upon emergence of a single certificate failure.

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- *Worst-case time-complexity* of output-sensitive algorithm:

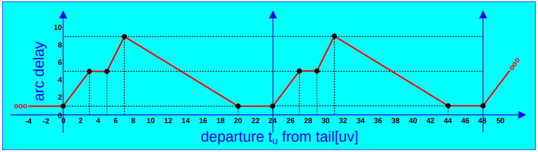
$$O\left(m \times \log^2 n \times (\text{PRIMBPs} + \text{MINBPs})\right)$$

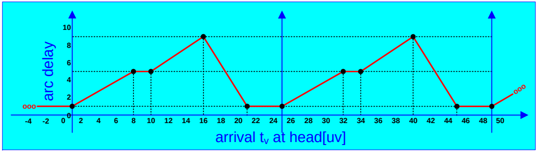
Poly-time Approximation Algorithms

$(1+\varepsilon)$–approximation of $D[o,d]$: Preliminaries

- Why focus on **shortest-travel-time** (delays) functions, and not on **earliest-arrival-time** functions?

- Arc/Path Delay Reversal: Easy task!!!

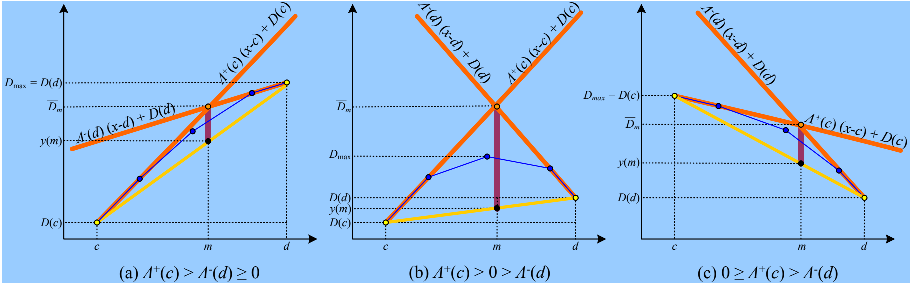




- $t_o = \overleftarrow{Arr}[o,v](t_v) = t_v - \overleftarrow{D}[o,v](t_v)$: **Latest-departure-time** from o to v, as a function of the arrival time t_v at v.

Approximating $D[o,d]$: Quality

- **Maximum Absolute Error:** A crucial quantity both for the time-complexity and for the space-complexity of the algorithm:



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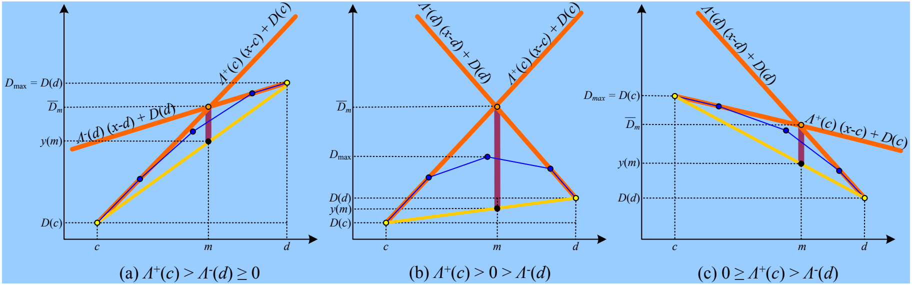

LEMMA: Closed Form of Maximum Absolute Error

[Kontogianis-Zaroliagis (2013)]

$$MAE(c,d) = (\Lambda^+(c) - \Lambda^-(d)) \cdot \frac{(m-c)\cdot(d-m)}{L} \leq \frac{L\cdot(\Lambda^+(c)-\Lambda^-(d))}{4}$$

Approximating $D[o,d]$: Basic Idea (I)

- Approximations of $D[o,d]$: For given $\varepsilon > 0$, and $\forall t \in [0,T)$,

$$\boxed{\boxed{\underline{D}[o,d](t) \leq D[o,d](t) \leq \overline{D}[o,d](t) \leq (1+\varepsilon)\cdot \underline{D}[o,d](t)}}$$

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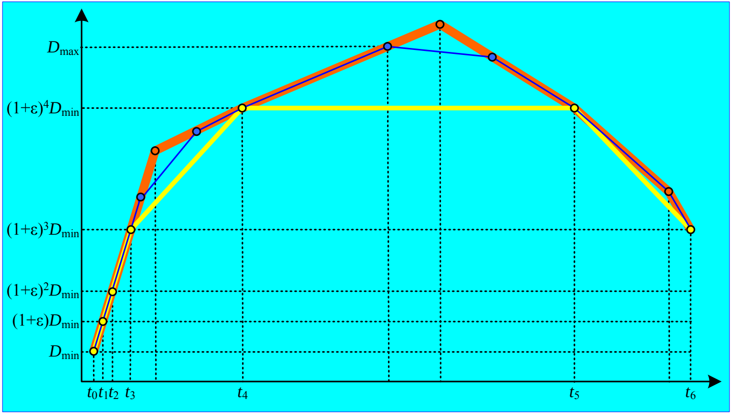
- FOCUS: **Linear** arc-delays. Later extend to pwl arc-delays.
- $D[o,d]$ lies entirely in a **bounding box** that we can easily determine, with only 3 TD-Djikstra probes.

Approximating $D[o,d]$: Basic Idea (II)



- Make the sampling so that
  $\forall t \in [0,T],\ \overline{D}[o,d](t) \leq (1+\varepsilon)\cdot \underline{D}[o,d](t)$.
- Keep sampling always the **fastest-growing axis** wrt to $D[o,d]$.

One-To-One Approximation: PHASE-1

[Foschini-Hershberger-Suri (2011)]

while slope of $D[o,d] \geq 1$ **do**

$$t_0 \xrightarrow{\text{Forward Dijkstra}} t_0 + D[o,d](t_0)$$

$$t_1 \xleftarrow{\text{Backward Dijkstra}} t_0 + (1+\varepsilon)^{1/2} D[o,d](t_0) = t_1 + D[o,d](t_1)$$

$$t_2 \xleftarrow{\text{Backward Dijkstra}} t_1 + (1+\varepsilon)^{1/2} D[o,d](t_1) = t_2 + D[o,d](t_2)$$

$$\vdots$$

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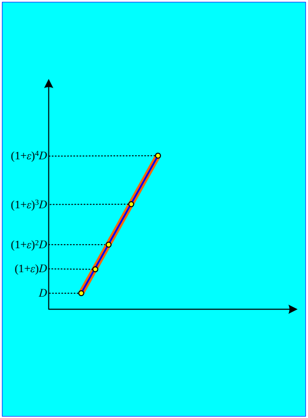
Bad Case for [Foschini-Hersberger-Suri (2011)] :



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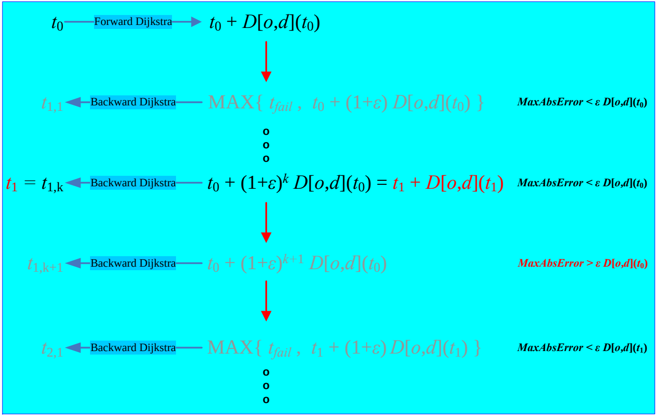

One-To-One Approximation: PHASE-2

[Foschini-Hershberger-Suri (2011)]

Slope of $D[o,d] \leq 1$:

repeat

Apply **BISECTION** to the remaining time-interval(s)

until desired approximation guarantee (wrt **Max Absolute Error**) is achieved.

One-To-All Approximaton via Bisection (I)

[Kontogiannis-Zaroliagis (2013)]

ASSUMPTION 1: Concavity of arc-delays. /* to be removed later */

- Implies concavity of the *unknown* function $D[o, d]$.

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- Implies concavity of the *unknown* function $D[o,d]$.

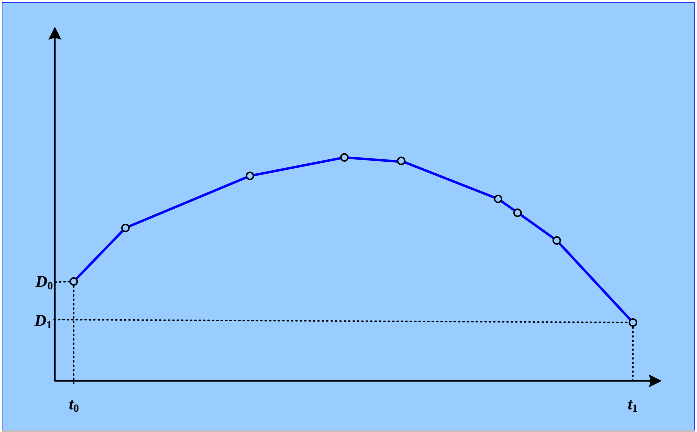
ASSUMPTION 2: Bounded Travel-Time Slopes. Small slopes of the (pwl) arc-delay functions.

- **Verified** by TD-traffic data for road network of Berlin [TomTom (February 2013)] that all arc-delay slopes are in $[-0.5, 0.5]$.
- Slopes of *shortest-travel-time* function $D[o,d]$ from $[-\Lambda_{\min}, \Lambda_{\max}]$, for some constants $\Lambda_{\max} > 0$, $\Lambda_{\min} \in [0,1)$.

One-To-All Approximaton via Bisection (II)

[Kontogiannis-Zaroliagis (2013)]

Under ASSUMPTIONS 1-2: Execute `Bisection` to *sample simultaneously* all distance values from o, at mid-points of time intervals, until required approximation guarantee is achieved *for each destination node*.



Example of Bisection Execution : INPUT = **UNKNOWN BLUE** function

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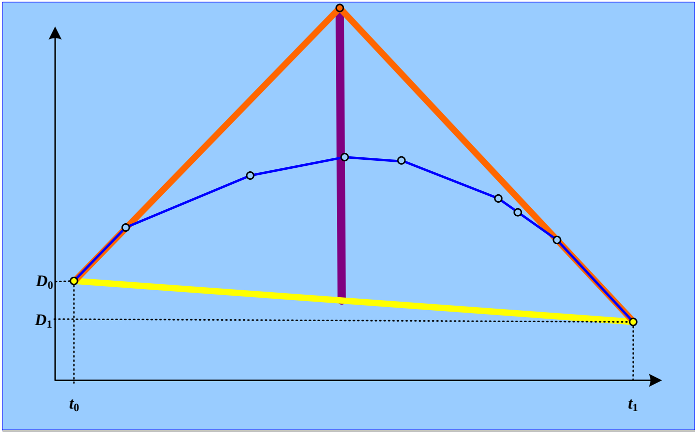

Example of Bisection Execution : **ORANGE** = Upper Bound, **YELLOW** = Lower Bound

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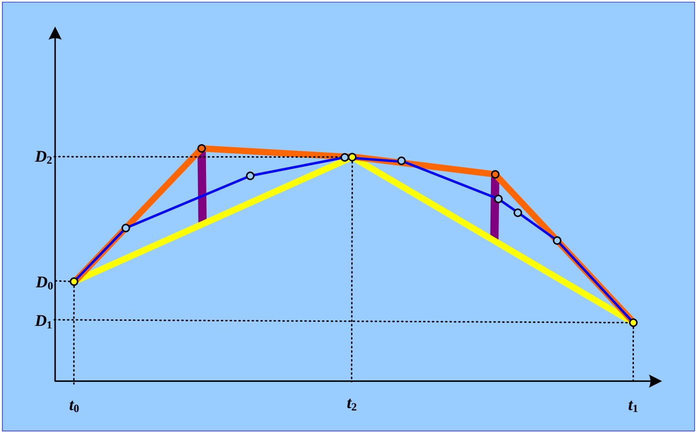

Example of Bisection Execution : Level-1 Recursion

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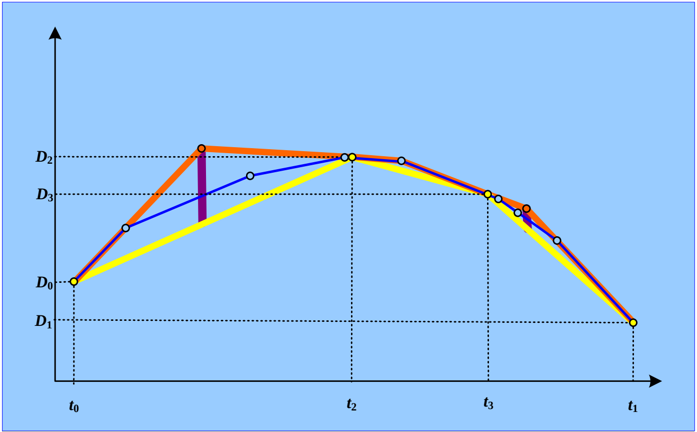


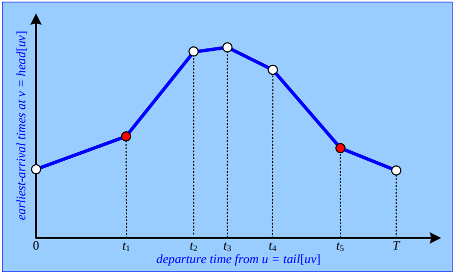
Example of Bisection Execution : Level-2 Recursion

One-To-All Approximaton via Bisection (III)

[Kontogiannis-Zaroliagis (2013)]

Only under ASSUMPTION 2: For continuous, pwl arc-delays.

1. Call `Reverse TD-Dijkstra` to project each concavity-spoiling PB to a PI of the origin o.
2. For each pair of *consecutive PIs* at o, run `Bisection` for the corresponding departure-times interval.



3. Return the *concatenation* of approximate distance summaries.

Approximating $D[o,d]$: Space/Time Complexity

THEOREM: Space Complexity [Kontogianis-Zaroliagis (2013)]

Let K^* be the total number of concavity-spoiling BPs among all the arc-delay functions in the instance.

Space Complexity: For a *given orign* $o \in V$ and *all* possible destinations $d \in V$, the following complexity bounds hold for creating all the approximation functions $\overline{D}[o,*] = (\overline{D}[o,d])_{d \in V}$:

1. $O\left(\frac{K^*}{\epsilon} \log\left(\frac{D_{\max}[o,*](0,T)}{D_{\min}[o,*](0,T)}\right)\right)$
2. In each interval of *consecutive* PIs, $|UBP[o,d]| \leq 4 \cdot$(minimum #BPs for any $(1+\varepsilon)$−approximation.

Time Complexity: The number of *shortest-path probes* executed for the computation of the approximate distance functions is:
$TDSP[o,d] \in O\left(\log\left(\frac{T}{\varepsilon \cdot D_{\min}[o,d]}\right) \cdot \frac{K^*}{\varepsilon} \log\left(\frac{D_{\max}[o,*](0,T)}{D_{\min}[o,*](0,T)}\right)\right)$

Implementation Issues wrt One-To-All Bisection

- `One-To-All Bisection` of [Kontogiannis-Zaroliagis (2013)] is a **label-setting** approximation method that provably works *space/time optimally* (within constant factors) wrt **concave** continuous pwl arc-delay functions.

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- A novel **one-to-all** (again **label-setting**) approximation technique, called the `Trapezoidal` method ([Kontogiannis-Wagner-Zaroliagis (2014)]) avoids entirely the dependence of the required space from the network structure (and, of course, the degree of disconcavity).

Any Questions?

Next Lecture
Time-Dependent Oracles
Wednesday, June 18 2014, 11:30

Distance Oracles

Distance Oracles

A Success Story in Static Graphs

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3. Query Algorithm: Respond fast to queries, based on the (hierarchical?) *distance-independent* division and/or the *distance-dependent* summaries.

Distance Oracles

- Extremely successful theme in static graphs.
  - In theory:
    - P-Space: Subquadratic (sometimes quasi-linear).
    - Q-Time: Constant.
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IN THIS TALK

The focus is on **time-dependent oracles**, with **provably good** preprocessing-space / query-time / stretch tradeoffs.

Theoretical Bounds for Static Graphs

Reference	Setting	Stretch	Query	Space
[TZ05]	weighted graph	$2k-1$, $k \geq 2$	$O(k)$	$O\left(kn^{1+1/k}\right)$
[WN13]	weighted graph	$2k-1$, $k \geq 2$	$O(\log(k))$	$O\left(kn^{1+1/k}\right)$
[Che13]	weighted graph	$2k-1$, $k \geq 2$	$O(1)$	$O\left(kn^{1+1/k}\right)$
[AG13]	sparse weighted graph	$1+\epsilon$	$o(n)$	$o\left(n^2\right)$
[Kle02] [Tho04]	planar weighted digraph	$1+\epsilon$	$O\left(\epsilon^{-1}\right)$	$O\left(\frac{n\log(n)}{\epsilon}\right)$
[MN06]	metric	$O(k)$	$O(1)$	$O\left(kn^{1+1/k}\right)$
[BGKRL11]	Doubling metric, dynamic	$1+\epsilon$	$O(1)$	$\epsilon^{-O(ddim)}n + 2^{O(ddim\log(ddim))}n$

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Is it a Success Story in Time-Dependent Graphs?

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FLAT TD-Oracle

FLAT TD-Oracle: Overall Idea

1. Choose a set L of **landmarks**.
   - In theory: Each vertex $v \in V$ is chosen, *independently of other vertices*, to be included in the landmark set L w.p. $\rho \in (0, 1)$.
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2. Preprocess $(1 + \epsilon)$-approximate **distance summaries** (functions) $\overline{D}[\ell, v]$ *from* every **landmark** $\ell \in L$ *towards* each destination $v \in V$.
   - Label-setting approach.
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3. Provide **query algorithms** (FCA/RQA) that return constant / $(1+\sigma)$-approximate distance values, for arbitrary query (o, d, t_o).

FLAT TD-Oracle

selection & preprocessing of landmarks

Landmark Selection and Preprocessing (I)

- Select each vertex *independently and uniformly at random* w.p. $\rho \in (0, 1)$ for the **landmark set** $L \subseteq V$.
- `Preprocessing:` $\forall \ell \in L$, precompute $(1 + \epsilon)-$approximate distance functions $\Delta[\ell, v]$ to all destinations $v \in V$.

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THEOREM: [Kontogiannis-Zaroliagis (2013)]

Using `Bisection` for computing approximate distance summaries:

- Pre-Space:

$$\boxed{O\left(\frac{K^* \cdot |L| \cdot |V|}{\epsilon} \cdot \max_{(\ell,v) \in L \times V}\left\{\log\left(\frac{\overline{D}[\ell,v](0,T)}{\underline{D}[\ell,v](0,T)}\right)\right\}\right)}$$

- Pre-Time (in number of TDSP-Probes):

$$\boxed{O\left(\max_{(\ell,v)}\left\{\log\left(\frac{T \cdot (\Lambda_{\max}+1)}{\epsilon \underline{D}[\ell,v](0,T)}\right)\right\} \cdot \frac{K^* \cdot |L|}{\epsilon} \max_{(\ell,v)}\left\{\log\left(\frac{\overline{D}[\ell,v](0,T)}{\underline{D}[\ell,v](0,T)}\right)\right\}\right)}$$

Landmark Selection and Preprocessing (II)

A recent development: Improved preprocessing time/space.

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A recent development: Improved preprocessing time/space.

THEOREM: [Kontogiannis-Wagner-Zaroliagis (2014)]

Using both `Bisection` (for *nearby* nodes) and `Trapezoidal` (for *faraway* nodes):

- Pre-Space:

$$\boxed{\mathbb{E}\left[S_{\texttt{BIS+TRAP}}\right] \in O\left(T\left(1+\frac{1}{\epsilon}\right)\Lambda_{\max}\cdot\rho n^2\,\mathrm{polylog}(n)\right)}$$

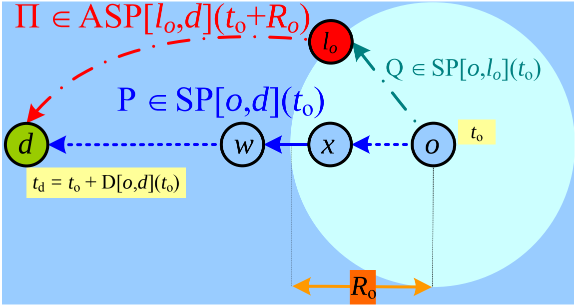
- Pre-Time:

$$\boxed{\mathbb{E}\left[P_{\texttt{BIS+TRAP}}\right] \in O\left(T\left(1+\frac{1}{\epsilon}\right)\Lambda_{\max}\cdot\rho n^2\,\mathrm{polylog}(n)\log\log(K_{\max})\right)}$$

FLAT TD-Oracle

FCA: constant-approximation query

FCA: A constant-approximation query algorithm (I)



Forward Constant Approximation: $FCA\left(o,d,t_o,(\Delta[\ell,v])_{(\ell,v)\in L\times V}\right)$

1. **Exploration:** Grow a TD-Dijkstra forward ball $B(o,t_o)$ until the closest landmark ℓ_o is settled.

2. **return** $sol_o = D[o,\ell_o](t_o) + \Delta[\ell_o,d](t_o + D[o,\ell_o](t_o))$.

FCA: A constant-approximation query algorithm (II)

- ASSUMPTION 3: Bounded Opposite Trips.
  $\exists\zeta \geq 1 : \forall(o,d) \in V \times V, \ \forall t \in [0,T], \ D[o,d](t) \leq \zeta \cdot D[d,o](t_o)$.

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THEOREM: FCA Performance

Under ASSUMPTIONS 2-3, and any route planning request (o,d,t_o), `FCA` achieves the following performance:

- Approximation guarantee:

$$\begin{aligned} D[o,d](t_o) &\leq R_o + \Delta[\ell_o,d](t_o+R_o) \leq (1+\epsilon)D[o,d](t_o) + \psi R_o \\ &\leq \left(1+\epsilon+\psi\cdot\frac{R_o}{D[o,d](t_o)}\right)\cdot D[o,d](t_o) \end{aligned}$$

where $\psi = 1 + \Lambda_{\max}(1+\epsilon)(1+2\zeta+\Lambda_{\max}\zeta) + (1+\epsilon)\zeta$.

FCA: A constant-approximation query algorithm (II)

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- Query-time complexity:
  - $\mathbb{E}\left[\mathcal{Q}_{FCA}\right] \in O\left(\frac{1}{\rho} \cdot \ln\left(\frac{1}{\rho}\right)\right)$
  - $\mathbb{P}\left[\mathcal{Q}_{FCA} \in \Omega\left(\frac{1}{\rho} \cdot \ln^2\left(\frac{1}{\rho}\right)\right)\right] \in O(\rho)$

FLAT TD-Oracle

RQA: boosting approximation guarantee

RQA: Overview

Recursive Query Approximation: $RQA\left(o, d, t_o, (\Delta[\ell, v])_{(\ell,v)\in L\times V}, R\right)$

1. **while** recursion budget R not exhausted **do**
2. Exploration: Grow a TD-Dijkstra forward-ball $B(w_i, t_i)$ until the closest landmark ℓ_i is settled.
3. $sol_i = D[o, w_i](t_o) + D[w_i, \ell_i](t_i) + \Delta[\ell_i, d](t_i + D[w_i, \ell_i](t_i))$.
4. Recursion: Execute RQA centered at *each boundary node* of $B(w_i, t_i)$ with recursion budget $R - 1$.
5. **endwhile**
6. **return** best possible solution found.

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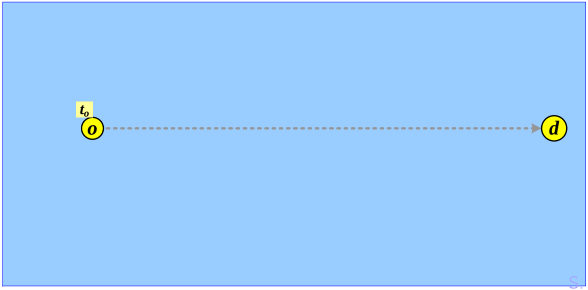

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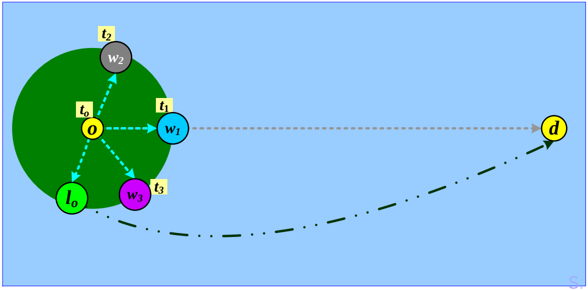

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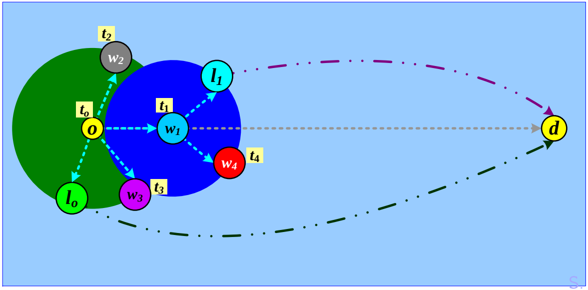

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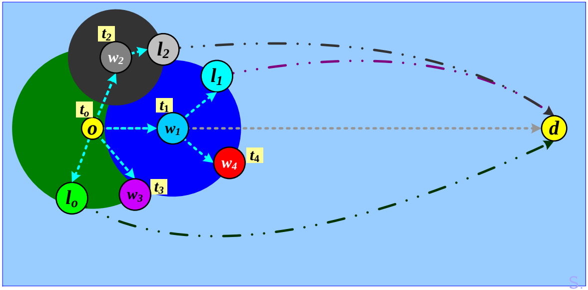

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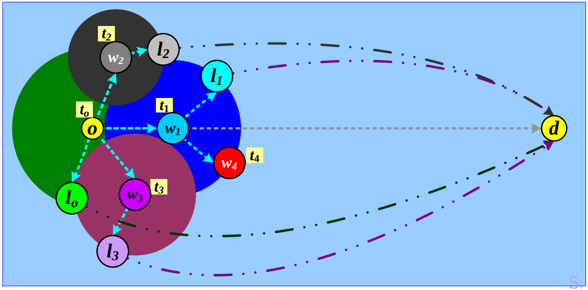

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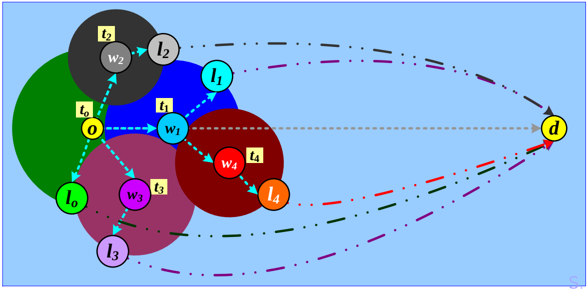

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- Growing level-1 balls...
- Growing level-2 balls...

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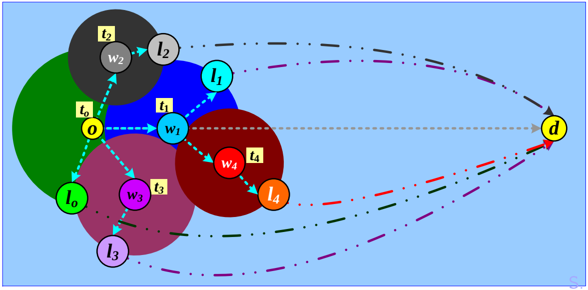

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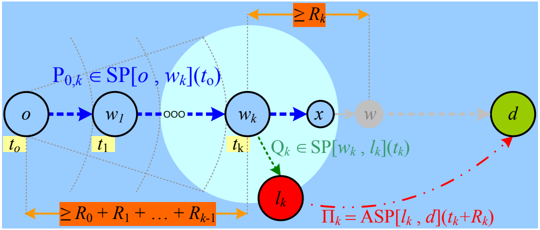
RQA: Why Does Recursion Boost Approximation?



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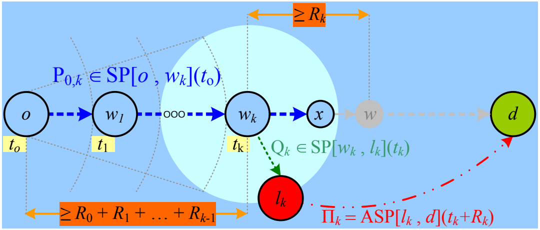

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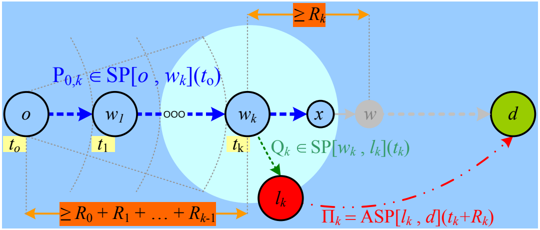

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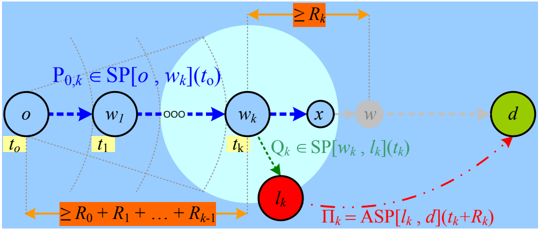

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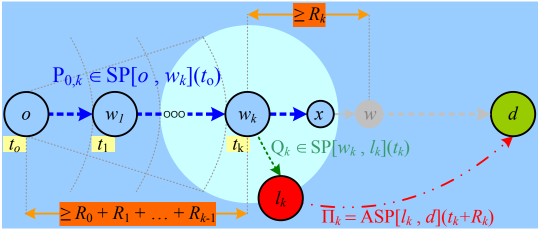

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3. Quality of approximation guarantee of `FCA` (per ball) for remaining **suffix subpath** to the destination depends on *ball radius* (distance from the closest landmark to the ball center).
4. A constant number of recursion depth R suffices to assure guarantee close to $1+\epsilon$.

RQA: Performance

THEOREM: Complexity of RQA

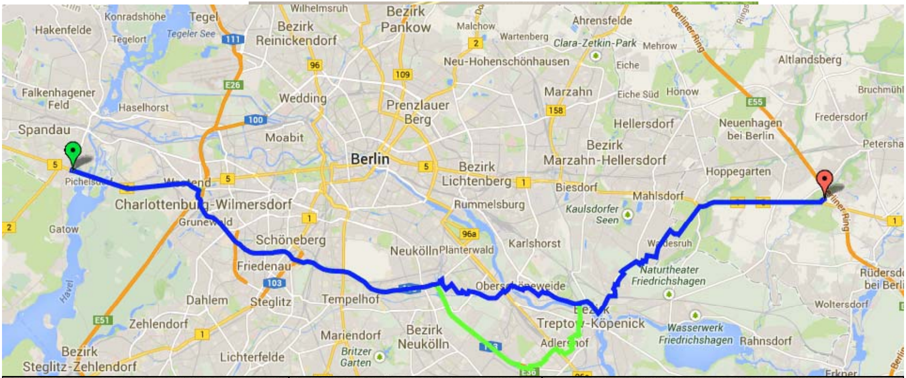
For sparse networks (i.e., having $\mu = |A|/|V| \in O(1)$), the complexity of `RQA` with recursion budget R for obtaining $(1+\sigma)-$approximate distances (for any constant $\sigma > \epsilon$) to arbitrary (o, d, t_o) queries, is:

- $\mathbb{E}\left[\mathcal{Q}_{RQA}\right] \in O\left(\left(\frac{1}{\rho}\right)^{R+1} \cdot \ln\left(\frac{1}{\rho}\right)\right)$.
- $\mathbb{P}\left[\mathcal{Q}_{RQA} \in O\left(\left(\frac{\ln(n)}{\rho}\right)^{R+1} \cdot \left[\ln\ln(n) + \ln\left(\frac{1}{\rho}\right)\right]\right)\right] \in 1 - O\left(\frac{1}{n}\right)$.

TD Distance Oracle: Recap

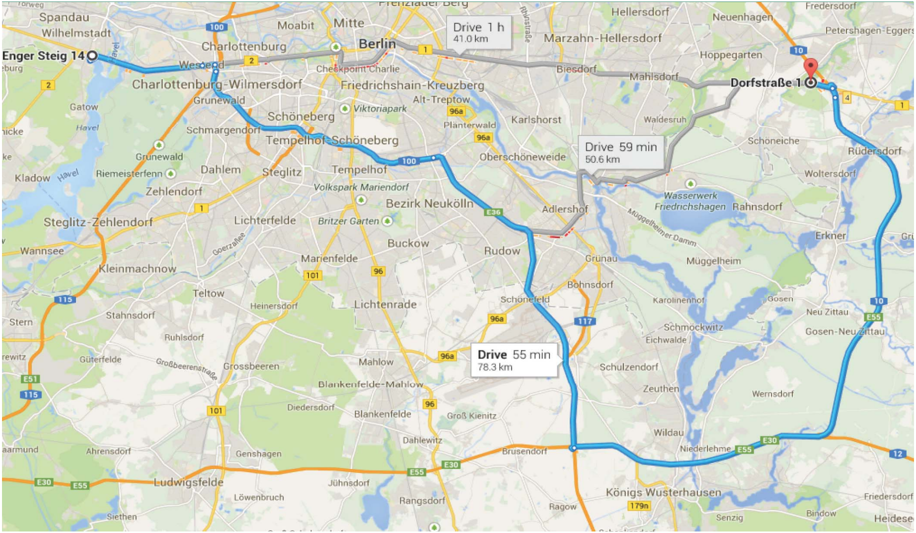
what is preprocessed	space : $\mathbb{E}[\mathcal{S}]$	preprocessing : $\mathbb{E}[\mathcal{P}]$	query : $\mathbb{E}[\mathcal{Q}_{RQA}]$
All-To-All	$O\left((K^* + 1)n^2 U\right)$	$O\left(\begin{array}{c} n^2 \log(n) \\ \cdot \log\log(K_{\max}) \\ \cdot(K^* + 1)TDP \end{array}\right)$	$O(\log\log(K^*))$
Nothing	$O(n + m + K)$	$O(1)$	$O\left(\begin{array}{c} n\log(n) \cdot \\ \log\log(K_{\max}) \end{array}\right)$
Landmarks-To-All	$O\left(\rho n^2 (K^* + 1)U\right)$	$O\left(\begin{array}{c} \rho n^2 \log(n) \\ \cdot \log\log(K_{\max}) \\ \cdot(K^* + 1)TDP \end{array}\right)$	$O\left(\begin{array}{c} \left(\frac{1}{\rho}\right)^{R+1} \cdot \log\left(\frac{1}{\rho}\right) \\ \cdot \log\log(K_{\max}) \end{array}\right)$
$K_{\max} \in O(1)$ $\rho = n^{-\alpha}$ $U, TDP \in O(1)$ $K^* \in O(\mathrm{polylog}(()n))$	$\tilde{O}\left(n^{2-\alpha}\right)$	$\tilde{O}\left(n^{2-\alpha}\right)$	$\tilde{O}\left(n^{(r+1)\cdot\alpha}\right)$

TD Distance Oracle: Towards Implementation...



Departure	**8:00 – 20:00**	**20:00 – 08:00**
Travel Time	**46.5 mins**	**45.1 mins**
FCA Response Time	**< 0.2 ms**	**< 0.2 ms**
RQA(1) Repsponse Time	**<1 ms**	**<1 ms**
P2P Dist.Summary Construction	**120 ms**	
One-To-All Dist.Summary Construction	**< 40 sec**	

TD Distance Oracle: Towards Implementation...



Related Literature

- [Dreyfus (1969)] S. E. Dreyfus. An appraisal of some shortest-path algorithms. In Operations Research, 17(3):395–412, 1969.
- [Orda-Rom (2000)] A. Orda, R. Rom. Shortest-path and minimum delay algorithms in networks with time-dependent edge-length. In J. ACM, 37(3):607–625, 1990.
- [Dean (2004)] B. C. Dean. Shortest paths in FIFO time-dependent networks: Theory and algorithms. Technical report. MIT, 2004.
- [Dehne-Omran-Sack (2010)] F. Dehne, O. T. Masoud, J. R. Sack. Shortest paths in time-dependent FIFO networks. In Algorithmica, 62(1-2):416–435, 2012.
- [Foschini-Hershberger-Suri (2011)] L. Foschini, J. Hershberger, S. Suri. On the complexity of time-dependent shortest paths. In Algorithmica, 68(4), pp. 1075–1097, 2014. Prelim. version in SODA 2011.
- [Kontogiannis-Zaroliagis (2013)] 14. S. Kontogiannis, C. Zaroliagis. Distance oracles for time dependent networks. In Automata, Languages and Programming (ICALP-A 2014), Springer, 2014.

Any Questions?

Next Lecture
Time-Dependent Speed-up Techniques
Wednesday, July 2 2014, 11:30