Vorlesung Graphenzeichnen:

Order in the Underground *or* How to Automate the Drawing of Metro Maps

Martin Nöllenburg

Lehrstuhl für Algorithmik I

26.06.2008

Outline

- Modeling the Metro Map Problem
  - What is a Metro Map?
  - Hard and Soft Constraints
- NP-Hardness: Bad News—Nice Proof
  - Rectilinear vs. Octilinear Drawing
  - Reduction from PLANAR 3-SAT
- MIP Formulation & Experiments
  - Mixed-Integer Programming Formulation
  - Experiments
  - Labeling

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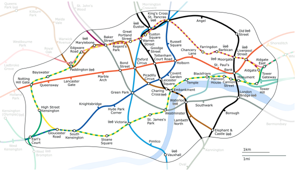

- schematic diagram for public transport

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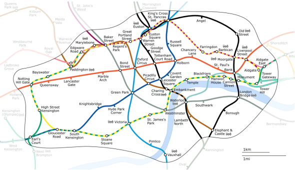

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- visualizes lines and stations

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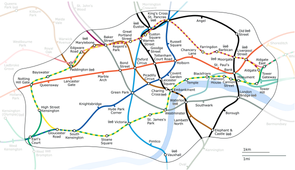

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  - “How do I get from A to B?”
  - “Where to get off and change trains?”

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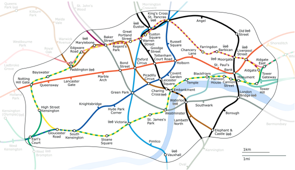

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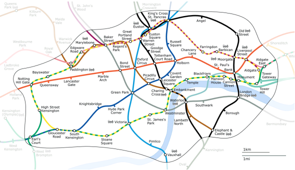

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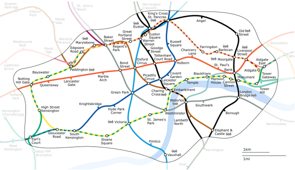

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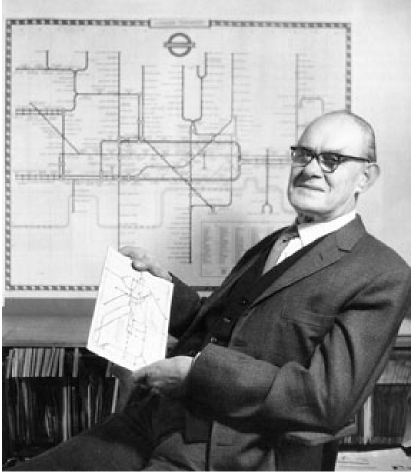
- improves readability
- compromise between
  schematic road map ↔ abstract graph

Why Automate Drawing Metro Maps?

- current maps designed manually



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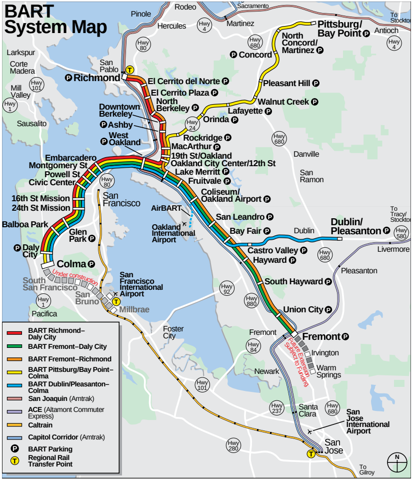

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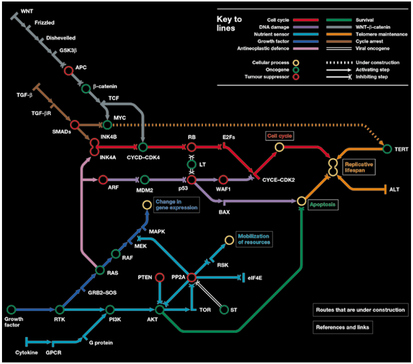

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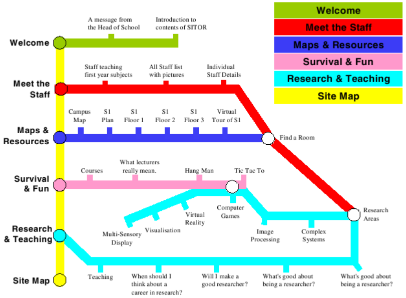

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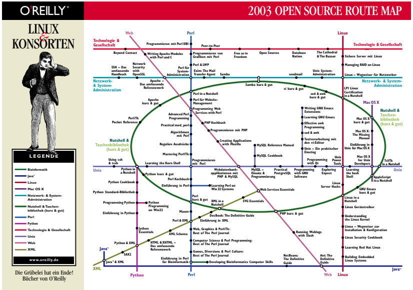

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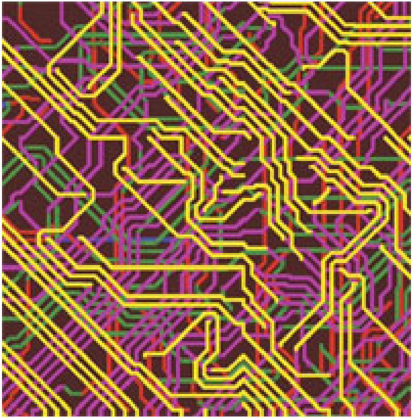

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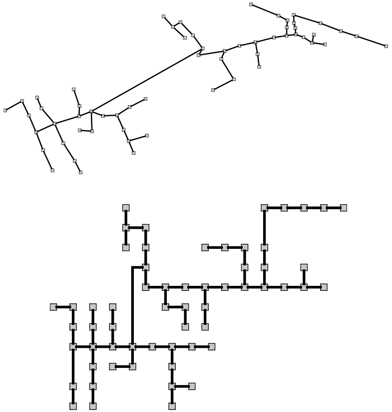
- VLSI: X-architecture
- drawing sketches
  [Brandes et al. '03]



More Formally

The Metro Map Problem

Given: planar embedded graph $G = (V, E)$, $V \subset \mathbb{R}^2$,
line cover $\mathcal{L}$ of paths or cycles in G (the metro lines),

Goal: draw G and $\mathcal{L}$ nicely.

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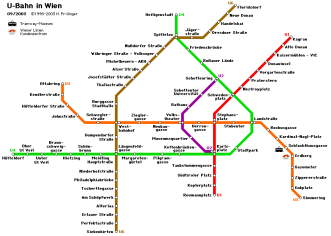
  - *hard* constraints – must be fulfilled,
  - *soft* constraints – should hold as tightly as possible.

Hard Constraints

(H1) preserve embedding of G



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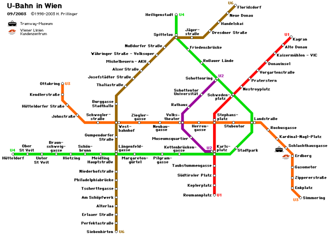
i.e. horizontal, vertical or diagonal (45 degrees)



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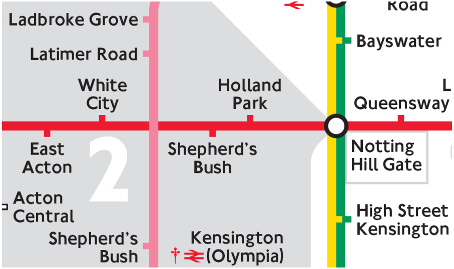
(H4) keep edges d_{min} away from non-incident edges

Soft Constraints

(S1) draw metro lines with few bends



Soft Constraints

(S1) draw metro lines with few bends

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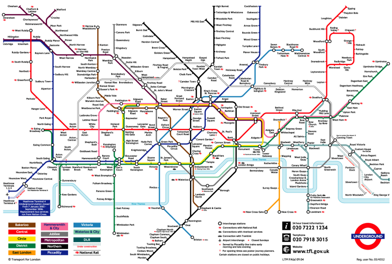
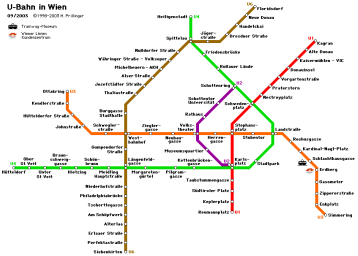


Soft Constraints

(S1) draw metro lines with few bends

(S2) keep total edge length small

(S3) draw each octilinear edge similar to its geographical orientation: keep its relative position



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RECTILINEARGRAPHDRAWING Decision Problem

Given a planar embedded graph G with max degree 4.
Is there a drawing of G that

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By reduction to a flow problem. □

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Proof.

By reduction to a flow problem. □

Our Problem

METROMAPLAYOUT Decision Problem

Given a planar embedded graph G with max degree 8.
Is there a drawing of G that

- preserves the embedding,
- uses straight-line edges,
- is octilinear?

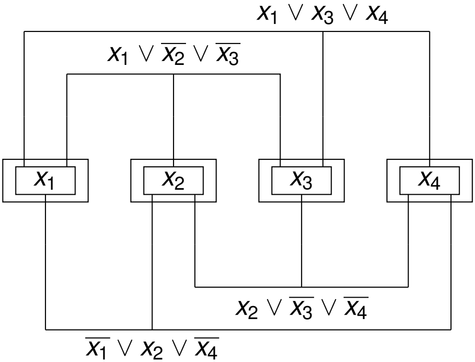
Theorem (Nöllenburg MSc'05)

METROMAPLAYOUT *is NP-hard.*

Proof.

By Reduction from PLANAR 3-SAT to METROMAPLAYOUT. □

Outline of the Reduction



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$(x_1 \vee x_3 \vee x_4) \wedge (x_1 \vee \overline{x_2} \vee \overline{x_3}) \wedge \dots$

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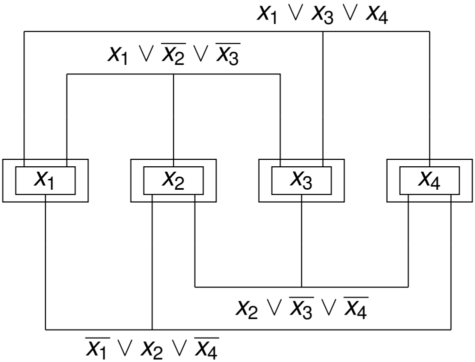

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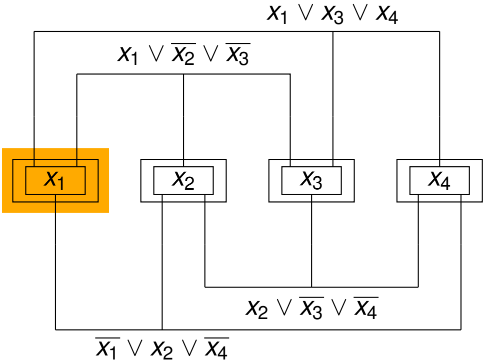
Goal: planar embedded graph G_φ with: G_φ has a metro map drawing $\Leftrightarrow$ φ satisfiable.

Outline of the Reduction



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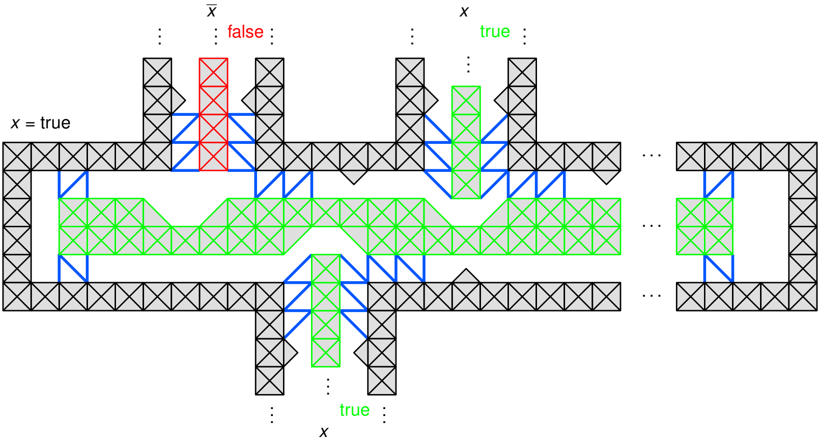

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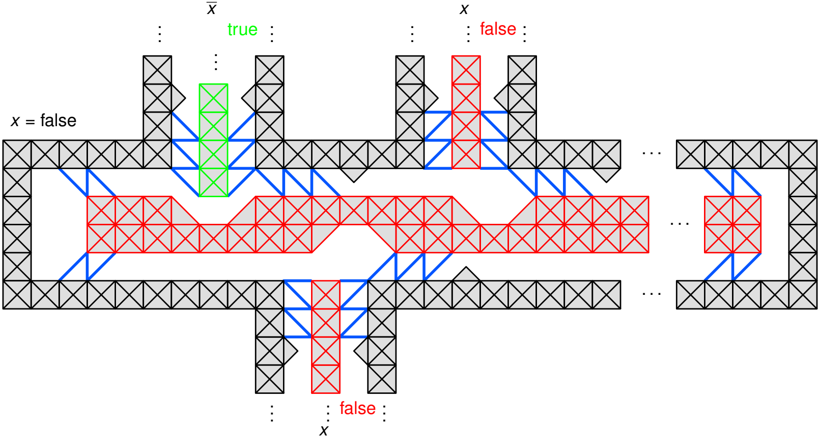

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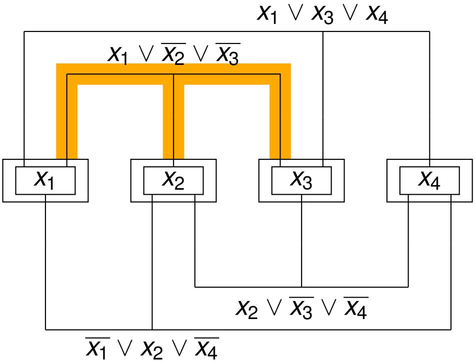

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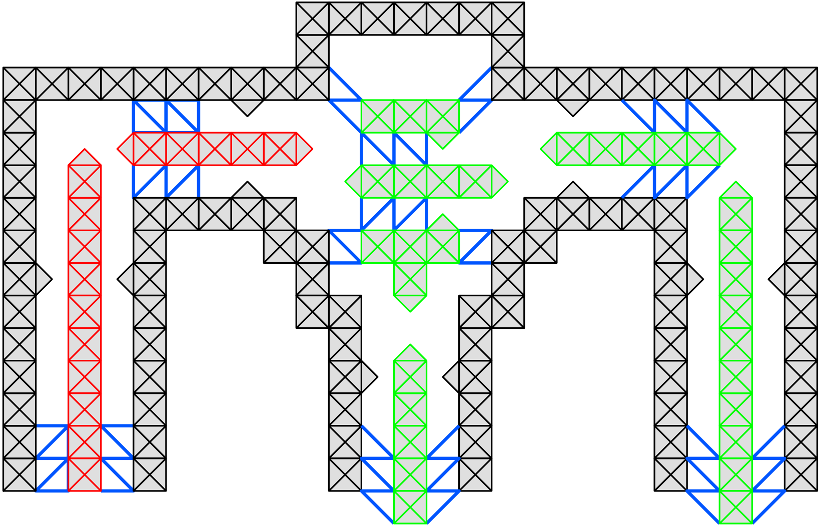

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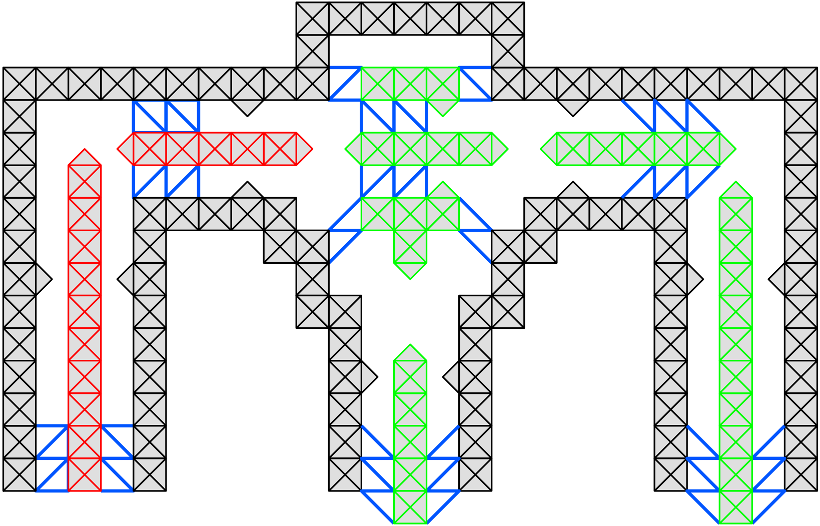

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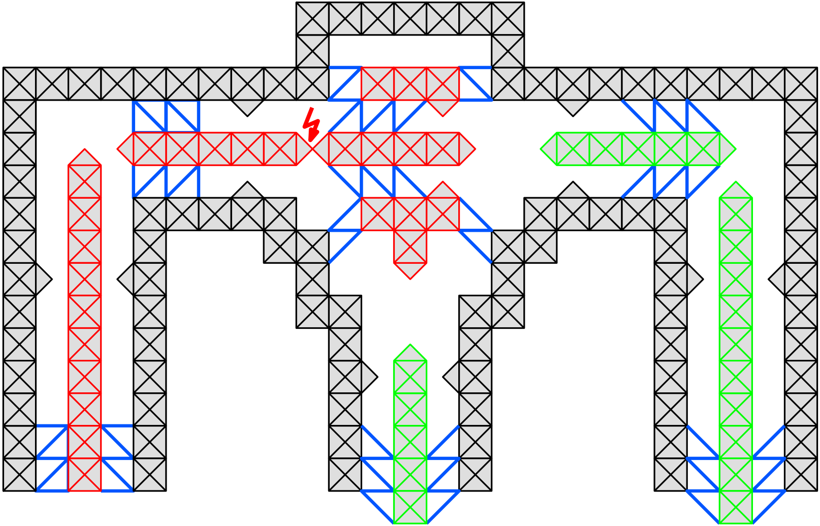

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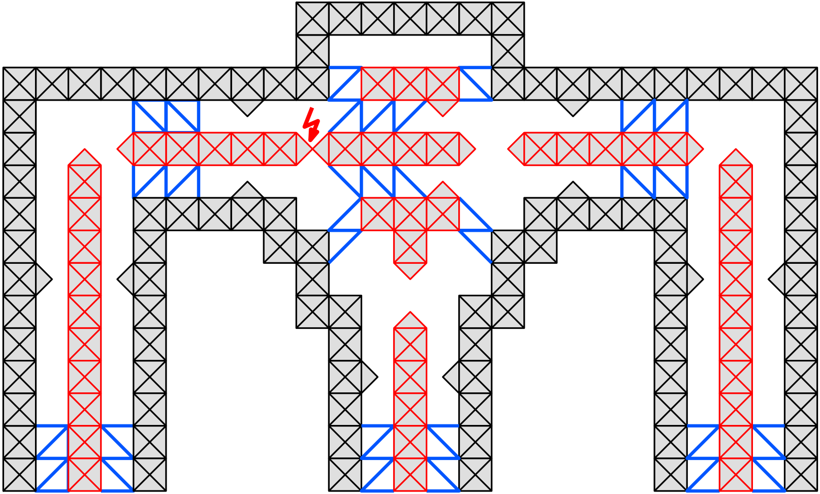

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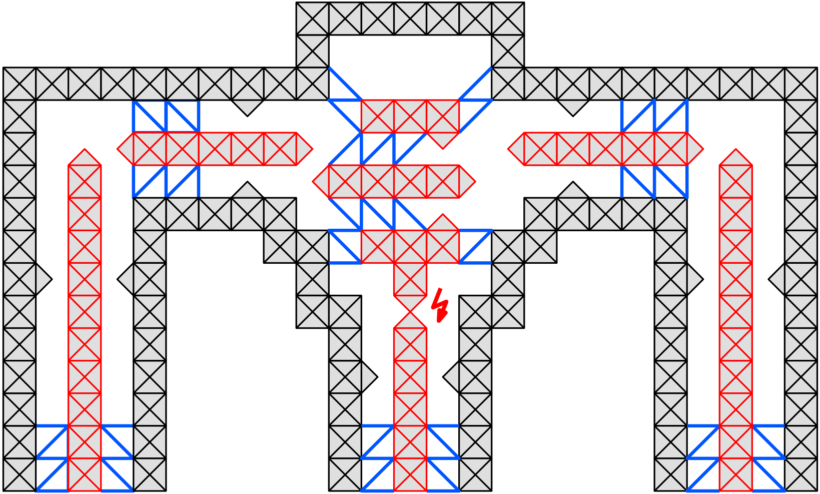

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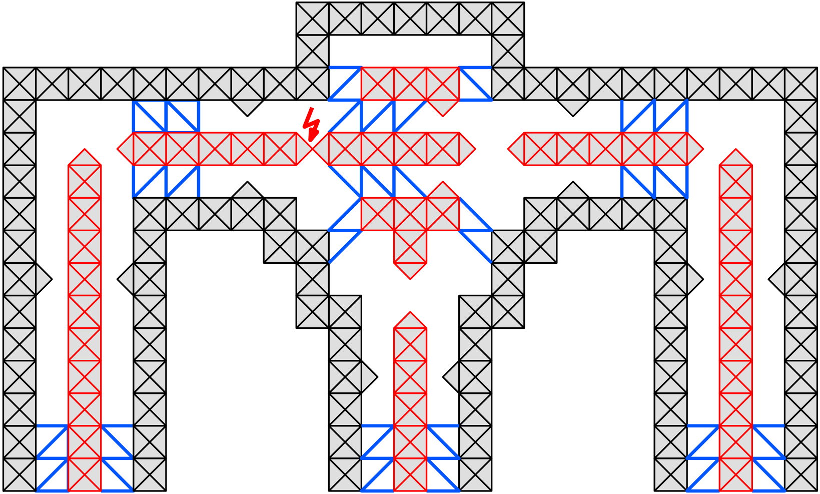

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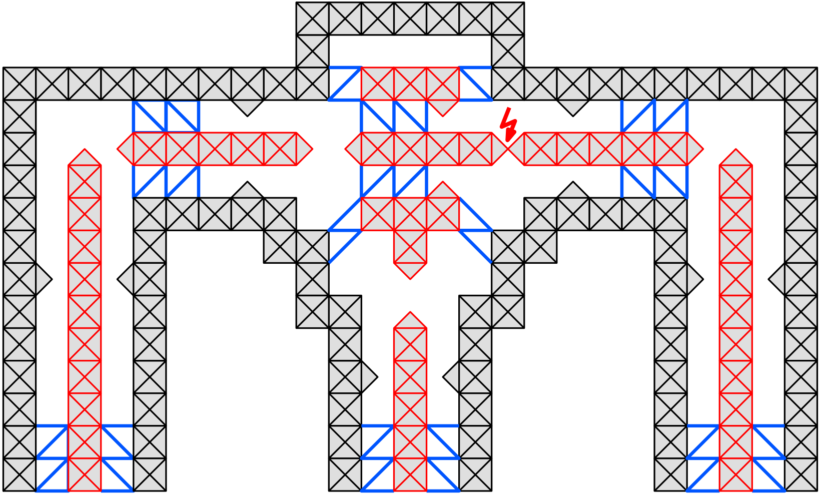
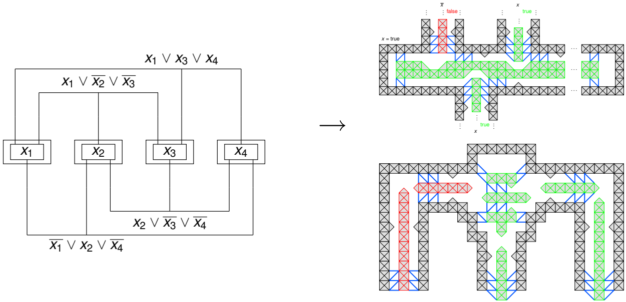


Summary of the Reduction



- Indeed we have:
  - φ satisfiable $\Rightarrow$ corresponding MM drawing of G_φ
  - G_φ has MM drawing $\Rightarrow$ satisfying truth assignment of φ

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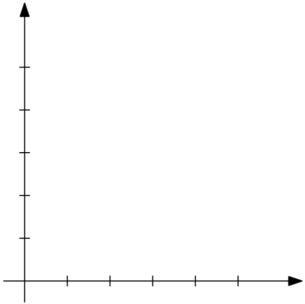
  - real-valued variables



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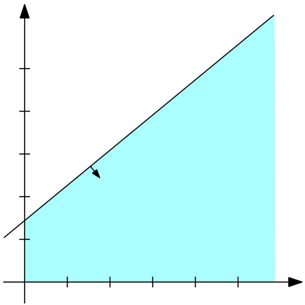

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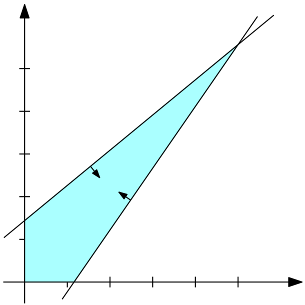

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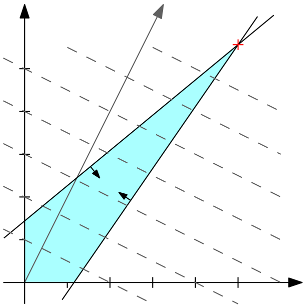

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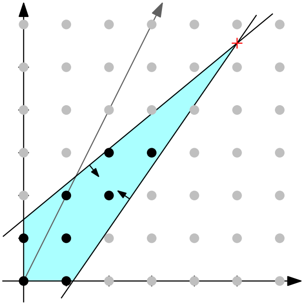
  - NP-hard in general
  - still practical method for many hard optimization problems



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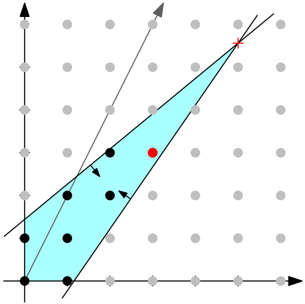

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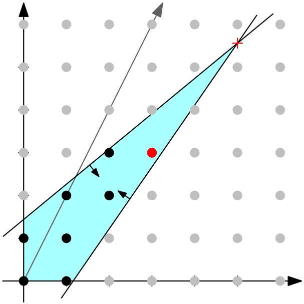
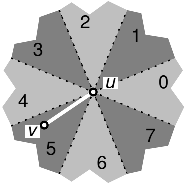


Theorem (Nöllenburg & Wolff GD'05)

The problem MetroMapLayout can be formulated as a MIP s.th.

hard constraints → linear constraints
soft constraints → objective function

Definitions: Sectors and Coordinates



Sectors

– for each vtx. u partition plane into sectors 0–7

- here: $\mathrm{sec}(u, v) = 5$ (input)

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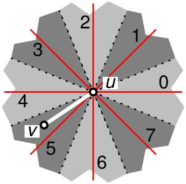

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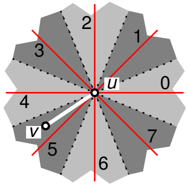

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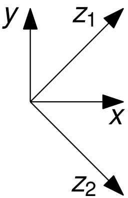


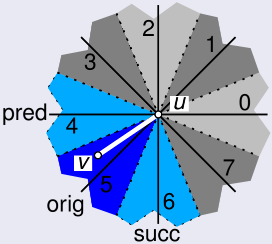
Coordinates

assign z_1- and z_2-coordinates to each vertex v:

- $z_1(v) = x(v) + y(v)$
- $z_2(v) = x(v) - y(v)$

Octilinearity and Relative Position

Goal



Draw edge *uv*

- octilinearly
- with minimum length ℓ_{uv}
- restricted to 3 directions

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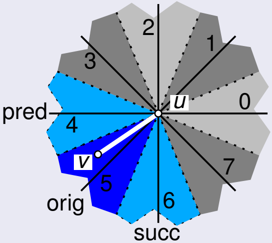

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How to model this using linear constraints?

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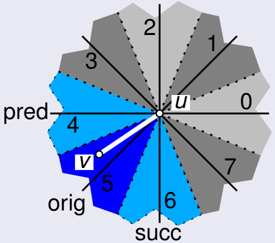

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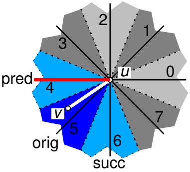
How to model this using linear constraints?

Binary Variables

$$\alpha_{\text{pred}}(u, v) + \alpha_{\text{orig}}(u, v) + \alpha_{\text{succ}}(u, v) = 1$$

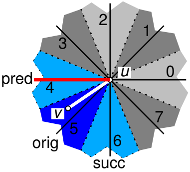
Octilinearity and Relative Position



Predecessor Sector

$$
\begin{aligned}
y(u) - y(v) &\leq M(1 - \alpha_{\text{pred}}(u, v)) \\
-y(u) + y(v) &\leq M(1 - \alpha_{\text{pred}}(u, v)) \\
x(u) - x(v) &\geq -M(1 - \alpha_{\text{pred}}(u, v)) + \ell_{uv}
\end{aligned}
$$

Octilinearity and Relative Position



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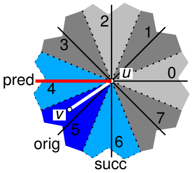
How does this work?

Octilinearity and Relative Position



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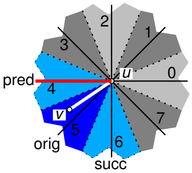
\end{aligned}
$$

How does this work?

Case 1: $\alpha_{\text{pred}}(u, v) = 0$

$$
\begin{aligned}
y(u) - y(v) &\leq M \\
-y(u) + y(v) &\leq M \\
x(u) - x(v) &\geq \ell_{uv} - M
\end{aligned}
$$

Octilinearity and Relative Position



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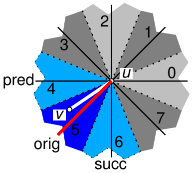
$$\begin{aligned} y(u) - y(v) &\leq M(1 - \alpha_{\text{pred}}(u,v)) \\ -y(u) + y(v) &\leq M(1 - \alpha_{\text{pred}}(u,v)) \\ x(u) - x(v) &\geq -M(1 - \alpha_{\text{pred}}(u,v)) + \ell_{uv} \end{aligned}$$

How does this work?

Case 2: $\alpha_{\text{prev}}(u,v) = 1$

$$\begin{aligned} y(u) - y(v) &\leq 0 \\ -y(u) + y(v) &\leq 0 \\ x(u) - x(v) &\geq \ell_{uv} \end{aligned}$$

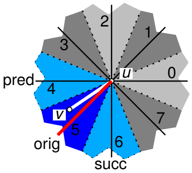
Octilinearity and Relative Position



Original Sector

$$\begin{aligned} z_2(u) - z_2(v) &\leq M(1 - \alpha_{\text{orig}}(u,v)) \\ -z_2(u) + z_2(v) &\leq M(1 - \alpha_{\text{orig}}(u,v)) \\ z_1(u) - z_1(v) &\geq -M(1 - \alpha_{\text{orig}}(u,v)) + 2\ell_{uv} \end{aligned}$$

Octilinearity and Relative Position



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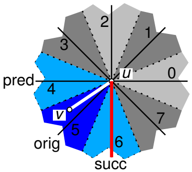


Successor Sector

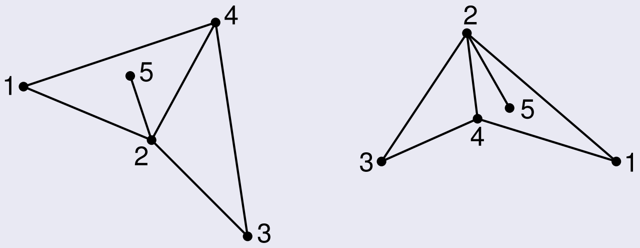
$$
\begin{aligned}
x(u) - x(v) &\leq M(1 - \alpha_{\mathrm{succ}}(u, v)) \\
-x(u) + x(v) &\leq M(1 - \alpha_{\mathrm{succ}}(u, v)) \\
y(u) - y(v) &\geq -M(1 - \alpha_{\mathrm{succ}}(u, v)) + \ell_{uv}
\end{aligned}
$$

Preserving the Embedding

Definition

Two planar drawings of G have the same *embedding* if the induced orderings on the neighbors of each vertex are equal.

Same Embedding



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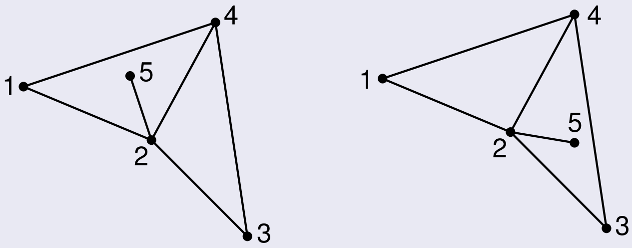
Different Embeddings



Preserving the Embedding

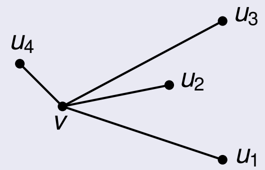
Constraints (Example)

- $N(v) = \{u_1, u_2, u_3, u_4\}$
- circular input order: $u_1 < u_2 < u_3 < u_4 < u_1$

All but one of the following inequalities must hold

$$\mathrm{dir}(v, u_1) < \mathrm{dir}(v, u_2) < \mathrm{dir}(v, u_3) < \mathrm{dir}(v, u_4) < \mathrm{dir}(v, u_1)$$

Input



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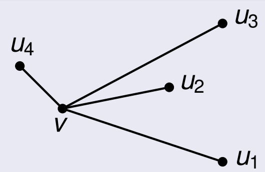
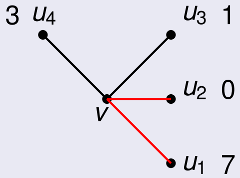


Output

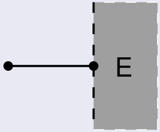


Planarity

Observation

For octilinear, straight edge e_1
non-intersecting edge e_2 must be placed

- east, northeast, north, northwest, west, southwest, south, or southeast



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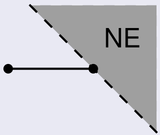

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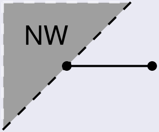

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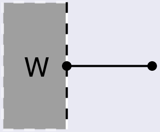

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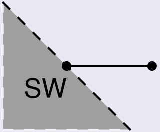

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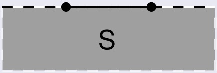

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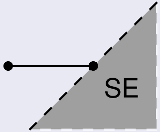

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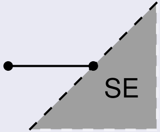

Constraints

- model as MIP with binary variables
- need planarity constraints for each pair of non-incident edges

Objective Function

Objective Function

- corresponds to soft constraints (S1)–(S3)
- weighted sum of individual cost functions

minimize $\lambda_{\text{bends}}\ \text{cost}_{\text{bends}} + \lambda_{\text{length}}\ \text{cost}_{\text{length}} + \lambda_{\text{relpos}}\ \text{cost}_{\text{relpos}}$

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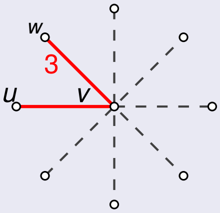
Line Bends (S1)



Edges *uv* and *vw* on a line $L \in \mathcal{L}$

- draw as straight as possible
- increasing cost $bend(u, v, w)$ for increasing acuteness of $\angle(\overline{uv}, \overline{vw})$

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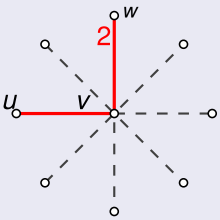

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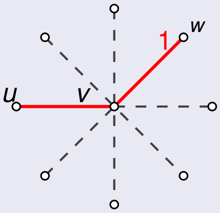

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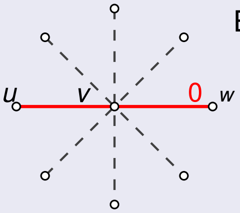

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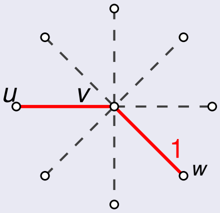

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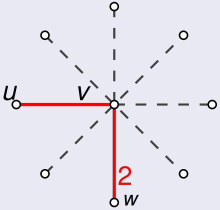

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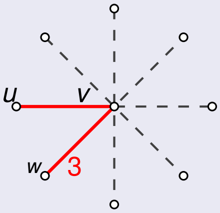

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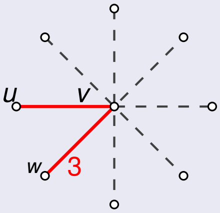

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$$\text{cost}_{\text{bends}} = \sum_{L \in \mathcal{L}} \sum_{uv, vw \in L} \text{bend}(u, v, w)$$

Objective Function

Total Edge Length (S2)

$$\text{cost}_{\text{length}} = \sum_{uv \in E} \text{length}(\overline{uv})$$

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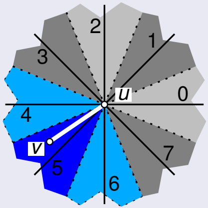

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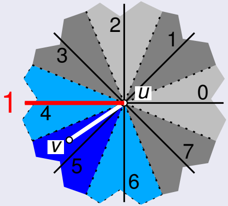

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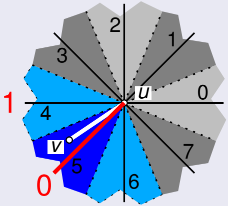

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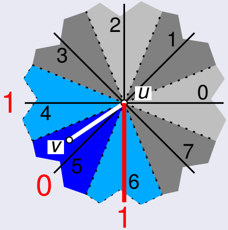

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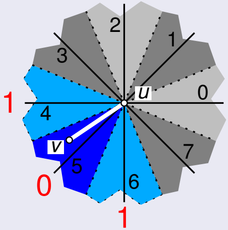

- only three directions possible
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$$\text{cost}_{\text{relpos}} = \sum_{uv \in E} \text{relpos}(uv)$$

Summary of the MIP

- hard constraints:
  - octilinearity
  - minimum edge length
  - (partially) relative position
  - preservation of embedding
  - planarity

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- models METROMAPLAYOUT as MIP
- in total $O(|V|^2)$ constraints and variables

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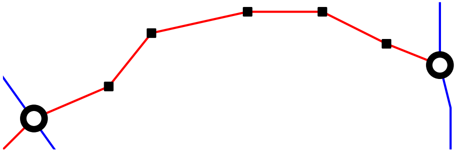
  - minimum edge length
  - (partially) relative position
  - preservation of embedding
  - planarity
- soft constraints:

  minimize $\lambda_{\text{bends}}\,\text{cost}_{\text{bends}} + \lambda_{\text{length}}\,\text{cost}_{\text{length}} + \lambda_{\text{relpos}}\,\text{cost}_{\text{relpos}}$

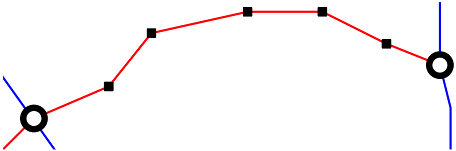
- models METROMAPLAYOUT as MIP
- in total $O(|V|^2)$ constraints and variables

Speed-Up Techniques: Reduce Graph Size



- metro graphs have many degree-2 vertices
- want to optimize line straightness

Speed-Up Techniques: Reduce Graph Size



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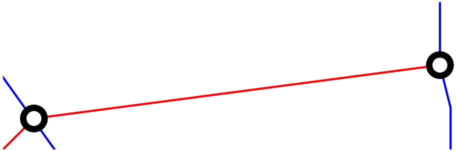
Speed-Up Techniques: Reduce Graph Size



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Speed-Up Techniques: Reduce Graph Size

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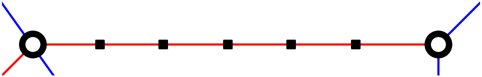

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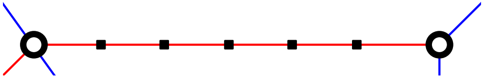
Speed-Up Techniques: Reduce Graph Size



- metro graphs have many degree-2 vertices
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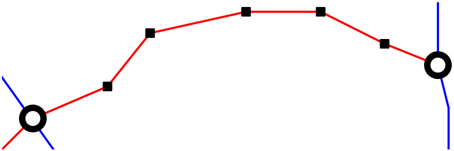
Idea 1 collapse all degree-2 vertices

- low flexibility

Speed-Up Techniques: Reduce Graph Size

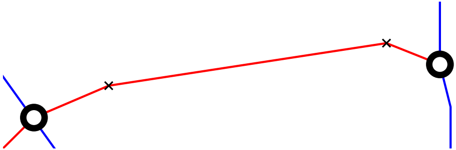


- metro graphs have many degree-2 vertices
- want to optimize line straightness

Idea 1 collapse all degree-2 vertices
- low flexibility

Idea 2 keep two *joints*

Speed-Up Techniques: Reduce Graph Size



- metro graphs have many degree-2 vertices
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Idea 1 collapse all degree-2 vertices
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Speed-Up Techniques: Reduce Graph Size

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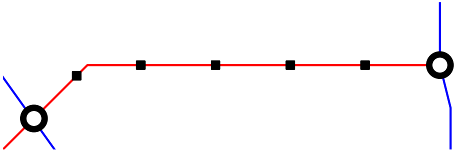
Speed-Up Techniques: Reduce Graph Size



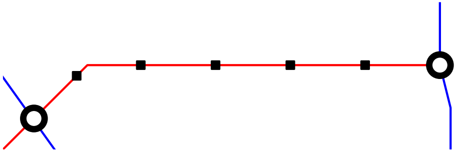
- metro graphs have many degree-2 vertices
- want to optimize line straightness

Idea 1 collapse all degree-2 vertices

- low flexibility

Idea 2 keep two *joints*

Speed-Up Techniques: Reduce Graph Size



- metro graphs have many degree-2 vertices
- want to optimize line straightness

Idea 1 collapse all degree-2 vertices
- low flexibility

Idea 2 keep two *joints*
- higher flexibility
- more similar to input

Speed-Up Techniques: Reduce MIP Size

- $O(|V|^2)$ planarity constraints (for each pair of edges...)
- in practice 95–99% of constraints

Speed-Up Techniques: Reduce MIP Size

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Observation 1

- consider only pairs of edges incident to the same face
- still $O(|V|^2)$ constraints

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Observation 2

- in practice no or only few crossings due to soft constraints

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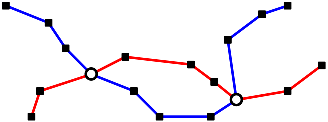

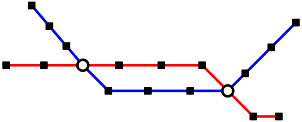

Speed-Up Techniques: Reduce MIP Size

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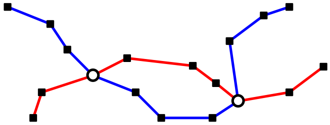
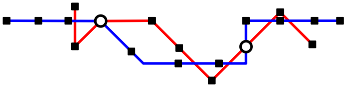




Speed-Up Techniques: Callback Functions

- MIP optimizer CPLEX offers advanced callback functions
- add required planarity constraints on the fly

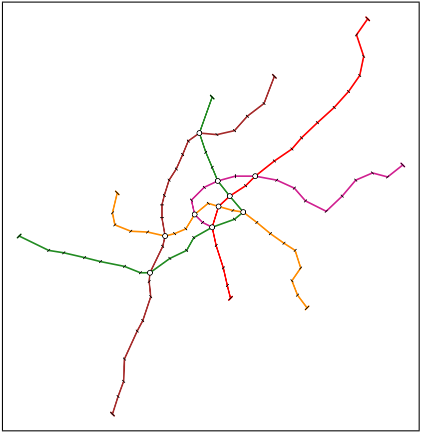
Algorithm

1. start solving MIP without planarity constraints
2. for each new solution s
    1. interrupt CPLEX
    2. if s is not planar
        - add planarity constraints for edges that intersect in s
        - reject s

       else
        - accept s
    3. continue solving the MIP (until optimal)

Results – Vienna

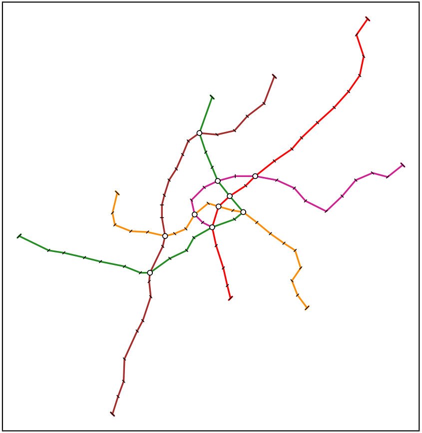
Input



Input	$\|V\|$	$\|E\|$	fcs.	$\|\mathcal{L}\|$
normal	90	96	8	5
reduced	44	50		

Results – Vienna

Input



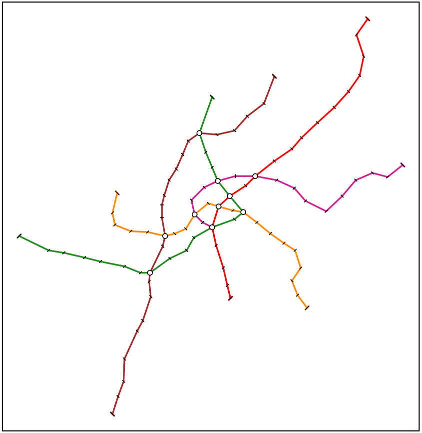
Input	$\lvert V \rvert$	$\lvert E \rvert$	fcs.	$\lvert \mathcal{L} \rvert$
normal	90	96	8	5
reduced	44	50		

$\downarrow$

MIP	constr.	var.
normal	39,363	9,960
faces	23,226	6,048
callback	1,875	872

Results – Vienna

Input



Input	$\|V\|$	$\|E\|$	fcs.	$\|\mathcal{L}\|$
normal	90	96	8	5
reduced	44	50		

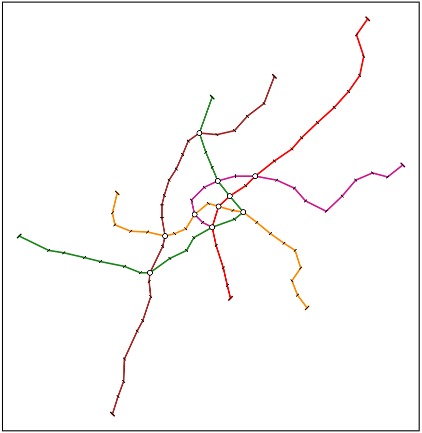
↓

MIP	constr.	var.
normal	39,363	9,960
faces	23,226	6,048
callback*	1,875	872

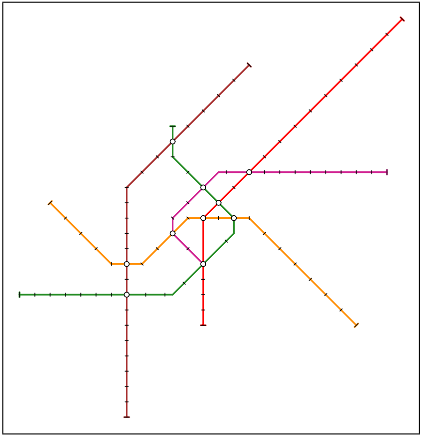
*) 21 seconds w/o proof of opt.

Results – Vienna

Input

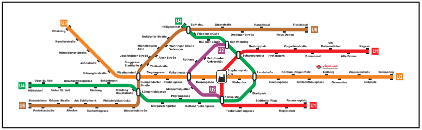


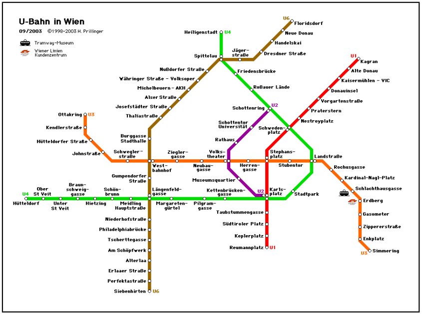
Output (21 sec.)



Results – Vienna

Official maps





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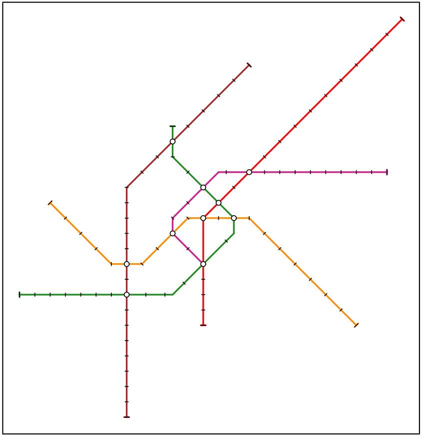
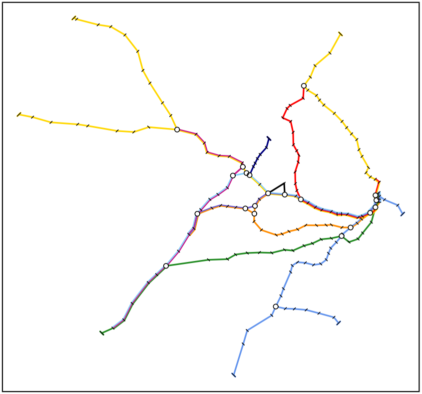


Results – Sydney

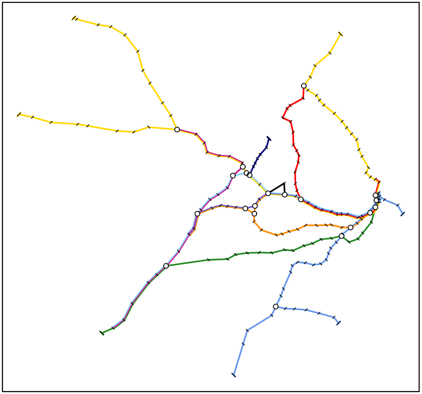
Input



Input	$\|V\|$	$\|E\|$	fcs.	$\|\mathcal{L}\|$
normal	174	183	11	10
reduced	67	76		

Results – Sydney

Input



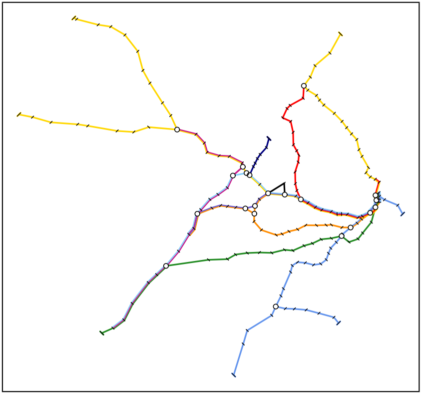
Input	$\|V\|$	$\|E\|$	fcs.	$\|\mathcal{L}\|$
normal	174	183	11	10
reduced	67	76		

↓

MIP	constr.	var.
normal	93,620	23,389
faces	52,568	13,437
callback	4,147	6,741

Results – Sydney

Input



Input	$\lvert V\rvert$	$\lvert E\rvert$	fcs.	$\lvert\mathcal{L}\rvert$
normal	174	183	11	10
reduced	67	76		

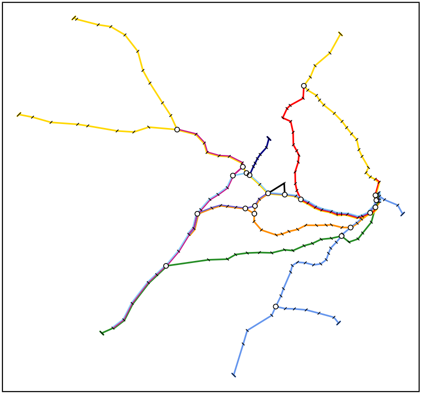
↓

MIP	constr.	var.
normal	93,620	23,389
faces	52,568	13,437
callback*	4,147	6,741

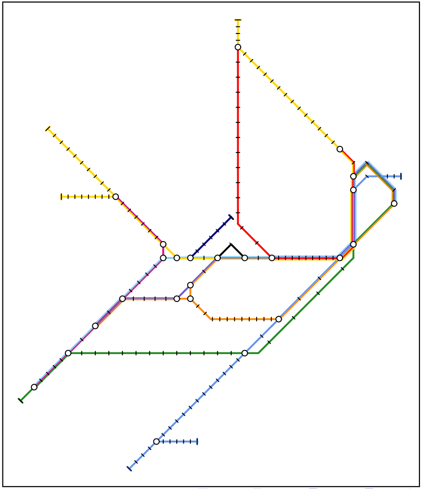
*) 33 seconds w/o proof of opt.
constr. of 4 edge pairs added

Results – Sydney

Input

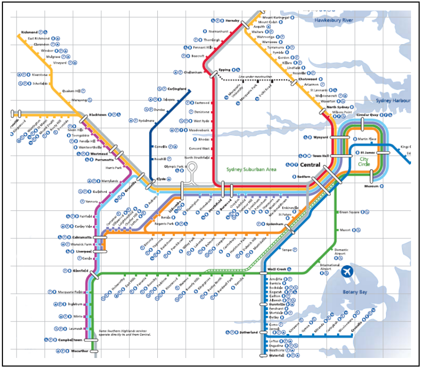


Output (33 sec.)

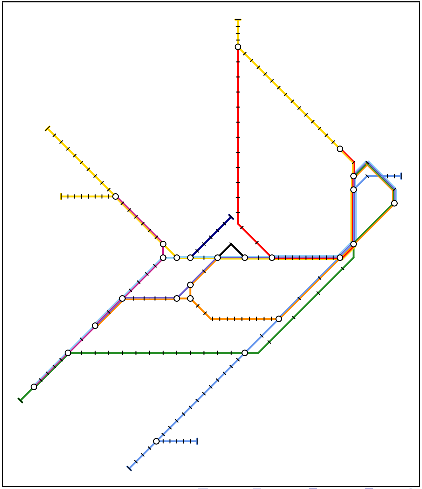


Results – Sydney

Official map

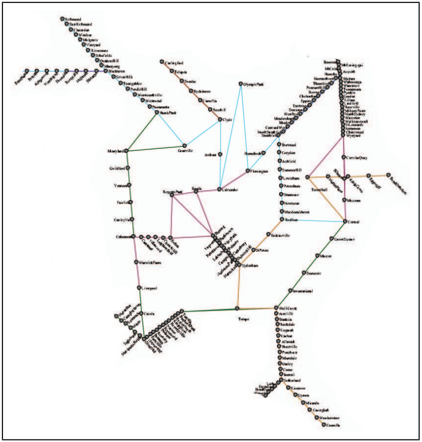


Output (33 sec.)

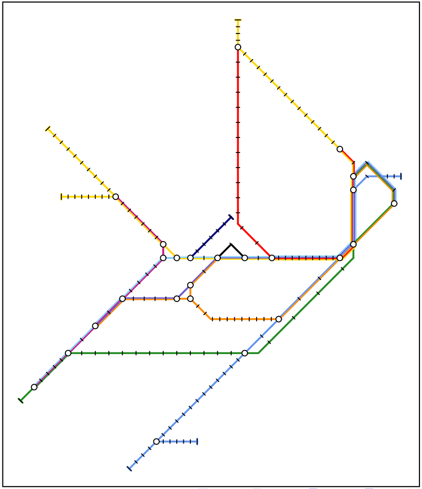


Sydney: Related Work

[Hong et al. GD'04] (7.6 sec.)

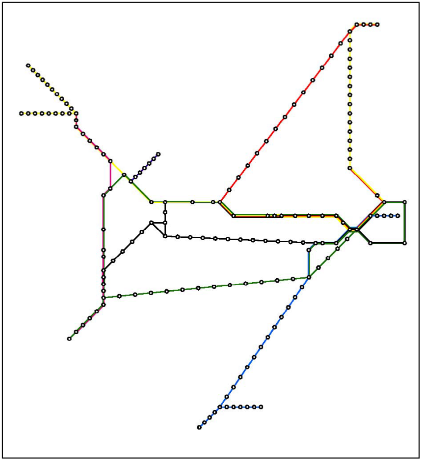


Our output (33 sec.)

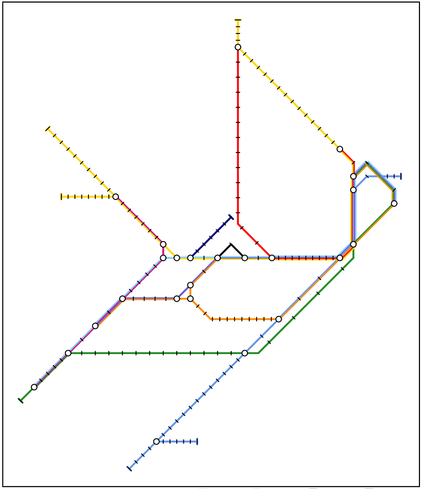


Sydney: Related Work

[Stott, Rodgers IV'04] (4 min.)

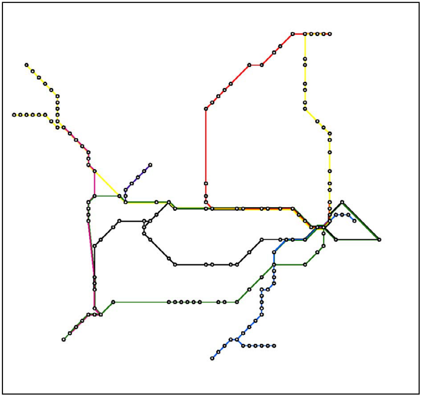


Our output (33 sec.)

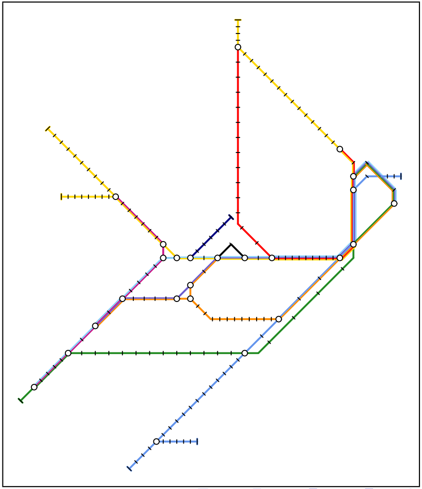


Sydney: Related Work

[Stott, Rodgers IV'04] (28 min.)

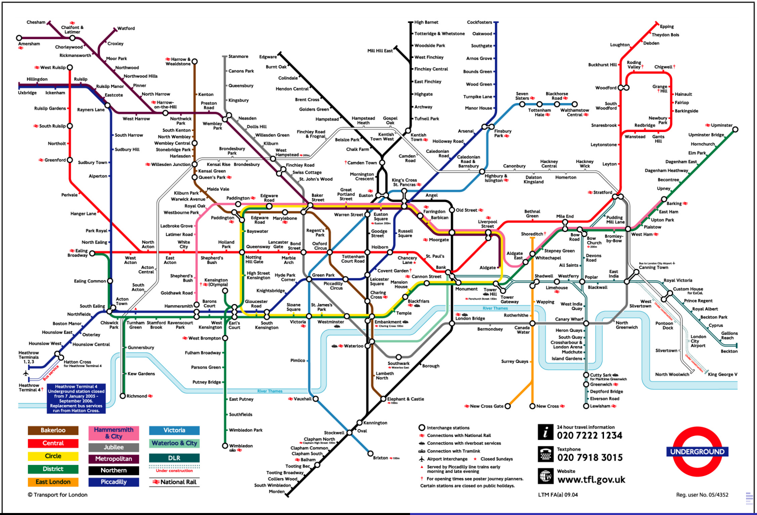


Our output (33 sec.)



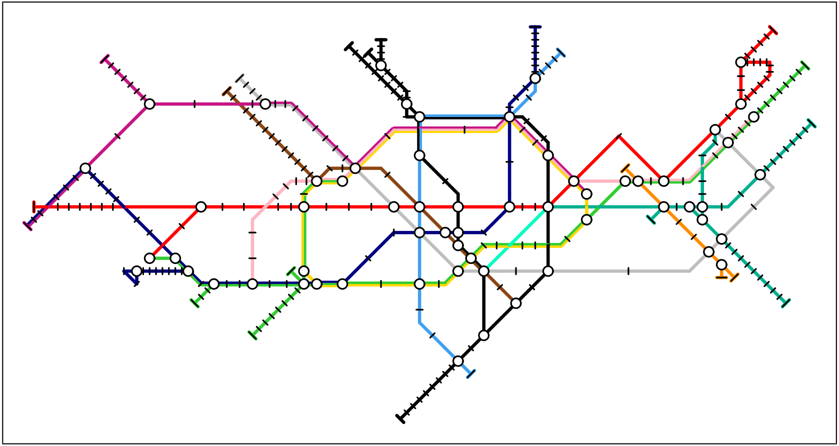
Large Maps

London



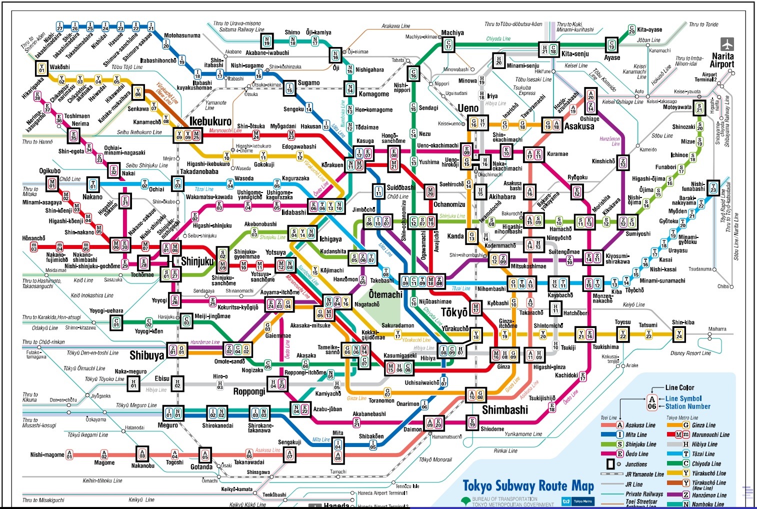
Large Maps

London (16 min.)



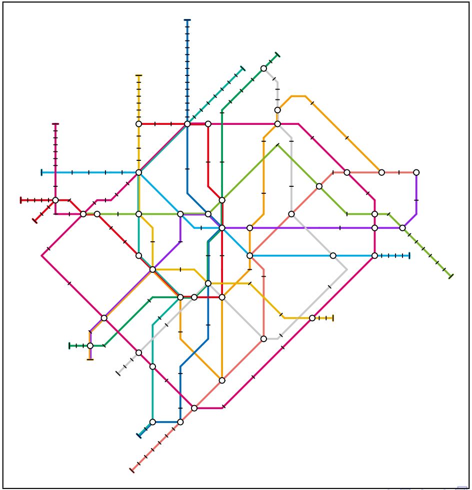
Large Maps

Tokyo



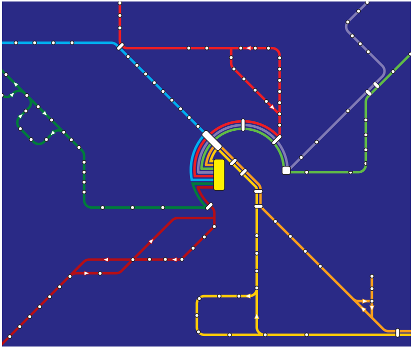
Large Maps

Tokyo (4:50 hrs.)



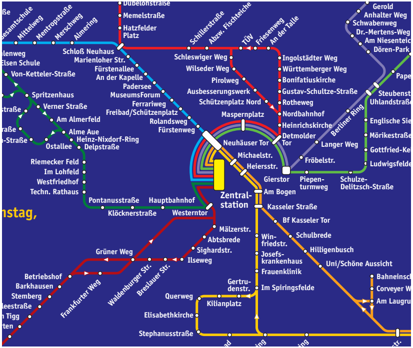
Labeling

- unlabeled metro map of little use in practice



Labeling

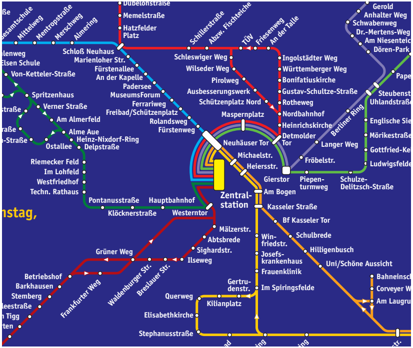
- unlabeled metro map of little use in practice
- labels
  - occupy space
  - may not overlap



Labeling

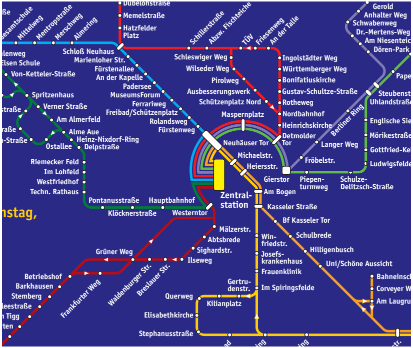
- unlabeled metro map of little use in practice
- labels
  - occupy space
  - may not overlap
- static map labeling is NP-hard

  [Tollis, Kakoulis ’01]



Labeling

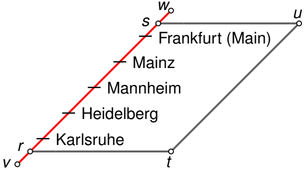
- unlabeled metro map of little use in practice
- labels
  - occupy space
  - may not overlap
- static map labeling is NP-hard
  [Tollis, Kakoulis '01]
- want to combine layout and labeling for better results



Modeling Labels

Model labels as special metro lines:

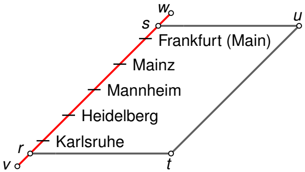
- put all labels between a pair of interchange stations into one parallelogram,



Modeling Labels

Model labels as special metro lines:

- put all labels between a pair of interchange stations into one parallelogram,
- allow parallelograms to change sides,



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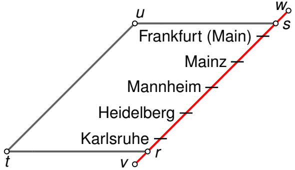
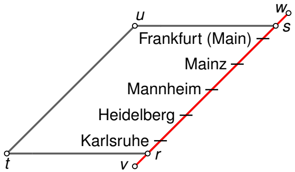


Modeling Labels

Model labels as special metro lines:

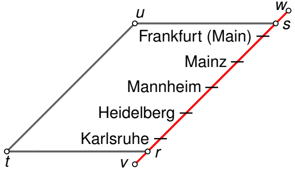
- put all labels between a pair of interchange stations into one parallelogram,
- allow parallelograms to change sides,
- bad news: a **lot** more planarity constraints



Modeling Labels

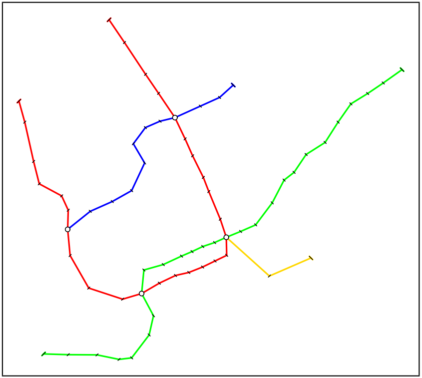
Model labels as special metro lines:

- put all labels between a pair of interchange stations into one parallelogram,
- allow parallelograms to change sides,
- bad news: a **lot** more planarity constraints
- good news: callback method helps



Labeling – Montréal

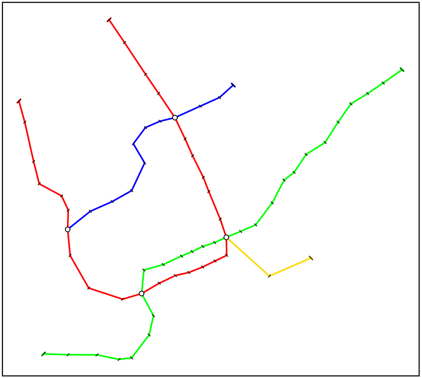
Input



Input	$\|V\|$	$\|E\|$	fcs.	$\|\mathcal{L}\|$
normal	65	66	3	4
reduced	20	21		
labeled	62	74	14	

Labeling – Montréal

Input



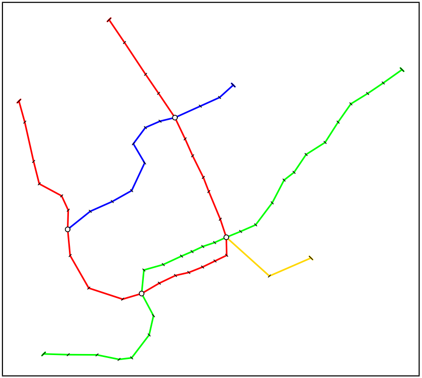
Input	$\lvert V\rvert$	$\lvert E\rvert$	fcs.	$\lvert\mathcal{L}\rvert$
normal	65	66	3	4
reduced	20	21		
labeled	62	74	14	

↓

MIP	constr.	var.
normal	86,493	21,209
faces	32,208	8,049
callback	4,400	8,049

Labeling – Montréal

Input



Input	$\|V\|$	$\|E\|$	fcs.	$\|\mathcal{L}\|$
normal	65	66	3	4
reduced	20	21		
labeled	62	74	14	

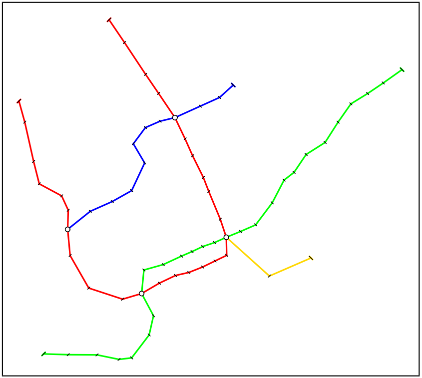
↓

MIP	constr.	var.
normal	86,493	21,209
faces	32,208	8,049
callback*	4,400	8,049

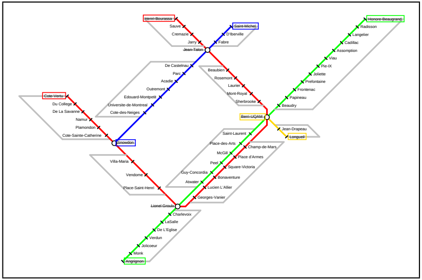
*) 17 minutes w/o proof of opt.
constr. of 60 edge pairs added

Labeling – Montréal

Input



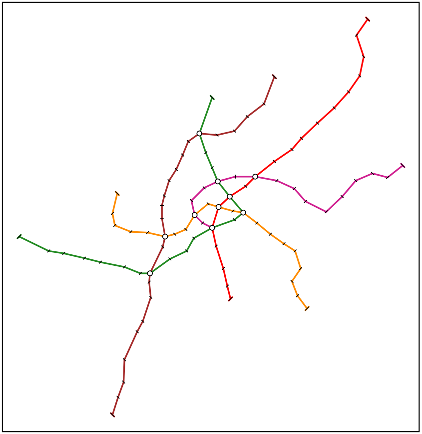
Output (17 min.)



Labeling – Montréal

Labeling – Vienna

Input



<table>
<tr><th>Input</th><th>$|V|$</th><th>$|E|$</th><th>fcs.</th><th>$|\mathcal{L}|$</th></tr>
<tr><td>normal</td><td>90</td><td>96</td><td rowspan="2">8</td><td rowspan="3">5</td></tr>
<tr><td>reduced</td><td>44</td><td>50</td></tr>
<tr><td>labeled</td><td>98</td><td>117</td><td>21</td></tr>
</table>

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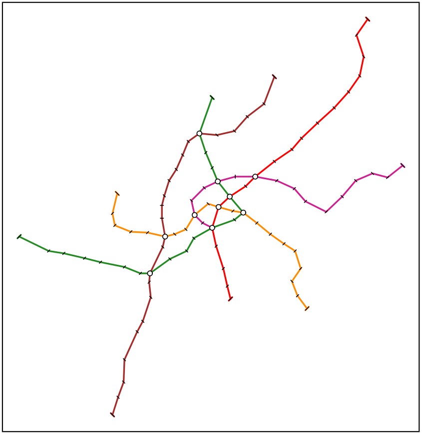
Input



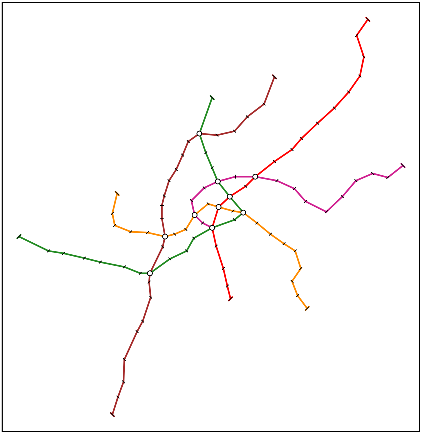
Input	$\|V\|$	$\|E\|$	fcs.	$\|\mathcal{L}\|$
normal	90	96	8	5
reduced	44	50		
labeled	98	117	21	

↓

MIP	constr.	var.
normal	219,064	53,538
faces	51,160	12,834
callback	9,144	12,834

Labeling – Vienna

Input



Input	$\|V\|$	$\|E\|$	fcs.	$\|\mathcal{L}\|$
normal	90	96	8	
reduced	44	50		5
labeled	98	117	21	

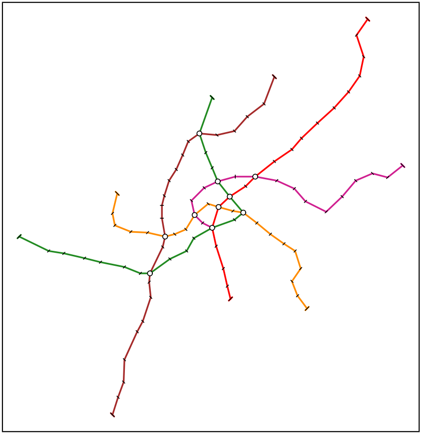
$\downarrow$

MIP	constr.	var.
normal	219,064	53,538
faces	51,160	12,834
callback*	9,144	12,834

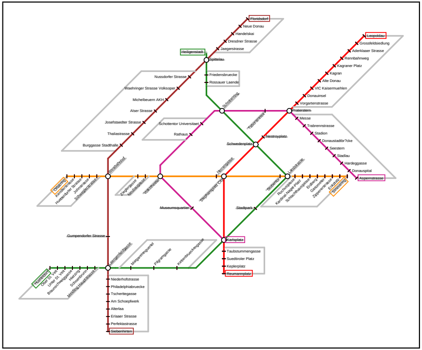
*) 1 day w/o proof of opt.
constr. of 160 edge pairs added

Labeling – Vienna

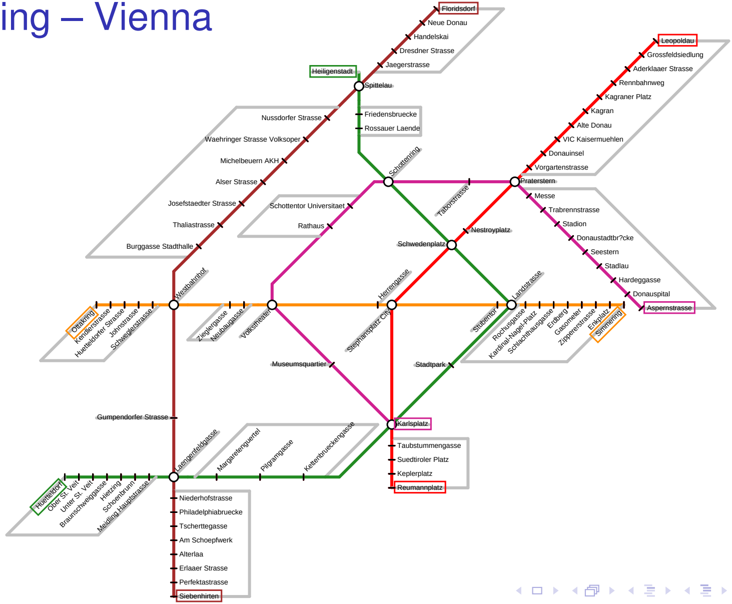
Input



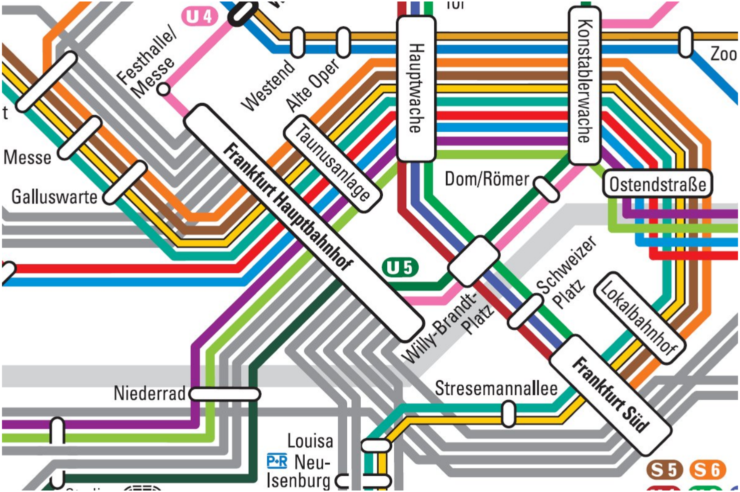
Output (1 day)



Labeling – Vienna



To Do: Rectangular Stations & Multi-Edges



Summary (Metro Maps)

- METROMAPLAYOUT is NP-hard.
- Formulated and implemented MIP.
- Results comparable to manually designed maps.
- Reduced MIP size & runtime drastically.
- Combined layout and labeling.
- Our MIP can draw *any* kind of sketch “nicely”.

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To Do

- rectangular stations
- multi-edges
- user interaction (e.g. fixing certain edge directions)

Thank you for the attention.

Any questions?