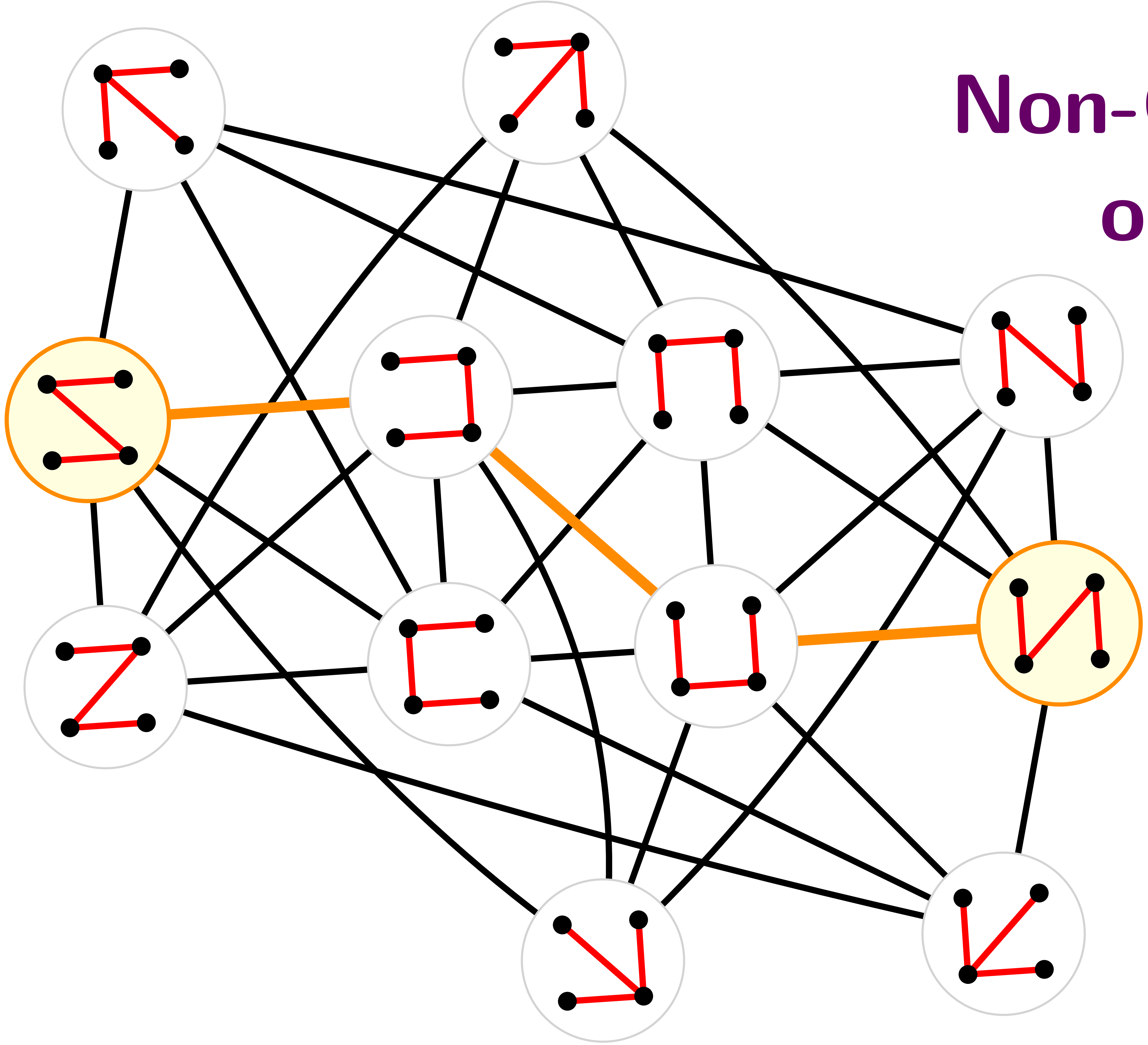

Flipping Non-Crossing Spanning Trees on Convex Point Sets

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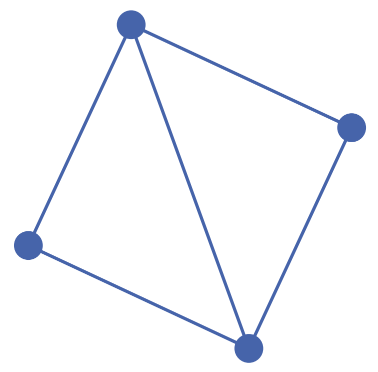
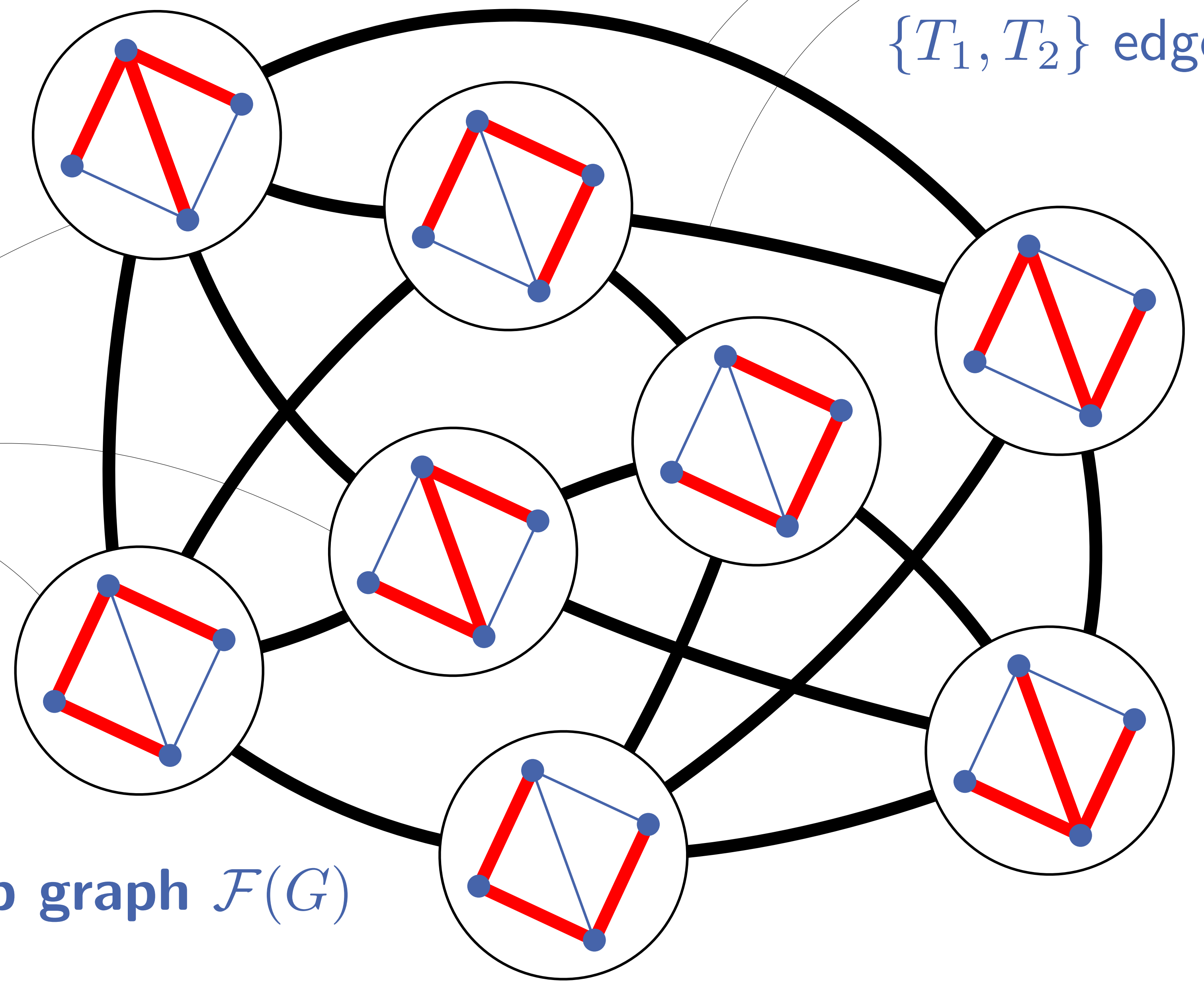
Order and Geometry
Lutherstadt Wittenberg, Germany
September 11, 2024

Flipping Spanning Trees



G graph



edges $=$ flips

$\{T_1, T_2\}$ edge $\Leftrightarrow |T_1 - T_2| = 1$

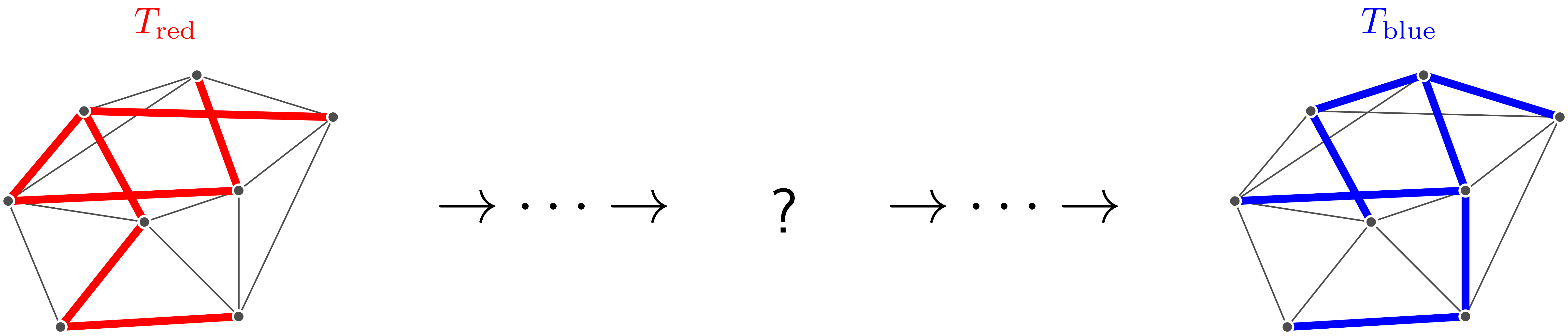
vertices
$=$ spanning trees

flip graph $\mathcal{F}(G)$

Questions

- ▷ connectivity
- ▷ min / max degree
- ▷ **flip distance**
- ▷ radius
- ▷ **diameter**
- ▷ mixing properties
- ▷ expander properties
- ▷ Hamiltonicity
- ▷ enumeration
- ▷ generation
- ▷ counting
- ▷ . . .

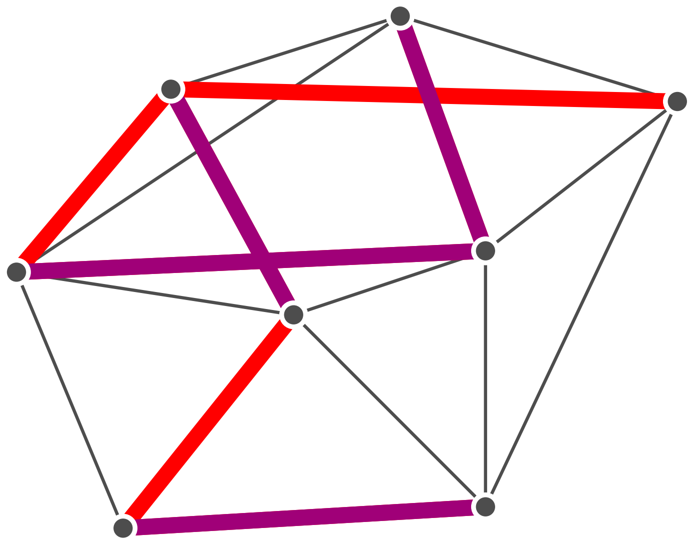
Flip Distance



flip distance $\text{dist}(T_{\text{red}}, T_{\text{blue}})$ = minimum number of flips required $T_{\text{red}} \to T_{\text{blue}}$

Flip Distance

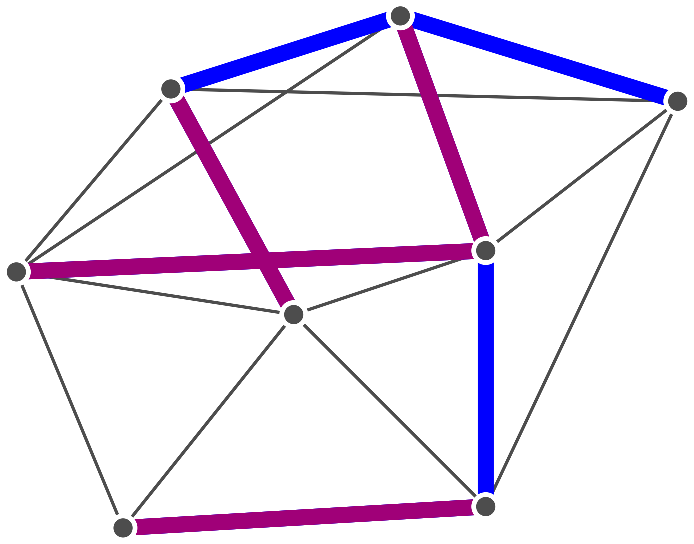
T_{red}



$D = 3$

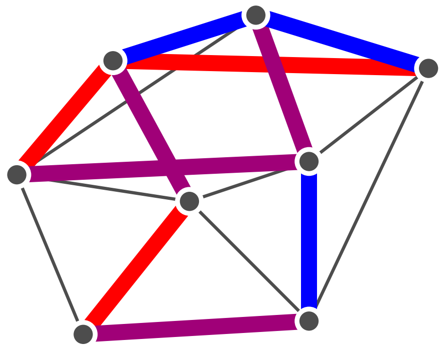
$\rightarrow \cdots \rightarrow \quad ? \quad \rightarrow \cdots \rightarrow$

T_{blue}



flip distance $\text{dist}(T_{\text{red}}, T_{\text{blue}})$ = minimum number of flips required $T_{\text{red}} \rightarrow T_{\text{blue}}$

▷ easy lower bound: $\text{dist}(T_{\text{red}}, T_{\text{blue}}) \geq |T_{\text{red}} - T_{\text{blue}}| = |T_{\text{blue}} - T_{\text{red}}| = D$



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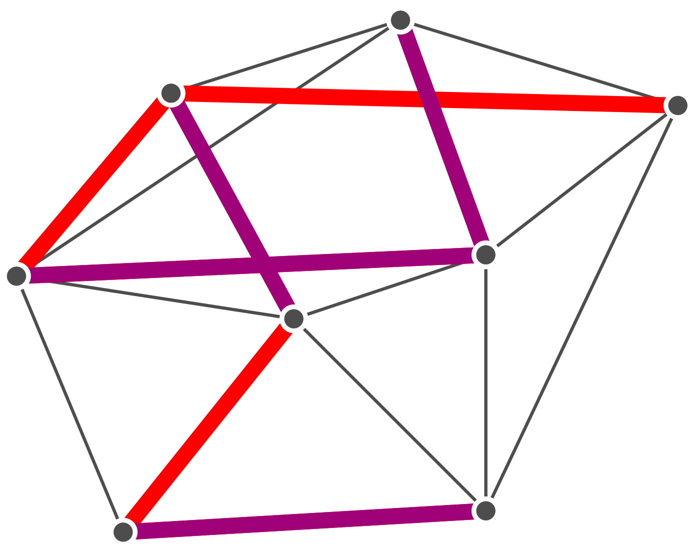

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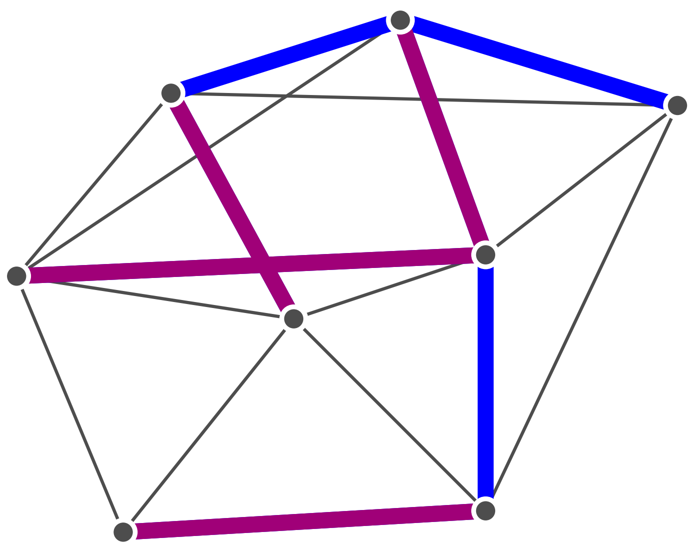
$\rightarrow \cdots \rightarrow \quad ? \quad \rightarrow \cdots \rightarrow$

T_{blue}



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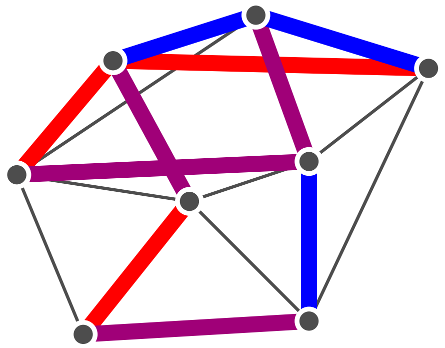

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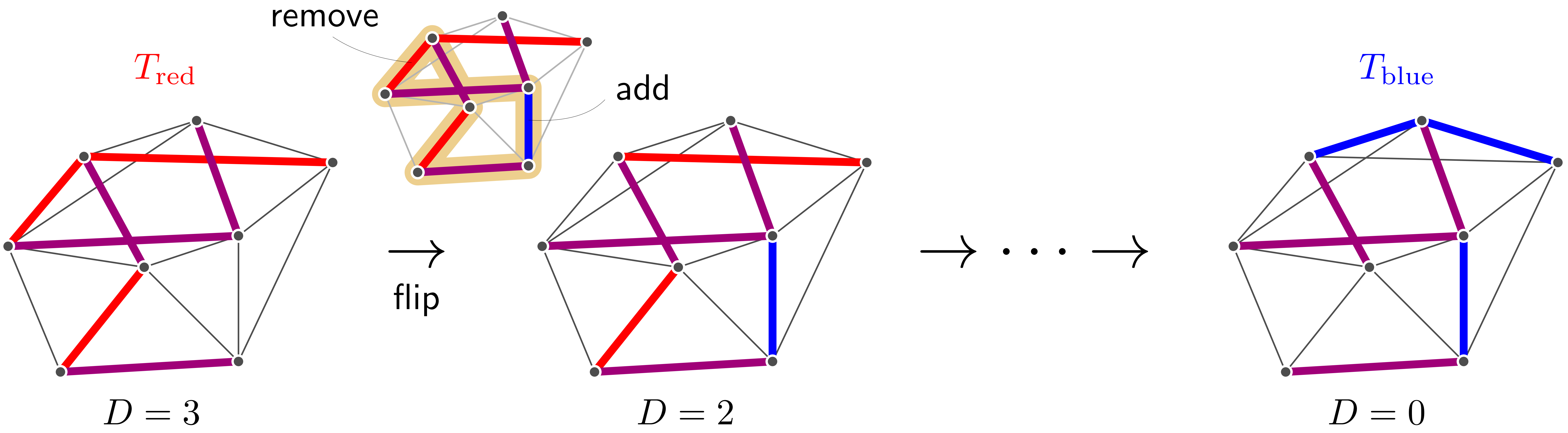

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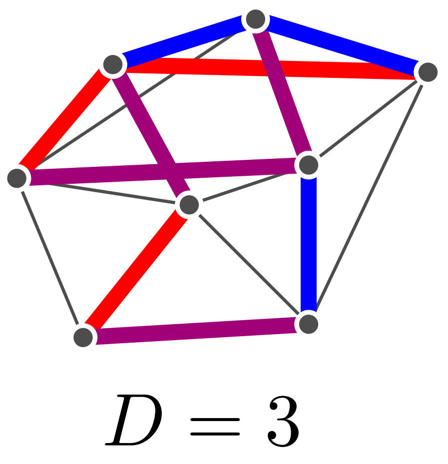
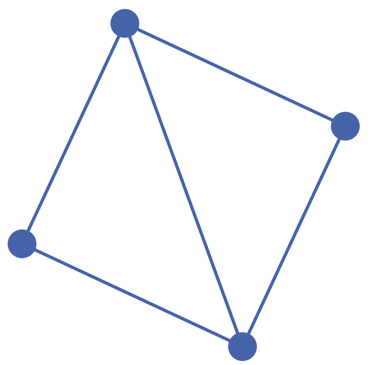


Radius and Diameter

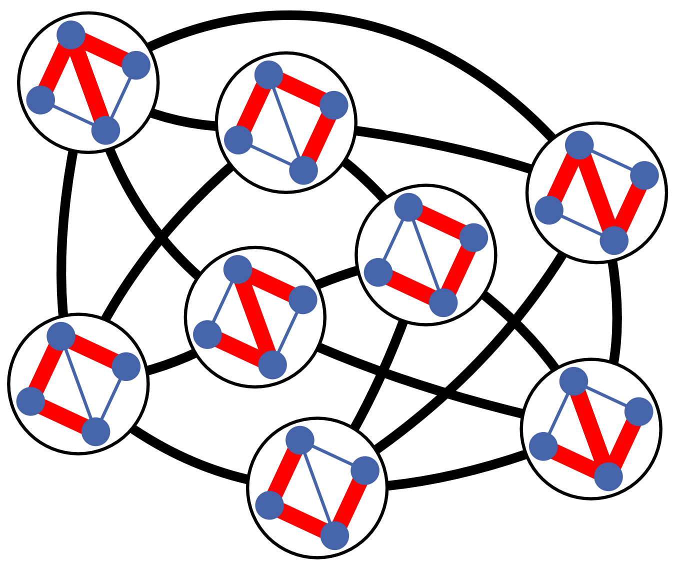
$\mathrm{dist}(T_{\mathrm{red}}, T_{\mathrm{blue}}) = D = |T_{\mathrm{red}} - T_{\mathrm{blue}}|$ for all $T_{\mathrm{red}}, T_{\mathrm{blue}}$



G graph

diameter of $\mathcal{F}(G) = \max_{T_{\mathrm{red}}, T_{\mathrm{blue}} \subseteq G} \{D = |T_{\mathrm{red}} - T_{\mathrm{blue}}|\}$

- ▷ $\mathrm{diam}(\mathcal{F}(K_n)) = n - 1$
- ▷ $\mathrm{diam}(\mathcal{F}(G)) \leq n - 1$ for all graphs G
- ▷ determining $\mathrm{diam}(\mathcal{F}(G))$ possible in poly-time



flip graph $\mathcal{F}(G)$

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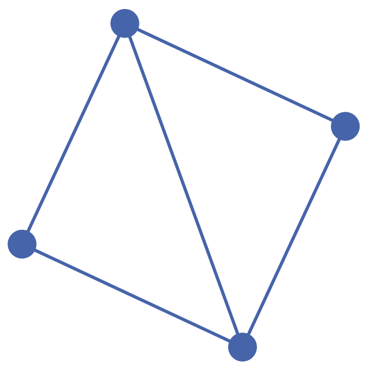

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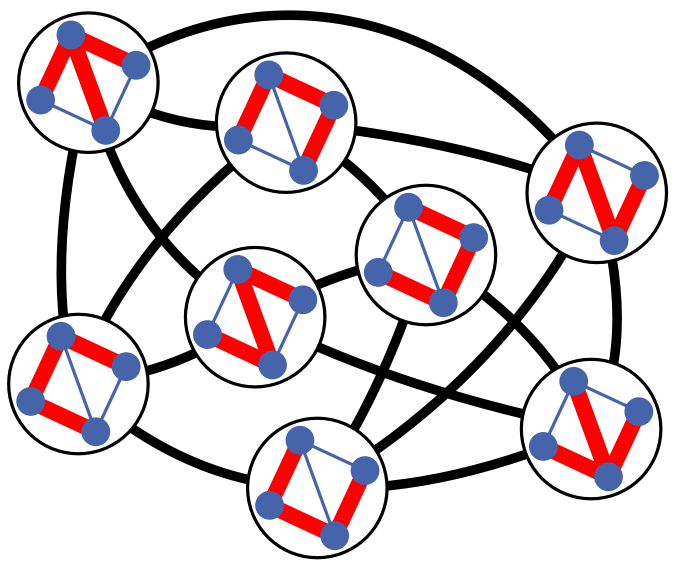

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- ▷ determining $\mathrm{rad}(\mathcal{F}(G))$ is NP-hard

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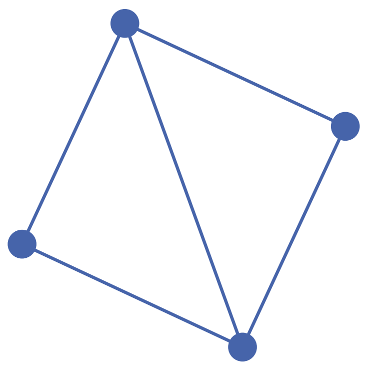

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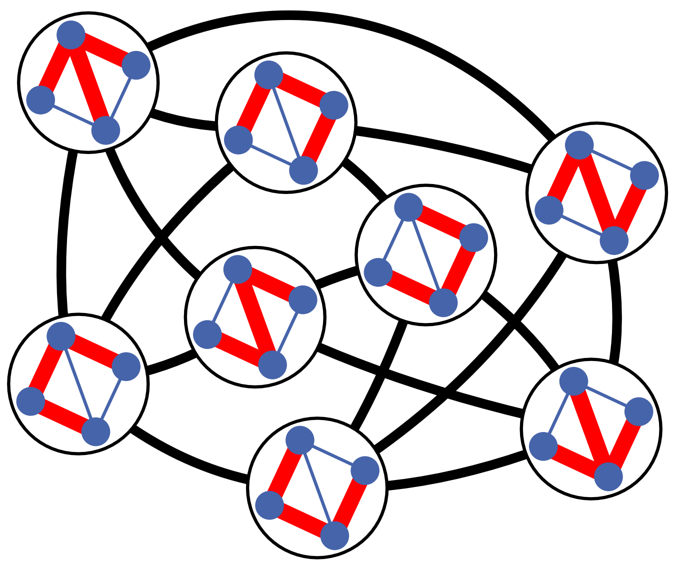

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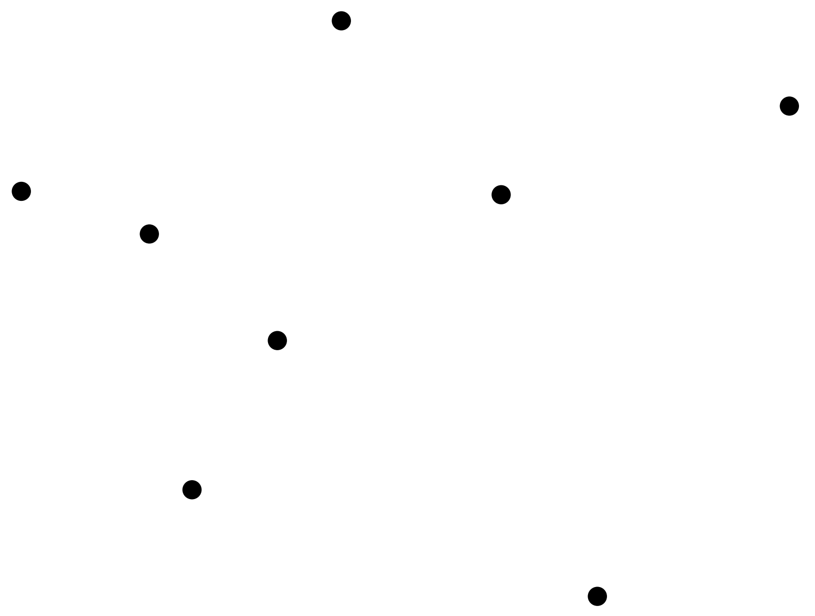
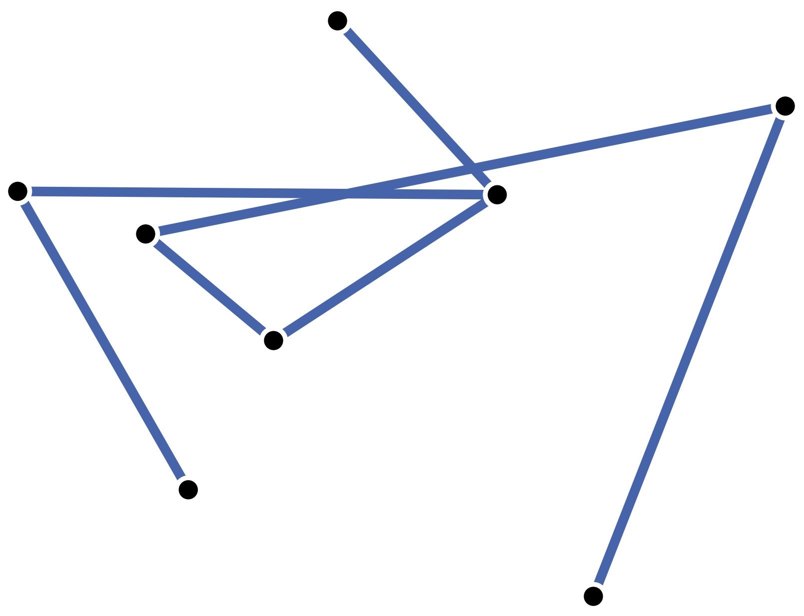
$$\mathrm{rad}(\mathcal{F}(G)) \leq \mathrm{diam}(\mathcal{F}(G)) \leq 2 \cdot \mathrm{rad}(\mathcal{F}(G))$$

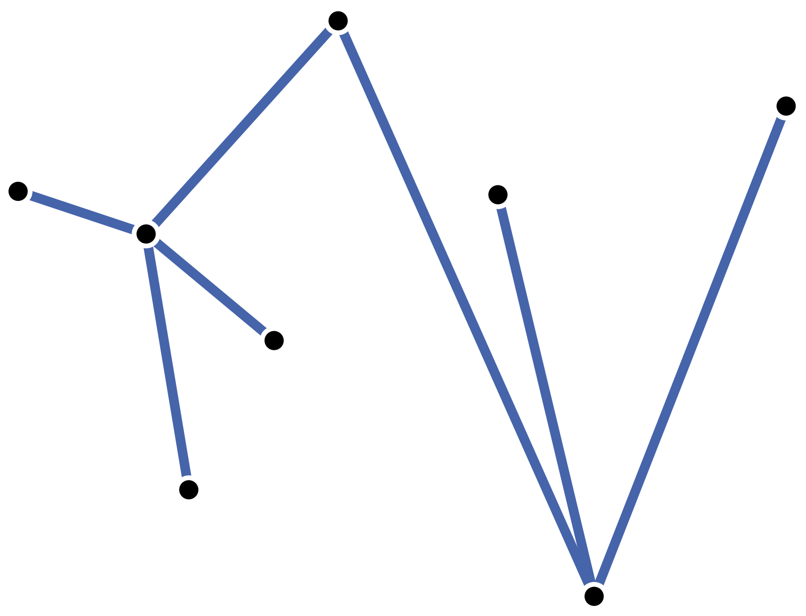
Spanning Trees on Point Sets



point set P
(general position)



spanning tree
(straightline segments)



non-crossing spanning tree
(no crossings)

$\longrightarrow$ **flip graph** $\mathcal{F}(P)$
on all non-crossing spanning trees

Questions

- ▷ **flip distance**
- ▷ radius
- ▷ **diameter**

largest for $|P| = n$?

State of the Art – Radius

- ▷ $\mathcal{F}(P)$ is induced subgraph of $\mathcal{F}(K_n)$ $\quad (|P| = n)$
- ▷ $\mathcal{F}(P)$ is connected for every point set P

Theorem (Avis and Fukuda, 1996)**.**
$\mathrm{rad}(\mathcal{F}(P)) \leq n - 2$ for every point set P

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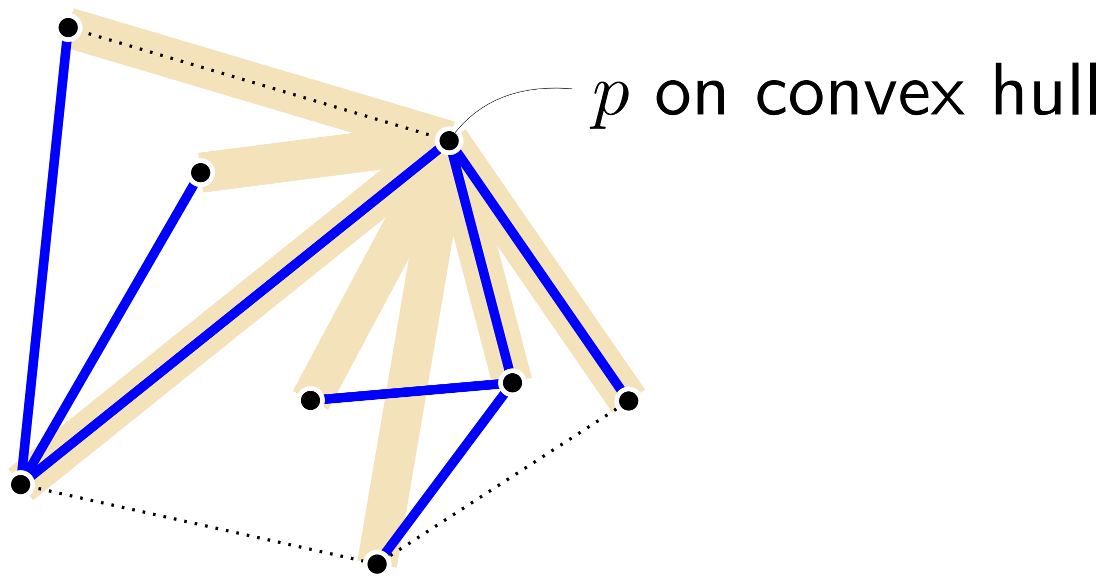
Lemma.
For every non-crossing spanning tree T
and every $p \in P$ on the convex hull,
there exists an uncrossed edge $e \notin T$ at p.



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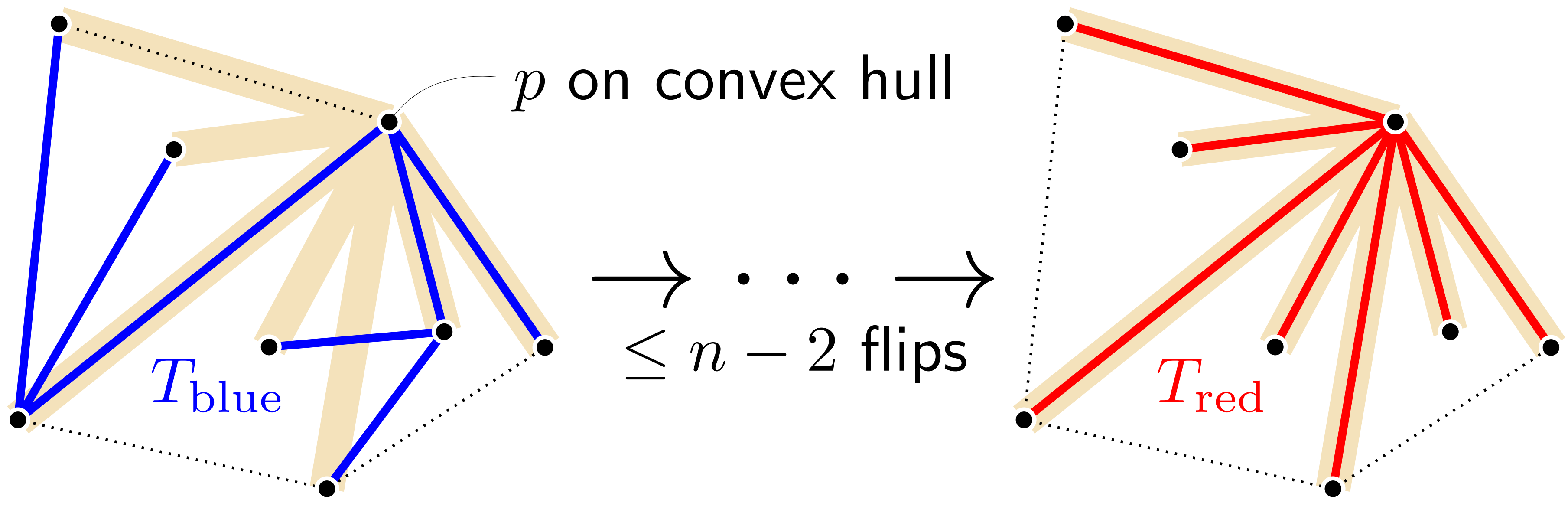

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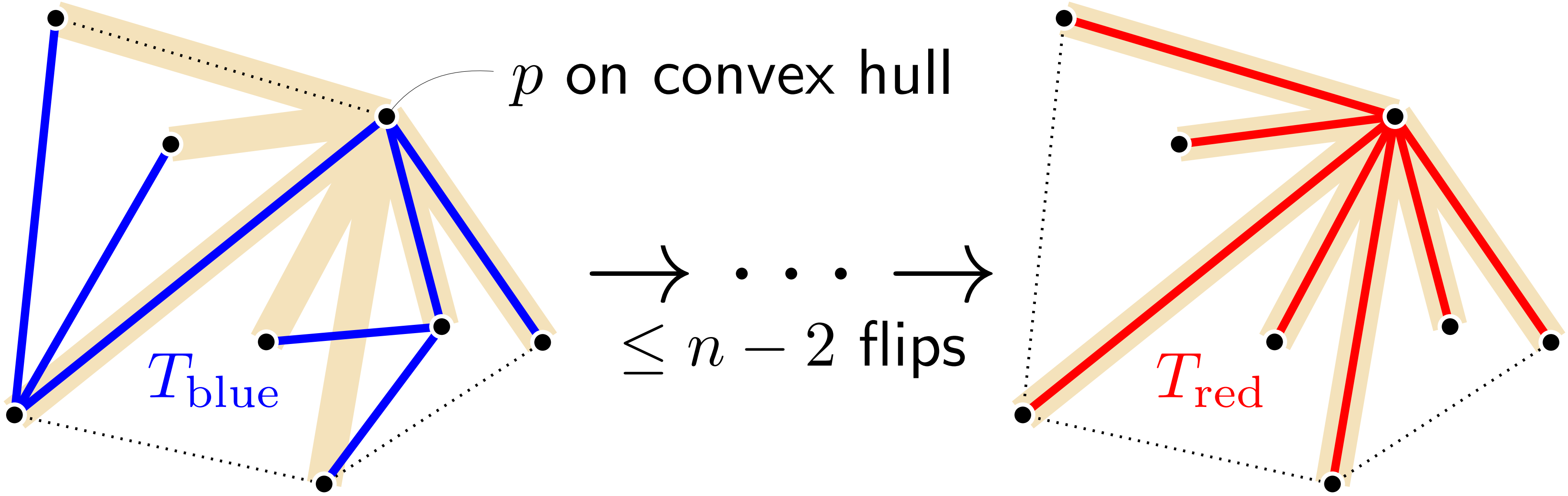

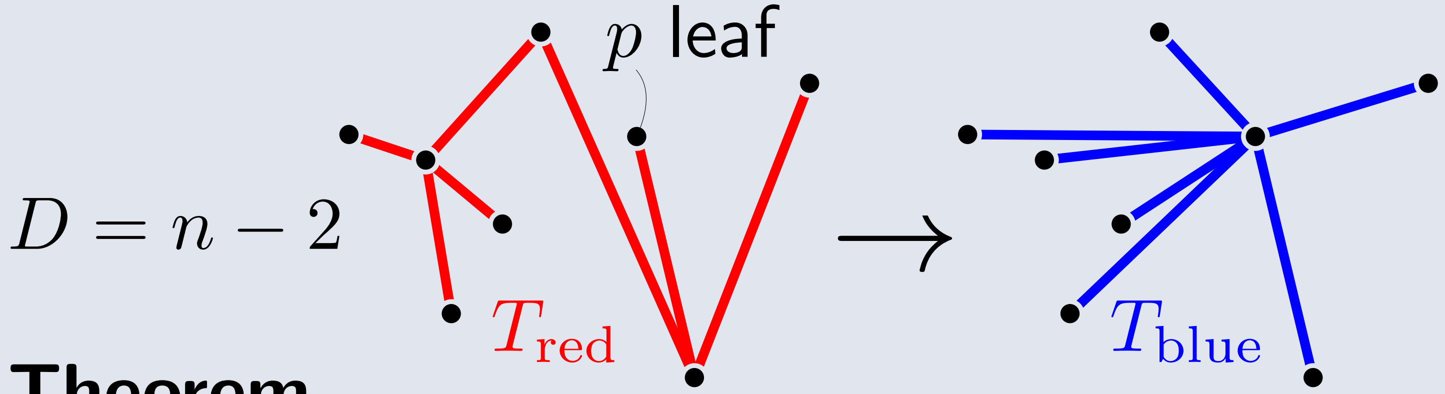

Theorem.
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State of the Art – Radius

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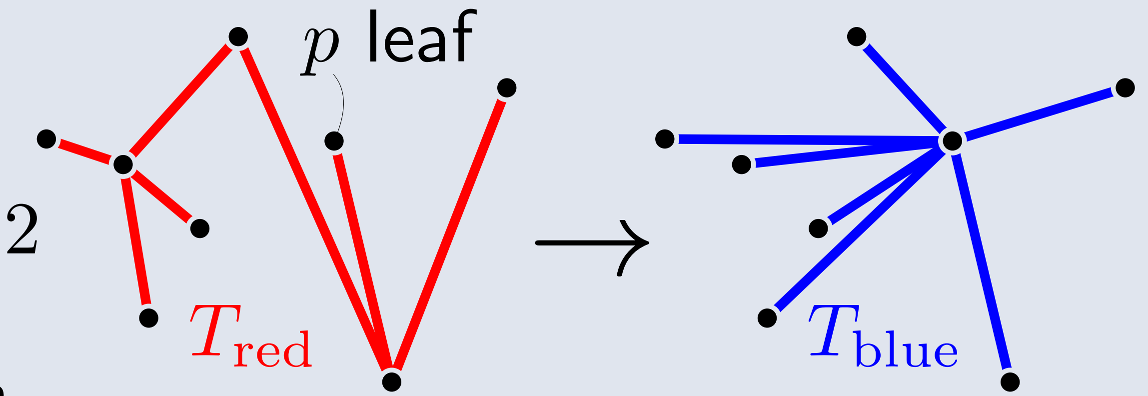
$D = n - 2$



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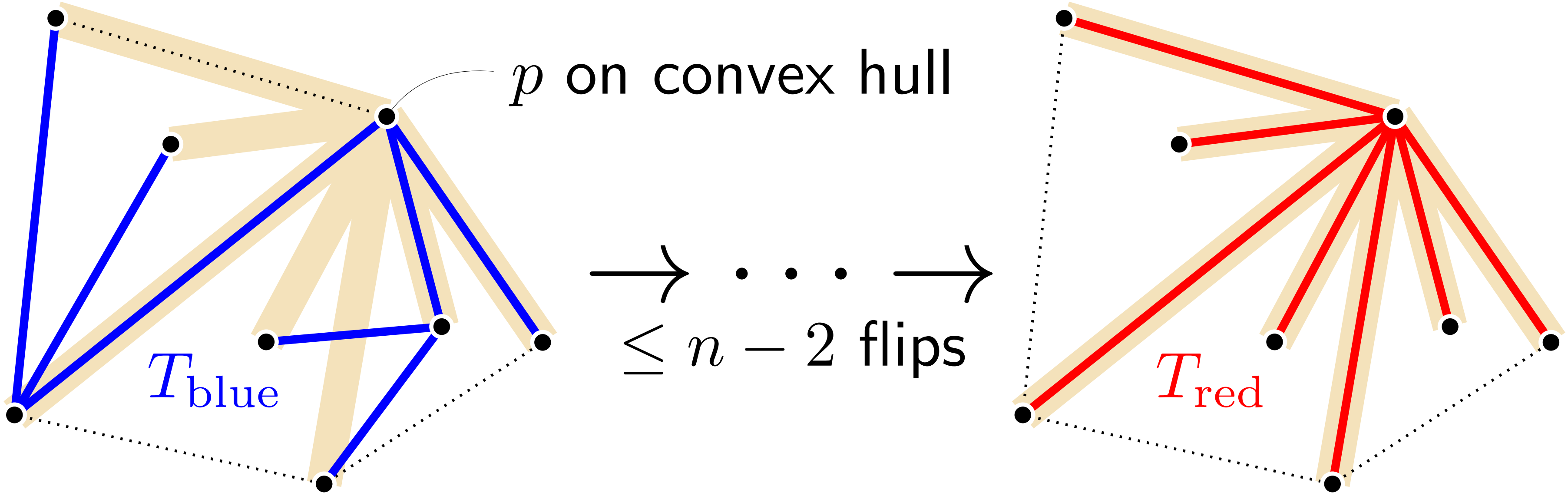

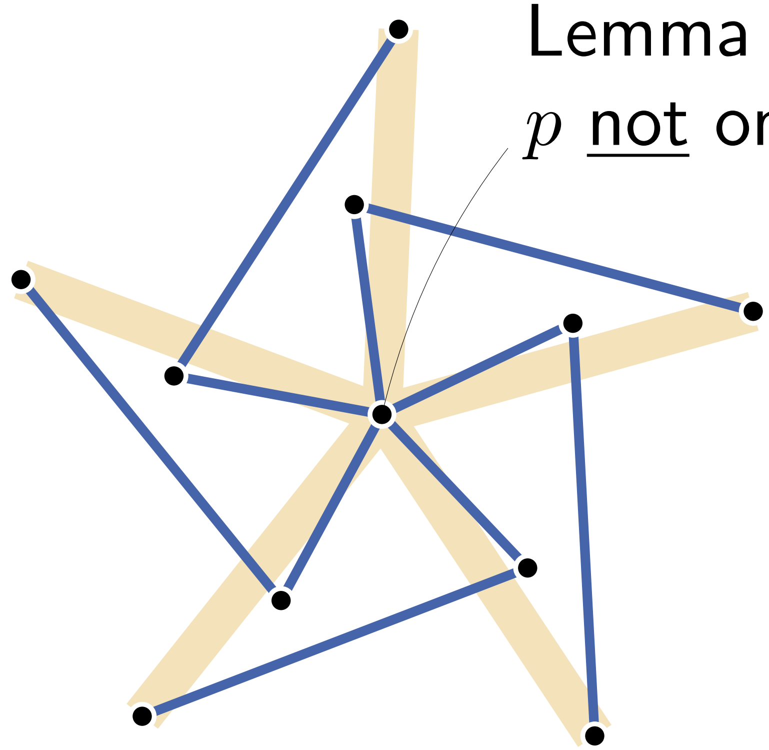

Lemma not true for p not on convex hull

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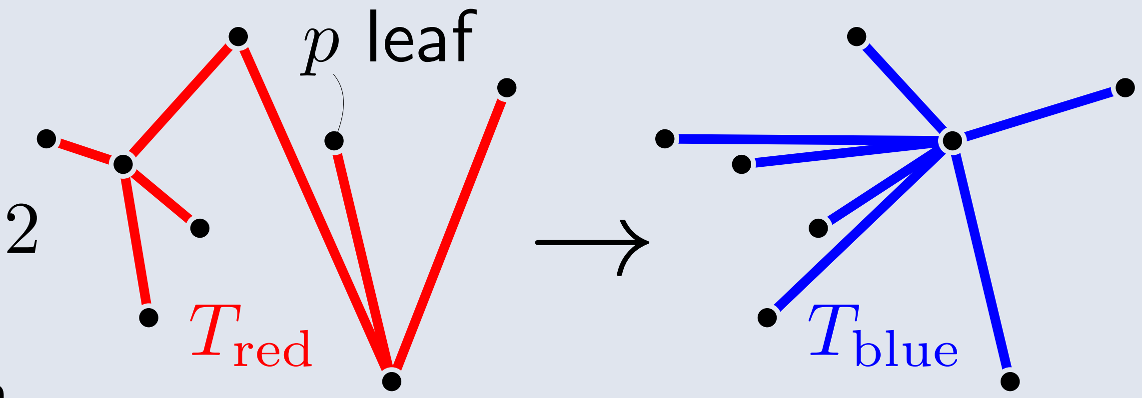

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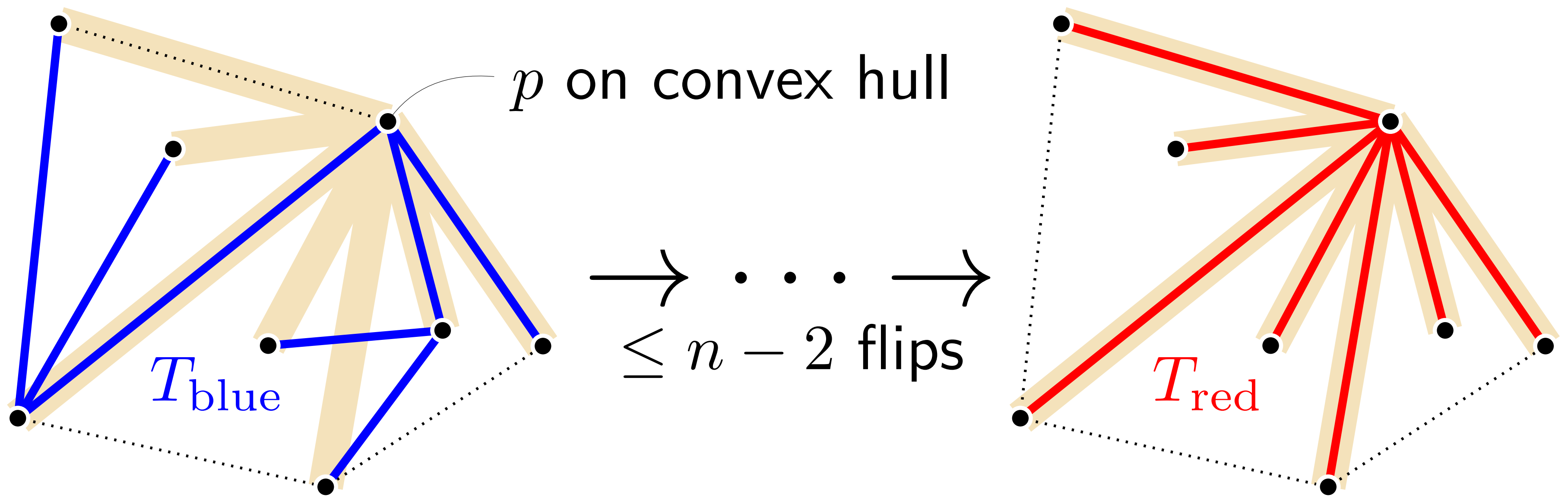

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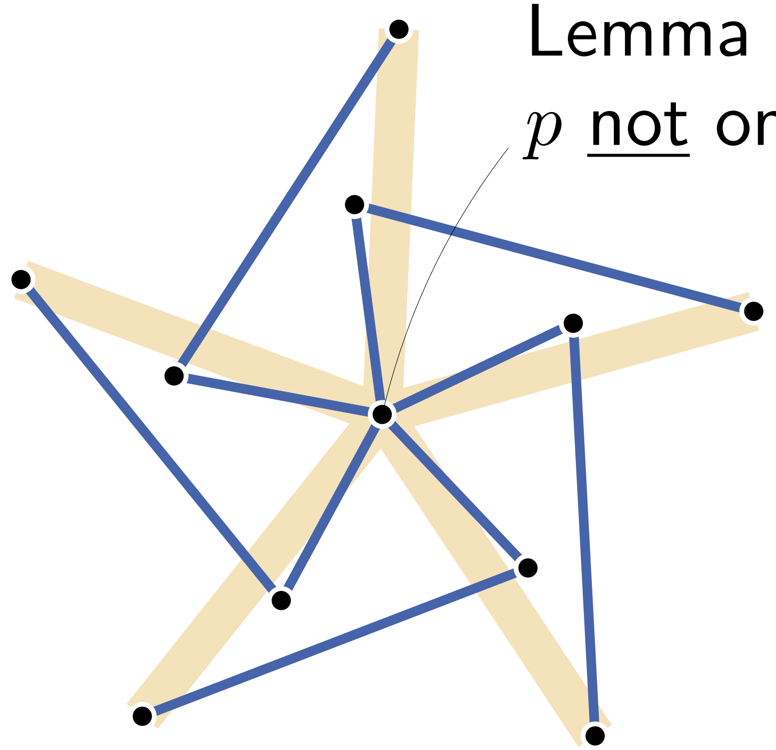

▷ $\mathrm{dist}(T_{\text{red}}, T_{\text{blue}}) > D = |T_{\text{red}} - T_{\text{blue}}|$ possible

State of the Art – Diameter

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- ▷ $\mathrm{rad}(\mathcal{F}(P)) \leq \mathrm{diam}(\mathcal{F}(P)) \leq 2 \cdot \mathrm{rad}(\mathcal{F}(P))$

$$n - 2 \ \leq \ \mathrm{diam}(\mathcal{F}(P)) \ \leq \ 2n - 4$$

Question. What is $\max\{\mathrm{diam}(\mathcal{F}(P))\colon |P| = n\}$?

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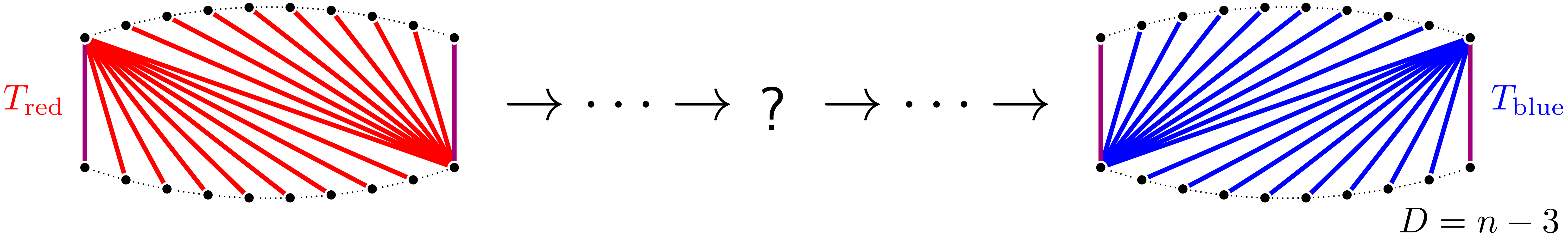
Theorem (Hernando et al., 1999).

$$\mathrm{diam}(\mathcal{F}(P)) \geq \tfrac{3}{2}n - 5$$

for point set P in convex position



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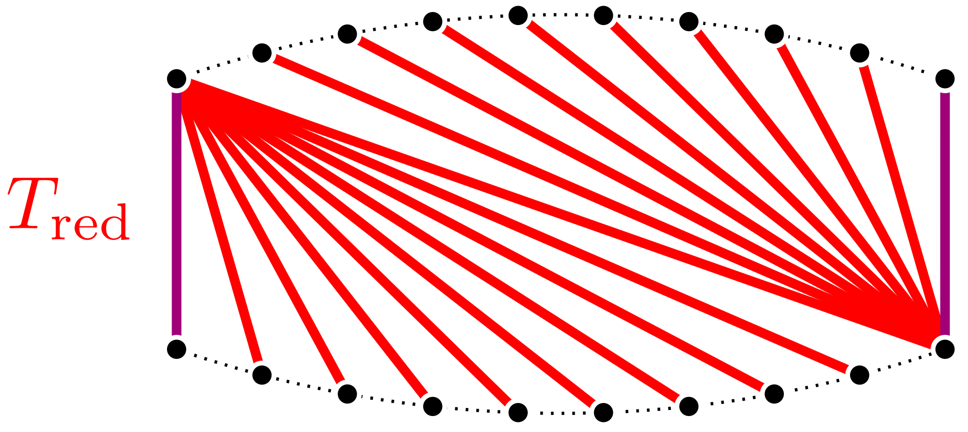
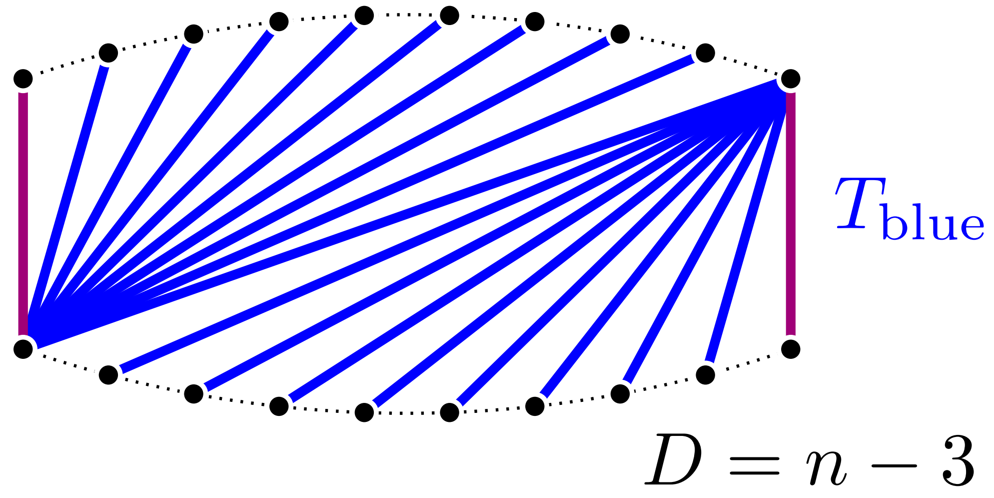


- ▷ every blue edge crosses $\ge \frac{n}{2} - 1$ red edges
- ▷ at least $\frac{n}{2} - 1$ flips until first blue edge introduced



State of the Art – Diameter

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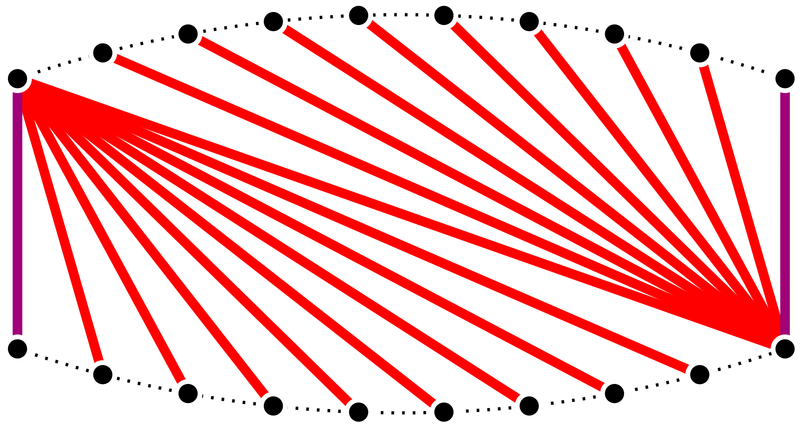
for point set P in convex position

The case of Convex Position

- $\triangleright$ Bousquet et al. 2023:
  $\mathrm{diam}(\mathcal{F}_n) \leq 2n - \Omega(\sqrt{n})$
- $\triangleright$ Aichholzer et al. 2022:
  $\mathrm{dist}(T_{\mathrm{red}}, T_{\mathrm{blue}}) \leq 2D - \Omega(\log D)$
- $\triangleright$ Bousquet et al. 2024:
  $\mathrm{dist}(T_{\mathrm{red}}, T_{\mathrm{blue}}) \leq 1.96D$
  $\mathrm{diam}(\mathcal{F}_n) \leq 1.96n$

T_{red}



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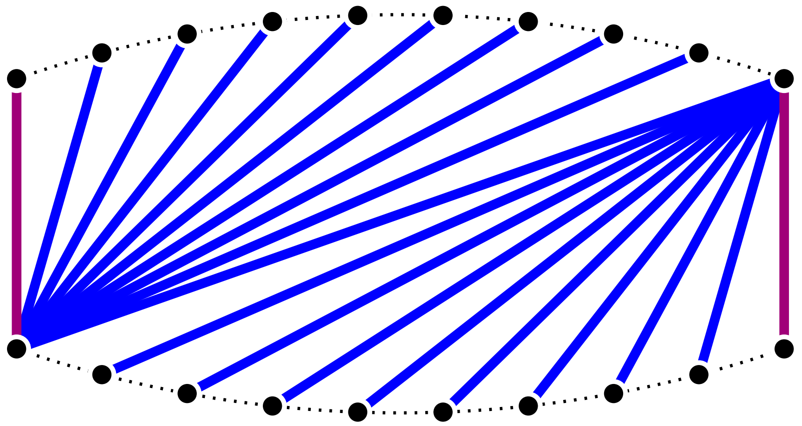

T_{blue}

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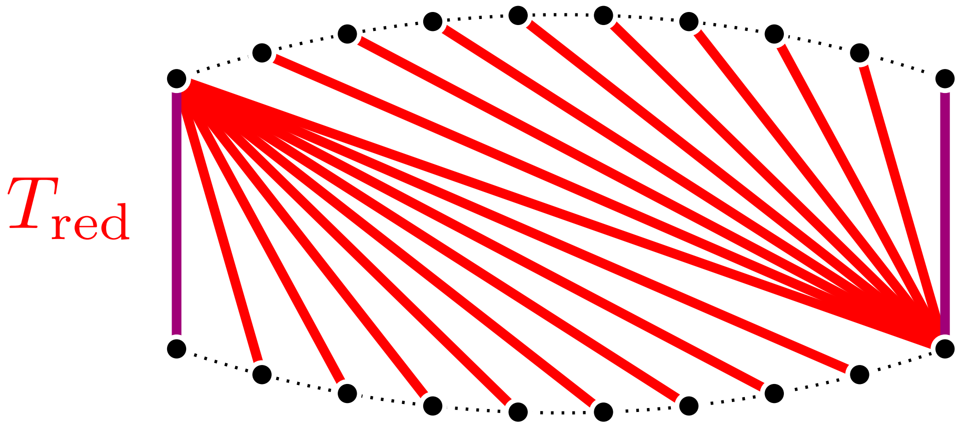
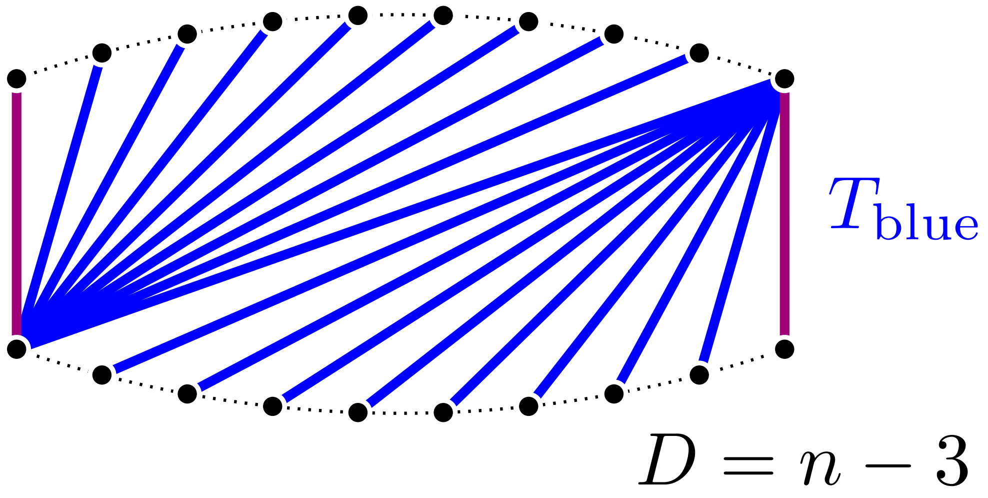
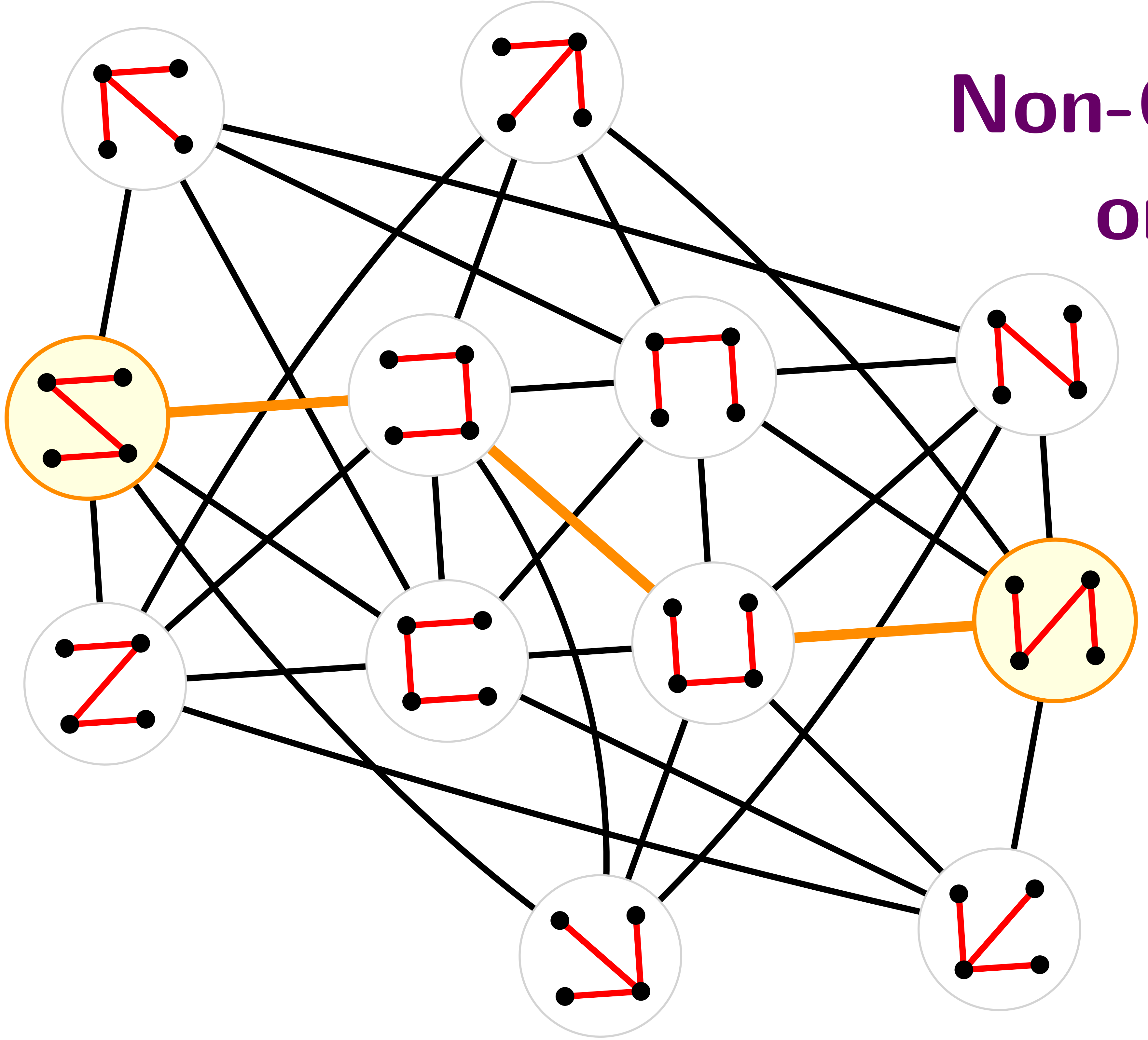
NEW: $1.\overline{5}n \leq \mathrm{diam}(\mathcal{F}_n) \leq 1.\overline{6}n$



- $\triangleright$ every blue edge crosses $\geq \frac{n}{2} - 1$ red edges
- $\triangleright$ at least $\frac{n}{2} - 1$ flips until first blue edge introduced





Flipping Non-Crossing Spanning Trees on Convex Point Sets

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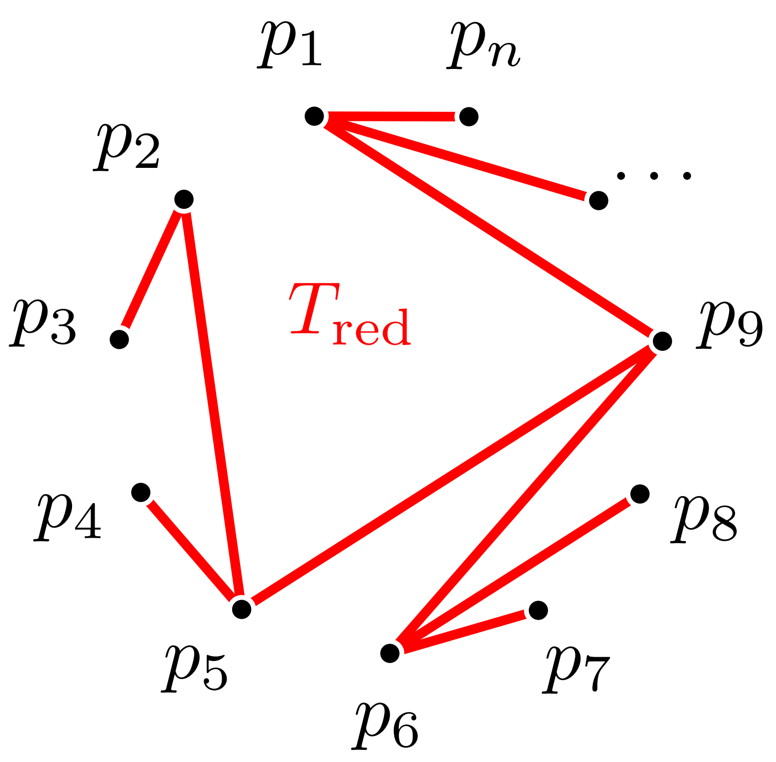
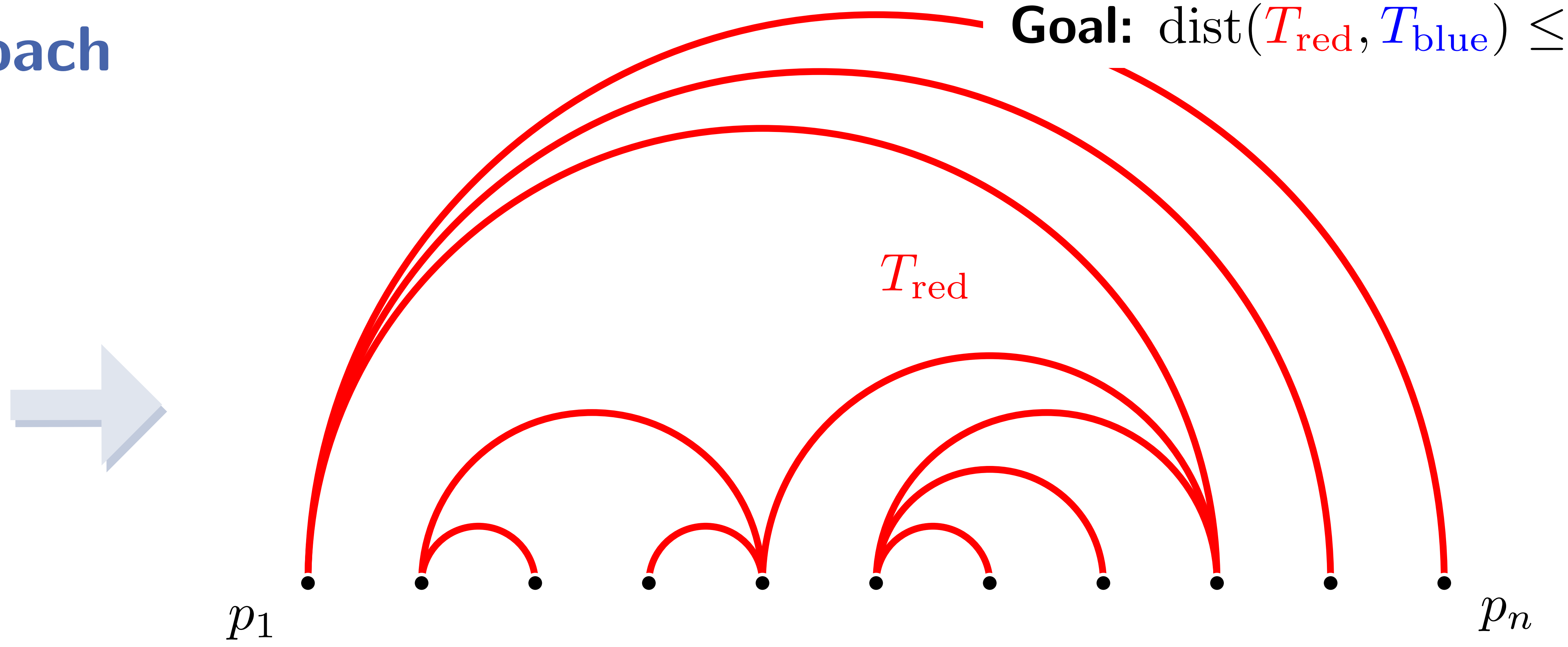
Order and Geometry
Lutherstadt Wittenberg, Germany
September 10, 2024

Overview of Approach

Goal: $\text{dist}(T_{\text{red}}, T_{\text{blue}}) \leq \frac{5}{3}n$

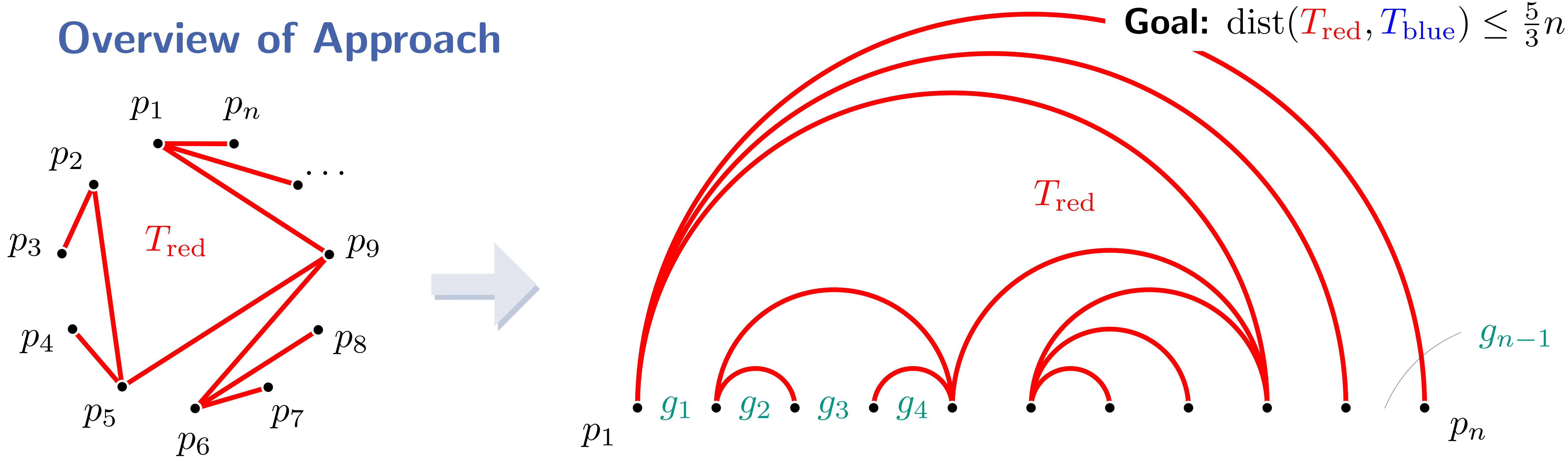




▷ open to linear order $p_1, \dots, p_n$

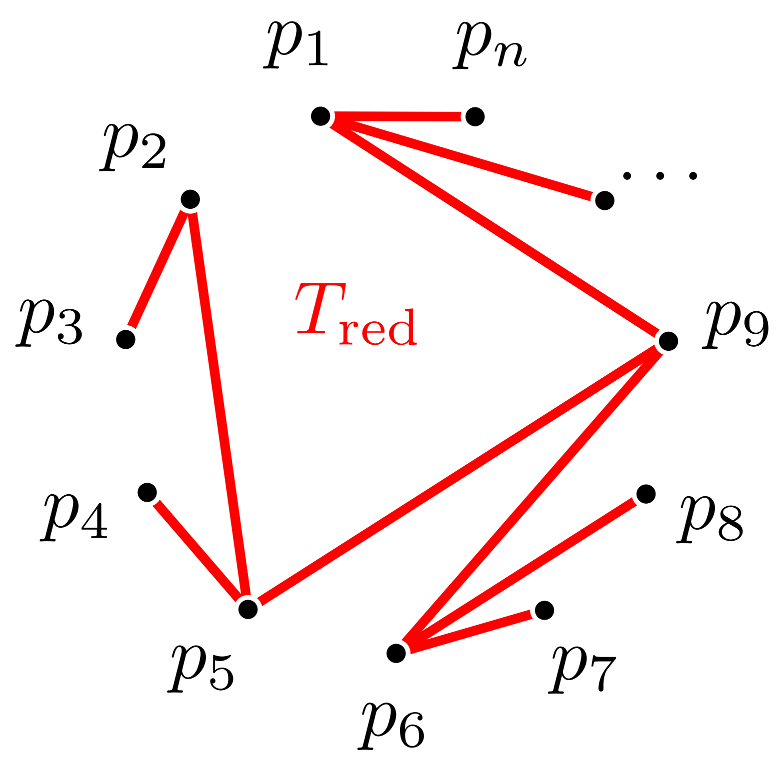
Overview of Approach

Goal: $\mathrm{dist}(T_{\mathrm{red}}, T_{\mathrm{blue}}) \leq \frac{5}{3}n$

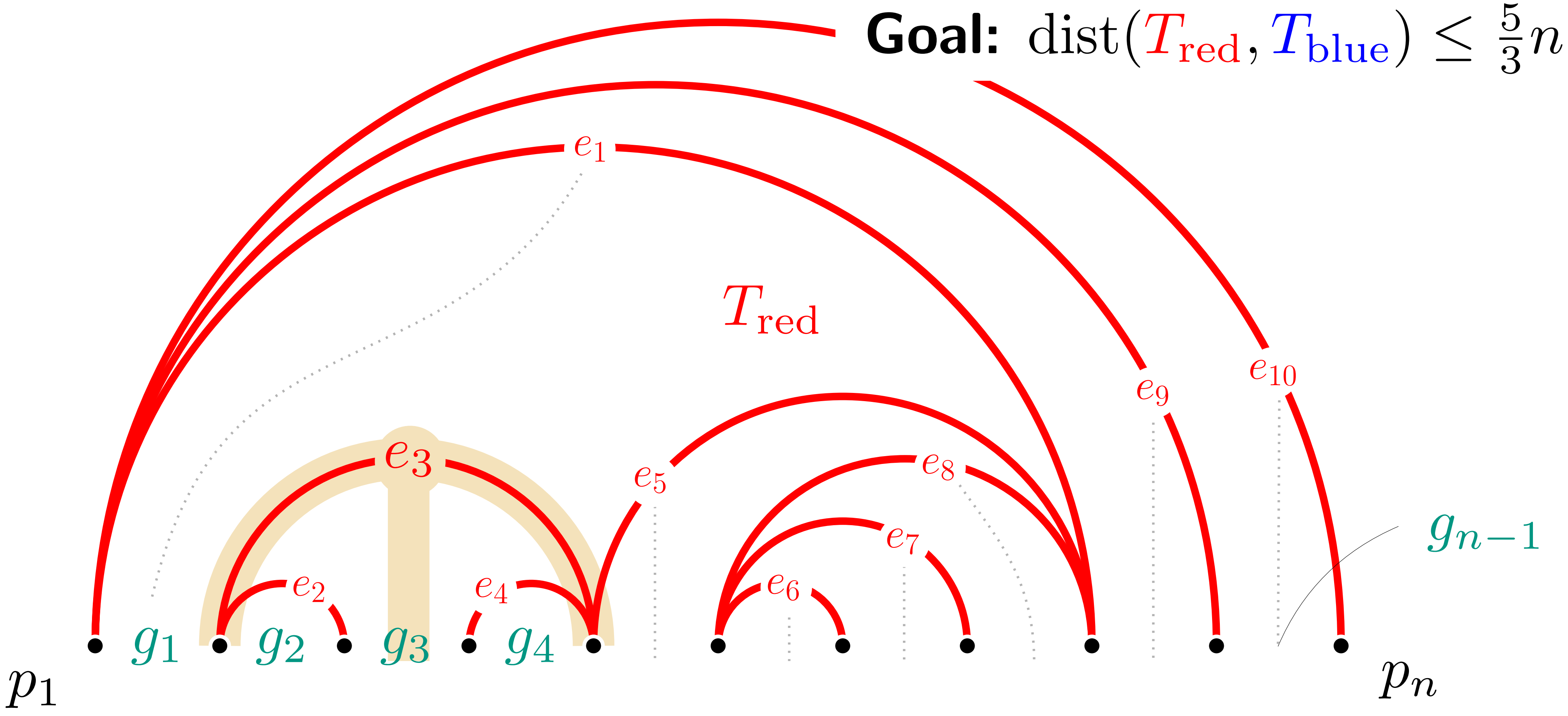


- ▷ open to linear order $p_1, \dots, p_n$
- ▷ gaps $g_1, \dots, g_{n-1}$

Overview of Approach



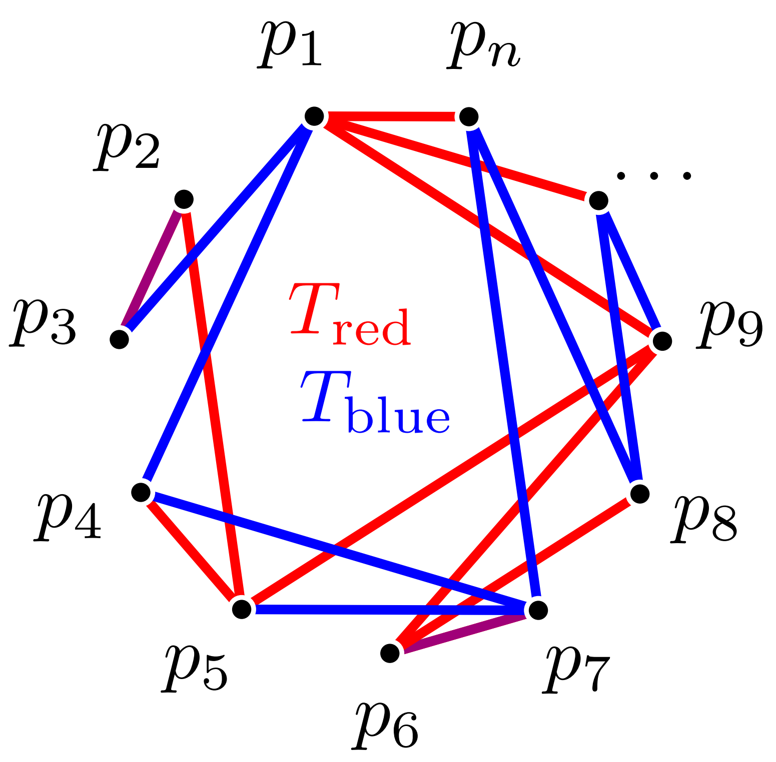
Goal: $\text{dist}(T_{\text{red}}, T_{\text{blue}}) \leq \frac{5}{3}n$

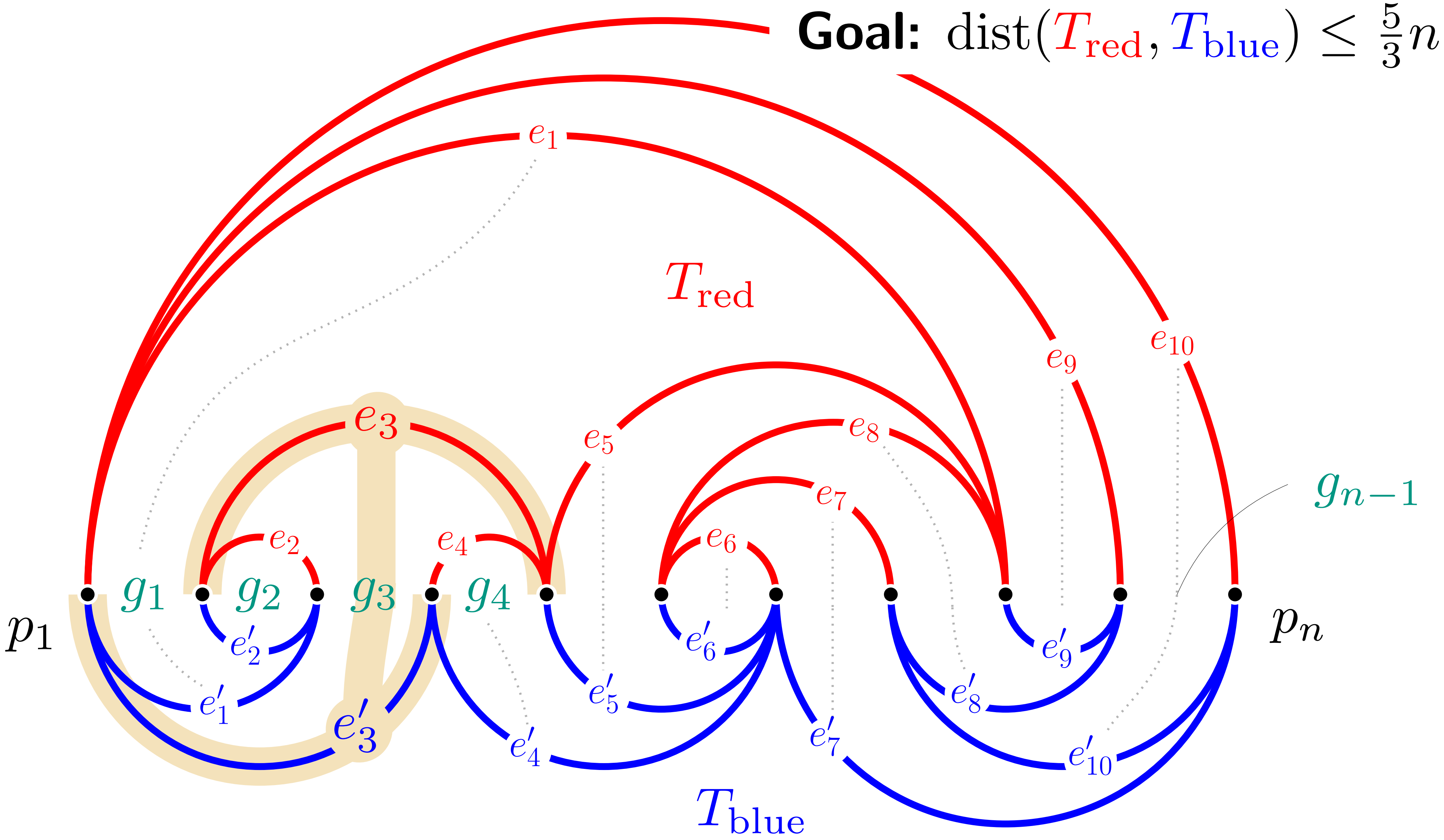


- ▷ open to linear order $p_1, \dots, p_n$
- ▷ gaps $g_1, \dots, g_{n-1}$
- ▷ corresponding edges $e_1, \dots, e_{n-1}$ of T_{red}

Overview of Approach

Goal: $\mathrm{dist}(T_{\mathrm{red}}, T_{\mathrm{blue}}) \leq \frac{5}{3}n$

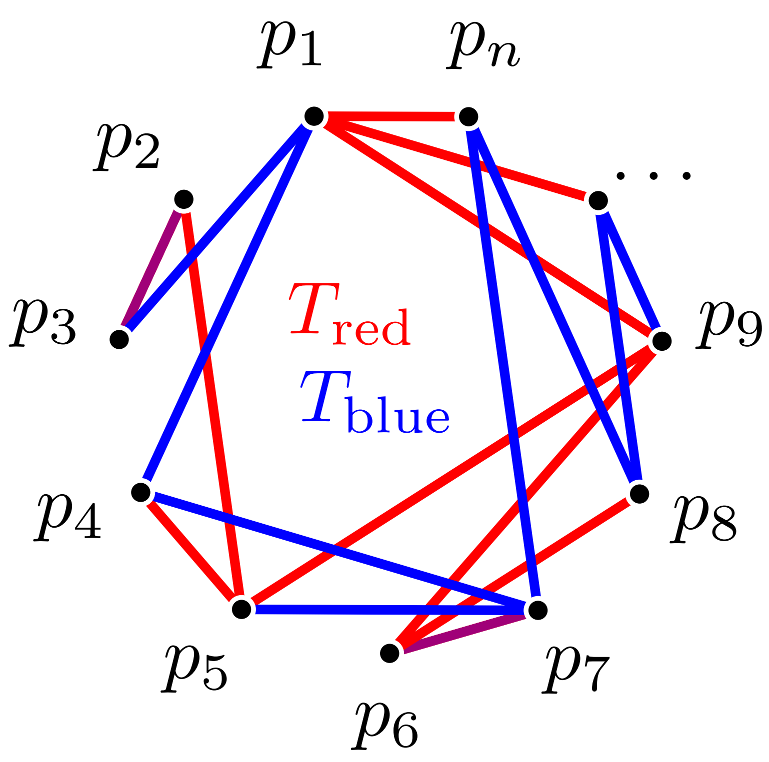


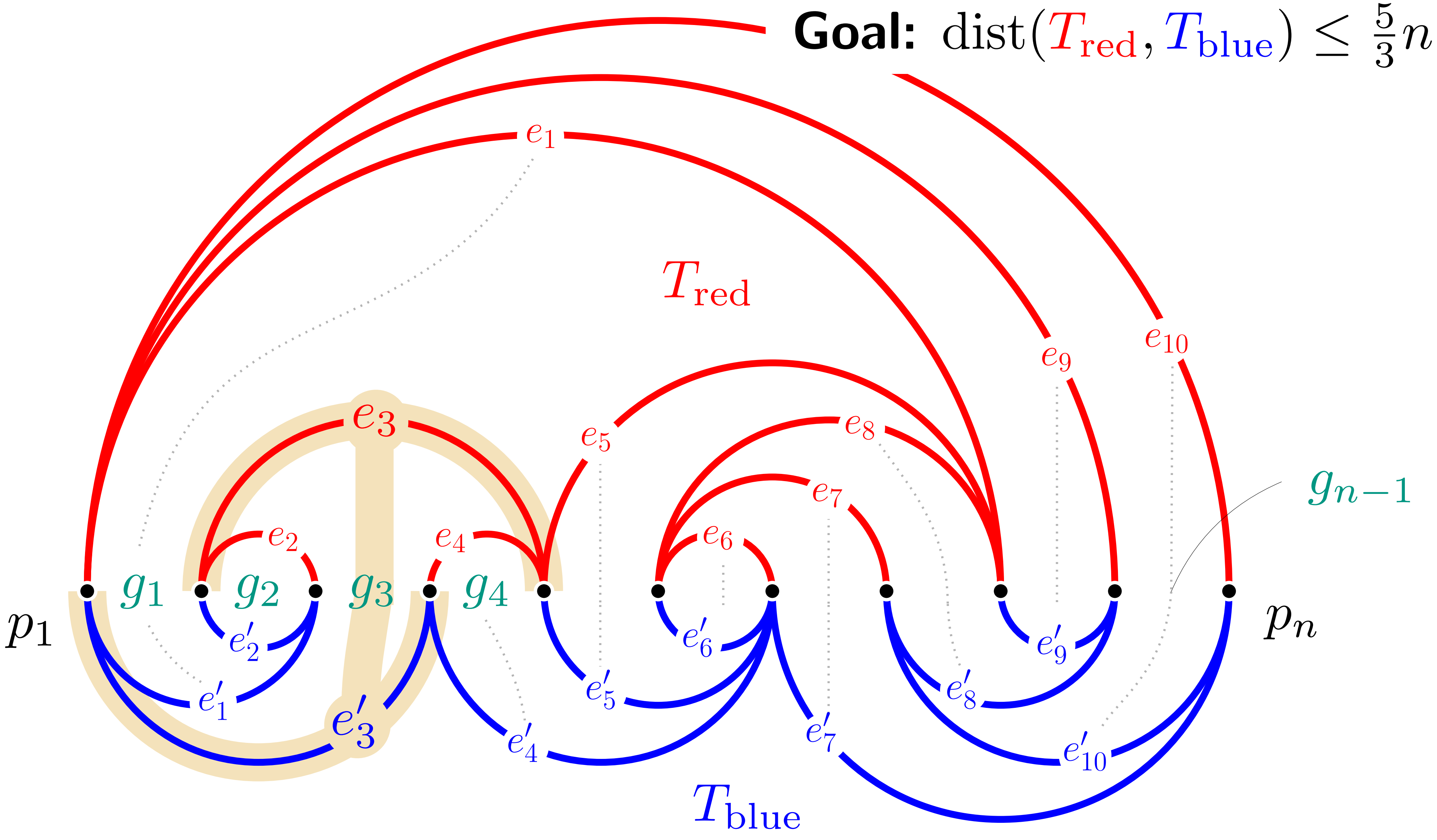


- ▷ open to linear order $p_1, \ldots, p_n$
- ▷ gaps $g_1, \ldots, g_{n-1}$
- ▷ corresponding edges $e_1, \ldots, e_{n-1}$ of T_{red}
- ▷ same for T_{blue} gives $e'_1, \ldots, e'_{n-1}$
- ▷ edge pairs $(e_1, e'_1), \ldots, (e_{n-1}, e'_{n-1})$

Overview of Approach

Goal: $\text{dist}(T_{\text{red}}, T_{\text{blue}}) \leq \frac{5}{3}n$





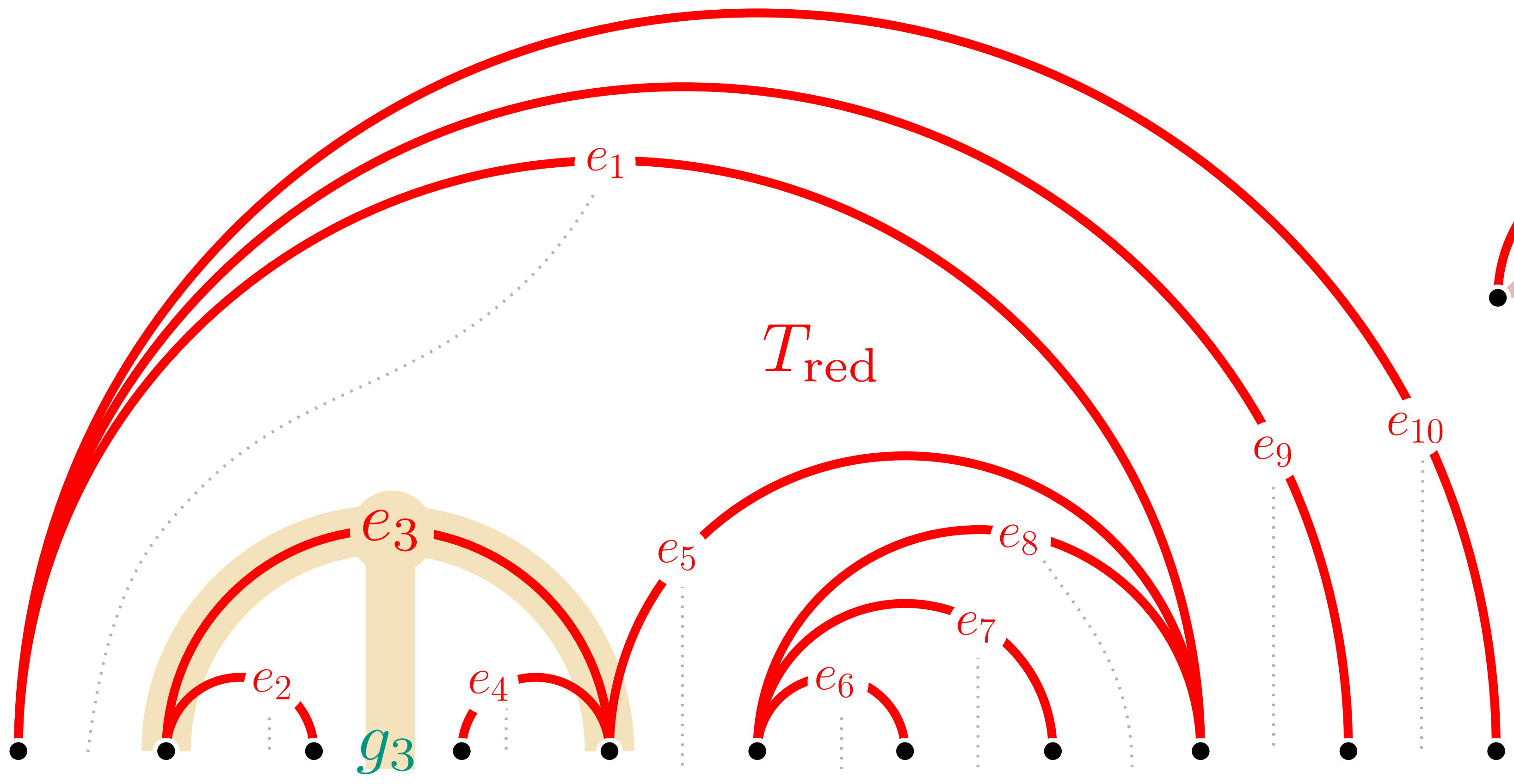
- ▷ open to linear order $p_1, \dots, p_n$
- ▷ gaps $g_1, \dots, g_{n-1}$
- ▷ corresponding edges $e_1, \dots, e_{n-1}$ of T_{red}
- ▷ same for T_{blue} gives $e'_1, \dots, e'_{n-1}$
- ▷ edge pairs $(e_1, e'_1), \dots, (e_{n-1}, e'_{n-1})$

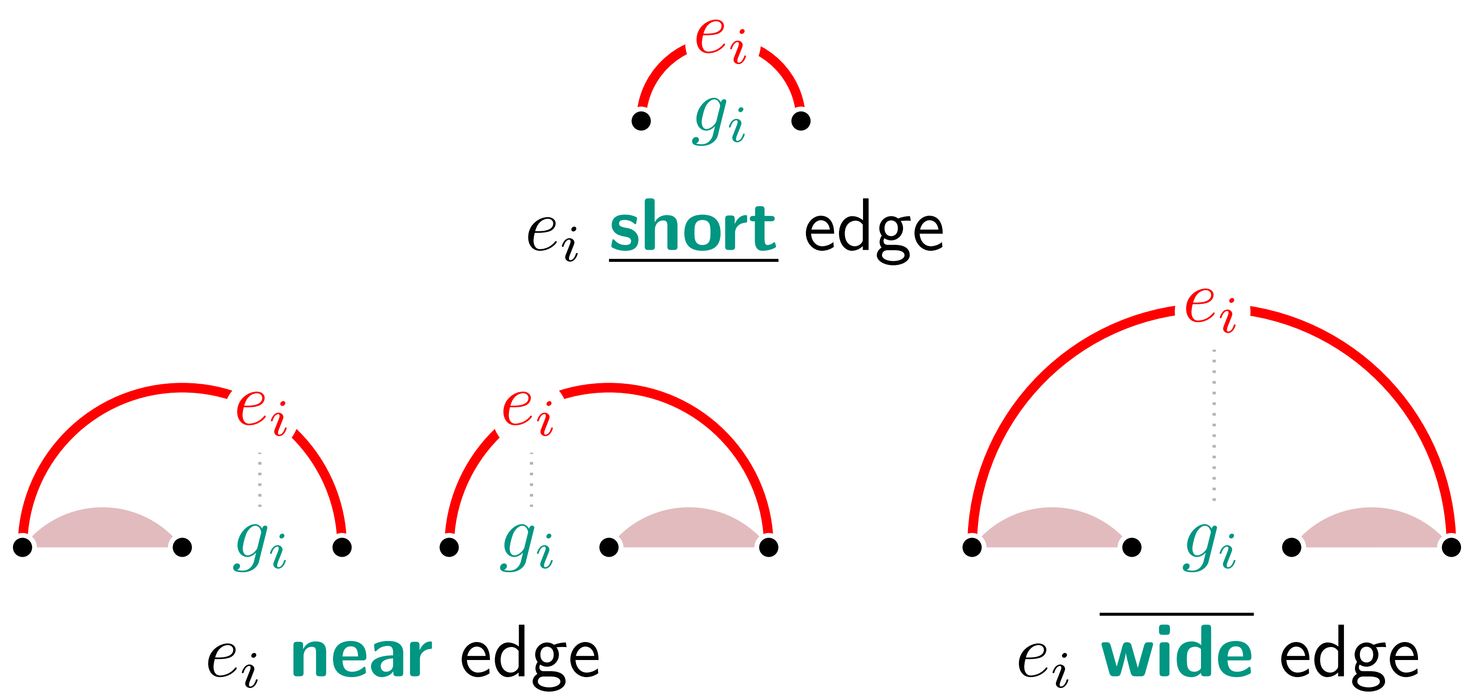
direct flip $e_i \to e'_i$

indirect flip $e_i \to g_i \to e'_i$

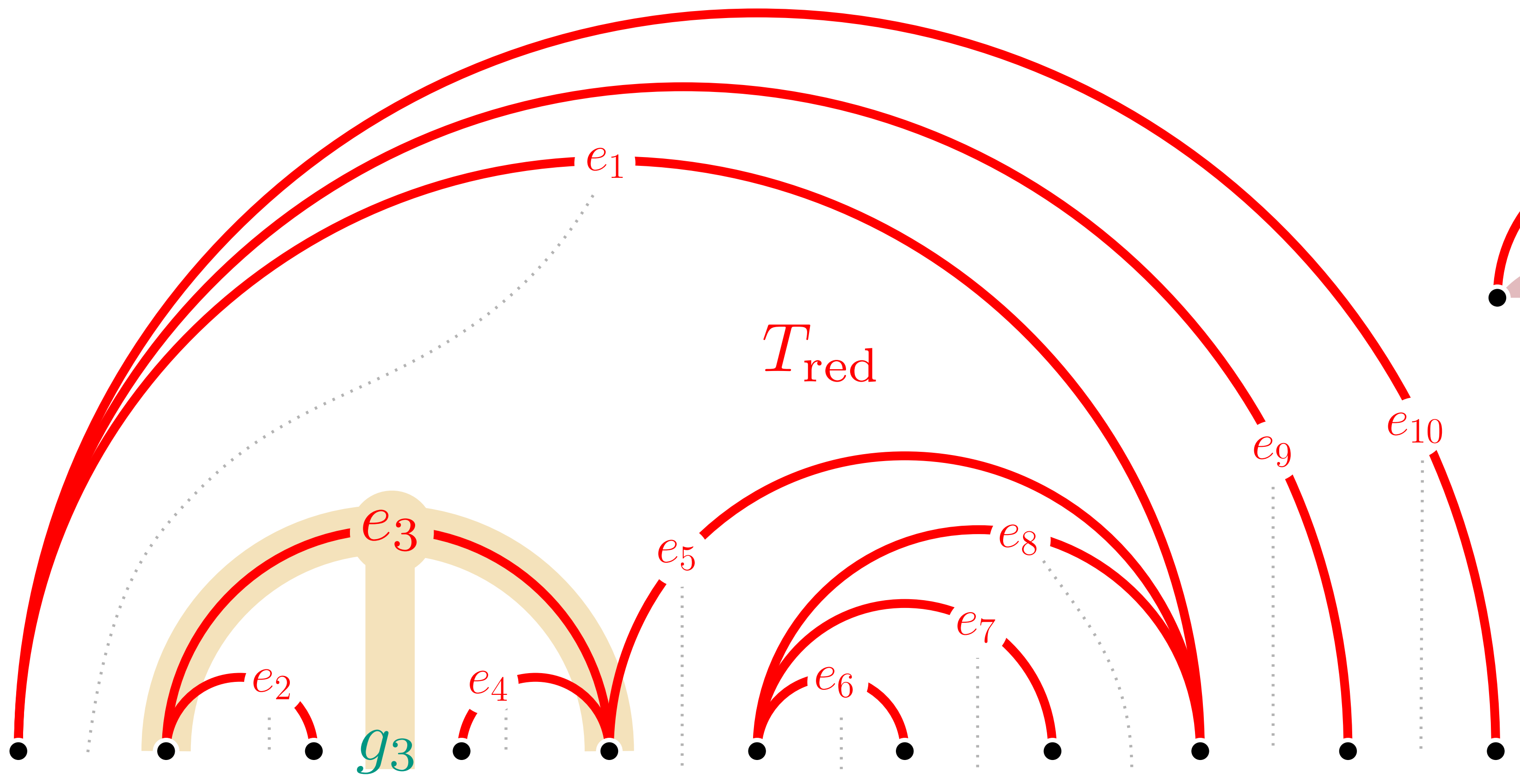
$\text{dist}(T_{\text{red}}, T_{\text{blue}}) \leq 2n -$ #**direct** flips

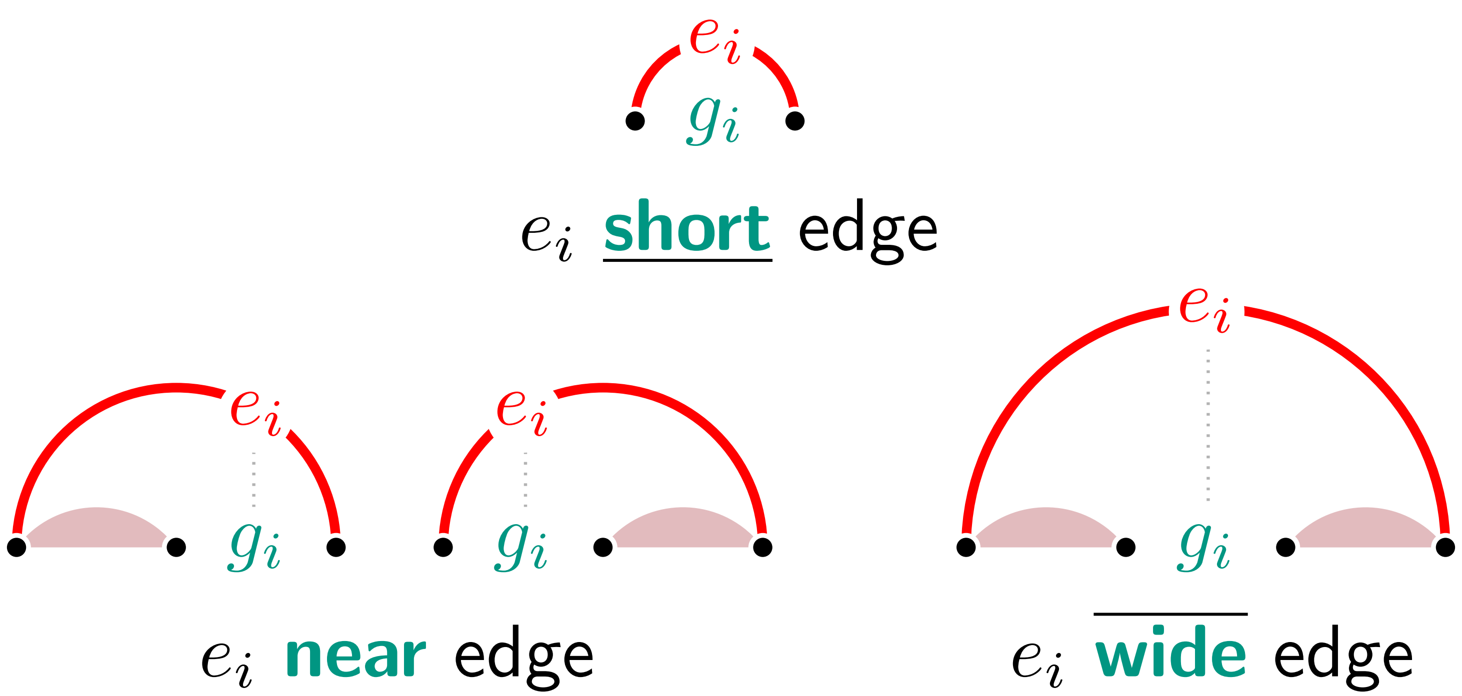
Short, Near and Wide Edges





Short, Near and Wide Edges



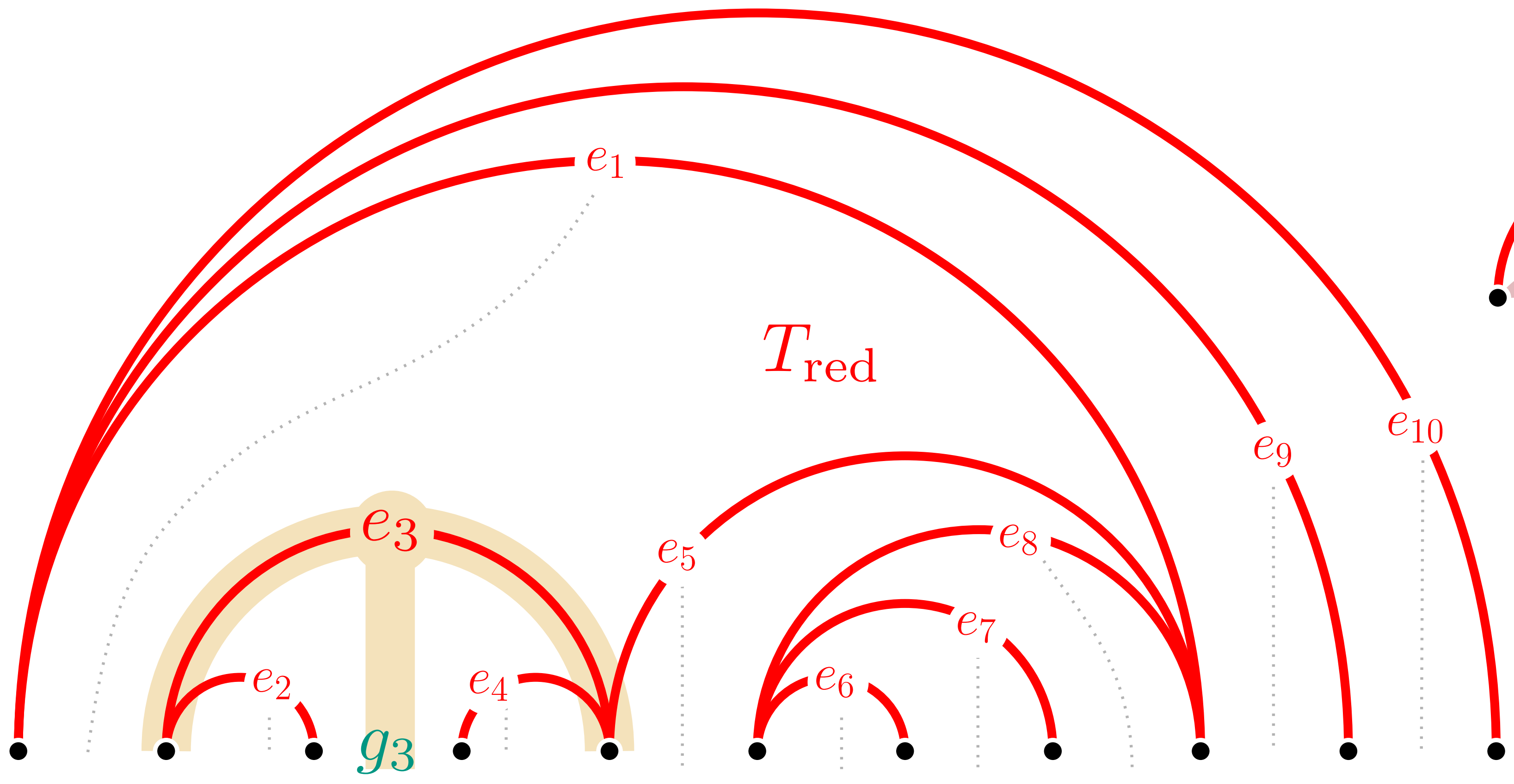


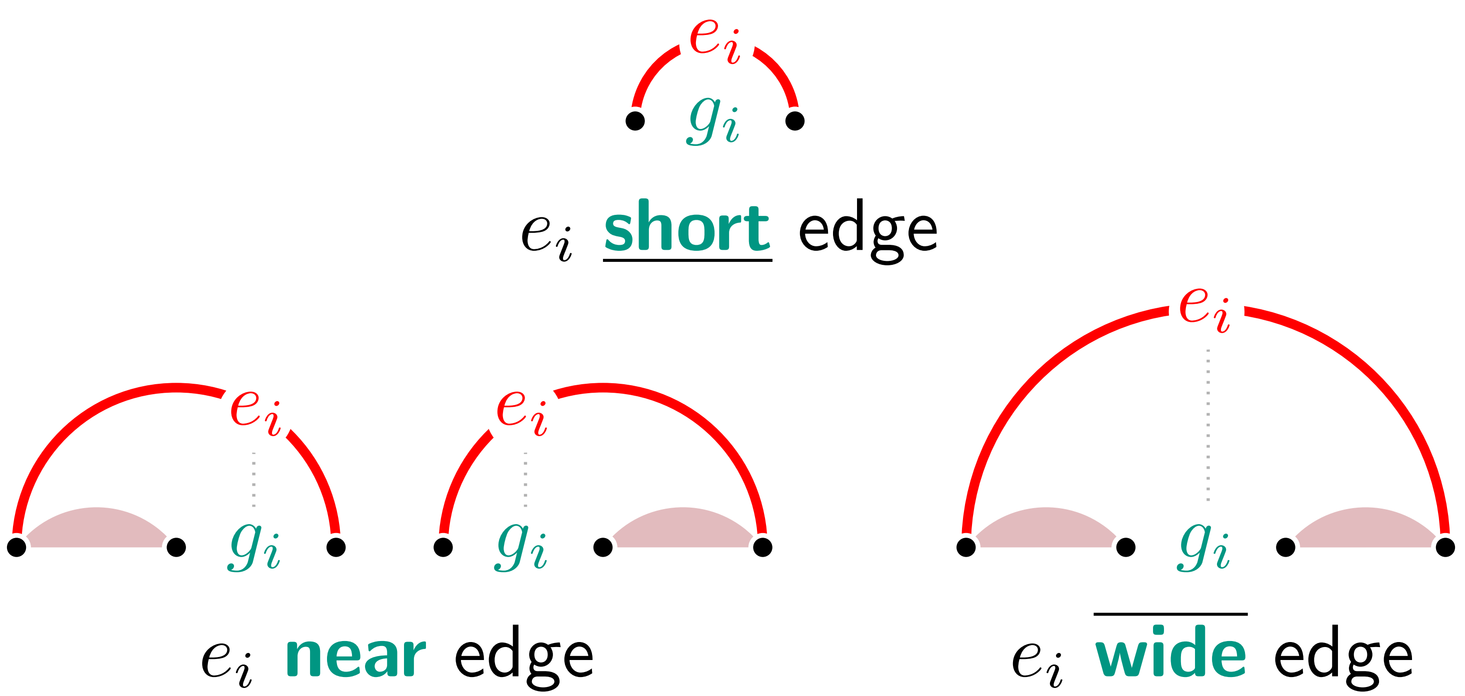
no flip $e_i = e'_i$

direct flip $e_i \to e'_i$

indirect flip $e_i \to g_i \to e'_i$

Short, Near and Wide Edges



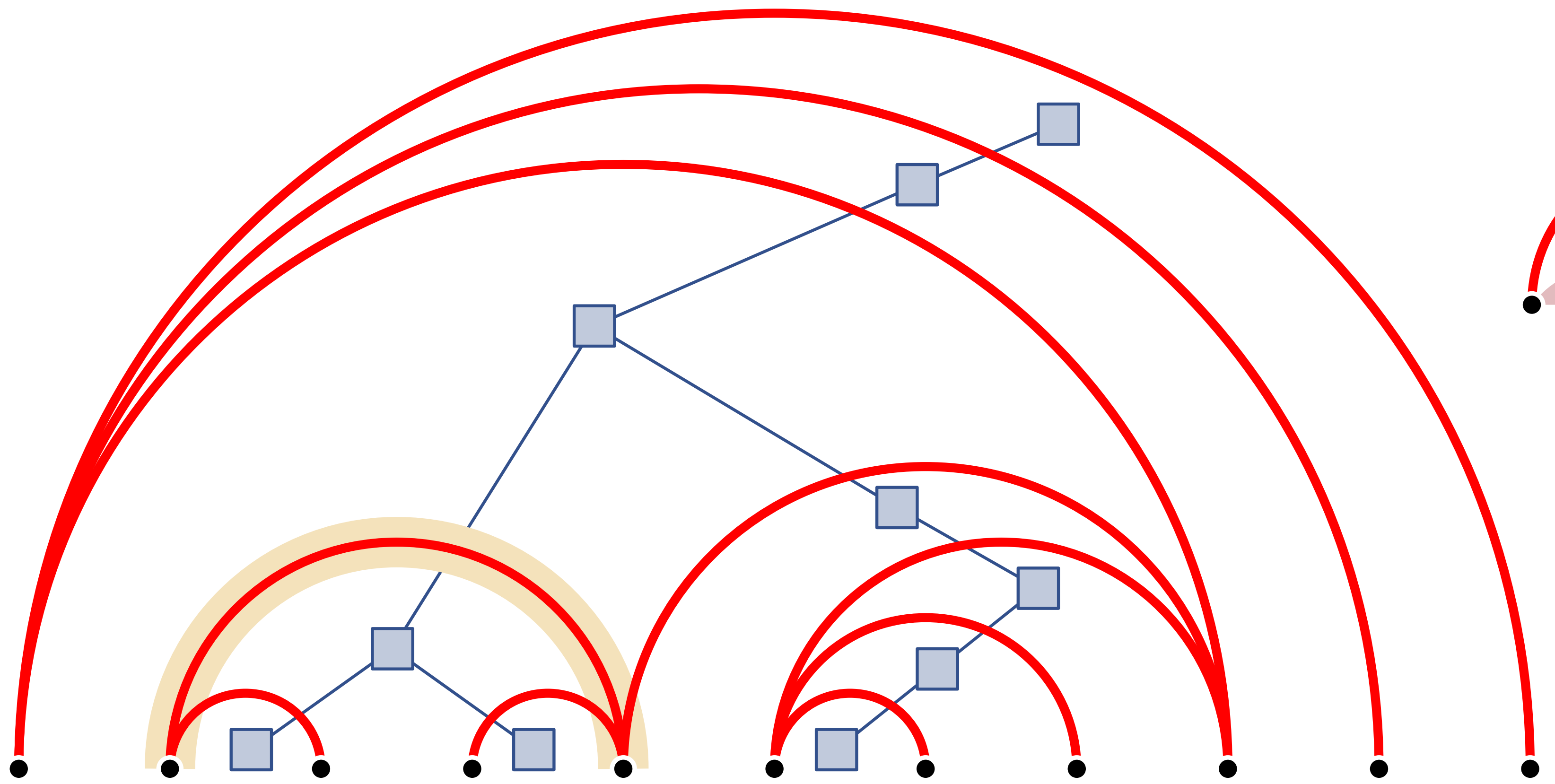


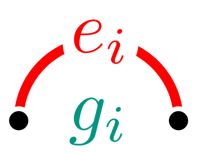
no flip $e_i = e_i'$: short-short pairs (e_i, e_i')

direct flip $e_i \to e_i'$: short-near and short-wide pairs (e_i, e_i')

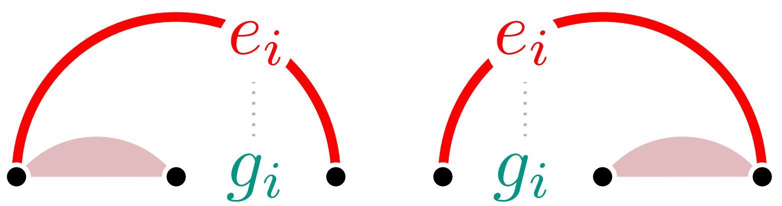
indirect flip $e_i \to g_i \to e_i'$: wide-near and wide-wide pairs (e_i, e_i')

Short, Near and Wide Edges

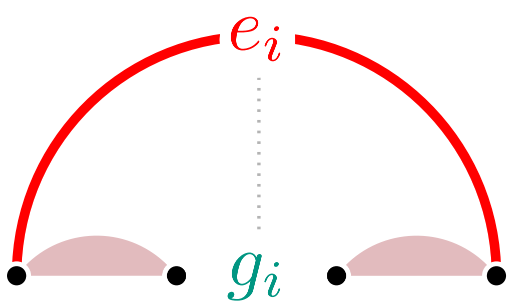




e_i **short** edge



e_i **near** edge



e_i **wide** edge

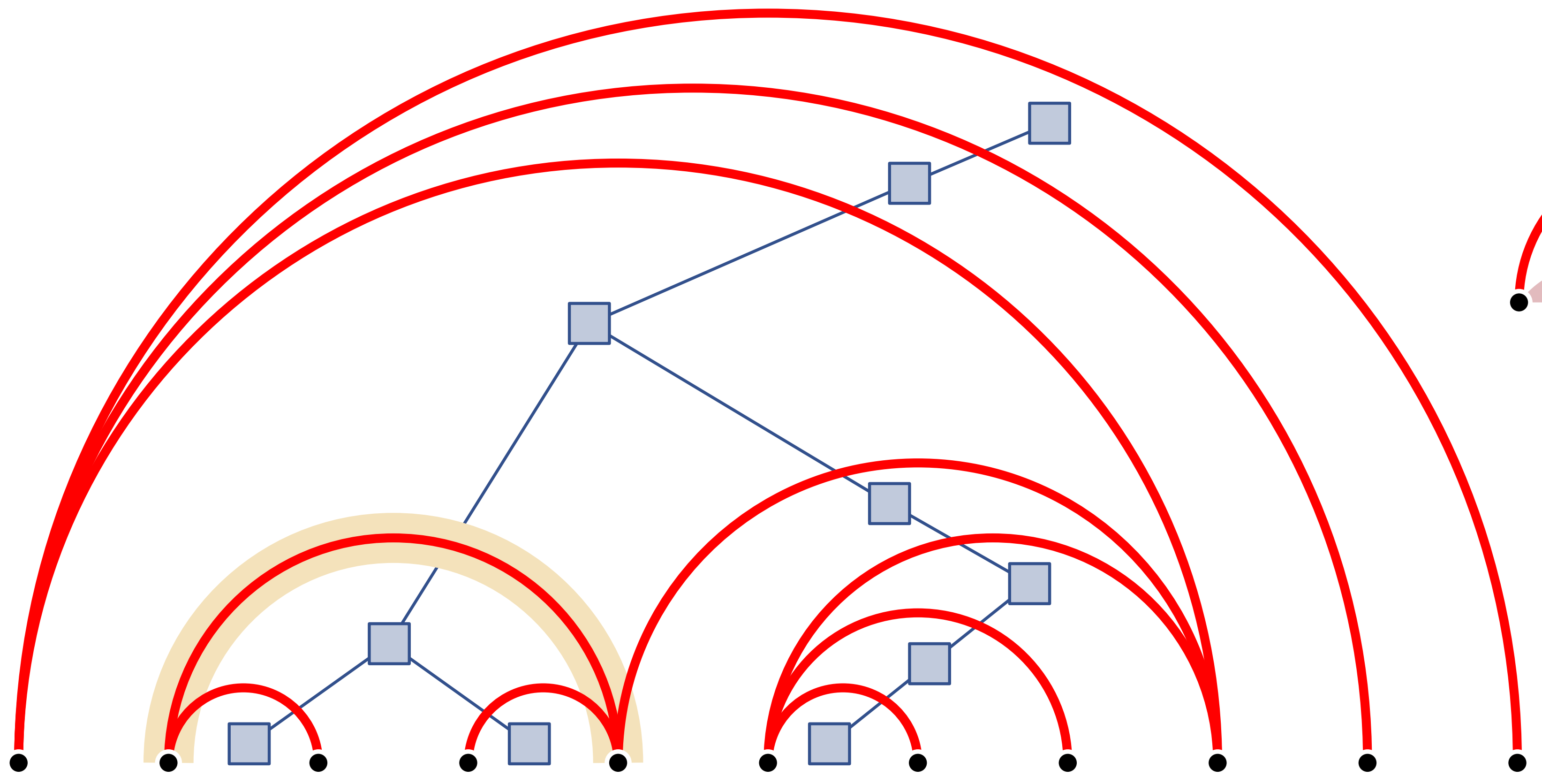
Lemma.
#**short** edges $>$ #**wide** edges

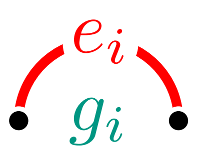
no flip $e_i = e'_i$: **short**-**short** pairs (e_i, e'_i)

direct flip $e_i \to e'_i$: **short**-**near** and **short**-**wide** pairs (e_i, e'_i)

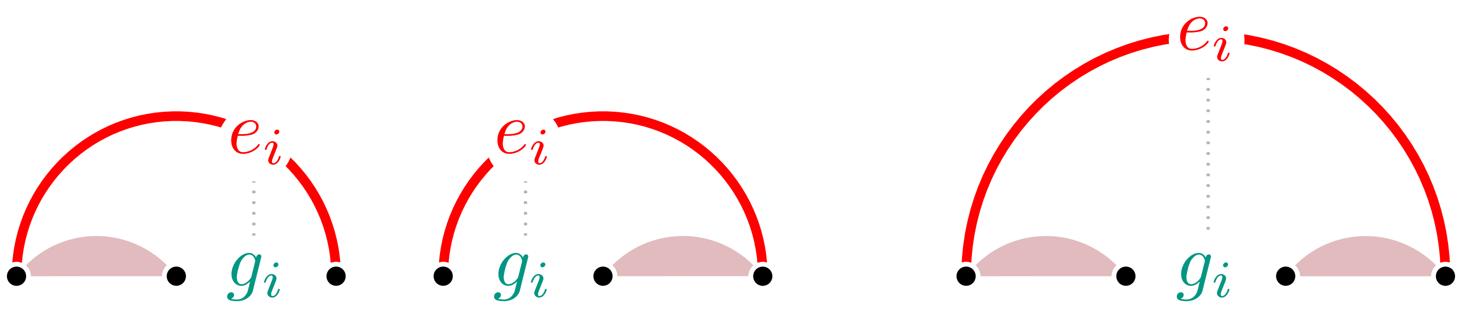
indirect flip $e_i \to g_i \to e'_i$: **wide**-**near** and **wide**-**wide** pairs (e_i, e'_i)

Short, Near and Wide Edges





e_i **short** edge



e_i **near** edge

e_i **wide** edge

Lemma.
#**short** edges > #**wide** edges

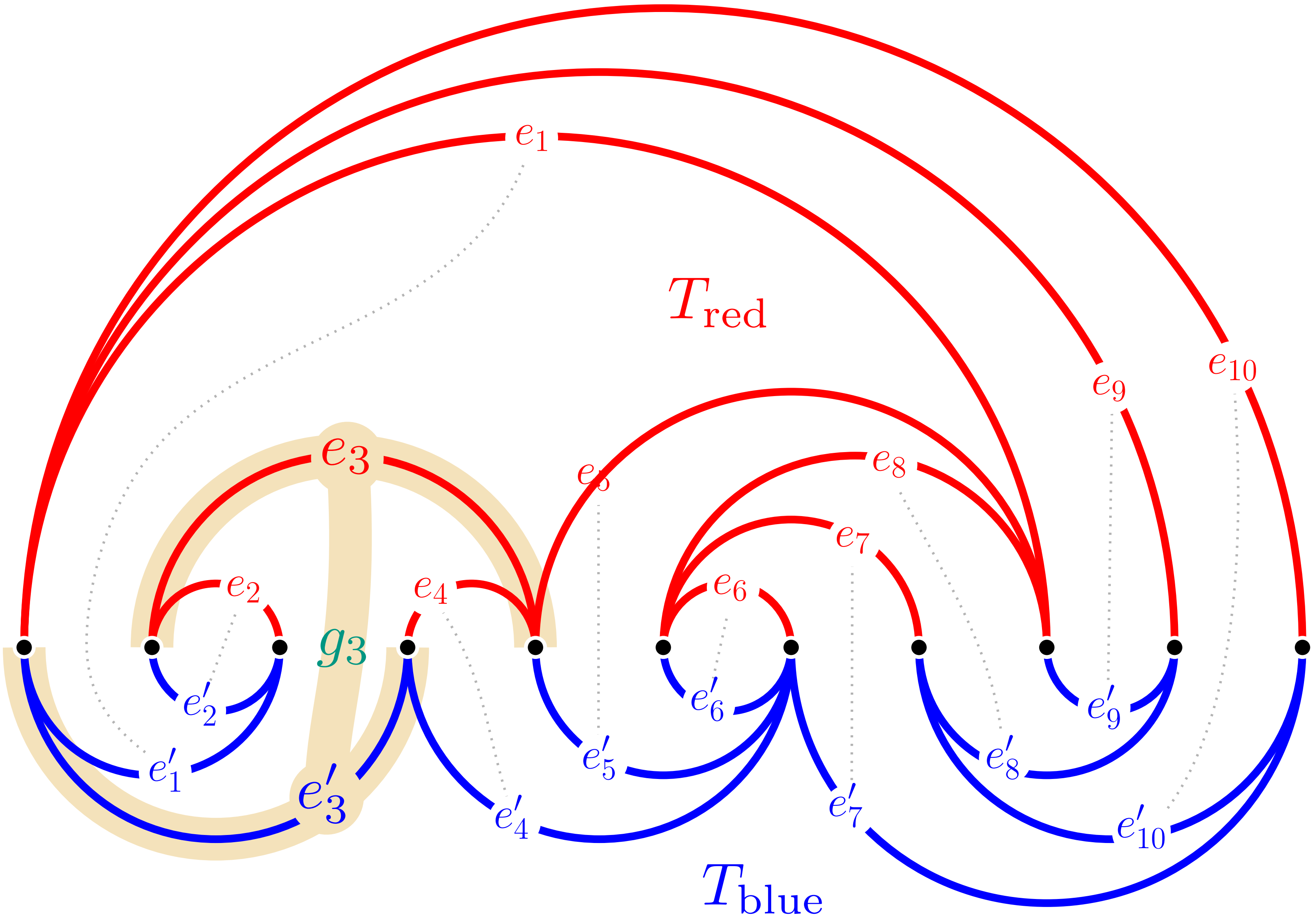
$\Rightarrow$ #flips so-far $\leq \frac{3}{2} \cdot$ #pairs so-far

no flip $e_i = e'_i$	:	**short-short** pairs (e_i, e'_i)
direct flip $e_i \to e'_i$	:	**short-near** and **short-wide** pairs (e_i, e'_i)
indirect flip $e_i \to g_i \to e'_i$	:	**wide-near** and **wide-wide** pairs (e_i, e'_i)

remaining:
near-near pairs

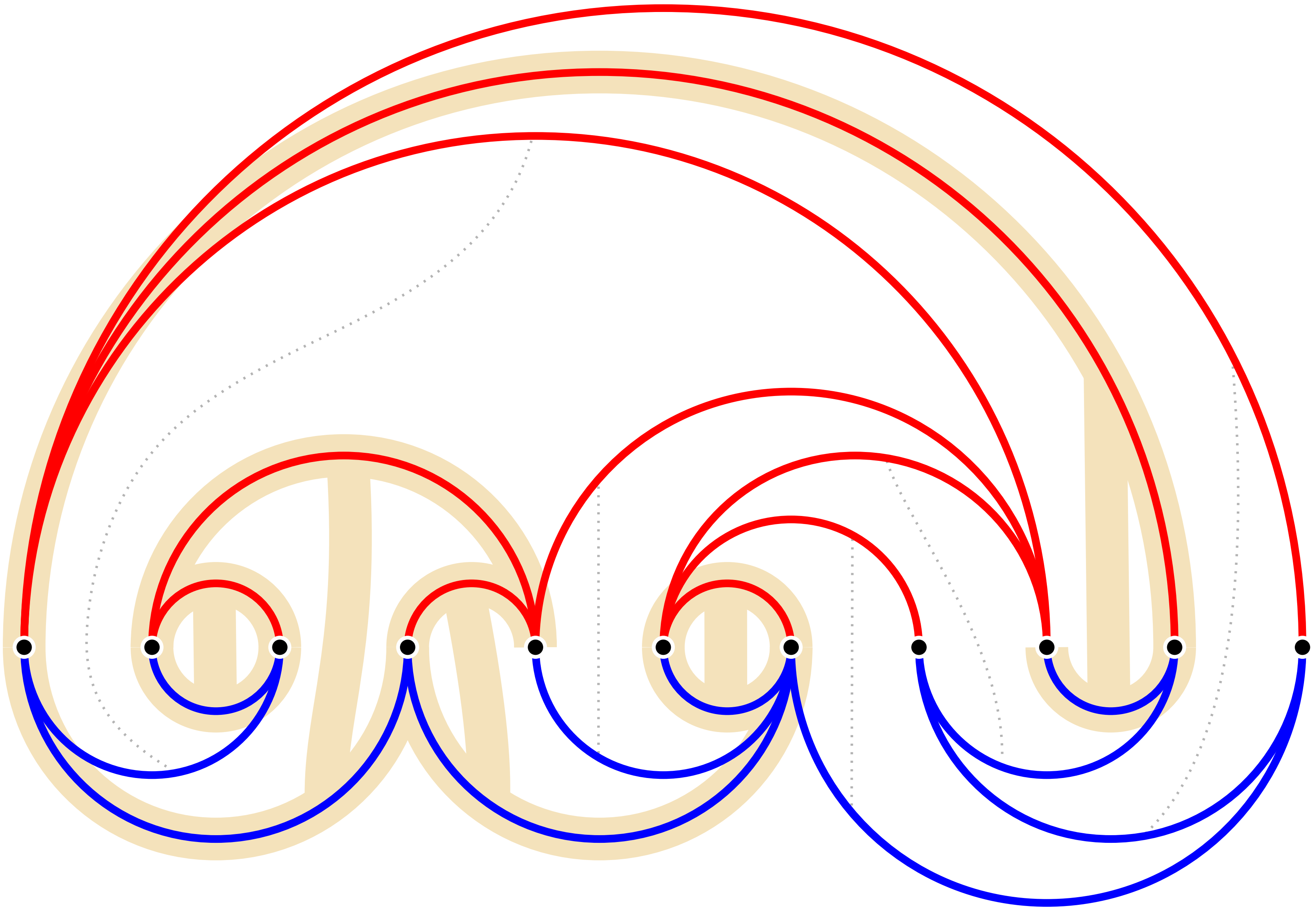
Conflicts

Goal: **direct** flip $e_i \to e'_i$ for many **near-near** pairs (e_i, e'_i)



Conflicts

Goal: **direct** flip $e_i \to e'_i$ for many **near**-**near** pairs (e_i, e'_i)



▷ treat non-**near**-**near** pairs

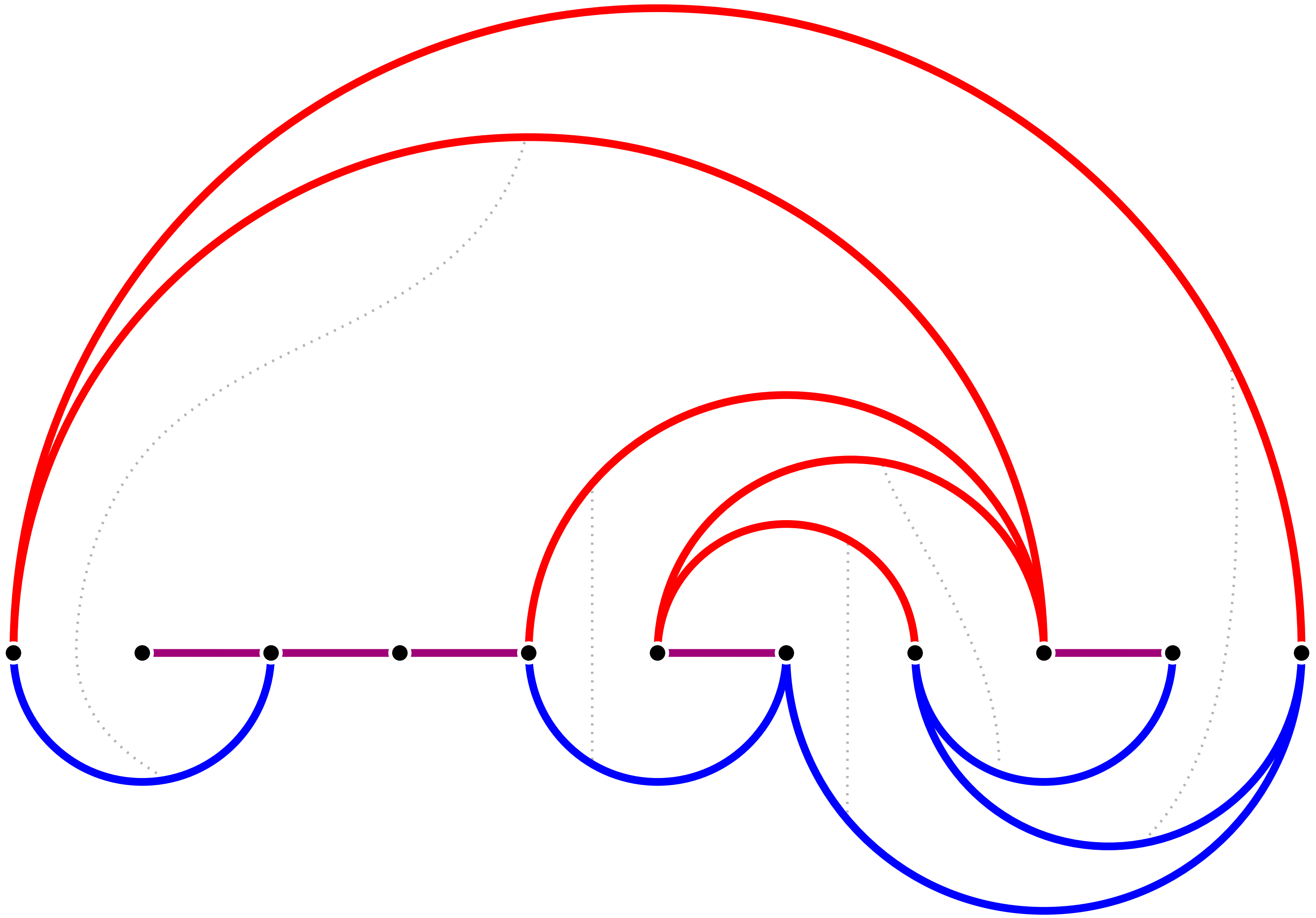
…spending 1.5 flips per pair on average

Conflicts

Goal: **direct** flip $e_i \to e'_i$ for many **near-near** pairs (e_i, e'_i)

▷ treat non-**near-near** pairs

…spending 1.5 flips per pair on average



Conflicts

Goal: **direct** flip $e_i \to e'_i$ for many **near-near** pairs (e_i, e'_i)

▷ treat non-**near-near** pairs

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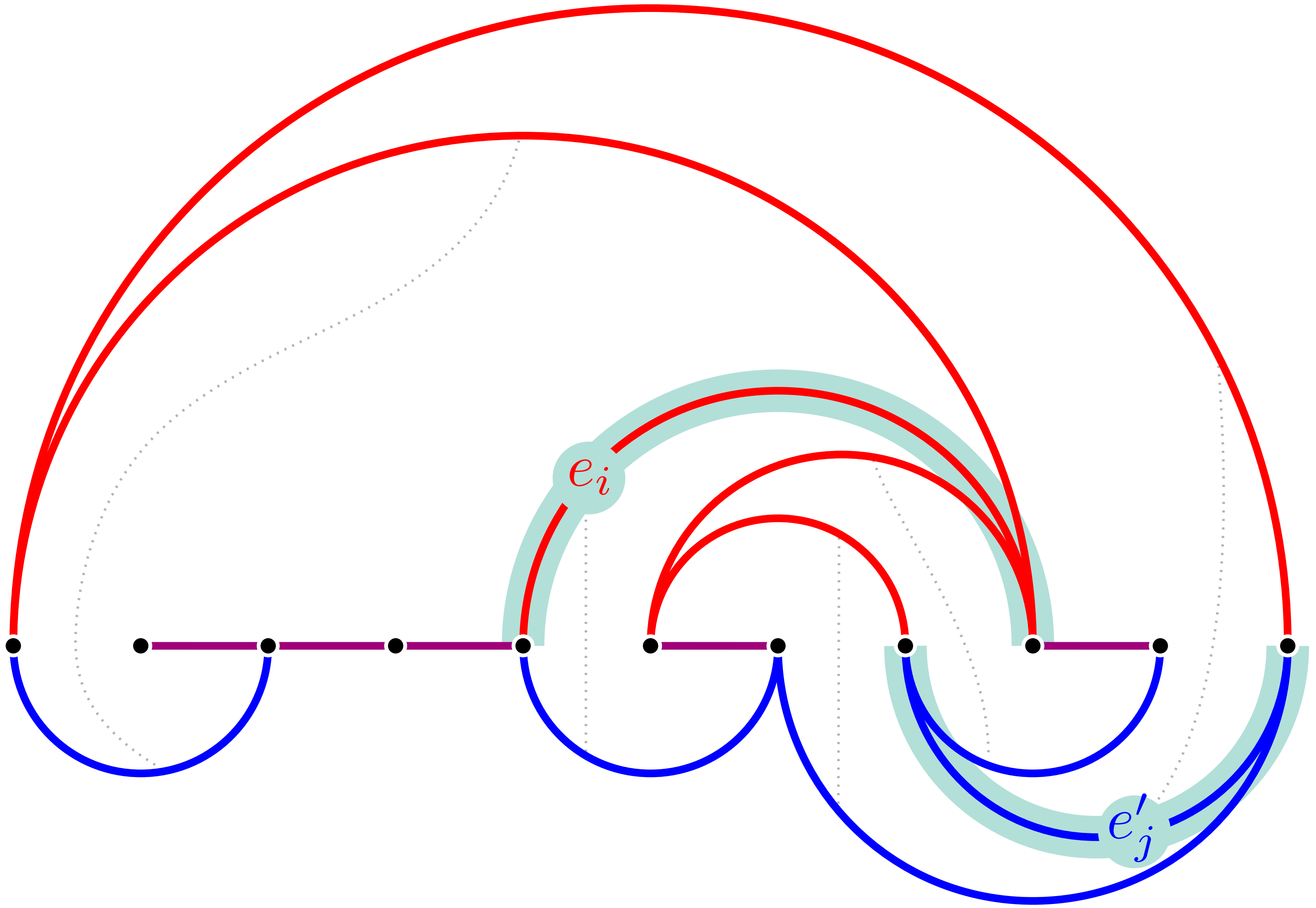
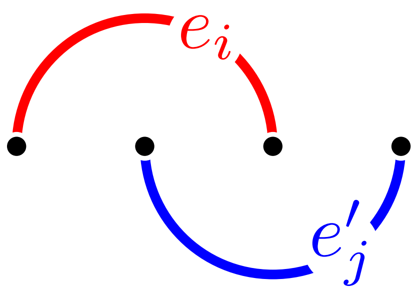


When (e_j, e'_j) must **wait** for (e_i, e'_i)

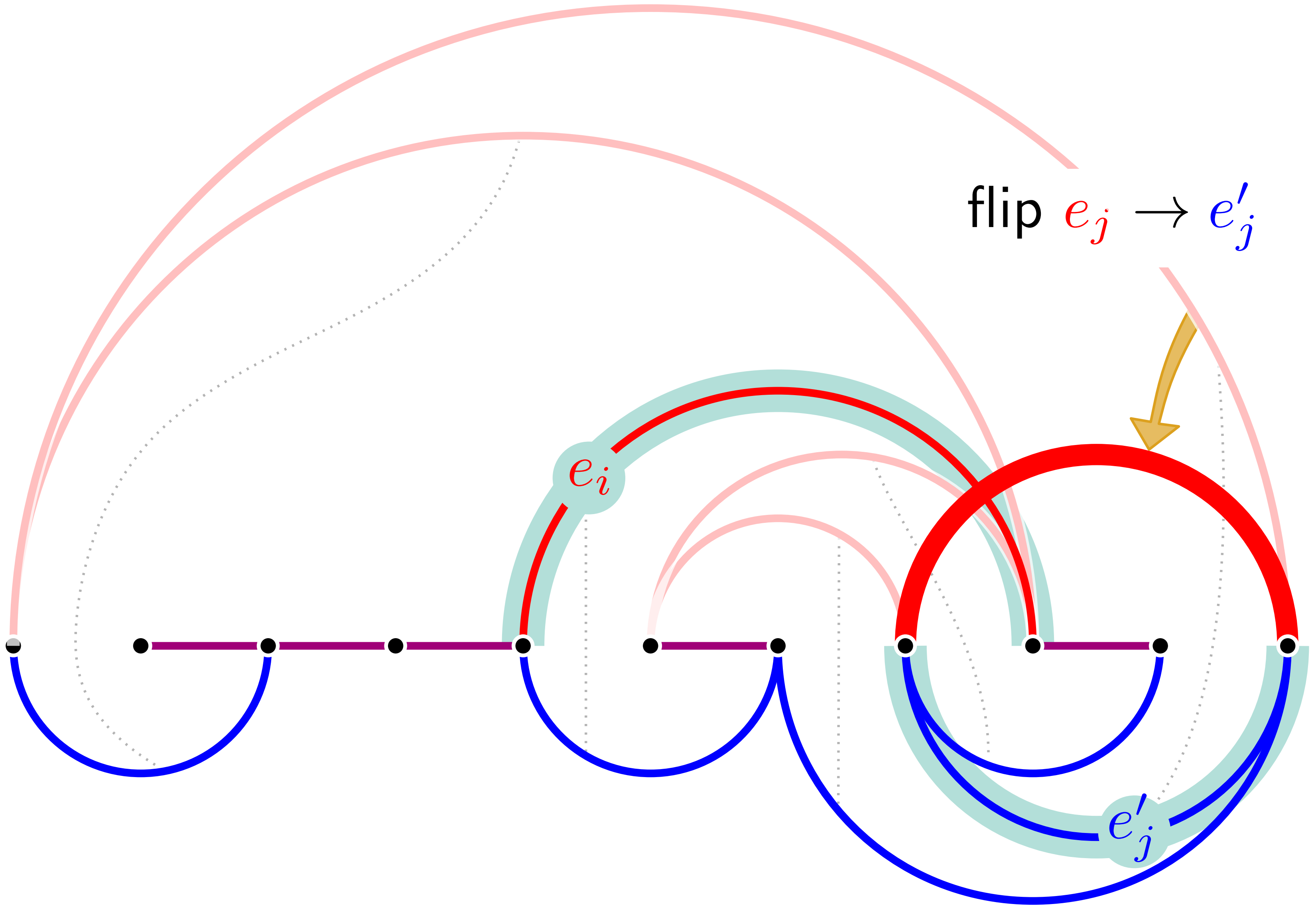


Conflicts

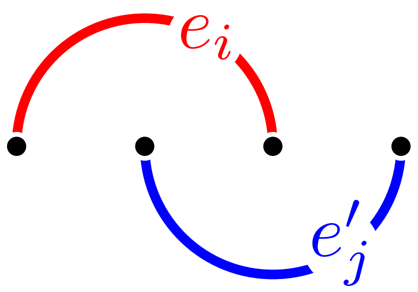
Goal: **direct** flip $e_i \to e'_i$ for many **near-near** pairs (e_i, e'_i)

▷ treat non-**near-near** pairs

…spending 1.5 flips per pair on average



When (e_j, e'_j) must **wait** for (e_i, e'_i)

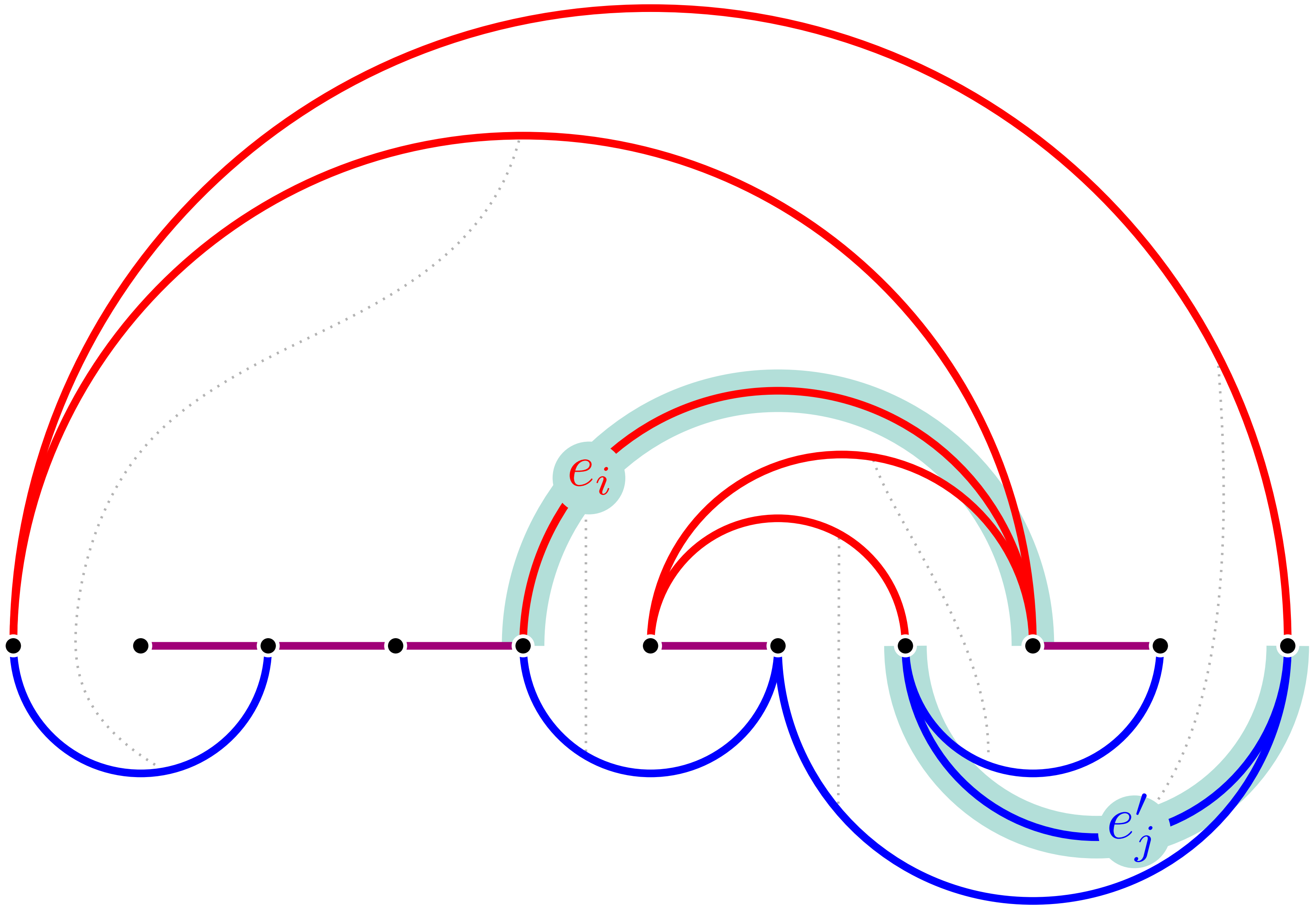


Conflicts

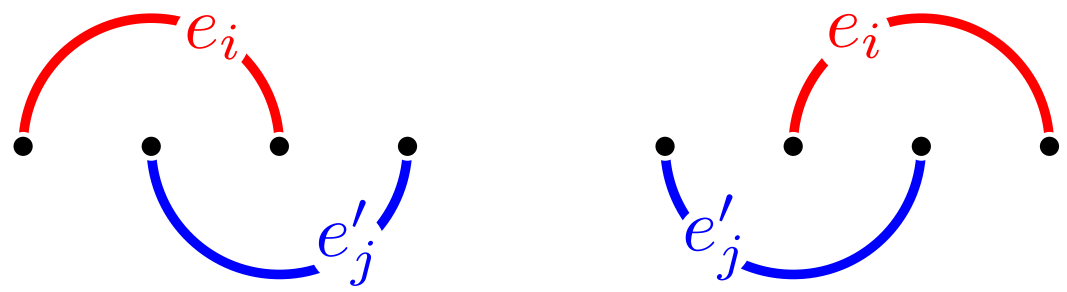
Goal: **direct** flip $e_i \rightarrow e'_i$ for many **near**-**near** pairs (e_i, e'_i)

▷ treat non-**near**-**near** pairs

…spending 1.5 flips per pair on average



When (e_j, e'_j) must **wait** for (e_i, e'_i)

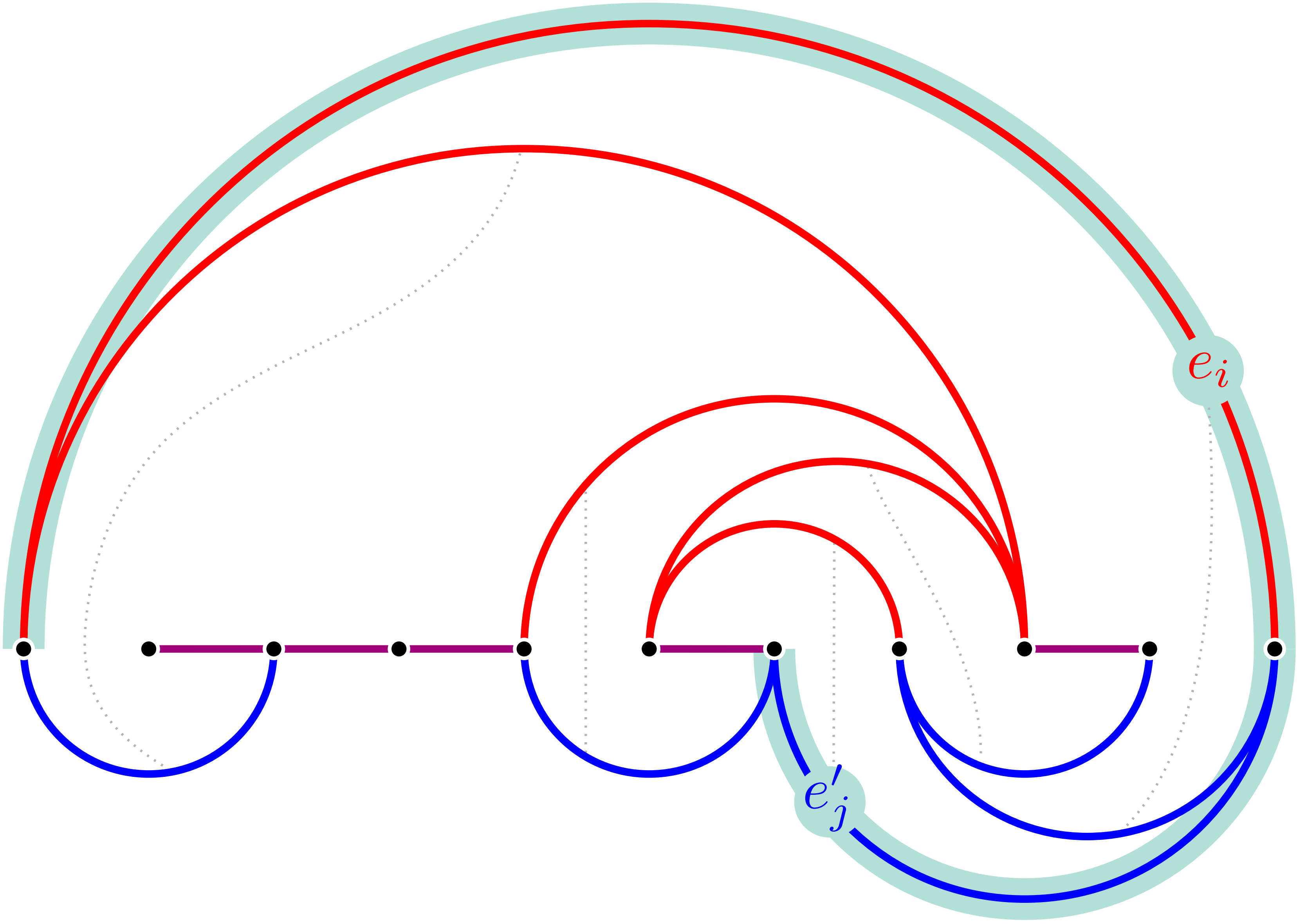


Conflicts

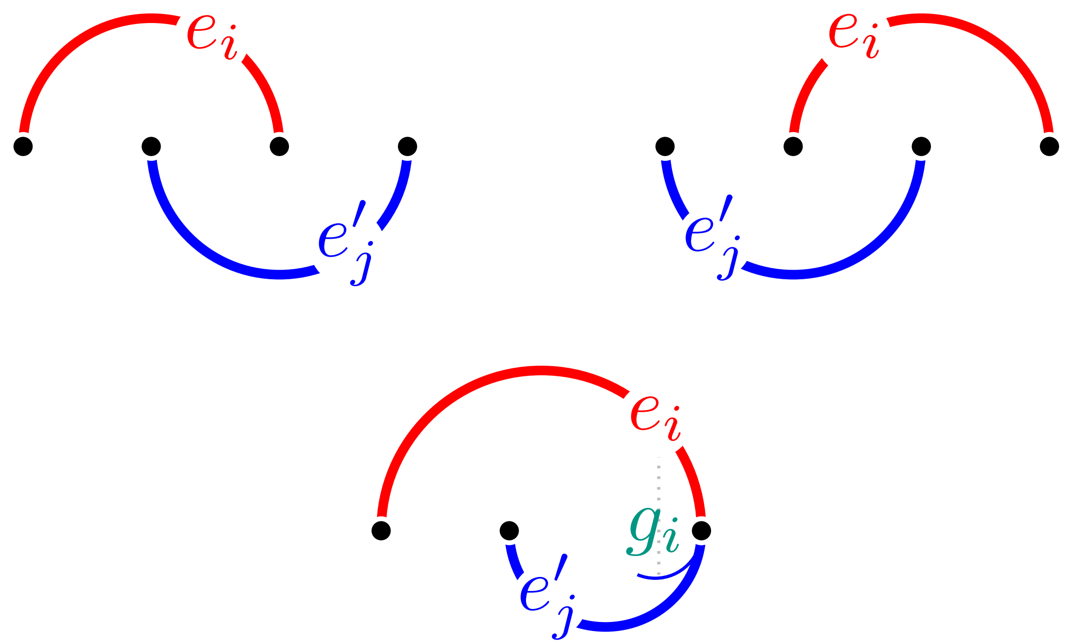
Goal: **direct** flip $e_i \to e'_i$ for many **near-near** pairs (e_i, e'_i)

▷ treat non-**near-near** pairs

...spending 1.5 flips per pair on average



When (e_j, e'_j) must **wait** for (e_i, e'_i)

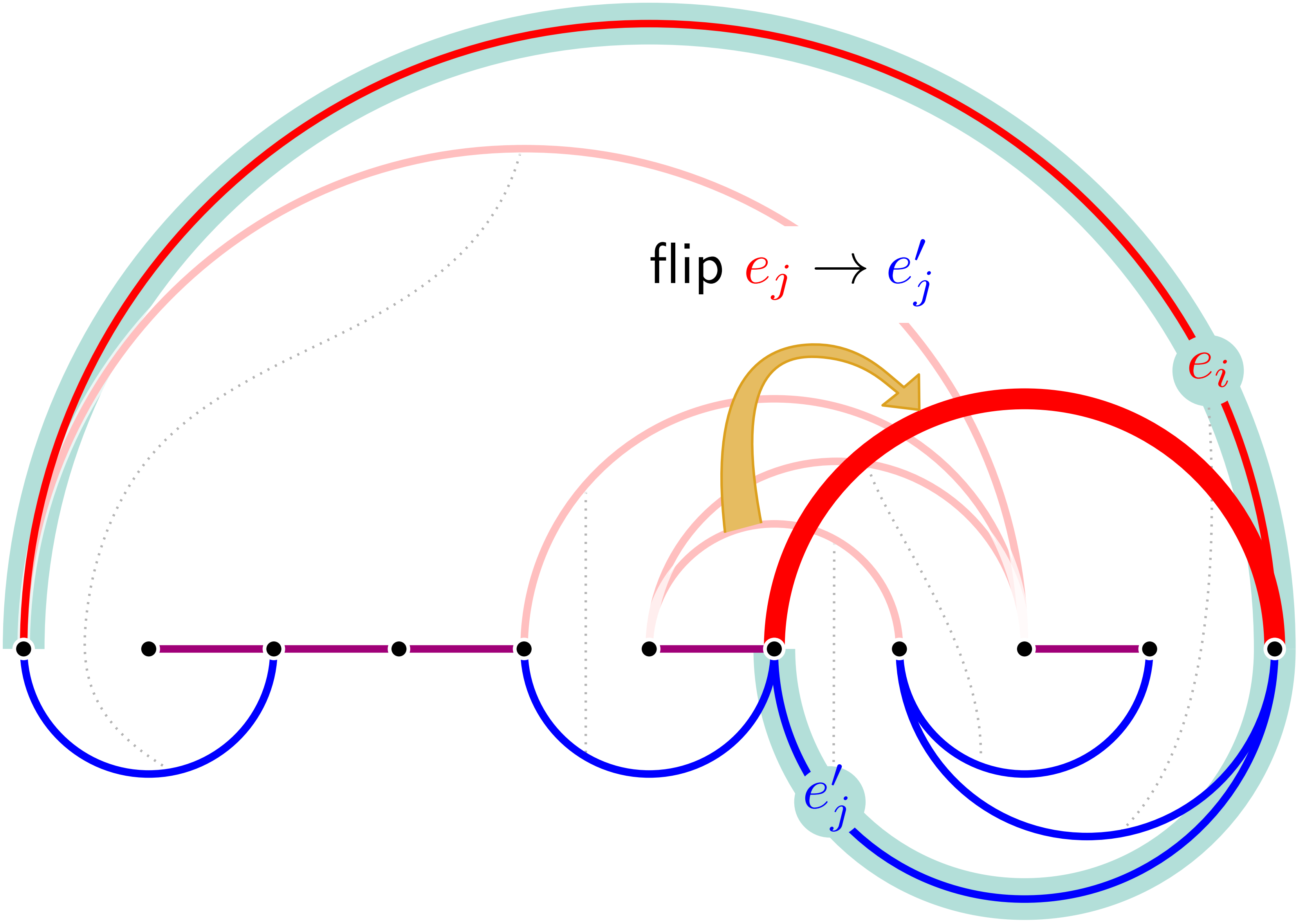


Conflicts

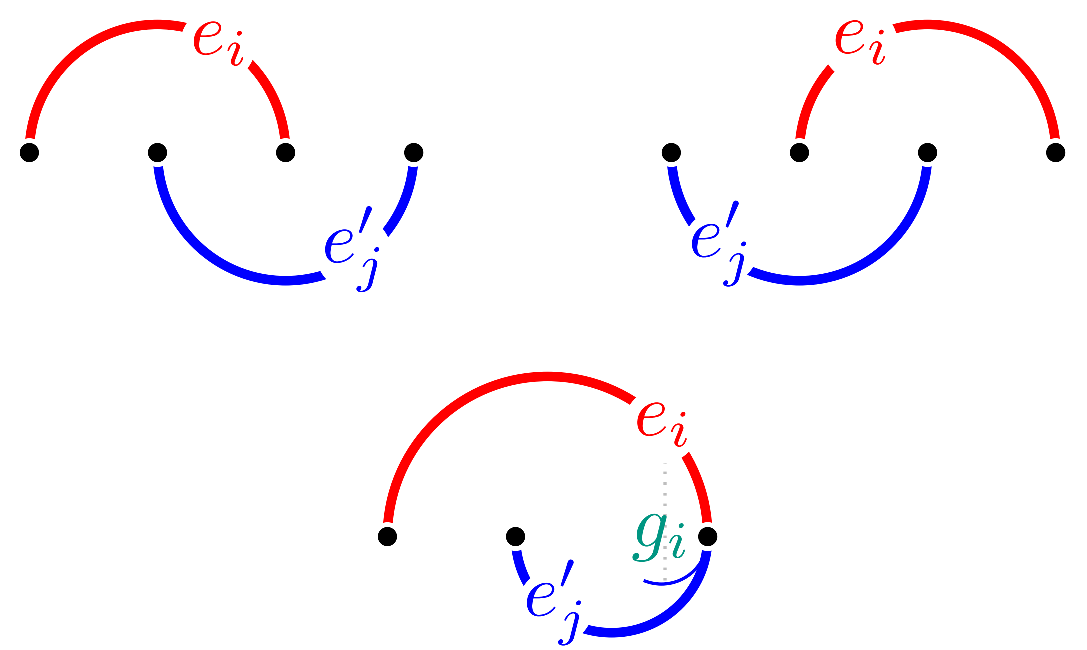
Goal: **direct** flip $e_i \to e'_i$ for many **near-near** pairs (e_i, e'_i)

▷ treat non-**near-near** pairs

…spending 1.5 flips per pair on average



When (e_j, e'_j) must **wait** for (e_i, e'_i)

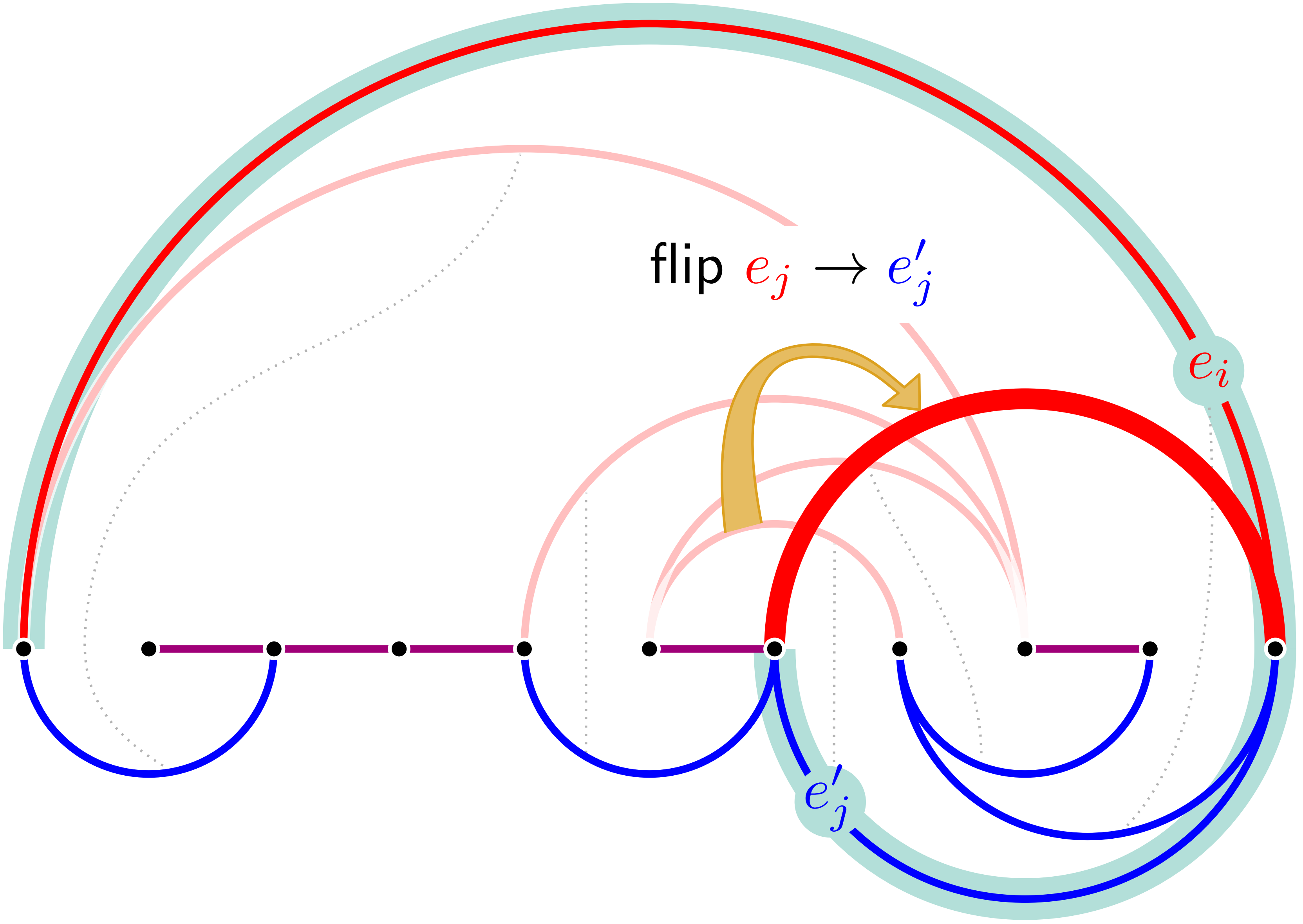


Conflicts

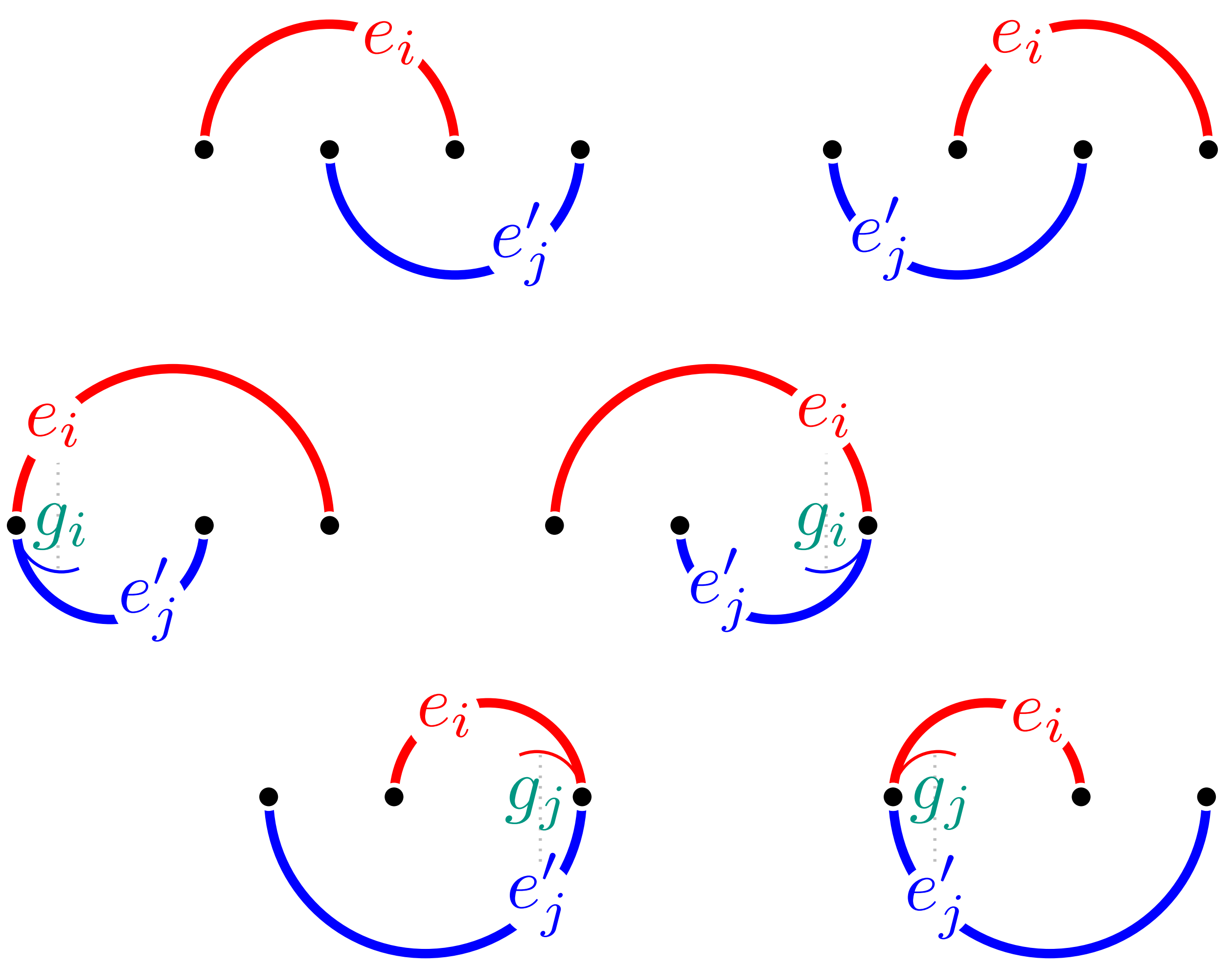
Goal: **direct** flip $e_i \to e'_i$ for many **near-near** pairs (e_i, e'_i)

▷ treat non-**near-near** pairs

…spending 1.5 flips per pair on average



When (e_j, e'_j) must **wait** for (e_i, e'_i)

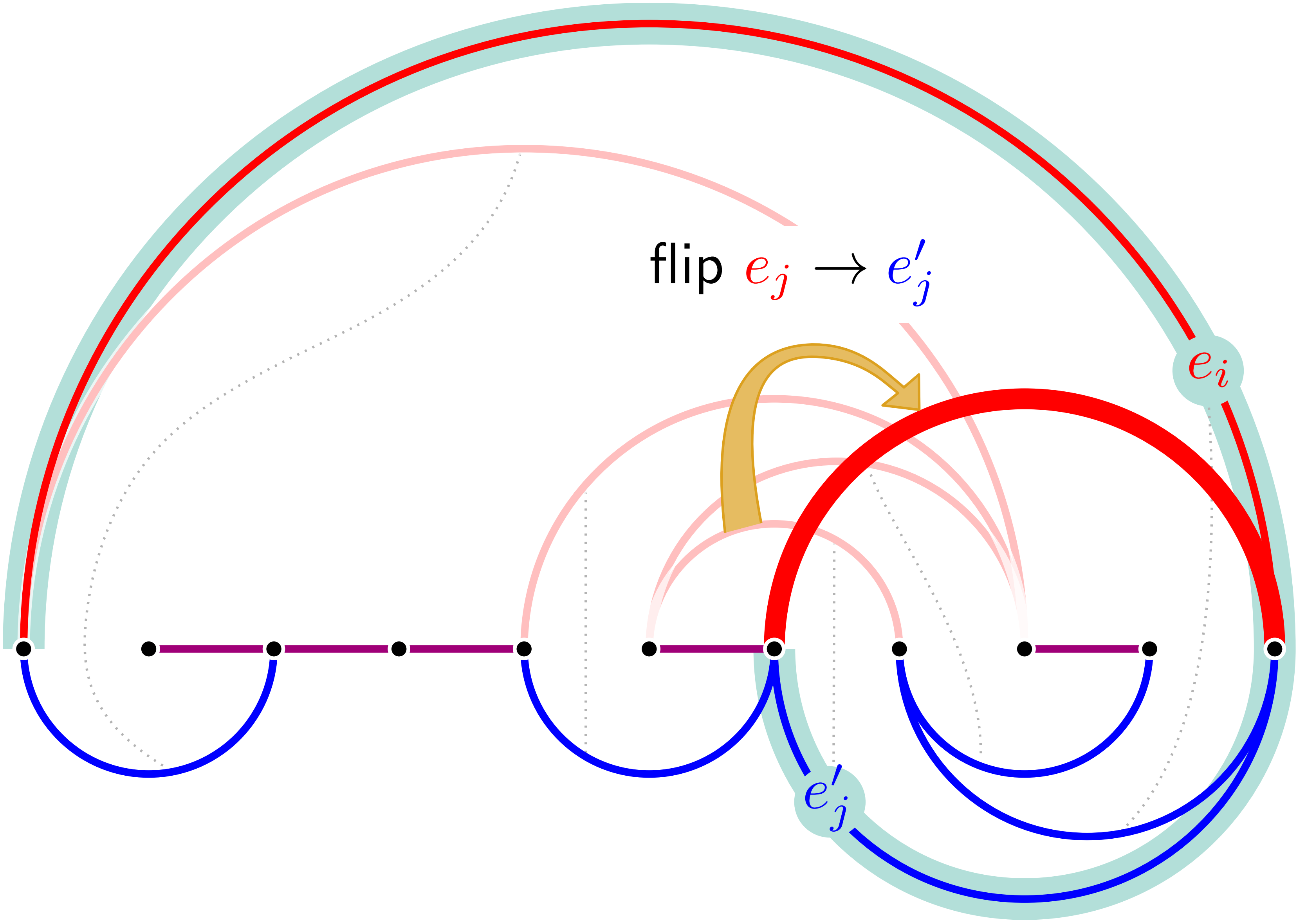


Conflicts

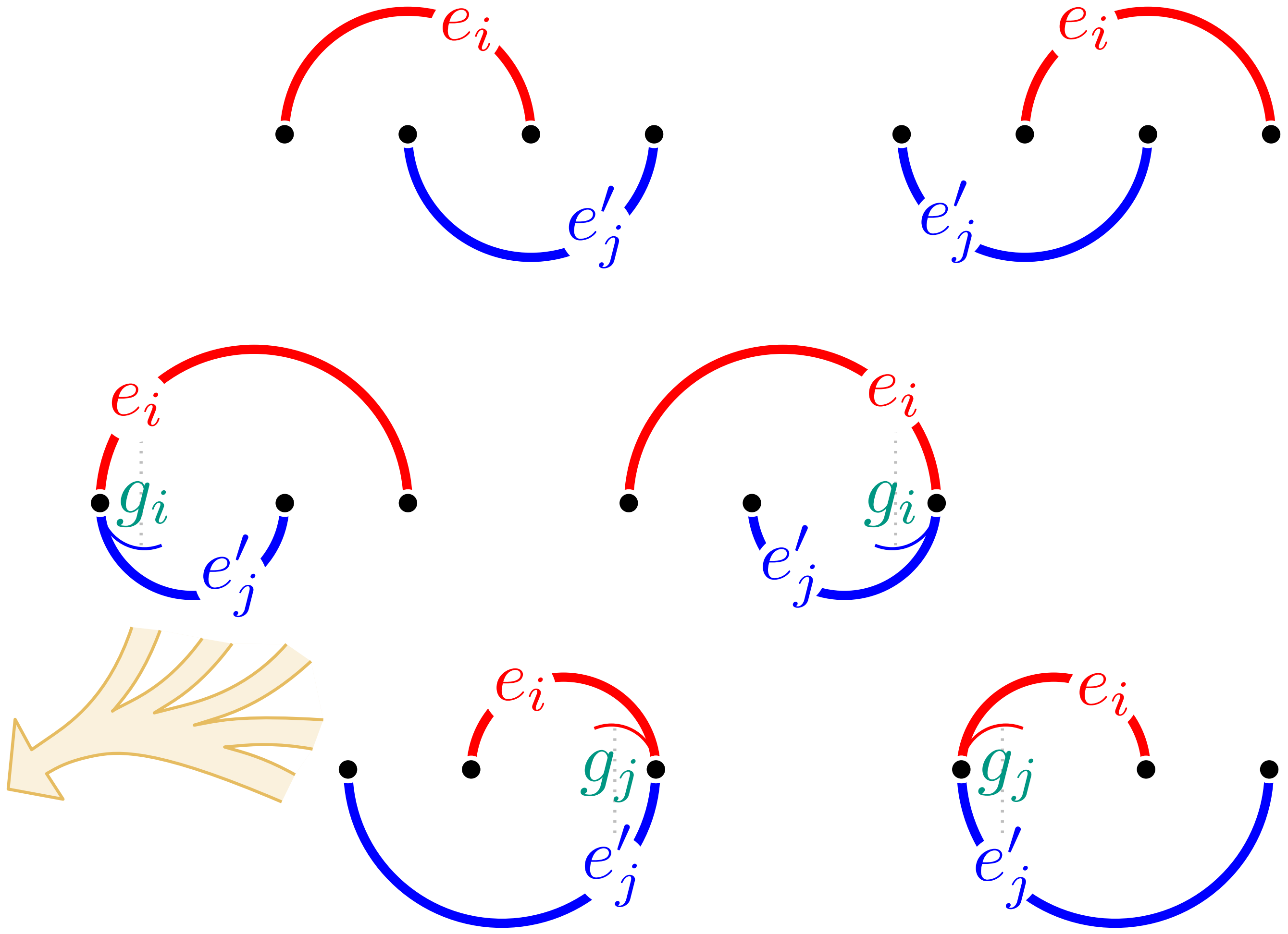
Goal: **direct** flip $e_i \to e'_i$ for many **near-near** pairs (e_i, e'_i)

▷ treat non-**near-near** pairs

. . . spending 1.5 flips per pair on average

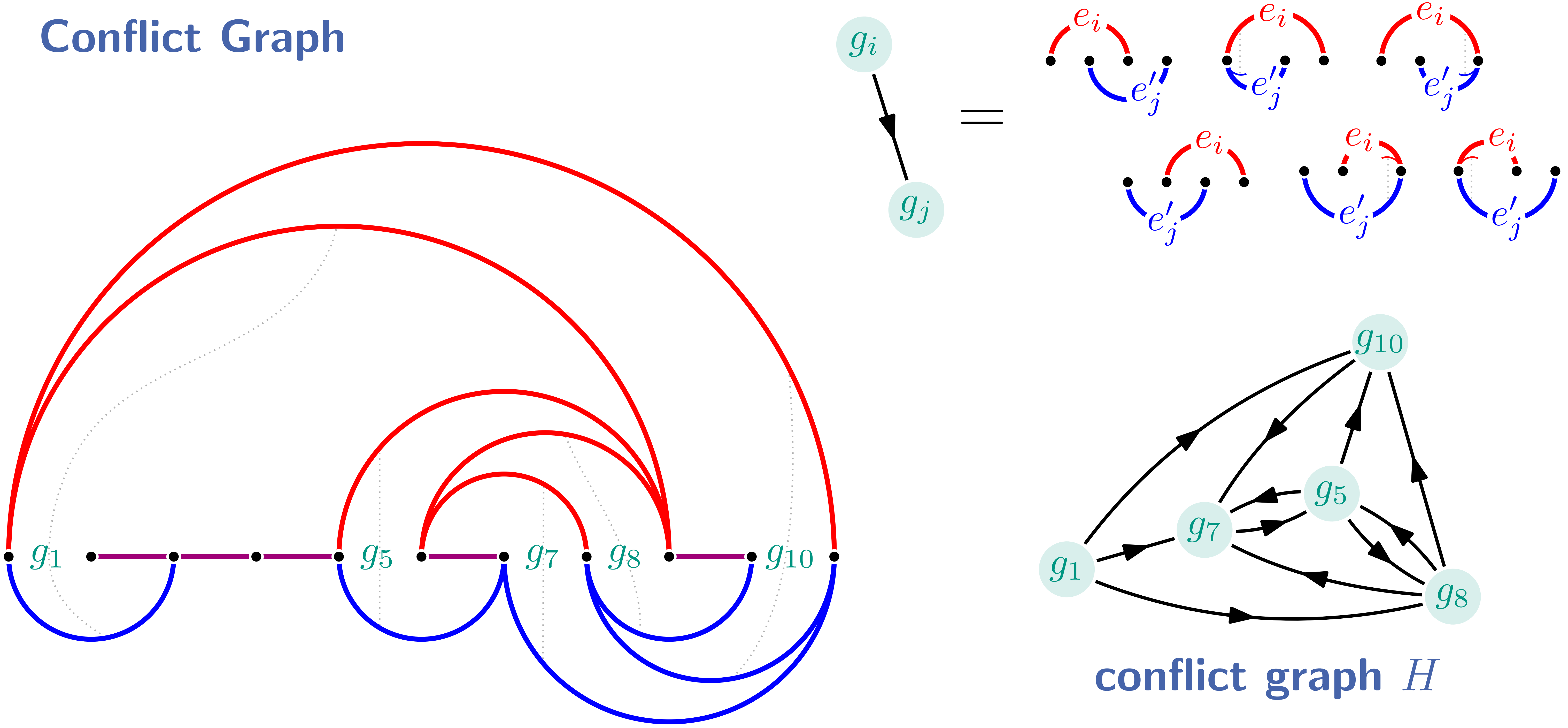


When (e_j, e'_j) must **wait** for (e_i, e'_i)

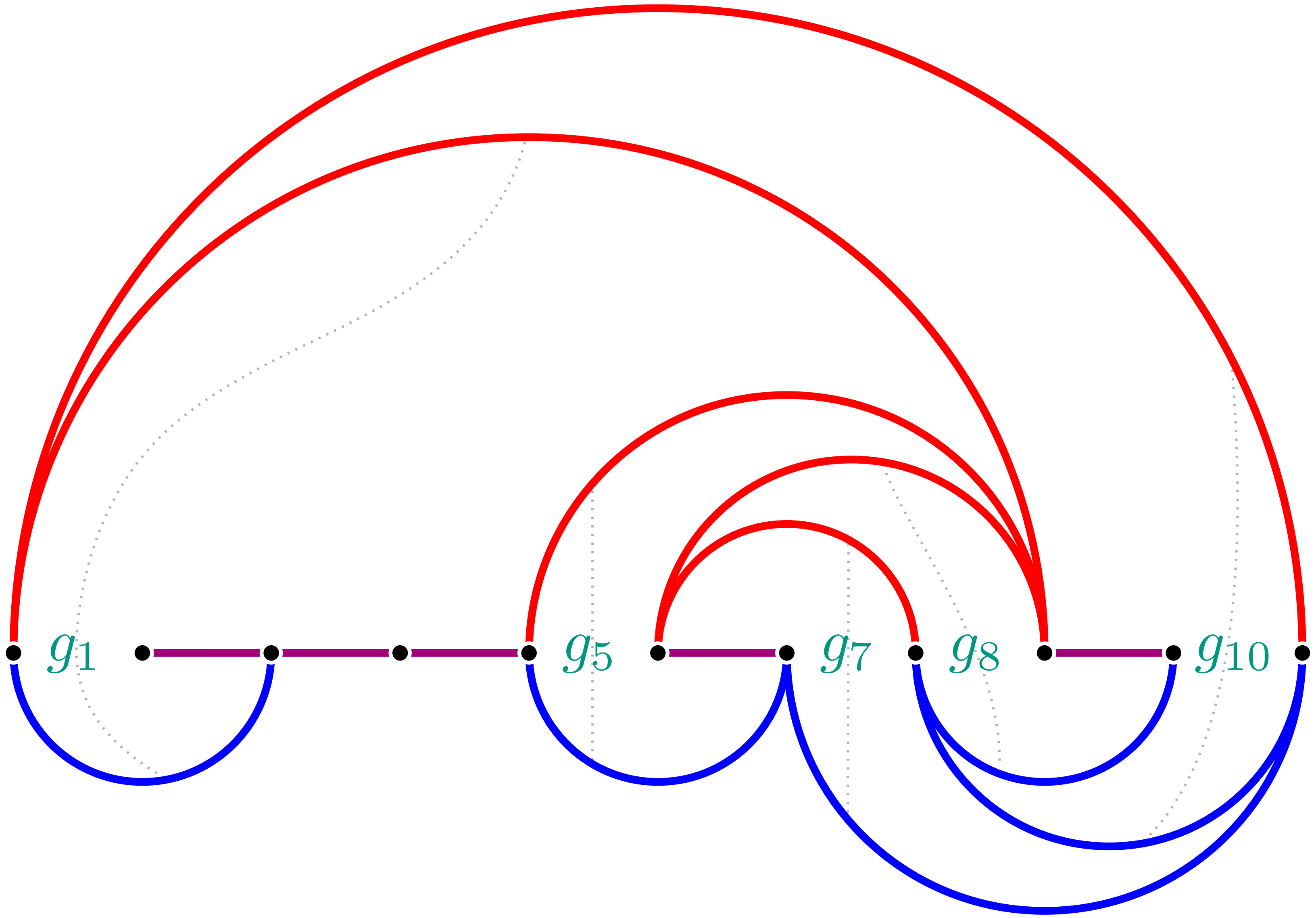


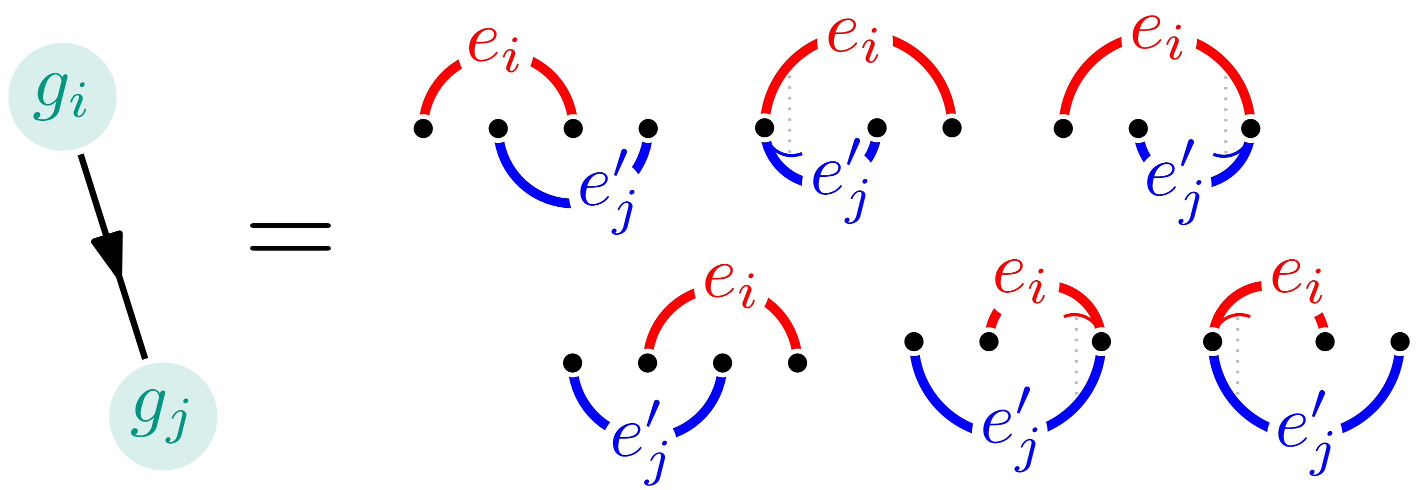
directed edge $\overrightarrow{g_i g_j}$ in **conflict graph**

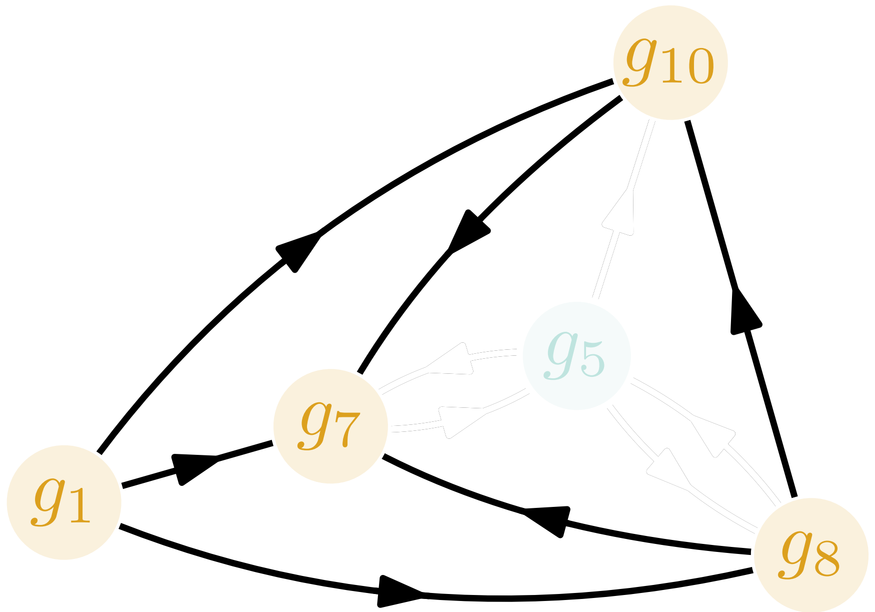
Conflict Graph



Conflict Graph



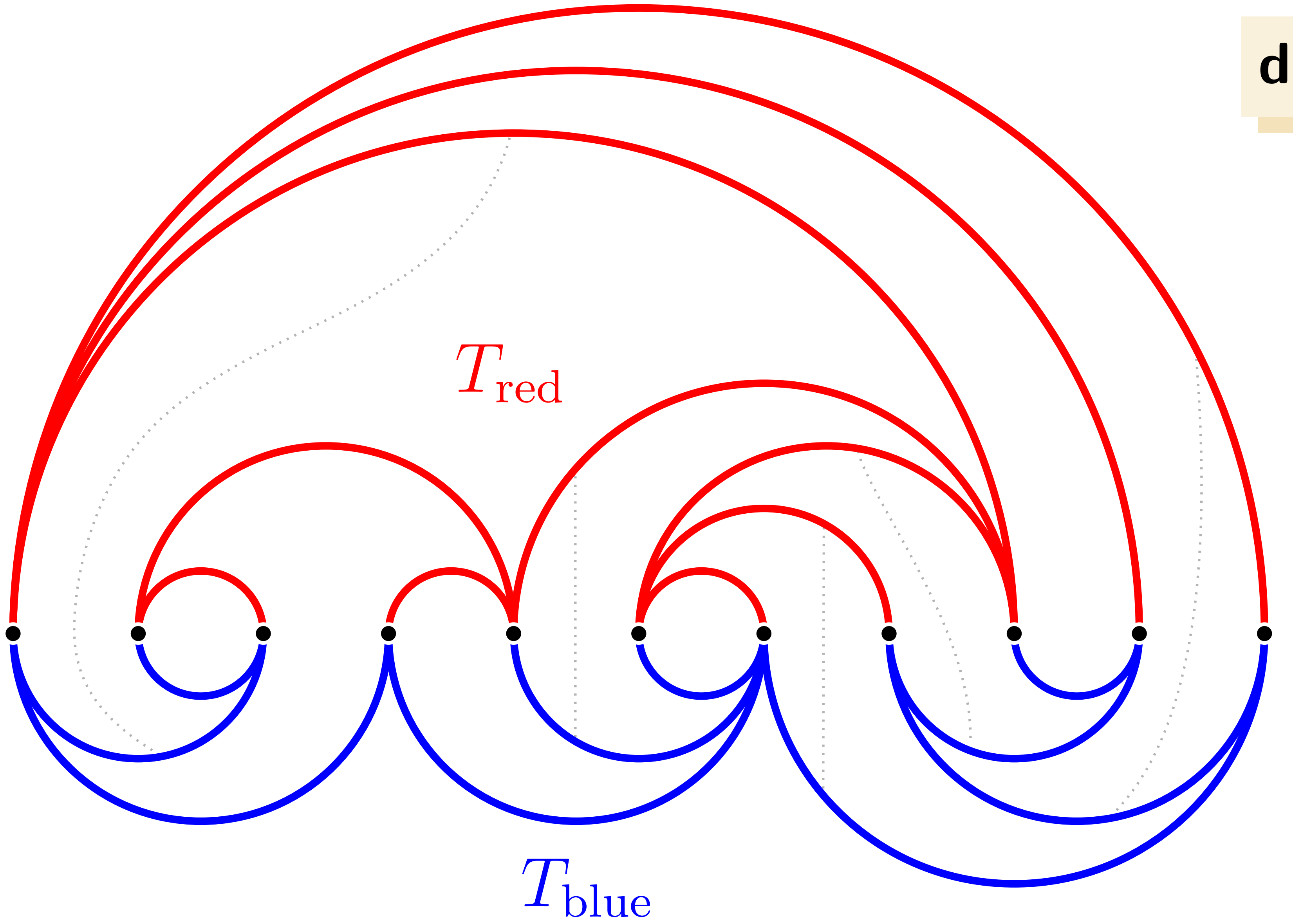




conflict graph H

near-near pairs can be **direct** flips $\iff$ **acyclic subset** in conflict graph

Overview



▷ open to linear order $p_1, \ldots, p_n$

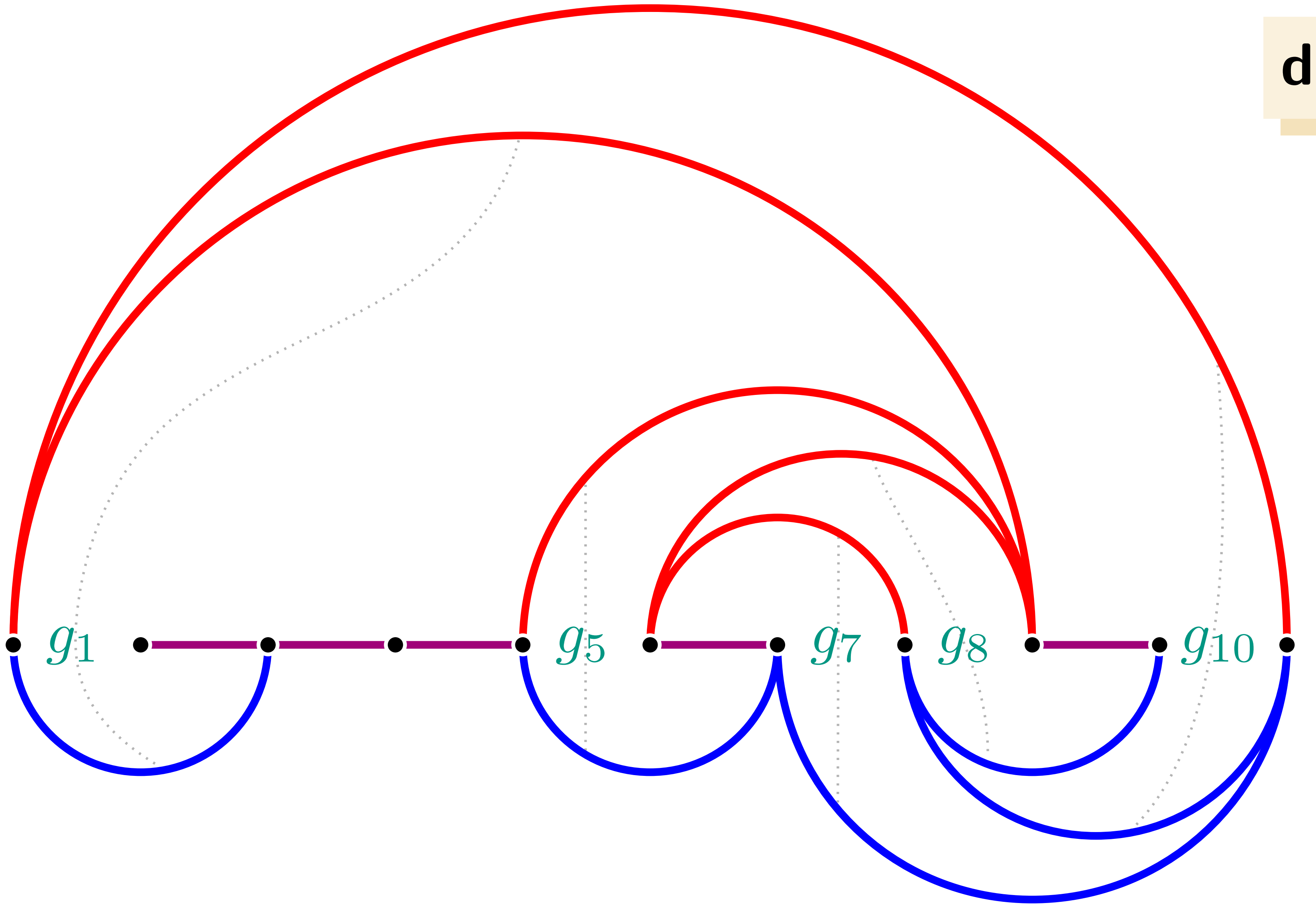
▷ edge pairs $(e_1, e'_1), \ldots, (e_{n-1}, e'_{n-1})$

direct flip $e_i \to e'_i$

indirect flip $e_i \to g_i \to e'_i$

$\text{dist}(T_{\text{red}}, T_{\text{blue}}) \leq 2n -$ #**direct** flips

Overview



- open to linear order $p_1, \ldots, p_n$
- edge pairs $(e_1, e'_1), \ldots, (e_{n-1}, e'_{n-1})$

direct flip $e_i \to e'_i$

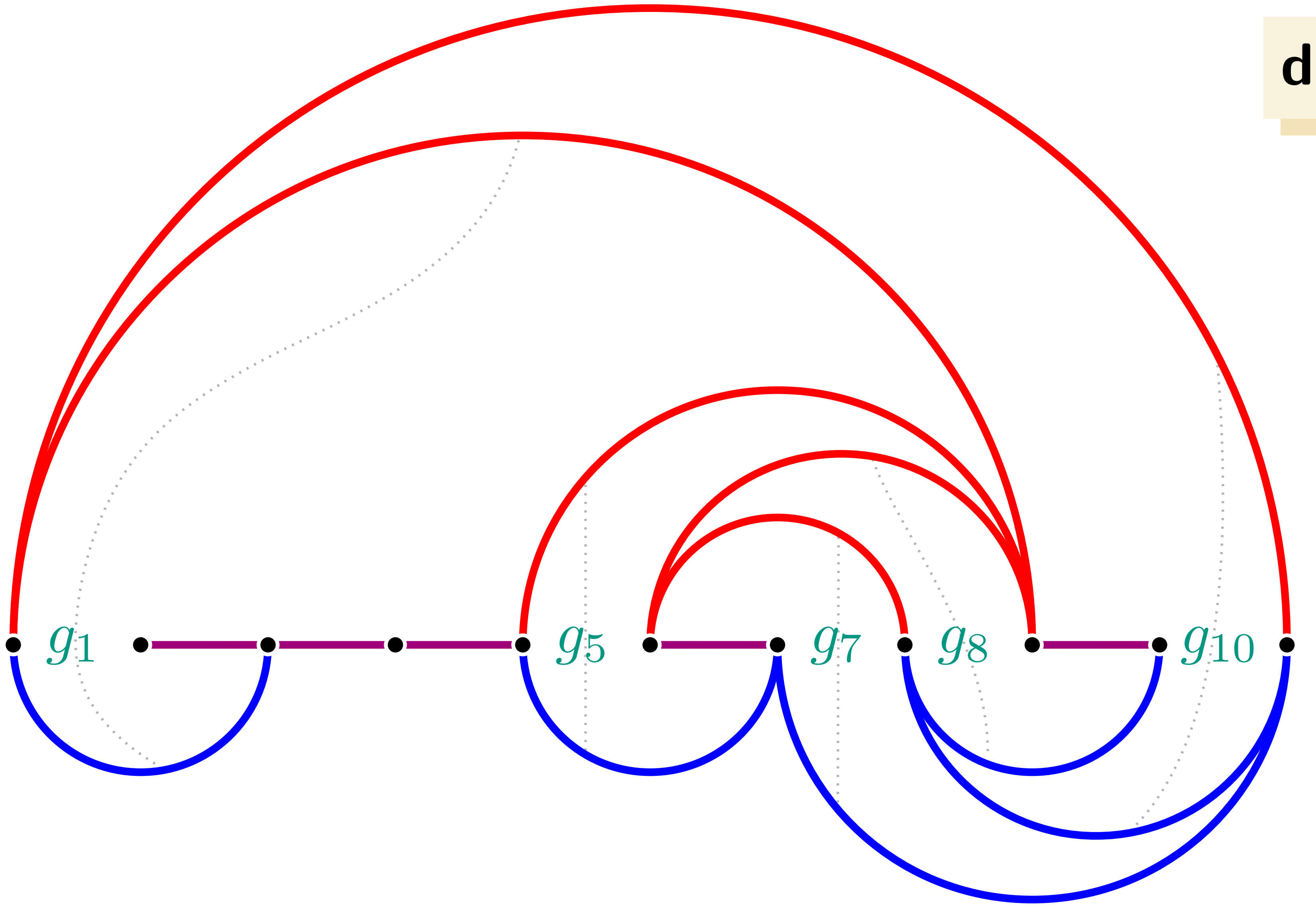
indirect flip $e_i \to g_i \to e'_i$

$$\mathrm{dist}(T_{\mathrm{red}}, T_{\mathrm{blue}}) \leq 2n - \#\textbf{direct flips}$$

- treat non-**near**-**near** pairs

  . . . **direct** flip half of the time

Overview



- ▷ open to linear order $p_1, \ldots, p_n$
- ▷ edge pairs $(e_1, e'_1), \ldots, (e_{n-1}, e'_{n-1})$

direct flip $e_i \to e'_i$

indirect flip $e_i \to g_i \to e'_i$

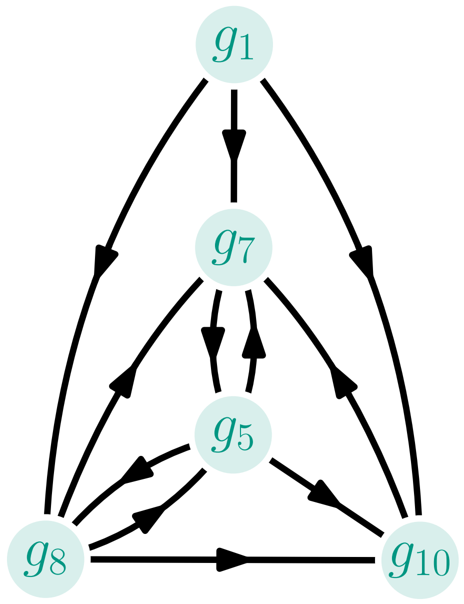
$$\mathrm{dist}(T_{\mathrm{red}}, T_{\mathrm{blue}}) \leq 2n - \#\textbf{direct} \text{ flips}$$

- ▷ treat non-**near**-**near** pairs

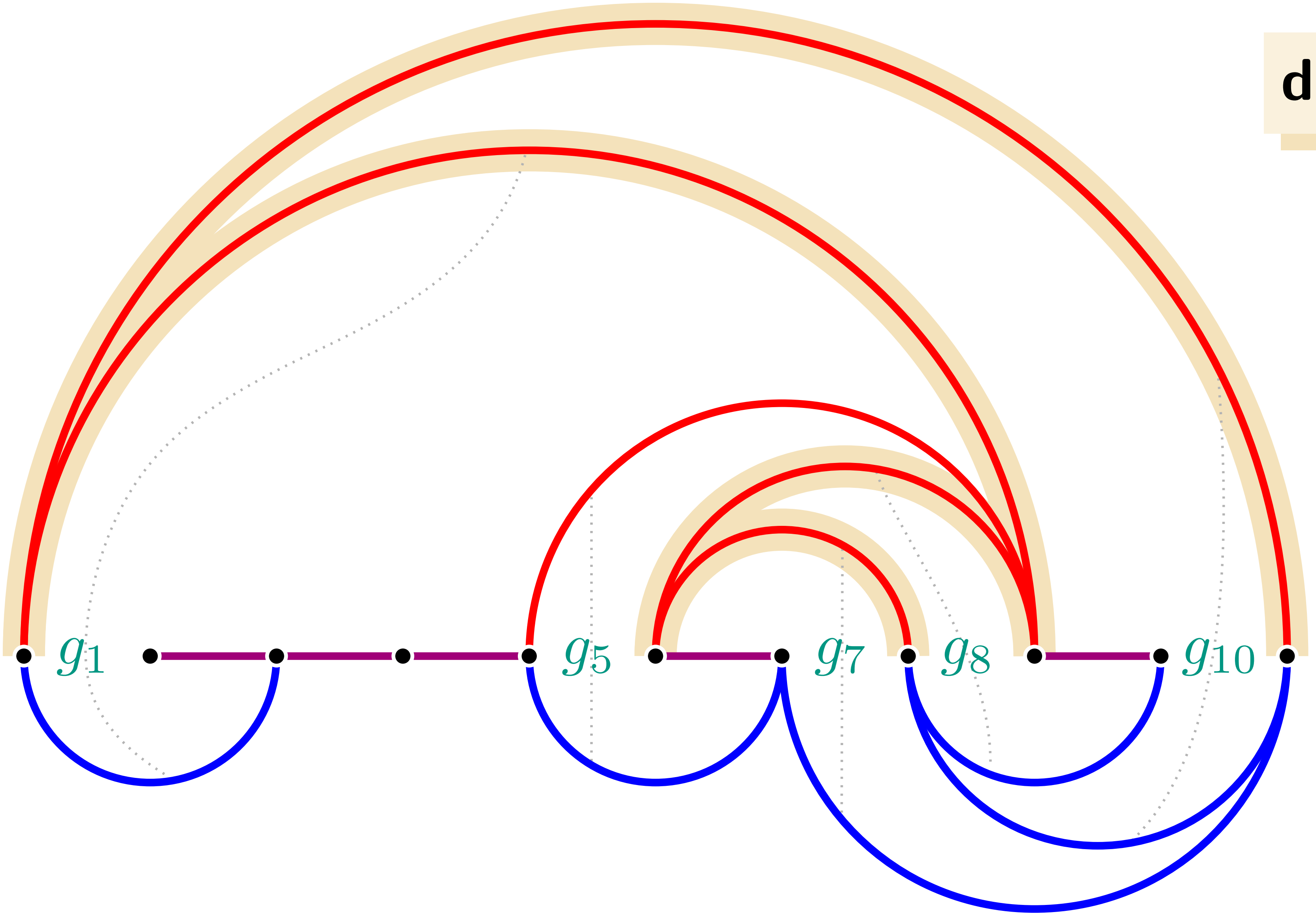
  . . . **direct** flip half of the time

- ▷ **conflict graph** H

  . . . on **near**-**near** pairs



Overview



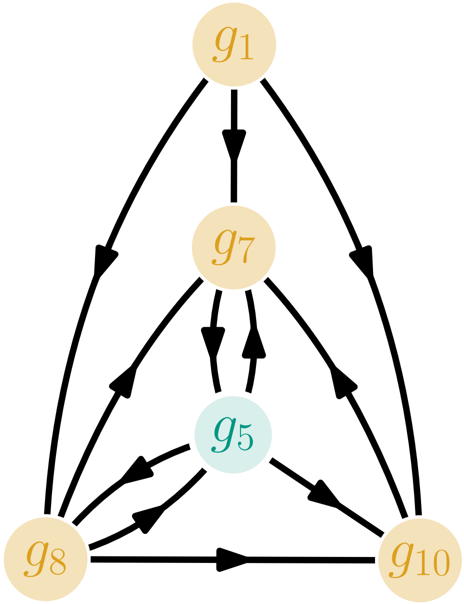
- ▷ open to linear order $p_1, \ldots, p_n$
- ▷ edge pairs $(e_1, e'_1), \ldots, (e_{n-1}, e'_{n-1})$

direct flip $e_i \to e'_i$

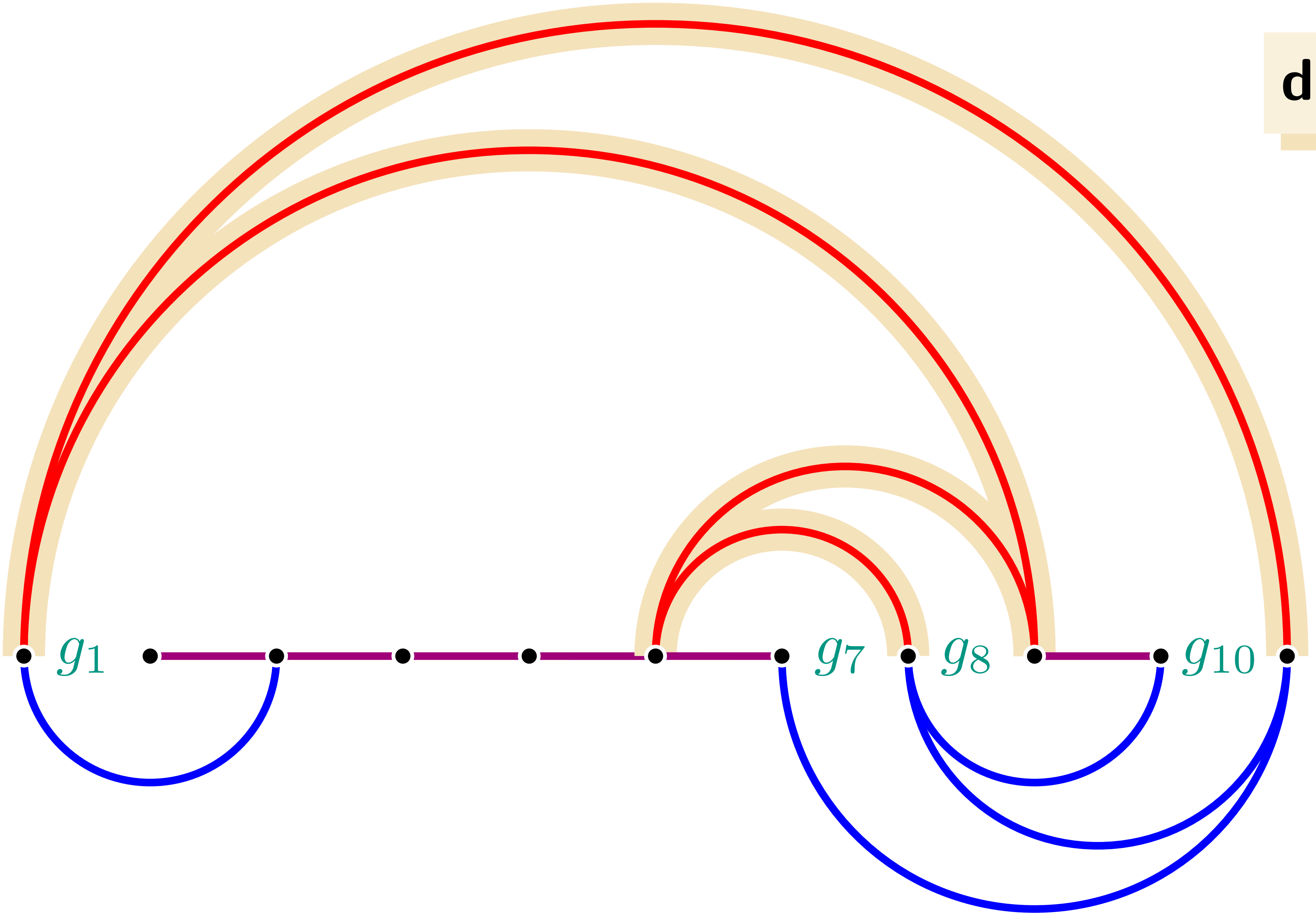
indirect flip $e_i \to g_i \to e'_i$

$\text{dist}(T_{\text{red}}, T_{\text{blue}}) \leq 2n -$ #**direct** flips

- ▷ treat non-**near**-**near** pairs
  . . . **direct** flip half of the time
- ▷ **conflict graph** H
  . . . on **near**-**near** pairs
- ▷ find **acyclic subset** X
- ▷ **direct** flips for X



Overview



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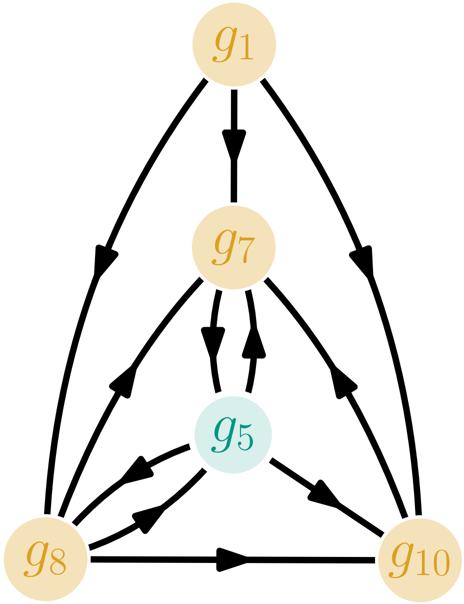

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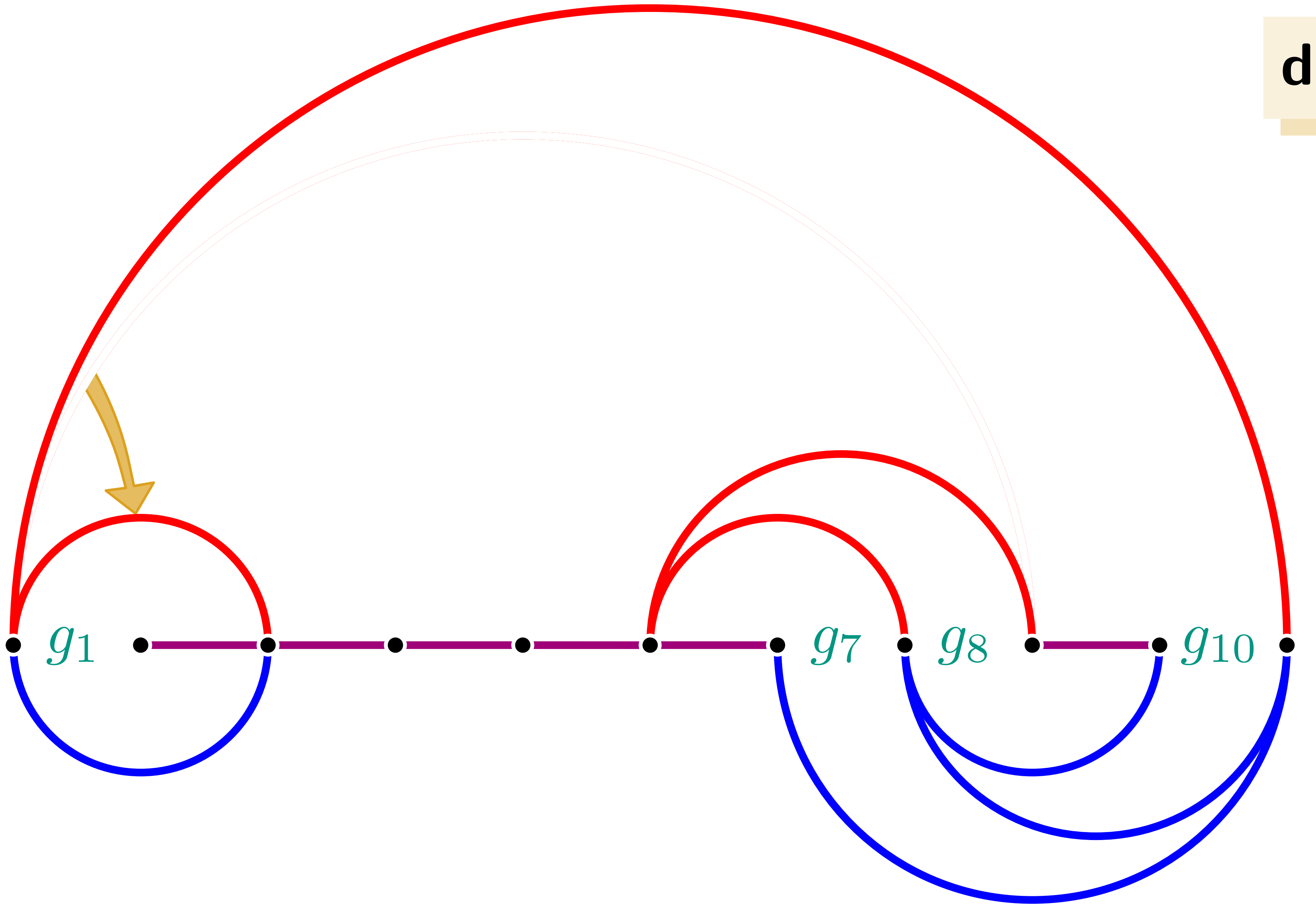

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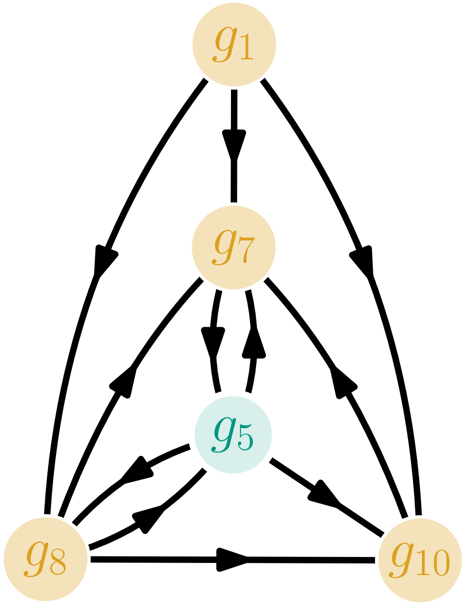

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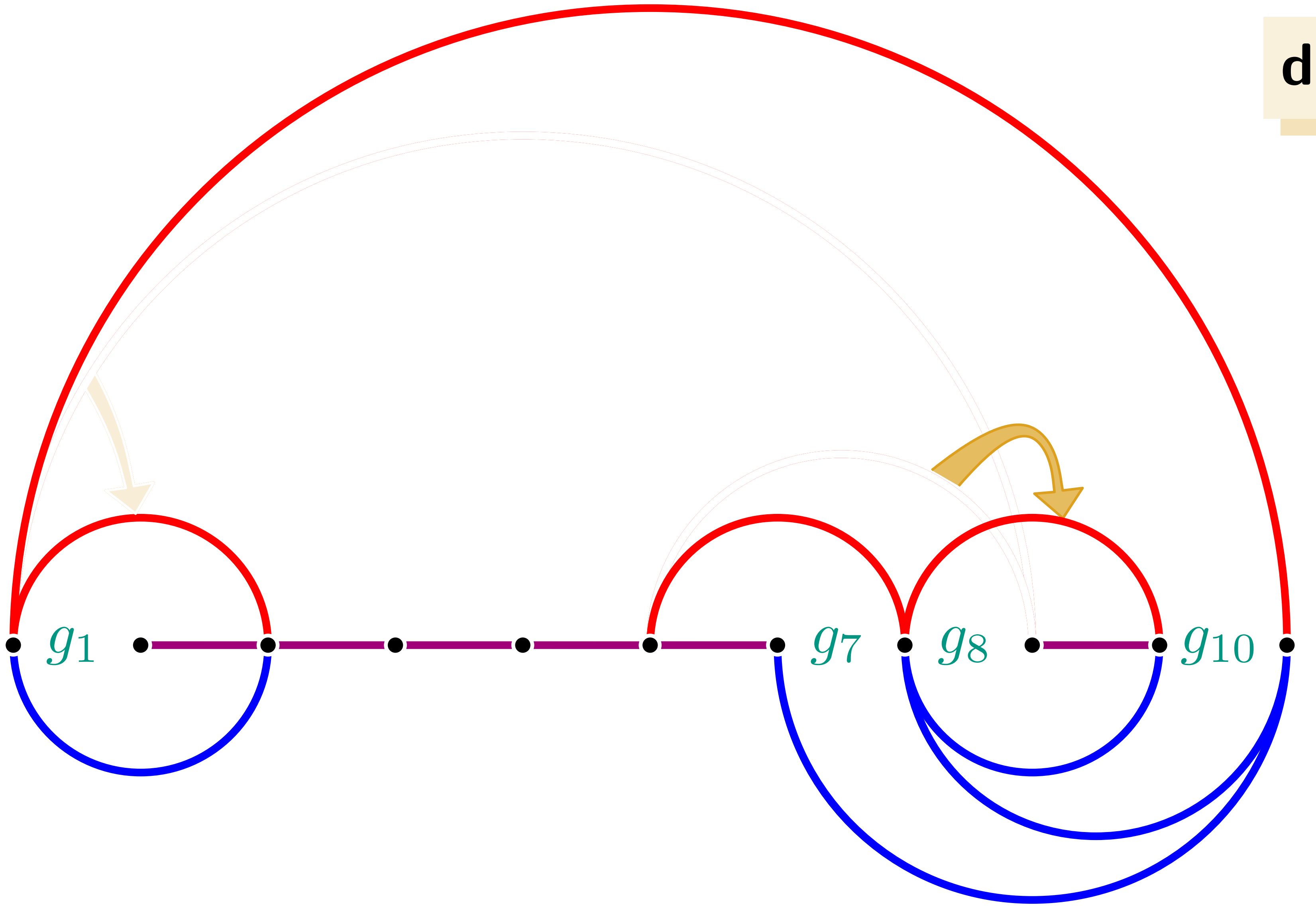

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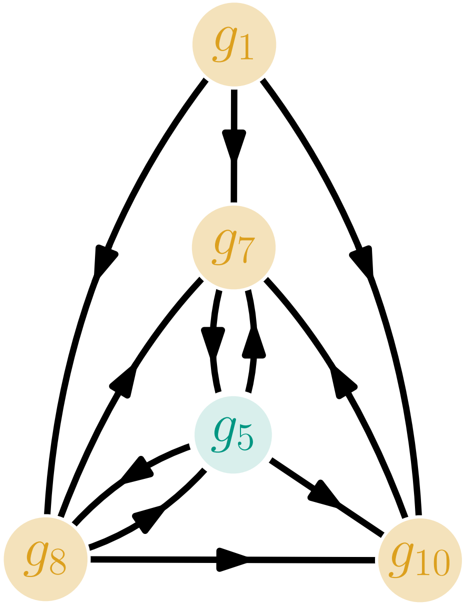

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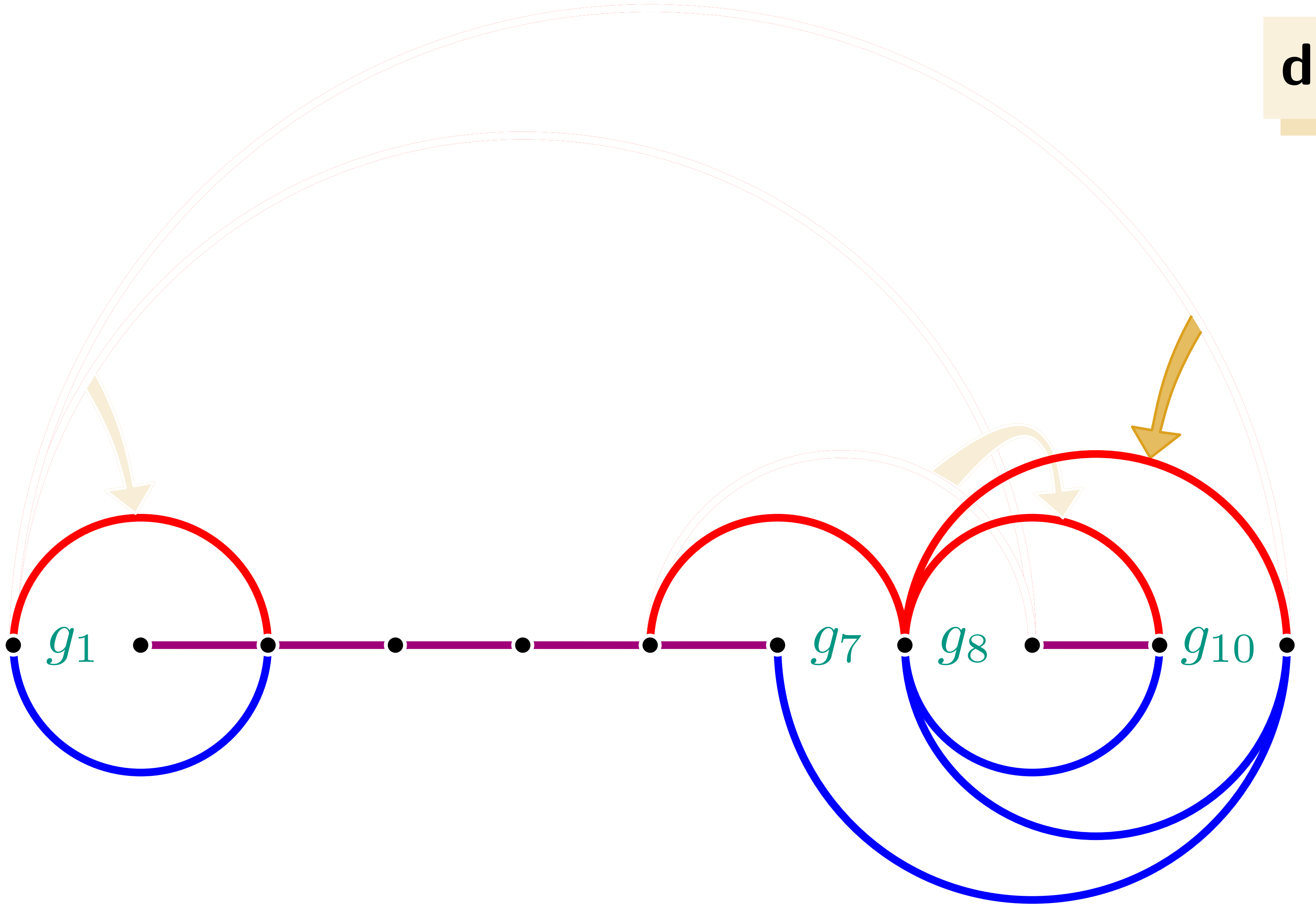

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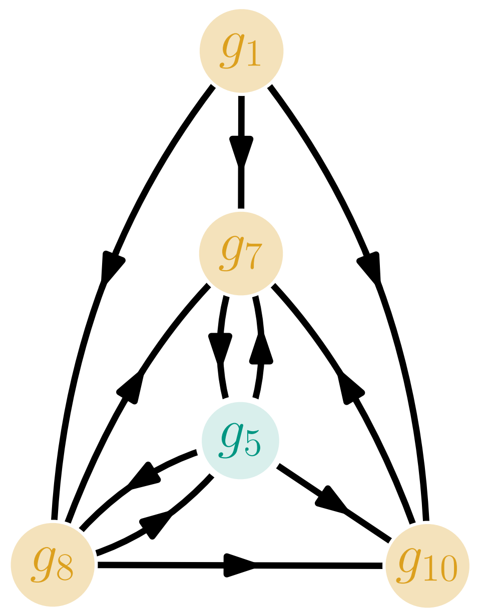

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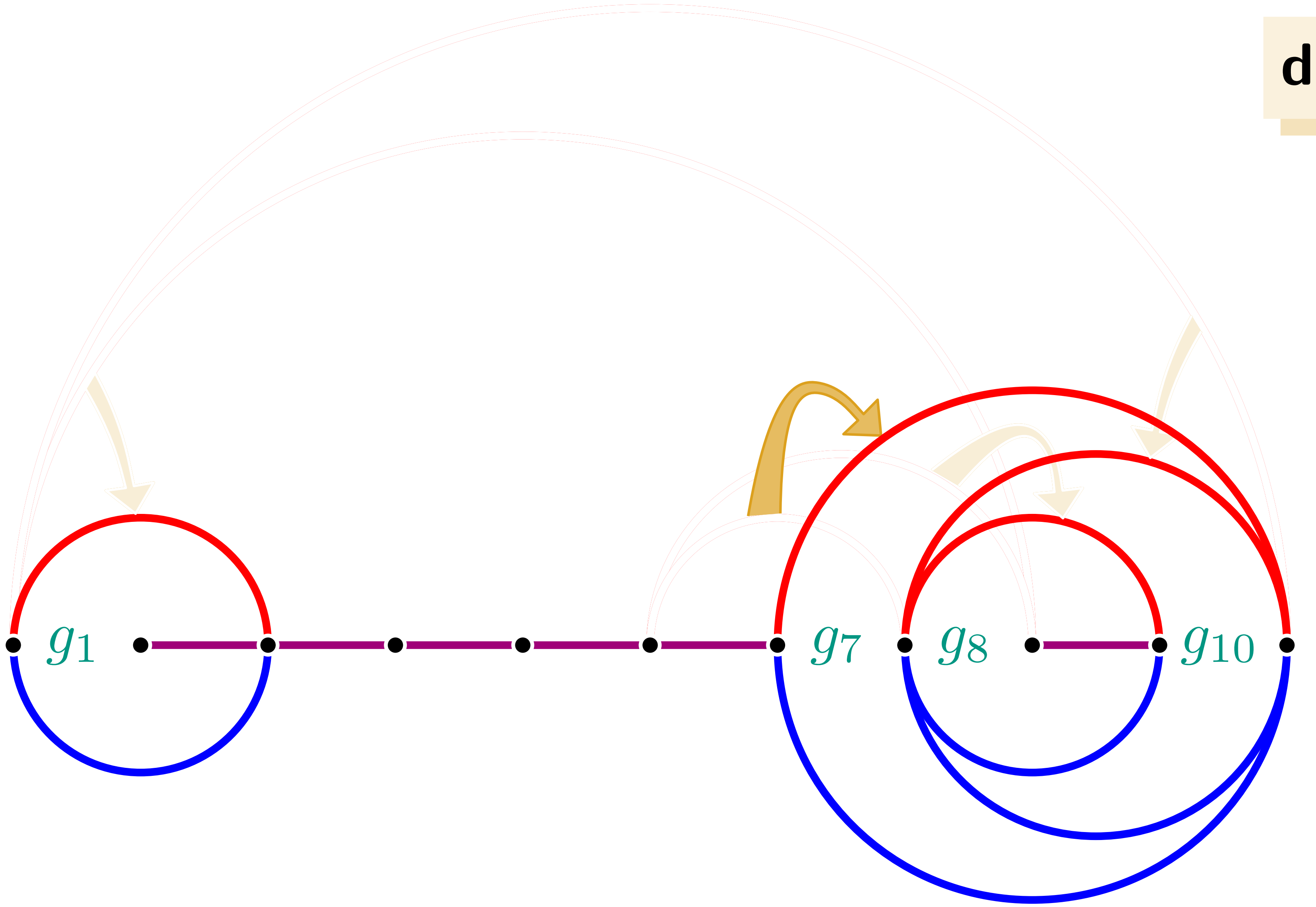

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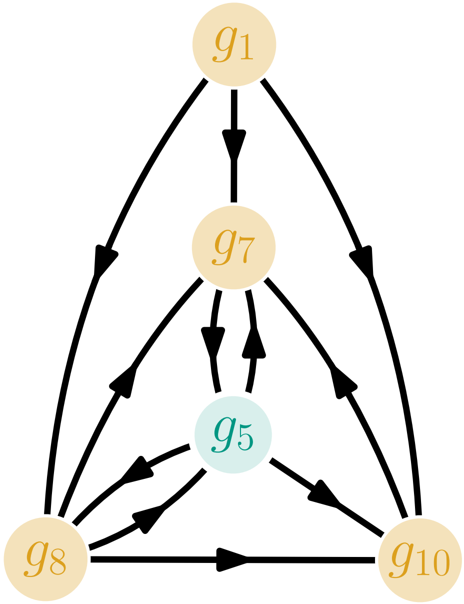

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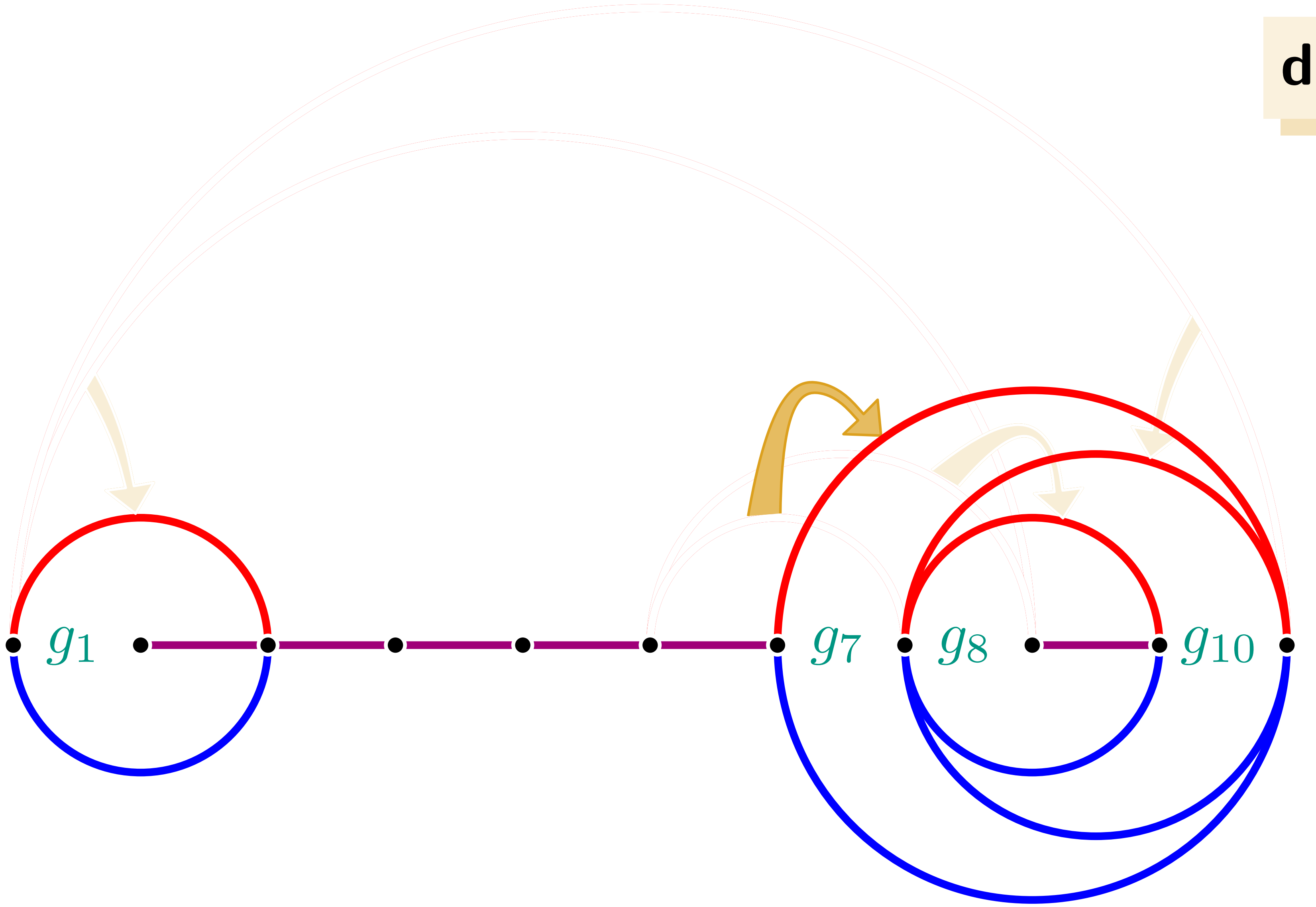

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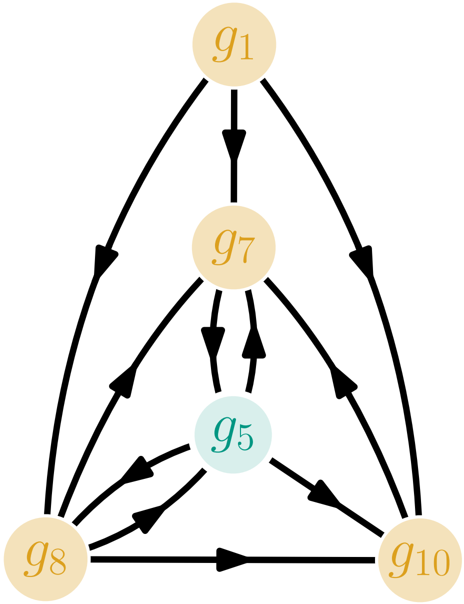

remaining: **largest** **acyclic subset** in **conflict graph**?

Largest Acyclic Subset in Conflict Graphs

$\mathrm{dist}(T_{\mathrm{red}}, T_{\mathrm{blue}}) \leq 2n - \#$**direct** flips

conflict graph H

- $V_H = \{$**near-near** pairs $(e_i, e'_i)\}$
- $\mathrm{ac}(H) = \max\{|X| : X \subseteq V_H$ **acyclic**$\}$
- **acyclic subset** $\iff$ pairs can be **direct** flips

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Lemma.
V_H can be partitioned into three acyclic subsets.

$\implies \mathrm{ac}(H) \geq \frac{1}{3}|V_H|$

$\implies$ at least $\frac{1}{3}$ of near-near pairs with **direct** flips

$\implies \mathrm{dist}(T_{\mathrm{red}}, T_{\mathrm{blue}}) \leq 2n - \frac{1}{3}n = \frac{5}{3}n$

Largest Acyclic Subset in Conflict Graphs

$\mathrm{dist}(T_{\mathrm{red}}, T_{\mathrm{blue}}) \leq 2n - \#$**direct** flips

conflict graph H

- $V_H = \{$**near-near** pairs $(e_i, e_i')\}$
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- **acyclic subset** $\iff$ pairs can be **direct** flips

> **Lemma.**
> V_H can be partitioned into three **acyclic subsets**.

$\implies \mathrm{ac}(H) \geq \frac{1}{3}|V_H|$

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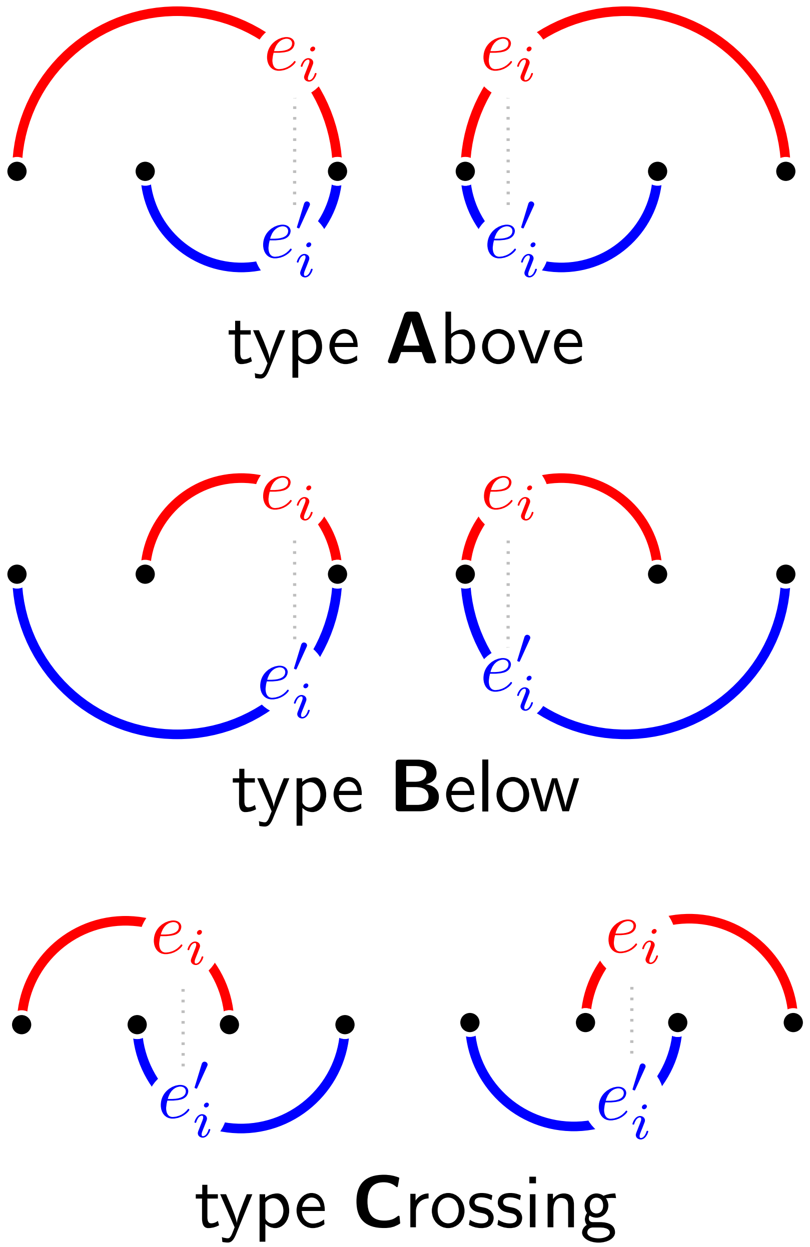
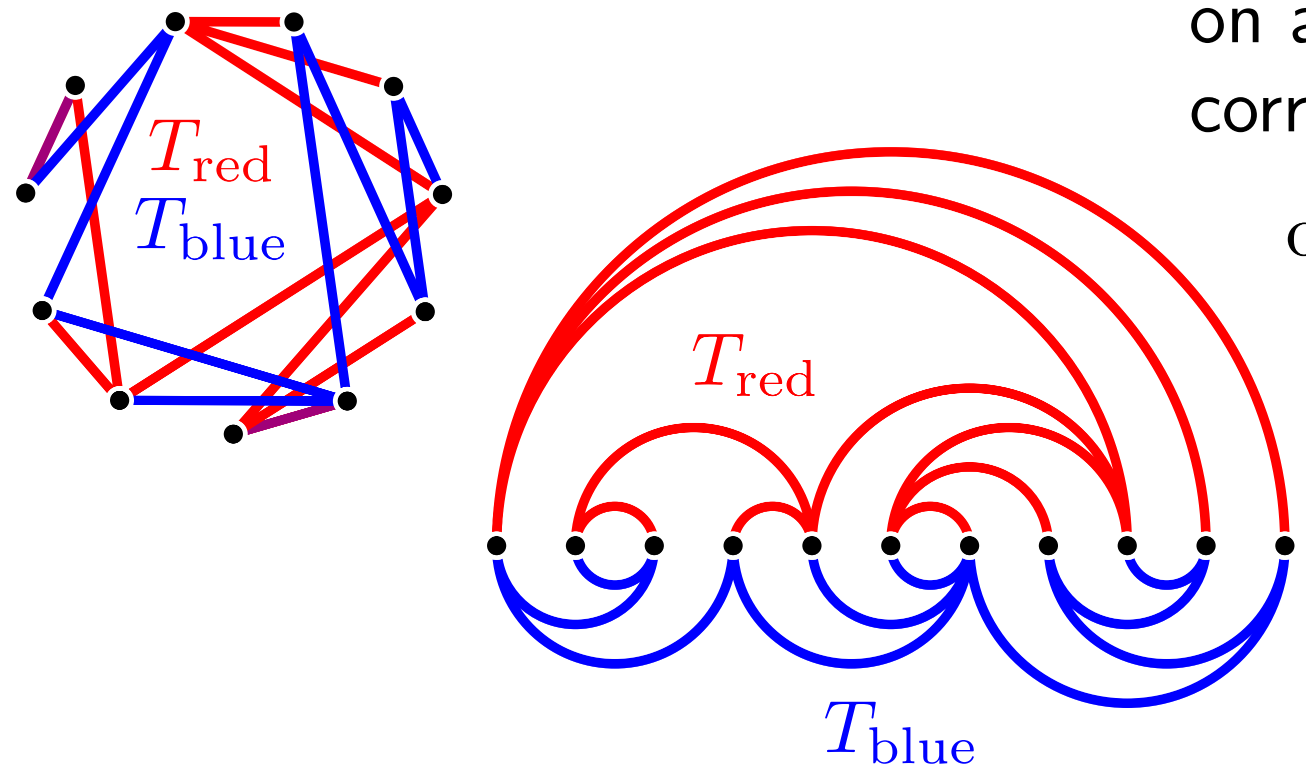
"Proof."



- each of **A**, **B**, **C** is **acyclic**

□

Main Theorem

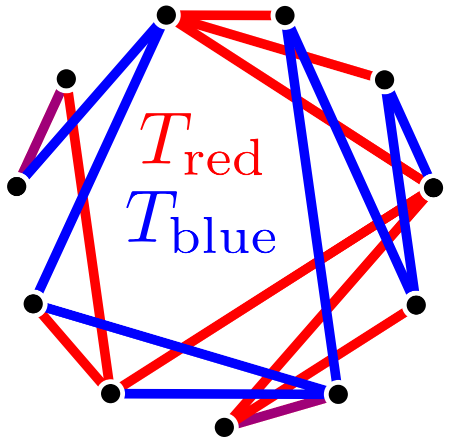


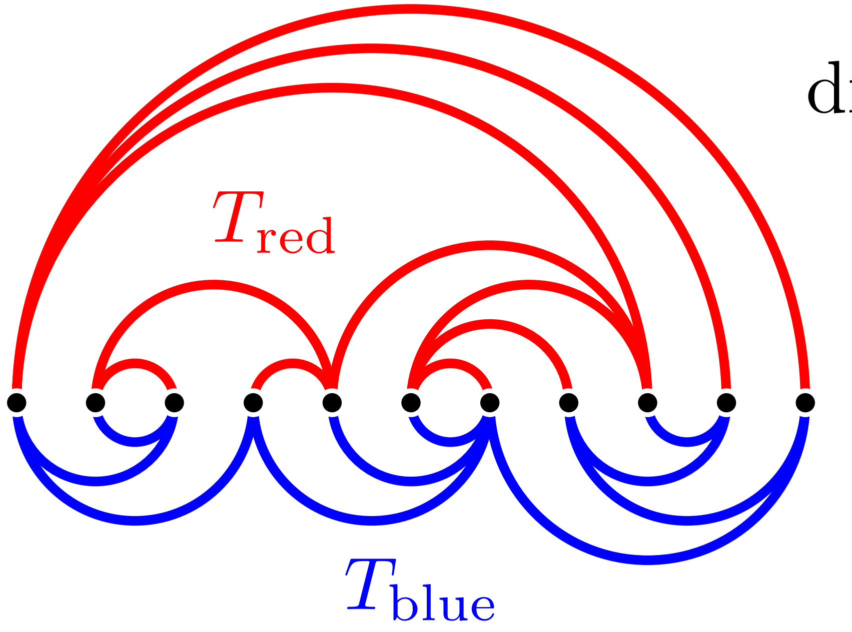
Theorem.

For any two non-crossing spanning trees $T_{\text{red}}, T_{\text{blue}}$ on a set of n points in convex position with corresponding **conflict graph** H we have:

$$\text{dist}(T_{\text{red}}, T_{\text{blue}}) \leq \max\left(\frac{3}{2}, 2 - \frac{\text{ac}(H)}{|V_H|}\right) \cdot n \leq \frac{5}{3} \cdot n$$

Main Theorem





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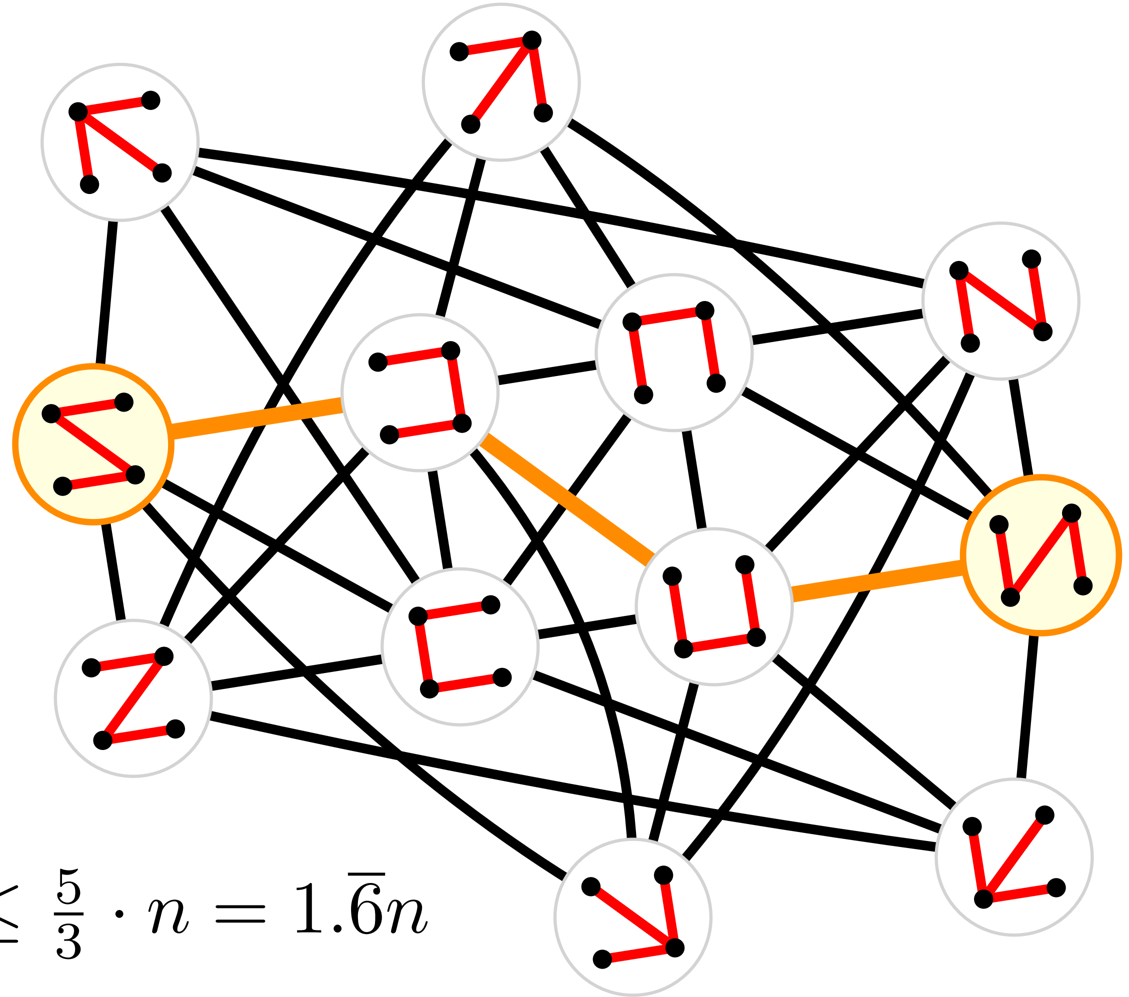
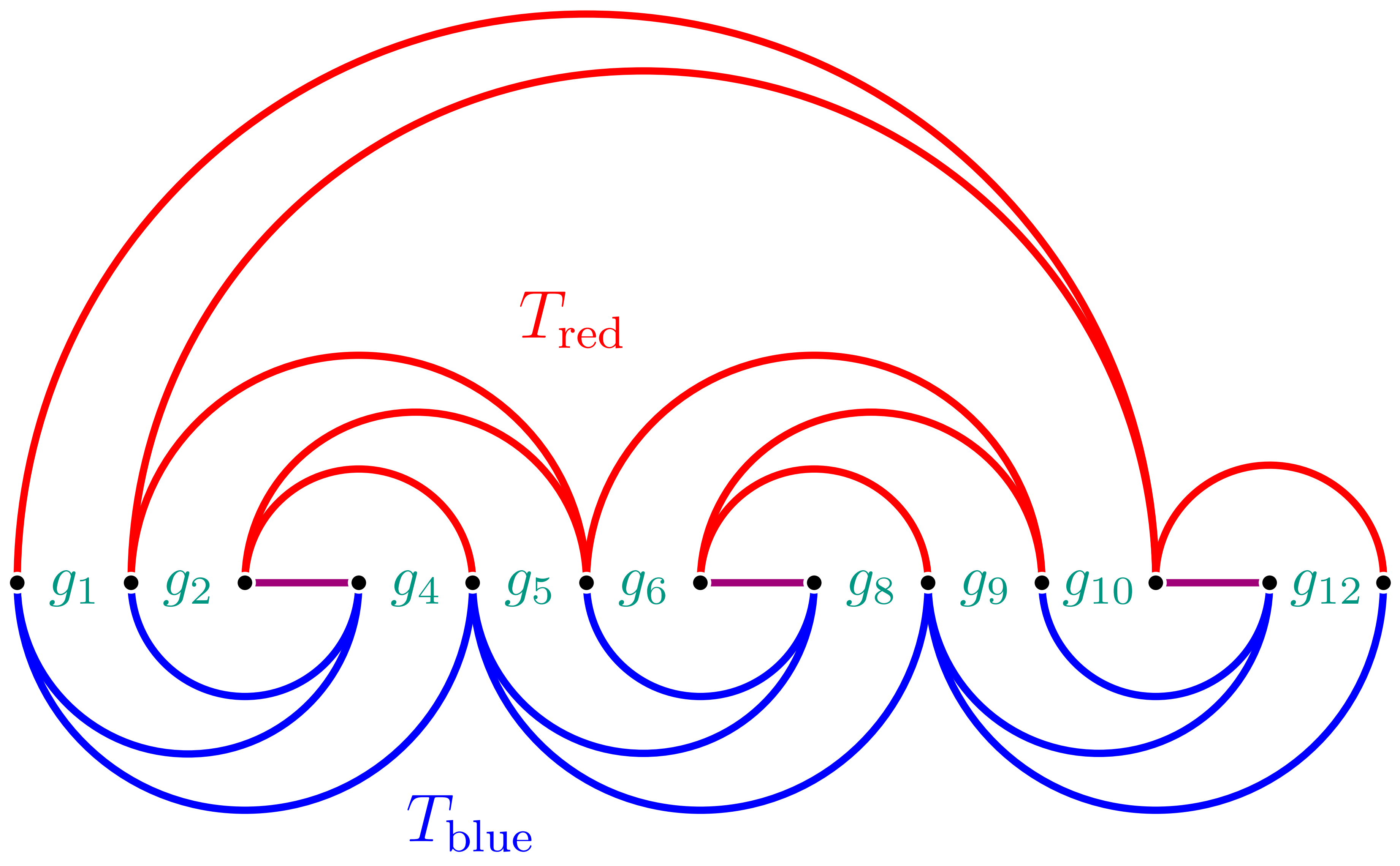


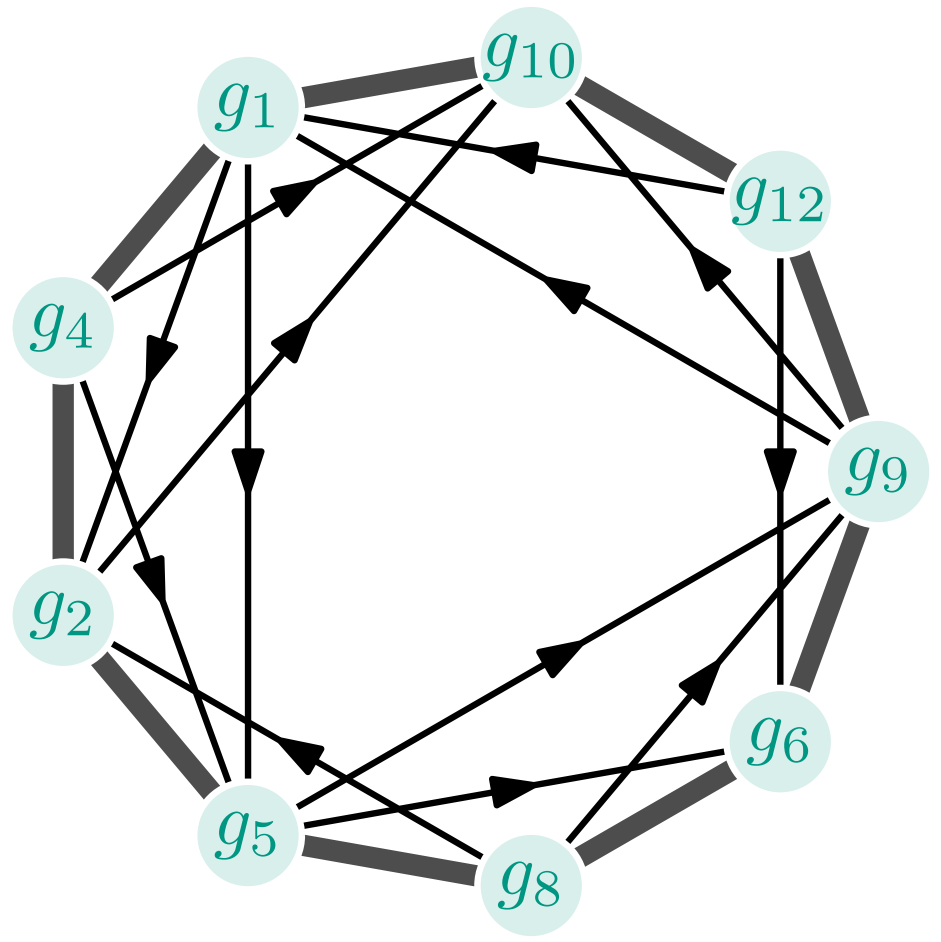
Theorem.

For the diameter $\text{diam}(\mathcal{F}_n)$ of the **flip graph** $\mathcal{F}_n$ of non-crossing spanning trees on a set of n points in convex position we have:

$$\text{diam}(\mathcal{F}_n) \leq \max\left\{2 - \frac{\text{ac}(H)}{|V_H|} : H \textbf{ conflict graph}\right\} \cdot n \leq \frac{5}{3} \cdot n = 1.\overline{6}n$$

the Worst Example we know of





conflict graph H

$|V_H| = 9 \qquad \mathrm{ac}(H) = 4$

$$\frac{\mathrm{ac}(H)}{|V_H|} = \frac{4}{9}$$

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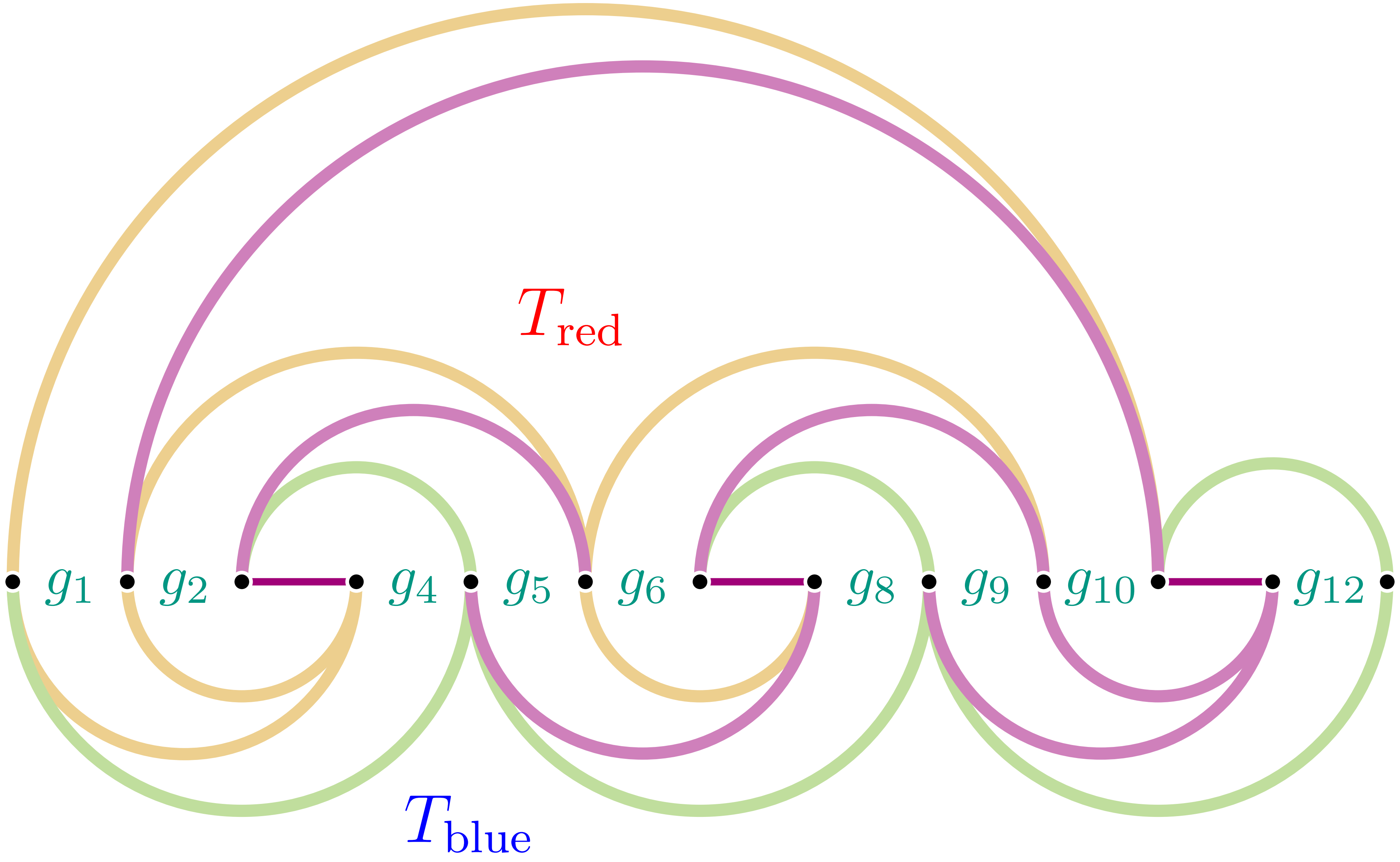

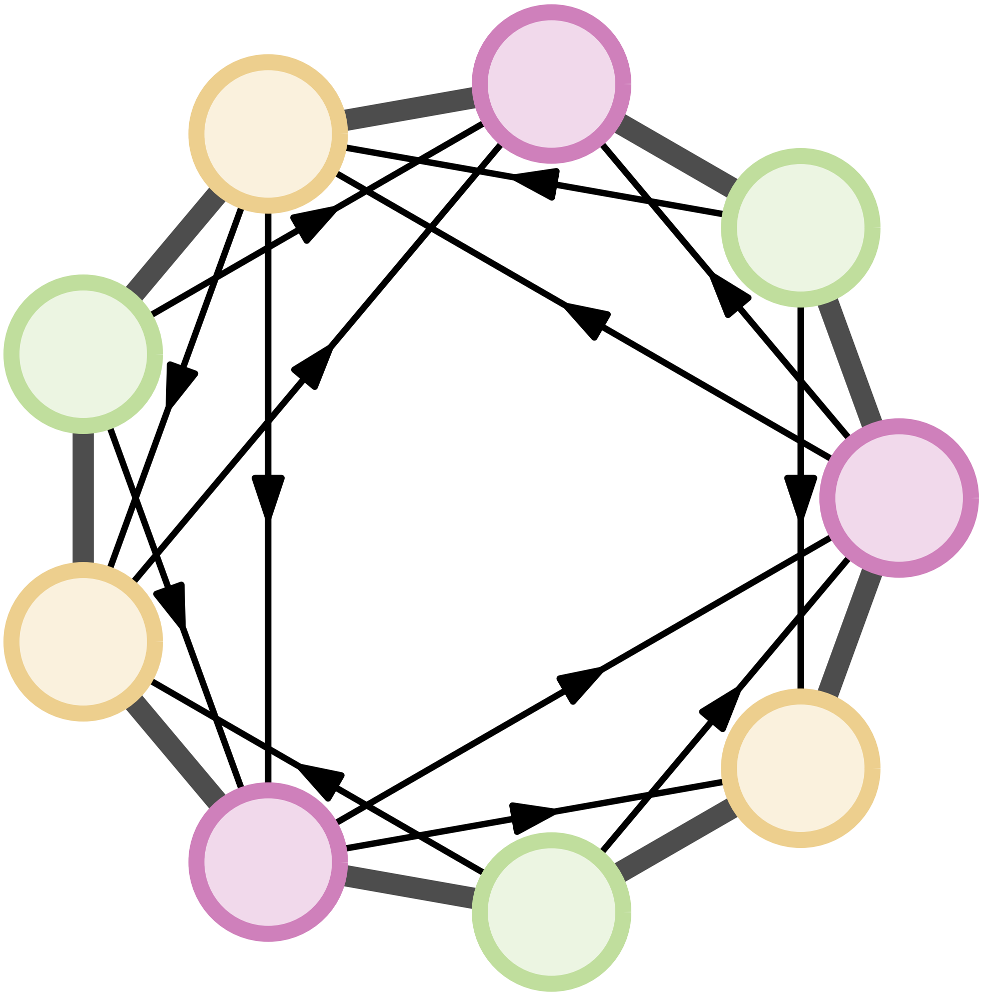

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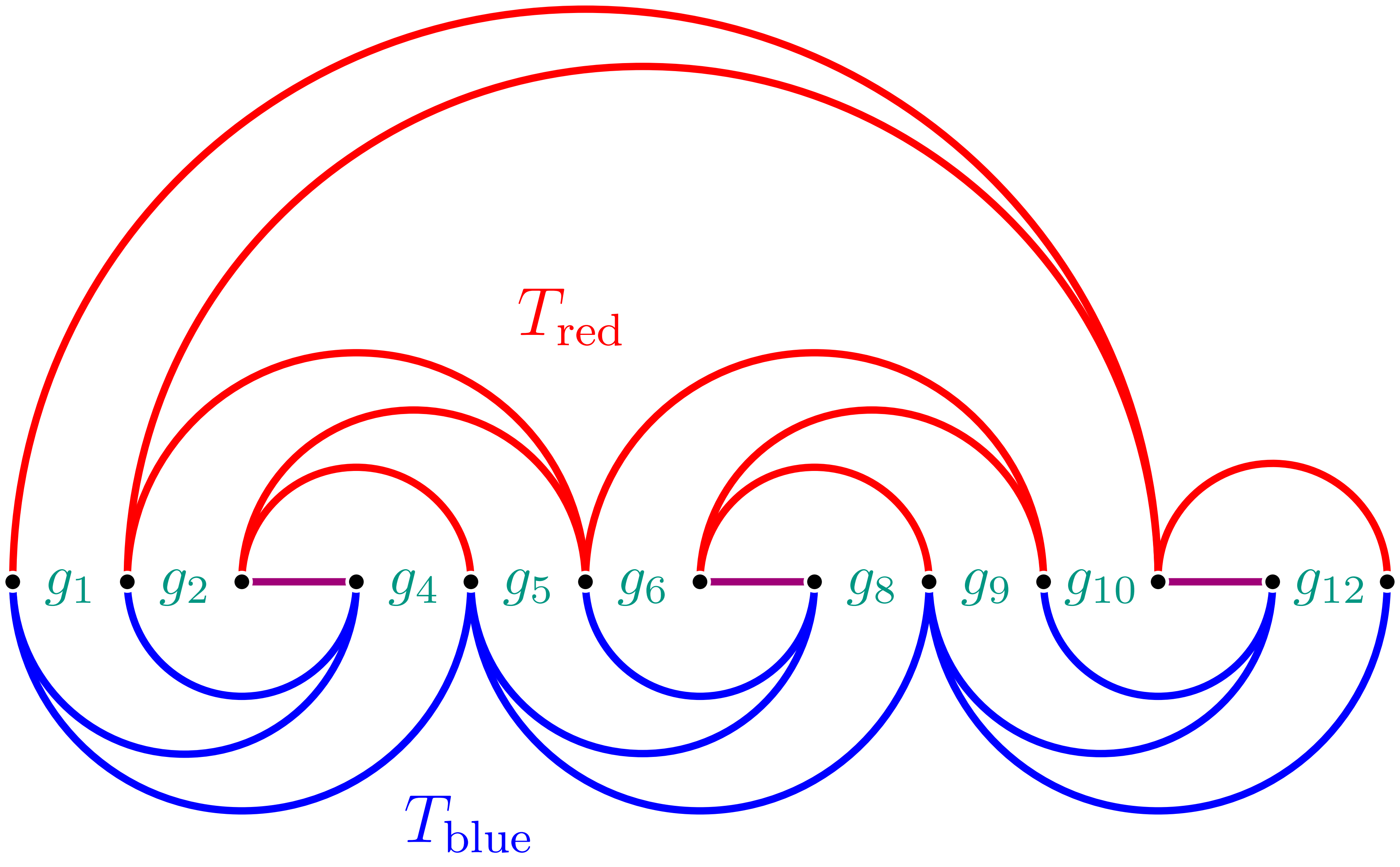

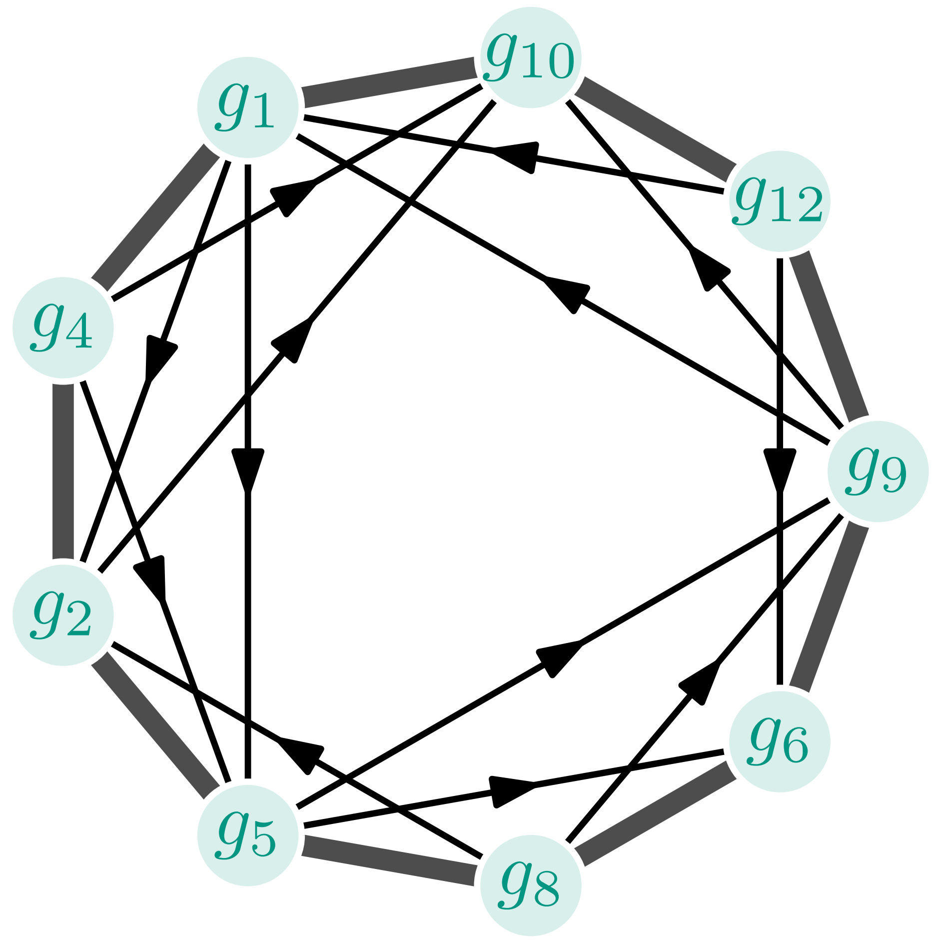

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$$\mathrm{dist}(T_{\text{red}}, T_{\text{blue}}) \leq \left(2 - \frac{\mathrm{ac}(H)}{|V_H|}\right) \cdot n \leq \frac{14}{9} \cdot n = 1.\overline{5}n$$

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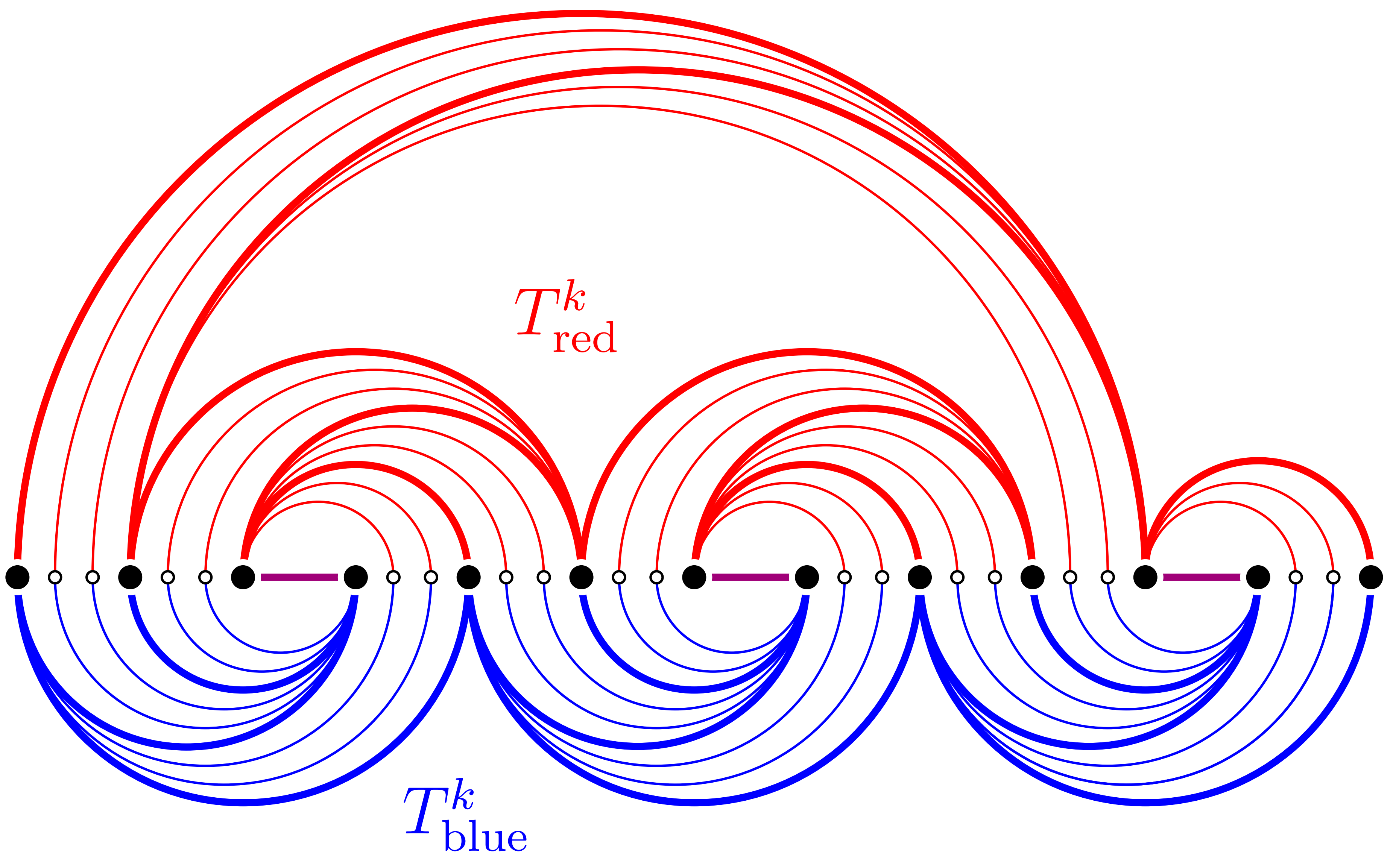
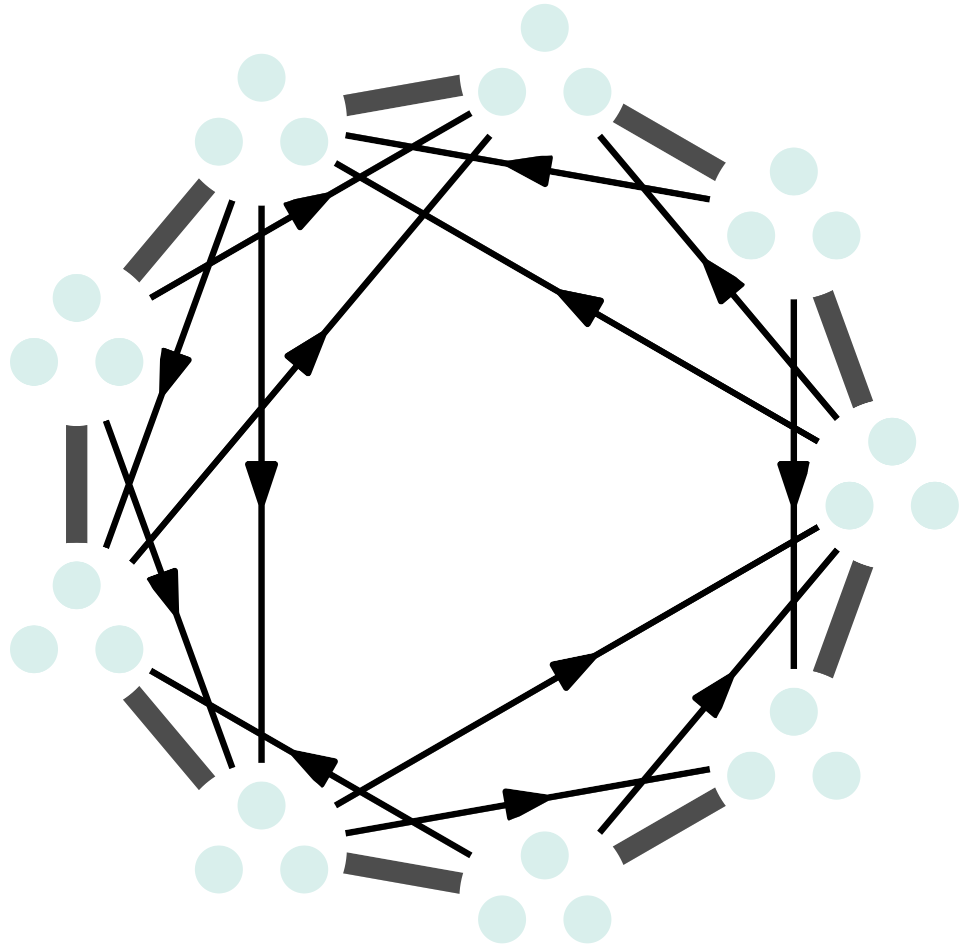




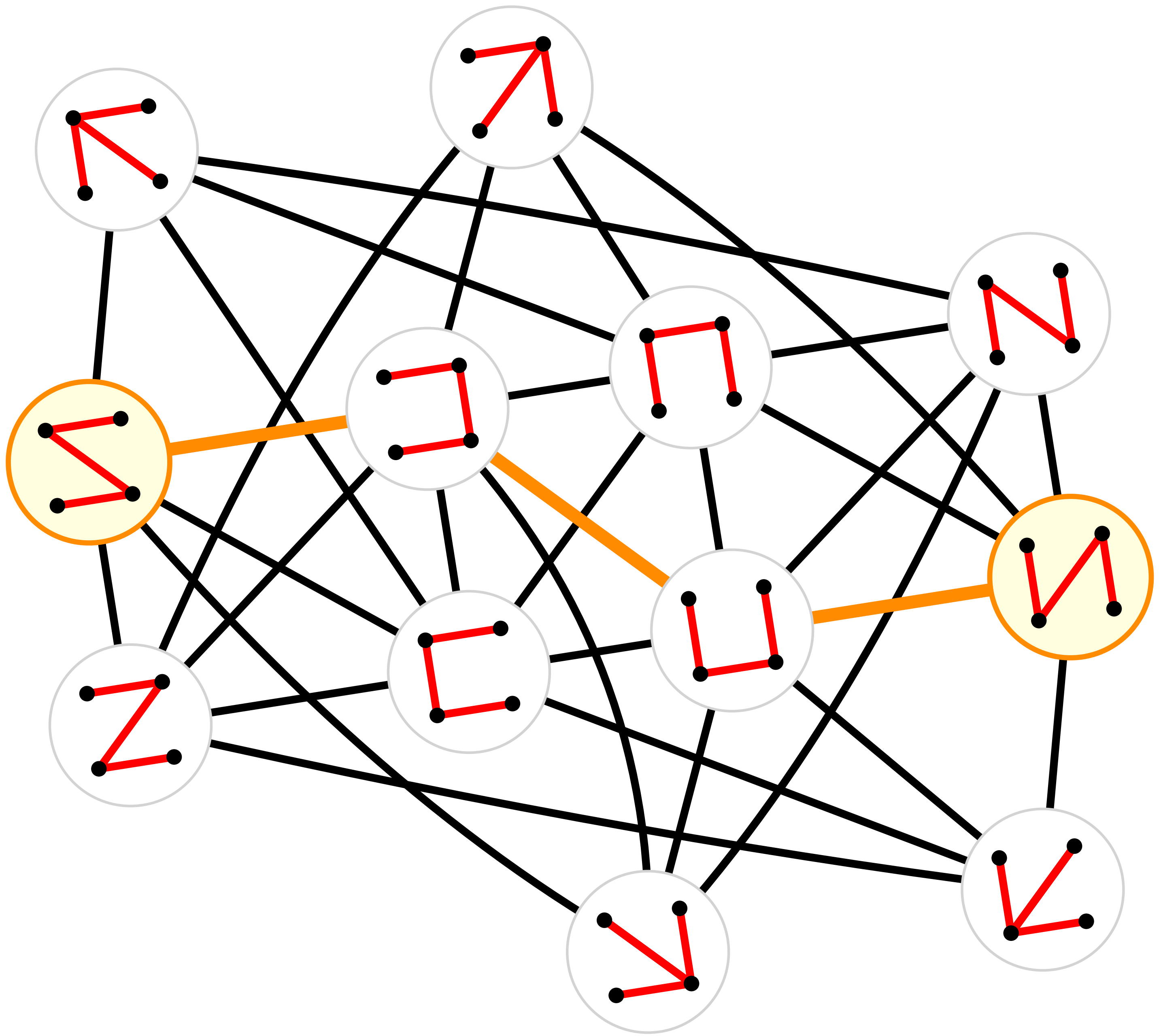
conflict graph H_k

$|V_{H_k}| = 9k$ $\quad$ $\text{ac}(H_k) = 4k$

$$\text{dist}(T^k_{\text{red}}, T^k_{\text{blue}}) \geq \left(2 - \frac{\text{ac}(H)}{|V_H|}\right) \cdot n - C = 1.\overline{5}n - C$$

$$\frac{\text{ac}(H_k)}{|V_{H_k}|} = \frac{4}{9}$$

Conclusion



Previous bounds:

$$1.5n - 5 \;\leq\; \operatorname{diam}(\mathcal{F}_n) \;\leq\; 1.96n$$

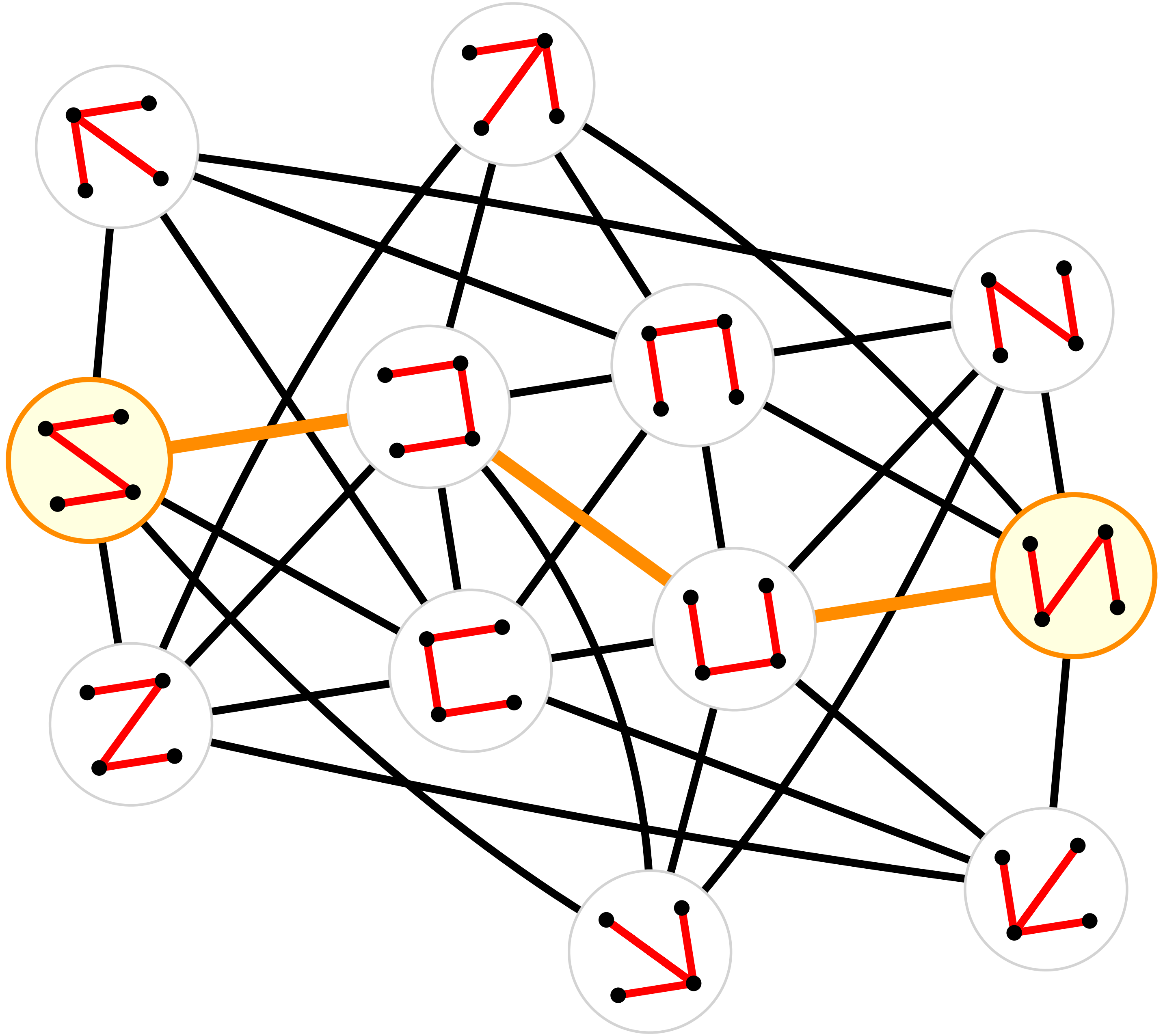
Theorem.

For the diameter $\operatorname{diam}(\mathcal{F}_n)$ of the **flip graph** $\mathcal{F}_n$ of non-crossing spanning trees on a set of n points in convex position we have:

$$1.\overline{5} = \frac{14}{9} \leq \lim_{n\to\infty} \frac{\operatorname{diam}(\mathcal{F}_n)}{n} = \sup\left\{2 - \frac{\operatorname{ac}(H)}{|V_H|} : H \text{ conflict graph}\right\} \leq \frac{5}{3} = 1.\overline{6}$$

Conclusion

Thank you!



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