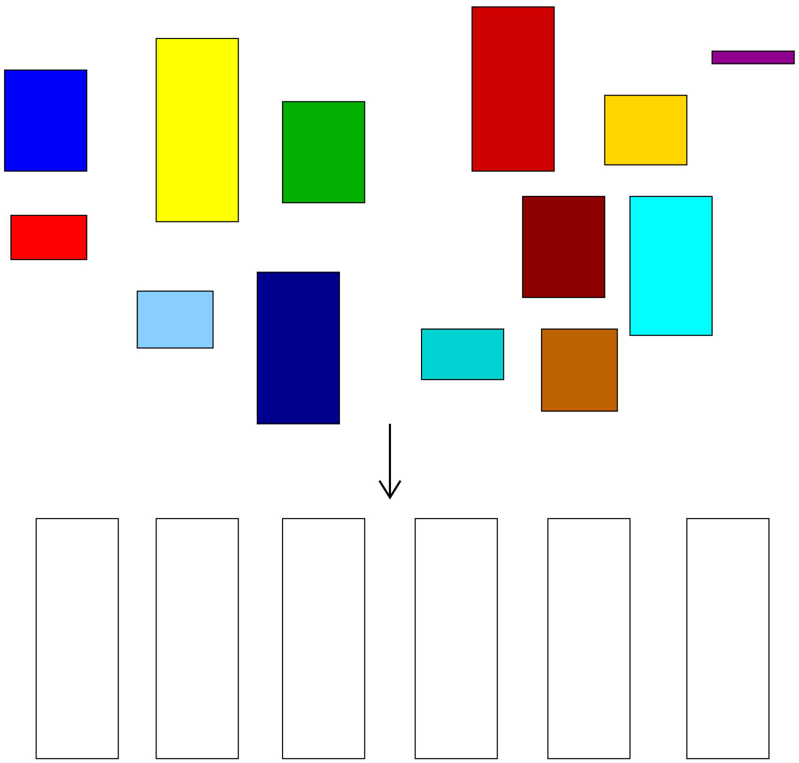
Online bin packing

- Problem definition
- First Fit and other algorithms
- The asymptotic performance ratio
- Weighting functions
- Lower bounds

Bin packing: problem definition (repeat)

- Input: n items with sizes $a_1, \ldots, a_n \in (0, 1]$
- Goal: pack these items into a minimal number of bins
- Each bin has size 1



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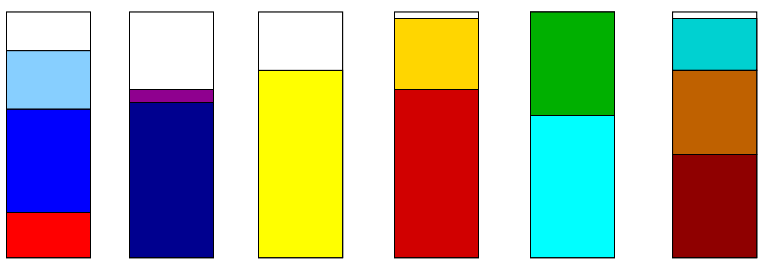
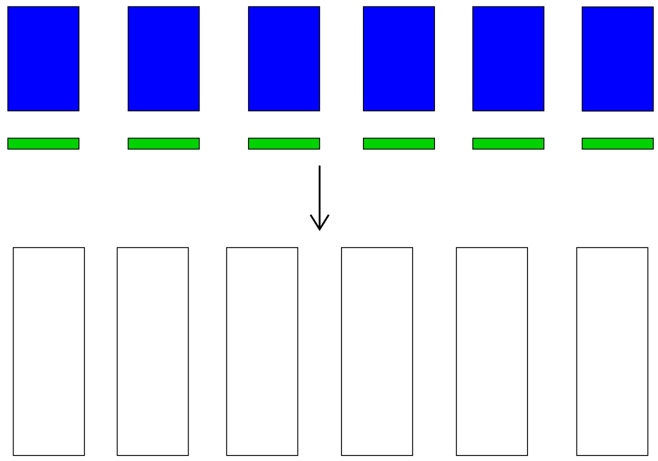


Next Fit (repeat)

- Keep a single open bin
- When an item does not fit, close this bin and open a new bin
- Closed bins are never used again
- Approximation ratio is 2
  - each pair of successive bins contains more than 1
  - matching lower bound



First Fit, Best Fit, Worst Fit

- Keep all bins open
- An item is assigned to the first / best / worst bin in which it fits

  best (worst) bin = bin in which the item fits and where it leaves the smallest (largest) empty space
- If item does not fit in any open bin, open a new bin

First Fit and Best Fit are better than Next Fit

Disadvantage: no bin is ever closed

Asymptotic performance ratio

- A performance measure for bin packing algorithms
- Similar to competitive ratio
- Idea: compare number of bins used to optimal number of bins, for large inputs
- Definition:

$$R_A^\infty = \limsup_{n\to\infty} \sup_\sigma \left\{ \frac{A(\sigma)}{OPT(\sigma)} \middle| OPT(\sigma) = n \right\}.$$

Asymptotic performance ratio

- It is the infimum R for which

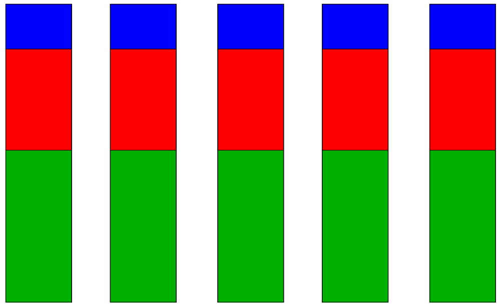
$$A(\sigma) \leq R \cdot \mathrm{OPT}(\sigma) + c$$

  for every input sequence σ

- Here c is a non-negative constant which does not depend on the input
- We still calculate the ratio between the costs of ALG and the costs of OPT
- However, we give ALG several bins for free

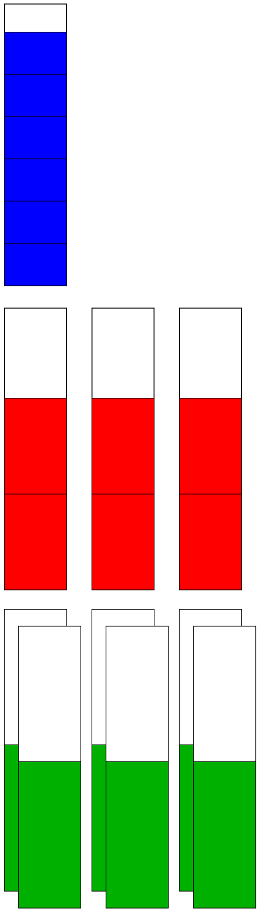
Performance ratio of FF and BF

- These algorithms have a ratio of 1.7
- We show a simple lower bound
- Asymptotic ratio $\Rightarrow$ we need an input sequence of arbitrary length n
- Sequence: $6n$ items of size 0.15, $6n$ items of size 0.34, $6n$ items of size 0.51
- These items can be packed into $6n$ bins by OPT



How do FF and BF pack this input?

- $6n$ items of size 0.15 $\Rightarrow$ packed in n bins
  All these bins are 0.9 full
- $6n$ items of size 0.34 $\Rightarrow$ packed in $3n$ bins
  All these bins are 0.68 full
- $6n$ items of size 0.51 $\Rightarrow$ packed in $6n$ bins
- Total $10n$ bins used
- Lower bound of $10n/6n = 5/3$



Bounded space algorithms

- keep only a constant number of bins open at any time
- gives a constant stream of output (closed bins)
- Idea: pack similar items together
- NF is bounded space, but FF, BF and WF are not

The HARMONIC algorithm

This algorithm classifies items into types according to their size

- Size $\in (\frac{1}{2}, 1]$: type 1, pack 1 per bin
- Size $\in (\frac{1}{3}, \frac{1}{2}]$: type 2, pack 2 per bin
- $\ldots$
- Size $\in (\frac{1}{k}, \frac{1}{k-1}]$: type $k-1$, pack $k-1$ per bin
- Size $\in (0, \frac{1}{k}]$: use Next Fit

The HARMONIC algorithm

- Items of type i are packed i per bin for $i = 1, \ldots, k-1$
- Items of type k are packed using Next Fit: use one bin until next item does not fit, then start a new bin
- How do we analyze such an algorithm?

Notes:

- Items of different types are packed independently
- Size of an item is irrelevant, only its type matters

Of course this is not true for the optimal solution.

Weighting functions

- Idea: give a weight to every type
- The weight represents the **amount of bin space** that this type occupies
- For a given input, the total weight W is then the total number of bins used
- OPT needs to pack this same amount of weight (W)
- OPT seeks to minimize the number of bins $\Leftrightarrow$ maximize the weight per offline bin

Weights for HARMONIC

- Items of type i are packed i per bin for $i = 1, \ldots, k-1 \Rightarrow$ weight is $1/i$
- Items of type k are packed using Next Fit. When an item does not fit in a bin, the bin is at least $\frac{k-1}{k}$ full (size of each item $\leq \frac{1}{k}$) $\Rightarrow$ weight of item of size x is $\frac{k}{k-1}x$
- For any input sequence σ, the number of bins that HARMONIC uses is at most the total weight of all the items $+k$

Weighting functions (ctd.)

- Denote the total weight of a given input by W
- HARMONIC uses at most $W+k$ bins
- Denote the optimal number of bins by b
- We have
$$\frac{HARMONIC(\sigma)}{\mathrm{OPT}(\sigma)} = \frac{W+k}{b}$$
- Asymptotic performance ratio
  = maximum weight per offline bin
- How much weight can there be in an offline bin?

A bin with maximal weight

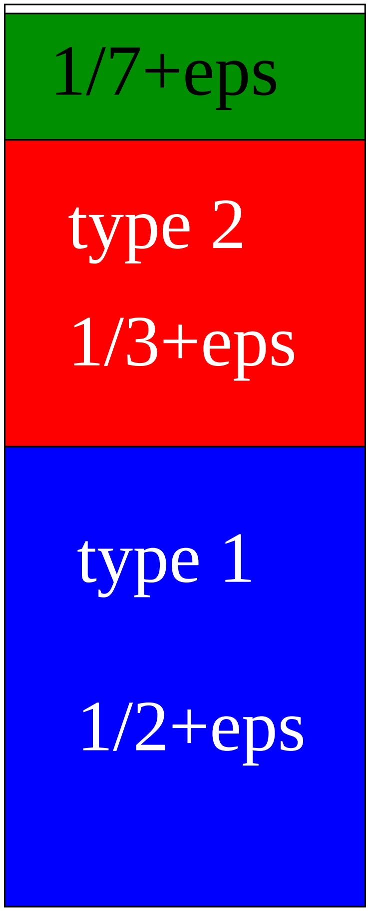
- We are looking for a set of items such that
  - their total size is at most 1
  - their weight is maximized
- We should use "smallest possible" size for each type: e.g. $\frac{1}{2} + \varepsilon$ for type 1, for $\varepsilon \to 0$
- Consider the ratio of weight to size for each type

The ratio of weight to size

- For type i, this is $(\frac{1}{i})/(\frac{1}{i+1}) = (i+1)/i$ for $i = 1,\ldots,k-1$
- For type k, this ratio is $\frac{k}{k-1}$ (item of size x has weight $\frac{k}{k-1}x$)
- These ratios are strictly decreasing from type 1 until type k
  Ratios are $2, 3/2, 4/3, \ldots, k/(k-1), k/(k-1)$
- We can find a maximum-weight set as follows:
  - start with an item of type 1 (i.e. size $\frac{1}{2}+\varepsilon$)
  - repeatedly add the largest item which will fit

A set of maximum weight

Type	size	weight	
1	1/2	1	
2	1/3	1/2	
3	1/4	–	(does not fit)
4	1/5	–	
5	1/6	–	
6	1/7	1/6	
...			
sum		1.691	



Lower bound for bounded space algorithms

- We have found the bin with maximum weight
- We can turn this into a lower bound for BS algorithms
- Each item in the bin occurs n times ($n \to \infty$)
- A bounded space algorithm must pack almost all items in bins that have one type of item – i.e., like HARMONIC!
- Thus, the number of bins needed for each n-tuple is n times the weight of this item
- Total amount of bins needed is n times total weight of input
- OPT can pack these items in n bins

Weighting functions: summary

A weighting function can be used:

- to find an upper bound for a bin packing algorithm
- to find a lower bound for bounded space algorithms

Notes:

- Finding the right weighting function can be hard
- For some problems, we can give an optimal bounded space algorithm, but *we do not know its performance ratio!*

A general lower bound

What about non-bounded space algorithms?

The previous lower bound construction does not work!

We will show a lower bound of 3/2.

A general lower bound

Consider the following input:

- n items of size $1/7+\varepsilon$ (small)
- n items of size $1/3+\varepsilon$ (medium-sized)
- n items of size $1/2+\varepsilon$ (large)

Depending on what the algorithm does, the input may already end after the first or second phase.

The small items can be packed in $n/6$ bins.

ALG is better than $3/2$-competitive? Then it uses at most $n/4$ bins for these items

General lower bound (2)

- We consider how ALG packs the small items
- It can use bins with $1, 2, \ldots, 6$ small items
- It wants to leave space for future items
- E.g. no bin will have 5 small items…
- The small and medium items ($1/3+\varepsilon$) together can be packed in $n/2$ bins
- Each bin contains two small and two medium items
- ALG may not use more than $3n/4$ bins for these items

General lower bound (3)

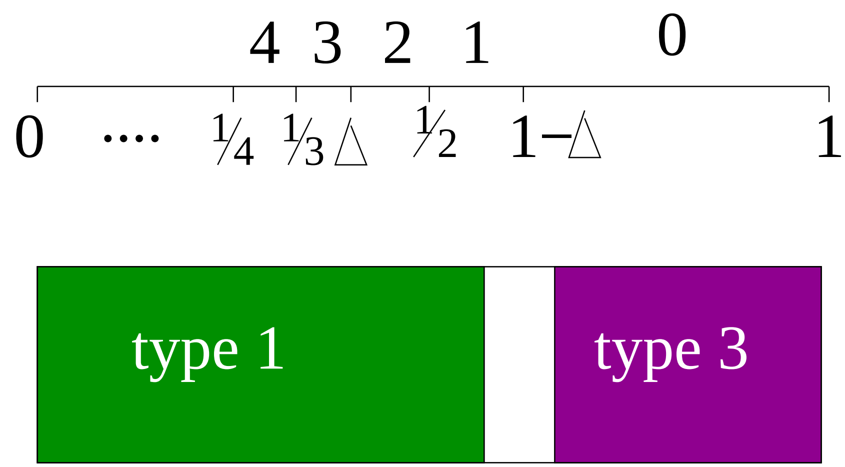
- Let a set of items of total size at most 1 be a configuration
- We can determine bounds on how often ALG uses each configuration
- E.g. it cannot use only bins with 1 small item in the first phase (then it needs n bins for them and the input stops)
- This gives us a system of equations
- Working it out, we find that ALG cannot be better than $3/2$-competitive

Improving on Harmonic

- The upper bound for Harmonic is given by a bin with maximum weight
- This weight is $1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{42} + \cdots = 1.691$
- The highest contribution is from type 1, items in $(1/2, 1]$
- A bin with an item of size $1/2 + \varepsilon$ is half empty
- We would like to avoid this situation and use the wasted space for other items
- Idea: subdivide the interval $(1/2, 1]$ in two types

Refined Harmonic

- Use a parameter $\Delta \in (1/3, 1/2)$
- Define extra types:
  0. size $> 1 - \Delta$
  1. size $\in (1/2, 1 - \Delta]$
  2. size $\in (\Delta, 1/2]$
  3. size $\in (1/3, \Delta]$
  4. size $\in (1/4, 1/3]$, etc.



- The idea is to combine type 3 with type 1
- However, we don't know how many items of any type will arrive

Refined Harmonic (2)

- We need an additional parameter α
- This parameter indicates which *fraction* of the type 3 items are placed alone in bins (red items)
- The remaining items are paired into bins (blue items)
- Whenever a type 3 item arrives, we determine its color by considering the current ratio of red to blue items ($\geq \alpha / \leq \alpha$)
- Red: place item in a bin with a type 1 item, or in a new bin
- Blue: place item in a bin with a blue item, or in a new bin

Refined Harmonic (3)

- Whenever a type 1 item arrives, we check if a bin with a red item is available
- If not, we pack it in a new bin (and keep it open for a possible red item)
- If we pick α too large, no items of type 1 will arrive
- If we pick α too small, many items of type 1 arrive
- We need to find good values for α and Δ
- How can we show an upper bound?

Weighting functions for RH

- Idea: define two weighting functions
- Let
  - x be the number of bins with red items
  - y be the number of type 1 items
  - z be the number of bins with other items
- Refined Harmonic uses

$$\max(x,y)+z = \max(x+z, y+z)$$

  bins in total
- We define one weighting function for each argument

Weighting functions for RH

- Suppose there are many type 1 items
- Then we can ignore the red items
- If there are N type 3 items, αN of them are red
- The rest are packed 2 per bin
- There are $(1-\alpha)N$ blue items each of weight $1/2$
- There are αN red items each of weight 0
- Average weight per item of type 3 is $(1-\alpha)/2$

Weighting functions for RH

- Suppose there are many red items
- Then we can ignore the type 1 items
- If there are N type 3 items, αN of them are red
- The rest are packed 2 per bin
- There are $(1-\alpha)N$ blue items each of weight $1/2$
- There are αN red items each of weight 1
- Average weight per item of type 3 is $(1+\alpha)/2$

Weighting functions for RH

Type	Interval	W	V
0	$(1-\Delta, 1]$	1	1
1	$(1/2, 1-\Delta]$	1	0
2	$(\Delta, 1/2]$	$1/2$	$1/2$
3	$(1/3, \Delta]$	$(1-\alpha)/2$	$(1+\alpha)/2$
3	$(1/4, 1/3]$	$1/3$	$1/3$
	$\vdots$		
18	$(1/19, 1/18]$	$1/18$	$1/18$
19	$(0, 1/19]$	$\frac{19}{18}x$	$\frac{19}{18}x$

Upper bound for RH

- For both weighting functions, we need to find a maximum weight bin
- The overall maximum weight will be an upper bound for RH
- We can choose α and Δ such that the maximum is minimized
- This gives an upper bound of 1.639
- With bounded space, better than 1.691 is impossible

Current state of research

- Using a longer lower bound structure, it is possible to show a lower bound of 1.5401 (van Vliet, 1992)
- Using a MUCH more complicated structure than RH, an upper bound of 1.58889 can be shown (Seiden, 2001: HARMONIC++)
- Improving either of these bounds is an open problem

Randomization

- Does not help!
- The lower bounds work in phases
- In each phase, only one type of item arrives
- The algorithm distributes them in some way
- Deterministic: fractions ($1/3$ of the items is packed in this way, $\ldots$)
- Randomized: expectations (The expected fraction of items packed in this way is $1/3, \ldots$)
- Final result is the same