Flipping Non-Crossing Spanning Trees

Håvard Bakke Bjerkevik
University at Albany

Linda Kleist
Universität Potsdam

Torsten Ueckerdt
Karlsruhe Institute of Technology

Birgit Vogtenhuber
TU Graz

Abstract

For a set P of n points in general position in the plane, the flip graph $\mathcal{F}(P)$ has a vertex for each non-crossing spanning tree on P and an edge between any two spanning trees that can be transformed into each other by one edge flip, i.e., the deletion and addition of exactly one edge. The diameter $\mathrm{diam}(\mathcal{F}(P))$ of this flip graph is subject of intensive study. For points P in general position, it is between $\lfloor 3/2 \cdot n \rfloor - 5$ and $2n - 4$, with no improvement for 25 years. For points P in convex position, $\mathrm{diam}(\mathcal{F}(P))$ lies between $\lfloor 3/2 \cdot n \rfloor - 5$ and $\approx 1.95n$, where the lower bound was conjectured to be tight up to an additive constant and the upper bound is a very recent breakthrough improvement over several previous bounds of the form $2n - o(n)$.

In this work, we provide new upper and lower bounds on the diameter of $\mathcal{F}(P)$ by mainly focusing on points P in convex position. We improve the lower bound even for this restricted case to $\mathrm{diam}(\mathcal{F}(P)) \geq 14/9 \cdot n - \mathcal{O}(1)$. This disproves the conjectured upper bound of $3/2 \cdot n$ for convex position, while also improving the long-standing lower bound for point sets in general position. In particular, we provide pairs T, T' of trees with flip distance $\mathrm{dist}(T, T') \geq 14/9 \cdot n - \mathcal{O}(1)$; in these examples, both trees T, T' have three convex hull edges. We complement this by showing that if one of T, T' has at most two convex hull edges, then $\mathrm{dist}(T, T') \leq 3/2 \cdot d < 3/2 \cdot n$, where $d = |T - T'|$ is the number of edges in one tree that are not in the other. This bound is tight up to additive constants.

Secondly, we significantly improve the upper bound on $\mathrm{diam}(\mathcal{F}(P))$ for n points P in convex position from $\approx 1.95n$ to $5/3 \cdot n - 3$. To prove both our lower and upper bound improvements, we introduce a new tool. Specifically, we convert the flip distance problem for given T, T' to the problem of a largest acyclic subset in an associated *conflict graph* $H(T, T')$. In fact, this method is powerful enough to determine the diameter of $\mathcal{F}(P)$ for points P in convex position up to lower-order terms. As such, conflict graphs are likely the key to a complete resolution of this and possibly also other reconfiguration problems.

2012 ACM Subject Classification Mathematics of computing → Discrete mathematics → Combinatorics → Combinatoric problems

Keywords and phrases flip graph, reconfiguration graph, spanning tree, non-crossing/crossing-free, convex point set

1 Introduction

Reconfiguration problems are important combinatorial problems with a high relevance in various settings and disciplines, e.g., robot motion planning, (multi agent) path finding, reconfiguration of data structures, sorting problems, string editing, in logistics, graph recoloring, token swapping, the Rubik's cube, or sliding puzzles, to name just a few. Given a collection of configurations and a set of allowed reconfiguration moves, each transforming one configuration into another, we naturally obtain a (directed) graph on the space of all configurations. When reconfiguration moves are reversible (then often called *flips*), this graph is undirected and called a *flip graph* $\mathcal{F}$. For example, the flip graph of the Rubik's cube has more than $43 \cdot 10^{18}$ vertices, each of degree 27.

A typical task is, for a pair A, B of input configurations, to find a sequence of flips that transform A into B – preferably fast. The distance of A and B in the flip graph $\mathcal{F}$ is the minimum number of required flips. As computing (or even storing) the entire flip graph is usually impractical, one often resorts to the structure of $\mathcal{F}$ to find a short flip sequence from A to B. However, even worst-case guarantees on the flip distance of A and B are mostly difficult to obtain. It took 29 years and 35 CPU-years donated by Google to determine the largest flip sequence between any two Rubik's cubes, that is, to determine the diameter of the corresponding flip graph. This elusive number is called God's number and equals 20 [35].

Flip graphs are a versatile structure with many potential applications. For example, they are used to obtain Markov chains to sample random configurations, or for Gray codes or reverse search algorithms to generate all configurations. We give more related work in Section 1.1, and refer to the survey articles [31, 37] for even more examples and applications of reconfiguration problems.

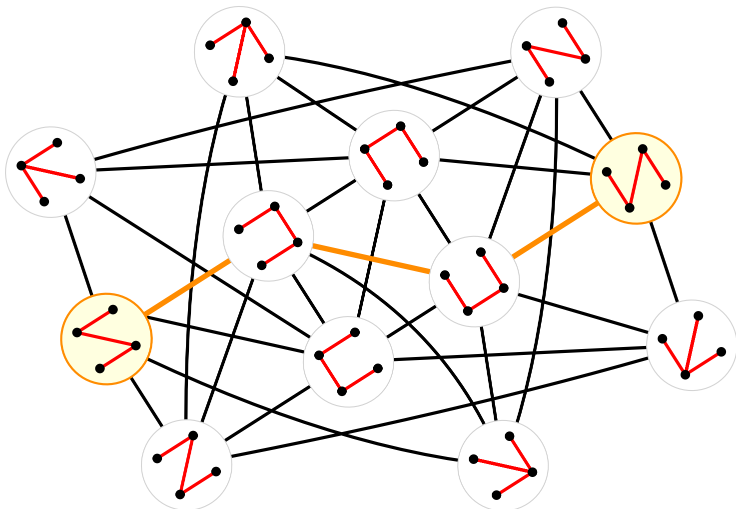


Figure 1 The flip graph $\mathcal{F}(P)$ on all non-crossing spanning trees on a set P of $n = 4$ points in convex position. A pair T, T' with $\mathrm{dist}(T, T') = \mathrm{diam}(\mathcal{F}(P)) = 3$ is highlighted.

A widely studied field concerns configuration of non-crossing straight-line graphs on a fixed point set in the plane. In this setting, a flip is usually the exchange of one edge with another edge. That is, two graphs A, B (i.e., configurations) are adjacent in the flip graph $\mathcal{F}$ if $|E(A) - E(B)| = |E(B) - E(A)| = 1$. Classical examples are triangulations [14, 16, 21, 22, 25, 26, 32, 33, 38], spanning trees [1, 2, 5, 7, 9, 18, 30], spanning paths [3, 4, 13, 24, 34], polygonizations [17], and matchings [20, 21, 28] on a fixed point set $P \subset \mathbb{R}^2$. For an overview, see the survey article [6].

Here, we study the flip graph of non-crossing spanning trees on a finite point set in the plane in general position. Throughout, let P denote a set of n points in $\mathbb{R}^2$ with no three collinear points. Consider a tree whose vertex-set is P and whose edges are pairwise non-crossing straight-line segments. Then a *tree T on P* is the edge-set of such a non-crossing spanning tree. E.g., Figures 2(a) and 2(c) show some trees on a set P in convex position.

Two trees T and T' on P are related by a *flip* if T can be obtained from T' by an exchange of one edge; i.e., there exist edges $e \in T$ and $e' \in T'$ such that $T' = T - e + e'$; see Figures 2(a) and 2(b) for an example. The *flip graph* $\mathcal{F}(P)$ of P has a vertex for each tree on P and an edge between any two trees that are related by a flip. A path from T to T' in $\mathcal{F}(P)$ corresponds to a *flip sequence* from T to T'. The length of a shortest flip sequence is the *flip distance* of T and T', denoted by $\operatorname{dist}(T, T')$. Finally, the *diameter* of $\mathcal{F}(P)$ is the largest flip distance of any two trees on P, i.e., the smallest D such that $\operatorname{dist}(T, T') \le D$ for all T, T'. In addition, the *radius* of $\mathcal{F}(P)$ is $\operatorname{rad}(\mathcal{F}(P)) = \min_T \max_{T'} \operatorname{dist}(T, T')$.

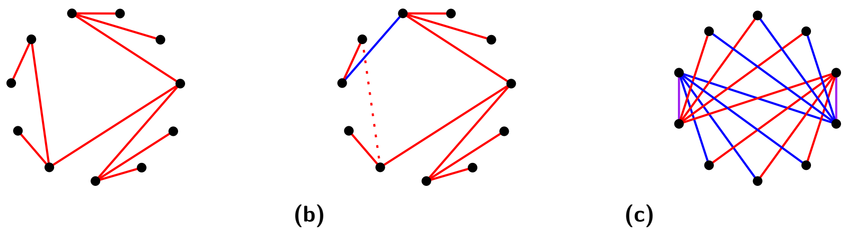


Figure 2 Some non-crossing trees on a point set in convex position.

The flip graph $\mathcal{F}(P)$ of trees on P has been considered since 1996, when Avis and Fukuda [5] showed that any tree T on P can be flipped to[1] any star T' on P whose central vertex lies on the boundary of the convex hull of P in $|T - T'| \le n - 2$ steps. This implies that $\mathcal{F}(P)$ has radius at most $n - 2$ and hence diameter at most $2n - 4$. However, the exact radius and diameter of $\mathcal{F}(P)$ remain unknown to this day. Also, it is unclear how much the diameter varies between different point sets of same cardinality, or which point sets P maximize the diameter of $\mathcal{F}(P)$ among all sets of n points. In 1999, Hernando et al. [18] provided a lower bound by constructing two trees T, T' on n points in convex position (for any $n \ge 4$) with flip distance $\operatorname{dist}(T, T') = \lfloor 3/2 \cdot n \rfloor - 5$; see Figure 2(c). In this example, each edge in $T' - T$ intersects roughly half the edges of T. Hence, every flip sequence from T to T' must flip away roughly $n/2$ edges of T before the first edge of $T' - T$ can be introduced, and thus $\operatorname{dist}(T, T')$ is at least roughly $3/2 \cdot n$. Yet, it remained open whether there is another pair of trees with larger flip distance. As the only matching upper bound, we know that $\operatorname{dist}(T, T') \le \lfloor 3/2 \cdot n \rfloor - 2$ in the special case that one of T, T' is an x-monotone path [2].

Note that the lower bound of $\lfloor 3/2 \cdot n \rfloor - 5$ uses a point set in *convex position*. Interestingly, already this restricted setting is very challenging. The lower bound of $\lfloor 3/2 \cdot n \rfloor - 5$ has not been improved for decades and the upper bound of $2n - 4$ only was gradually improved in recent years. In 2023, Bousquet et al. [8] showed that $\operatorname{dist}(T, T') \le 2n - \Omega(\sqrt{n})$ for any two trees T, T' on n points in convex position and conjectured that $3/2 \cdot n$ flips always suffice.

▶ **Conjecture 1** (Bousquet et al. [8]).
For any set P of n points in convex position, the flip graph $\mathcal{F}(P)$ has diameter at most $3/2 \cdot n$.

Conjecture 1 claims that every pair T, T' of trees on a convex point set P admits a flip

[1] We use terms like "flipping to a tree" or "flipping edges" in the (hopefully) natural way.

sequence from T to T' of length at most $3/2 \cdot n$. This is confirmed only for special cases, namely when one of T, T' is a path [2] or a so-called *separated caterpillar* [8] (defined below).

It is also natural to compare the flip distance $\mathrm{dist}(T, T')$ of two trees with the trivial lower bound given by the number of edges in which T and T' differ, formally defined as $d = d(T, T') = |T - T'|$. If P is in convex position, it is easy to show that $d \leq \mathrm{dist}(T, T') \leq 2d - 4$. In 2022, Aichholzer et al. [2] showed that in fact $\mathrm{dist}(T, T') \leq 2d - \Omega(\log d)$. Recently, Bousquet et al. [7] broke the barrier of 2 in the leading coefficient by showing that $\mathrm{dist}(T, T') \leq 1.96d < 1.96n$. They also give a pair T, T' with flip distance $\mathrm{dist}(T, T') \approx 5/3 \cdot d$. However, as their pair T, T' has $d = |T - T'| \approx n/2$, this is not a counterexample to Conjecture 1.

Our contribution. We consider non-crossing trees on sets P of n points in convex position. Our main results are significantly improved lower and upper bounds on the diameter of the corresponding flip graph $\mathcal{F}(P)$ in terms of n. As all n-element convex point sets P give the same flip graph $\mathcal{F}(P)$, let us denote it by $\mathcal{F}_n$ for brevity. Recall that it is known that the diameter $\mathrm{diam}(\mathcal{F}_n)$ of $\mathcal{F}_n$ lies between roughly $1.5n$ [18] and $1.95n$ [7].

We improve the upper bound to $5/3 \cdot n = 1.\overline{6}n$.

▶ **Theorem 2.** *For any set P of $n \geq 2$ points in convex position, the flip graph $\mathcal{F}(P)$ of non-crossing spanning trees on P has diameter at most $5/3 \cdot n - 3$. That is,* $\mathrm{diam}(\mathcal{F}_n) \leq 5/3 \cdot n - 3$.

Secondly, we improve the known lower bound to roughly $14/9 \cdot n = 1.\overline{5}n$.

▶ **Theorem 3.** *There is a constant C such that for any $n \geq 2$, there are non-crossing trees T_n, T'_n on n points in convex position with* $\mathrm{dist}(T_n, T'_n) \geq 14/9 \cdot n - C$. *That is,* $\mathrm{diam}(\mathcal{F}_n) \geq 14/9 \cdot n - C$.

Theorem 3 is the first improvement over $\mathrm{diam}(\mathcal{F}_n) \geq \lfloor 3/2 \cdot n \rfloor - 5$, as given 25 years ago by the example of Hernando et al. [18] depicted in Figure 2(c). Moreover, Theorem 3 disproves Conjecture 1 and also improves the lower bound on the largest diameter of $\mathcal{F}(P)$ among all point sets P in general (not necessarily convex) position.

The trees T_n, T'_n in Theorem 3 have three boundary edges. On the other hand, every non-crossing tree contains at least two boundary edges (provided $n \geq 3$), and trees on a convex point set P with exactly two boundary edges are called *separated caterpillars*. Complementing Theorem 3, we show that if at least one of T, T' is a separated caterpillar, then their flip distance $\mathrm{dist}(T, T')$ is at most $3/2 \cdot d(T, T')$. This improves on the recent upper bound of $\mathrm{dist}(T, T') \leq 3/2 \cdot n$ for the same setting in [8]. Further, the bound is tight up to an additive constant since the construction from [18] in Figure 2(c) consists of two separated caterpillars.

▶ **Theorem 4.** *Let T, T' be non-crossing trees on $n \geq 3$ points in convex position. Let T be a separated caterpillar and $d := |T - T'|$. Then $\mathrm{dist}(T, T') \leq 3/2 \cdot d$. Moreover, there exists a flip sequence from T to T' of length at most $3/2 \cdot d$ in which no common edges are flipped.*

Concerning sets P of n points in general (not necessarily convex) position, Aichholzer et al. [2, Open Problem 3] ask for the radius of the flip graph $\mathcal{F}(P)$, in particular for a lower bound of the form $n - C$ for some small constant C. Avis and Fukuda [5] show that the radius is at most $n - 2$. In fact, a matching lower bound is easily obtained.

▶ **Theorem 5.** *For any set P of n points in general position, the flip graph $\mathcal{F}(P)$ of non-crossing trees on P has radius at least (and thus exactly) $n - 2$.*

Proof. Let T be any tree on P, v be a leaf of T, and S_v be the star on P with central vertex v. Then $\mathrm{dist}(T, S_v) \geq d(T, S_v) = |T - S_v| = n - 2$. As T was arbitrary, the result follows. ◀

Organization of the paper. We give an outline of our approach in Section 2; in particular we explain our strategy of reducing the task of determining the diameter of $\mathcal{F}_n$ to finding largest acyclic subsets of an associated conflict graph. Our main tool is Theorem 6 (stated below). In Section 3, we define the conflict graphs and show how to derive Theorems 2 and 3 from Theorem 6. Then, Section 4 is devoted to the proof of Theorem 6. In Section 5 we refine our tools to obtain an improved upper bound on $\mathrm{dist}(T,T')$ depending $d(T,T') = |T - T'|$ and the number of boundary edges in $T \cap T'$. In Section 6, we study the case where one tree is a separated caterpillar and show Theorem 4. We conclude with a list of interesting open problems in Section 7.

1.1 Related Work

First, let us mention further graph properties of the flip graphs $\mathcal{F}(P)$ of non-crossing trees on point set P that have been investigated. For P in convex position, Hernando et al. [18] show that $\mathcal{F}(P)$ has radius $n-2$ and minimum degree $2n-4$, and that $\mathcal{F}(P)$ is Hamiltonian and $2n-4$-connected [18]. For point sets P in general (not necessarily convex) position, Felsner et al. [16] show that their flip graphs $\mathcal{F}(P)$ have so-called r-rainbow cycles for all $r = 1, \ldots, n-2$, which generalize Hamiltonian cycles.

Resticted variants of flips for spanning trees. Besides the general edge exchange flip (that we consider here), several more restricted flip operations have been investigated. There is the *compatible edge exchange* (where the exchanged edges are non-crossing), the *rotation* (where the exchanged edge are adjacent), and the *edge slide* (where the exchanged edges together with some third edge form an uncrossed triangle). Nichols et al. [30] provide a nice overview of the best known bounds for five studied flip types. Let us remark that for all five flip types, the best known lower bound in terms of n (in the convex setting) corresponds to the general edge exchange. Consequently, our Theorem 3 translates to all these settings. In terms of $d = |T - T'|$, Bousquet et al. [7] show a tight bound of $2d$ for point sets P in convex position. For a variant with edge labels (which are transferred in edge exchanges), Hernando et al. [19] show that the flip graph remains connected for any set P in general position.

Lastly, let us mention reconfiguration of spanning trees in combinatorial (instead of geometric) settings, such as with leaf constraints [10], or degree and diameter constraints [11].

Spanning paths. Much less is known when restricting $\mathcal{F}(P)$ only to the spanning paths on P. In fact, it is open for more than 16 years [4, 6] whether this subgraph $\mathcal{F}'(P)$ of $\mathcal{F}(P)$ is connected. Akl et al. [4] conjecture the answer to be positive, while confirming it if P is in convex position. In fact, Chang and Wu [13] prove that for n points P in convex position, $\mathcal{F}'(P)$ has diameter $2n-5$ for $n = 3, 4$ and $2n-6$ for all $n \geq 5$. It is also known that $\mathcal{F}'(P)$ is Hamiltonian [34] and has chromatic number $\chi(\mathcal{F}'(P)) = n$ [29].

For P in general position, $\mathcal{F}'(P)$ is known to be connected for so-called generalized double circles [3], and its diameter is at least $2n-4$ if P is a wheel of size n [3]. Kleist, Kramer, and Rieck [24] show that so-called *suffix-independent paths* induce a large connected subgraph in $\mathcal{F}'(P)$, and confirmed connectivity of $\mathcal{F}'(P)$ if P has at most two convex layers.

Triangulations. For (non-crossing) inner triangulations on a set P of n points in general position, a flip replaces a diagonal of a convex quadrilateral spanned by two adjacent inner faces by the other diagonal. When points in P are in convex position, the corresponding flip graph $\mathcal{T}(P)$ is the 1-skeleton of the $(n-3)$-dimensional associahedron. In fact, the vertices of $\mathcal{T}(P)$ are in bijection with binary trees and the flip operation with rotations of these trees.

The diameter of $\mathcal{T}(P)$ is known to be in $\Omega(n^2)$ for P in general position [22], and at most $2n-10$ (for $n \geq 9$) for P in convex position [36], where the latter is in fact tight [33].

Computing the flip distance of two triangulation on P is known to be NP-complete [26,32], also in the more general setting of graph associahedra [23]. Many further properties of the associahedron have been investigated, such as geometric realizations [12], Hamiltonicity [27], rainbow cycles [16], and expansion and mixing properties [15].

2 Outline of Our Approach

Let us outline our approach to tackle the diameter $\mathrm{diam}(\mathcal{F}_n)$ of the flip graph $\mathcal{F}_n$ for n points in convex position. Together, Theorems 2 and 3 state that

$$^{14}/_{9} \cdot n - \mathcal{O}(1) \leq \mathrm{diam}(\mathcal{F}_n) \leq {}^{5}/_{3} \cdot n = {}^{15}/_{9} \cdot n,$$

narrowing the gap from roughly $^{1}/_{2} \cdot n$ to only $^{1}/_{9} \cdot n$. We obtain both, the upper and the lower bound, by transferring the question for the diameter of the flip graph into a more approachable question about largest acyclic subsets in certain conflict graphs. Our corresponding result is stated in Theorem 6 below. While we defer the precise definitions to Section 3, let us provide here some background needed to understand Theorem 6 and explain how Theorem 6 could be used to determine $\mathrm{diam}(\mathcal{F}_n)$ exactly up to lower-order terms.

Given a pair T, T' of trees on a set P of n points in convex position, we define a canonical bijection between the edges in T and the edges in T', formalized as a set $\mathcal{P}$ of pairs (e, e') with $e \in T$ and $e' \in T'$. So, each $e \in T$ has a unique partner $e' \in T'$, and vice versa. We then restrict our attention to flip sequences from T to T' that respect this bijection in the sense that every $e \in T$ is flipped to its partner $e' \in T'$ in at most two steps. That is, either e is flipped to e' directly (a *direct* flip), or e is flipped to e' in two steps via one intermediate boundary edge (an *indirect* flip). The length of a flip sequence of this form is then #direct flips + $2 \cdot$ #indirect flips, and our task is to minimize the number of indirect flips.

We associate a directed *conflict graph* $H = H(T, T')$ whose vertices correspond to a subset of the pairs in $\mathcal{P}$. A directed edge $(e_1, e_1') \to (e_2, e_2')$ in H expresses that the direct flip $e_2 \to e_2'$ cannot occur before the direct flip $e_1 \to e_1'$, as otherwise it would create a cycle or a crossing. Let $\mathrm{ac}(H)$ denote the size of a largest subset of $V(H)$ that induces an acyclic subgraph. We then construct a flip sequence from T to T' with $\mathrm{ac}(H)$ direct flips. So, if $\mathrm{ac}(H)$ is large, then $\mathrm{dist}(T, T')$ is small. On the other hand, if $\mathrm{ac}(H)$ is small, we can derive a good asymptotic lower bound on $\mathrm{diam}(\mathcal{F}_n)$. The precise statements go as follows.

▶ **Theorem 6.** *Let T, T' be two non-crossing trees on linearly ordered vertices $p_1, \ldots, p_n$ with corresponding conflict graph $H = H(T, T')$.*

(i) *If H is non-empty, then* $\mathrm{dist}(T, T') \leq \max\left\{\frac{3}{2}, 2 - \frac{\mathrm{ac}(H)}{|V(H)|}\right\}(n-1)$.
If H is empty, then $\mathrm{dist}(T, T') \leq \frac{3}{2}(n-1)$.

(ii) *If H is non-empty, then there is a constant C such that for all $N \geq n$, we have* $\mathrm{diam}(\mathcal{F}_N) \geq \left(2 - \frac{\mathrm{ac}(H)}{|V(H)|}\right) N - C$.

Theorem 6 implies that there exists a constant $\gamma \in [^{3}/_{2}, 2]$ such that

$$\lim_{n \to \infty} \frac{\mathrm{diam}(\mathcal{F}_n)}{n} = \gamma = \sup_{H} \left(2 - \frac{\mathrm{ac}(H)}{|V(H)|}\right), \tag{1}$$

where the supremum is taken over all non-empty conflict graphs H arising from pairs of non-crossing trees. In the light of (1), the task of finding γ looks quite different. But, as evidenced by our own results, this is a major simplification: With Theorem 6(ii) at hand, we can prove a lower bound on $\text{diam}(\mathcal{F}_n)$ for all n quite easily. It is enough to construct a single example of two trees T, T' on a point set P in convex position, and to compute a largest acyclic subset of the corresponding conflict graph H. In fact, all we do to prove Theorem 3 is exhibit an example of two trees on 13 vertices, compute their conflict graph H on 9 vertices, and write a two-line proof that $\text{ac}(H) \leq 4$; see Lemma 11. To further improve on our lower bound (if possible), one simply needs to do the same with a better example pair of trees.

And with Theorem 6(i) at hand, we also prove our upper bound in Theorem 2 through conflict graphs. We divide the vertices of the conflict graph H arising from an arbitrary pair T, T' of trees into three sets, and show that each set induces an acyclic subgraph of H; see Lemma 9. Also showing this acyclicity requires only a short argument.

Thus, Theorem 6 allows succinct proofs of upper and lower bounds on $\text{diam}(\mathcal{F}_n)$. Any improved lower bounds on $\frac{\text{ac}(H)}{|V(H)|}$, or examples of conflict graphs H with $\frac{\text{ac}(H)}{|V(H)|} < 4/9$, will give improved bounds on $\text{diam}(\mathcal{F}_n)$. This is a promising avenue towards determining the exact value of γ. Moreover, the conflict graph might be useful in other reconfiguration problems.

3 Conflict Graphs, Acyclic Subsets, and the Diameter of $\mathcal{F}_n$

Throughout this section, let P be a set of n points in the plane in convex position. As the flip graph $\mathcal{F}(P) = \mathcal{F}_n$ only depends on n, we can imagine the points in P to lie equidistant on a circle and are circularly labeled as $p_1, \ldots, p_n$. Given a tree T with vertex-set P, we can represent T as a straight-line drawing on P. If this drawing has no crossing edges, then T is non-crossing and we simply call T a *tree on* P. For convenience, we treat each tree T as its set of edges. Let $\mathcal{T}_n$ denote the set of all trees on P. If for two trees T, T' on P we have $|T - T'| = 1$, then T and T' are related by a *flip*. The *flip graph* $\mathcal{F}_n$ has vertex-set $\mathcal{T}_n$ and an edge for any two trees on P that are related by a flip. Given two trees $T, T' \in \mathcal{T}_n$, the *flip distance* $\text{dist}(T, T')$ is the length of a shortest path from T to T' in $\mathcal{F}_n$.

Consider two fixed trees T, T' on P. We work with a *linear representation* as illustrated in Figure 3(b) with an example. Intuitively speaking, we cut open the circle between p_1 and p_n and unfold the circle into a horizontal line segment, usually called the *spine*. Each edge in $T \cup T'$ was a straight-line chord of the circle and can now be thought of as a semi-circle above or below the spine. For better readability, we usually put the edges of T above and the edges of T' below the spine, see again Figure 3.

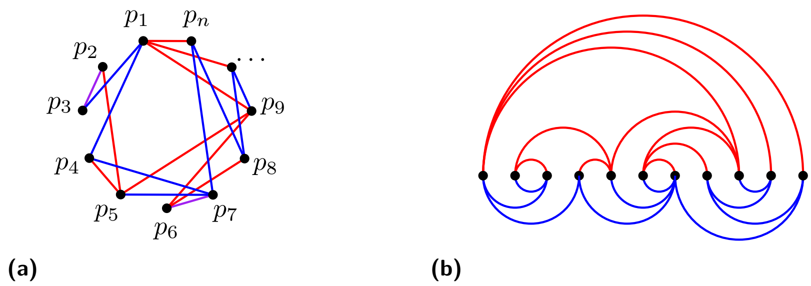


Figure 3 (a) Two non-crossing trees T, T' on a circularly labeled point sets in convex position and (b) its linear representation with T above and T' below the horizontal spine.

By the linear order $p_1, \ldots, p_n$ we have also a natural notion of the length of an edge. That is, if edge e has endpoints p_i and p_j, then the *length* of e is $|i - j|$. Moreover, we say that an edge e with endpoints p_i and p_j, $i < j$, *covers* a vertex p_k if $i \leq k \leq j$. Especially, each edge covers both of its endpoints. An edge e *covers* an edge f if e covers both endpoints of f. If no edge $e \in T - f$ covers the edge $f \in T$, then we say that f is an *uncovered edge* in T.

The linear order of the n points also defines $n-1$ *gaps* $g_1, \dots g_{n-1}$, where gap g_i simply is the (open) segment along the spine with endpoints p_i and p_{i+1}. To introduce a few crucial properties, let us consider any set S of non-crossing edges on $p_1, \dots, p_n$ (not necessarily forming a tree). For each gap g that is covered by at least one edge of S, let $\rho_S(g)$ be the shortest edge of S covering g. Spanning trees cover all gaps. The following lemma shows that ρ_S forms a bijection between gaps and edges in S if and only if S is a spanning tree.

▶ **Lemma 7.** *Let S be a set of non-crossing edges on a linearly labeled point set with each gap covered by at least one edge in S. Then ρ_S defines a bijection from the set of gaps to S if and only if S is a tree.*

Proof. Suppose first that ρ_S is a bijection. We argue by induction on $|S|$ that S is a tree. Pick an edge $e \in S$ that is not covered by any other edge in S, and let $g = \rho_S^{-1}(e)$ be the corresponding gap. Then $S - e$ has two connected components, one to the left of gap g, and one to the right. Restricted to either side, ρ is again a bijection, and hence by induction we have a tree on either side of g. Now, e connects the two trees, showing the S is a tree itself.

Now suppose ρ_S is not a bijection. We shall show that S is not a tree. This clearly holds if $|S| \neq n-1$. And if $|S| = n-1$, then ρ_S is not injective. Hence we have $e = \rho_S(g) = \rho_S(g')$ for different gaps g and g' and some $e \in S$. But then the vertices between g and g' form one or more separate connected components and S is not connected, i.e., not a tree. ◀

For a non-crossing tree T on a linearly ordered point set $p_1, \dots p_n$, we define $e_i := \rho_T(g_i)$ and categorize the edges of T into three types, depending on how many endpoints of an edge e_i are also endpoints of its corresponding gap g_i. For each $i \in [n-1]$, we say that the edge $e_i = \{u, v\}$ of T is a

- *short edge* if $\{u, v\} = \{p_i, p_{i+1}\}$,
- *near edge* if $|\{u, v\} \cap \{p_i, p_{i+1}\}| = 1$, and
- *wide edge* if $\{u, v\} \cap \{p_i, p_{i+1}\} = \varnothing$.

The set of all short, near, and wide edges of T is denoted by T_S, T_N, and T_W, respectively. Note that the short edges of T are the boundary edges of T different from $p_n p_1$, or in other words, the edges of length 1. Symmetrically, for the tree T', we denote the edge corresponding to gap g_i by e'_i.

Pairing. Given T, T' and a linear representation, we define $\mathcal{P} = \{(e_i, e'_i) \mid i = 1, \dots, n-1\}$ to be the natural pairing of the edges in T with those in T' according to their corresponding gap. That is, $(e, e') \in \mathcal{P}$ for $e \in T$ and $e' \in T'$ if and only if $\rho_T^{-1}(e) = \rho_{T'}^{-1}(e')$. Note that e_i and e'_i might coincide, i.e., $e_i = e'_i$; particularly, this happens if e_i is a short edge in $T \cap T'$. Next we partition the set $\mathcal{P}$ of edge pairs as follows:

$$\mathcal{P}_= = \{(e, e') \in \mathcal{P} \mid e = e'\},$$
$$\mathcal{P}_N = \{(e, e') \in \mathcal{P} \mid e \neq e' \text{ and } e \in T_N \text{ and } e' \in T'_N\},$$
$$\mathcal{P}_R = \mathcal{P} - (\mathcal{P}_= \cup \mathcal{P}_N).$$

Clearly, $|\mathcal{P}_=| + |\mathcal{P}_N| + |\mathcal{P}_R| = |\mathcal{P}| = n-1$. As it turns out, we will spend no flips on pairs in $\mathcal{P}_=$ and it will be enough to spend in total at most $\frac{3}{2}|\mathcal{P}_R| + |\mathcal{P}_=|$ flips on pairs in $\mathcal{P}_R$. The more difficult part will be the pairs in $\mathcal{P}_N$, namely, the near-near pairs. The aim is to find a large subset of $\mathcal{P}_N$ which only needs one flip per edge.

Conflict graph. We want to find a large set of near-near pairs that can be flipped directly. However, two near-near pairs (e_i, e'_i) and (e_j, e'_j) could be so interlocked that it is impossible to have both as direct flips in any flip sequence from T to T'. This is for example the case if e_i crosses e'_j and e_j crosses e'_i. To capture all these dependencies we define a directed auxiliary graph which we call the *conflict graph* H of T, T'. Let $I_=, I_R, I_N$ denote the subsets of gaps corresponding to $\mathcal{P}_=, \mathcal{P}_R, \mathcal{P}_N$, respectively.

▶ **Definition 8** (Conflict graph).
The conflict graph $H = H(T, T')$ *is the directed graph defined by*

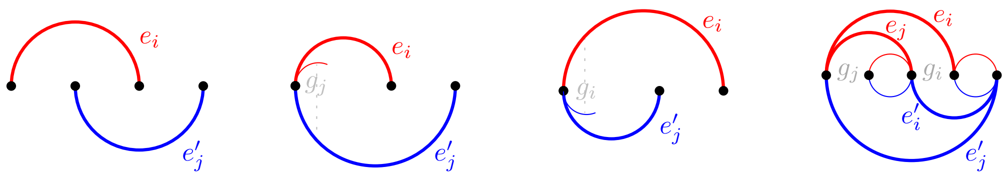
- $V(H) := I_N$; *i.e., the vertices are the gaps corresponding to near-near pairs, and*
- *there is a directed edge in* $E(H)$ *from* g_i *to* g_j, *denoted* $\overrightarrow{g_i g_j}$, *if*

T1: e_i *crosses* e'_j, *or*

T2: e'_j *covers* e_i *and* e_i *covers* g_j, *or*

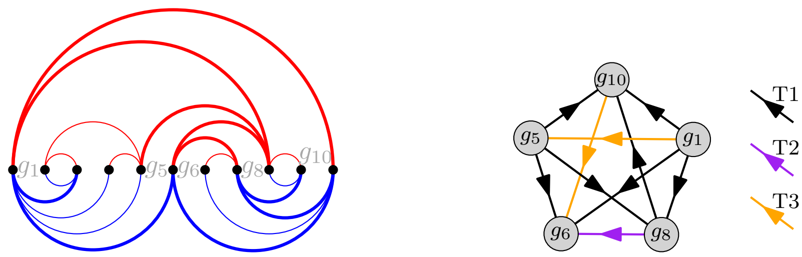
T3: e_i *covers* e'_j *and* e'_j *covers* g_i.

Figure 4 illustrates the three types of edges in H. Figure 5 depicts a full example of a linear representation of a pair T, T' and the corresponding conflict graph $H(T, T')$. Observe that $H(T', T)$ is obtained from $H(T, T')$ by reversing the direction of all edges.



(a) $\overrightarrow{g_i g_j}$ of type T1 **(b)** $\overrightarrow{g_i g_j}$ of type T2 **(c)** $\overrightarrow{g_i g_j}$ of type T3 **(d)** $\overrightarrow{g_i g_j}$ of type T2

Figure 4 Examples of directed edges in the conflict graph: (a) type T1 (b) type T2 (c) type T3. Mirroring the examples in (a)-(c) horizontally gives a complete list of all possibilities. (d) Example of a possible conflict of type T2: the direct flip $e_j \to e'_j$ in T (above, red) does not yield a tree.



(a) **(b)**

Figure 5 (a) The pair T, T' from Figure 3 with pairs in $\mathcal{P}_N$ in fat and (b) their conflict graph H.

A direct flip $e_j \to e'_j$ for a near-near pair (e_j, e'_j) may be invalid for two reasons, either because the introduced edge crosses an existing edge e_i (this corresponds in H to an edge $\overrightarrow{g_i g_j}$ of type T1) or because the new graph is not a tree (this is captured by incoming edges at g_j in H of type T2 and T3). In fact, we claim (and prove later) that a near-near edge pair (e_j, e'_j) admits a direct flip $e_j \to e'_j$ (after flipping all pairs of $\mathcal{P}_= \cup \mathcal{P}_R$ to the boundary) if and only if the according gap g_j has no incoming edge in H. In the remainder of this section, we show how Theorem 6 can be used to prove $^{14}/_{9} \cdot n - O(1) \le \operatorname{diam}(\mathcal{F}_n) \le {}^{5}/_{3}(n-1)$.

3.1 Upper bound on the flip distance via Theorem 6(i)

In this subsection, we assume that Theorem 6(i) holds, and show how to derive Theorem 2 from it. That is, we prove an upper bound of $5/3(n-1)$ on the flip distance of two non-crossing trees T and T' on n vertices by finding a large acyclic subset of the corresponding conflict graph H. In fact, by Theorem 6(i) we have $\operatorname{dist}(T,T') \le \max\left\{\frac{3}{2}, 2 - \frac{\operatorname{ac}(H)}{|V(H)|}\right\}(n-1)$. So we seek to prove that $\operatorname{ac}(H) \ge 1/3 \cdot |V(H)|$ whenever H is non-empty.

Recall that a near edge is incident to exactly one vertex at its corresponding gap. We can think of a near edge e_i (or e'_i) to "start" at gap i (either at p_i or p_{i+1}) and then "go" either left or right. Clearly, each near-near pair (e, e') starts at the same gap. Moreover, observe that e and e' go in the same direction if and only if e and e' are adjacent, i.e., have a common endpoint (which is then necessarily at the gap).

In order to prove a lower bound on $\operatorname{ac}(H)$, and hence an upper bound on $\operatorname{dist}(T,T')$, let us inspect the gaps more closely. We partition the set I_N of all near-near gaps into three subsets, distinguishing for each gap with a corresponding near-near pair, whether these two edges are adjacent, and (in case they are) which edge is longer. For each gap $g_i \in I_N$ with near-near pair $(e_i, e'_i) \in \mathcal{P}_N$, we say that

- g_i is *above* if e_i and e'_i are adjacent and e_i is longer than e'_i,
- g_i is *below* if e_i and e'_i are adjacent and e_i is shorter than e'_i, and
- g_i is *crossing* if e_i and e'_i are not adjacent.

We denote the set of all above (respectively below, crossing) gaps in I_N by A (respectively B, C). By definition, A, B, C are pairwise disjoint, and hence $|A| + |B| + |C| = |I_N| \le n - 3$.

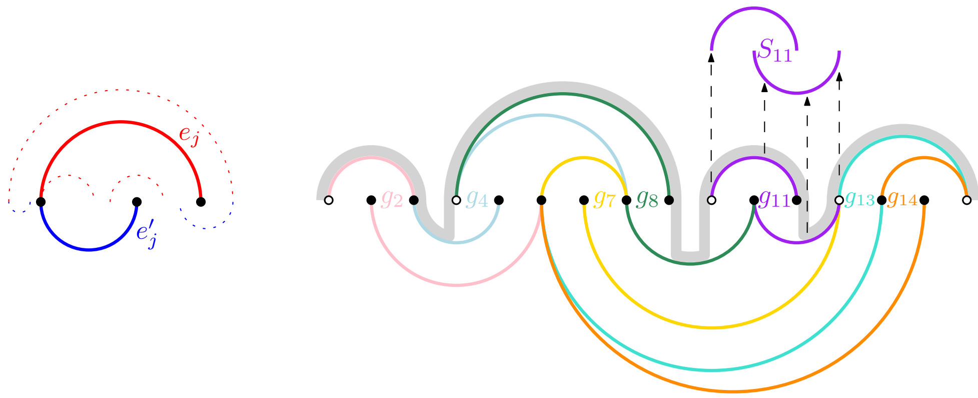
▶ **Lemma 9.** *Each of A, B, C is an acyclic subset of H. In particular,* $\operatorname{ac}(H) \ge 1/3 \cdot |V(H)|$.

Proof. To prove that $Y \in \{A, B, C\}$ is acyclic, we show that there is some gap $g^* \in Y$ without incoming edges in $H[Y]$. By removing g^* from Y and repeating the argument, it follows that Y is acyclic. We separately consider the three possible choices of Y.

Case $Y = A$: Consider a gap $g_j \in A$ such that the length of e_j is minimal. We claim that the gap g_j has no incoming edge in $H[A]$. Without loss of generality, we assume that e_j and e'_j share their left endpoint as illustrated in Figure 6(a); otherwise consider the mirror image. We show that g_j has no incoming edge in $H[A]$ in any of the three types T1,T2,T3. By the choice of g_j, e_j is not covering any edge e_i; otherwise e_i is shorter. This excludes incoming edges of type T1 and T2. For any $g_i \in A$ with e_i covering e_j and e'_j, then the gap g_i cannot be covered by e'_j. This excludes incoming edges of type T3.

Case $Y = B$: This is symmetric to the previous case by exchanging the roles of T and T'. In particular, the vertex $g^* = \arg\min_{g_j \in B}\{\text{length of } e'_j\}$ has no incoming edges in $H[B]$.

Case $Y = C$: Consider the linear representation of T and T' with horizontal spine, but restricted only to the edges corresponding to gaps in C. For each $g_i \in C$ with pair (e_i, e'_i) let $S_i \subseteq \mathbb{R}^2$ be the union of the two open semicircles for e_i and e'_i (without their endpoints). Clearly, the $\{S_i \mid g_i \in C\}$ are pairwise disjoint. Now let us try to move one S_j vertically up towards $(0, +\infty)$. Observe from Figures 4(a)–4(c) that if we could move S_j upwards without colliding with another S_i, then $g_j \in C$ has no incoming directed edges in $H[C]$. It remains to prove that at least one S_j can be moved upwards like this, i.e., is "fully visible from above". To this end, consider the upper envelope $\mathcal{E}$ of the S_i's; see the gray-shaded silhouette in Figure 6. Let the *left end* and *right end* of each S_i be its leftmost and rightmost point, respectively. Some left ends and right ends are on $\mathcal{E}$. Note that in Figure 6 the right end of S_7 is not on $\mathcal{E}$, as the right end of S_{11} is vertically above



(a) **(b)**

Figure 6 Illustration for the proof of Lemma 9. (a) Case $Y = A$. (b) Case $Y = C$. Gaps $g_2, g_4, g_7, g_8, g_{11}, g_{13}, g_{14}$ in C. On the upper envelope (gray) left-to-right we have left end S_2, left end S_8, left end S_{11}, right end S_{11}, right end S_{13}. Hence S_{11} can be moved vertically up and g_{11} has no incoming edges in $H[C]$.

it. Now consider the left and right ends on $\mathcal{E}$ from left to right. There is at least one right end on $\mathcal{E}$; at p_n at the latest.
Say the leftmost right end on $\mathcal{E}$ belongs to S_j. We claim that immediately to the left there is the left end of S_j and hence S_j is unobstructed to be moved upwards. Indeed, if some S_i would cover parts of S_j, then either S_i would cover also the right end of S_j or the right end of S_i would be further left than the right end of S_j; both being a contradiction to the choice of S_j. ◀

Together, Lemma 9 and Theorem 6(i) immediately imply an upper bound on $\operatorname{diam}(\mathcal{F}_n)$.

▶ **Corollary 10.** *Let T, T' be any pair of two non-crossing trees on a convex set of n points. Then the flip distance $\operatorname{dist}(T, T')$ is at most $5/3(n-1)$. In other words, $\operatorname{diam}(\mathcal{F}_n) \leq 5/3(n-1)$.*

Proof. If H is empty, then Theorem 6(i) guarantees a flip distance of at most $3/2(n-1)$. If H is non-empty, then Lemma 9 states that $\frac{\operatorname{ac}(H)}{|V(H)|} \geq 1/3$ and Theorem 6(i) implies an upper bound of $\max\{3/2, 2-1/3\}(n-1) = 5/3(n-1)$. ◀

3.2 Lower bound on the flip distance via Theorem 6(ii)

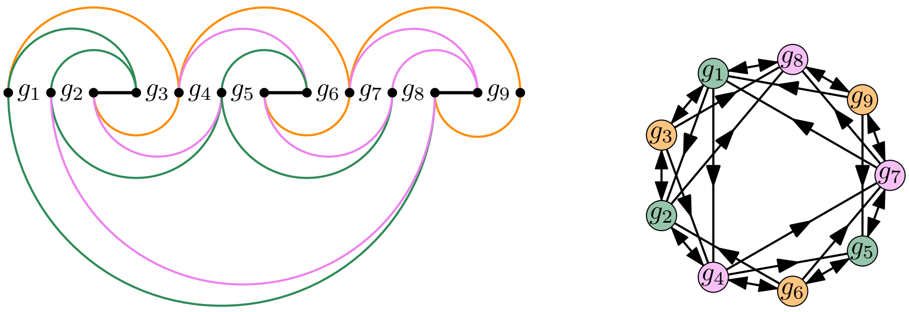
We present an example of two trees T, T' where the largest acyclic subset in the corresponding conflict graph H is comparatively small. By Theorem 6(ii) this then improves the lower bound on the diameter of $\mathcal{F}_n$ from $1.5n$ to $14/9 \cdot n - \mathcal{O}(1) = 1.\overline{5}n - \mathcal{O}(1)$.

▶ **Lemma 11.** *There exist trees T, T' on linearly ordered points $p_1, \ldots, p_{13}$ such that for their conflict graph H we have $\operatorname{ac}(H) \leq 4$ and $|V(H)| = 9$.*

Proof. Let T, T' be the trees depicted in Figure 7(a). Their conflict graph H is depicted in Figure 7(b) and contains a cycle of length 9 with all edges bi-directed. Consequently, any acyclic subset may contain at most every other gap and thus $\operatorname{ac}(H) \leq \lfloor 9/2 \rfloor = 4$. ◀

Together, Lemma 11 and Theorem 6(ii) imply Theorem 3:

▶ **Corollary** (Theorem 3). *There is a constant C such that for any $n \geq 1$, there are non-crossing trees T_n and T'_n on n vertices in convex position with $\operatorname{dist}(T_n, T'_n) \geq 14/9 \cdot n - C$. That is, $14/9 \cdot n - C \leq \operatorname{diam}(\mathcal{F}_n)$ for all $n \geq 1$.*



(a) **(b)**

Figure 7 A linear representation of two trees T, T' (a) and their conflict graph H (b). The coloring of edge pairs in (a) and gaps in (b) is according to the partition A, B, C (above, below, crossing) in Section 3.1.

4 Proof of Theorem 6

We now prove Theorem 6, which relates the size of acyclic subsets of conflict graphs with upper and lower bounds for $\text{diam}(\mathcal{F}_n)$, and is the key ingredient to Theorems 2, 3, and 25. We prove Theorem 6(i) in Section 4.1 and Theorem 6(ii) in Section 4.2.

4.1 Upper bound

Recall that we want a flip sequence that transforms T into T'. With each flip we remove an edge and replace it by a new edge. This way, we can trace each edge from its initial to its final position. In particular, every flip sequence naturally pairs the edges of T with those of T'. Our approach is to let $\mathcal{P}$ be this pairing, i.e., to convert each edge e of T into the edge e' of T' with $(e, e') \in \mathcal{P}$. For each pair $(e, e') \in \mathcal{P}$ we will do at most two flips. More precisely, in our flip sequence, every gap g_i, $i \in [n-1]$ and the corresponding pair $(e_i, e'_i) \in \mathcal{P}$ shall have exactly one of the following properties.

0-flip: $e_i = e'_i$ and the edge e_i is never replaced, keeping $e_i = e'_i$ in every intermediate tree.

1-flip: $e_i \neq e'_i$ and the edge e_i is replaced by e'_i in a single, direct flip.

2-flip: $e_i \neq e'_i$, the edge e_i is replaced by the boundary edge $p_i p_{i+1}$ in one flip, and $p_i p_{i+1}$ is replaced by e'_i in a later flip.

Clearly, the total number of flips in our flip sequence is then the number of 1-flips plus two times the number of 2-flips. Our goal is to have as few 2-flips as possible.

Recall the partition of $\mathcal{P}$ into $\mathcal{P}_=, \mathcal{P}_N, \mathcal{P}_R$. As mentioned before, we shall spend no flips on pairs in $\mathcal{P}_=$ and in total at most $\frac{3}{2}|\mathcal{P}_R| + |\mathcal{P}_=|$ flips on pairs in $\mathcal{P}_R$. For the pairs in $\mathcal{P}_N$, we shall do a 1-flip for those corresponding to an acyclic subset of the conflict graph H, and spend a 2-flip for the remaining pairs in $\mathcal{P}_N$. But first, let us present sufficient conditions for the validity of these flip.

Recall that every non-crossing edge-set S on linearly ordered vertices $p_1, \ldots, p_n$ that covers all gaps has a corresponding map $\rho_S \colon \{g_1, \ldots, g_{n-1}\} \to S$, and that by Lemma 7 S is a tree if and only if ρ_S is a bijection.

▶ **Lemma 12.** *Let T_1 be a non-crossing tree on linearly ordered vertices $p_1, \ldots, p_n$, and let $e_k = \rho_{T_1}(g_k)$ for $k = 1, \ldots, n-1$.*

Fix an edge $e_j \in T_1$ and consider an edge $e' = p_x p_y$ with $e' \notin T_1$, such that e' covers g_j, and e' does not cross any edge in $T_1 - e_j$, and there is no $e_i \in T_1 - e_j$ such that

(a) *e' covers e_i, and e_i covers g_j, or*
(b) *e_i covers e', and e' covers g_i.*

Then, for $T_2 := (T_1 - e_j) + e'$, each of the following holds.

(i) *T_2 is a non-crossing tree, i.e., ρ_{T_2} is a bijection.*
(ii) *Each edge $e \in T_1 \cap T_2 = T_1 - e_j$ we have $\rho_{T_1}^{-1}(e) = \rho_{T_2}^{-1}(e)$, i.e., e corresponds to the same gap in T_1 and T_2, while $e' = \rho_{T_2}(g_j)$, i.e., e' corresponds to g_j in T_2.*
(iii) *Each edge $e \in T_1 \cap T_2 = T_1 - e_j$ is short (respectively near, wide) in T_1 if and only if e is short (respectively near, wide) in T_2.*

Proof. We first show (i). Clearly, T_2 is non-crossing as e' crosses no edge in $T_1 - e_j$ by assumption. To show that T_2 is a tree, by Lemma 7, it suffices to show every gap is covered and $\rho_{T_2}\colon \{g_1, \dots, g_{n-1}\} \to T_2$ is a bijection. In fact, gap g_j is covered by e' and each $g_i \neq g_j$ is still covered by $e_i \in T_1 - e_j$. As $|T_2| = n-1$, it suffices to show that ρ_{T_2} is injective. By (a), $\rho_{T_2}(g_j) = e'$, and by (b), $\rho_{T_2}(g_i) = e_i$ for all $i \neq j$. Thus, ρ_{T_2} is a bijection and T_2 a tree.

In fact, we already know ρ_{T_2} explicitly, and can also conclude (ii). And (iii) follows from (ii), since the type of an edge e depends only on e and its associated gap. ◀

We use Lemma 12 in particular for two special cases, namely in 2-flips when e' is a boundary edge, and in 1-flips when the gap of e' has no incoming edges in a subgraph $H[Y]$ of H. For convenience, we show that the preconditions are fulfilled in these two cases.

▶ **Lemma 13.** *The preconditions of Lemma 12 are fulfilled if we choose e' as the boundary edge $p_j p_{j+1}$.*

Proof. Clearly, a boundary edge does not cross any edge of T_1. Moreover, e' is a short edge that covers no edge of T_1 and covers only one gap, namely g_j, proving that no edge $e_i \in T_1 - e_j$ has property (a) or (b). ◀

▶ **Lemma 14.** *The preconditions of Lemma 12 are fulfilled if $\mathcal{P}_R = \varnothing$, g_j has no incoming edge in H and e' is chosen such that $(e_j, e') \in \mathcal{P}_N$, i.e., we flip e_j for $e' = e'_j$.*

Proof. If e' crosses an edge e_i of $T_1 - e_j$, then there is the incoming edge $\overrightarrow{g_i g_j}$ of type T1 at g_j in H. Secondly, if an edge e_i of $T_1 - e_j$ has property (a), respectively (b), then there is the incoming edge $\overrightarrow{g_i g_j}$ of type T2, respectively T3, at g_j in H. ◀

Recall that for a gap g_i and non-crossing tree T, the edge $e_i = \rho_T(g_i)$ is wide if e_i is neither incident to p_i nor p_{i+1}. We next bound the number $|T_W|$ of wide edges of T in terms of the number $|T_S|$ of short edges in T.

▶ **Lemma 15.** *Let T be a non-crossing tree on linearly ordered vertices $p_1, \dots, p_n$. Let $k \geq 1$ be the number of edges of T that are not covered by any other edge of T. Then $|T_S| \geq k$ and $|T_W| \leq |T_S| - k$.*

Proof. Consider the cover relation of the edges of T (with respect to the given linear order). Let us write $e \preceq f$ whenever e is covered by f. Trivially, every edge $e \in T$ covers itself, i.e., $e \preceq e$. Since $e \preceq e'$ and $e' \preceq e''$ implies $e \preceq e''$, we have that $(T, \preceq)$ forms a partial order. Moreover, since T is non-crossing, it holds that for every $e \in T$, its upset $\{e' \in T \mid e \preceq e'\}$ is totally ordered, implying that the Hasse diagram of $(T, \preceq)$ is a rooted forest R. The roots of R are the uncovered edges of T. The leaves of R are the short edges of T. We will show that any wide edge of T has at least two children in R, which clearly implies the lemma.

Let $e = p_i p_j$ be a wide edge. Say $e = e_k = \rho_T(g_k)$ for the gap g_k between p_k and p_{k+1}. Because e is wide, we have $i < k$ and $k+1 < j$. Let $e_i = \rho_T(g_i)$ be the edge with gap g_i, and let

$e_{j-1} = \rho_T(g_{j-1})$ be the edge with gap g_{j-1}. The edges e_k, e_i, and e_{j-1} are all different, since they have pairwise different gaps. Since $i < k$ and $k+1 < j$, we have $e_i \preceq e_k$ and $e_{j-1} \preceq e_k$. Further, any edge $f \neq e$ that covers both e_i and e_{j-1} also covers the vertices p_k and p_{k+1}, and since g_k is the gap of e_k, it follows that f also covers e_k. It follows that e_k is the join of e_i and e_{j-1}, which means that $e = e_k$ has at least two children in R. To be specific, two children of e are the maximum of the totally ordered set $\{e' \in T \mid e' \neq e_k \text{ and } e_i \preceq e' \preceq e_k\}$ and the maximum of the totally ordered set $\{e' \in T \mid e' \neq e_k \text{ and } e_{j-1} \preceq e' \preceq e_k\}$. ◀

Recall that we plan to flip some edges e_i of T to the boundary edge $p_i p_{i+1}$ corresponding to the gap g_i of e_i. In general, for a subset I of gaps, let us denote by T_I the graph obtained from T by replacing, for each gap $g_i \in I$, the edge e_i of T by the corresponding boundary edge $p_i p_{i+1}$. If e_i is already short, then $e_i = p_i p_{i+1}$; i.e., this replacement does not change anything. Otherwise, $e_i \to p_i p_{i+1}$ always constitutes a valid flip by Lemma 13. In particular, Lemmas 12 and 13 assert that T_I is a non-crossing tree and that there is a valid flip sequence $T \to \cdots \to T_I$. The length of this flip sequence is the number of gaps in I *not* corresponding to short edges of T.

Recall that $I_=, I_R, I_N$ are the subsets of gaps corresponding to $\mathcal{P}_=, \mathcal{P}_R, \mathcal{P}_N$, respectively.

▶ **Proposition 16.** *Let T, T' be two non-crossing trees on linearly ordered vertices $p_1, \ldots, p_n$, and $X \subseteq I_N$ be any (possibly empty) subset of the gaps corresponding to the near-near pairs. Then there are flip sequences $T \to \cdots \to T_{I_R \cup X}$ and $T'_{I_R \cup X} \to \cdots \to T'$ with in total at most $\frac{3}{2}|I_R| + |I_=| + 2|X| - 1$ flips.*

Proof. Consider any $g_i \in I_R \cup X$ and the corresponding pair (e_i, e'_i). If $g_i \in I_R$, at most one of e_i, e'_i is short. If $g_i \in X \subseteq I_N$, none of e_i, e'_i is short. Let $S = \{g_i \in I_R \mid e_i \text{ is short}\}$ and $S' = \{g_i \in I_R \mid e'_i \text{ is short}\}$. Then $S \cap S' = \varnothing$, since short-short pairs are in $I_=$ only. For the first flip sequence $T \to \cdots \to T_{I_R \cup X}$ we do (in any order) for every $g_i \in (I_R \cup X) - S$ a flip that replaces e_i by $p_i p_{i+1}$. This is a valid flip sequence by Lemma 13, and clearly transforms T into $T_{I_R \cup X}$. It uses $|(I_R \cup X) - S| = |I_R - S| + |X| = |I_R| - |S| + |X|$ flips. Similarly, there is a valid flip sequence $T' \to \cdots \to T'_{I_R \cup X}$ that uses $|I_R| - |S'| + |X|$ flips. Its reverse is the desired flip sequence $T'_{I_R \cup X} \to \cdots \to T'$.

In total, both flip sequences have $2|I_R| - (|S| + |S'|) + 2|X|$ flips. It remains to prove that $|S| + |S'| \geq \frac{1}{2}|I_R| - |I_=| + 1$. To this end, note that every gap in $I_R - (S \cup S')$ involves at least one wide edge. Recall that T_W and T_S denote the set of all wide and all short edges in T, respectively. By Lemma 15, we have $|T_W| \leq |T_S| - 1$ and $|T'_W| \leq |T'_S| - 1$. Moreover, $|T_S| + |T'_S| \leq |S| + |S'| + 2|I_=|$. Together we conclude

$$|I_R| \leq |S| + |S'| + |T_W| + |T'_W| \leq |S| + |S'| + |T_S| + |T'_S| - 2 \leq 2(|S| + |S'| + |I_=|) - 2,$$

which gives the desired $|S| + |S'| \geq \frac{1}{2}|I_R| - |I_=| + 1$. ◀

Proposition 16 works for any subset $X \subseteq I_N$ of the near-near gaps. We content ourselves with spending a 2-flip on each gap in X (reflected by the $2|X|$ term in the bound of Proposition 16), but aim to do a direct 1-flip on each gap in $Y = I_N - X$. For larger $|Y|$ we obtain an overall shorter flip sequence. So we want a large set of near-near pairs that can all be done as 1-flips. These flips shall form a valid flip sequence $T_{I_R \cup X} \to \cdots \to T'_{I_R \cup X}$, connecting the two sequences obtained by Proposition 16. As all the edges for gaps in $I_R \cup X$ are flipped to boundary edges in $T_{I_R \cup X}$ and $T'_{I_R \cup X}$, we can "safely ignore" all pairs corresponding to gaps in $I_= \cup I_R \cup X$ and focus on the near-near pairs corresponding to Y.

▶ **Proposition 17.** *Let T, T' be two non-crossing trees on linearly ordered vertices $p_1, \dots, p_n$, and $Y \subseteq I_N$ be an acyclic subset in $H = H(T, T')$, and $X = I_N - Y$. Then there is a flip sequence $T_{I_R \cup X} \to \dots \to T'_{I_R \cup X}$ with $|Y|$ flips.*

Proof. Let $T_1 := T_{I_R \cup X}$ and $T_2 := T'_{I_R \cup X}$. Lemma 12 guarantees that near-near pairs of T, T' corresponding to gaps in Y are still near-near pairs of T_1, T_2. Moreover, we have $\mathcal{P}_R = \varnothing$ for T_1, T_2. In particular, the conflict graph of T_1, T_2 is $H[Y]$. Because $H[Y]$ is acyclic, there exists a topological ordering $\prec$ of $H[Y]$. The first gap g in $\prec$ has no incoming edges in $H[Y]$, and Lemmas 12 and 14 ensure that the direct flip of the corresponding pair (e, e') of g is valid and maintains all gap-assignments and types of edges. We repeat with direct flips for all pairs corresponding to Y in the order given by $\prec$, until we reach T_2. As we spent one flip per pair, the resulting flip sequence has length $|Y|$. ◀

4.1.1 Putting things together – Proof of Theorem 6(i)

Now, we show how to obtain a short flip sequence from a large acyclic subset.

▶ **Theorem** (corresponding to Theorem 6(i)).
Let T, T' be two non-crossing trees on linearly ordered vertices $p_1, \dots, p_n$ with conflict graph $H = (V(H), E(H))$. Then the flip distance $\operatorname{dist}(T, T')$ is at most $\max\left\{\frac{3}{2}, 2 - \frac{\operatorname{ac}(H)}{|V(H)|}\right\}(n-1)$ if H is non-empty, and at most $\frac{3}{2}(n-1)$ if H is empty.

Proof. First assume that H is non-empty. Let $Y \subseteq I_N = V(H)$ be an acyclic subset of H with $|Y| = \operatorname{ac}(H)$. Let $\prec$ be a topological ordering of $H[Y]$. Denoting $X = I_N - Y$, our flip sequence F from T to T' is composed of three parts:

F_1: $T \to \dots \to T_{I_R \cup X}$ replaces (in any order) each non-short edge $e_i \in T$ with $g_i \in I_R \cup X$ by the boundary edge $p_i p_{i+1}$.

F_2: $T_{I_R \cup X} \to \dots \to T'_{I_R \cup X}$ replaces in order according to $\prec$ each edge $e_i \in T$ with $g_i \in Y$ by the edge $e'_i \in T'$.

F_3: $T'_{I_R \cup X} \to \dots \to T'$ replaces (in any order) each boundary edge $p_i p_{i+1}$ with $g_i \in I_R \cup X$ and $p_i p_{i+1} \notin T'$ by the non-short edge $e'_i \in T'$.

By Proposition 16, the sequences F_1 and F_3 are valid and have a total length of $|F_1| + |F_3| = \frac{3}{2}|I_R| + |I_=| + 2|X| - 1$. Proposition 17 ensures that F_2 is valid and has length $|Y| = |I_N| - |X|$. With $|Y| = \operatorname{ac}(H)$ and $I_N = V(H)$ we conclude that

$$
\begin{aligned}
\operatorname{dist}(T, T') &\le |F_1| + |F_2| + |F_3| \le \frac{3}{2}|I_R| + |I_=| + |I_N| + |X| = \frac{3}{2}|I_R| + |I_=| + 2|I_N| - |Y| \\
&\le \frac{3}{2}(|I_R| + |I_=|) + \left(2 - \frac{|Y|}{|I_N|}\right)|I_N| = \frac{3}{2}(|I_R| + |I_=|) + \left(2 - \frac{\operatorname{ac}(H)}{|V(H)|}\right)|I_N| \\
&\le \max\left\{\frac{3}{2}, 2 - \frac{\operatorname{ac}(H)}{|V(H)|}\right\}(|I_R| + |I_=| + |I_N|) \le \max\left\{\frac{3}{2}, 2 - \frac{\operatorname{ac}(H)}{|V(H)|}\right\}(n-1).
\end{aligned}
$$

If H is empty, then $\mathcal{P}_N = \varnothing$, and hence, $|I_N| = |X| = 0$. Then the above with only F_1 and F_3 gives $\operatorname{dist}(T, T') \le \frac{3}{2}|I_R| + |I_=| \le \frac{3}{2}(n-1)$. ◀

4.2 Lower bound

In this section, we show that a single example of a pair T, T' of trees gives rise to a lower bound for $\operatorname{diam}(\mathcal{F}_n)$ for all n through properties of the conflict graph H of T, T'. To be precise, we show the following statement, which corresponds to Theorem 6(ii).

▶ **Theorem 18.** *Let T, T' be non-crossing trees on linearly ordered vertices $p_1, \dots, p_n$ with non-empty conflict graph $H = H(T, T')$. Then there is a constant C depending only on T and T', such that for all $N \geq n$ we have* $\operatorname{diam}(\mathcal{F}_N) \geq (2 - \frac{\operatorname{ac}(H)}{|V(H)|})N - C$.

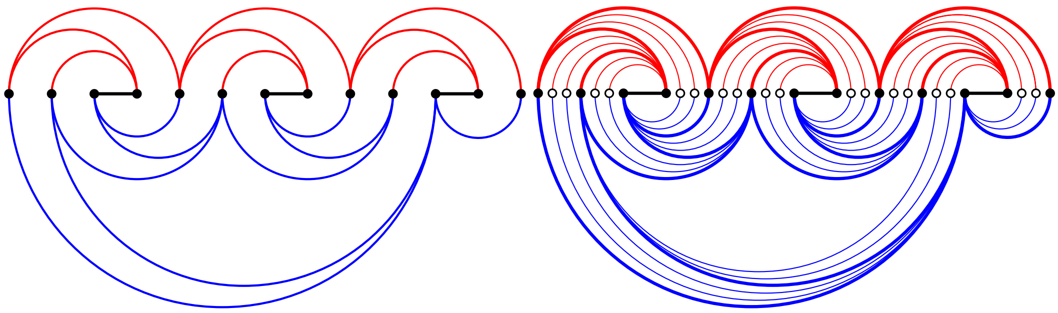
To this end, let us consider a pair of trees T and T' on n vertices with conflict graph H. We will construct a sequence of tree pairs $(T_k, T'_k)_{k \in \mathbb{N}}$ on n_k vertices each such that

$$\operatorname{dist}(T_k, T'_k) \geq \left(2 - \frac{\operatorname{ac}(H)}{|V(H)|}\right) n_k - \mathcal{O}(1).$$

We now explain how to construct these trees. We consider the edge pairs $\mathcal{P}$ of T and T' as before. Recall that $\mathcal{P}_N$ is the set of near-near pairs (e, e') of T and T' with $e \neq e'$, and that this set is assumed to be nonempty. Let N and N' be the sets of edges of T and T', respectively, appearing in pairs in $\mathcal{P}_N$. For $k \geq 1$, we define the *k-blowups* T_k and T'_k of T and T' by doing the following for each $(e, e') \in \mathcal{P}_N$ (for an illustration consider Figure 8).

- Insert a set $V(e)$ of k new vertices in the gap g associated to (e, e').
- In T_k, add an edge from each $v \in V(e)$ to the endpoint of e that is not adjacent to g, and similarly for T' and e'.
- Let $\Lambda(e)$ denote the set of the k edges added to T_k, and let $\Lambda(e')$ denote the set of the k edges added to T'_k.

This way, for each edge e appearing in a pair in $\mathcal{P}_N$ we add next to e a fan $\Lambda(e)$ of k edges ending at leaves. By construction, the blowups have $n_k := n + k|\mathcal{P}_n|$ vertices and $n_k - 1$ edges.



(a) **(b)**

Figure 8 A pair of trees (T, T') and their 2-blowup (T_2, T'_2).

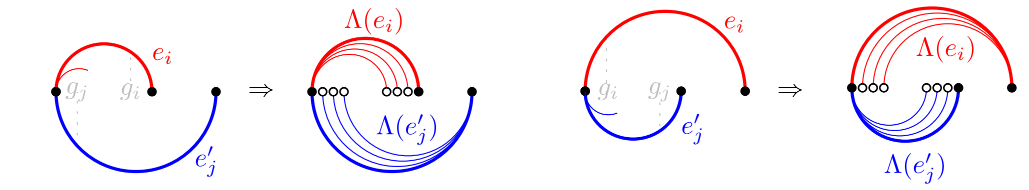
Here is the crucial connection between k-blowups and the conflict graph H of T, T'.

▶ **Observation 19.** *If $\overrightarrow{g_i g_j}$ is a directed edge in the conflict graph H of T, T', then in the k-blowups T_k, T'_k every edge in $\Lambda(e_i)$ crosses every edge in $\Lambda(e'_j)$.*

Proof. Indeed, this is clear if $\overrightarrow{g_i g_j}$ is of type T1, as then already e_i and e'_j cross. For $\overrightarrow{g_i g_j}$ of type T2 look at Figure 9(a), for type T3 look at Figure 9(b). ◀

Now, we consider any flip sequence F from T_k to T'_k and denote the intermediate trees by $T_k = T[0], T[1], \dots, T[\ell] = T'_k$. For each $(e, e') \in \mathcal{P}_N$, let $\operatorname{gone}(e)$ be the smallest index a such that $T[a]$ contains no edge in $\Lambda(e)$. Since T'_k contains no edge in $\Lambda(e)$, $\operatorname{gone}(e)$ is well-defined. Evidently, in F there is a flip $T[\operatorname{gone}(e) - 1] \to T[\operatorname{gone}(e)]$ that replaces the last remaining edge in $\Lambda(e)$ by an edge not in $\Lambda(e)$. We say that the pair (e, e') is *direct* if there is an $a \leq \operatorname{gone}(e)$ such that $T[a]$ contains an edge in $\Lambda(e')$, and *indirect* otherwise.

▶ **Lemma 20.** *Let $(e_i, e'_i) \neq (e_j, e'_j)$ be direct pairs in $\mathcal{P}_N$. If the conflict graph H of T, T' contains the directed edge $\overrightarrow{g_i g_j}$, then* $\operatorname{gone}(e_i) < \operatorname{gone}(e_j)$.



(a) $\overrightarrow{g_i g_j}$ of type T2 **(b)** $\overrightarrow{g_i g_j}$ of type T3

Figure 9 A directed edge $\overrightarrow{g_i g_j}$ of type T2 (a) or type T3 (b), makes $\Lambda(e_i)$ crossing $\Lambda(e'_j)$.

Proof. We show that if $\text{gone}(e_i) \geq \text{gone}(e_j)$, then there is no edge from g_i to g_j in H.

Since $\text{gone}(e_i) \geq \text{gone}(e_j)$, $T[\text{gone}(e_j) - 1]$ contains an edge of $\Lambda(e_i)$. The edge that is flipped away from $T[\text{gone}(e_j) - 1]$ in the flip $T[\text{gone}(e_j) - 1] \to T[\text{gone}(e_j)]$ is in $\Lambda(e_j)$, so since $\Lambda(e_j) \cap \Lambda(e_i) = \varnothing$ by construction, $T[\text{gone}(e_j)]$ also contains an edge in $\Lambda(e_i)$. Thus, $\text{gone}(e_i) > \text{gone}(e_j)$.

Now choose any $a \leq \text{gone}(e_j)$ such that $T[a]$ contains an edge in $f' \in \Lambda(e'_j)$. Since $a \leq \text{gone}(e_j) < \text{gone}(e_i)$, $T[a]$ contains at least one edge of $f \in \Lambda(e_i)$. But $T[a]$ is non-crossing, so we have found edges $f \in \Lambda(e_i)$ and $f' \in \Lambda(e'_j)$ that do not cross. By Observation 19 it follows that $\overrightarrow{g_i g_j}$ is not an edge in H. ◀

Let δ and $\bar{\delta}$ denote the number of direct and indirect pairs in $\mathcal{P}_N$ induced by the flip sequence F, respectively. Clearly, $\delta + \bar{\delta} = |\mathcal{P}_N|$. Lemma 20 implies the following crucial property.

▶ **Corollary 21.** *The conflict graph H has an acyclic subset of size δ, i.e., $\text{ac}(H) \geq \delta$.*

Proof. Let $(e_{i_1}, e'_{i_1}), \ldots, (e_{i_\delta}, e'_{i_\delta})$ be the direct pairs, sorted so that $\text{gone}(e_{i_1}) \leq \cdots \leq \text{gone}(e_{i_\delta})$. By Lemma 20, every edge of H between the gaps of two direct pairs points forward in that ordering. Hence, the subgraph of H corresponding to (the gaps of) direct pairs is acyclic. ◀

Now, we aim to show a lower bound on the flip sequence in terms of the largest acyclic subset in H. Intuitively speaking, we show that for each indirect pair (e, e'), the process of removing the k edges in $\Lambda(e)$ and adding the k edges in $\Lambda(e')$ in the flip sequence F must involve introducing almost k "intermediate" edges that are neither in T_k nor in T'_k, which then increases the length of F. Lemma 22 below shows that a single indirect pair gives rise to many intermediate edges, i.e., costs additional flips. Lemma 23 further below shows that costs for different indirect pairs add up. That is, we cannot "reuse" intermediate edges to reduce the cost in any effective way.

▶ **Lemma 22.** *Let $(e, e') \in \mathcal{P}_N$ be an indirect pair. Then there is a subgraph S of the tree $T[\text{gone}(e)]$ that contains $V(e)$, does not contain any edges in T_k or T'_k, and has at most $2n - 1$ connected components.*

Proof. Let $a = \text{gone}(e)$. Because (e, e') is indirect, $T[a] \cap \Lambda(e)$ and $T[a] \cap \Lambda(e')$ are empty. We first find a subset S' of $T[a]$ by doing the following for every edge $f \in T[a] \cap (T_k \cup T'_k)$:

- Since $f \notin \Lambda(e) \cup \Lambda(e')$, all the vertices of $V(e)$ lie on the same side of f.
- Delete from $T[a]$ all the vertices (and their incident edges) that are on the other side (without $V(e)$) of f, keeping the endpoints of f.

Note that these deletions do not disconnect $T[a]$, so the remaining subset $S' \subseteq T[a]$ is still connected. Further note that for every $f \in T \cup T'$ and its fan $\Lambda(f)$ in T_k or T'_k, no two edges of $f \cup \Lambda(f)$ lie in S'. Indeed, otherwise $V(e)$ lies on the same side of both these edges and one would be deleted when considering the other. Consequently, S' has at most $n-1$ edges of T_k and at most $n-1$ edges of T'_k, i.e., $|S' \cap (T_k \cup T'_k)| \le 2n-2$. Then $S = S' - (T_k \cup T'_k)$ is the desired subgraph of $T[a]$. ◀

▶ **Lemma 23.** *The flip sequence F from T_k and T'_k has length at least $(k-2n)(\bar{\delta} + |\mathcal{P}_N|)$.*

Proof. For each indirect pair $(e,e') \in \mathcal{P}_N$, let $S(e,e')$ be the corresponding subgraph of $T[\text{gone}(e)]$ guaranteed by Lemma 22. Let U be the union of all $S(e,e')$ over all indirect pairs $(e,e') \in \mathcal{P}_N$. By Lemma 22, U has at most $(2n-1)\bar{\delta}$ connected components and at least $k\bar{\delta}$ vertices, because it contains the vertices $V(e)$ for all indirect pairs (e,e'), and $|V(e)| = k$. Thus, U has at least $k\bar{\delta} - (2n-1)\bar{\delta} \ge (k-2n)\bar{\delta}$ edges, none of which is in T_k or T'_k.

Consequently, the flip sequence F has at least $|U| \ge (k-2n)\bar{\delta}$ flips that introduce an edge of U, as well as $|T'_k \smallsetminus T_k|$ additional flips that introduce an edge of $T'_k \smallsetminus T_k$. For each $(e,e') \in \mathcal{P}_N$, there are k edges in $\Lambda(e')$ that do not appear in T_k. Thus, $|T'_k \smallsetminus T_k| \ge k|\mathcal{P}_N|$. Adding all together, F has length at least $(k-2n)\bar{\delta} + k|\mathcal{P}_N| \ge (k-2n)(\bar{\delta} + |\mathcal{P}_N|)$. ◀

▶ **Lemma 24.** *For $k \to \infty$, we have*

$$\text{dist}(T_k, T'_k) \ge \left(2 - \frac{\text{ac}(H)}{|V(H)|}\right) n_k - \mathcal{O}(1).$$

Proof. By Lemma 23, any flip sequence from T_k to T'_k with $\bar{\delta}$ indirect flips has length at least $(k-2n)(\bar{\delta} + |\mathcal{P}_N|)$. By construction, T_k (and also T'_k) has $n_k = n + k|\mathcal{P}_N|$ vertices. We get

$$\text{dist}(T_k, T'_k) \ge (k-2n)(\bar{\delta} + |\mathcal{P}_N|) \ge k(\bar{\delta} + |\mathcal{P}_N|) - \mathcal{O}(1) \ge n_k \frac{\bar{\delta} + |\mathcal{P}_N|}{|\mathcal{P}_N|} - \mathcal{O}(1). \tag{2}$$

The vertices of H are in bijection with the pairs in $\mathcal{P}_N$, and by Corollary 21, the number δ of direct pairs in $\mathcal{P}_N$ is at most $\text{ac}(H)$. Thus,

$$\bar{\delta} = |\mathcal{P}_N| - \delta \ge |\mathcal{P}_N| - \text{ac}(H) = |V(H)| - \text{ac}(H).$$

Plugging this with $|V(H)| = |\mathcal{P}_N|$ into Equation (2), we get

$$\text{dist}(T_k, T'_k) \ge n_k \frac{2|V(H)| - \text{ac}(H)}{|V(H)|} - \mathcal{O}(1) = n_k \left(2 - \frac{\text{ac}(H)}{|V(H)|}\right) - \mathcal{O}(1). \quad ◀$$

With Lemma 24 at hand, we are finally ready to prove Theorem 18, which corresponds to Theorem 6(ii), i.e., our tool to prove a lower bound on $\text{diam}(\mathcal{F}_n)$.

▶ **Theorem 18.** *Let T, T' be non-crossing trees on linearly ordered vertices $p_1, \ldots, p_n$ with non-empty conflict graph $H = H(T,T')$. Then there is a constant C depending only on T and T', such that for all $N \ge n$ we have $\text{diam}(\mathcal{F}_N) \ge (2 - \frac{\text{ac}(H)}{|V(H)|})N - C$.*

Proof. By Lemma 24, we have a family of pairs of trees $(T_k, T'_k)_{k \ge 1}$ showing the desired lower bound on $\text{diam}(\mathcal{F}_N)$ for each N of the form $N = n_k := n + k|V(H)|$ for some $k \ge 1$. Since $n_{k+1} - n_k$ is constant, it suffices to show that $\text{diam}(\mathcal{F}_N) \ge \text{diam}(\mathcal{F}_{n_k})$ for $n_k \le N < n_{k+1}$. We will show the slightly stronger statement that $\text{diam}(\mathcal{F}_N) \ge \text{diam}(\mathcal{F}_{N'})$ for any $N \ge N'$.

To this end, let (T, T') be any pair of trees on N' vertices $p_1, \ldots, p_{N'}$ with $\text{dist}(T,T') = \text{diam}(\mathcal{F}_{N'})$. We construct T_N and T'_N by adding vertices $p_{N'+1}, \ldots, p_N$ and edges $\{p_i, p_{i+1}\}$ to T and T' for $i = N', \ldots, N-1$. By [2, Corollary 18], there is a shortest flip sequence F from T_N to T'_N that does not flip any of the added edges $\{p_i, p_{i+1}\}$ (since they are in both T_N and T'_N). By collapsing all the vertices $p_{N'}, \ldots, p_N$ to one and removing the edges between them, we get a flip sequence from T to T' that is not longer than F. Thus, $\text{dist}(T_N, T'_N) \ge \text{dist}(T,T') = \text{diam}(\mathcal{F}_{N'})$, so $\text{diam}(\mathcal{F}_N) \ge \text{diam}(\mathcal{F}_{N'})$. ◀

5 Keeping common edges and improving the upper bound

In this section, we aim to further improve the upper bound of $5/3(n-1)$ from Corollary 10 to also depend on the number of edges that the two trees share. Moreover, we want to obtain a flip sequence that avoids flipping any edge that is already in both trees.

We distinguish two different types of edges in a tree on a convex point set P, namely, *boundary edges*, which are edges connecting two consecutive points along the convex hull of P, and *chords*, which are edges connecting two non-consecutive points along the convex hull of P. For a pair T, T' of trees on P let $b = b(T, T')$ denote the number of *common edges* (edges in $T \cap T'$) that are also boundary edges. Clearly $d + b \leq n - 1$. Theorem 2 is implied by the following stronger statement.

▶ **Theorem 25.** *Let T, T' be two non-crossing spanning trees on a set of $n \geq 2$ points in convex position. Let $d = |T - T'|$ and let b be the number of common boundary edges of T and T'. Then $\mathrm{dist}(T, T') \leq 5/3 \cdot d + 2/3 \cdot b - 4/3$. Moreover, there exists a flip sequence from T to T' of at most that length in which no common edges are flipped.*

The high level proof idea for Theorem 25 is the following: We will "cut" the instance along common chords, by this obtaining sub-instances where all common edges are boundary edges and which we handle independently. For each sub-instance, we will identify a "good" linear order by the following observation.

▶ **Observation 26.** *Let T and be a non-crossing spanning tree on a set P of n points in convex position. Then for any edge p_1p_n of the convex hull of P that is not a boundary edge of T, the tree T with linear order $p_1, \ldots, p_n$ has at least two uncovered edges.*

The order obtained by Observation 26 will avoid flipping common edges and will facilitate obtaining the upper bound of Theorem 25 for each sub-instance as well as in total.

To use Observation 26, we need to identify a gap that is not a boundary edge in any of the trees. Note that, in particular after cutting along common edges, it is easy to see that one can perform a flip which introduces a boundary edge from $T' - T$ (or $T - T'$) and removes a non-boundary edge, unless both trees consist of boundary edges only, see also Bousquet et al. [8, Claim 2]. Hence we have the following observation.

▶ **Observation 27.** *Consider two non-crossing spanning trees T, T' on a convex point set. If $T \cup T'$ contain all boundary edges, then there exists a flip sequence of length $|T \setminus T'|$.*

We are now ready to prove Theorem 25.

Proof of Theorem 25. Let $B = B(T, T')$ be the set of common boundary edges of T and T', let $C = C(T, T')$ be the set of common chords of T and T', and let $D(T) = T - T'$ (resp. $D(T') = T' - T$) be the edges that are only in T (resp. T'). Then $c := |C| = n - 1 - b - d$.

Assume first that $C(T, T') = \varnothing$, that is, that T and T' do not have any common chords.

By Observation 27, we may choose an arbitrary edge p_1p_n of the convex hull of P that is neither a boundary edge of T nor of T' to cut the cyclic order of the points of P into a linear order $p_1, \ldots, p_n$. With this linear order, each of T and T' has at least two uncovered edges by Observation 26.

Consider the set $\mathcal{P}$ of pairs of edges of T and T' that are induced by the gaps, its partition into $\mathcal{P}_=, \mathcal{P}_N, \mathcal{P}_R$, and the according sets $I_=, I_N, I_R$ of gaps. Because the set $I_=$ consists of all gaps that correspond to short-short pairs, we have $|I_=| = |B| = b$.

The set I_R contains the gaps corresponding to all other pairs that contain at least one short or wide edge. Hence $|I_R| \leq |T_W| + |T'_W| + |T_S| + |T'_S| - 2b$. By Lemma 15, we have $|T_W| + |T'_W| \leq |T_S| - 2 + |T'_S| - 2 = |T_S| + |T'_S| - 4$ and hence $|I_R| \leq 2|T_S| + 2|T'_S| - 2b - 4$. We remark that exactly $|T_S| + |T'_S| - 2b$ of the gaps in $|I_R|$ correspond to pairs with one short edge and hence require only one flip.

Recall that the set I_N consists exclusively of gaps with near-near pairs and consider the conflict graph H with vertex set $V(H) = I_N$. By Lemma 9, a maximum acyclic subset Y of H has size at least $1/3|V(H)|$. Let $\prec$ be a topological ordering of $H[Y]$ and let $X = I_N \setminus Y$.

We use one flip for each of the $|T_S| + |T'_S| - 2b$ gaps in $|I_R|$ corresponding to pairs with one short edge, two flips for all other each edge pairs corresponding to a gaps $X \cup I_R$ and one flip for each pair corresponding to a gap in Y. To ease the counting, we split the total flip sequence $T \to \cdots \to T'$ into five parts:

- $T \to \cdots \to T_{I_R}$ replaces (in any order) each edge $e_i \in T$ with $g_i \in I_R$ by $p_i p_{i+1}$.
- $T_{I_R} \to \cdots \to T_{I_R \cup X}$ replaces (in any order) each edge $e_i \in T$ with $g_i \in X$ by $p_i p_{i+1}$.
- $T_{I_R \cup X} \to \cdots \to T'_{I_R \cup X}$ replaces (in order according to $\prec$) each edge $e_i \in T$ with $g_i \in Y$ by the edge $e'_i \in T'$.
- $T'_{I_R \cup X} \to \cdots \to T'_{I_R}$ replaces (in any order) each edge $p_i p_{i+1}$ with $g_i \in X$ by the edge $e'_i \in T'$.
- $T'_{I_R} \to \cdots \to T'$ replaces (in any order) each edge $p_i p_{i+1}$ with $g_i \in I_R$ by the edge $e'_i \in T'$.

Note that the flip sequence is valid by Lemmas 12 and 13 and Proposition 17.

It remains to compute the total length of the flip sequence. Let $d_1 = |T_S| + |T'_S| - 2b$ be the number of 1-flips for I_R, let $d_2 = |I_R| - d_1$ be the number of 2-flips for I_R and let $d_3 = |I_N|$. Note that $d = d_1 + d_2 + d_3$.

The first and last step of the sequence require a total of $d_1 + 2d_2$ flips. The middle three steps together require $2d_3 - |Y| \leq 5/3 \cdot d_3$ flips.

Since $|I_R| \leq |T_S| + |T'_S| - 4 + |T_S| + |T'_S| - 2b = 2d_1 + 2b - 4$, we have $d_2 \leq d_1 + 2b - 4$.

Altogether we obtain

$$\begin{aligned}
\operatorname{dist}(T, T') &\leq d_1 + 2d_2 + \frac{5}{3}d_3 = d_1 + \frac{5}{3}d_2 + \frac{1}{3}d_2 + \frac{5}{3}d_3 \\
&\leq d_1 + \frac{5}{3}d_2 + \frac{1}{3}(d_1 + 2b) - \frac{4}{3} + \frac{5}{3}d_3 \\
&= \frac{4}{3}d_1 + \frac{5}{3}(d_2 + d_3) + \frac{2}{3}b - \frac{4}{3} \leq \frac{5}{3}d + \frac{2}{3}b - \frac{4}{3}.
\end{aligned}$$

We now turn to the case $C(T, T') \neq \emptyset$. Consider the $c + 1$ bounded cells $F_0, \ldots F_c$ of the convex hull of P that are induced by the set $C(T, T')$. For each closed cell F_i with n_i points of P, $T_i = T \cap F_i$ and $T'_i = T' \cap F_i$ are non-crossing spanning trees on the n_i points of F_i, with $C(T_i, T'_i) = \emptyset$ and $b_i = |B(T_i, T'_i)|$ common boundary edges.

Consider again the edges of T and T'. Every edge of $B(T, T')$ contributes to exactly one of the b_i's and every edge of $C(T, T')$ to exactly two of them. Hence $\sum_{i=0}^{c} b_i = b + 2c$. On the other hand, every edge of $D(T)$ (resp. $D(T')$) lies in exactly one cell F_i. Thus, with $d_i = |D(T_i)| = |D(T'_i)|$, we have that $\sum_{i=0}^{c} d_i = d$. Applying the above flip process to each of the tree pairs (T_i, T'_i) independently, we obtain the first part of the theorem.

$$\begin{aligned}
\operatorname{dist}(T, T') \leq \sum_{i=0}^{c} \operatorname{dist}(T_i, T'_i) \leq \sum_{i=0}^{c} \left(\frac{5}{3}d_i + \frac{2}{3}b_i - \frac{4}{3} \right) &= \frac{5}{3}d + \frac{2}{3}(b + 2c) - \frac{4}{3}(c + 1) \\
&= \frac{5}{3}d + \frac{2}{3}b - \frac{4}{3}.
\end{aligned}$$

Since $d + b \leq n - 1$, the second part then follows directly. ◀

6 Separated Caterpillars – Proof of Theorem 4

In this section, we improve the upper bound for the case where one tree has a special structure, namely, if it is a separated caterpillar.

We call a tree T on a convex point set a *separated caterpillar* if the weak dual graph of T and all convex hull edges forms a path. For an example, consider Figure 10.

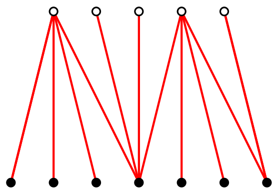


Figure 10 A separated caterpillar.

In fact, there are a few equivalent definitions.

▶ **Observation 28.** *Let T be a tree on a convex point set (with $n \geq 3$ points) with a valid 2-coloring of the vertices. Then the following statements are equivalent.*

- *T is a separated caterpillar.*
- *The weak dual graph of T and the convex hull edges is a path.*
- *Each color class forms a consecutive interval (along the boundary of the convex hull).*
- *The color classes can be separated by a line.*
- *T contains exactly two convex hull edges.*
- *There is linear vertex labeling such that poset defined by the edge cover relation of the edges is a total order.*
- *For every linear vertex labeling, the poset defined by the edge cover relation consists of (at most) two chains.*
- *For every linear vertex labeling, T has no wide edge.*

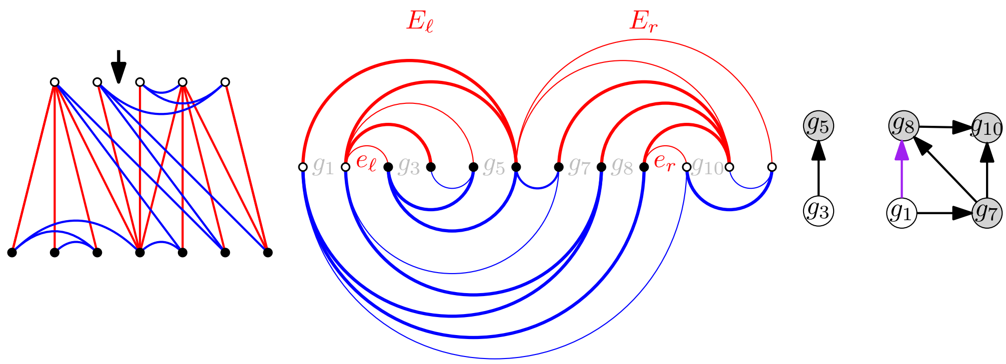
We note that Bousquet et al. [8] have considered separated caterpillars (under the name of *nice caterpillars*). They show that if one of two trees is a nice caterpillar, then their flip distance is at most $3/2 \cdot n$. We show that this even holds in terms of d. Note that the lower bound examples, illustrated in Figure 2(c), are in fact separated caterpillars. Thus, the bound is tight up to additive constants.

▶ **Theorem 4.** *Let T, T' be non-crossing trees on $n \geq 3$ points in convex position. Let T be a separated caterpillar and $d := |T - T'|$. Then $\text{dist}(T, T') \leq 3/2 \cdot d$. Moreover, there exists a flip sequence from T to T' of length at most $3/2 \cdot d$ in which no common edges are flipped.*

In the following, we assume without loss of generality that T and T' have no common chords. To this end, note that we can split an instance at a common chord into two subinstances where the common chord will turn into a common boundary edge in each part. By repeated application, we obtain a collection of subinstances T_i, T'_i without common chords. Defining $d_i := |T_i - T'_i|$, we clearly have $\sum_i d_i = d$. Hence, when guaranteeing at most $3/2 \cdot d_i$ in each subinstance, the claim follows.

By Observation 27, we may label the vertices by $p_1, \dots, p_n$ such that neither T nor T' has the edge $p_1 p_n$ and consider the linear representation of T and T'; otherwise, d flips suffice and we are done. It follows that both trees have at least two maximal edges, so by Lemma 15, both T and T' have at least two more wide than short edges.

Now we use the fact that T is a separated caterpillar, and thus has a special structure. In particular, by Observation 28, its edges form two chains by the cover relation. We define E_ℓ as the set of all edges covering the leftmost short edge e_ℓ and E_r as the set of all edges covering the rightmost short edge e_r; we have $e_\ell \in E_\ell$ and $e_r \in E_r$. Clearly, we have $T = E_\ell \cup E_r$. For an illustration, consider Figure 11. Moreover, note that T, besides the two short edges e_ℓ and e_r, has only near edges.



(a) **(b)** **(c)**

Figure 11 Illustration for the proof of Theorem 4.

We pair the edges of T and T' via the shortest edge covering a gap as explained in Section 3 and partition the pairs into the sets $\mathcal{P}_=, \mathcal{P}_N, \mathcal{P}_R$. For the gaps I_N corresponding to near-near pairs, we define $A, B \subseteq I_N$ as follows: For a gap g with associated pair $(e, e') \in \mathcal{P}_N$ where $e \in E_\ell$, we let $g \in A$ if e covers e', and $g \in B$ otherwise. If $e \in E_r$, then $g \in B$ if e covers e', and $g \in A$ otherwise. Clearly, $A \cup B = I_N$.

▶ **Lemma 29.** *$H[A]$ and $H[B]$ are acyclic.*

Proof. By left-right symmetry, it suffices to show that $H[A]$ is acyclic. We say that a gap $g_i \in A$ with pair (e_i, e_i') comes *before* a gap g_j with pair $(e_j, e_j') \in A$ if

(i) $e_i \in E_\ell$ and $e_j \in E_r$, or

(ii) $e_i, e_j \in E_\ell$ and e_j covers e_i, or

(iii) $e_i, e_j \in E_r$ and e_i covers e_j.

This gives a total order on A.

For $g_i, g_j \in A$, we show that if g_i comes before g_j, then there is no edge in the conflict graph H from g_j to g_i. We consider the cases (i)-(iii) separately. In case (i), since $g_i \in A$, e_i' and e_j do not intersect, nor does one cover the other. Thus, there is no edge $g_j \to g_i$ in H. In case (ii), e_j covers e_i, which covers e_i'. This immediately excludes cases 1 and 2 in Definition 8. Moreover, since e_j covers e_i, e_i does not cover g_j, so neither can e_i', which excludes case 3. In case (iii), we have that e_i does not cover e_i'. Both e_i and e_i' cover g_i, which means that either (a) e_i' covers e_i, or (b) e_i and e_i' cross. If (a), e_i' covers e_i, which covers e_j, so cases 1 and 3 of Definition 8 are excluded. Since e_i covers e_j, e_j does not cover g_i, so also case 2 is excluded. If (b), then since e_i and e_i' are near, the only gap covered by both is g_i. This gap is not covered by e_j, so there is no gap covered by both e_i' and e_j. It follows that there is no edge $g_j \to g_i$ in H in either of the cases (i)-(iii) and thus $H[A]$ is acyclic. ◀

We now have all tools to present the flip sequence.

▶ **Lemma 30.** *There exists a flip sequence F from T to T' of length at most $3/2 \cdot d$.*

Proof. We start by describing our flip sequence F which consists of four parts. Choose Y as a largest acyclic subset of H among A and B and let $X = I_N - Y$. Recall that $|T' \cap T| = |\{e_\ell, e_r\}| = 2$. Hence, we have $d = n - 3$.

F_1: For each $(e_i, e'_i) \in \mathcal{P}_R$ with gap g_i where e' is short or wide, flip e to $p_i p_{i+1}$. Clearly, $|F_1| = |T'_S| + |T'_W| - 2$; recall that T' contains the two short edges e_ℓ, e_r which belong to pairs in $\mathcal{P}_=$.

F_2: For each $g_i \in X$, let $(e_i, e'_i) \in \mathcal{P}_N$ denote the corresponding pair. We flip e to $p_i p_{i+1}$. Clearly, $|F_2| = 2|X|$.

F_3: For each $g_i \in Y$, let $(e_i, e'_i) \in \mathcal{P}_N$ denote the corresponding pair. We flip e_i to e'_i. Clearly, $|F_3| = |Y|$.

F_4: For each $e' \in T'_W$ with corresponding gap g_k, perform flip $p_k p_{k+1} \to e'$. Clearly, $|F_4| = |T'_W|$.

The validity of the flip sequences in F_1, F_2, and F_4 follow from Lemma 13 as we introduce a boundary edge or remove a boundary edge covering the same gap as its partner edge. Let us denote the tree resulting from applying F_1 and F_2 to T by T_1 and F_4 to T' by T_2; note that all these flips can be applied in any order, but you might think about applying F_4, reversely. Then, for T_1 and T_2, $\mathcal{P}_R = \varnothing$, $\mathcal{P}_N$ corresponds to Y, and $H(T_1, T_2)[Y]$ is acyclic. Hence, Proposition 17 guarantees a flip sequence of length $|Y|$.

It remains to discuss the total length. By Lemma 15 and the fact that T' has at least two uncovered edges, we have $|T'_W| \le |T'_S| - 2$. Hence,

$$|F_1| + |F_4| = |T'_S| + 2|T'_W| - c \le 3/2(|T'_S| + |T'_W| - 2).$$

By Lemma 29, we have $|Y| \ge 1/2|I_N| = 1/2|T'_N|$ and thus $|F_2| + |F_3| = 2|X| + |Y| = 3/2|T'_N|$. Therefore, we obtain the following bound

$$|F| = |F_1| + |F_2| + |F_3| + |F_4| \le 3/2(|T'_S| + |T'_W| + |T'_N| - 2) = 3/2(n-3) = 3/2 \cdot d,$$

which concludes the proof. ◀

7 Discussion and open problems

In this work, we improved the lower and upper bounds on the diameter of $\mathcal{F}_n$. Together, Theorems 3 and 25 yield

$$14/9 \cdot n - \mathcal{O}(1) \le \operatorname{diam}(\mathcal{F}_n) \le 5/3 \cdot n - 3 = 15/9 \cdot n - 3.$$

Thus, the gap between the upper and lower bounds on $\operatorname{diam}(\mathcal{F}_n)$ has been tightened from about $0.45n$ to just $1/9 \cdot n + \mathcal{O}(1)$. With Theorem 6 at hand, closing the gap can be achieved by improving the lower bound $\frac{\operatorname{ac}(H)}{|V(H)|} \ge 1/3$ for all conflict graphs H, or by presenting a conflict graph H with $\frac{\operatorname{ac}(H)}{|V(H)|} < 4/9$. We therefore believe that our techniques have potential to help determining $\operatorname{diam}(\mathcal{F}_n)$ completely.

Let us note that the new lower bound of $14/9 \cdot n$ for the convex setting actually improves upon the best known lower bounds not only for points in general position, but also for more restricted flip operations, e.g., the *compatible edge exchange* (where the exchanged edges are non-crossing), the *rotation* (where the exchanged edge are adjacent), and the *edge slide* (where the exchanged edges together with some third edge form an uncrossed triangle). For an overview of best known bounds for five studied flip types, we refer to Nichols et al. [30].

We also considered bounding the flip distance $\text{dist}(T, T')$ of two trees T, T' in terms of $d = |T, T'|$. Our Theorem 25 is somewhat halfway between an upper bound on $\text{dist}(T, T')$ in terms of n and one in terms of d: common chords do not contribute at all, and common boundary edges (their number is b) contribute less than the edges in the symmetric difference. If $2/3 \cdot b - 4/3$ can be removed from the bound in Theorem 25, then this would give a tight upper bound in terms of d. In fact, Bousquet et al. [7, Theorem 4], present graphs T_d and T'_d with symmetric difference $2d$ and $\text{dist}(T_d, T'_d) = 5/3 \cdot d$ (for all d divisible by 3).

Besides determining the maximum flip distance in terms of n or d for the mentioned settings, it is also interesting to investigate the computational complexity of computing a shortest flip sequence for two given non-crossing trees. Is it NP-complete or polynomial-time solvable? The question is open for both settings of convex and general position.

Moreover, is it true that for any two trees T, T' there exists a flip sequence of length $\text{dist}(T, T')$, such that common edges (so called *happy edges*) are not flipped. Aichholzer et al. [2, Conjecture 16] conjecture that this is the case for the convex setting.

References

1. Oswin Aichholzer, Franz Aurenhammer, and Ferran Hurtado. Sequences of spanning trees and a fixed tree theorem. *Computational Geometry: Theory & Applications*, 21(1-2):3–20, 2002. `doi:10.1016/S0925-7721(01)00042-6`.
2. Oswin Aichholzer, Brad Ballinger, Therese Biedl, Mirela Damian, Erik D. Demaine, Matias Korman, Anna Lubiw, Jayson Lynch, Josef Tkadlec, and Yushi Uno. Reconfiguration of non-crossing spanning trees, 2022. `doi:10.48550/arXiv.2206.03879`.
3. Oswin Aichholzer, Kristin Knorr, Wolfgang Mulzer, Johannes Obenaus, Rosna Paul, and Birgit Vogtenhuber. Flipping plane spanning paths. In *International Conference and Workshops on Algorithms and Computation (WALCOM)*, pages 49–60, 2023. `doi:10.1007/978-3-031-27051-2_5`.
4. Selim G. Akl, Md Kamrul Islam, and Henk Meijer. On planar path transformation. *Information Processing Letters*, 104(2):59–64, 2007. `doi:10.1016/j.ipl.2007.05.009`.
5. David Avis and Komei Fukuda. Reverse search for enumeration. *Discrete Applied Mathematics*, 65(1-3):21–46, 1996. `doi:10.1016/0166-218X(95)00026-N`.
6. Prosenjit Bose and Ferran Hurtado. Flips in planar graphs. *Computational Geometry: Theory & Applications*, 42(1):60–80, 2009. `doi:10.1016/j.comgeo.2008.04.001`.
7. Nicolas Bousquet, Lucas de Meyer, Théo Pierron, and Alexandra Wesolek. Reconfiguration of Plane Trees in Convex Geometric Graphs. In *International Symposium on Computational Geometry (SoCG 2024)*, volume 293 of *LIPIcs*, pages 22:1–22:17, 2024. `doi:10.4230/LIPIcs.SoCG.2024.22`.
8. Nicolas Bousquet, Valentin Gledel, Jonathan Narboni, and Théo Pierron. A note on the flip distance between non-crossing spanning trees. *Computing in Geometry and Topology*, 2(1):8:1–8:7, 2023. `doi:10.57717/cgt.v2i1.36`.
9. Nicolas Bousquet, Valentin Gledel, Jonathan Narboni, and Théo Pierron. A note on the flip distance between non-crossing spanning trees, 2023. `doi:10.48550/arXiv.2303.07710`.
10. Nicolas Bousquet, Takehiro Ito, Yusuke Kobayashi, Haruka Mizuta, Paul Ouvrard, Akira Suzuki, and Kunihiro Wasa. Reconfiguration of Spanning Trees with Many or Few Leaves. In *European Symposium on Algorithms (ESA 2020)*, volume 173 of *LIPIcs*, pages 24:1–24:15, 2020. `doi:10.4230/LIPIcs.ESA.2020.24`.

11 Nicolas Bousquet, Takehiro Ito, Yusuke Kobayashi, Haruka Mizuta, Paul Ouvrard, Akira Suzuki, and Kunihiro Wasa. Reconfiguration of spanning trees with degree constraints or diameter constraints. *Algorithmica*, 85(9):2779–2816, 2023. doi:10.1007/s00453-023-01117-z.

12 Cesar Ceballos, Francisco Santos, and Günter M. Ziegler. Many non-equivalent realizations of the associahedron. *Combinatorica*, 35(5):513–551, 2015. doi:10.1007/s00493-014-2959-9.

13 Jou-Ming Chang and Ro-Yu Wu. On the diameter of geometric path graphs of points in convex position. *Information Processing Letters*, 109(8):409–413, 2009. doi:10.1016/j.ipl.2008.12.017.

14 David Eppstein. Happy endings for flip graphs. *Journal of Computational Geometry*, 1(1):3–28, 2010. doi:10.20382/JOCG.V1I1A2.

15 David Eppstein and Daniel Frishberg. Improved Mixing for the Convex Polygon Triangulation Flip Walk. In *International Colloquium on Automata, Languages, and Programming (ICALP 2023)*, volume 261 of *LIPIcs*, pages 56:1–56:17, 2023. doi:10.4230/LIPIcs.ICALP.2023.56.

16 Stefan Felsner, Linda Kleist, Torsten Mütze, and Leon Sering. Rainbow cycles in flip graphs. *SIAM Journal on Discrete Mathematics*, 34(1):1–39, 2020. doi:10.1137/18M1216456.

17 Carmen Hernando, Michael E. Houle, and Ferran Hurtado. On local transformation of polygons with visibility properties. *Theoretical Computer Science*, 289(2):919–937, 2002. doi:10.1016/S0304-3975(01)00409-1.

18 Carmen Hernando, Ferran Hurtado, Alberto Márquez, Mercè Mora, and Marc Noy. Geometric tree graphs of points in convex position. *Discrete Applied Mathematics*, 93(1):51–66, 1999. doi:10.1016/S0166-218X(99)00006-2.

19 Carmen Hernando, Ferran Hurtado, Mercè Mora, and Eduardo Rivera-Campo. Grafos de árboles etiquetados y grafos de árboles geométricos etiquetados. *Proc. X Encuentros de Geometra Computacional*, pages 13–19, 2003.

20 Carmen Hernando, Ferran Hurtado, and Marc Noy. Graphs of non-crossing perfect matchings. *Graphs and Combinatorics*, 18(3):517–532, 2002. doi:10.1007/S003730200038.

21 Michael E. Houle, Ferran Hurtado, Marc Noy, and Eduardo Rivera-Campo. Graphs of triangulations and perfect matchings. *Graphs and Combinatorics*, 21(3):325–331, 2005. doi:10.1007/S00373-005-0615-2.

22 Ferran Hurtado, Marc Noy, and Jorge Urrutia. Flipping edges in triangulations. *Discrete & Computational Geometry*, 22(3):333–346, 1999. doi:10.1007/PL00009464.

23 Takehiro Ito, Naonori Kakimura, Naoyuki Kamiyama, Yusuke Kobayashi, Shun-ichi Maezawa, Yuta Nozaki, and Yoshio Okamoto. Hardness of Finding Combinatorial Shortest Paths on Graph Associahedra. In *International Colloquium on Automata, Languages, and Programming (ICALP 2023)*, volume 261 of *LIPIcs*, pages 82:1–82:17, 2023. doi:10.4230/LIPIcs.ICALP.2023.82.

24 Linda Kleist, Peter Kramer, and Christian Rieck. On the connectivity of the flip graph of plane spanning paths. In *International Workshop on Graph-Theoretic Concepts in Computer Science*, 2024.

25 Charles L. Lawson. Transforming triangulations. *Discrete Mathematics*, 3(4):365–372, 1972. doi:10.1016/0012-365X(72)90093-3.

26 Anna Lubiw and Vinayak Pathak. Flip distance between two triangulations of a point set is NP-complete. *Computational Geometry: Theory & Applications*, 49:17–23, 2015. doi:10.1016/J.COMGEO.2014.11.001.

27 Joan M Lucas. The rotation graph of binary trees is hamiltonian. *Journal of Algorithms*, 8(4):503–535, 1987. doi:10.1016/0196-6774(87)90048-4.

28 Marcel Milich, Torsten Mütze, and Martin Pergel. On flips in planar matchings. *Discrete Applied Mathematics*, 289:427–445, 2021. doi:10.1016/J.DAM.2020.10.018.

29 Ruy Fabila Monroy, David Flores-Peñaloza, Clemens Huemer, Ferran Hurtado, David R. Wood, and Jorge Urrutia. On the chromatic number of some flip graphs. *Discrete Mathematics & Theoretical Computer Science*, 11(2):47–56, 2009. doi:10.46298/DMTCS.460.

30 Torrie L Nichols, Alexander Pilz, Csaba D Tóth, and Ahad N Zehmakan. Transition operations over plane trees. *Discrete Mathematics*, 343(8):111929, 2020. doi:10.1016/j.disc.2020.111929.

31 Naomi Nishimura. Introduction to reconfiguration. *Algorithms*, 11(4):52, 2018. doi:10.3390/a11040052.

32 Alexander Pilz. Flip distance between triangulations of a planar point set is apx-hard. *Computational Geometry*, 47(5):589–604, 2014. doi:10.1016/j.comgeo.2014.01.001.

33 Lionel Pournin. The diameter of associahedra. *Advances in Mathematics*, 259:13–42, 2014. doi:10.1016/j.aim.2014.02.035.

34 Eduardo Rivera-Campo and Virginia Urrutia-Galicia. Hamilton cycles in the path graph of a set of points in convex position. *Computational Geometry: Theory & Applications*, 18(2):65–72, 2001. doi:10.1016/S0925-7721(00)00026-2.

35 Tomas Rokicki, Herbert Kociemba, Morley Davidson, and John Dethridge. God's number is 20. last accessed: July 2024. URL: http://www.cube20.org.

36 Daniel Dominic Sleator, Robert Endre Tarjan, and William P. Thurston. Rotation distance, triangulations, and hyperbolic geometry. In *Symposium on Theory of Computing (STOC)*, pages 122–135, 1986. doi:10.1145/12130.12143.

37 Jan van den Heuvel. The complexity of change. *Surveys in Combinatorics 2013*, pages 127–160, 2013. doi:10.1017/cbo9781139506748.005.

38 Uli Wagner and Emo Welzl. Connectivity of triangulation flip graphs in the plane. *Discrete & Computational Geometry*, 68(4):1227–1284, 2022. doi:10.1007/S00454-022-00436-2.