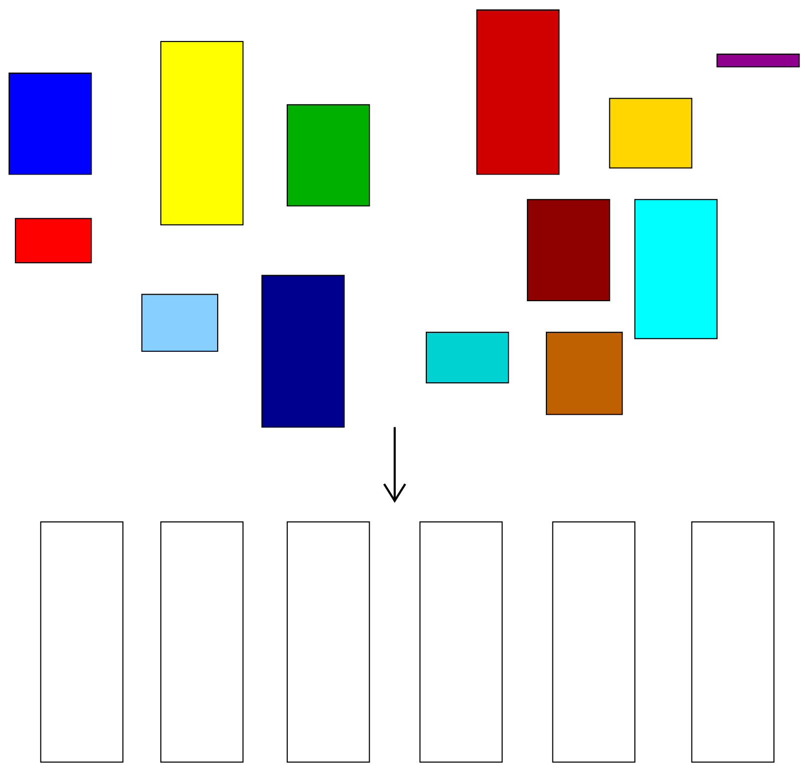
Bin packing and scheduling

Overview

- Bin packing: problem definition
- Simple 2-approximation (Next Fit)
- Better than 3/2 is not possible
- Asymptotic PTAS
- Scheduling: minimizing the makespan (repeat)
- PTAS

Bin packing: problem definition

- Input: n items with sizes $a_1, \ldots, a_n \in (0,1]$
- Goal: pack these items into a minimal number of bins
- Each bin has size 1



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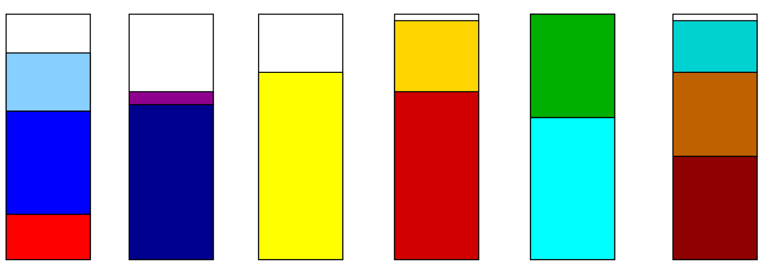
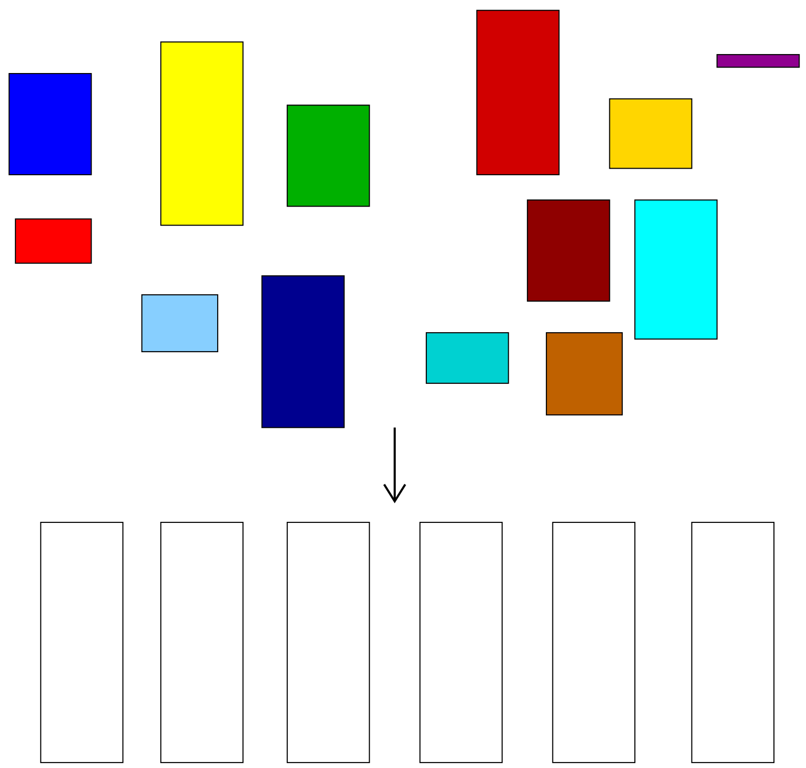


A simple (online!) algorithm is **Next Fit**

- Place items in a bin until next item does not fit
- Then, close the bin and start a new bin
- Approximation ratio is 2 (and competitive ratio is also 2)



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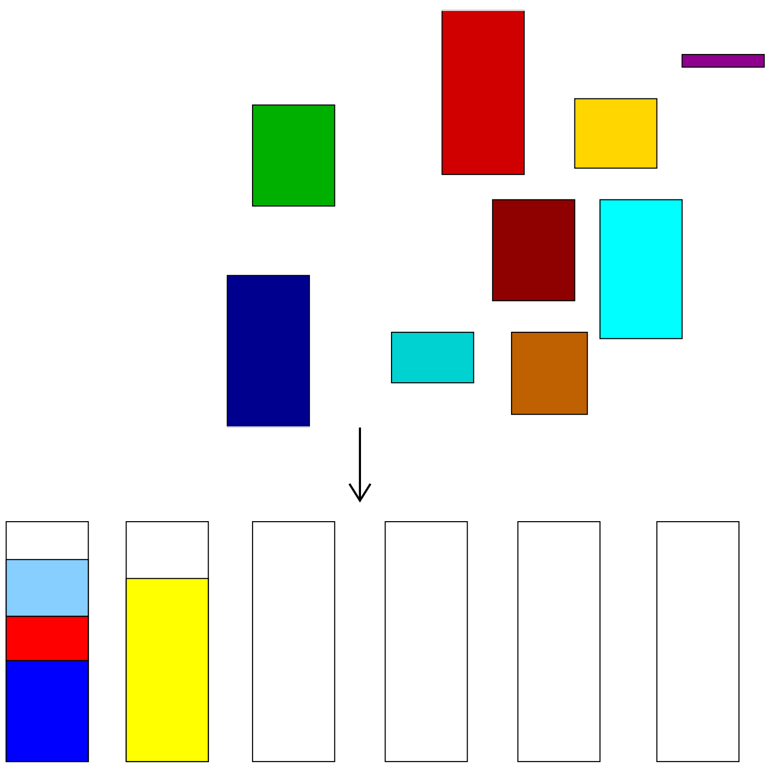

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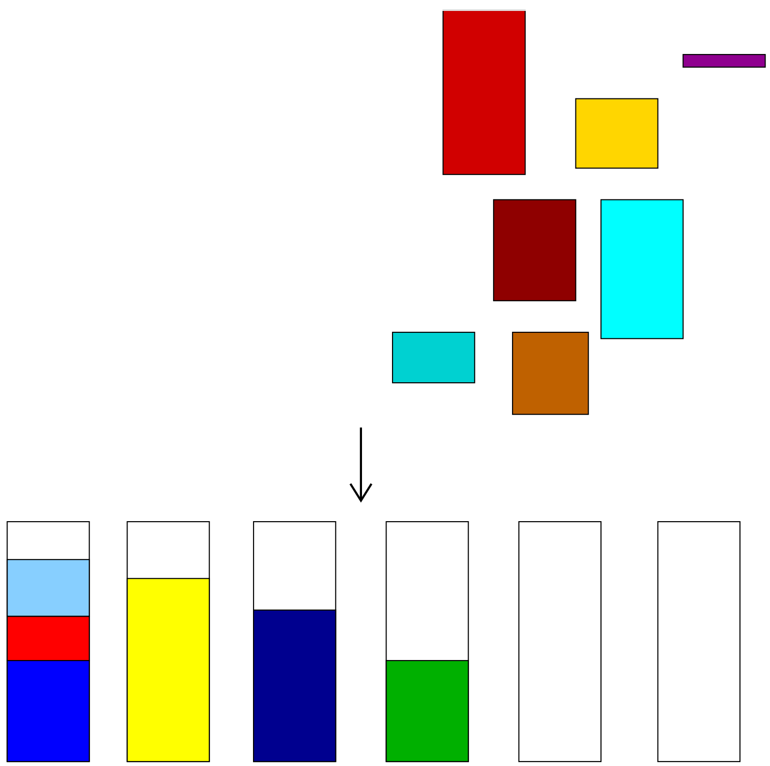

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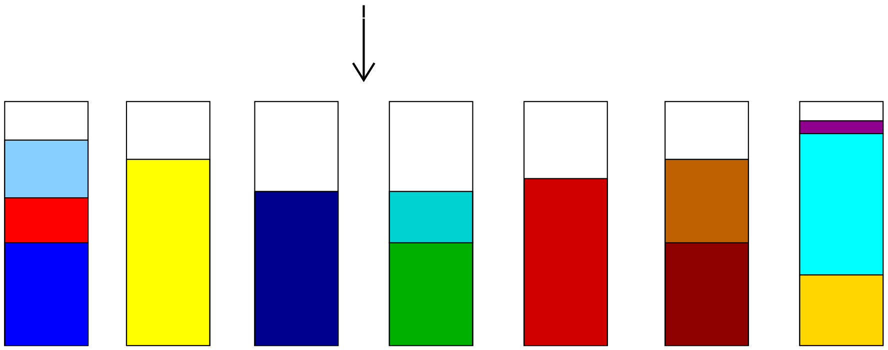


A general lower bound

It is possible to improve on Next Fit, for instance by using First Fit.

However...

Lemma 1. *There is no algorithm with approximation ratio below $3/2$, unless P=NP*

Proof: reduction from PARTITION

PARTITION = given a set of items of total size B, can you split them into two subsets of equal size?

This problem is known to be NP-hard

The reduction

- Input is a set of items of total size 2
- Does this input fit in two bins?
- An algorithm with approximation ratio $< 3/2$ must give a packing in two bins (not three) if one exists
- Thus, it must solve PARTITION, which is NP-hard

The asymptotic performance ratio

- This result deals with “small” inputs
- What about more reasonable instances?
- For a given input I, let $\mathrm{OPT}(I)$ denote the optimal number of bins needed to pack it
- Idea: we are interested in the worst ratio for large inputs

The asymptotic performance ratio

$$\frac{\mathcal{A}(I)}{\mathrm{OPT}(I)}$$

The asymptotic performance ratio

$$\sup_I \frac{\mathcal{A}(I)}{\mathrm{OPT}(I)}$$

The asymptotic performance ratio

$$R_{\mathcal{A}}^{\infty} = \limsup_{n\to\infty} \sup_{I} \left\{ \frac{\mathcal{A}(I)}{\text{OPT}(I)} \,\middle|\, \text{OPT}(I) = n \right\}.$$

Note: we also use this measure to compare online algorithms

A positive result

We can show the following theorem:

Theorem 1. *For any* $\varepsilon > 0$, *there is an algorithm* $\mathcal{A}_\varepsilon$ *that runs in time polynomial in n and for which*

$$\mathcal{A}_\varepsilon(I) \leq (1+2\varepsilon)\textsc{opt}(I)+1 \qquad \forall I$$

Meaning: you can get as close to the optimal solution as you want

The degree of the polynomial depends on ε: the closer you want to get to the optimum, the more time it takes

A simple case

- All items have size at least ε
  $\rightarrow$ at most $M = \lfloor 1/\varepsilon \rfloor$ items fit in a bin
- There are only K different item sizes
  $\rightarrow$ at most $R = \binom{M+K}{M}$ bin types
  (M "items" in a bin, $K+1$ options per item)
- We know that at most n bins are needed to pack all items
  $\rightarrow$ at most $\binom{n+R}{R}$ feasible packings need to be checked
- We can do this in polynomial (in n) time

Note: this is **extremely** impractical

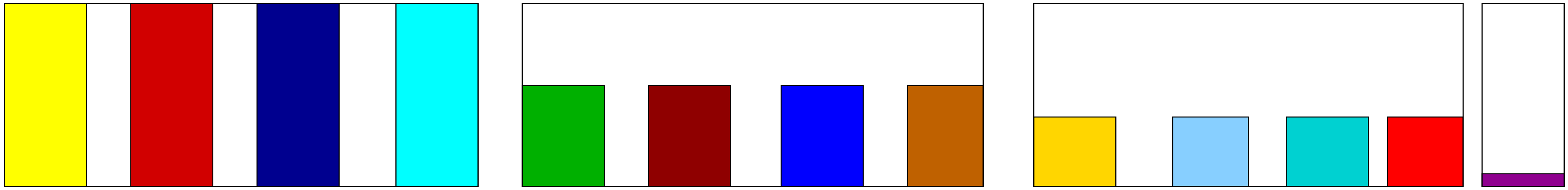
Example: $n = 50, K = 6, \varepsilon = 1/3$, then $1.98 \cdot 10^{37}$ options

Generalizing the simple case (1)

Suppose there are more different item sizes (at most n).

Do the following:

- Sort items
- Make groups containing $\lfloor n\varepsilon^2 \rfloor$ items
- In each group, round sizes up to largest size in group



Generalizing the simple case (2)

So far we had a lower bound of ε on the item sizes.

How do we pack instances that also contain such small items?

- Ignore items $< \varepsilon$ (small items) at the start
- Apply algorithm on remaining items
- Fill up bins with small items

The small items

- If all small items fit in the bins used to pack L, we use no more than $\text{OPT}(L)$ bins
- Else, all bins except the last are full by at least

$$1-\varepsilon$$

- $\text{OPT}(I)$ is at least the total size of all the items

$$\text{OPT}(I) \geq \frac{\text{ALG}(I)-1}{1-\varepsilon} \to \text{ALG}(I) \leq (1+2\varepsilon)\text{OPT}(I)+1$$

This proves the theorem.

A better solution

- The core algorithm is very much brute force
- We can improve by using dynamic programming
- We no longer need a lower bound on the sizes
- There are k different item sizes
- An input is of the form $(n_1, \ldots, n_k)$
- We want to calculate $\text{OPT}(n_1, \ldots, n_k)$, the optimal number of bins to pack this input

The base case

- Consider an input $(n_1,\ldots,n_k)$ with $n=\sum n_j$ items
- Determine set of k-tuples (subsets of the input) that can be packed into a single bin
- That is, all tuples $(q_1,\ldots,q_k)$ for which $\text{OPT}(q_1,\ldots,q_k)=1$ and for which $0\le q_j\le n_j$ for all j
- There are at most n^k such tuples, each tuple can be checked in linear time
- (Exercise: there are at most $(n/k)^k$ such tuples)
- Denote this set by Q

Dynamic programming

- For each k-tuple $q \in Q$, we have $\mathrm{OPT}(q) = 1$
- Calculate remaining values by using the recurrence

$$\mathrm{OPT}(i_1, \ldots, i_k) = 1 + \min_{q \in Q} \mathrm{OPT}(i_1 - q_1, \ldots, i_k - q_k)$$

- Exercise: think about the order in which we can calculate these values
- Each value takes $O(n^k)$ time, so we can calculate all values in $O(n^{2k})$ time
- This gives us in the end the value of $\mathrm{OPT}(n_1, \ldots, n_k)$

Advantages

- Much faster than simple brute force
- Can be used to create PTAS for load balancing!

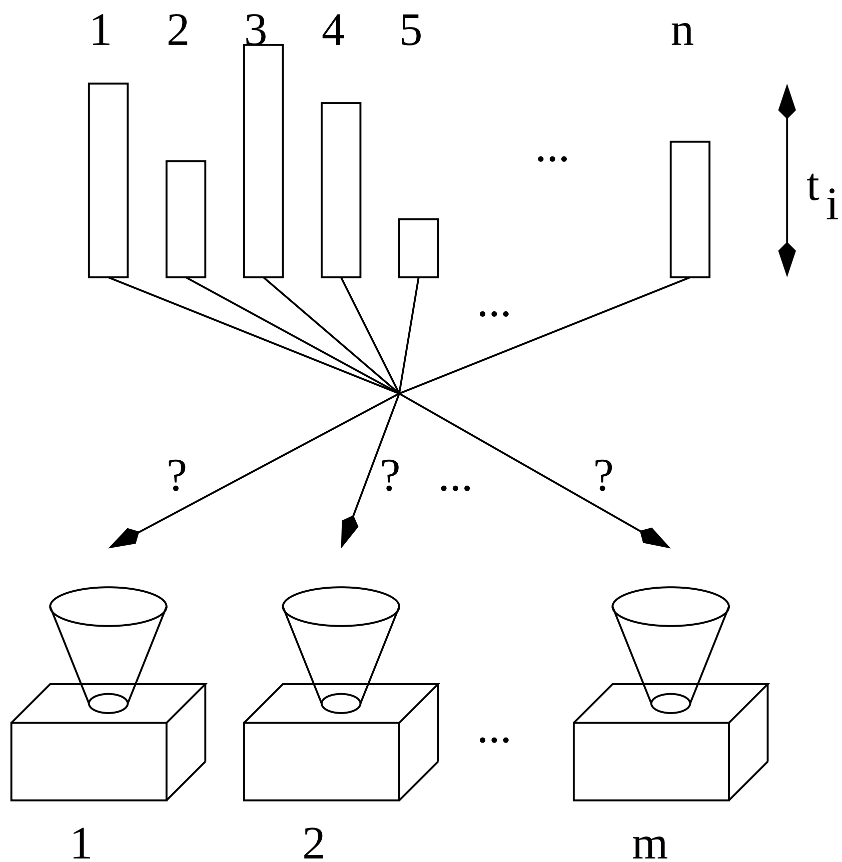
PTAS from *Algorithmentechnik*:

- separate ℓ largest jobs
- assign them optimally
- add smallest jobs greedily

Time $O(m^\ell + n)$. For $\varepsilon = 1/3, m = 15$ we have $m^\ell = 8.5 \cdot 10^{32}$.

This PTAS could only be used for very small m and large ε.

Scheduling Independent Weighted Jobs on Parallel Machines



$\mathbf{x}(j)$: Machine where job j is executed

L_i: $\sum_{\mathbf{x}(j)=i} t_j$, load of machine i

Objective: Minimize Makespan $L_{\max} = \max_i L_i$

Details: **Identical** machines, independent jobs, known processing times, offline

NP-hard

Old results

- Greedy algorithm is $(2-\frac{1}{m})$-approximation
- LPT is $(4/3-\frac{1}{3m})$-approximation

New result: PTAS for load balancing

Idea: find optimal makespan using binary search

A step in the binary search

- Let current guess for the makespan be t

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- Remove “small” items: smaller than $t\varepsilon$
- Round remaining sizes down using geometric rounding
- Find optimal solution in bins of size t
- Extend to near-optimal solution for entire input
- More than m bins needed: increase t
- At most m bins needed: decrease t

Geometric rounding

- Each large item is rounded down so that its size is of the form

$$t\varepsilon(1+\varepsilon)^i$$

  for some $i \geq 0$

- Since large items have size at least $t\varepsilon$, this leaves $k = \lceil \log_{1+\varepsilon} \frac{1}{\varepsilon} \rceil$ different sizes

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We add the small items to those bins (and to new bins if needed)

Comparing to the optimal solution

We use bins of size $t(1+\varepsilon)$

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Proof: If we need no extra bins for the small items, we have found an optimal packing for the rounded down items in bins of size t

Else, all bins (except maybe the last one) are full by at least t $\square$

Connection between bin packing and scheduling

- We look for the smallest t such that we can pack the items in m bins (machines).
- Suppose that we can find the exact value of t
- Then, OPT also needs m bins of size t to pack these items
- In other words, the makespan on m machines is at least t. (For smaller t, the items cannot all be placed below a level of t.)

The binary search

- We start with the following lower bound on OPT:

$$LB = \max\left\{\sum t_j/m, \max_j t_j\right\}$$

- Greedy gives a schedule which is at most twice this value, this is an upper bound for OPT
- Each step of the binary search halves this interval
- We repeat until the length of the interval is at most $\varepsilon \cdot LB$
- Let T be the upper bound of this interval
- Then $T \leq \text{OPT} + \varepsilon \cdot LB \leq (1+\varepsilon)\cdot\text{OPT}$
- The makespan of our algorithm is at most $(1+\varepsilon)T$

Conclusion

Theorem 2. *For any* $\varepsilon > 0$, *there is an algorithm* $\mathcal{A}_\varepsilon$ *which works in polynomial time in n and which gives a schedule with makespan at most* $(1+\varepsilon)^2\text{OPT} < (1+3\varepsilon)\text{OPT}$.

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$n = 50, \varepsilon = 1/3 \rightarrow 7.8 \cdot 10^{13}$ options

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- The running time of our algorithm is $O(\lceil \log_2 \frac{1}{\varepsilon} \rceil n^{2k})$
- There is no FPTAS for this problem