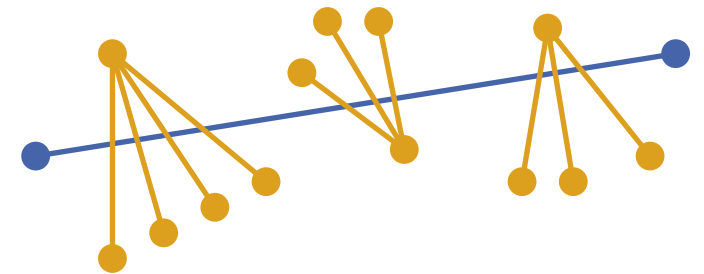
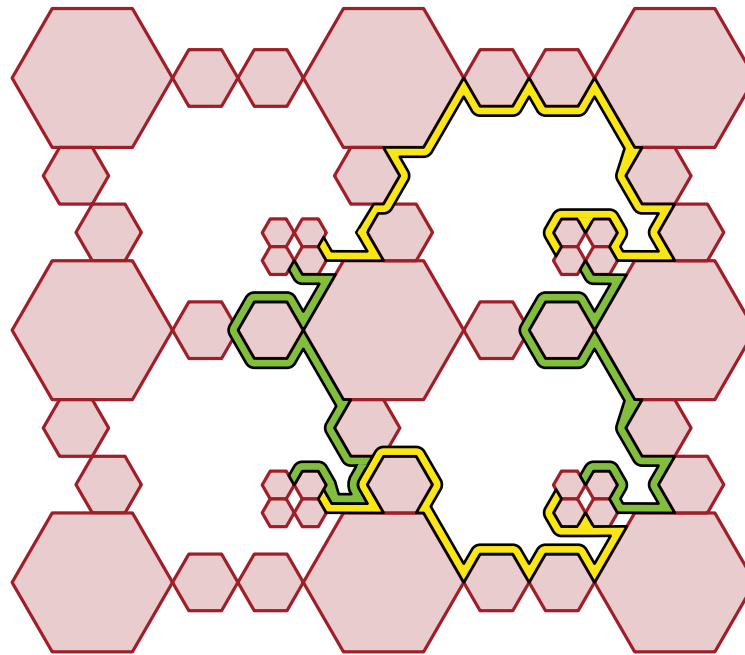
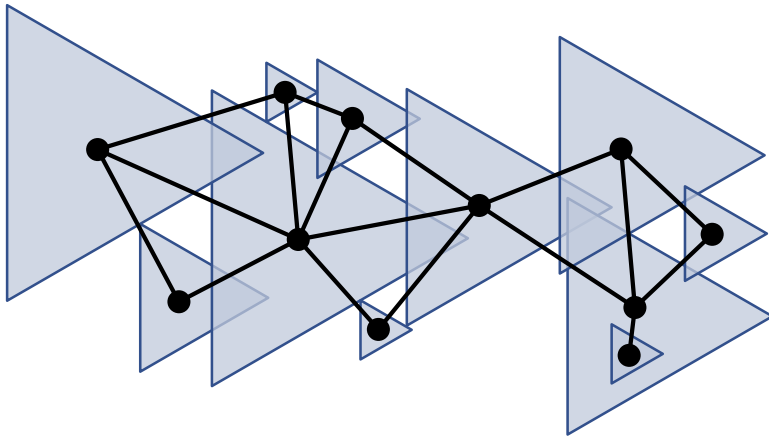


Intersection Graphs with and without Product Structure

GD 2024 · 19.09.2024

Laura Merker, Lena Scherzer, **Samuel Schneider**, Torsten Ueckerdt

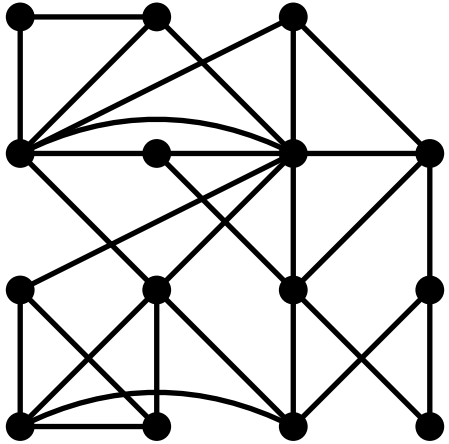


Product Structure

A graph class \mathcal{G} has product structure if there exists a $c \in \mathbb{N}$ such that:

Product Structure

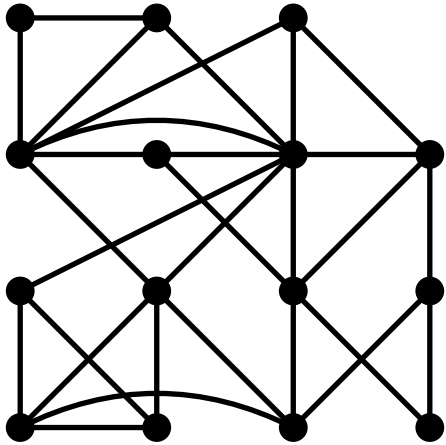
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For every $G \in \mathcal{G}$

Product Structure

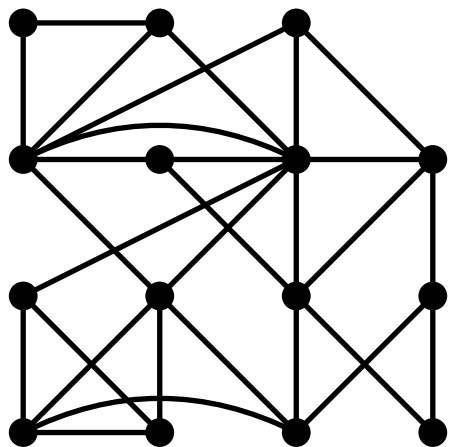
A graph class \mathcal{G} has product structure if there exists a $c \in \mathbb{N}$ such that:



For every $G \in \mathcal{G}$ there are H and P

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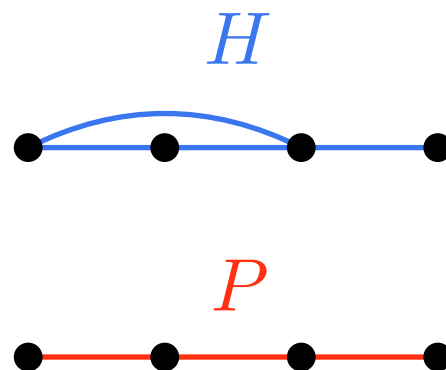
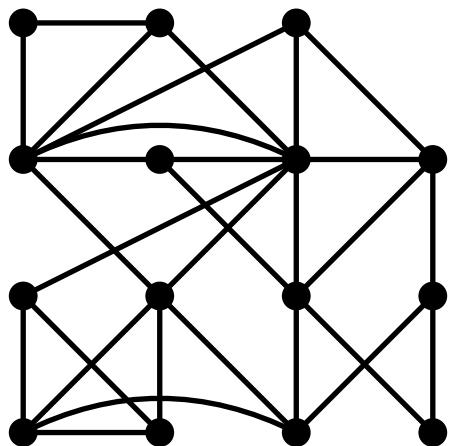
there are H and P

treewidth c

path

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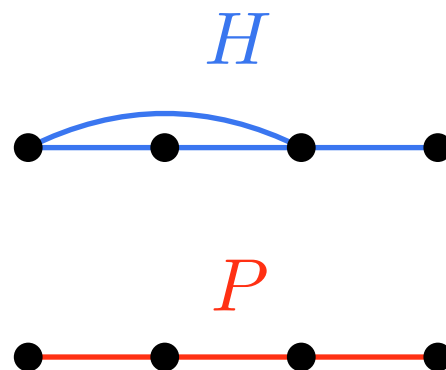
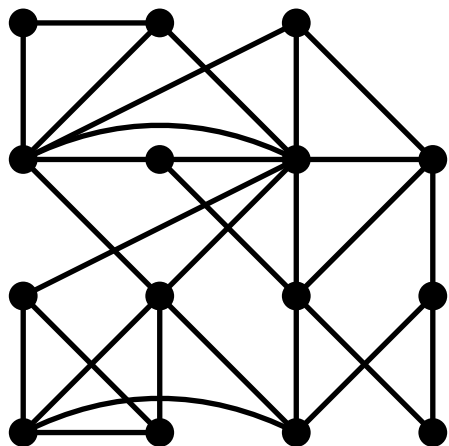
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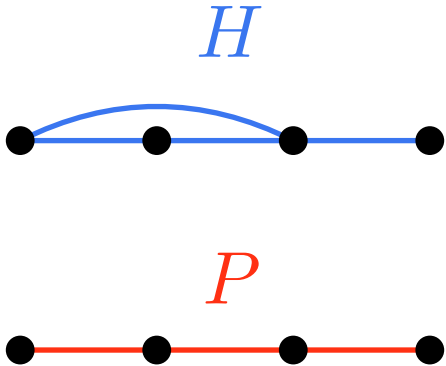
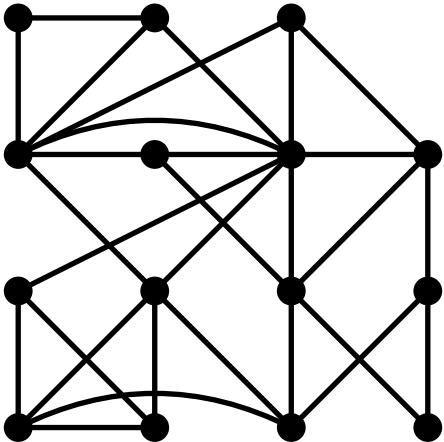
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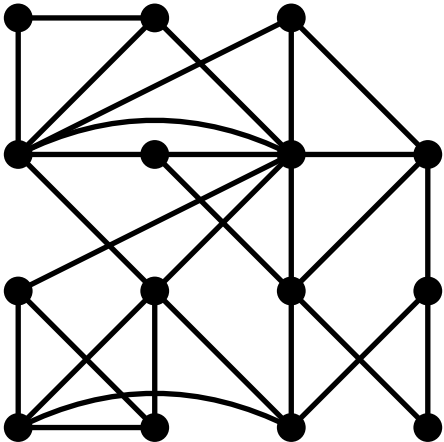
treewidth c

path

strong product

Product Structure

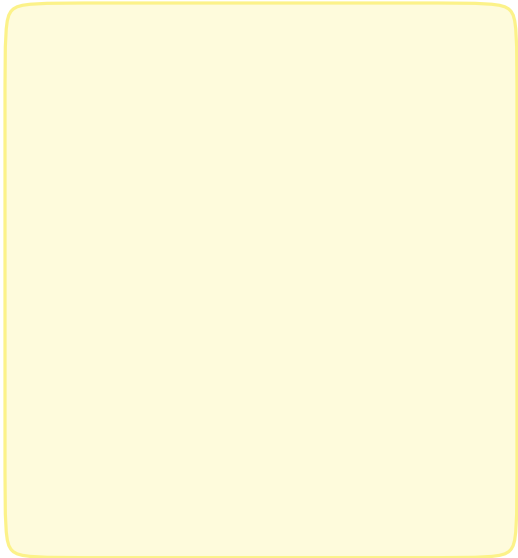
A graph class \mathcal{G} has product structure if there exists a $c \in \mathbb{N}$ such that:



P



H



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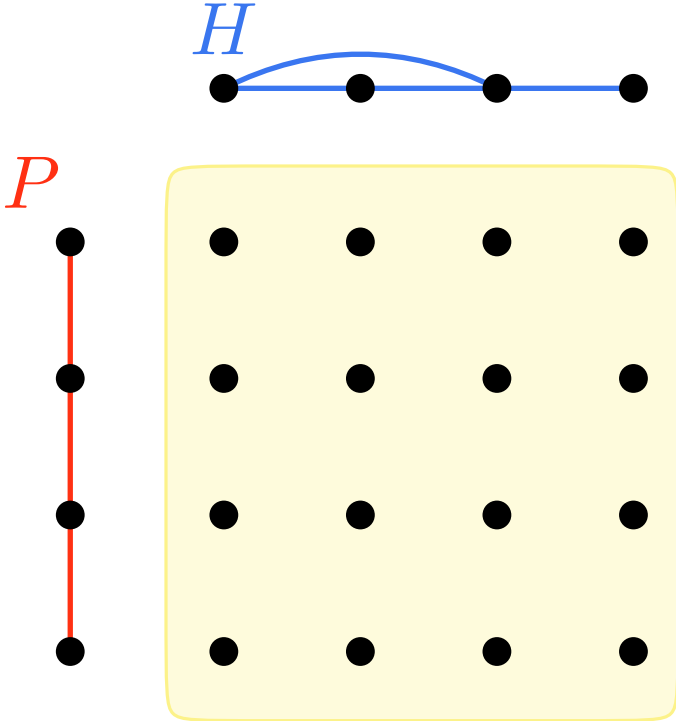
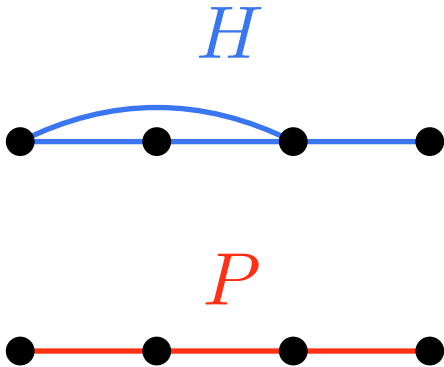
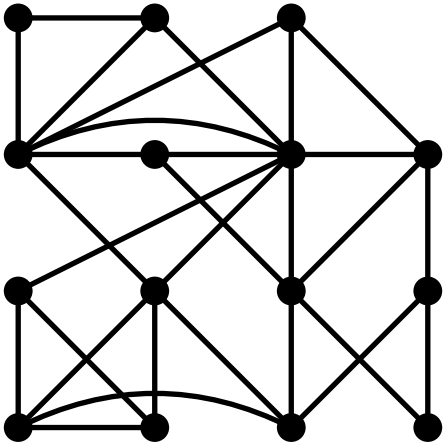
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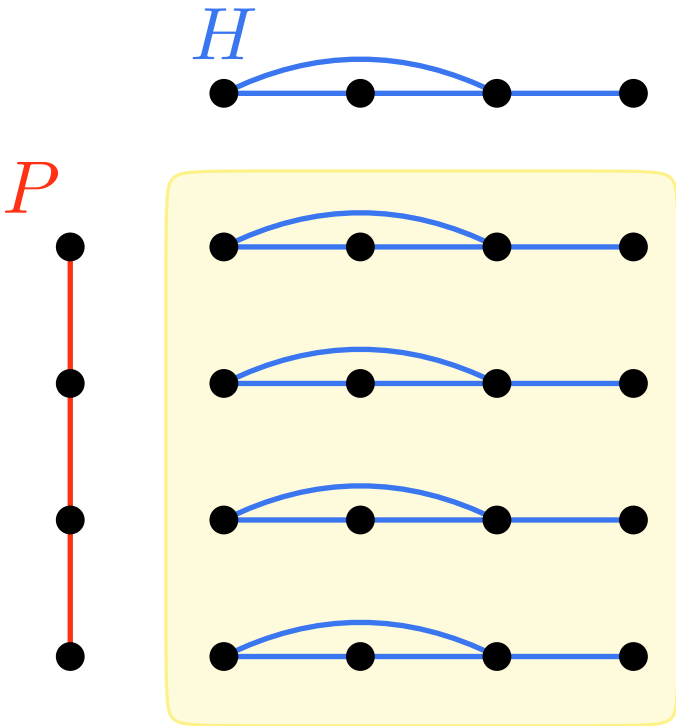
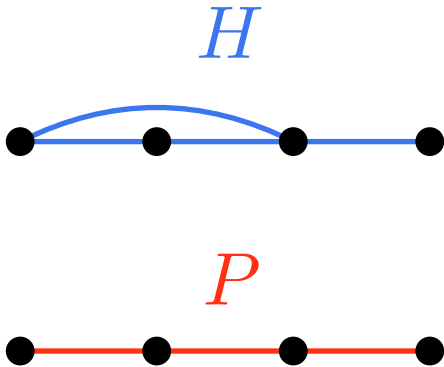
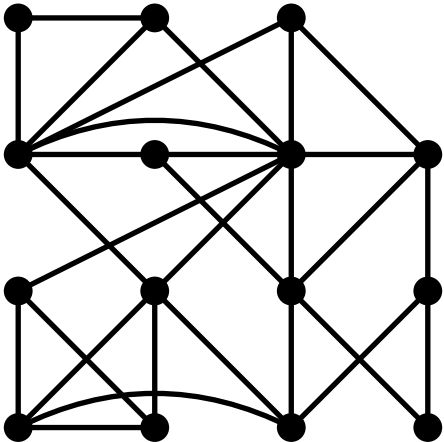
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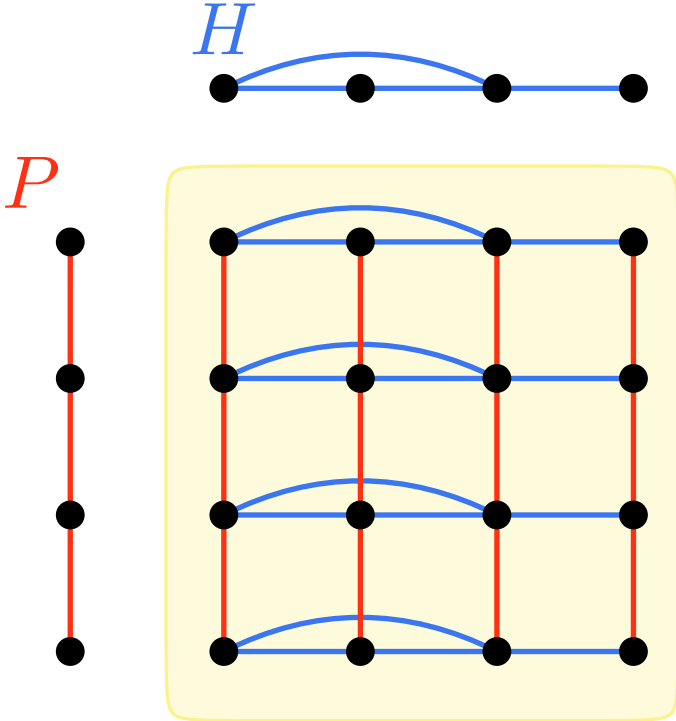
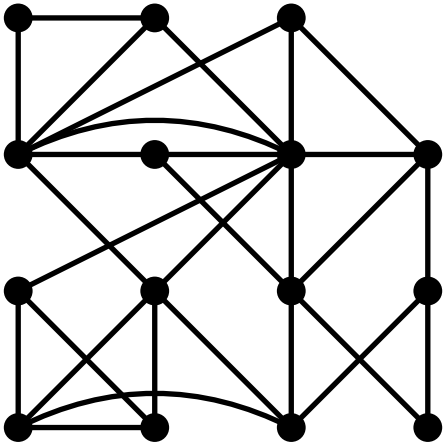
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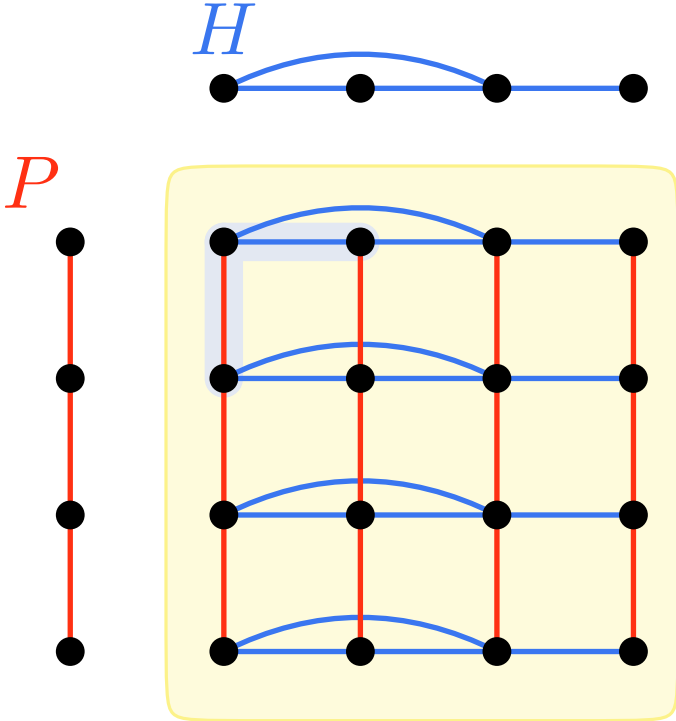
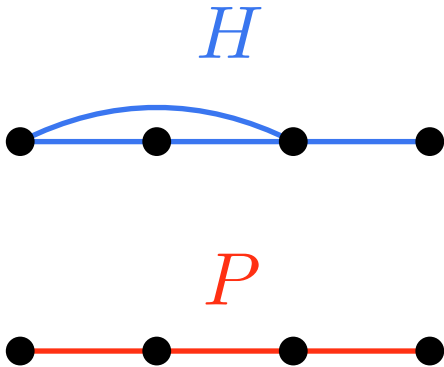
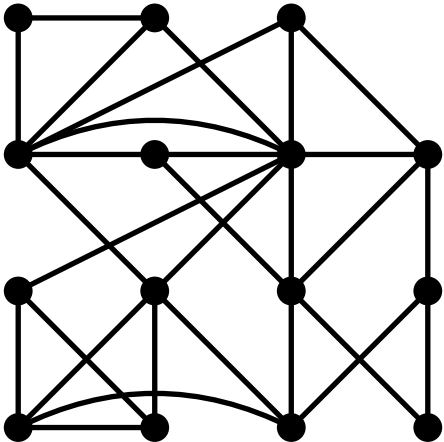
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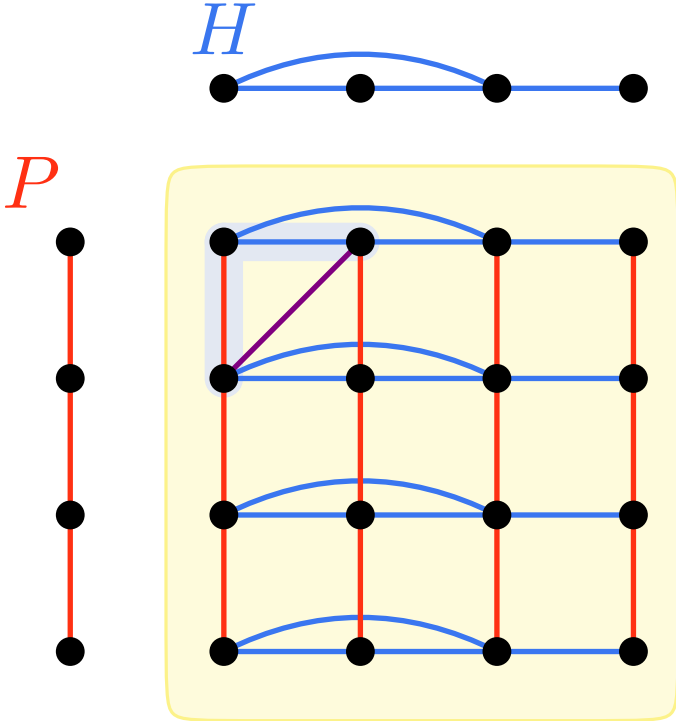
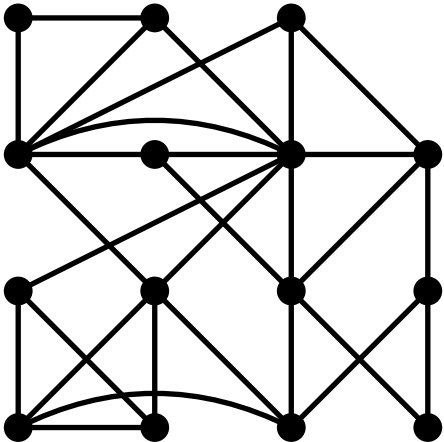
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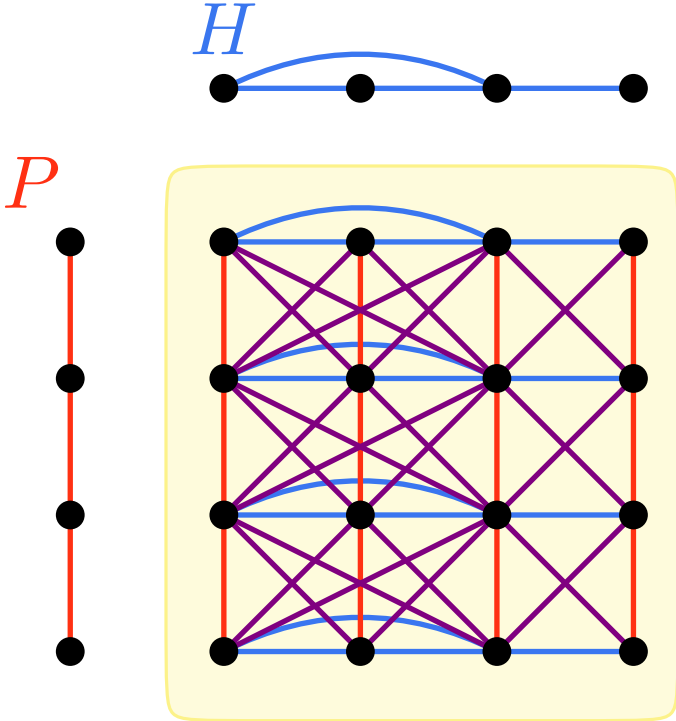
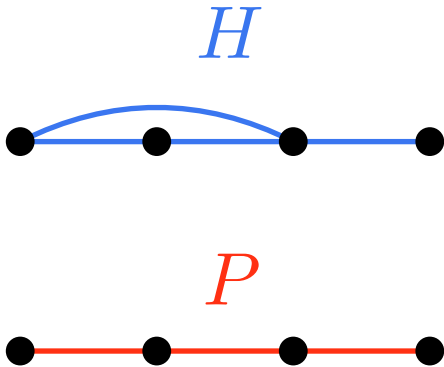
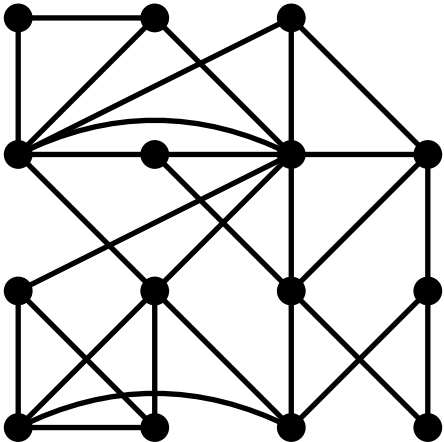
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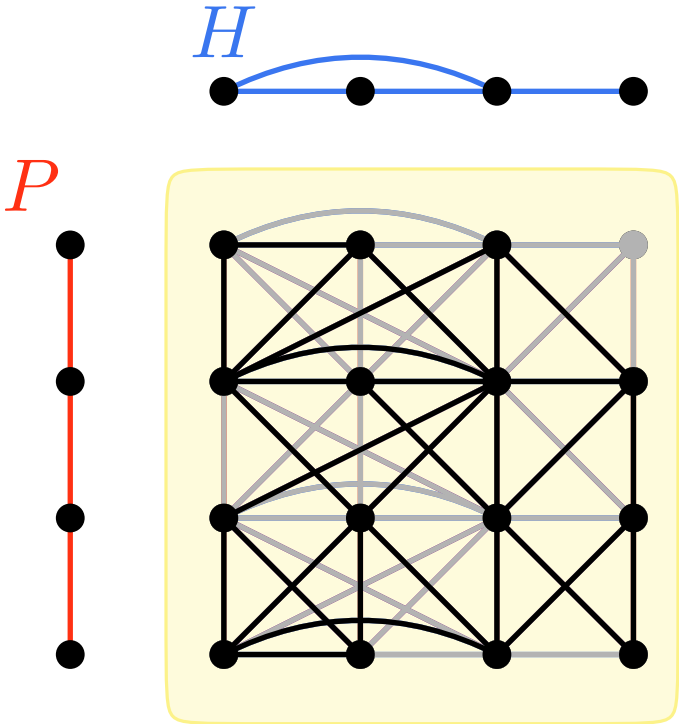
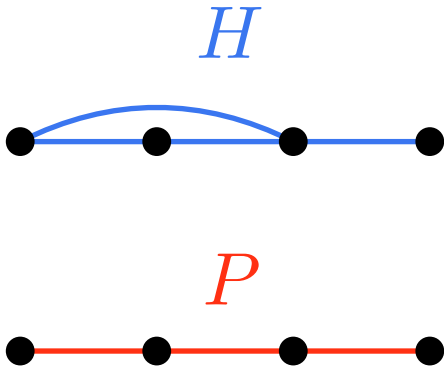
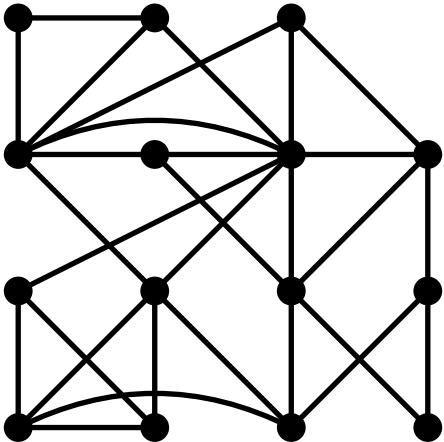
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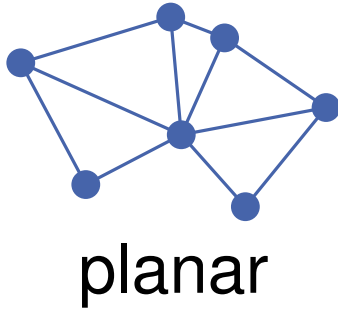
strong product

Existence of Product Structure

- Many beyond-planar graph classes have product structure.

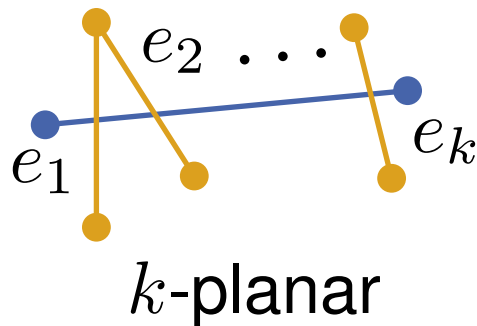
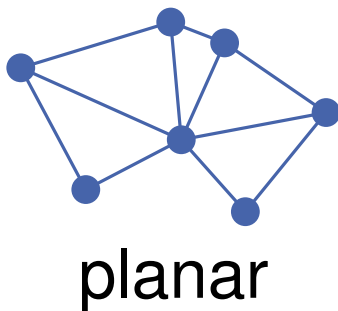
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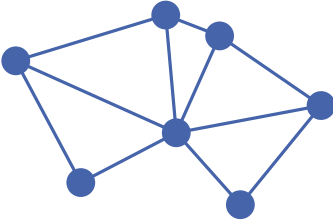
- Many beyond-planar graph classes have product structure.



[Dujmović et al.]

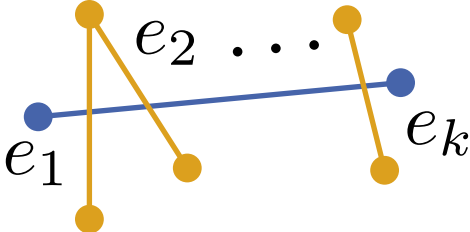
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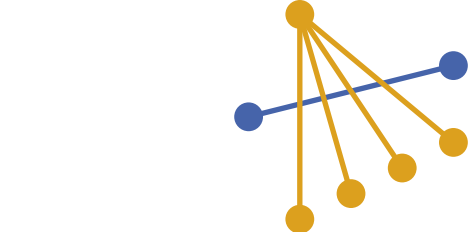


planar

[Dujmović et al.]



k -planar

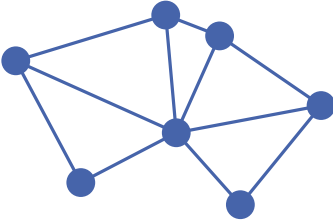


fan-planar

[Hickingbotham and Wood]

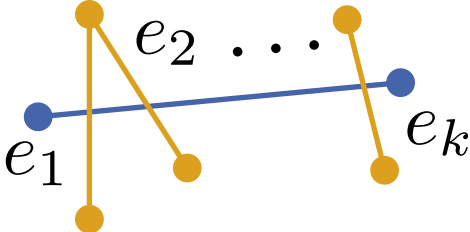
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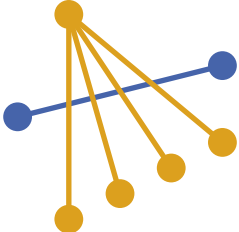


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k -planar



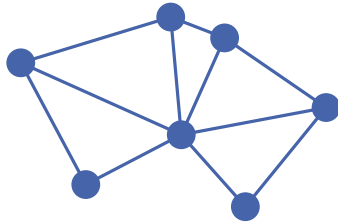
fan-planar

[Hickingbotham and Wood]

- Large cliques \implies **no** product structure.

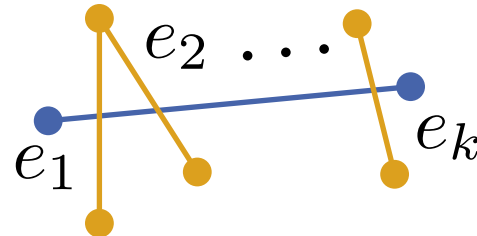
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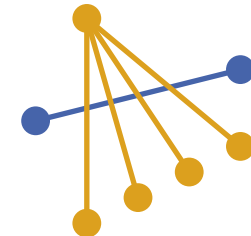


planar

[Dujmović et al.]



k -planar



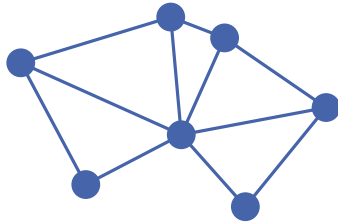
fan-planar

[Hickingbotham and Wood]

-
- Large cliques \implies **no** product structure.
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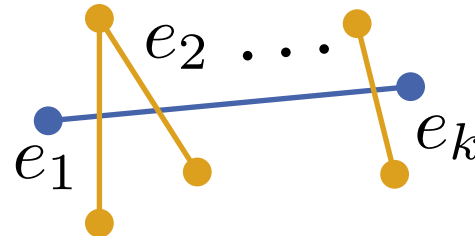
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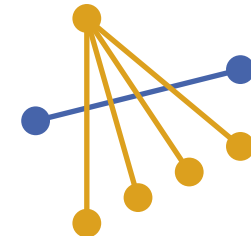


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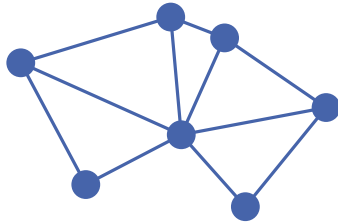
fan-planar

[Hickingbotham and Wood]

-
- Large cliques \implies **no** product structure.
 - Large treewidth in neighborhood \implies **no** product structure.
 - **No linear local treewidth** \implies **no** product structure.

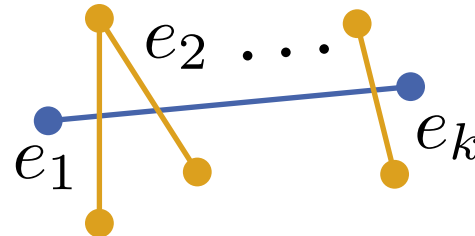
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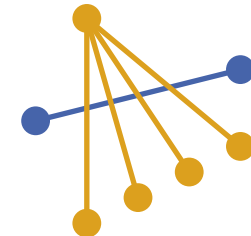


planar

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k -planar



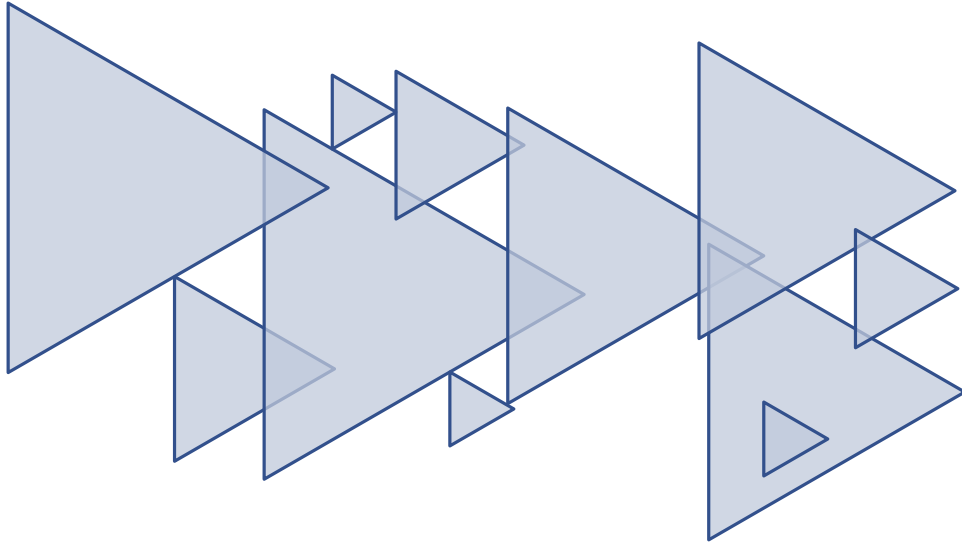
fan-planar

[Hickingbotham and Wood]

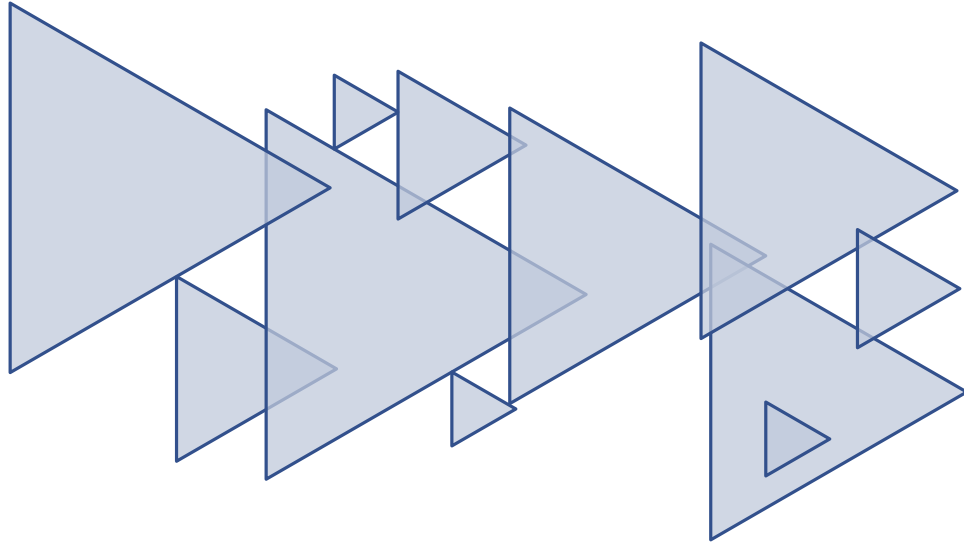
— Where is the border between product structure and no product structure? —

- Large cliques \implies **no** product structure.
- Large treewidth in neighborhood \implies **no** product structure.
- **No linear local treewidth** \implies **no** product structure.

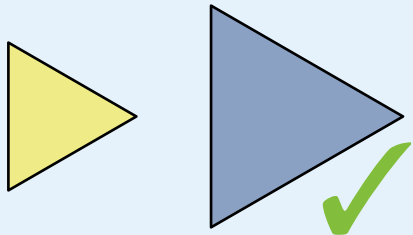
Intersection Graphs of Homothetic n -Gons



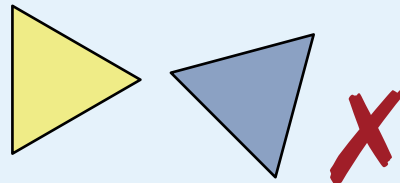
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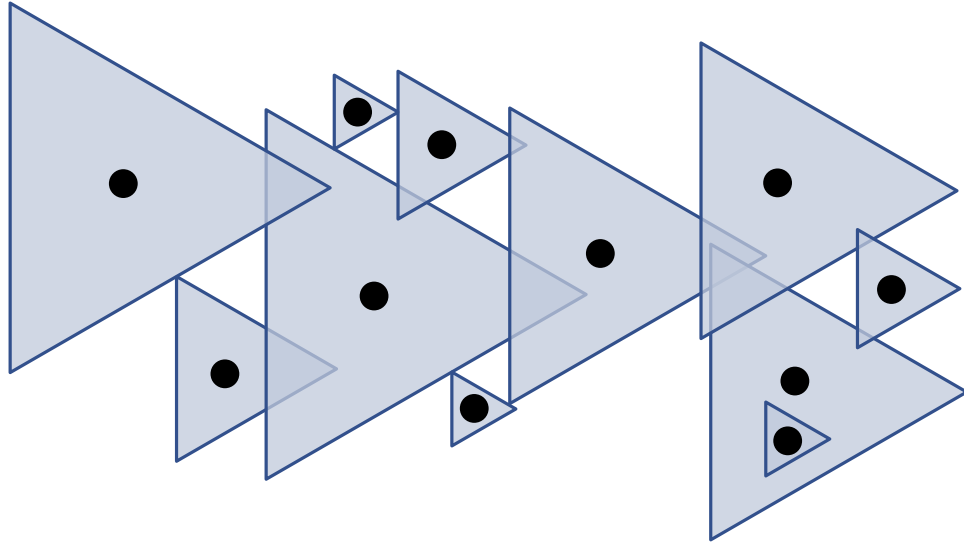
homothetic



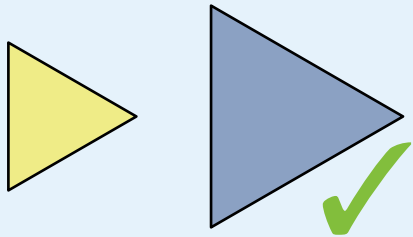
not homothetic



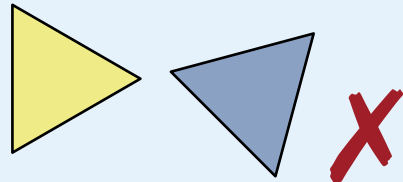
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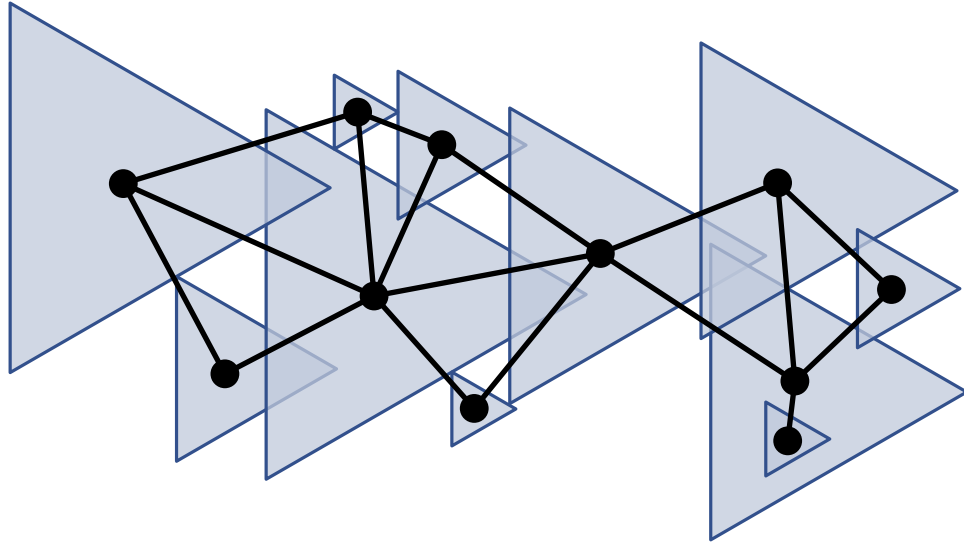
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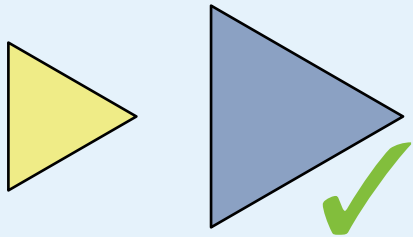
not homothetic



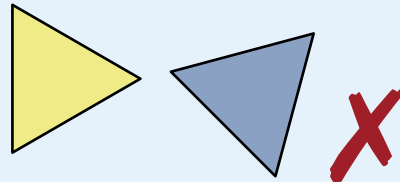
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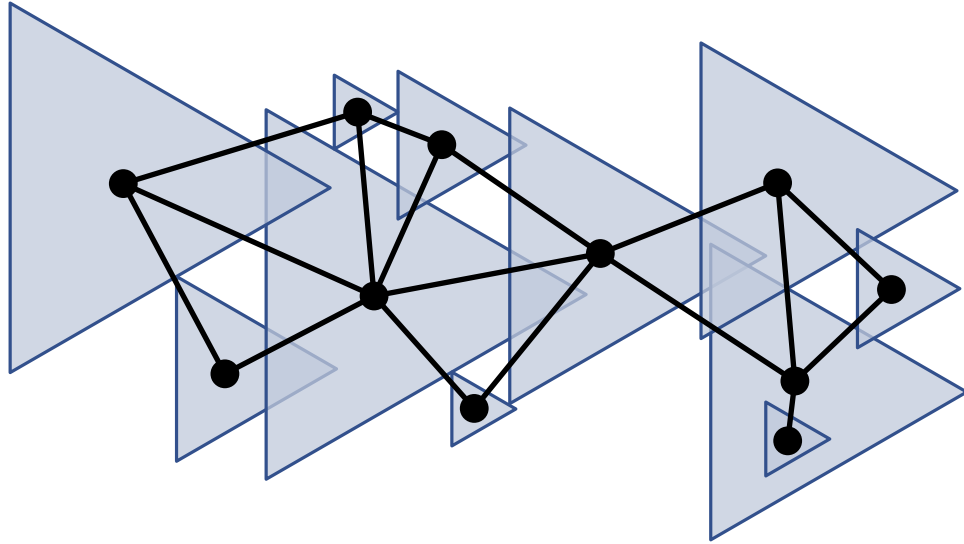
homothetic



not homothetic

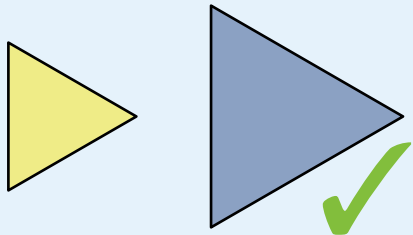


Intersection Graphs of Homothetic n -Gons

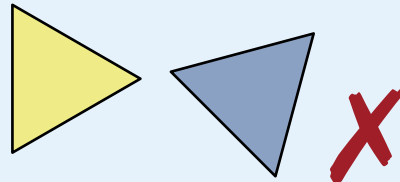


Problems for product structure:

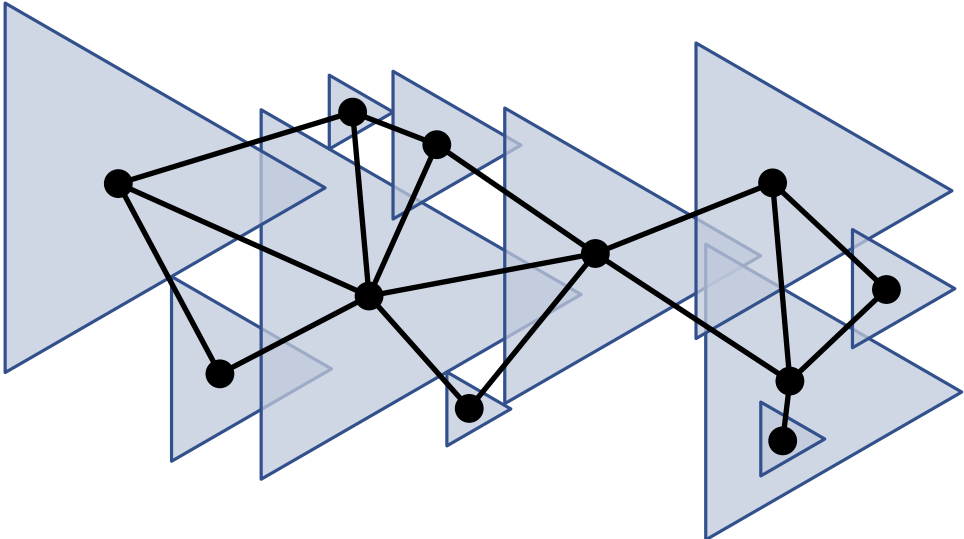
homothetic



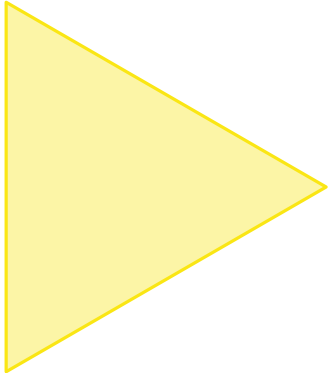
not homothetic



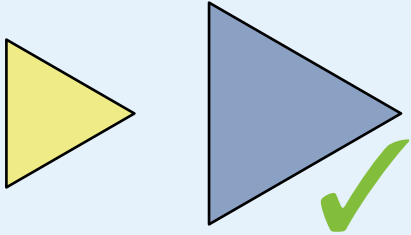
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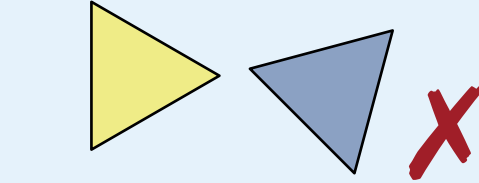
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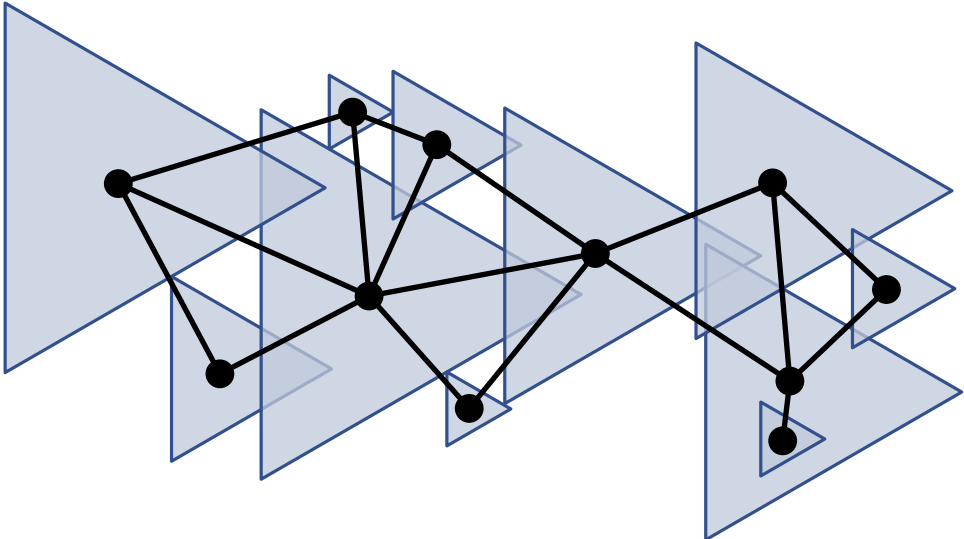
homothetic



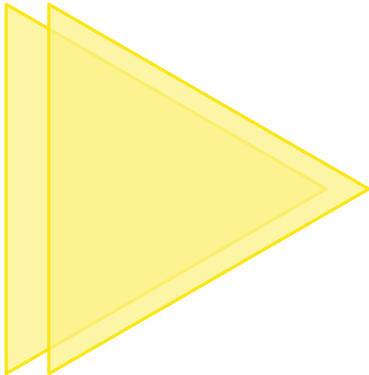
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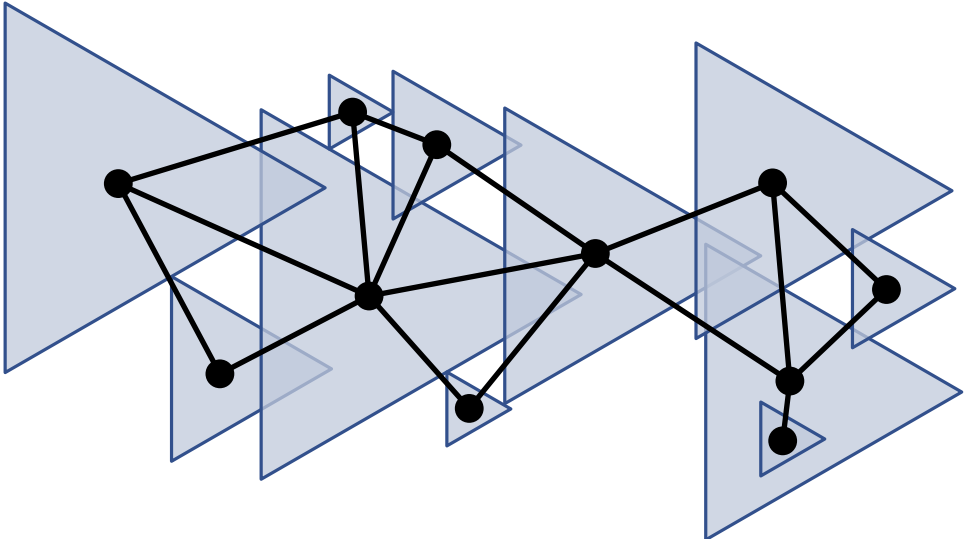
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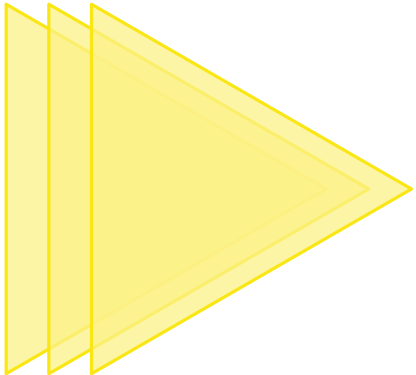
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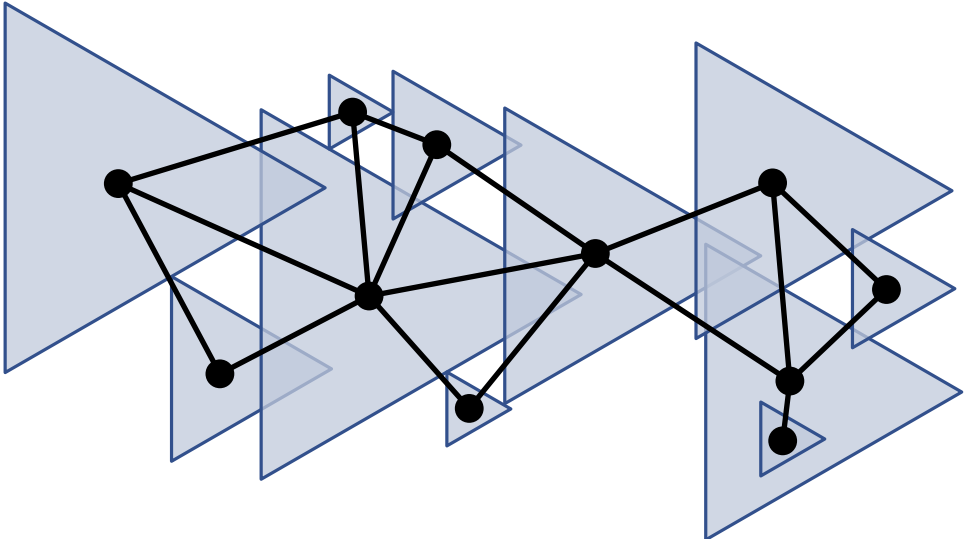
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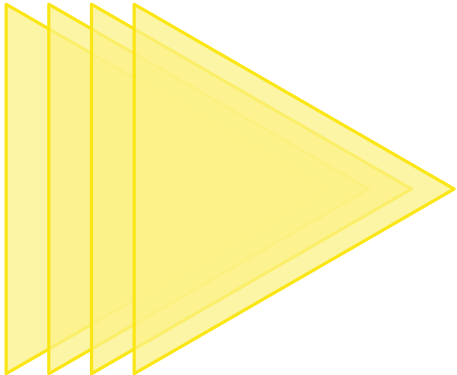
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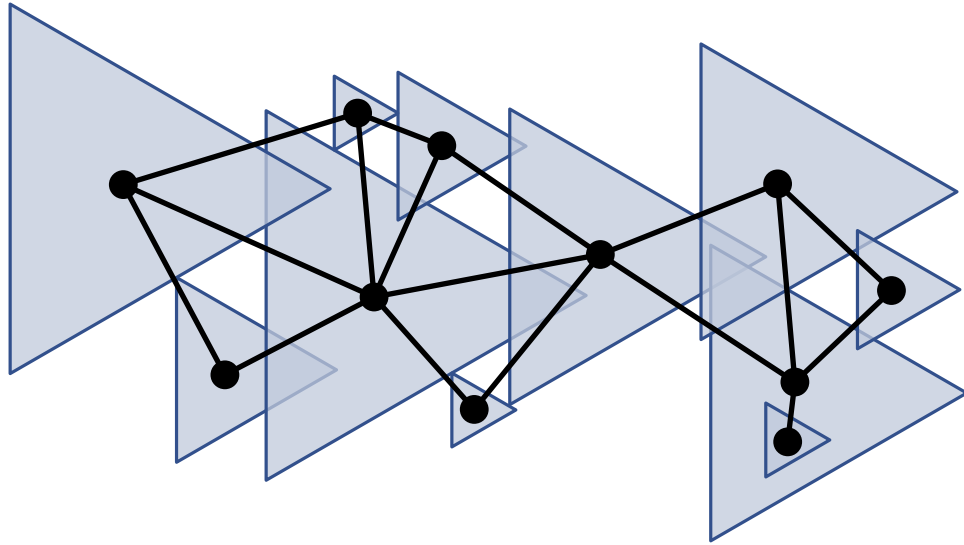
Problems for product structure:



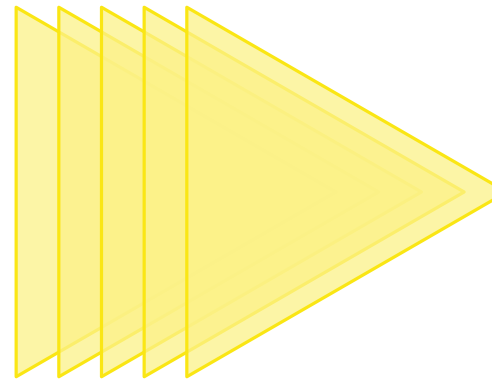
homothetic

not homothetic

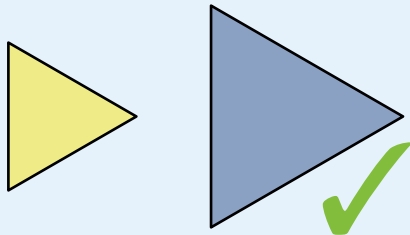
Intersection Graphs of Homothetic n -Gons



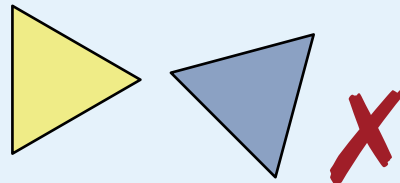
Problems for product structure:



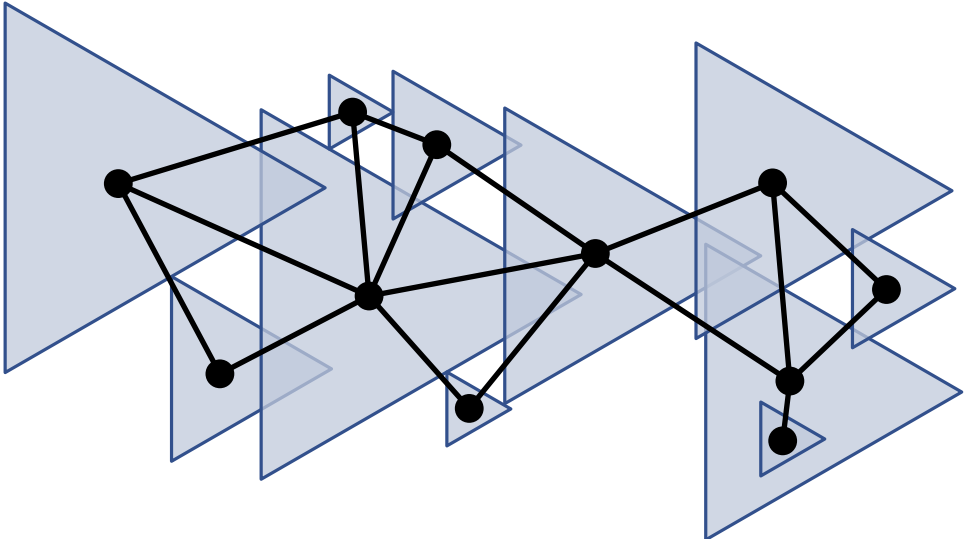
homothetic



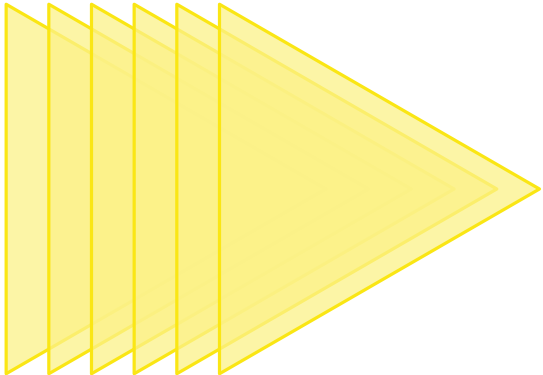
not homothetic



Intersection Graphs of Homothetic n -Gons



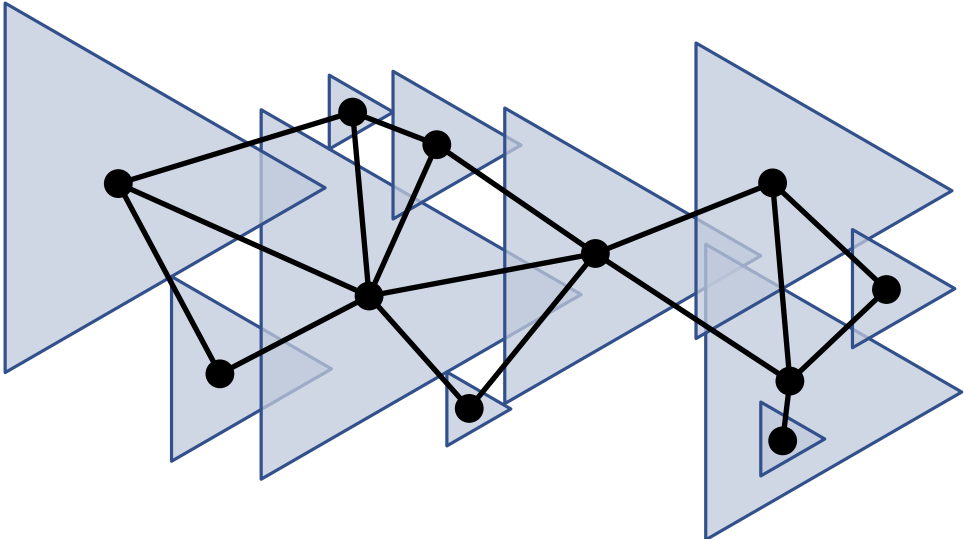
Problems for product structure:



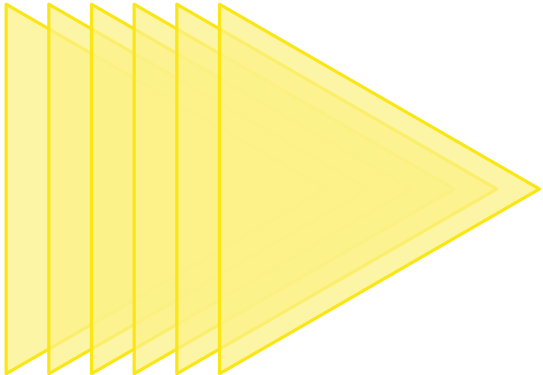
homothetic

not homothetic

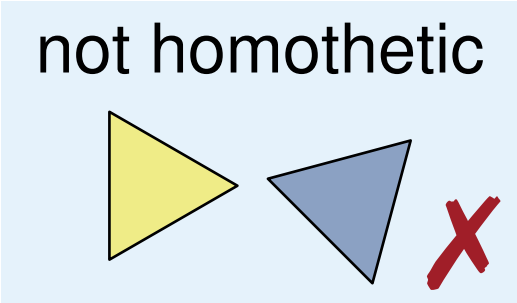
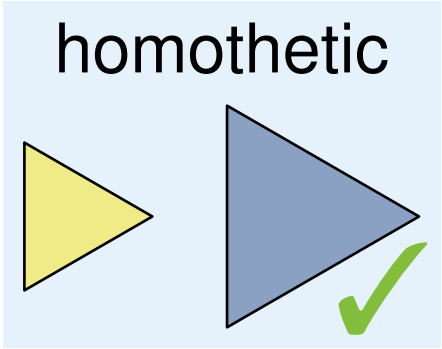
Intersection Graphs of Homothetic n -Gons



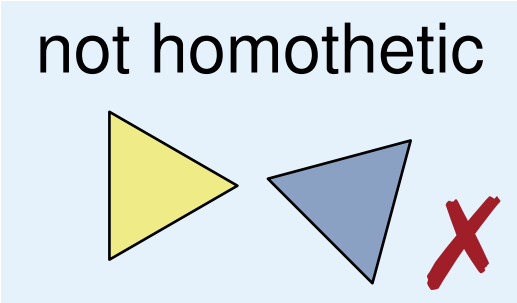
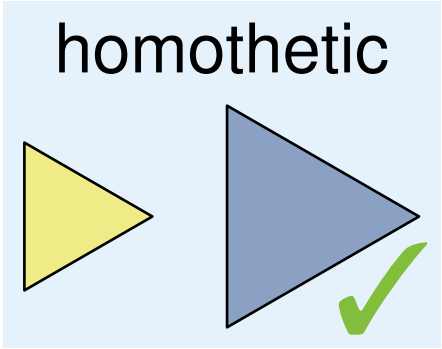
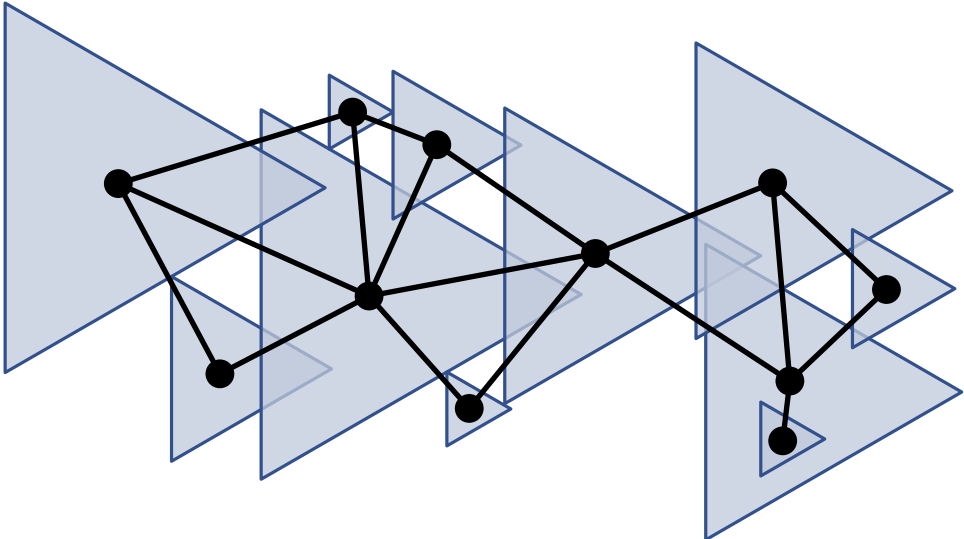
Problems for product structure:



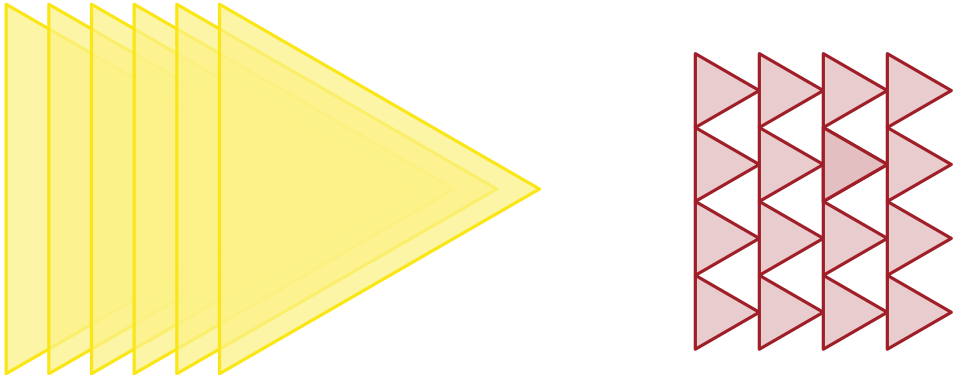
large cliques



Intersection Graphs of Homothetic n -Gons

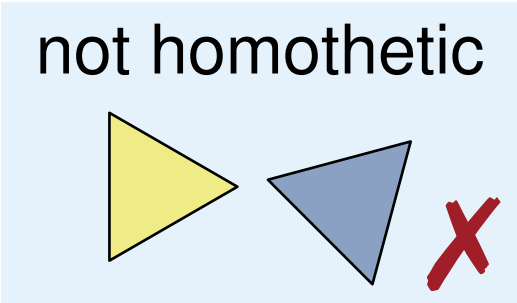
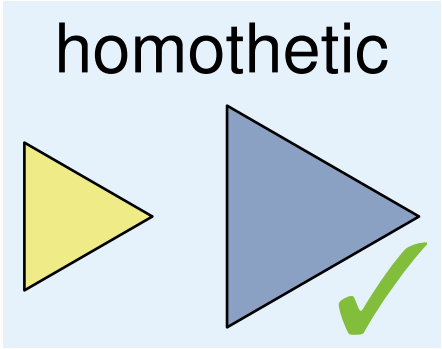
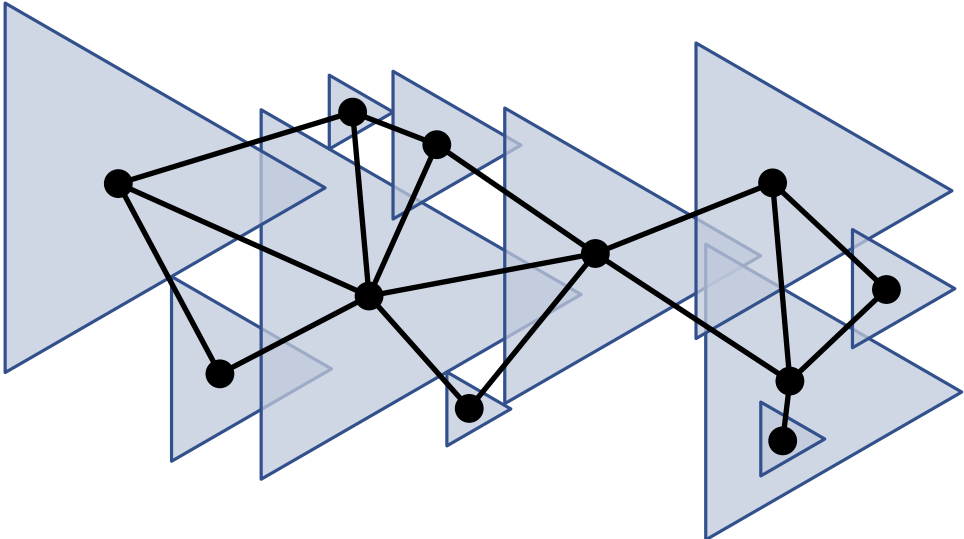


Problems for product structure:



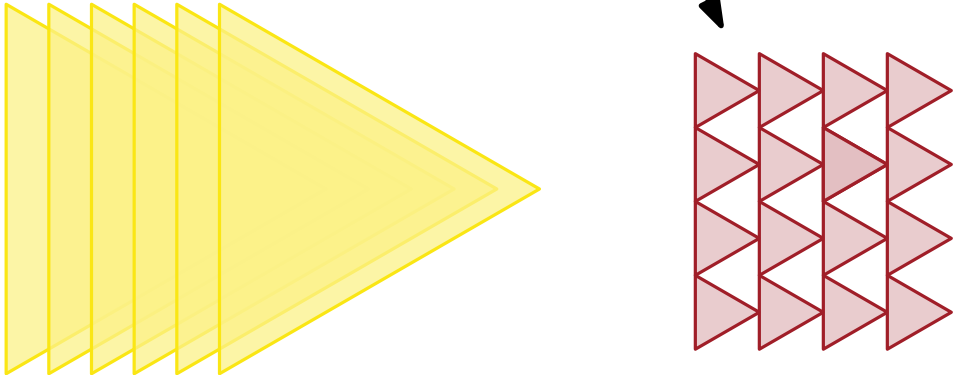
large cliques

Intersection Graphs of Homothetic n -Gons



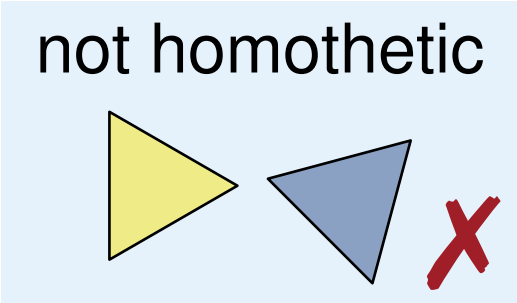
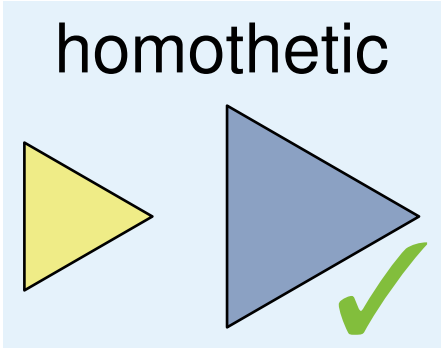
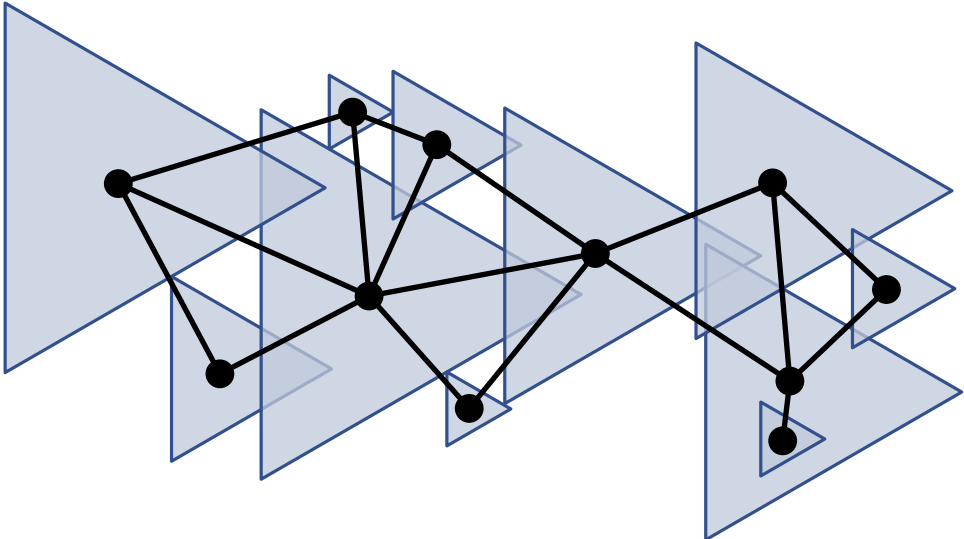
Problems for product structure:

large treewidth



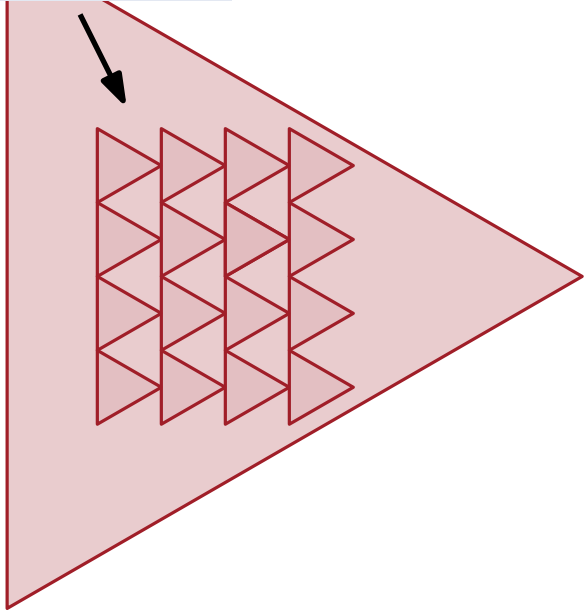
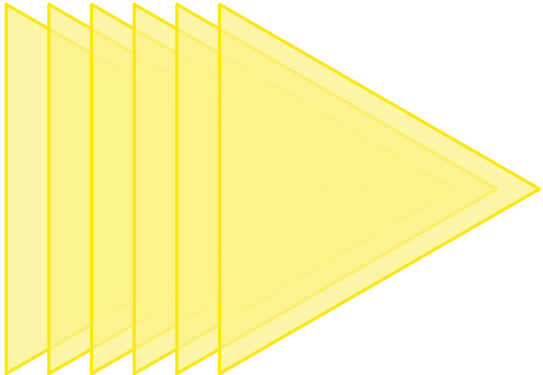
large cliques

Intersection Graphs of Homothetic n -Gons



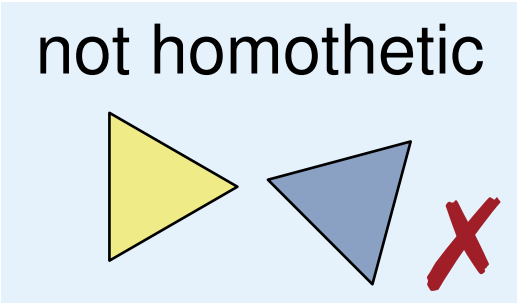
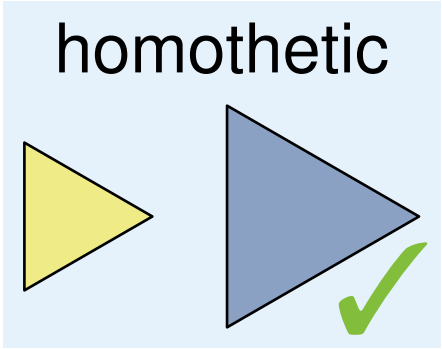
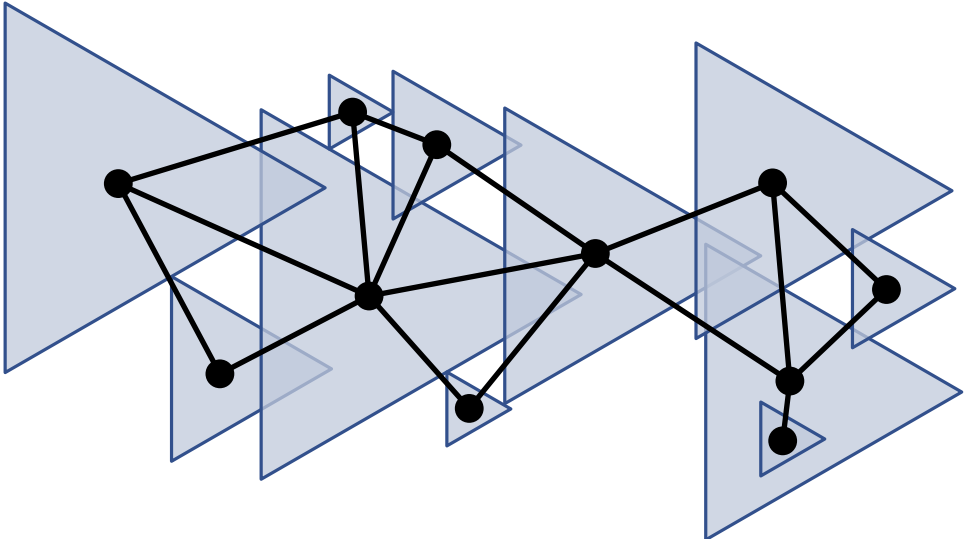
Problems for product structure:

large treewidth



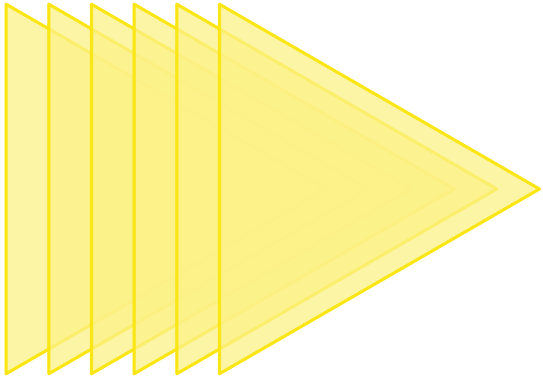
large cliques

Intersection Graphs of Homothetic n -Gons

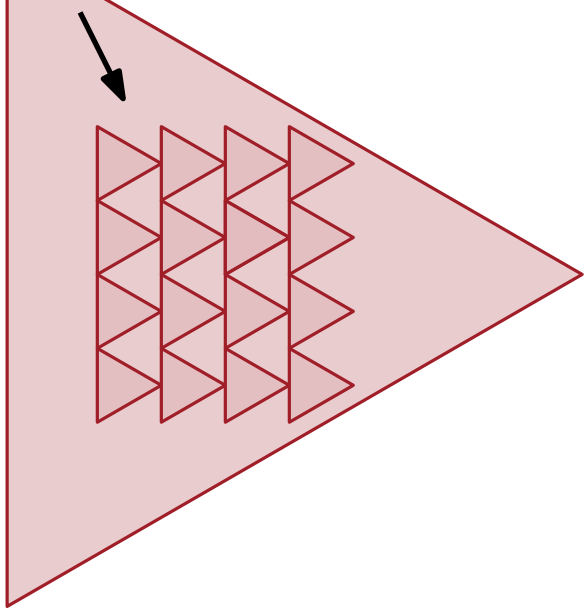


Problems for product structure:

large treewidth



large cliques



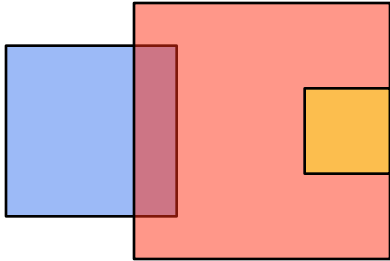
grids in neighborhood

α -free Intersection Graphs

A set \mathcal{S} of shapes in the plane is α -free with $\alpha \in [0, 1]$ if every shape $S \in \mathcal{S}$ has at least α of its area disjoint from all other shapes.

α -free Intersection Graphs

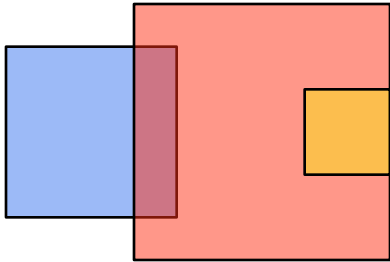
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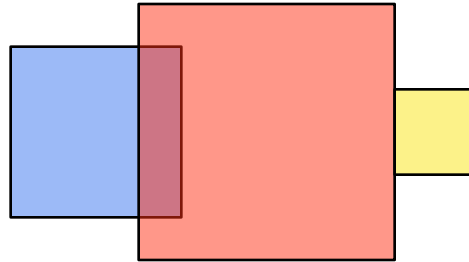
0-free

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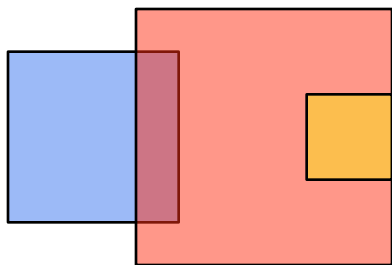
0-free



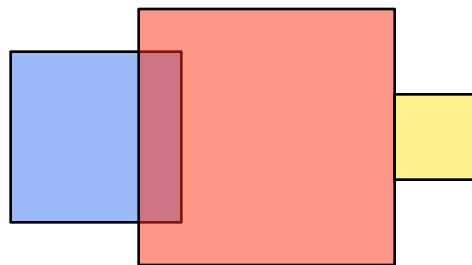
$\frac{3}{4}$ -free

α -free Intersection Graphs

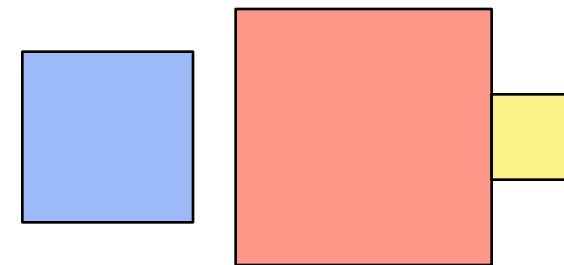
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0-free



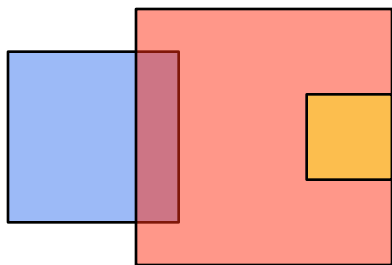
$\frac{3}{4}$ -free



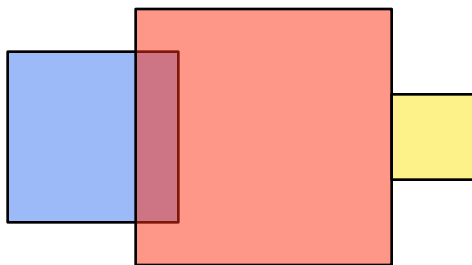
1-free

α -free Intersection Graphs

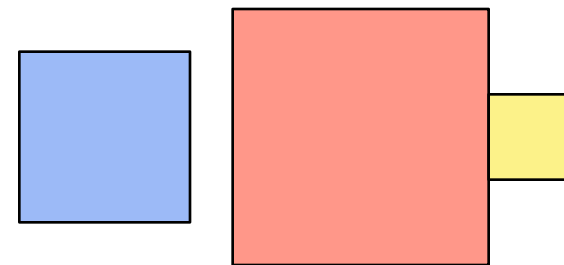
A set \mathcal{S} of shapes in the plane is α -free with $\alpha \in [0, 1]$ if every shape $S \in \mathcal{S}$ has at least α of its area disjoint from all other shapes.



0-free



$\frac{3}{4}$ -free

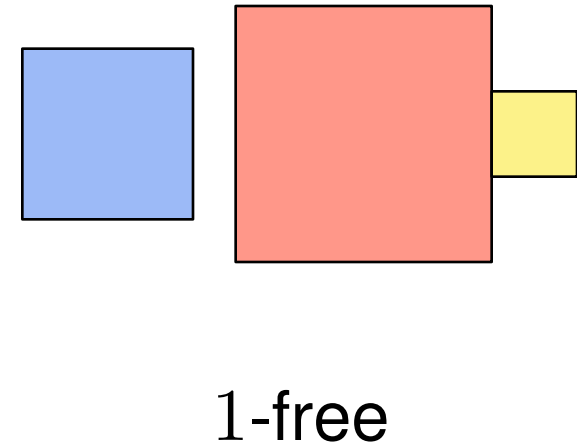
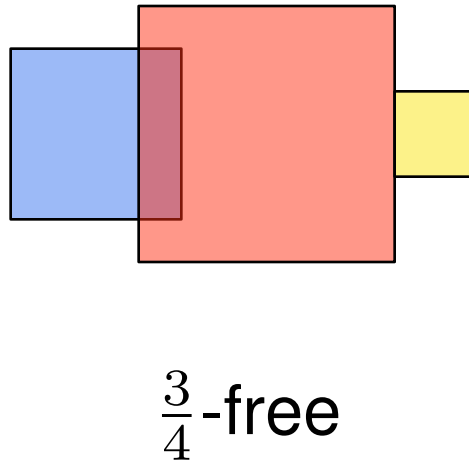
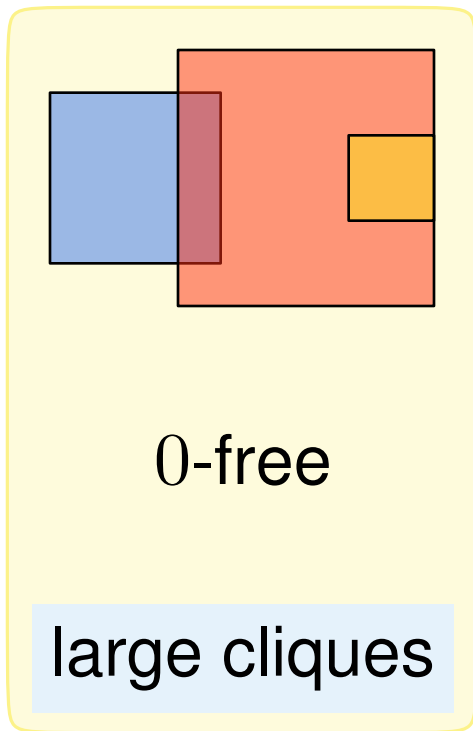


1-free

large cliques

α -free Intersection Graphs

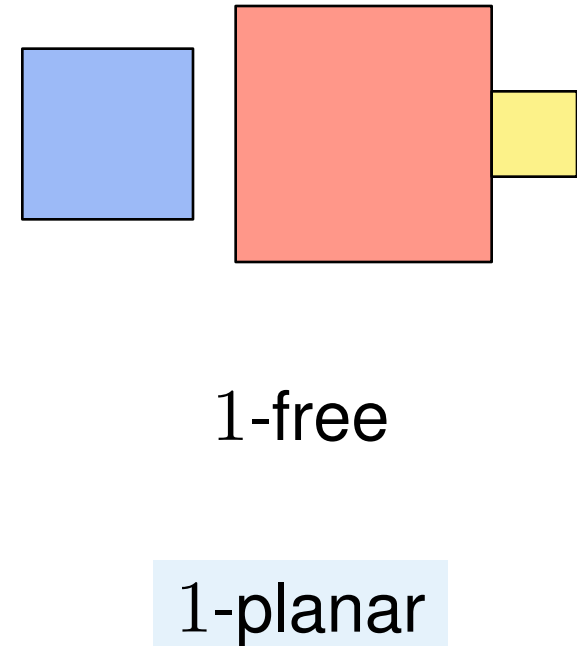
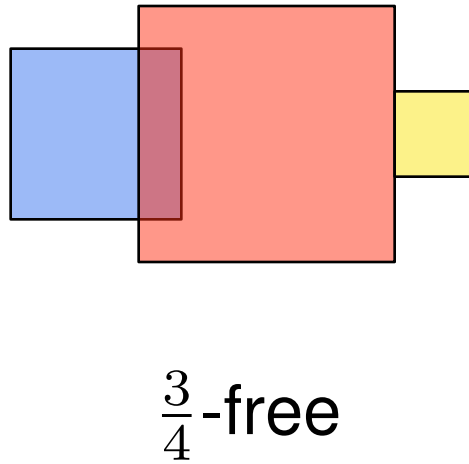
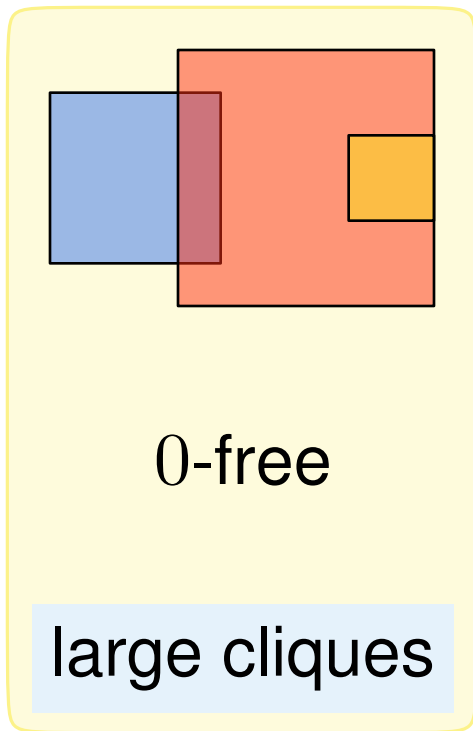
A set \mathcal{S} of shapes in the plane is α -free with $\alpha \in [0, 1]$ if every shape $S \in \mathcal{S}$ has at least α of its area disjoint from all other shapes.



no product structure

α -free Intersection Graphs

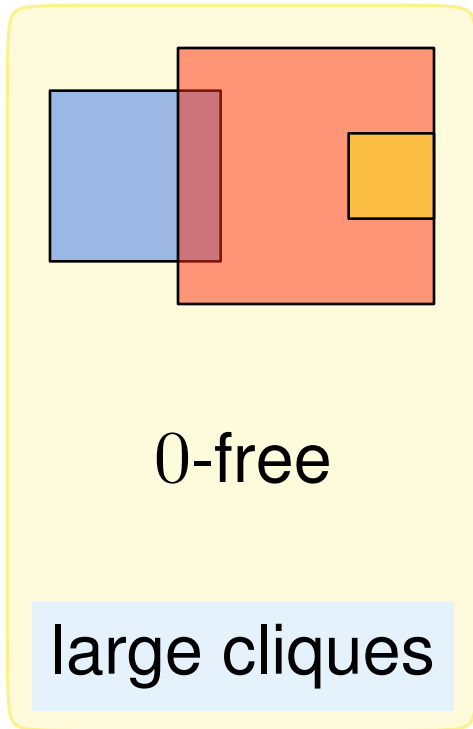
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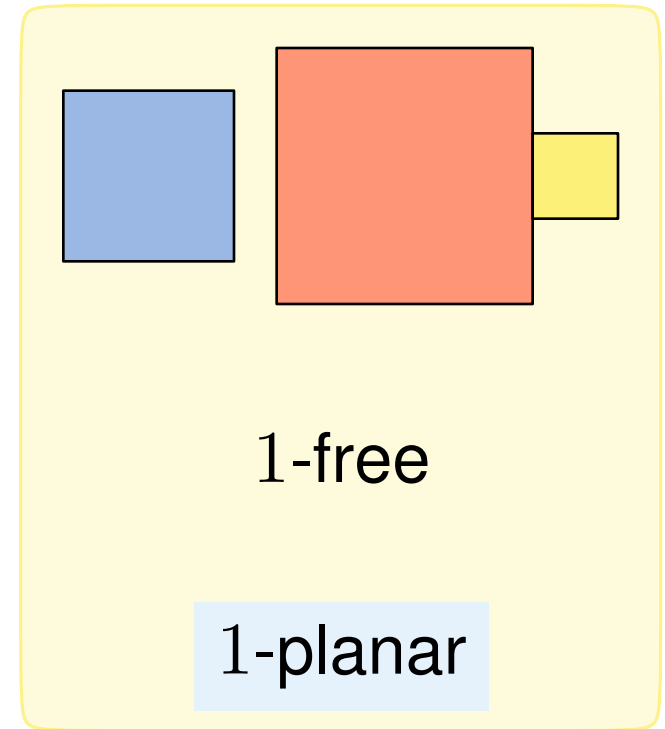
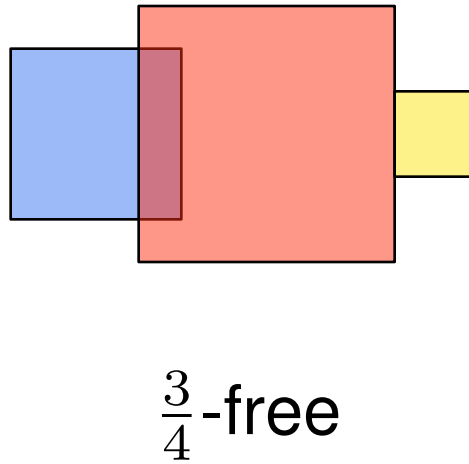
no product structure

α -free Intersection Graphs

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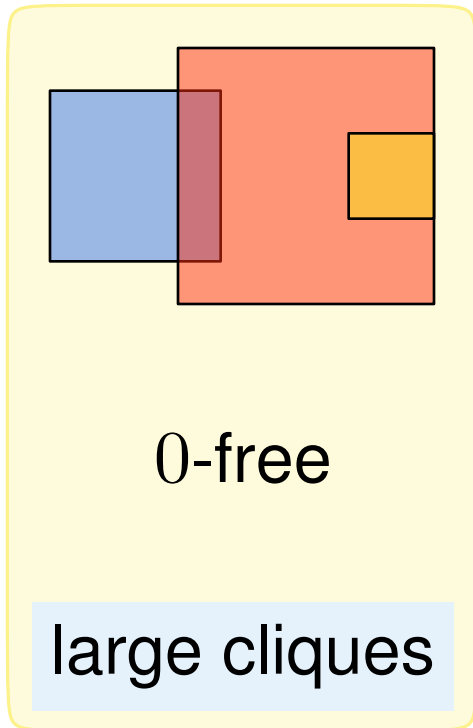
no product structure



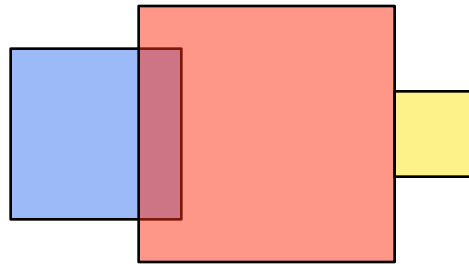
product structure

α -free Intersection Graphs

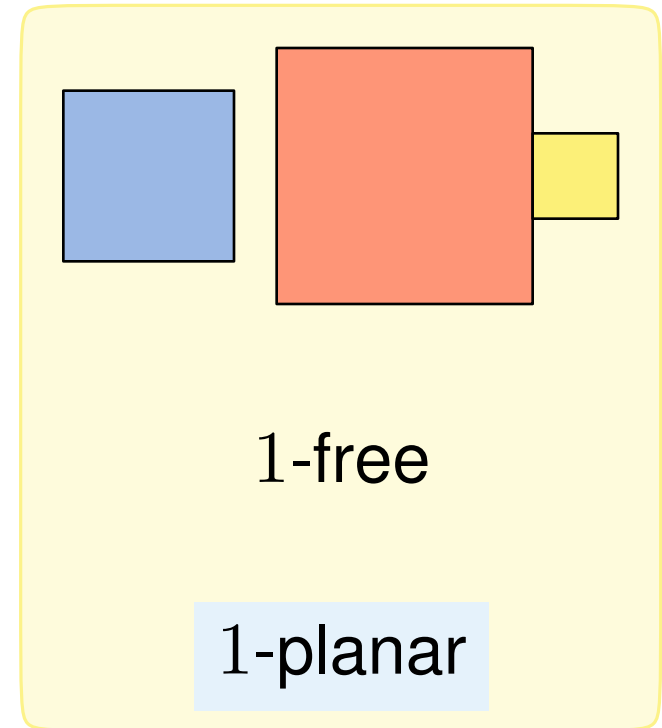
A set \mathcal{S} of shapes in the plane is α -free with $\alpha \in [0, 1]$ if every shape $S \in \mathcal{S}$ has at least α of its area disjoint from all other shapes.



no product structure



What about $\alpha \in (0, 1)$?



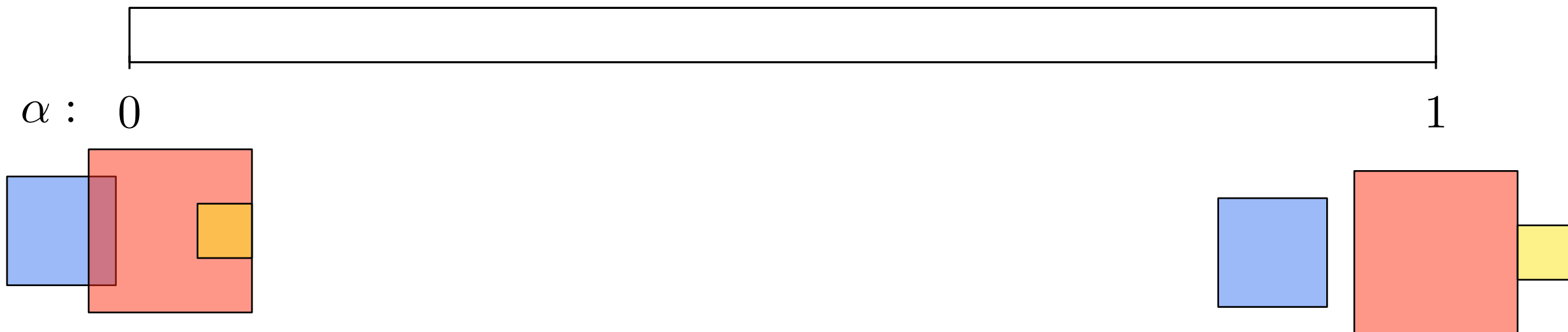
product structure

Our Results for Regular n -gons

■ For  and  : **no** product structure for $\alpha < 1$.

Our Results for Regular n -gons

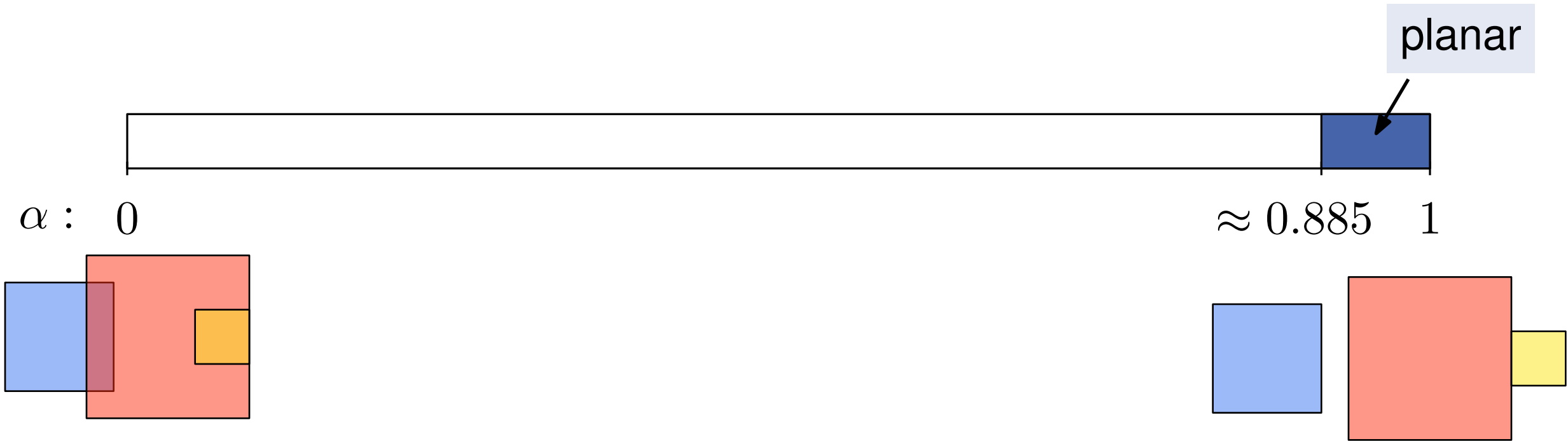
- For  and  : **no** product structure for $\alpha < 1$.
- For even $n \geq 6$:



Our Results for Regular n -gons

■ For  and  : **no** product structure for $\alpha < 1$.

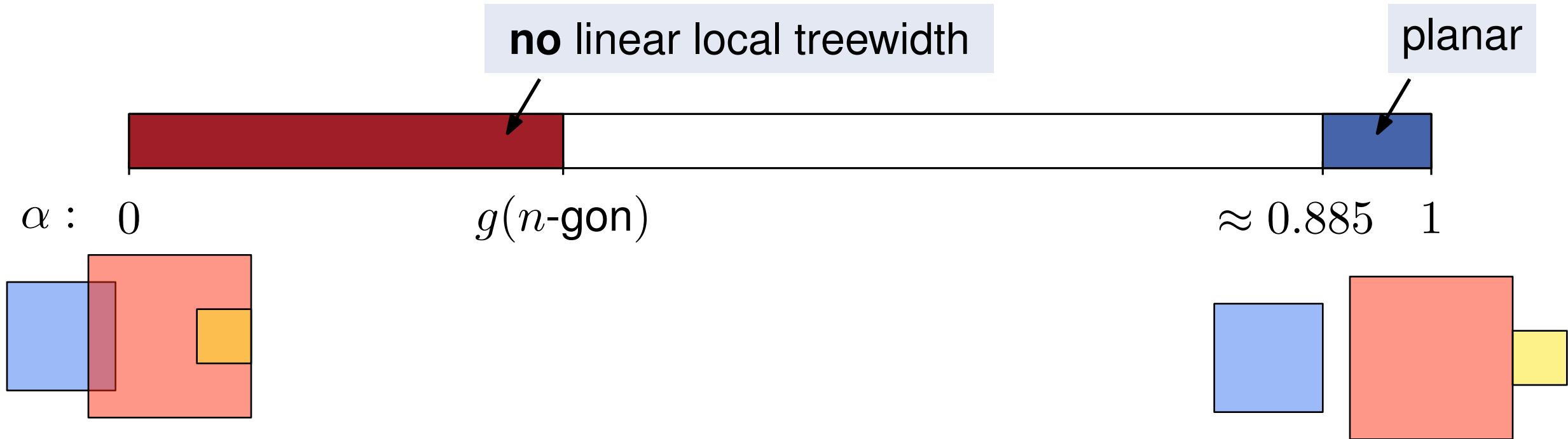
■ For even $n \geq 6$:



Our Results for Regular n -gons

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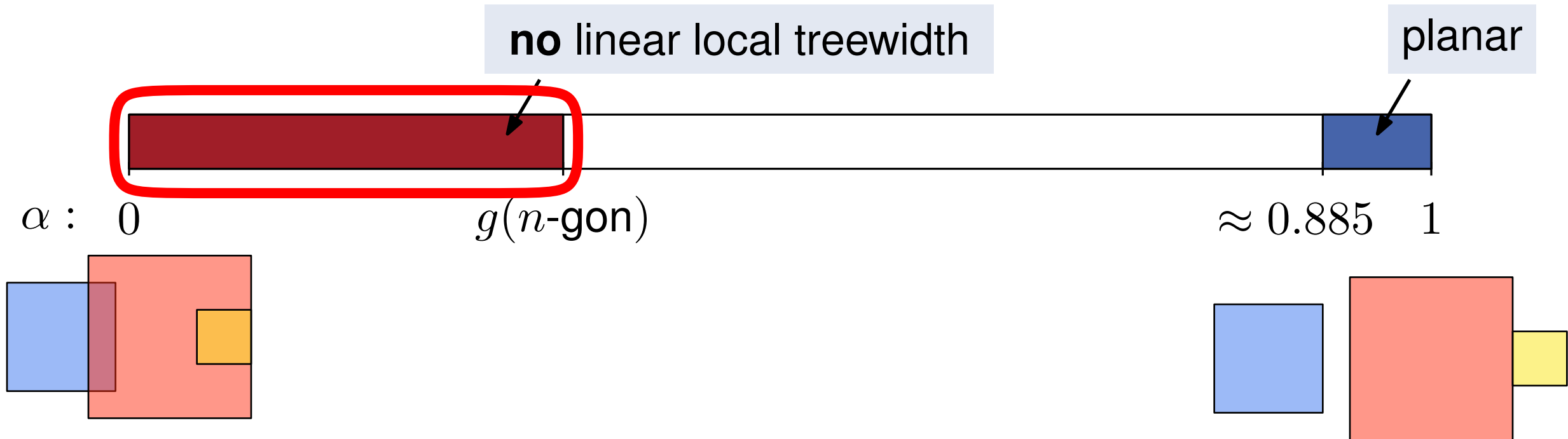
■ For even $n \geq 6$:



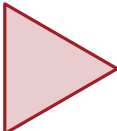
Our Results for Regular n -gons

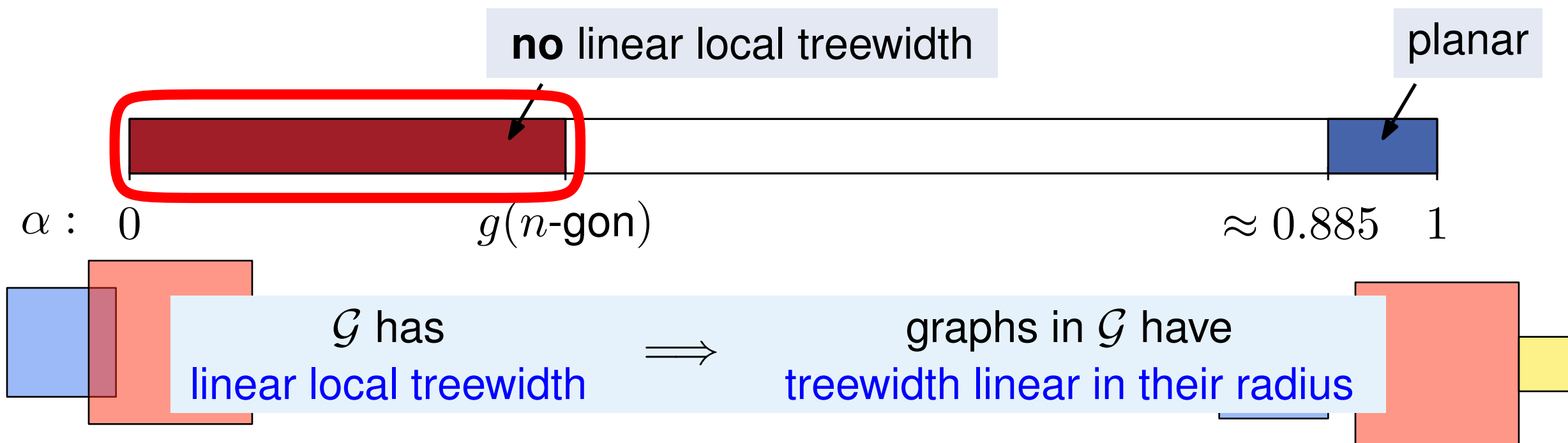
■ For  and  : **no** product structure for $\alpha < 1$.

■ For even $n \geq 6$:

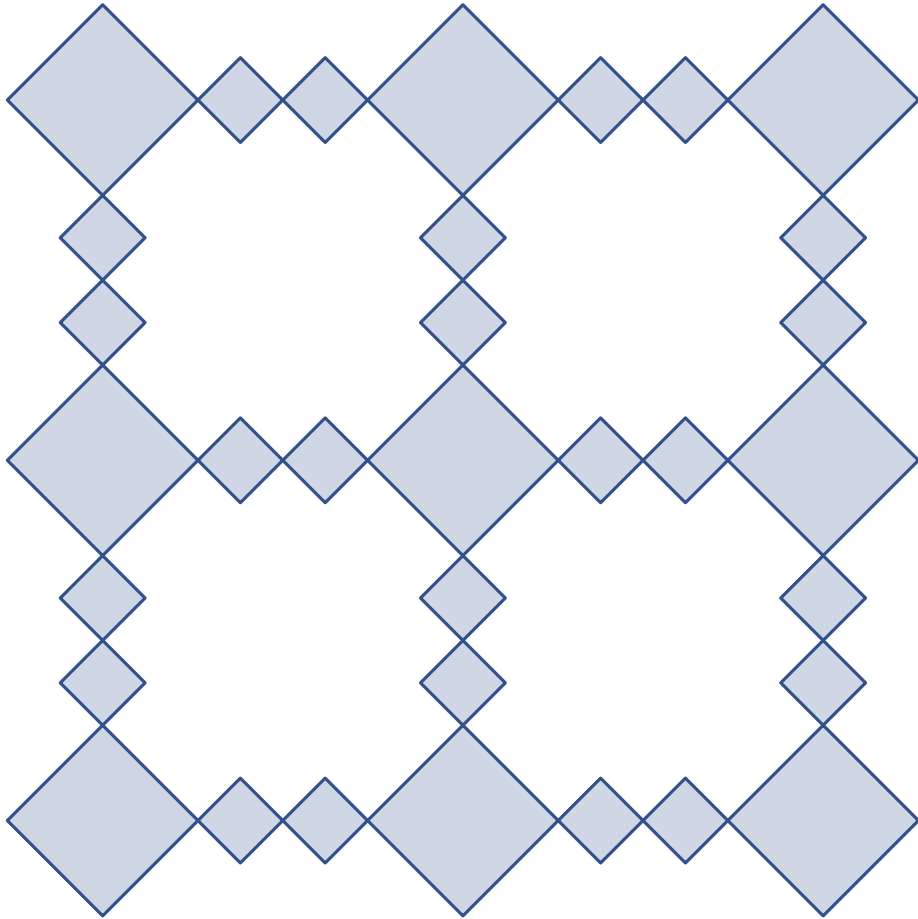


Our Results for Regular n -gons

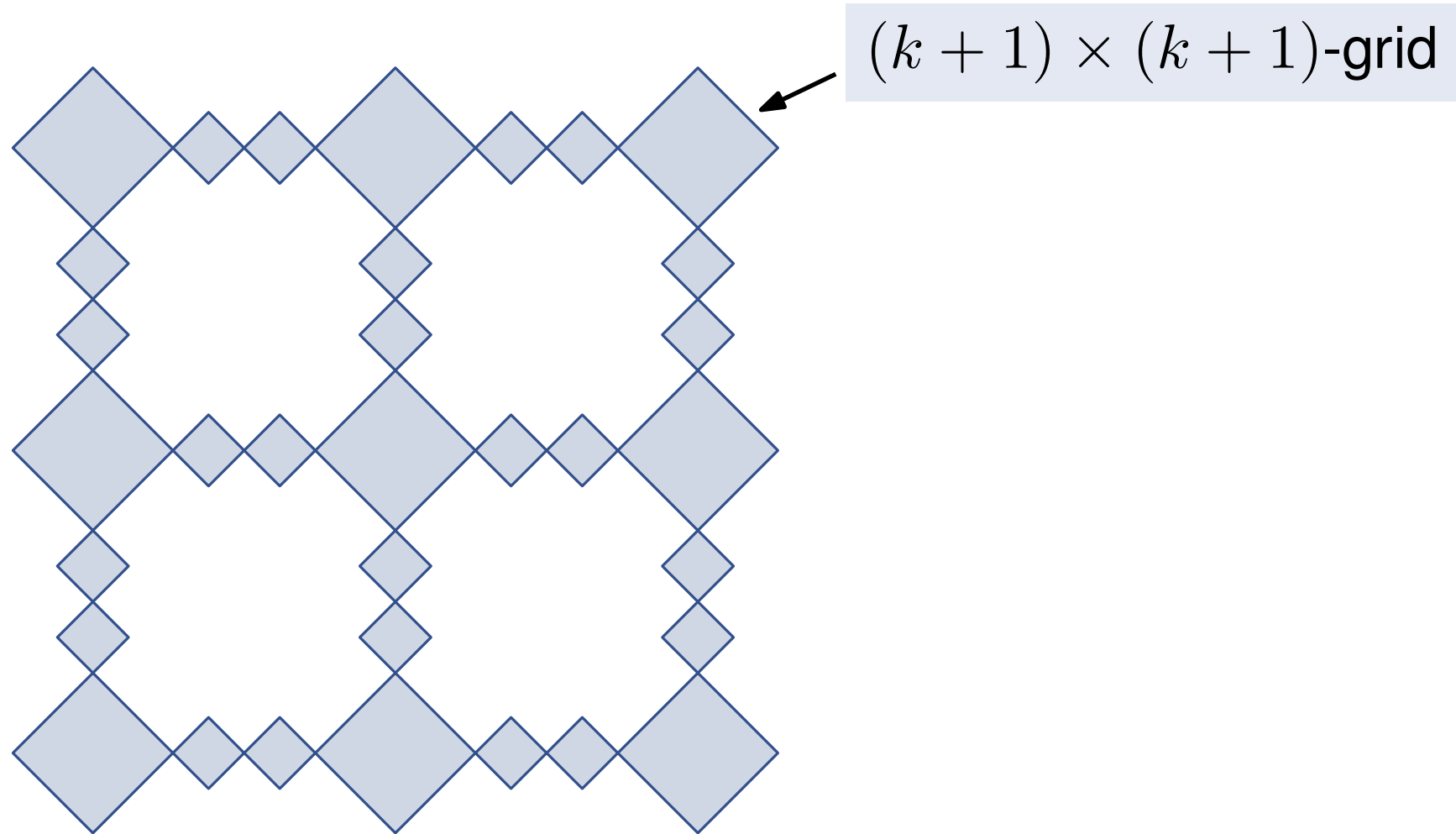
- For  and  : **no** product structure for $\alpha < 1$.
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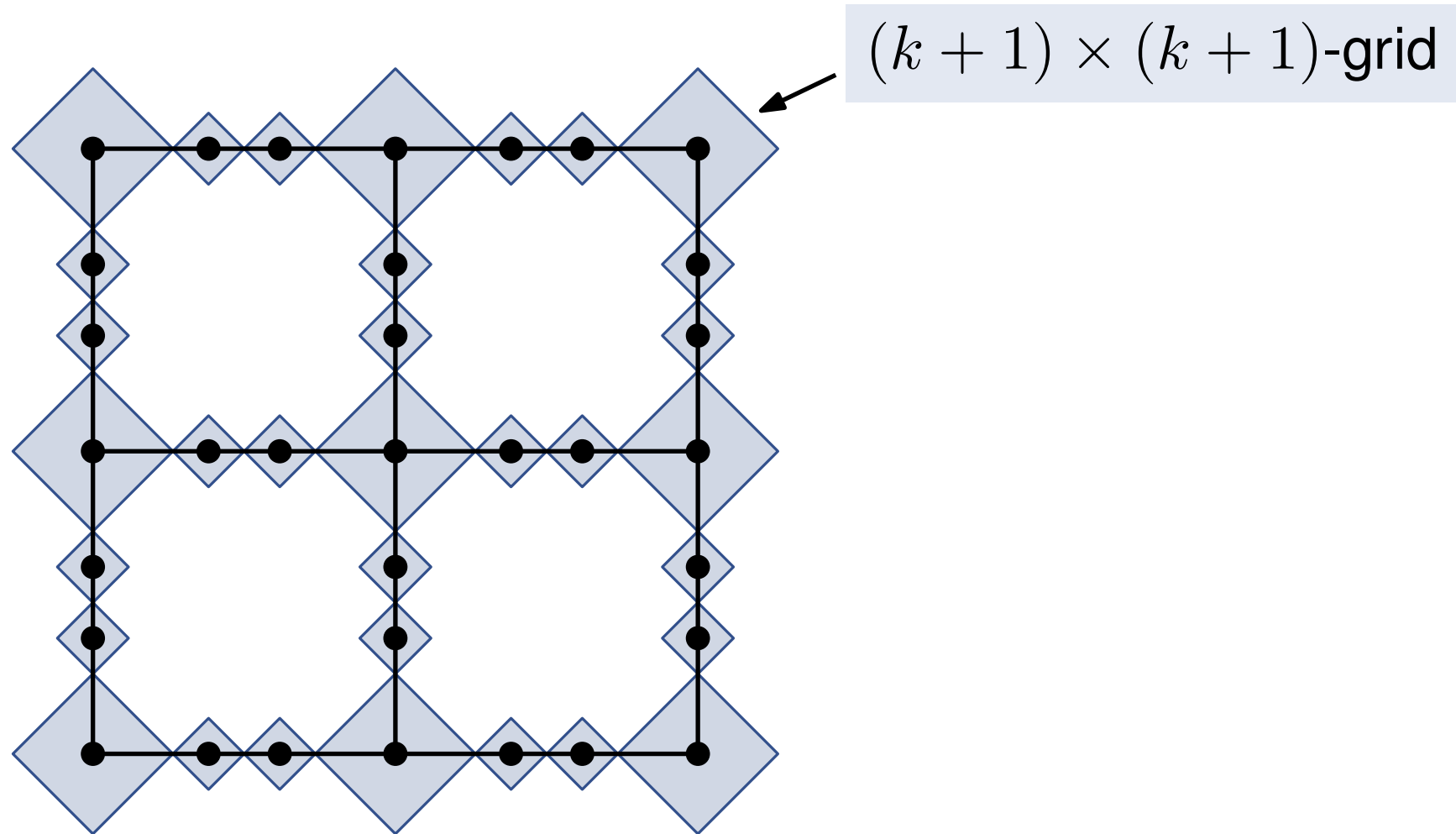
Constructing Graphs with Radius $\mathcal{O}(k)$ and Treewidth $\Omega(k^2)$



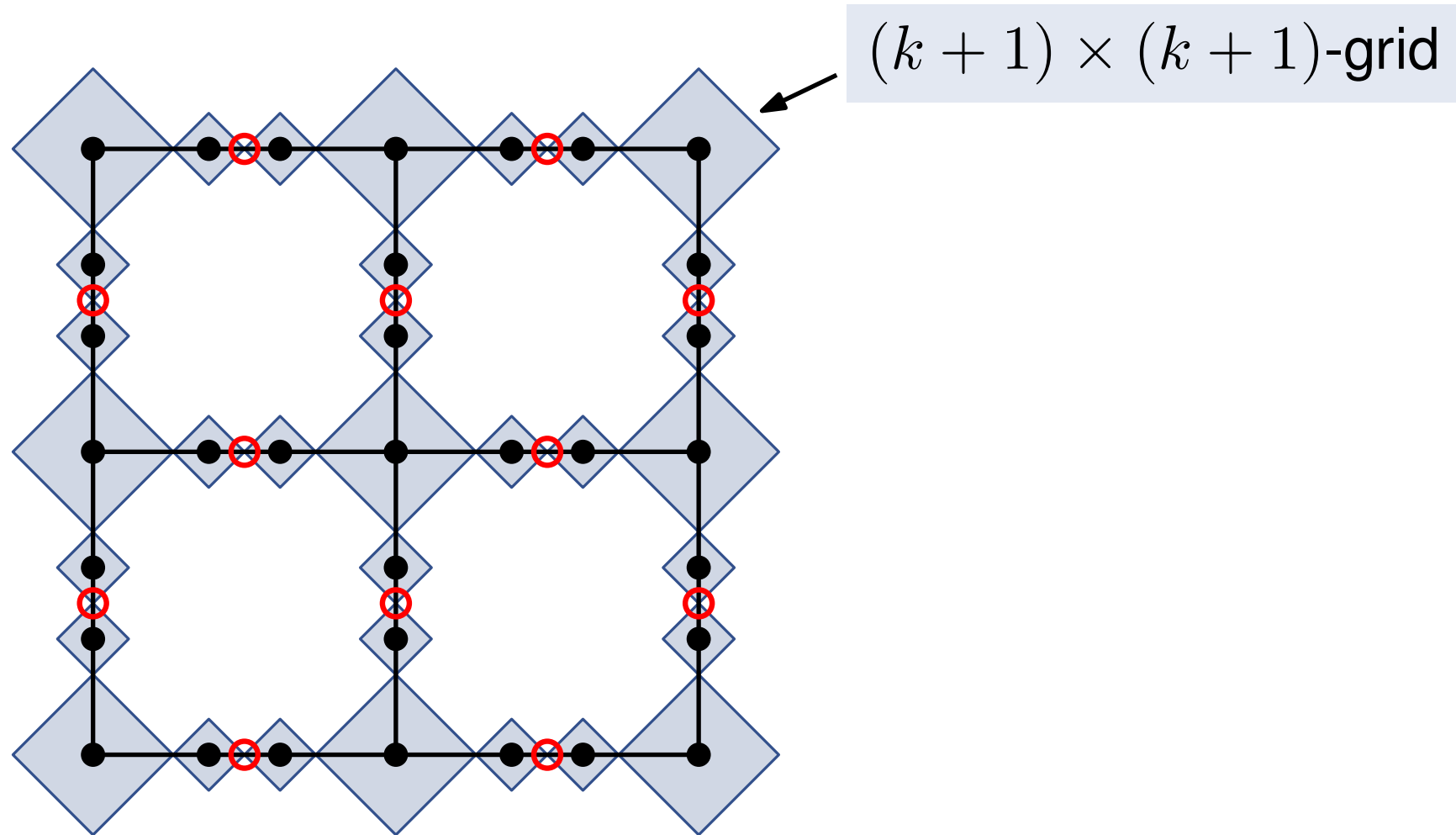
Constructing Graphs with Radius $\mathcal{O}(k)$ and Treewidth $\Omega(k^2)$



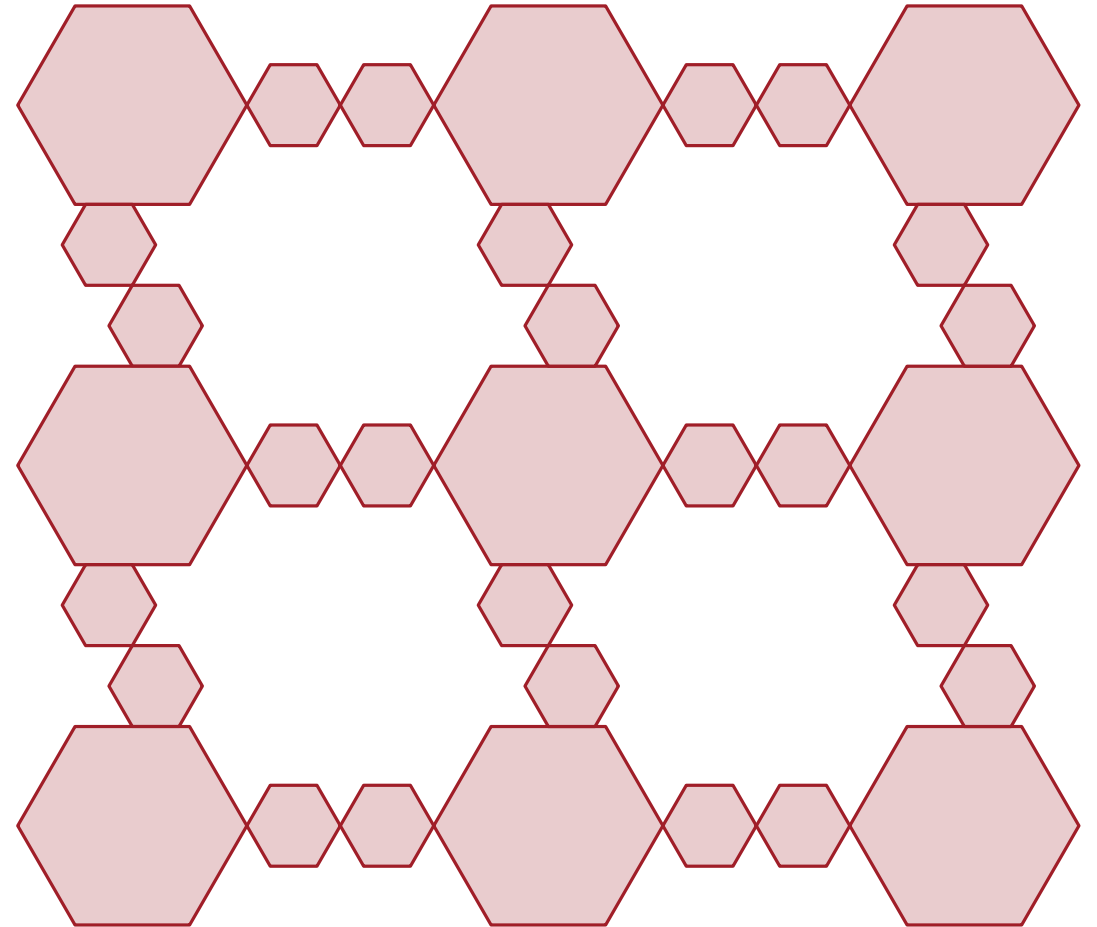
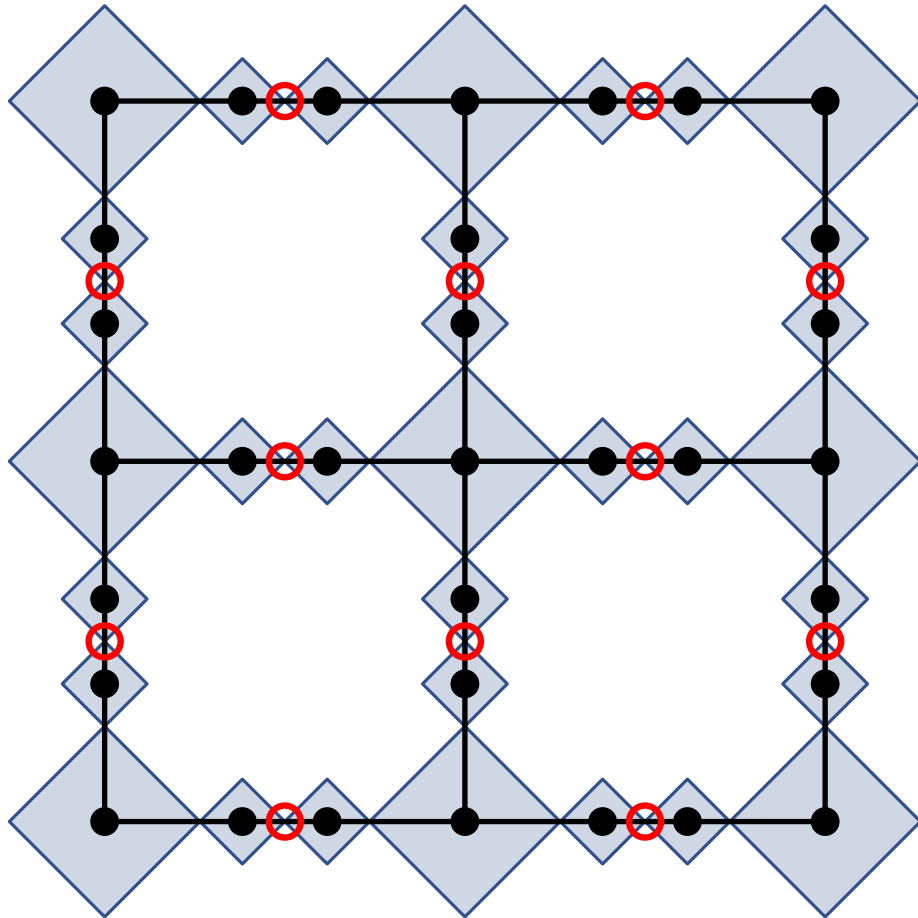
Constructing Graphs with Radius $\mathcal{O}(k)$ and Treewidth $\Omega(k^2)$



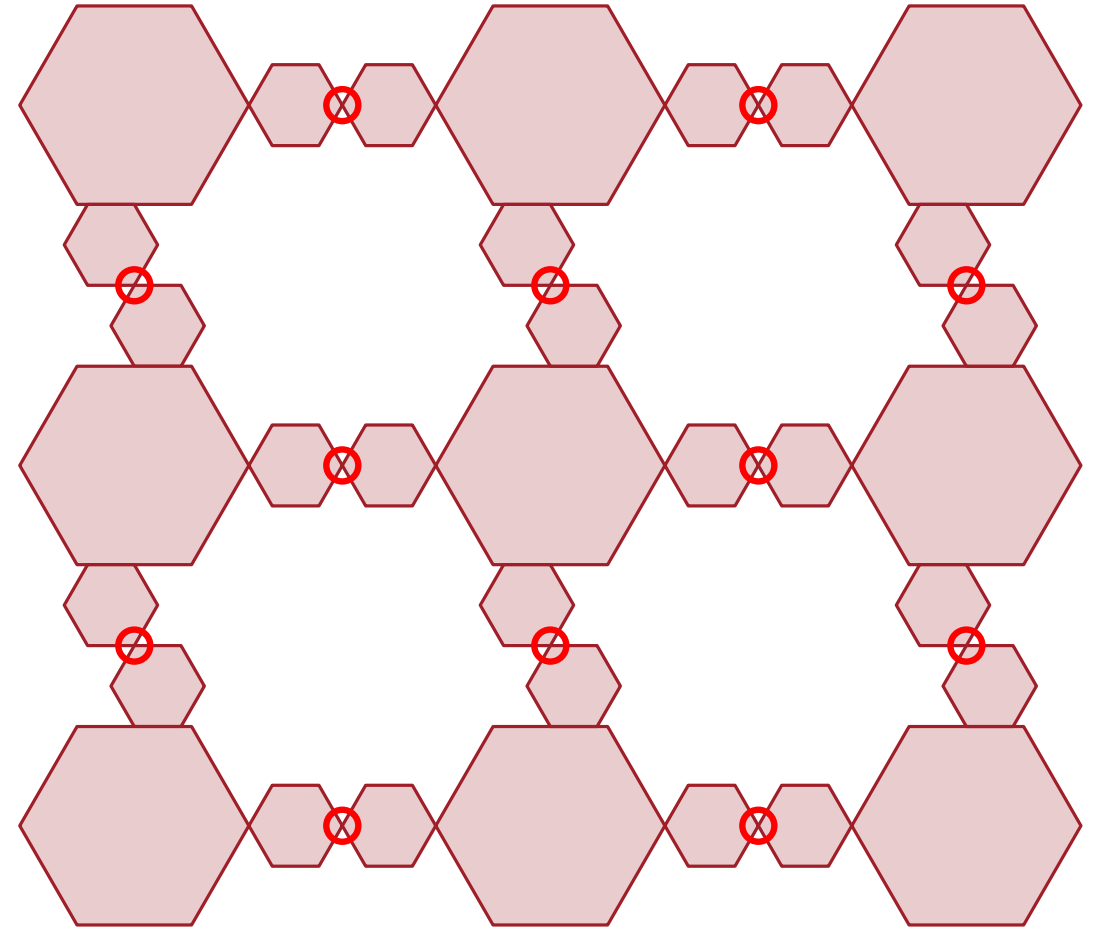
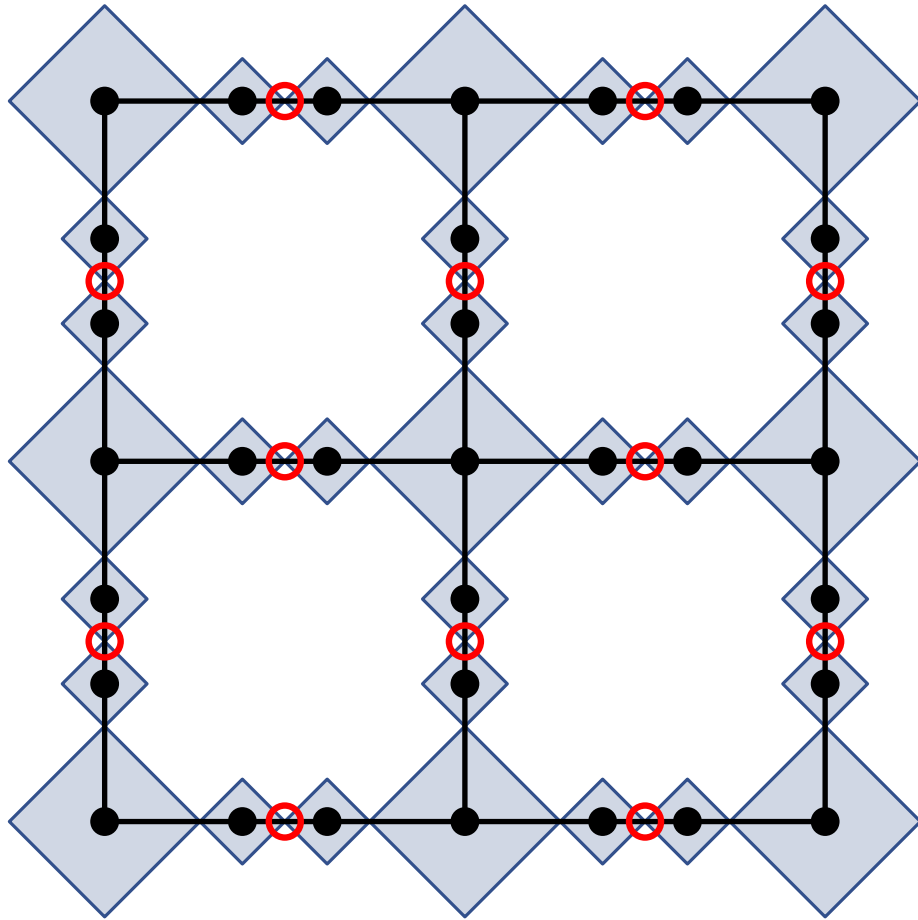
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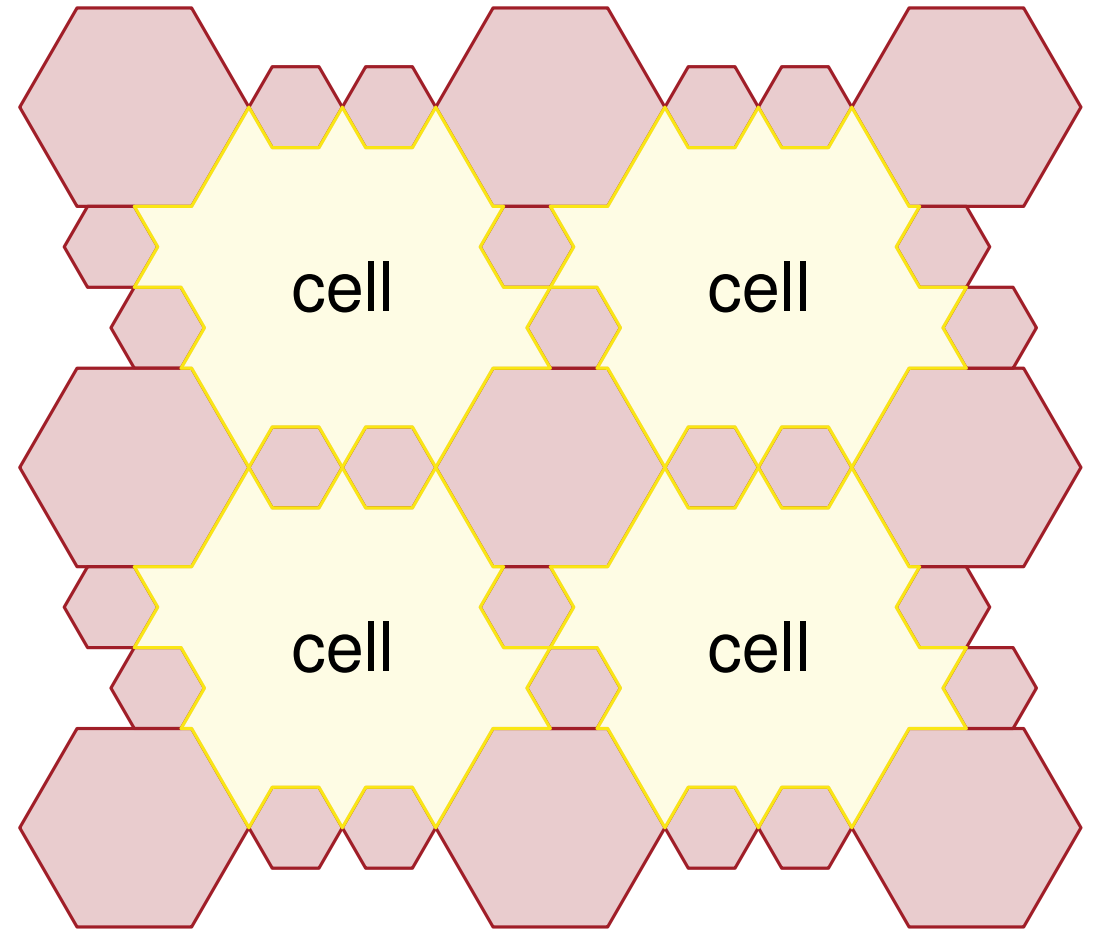
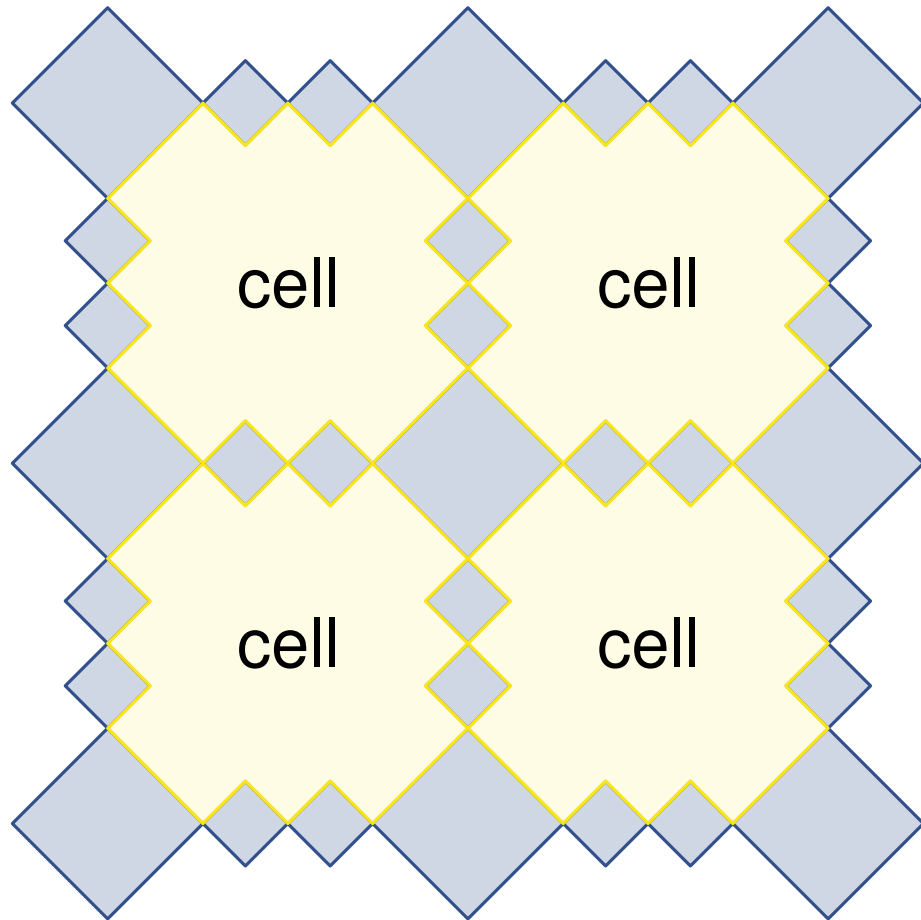
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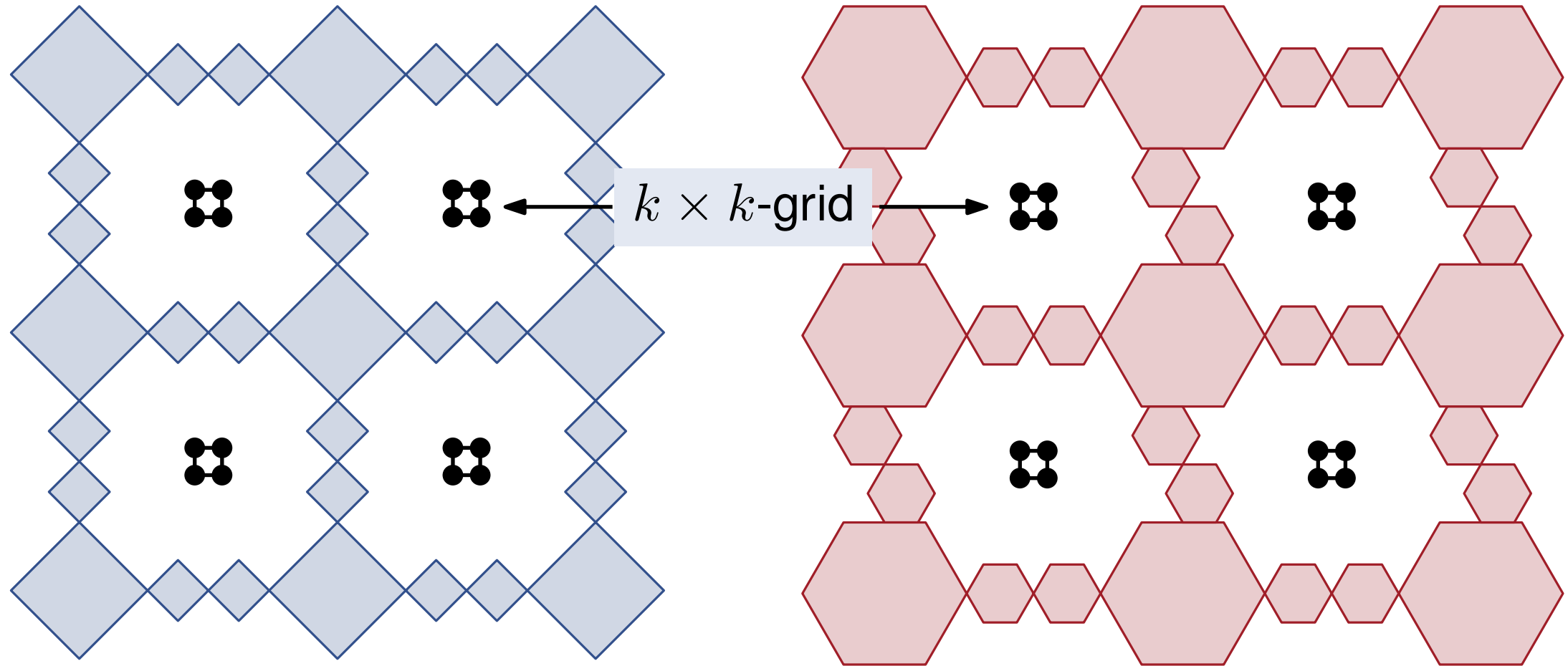
Constructing Graphs with Radius $\mathcal{O}(k)$ and Treewidth $\Omega(k^2)$



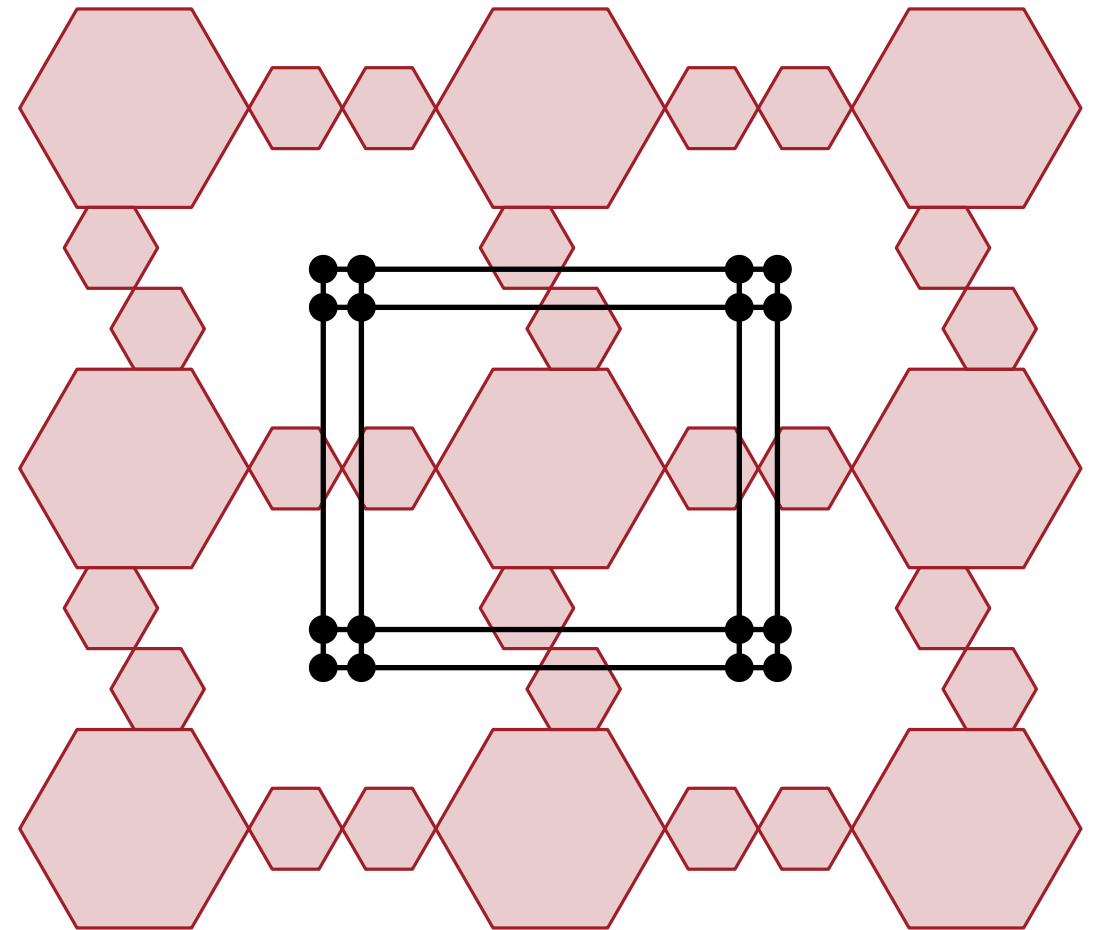
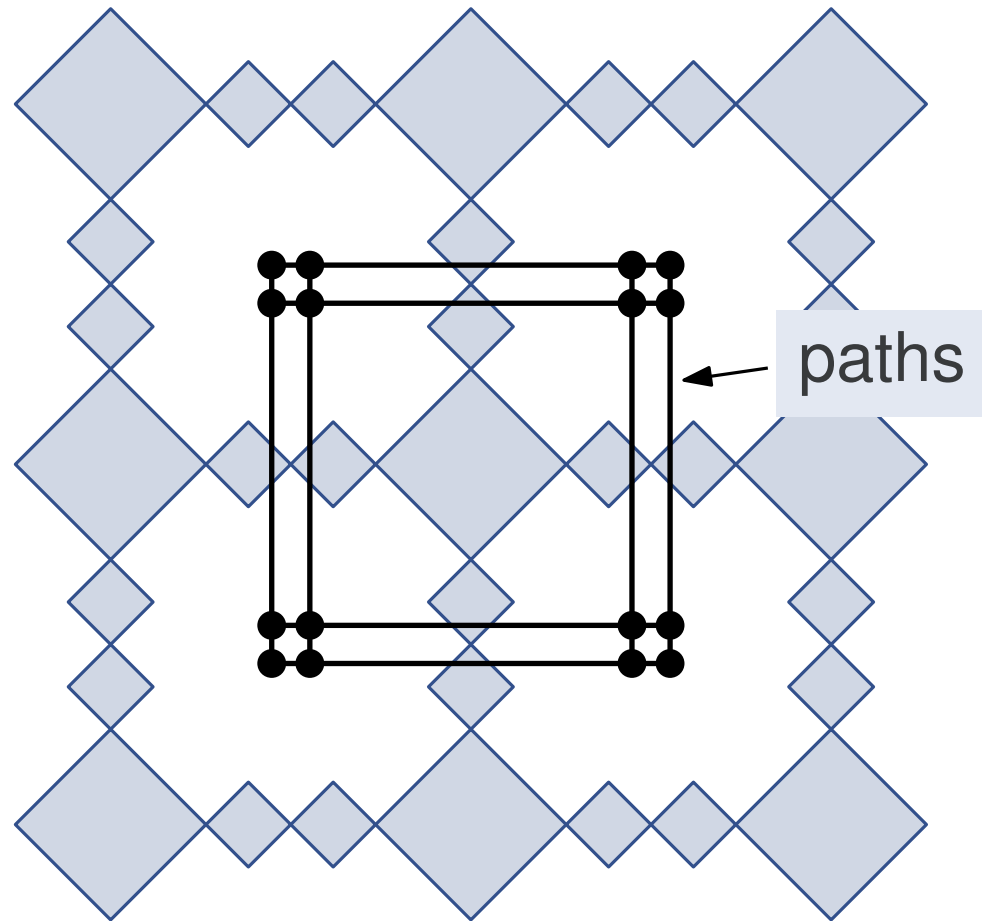
Constructing Graphs with Radius $\mathcal{O}(k)$ and Treewidth $\Omega(k^2)$



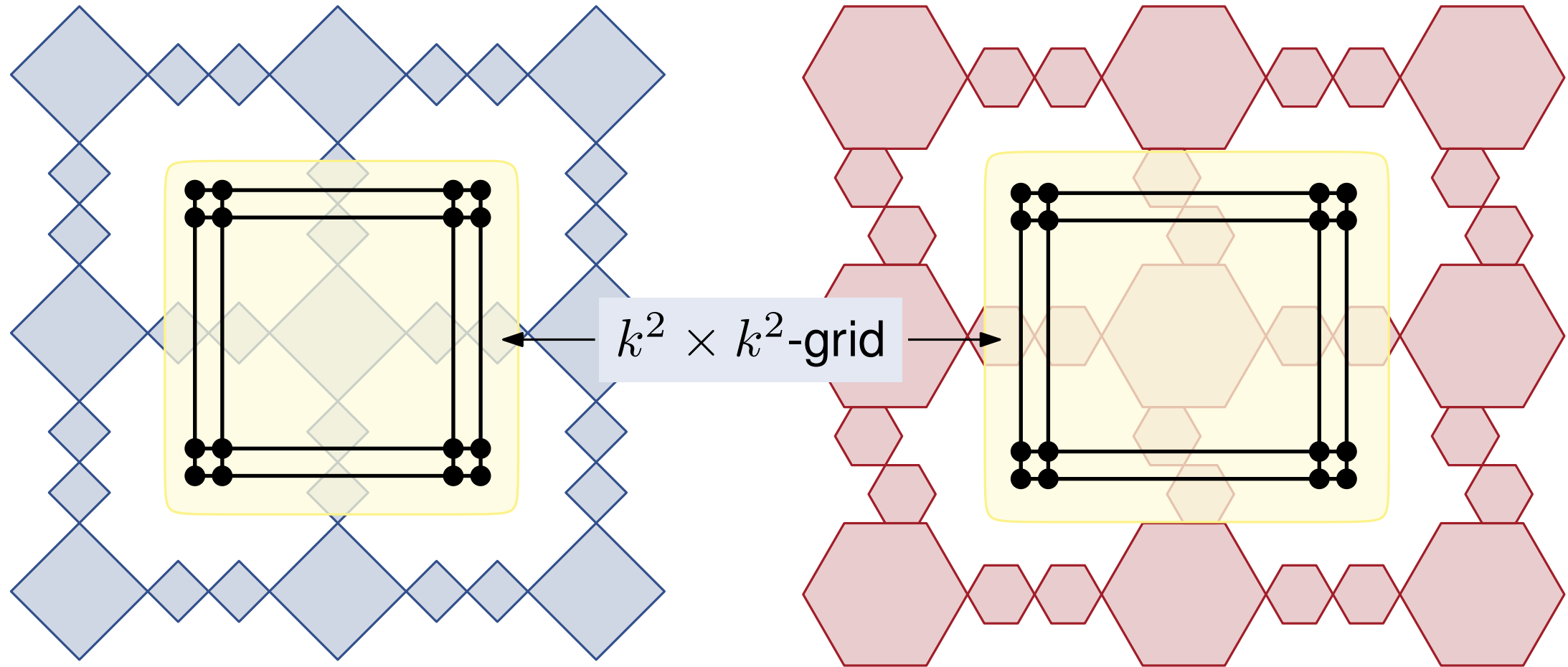
Constructing Graphs with Radius $\mathcal{O}(k)$ and Treewidth $\Omega(k^2)$



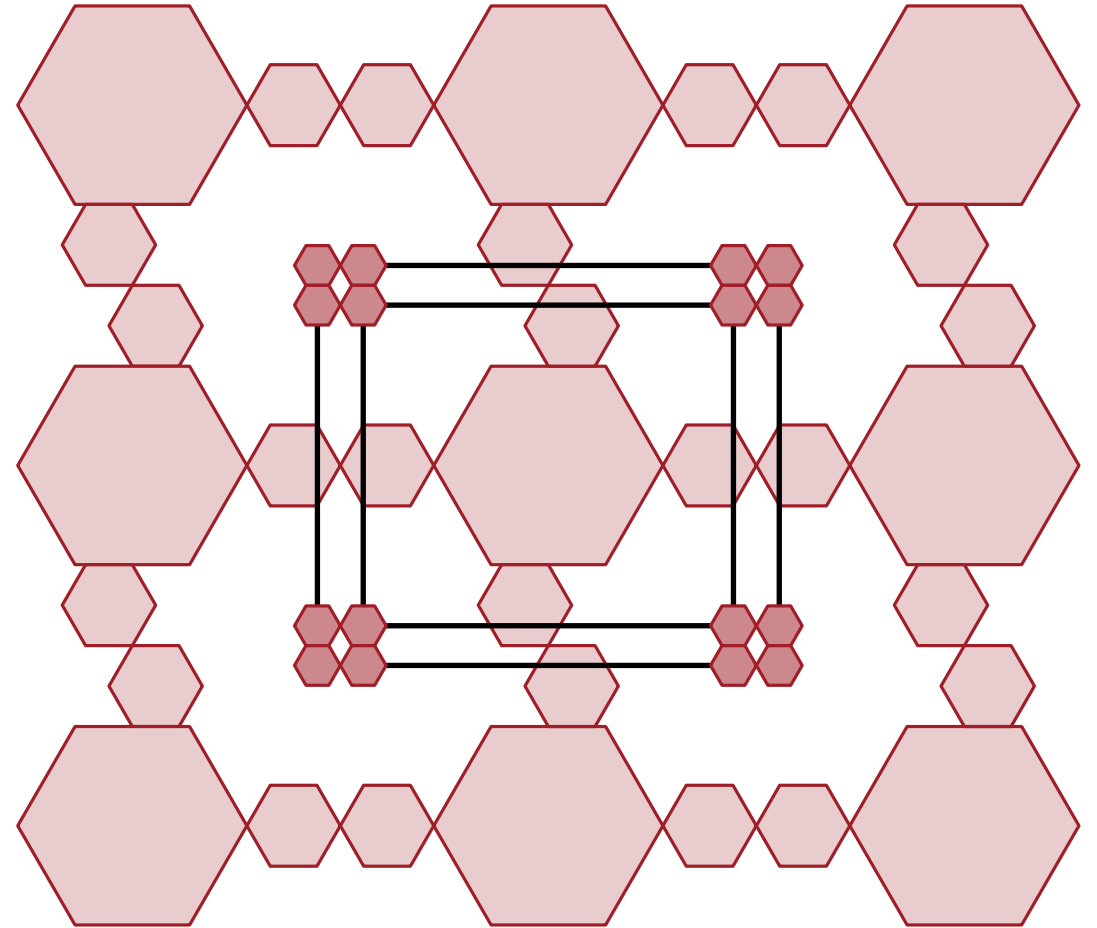
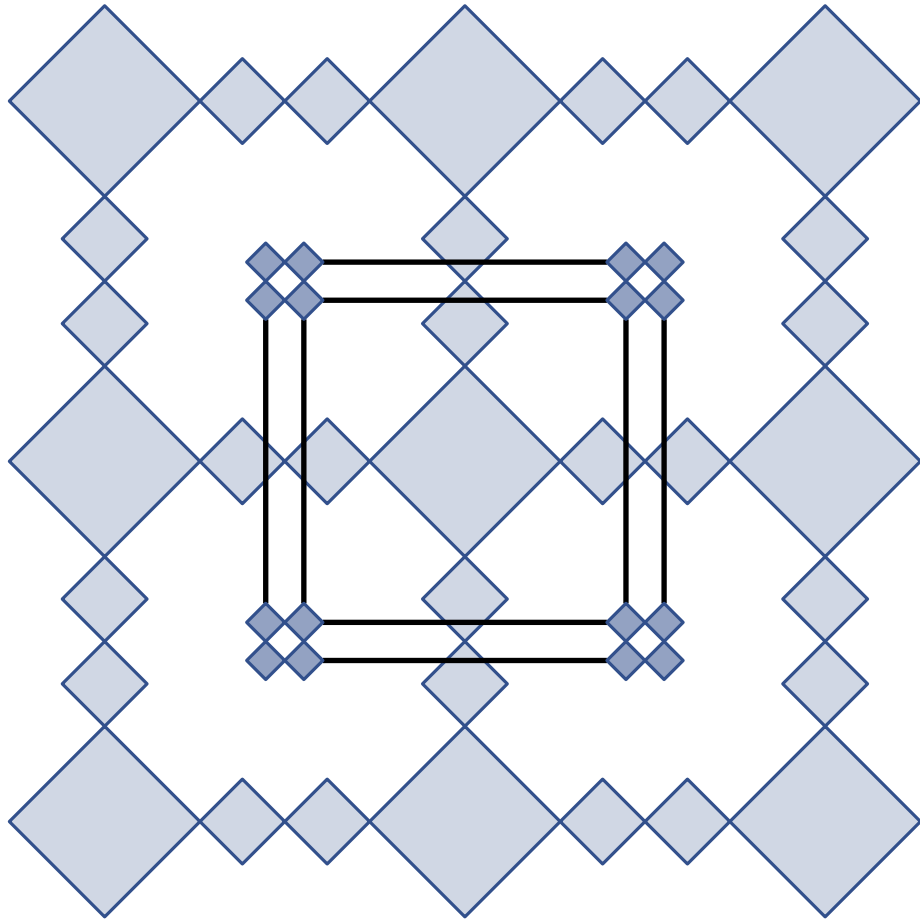
Constructing Graphs with Radius $\mathcal{O}(k)$ and Treewidth $\Omega(k^2)$



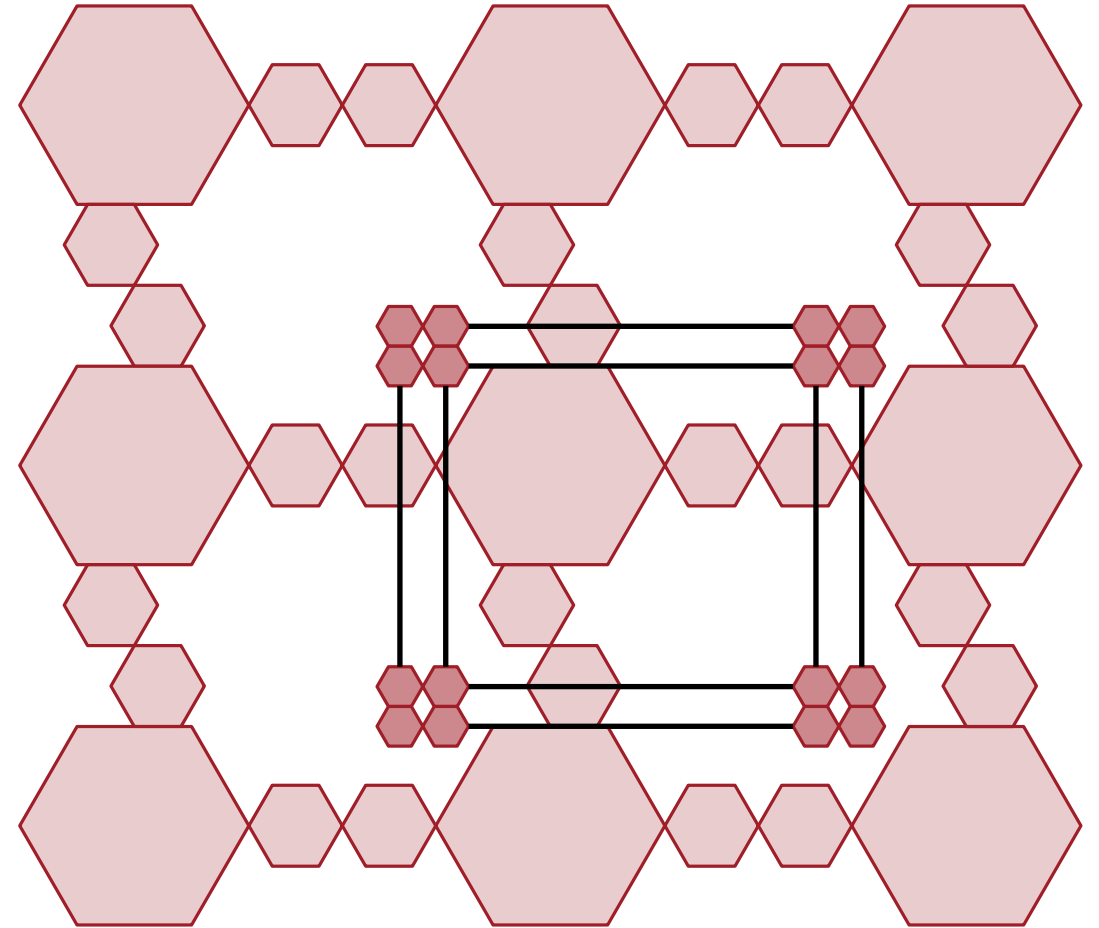
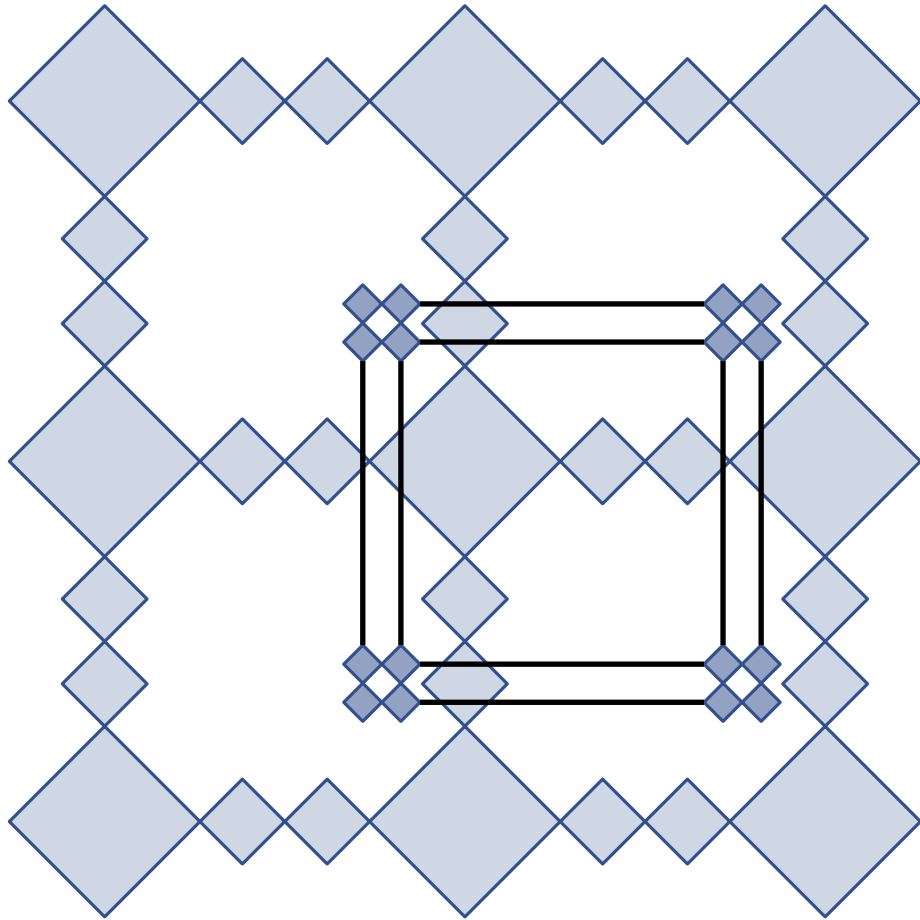
Constructing Graphs with Radius $\mathcal{O}(k)$ and Treewidth $\Omega(k^2)$



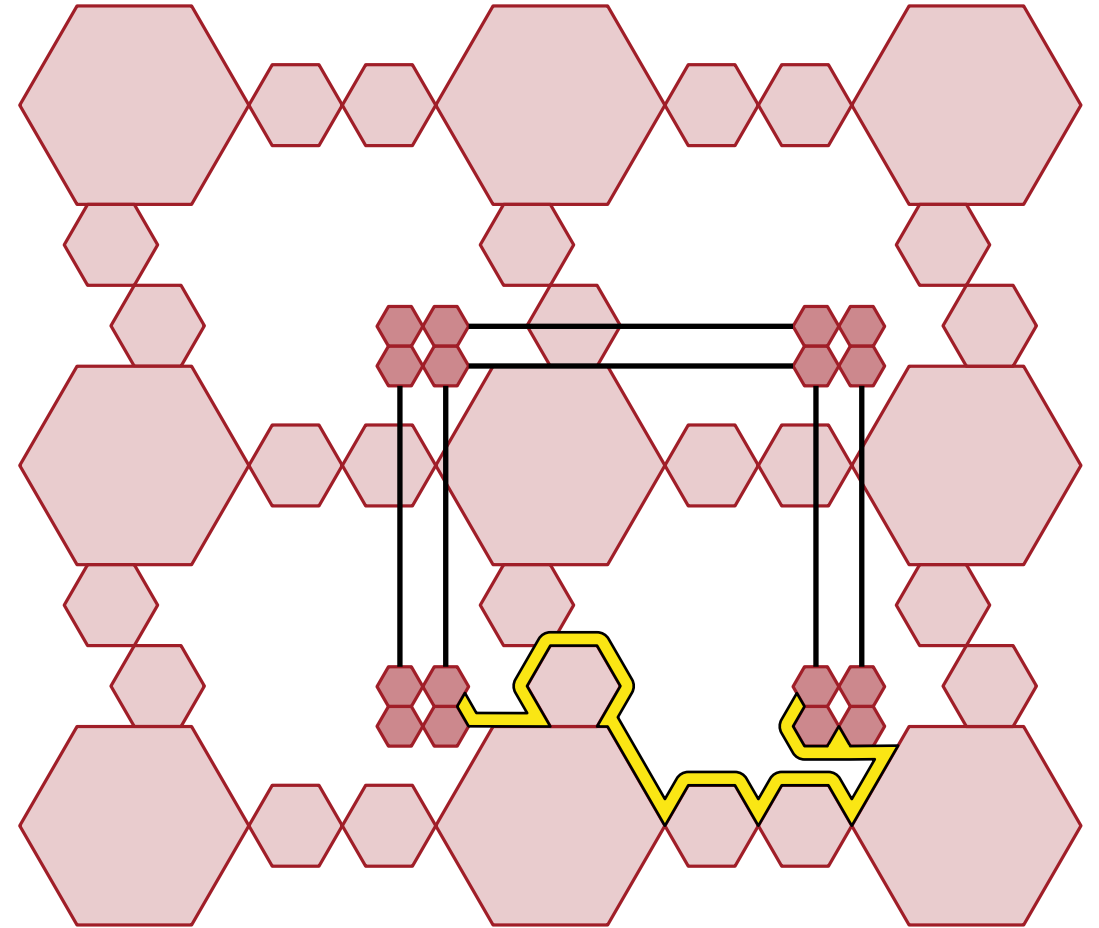
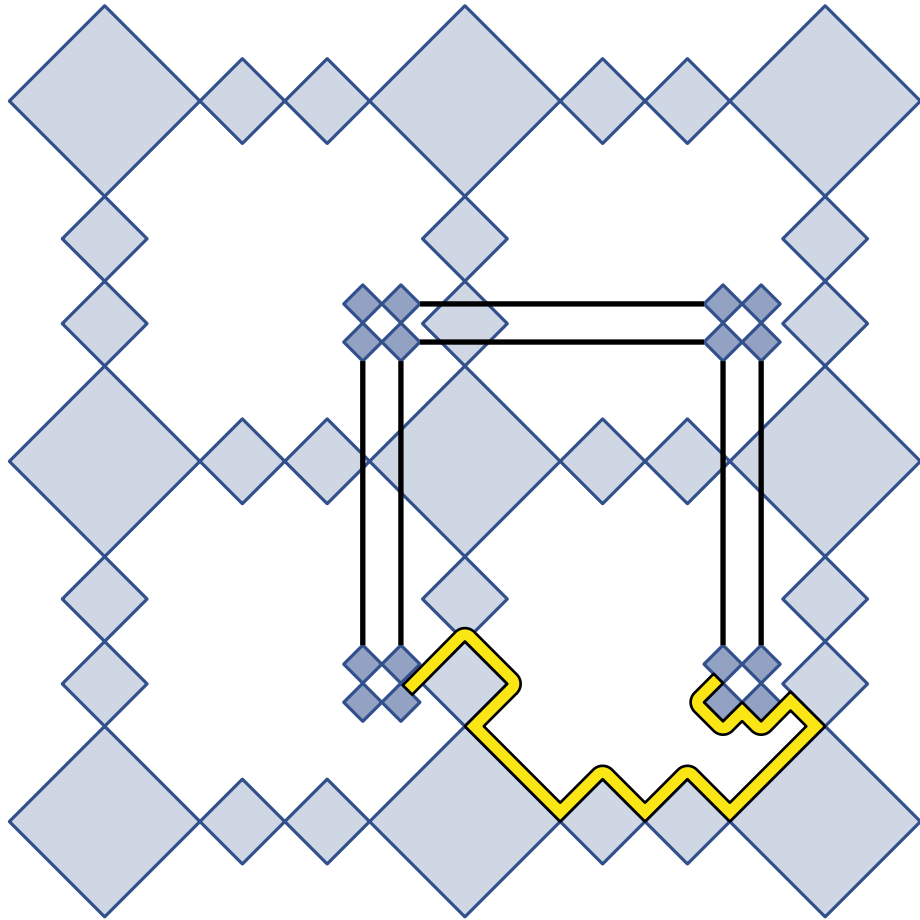
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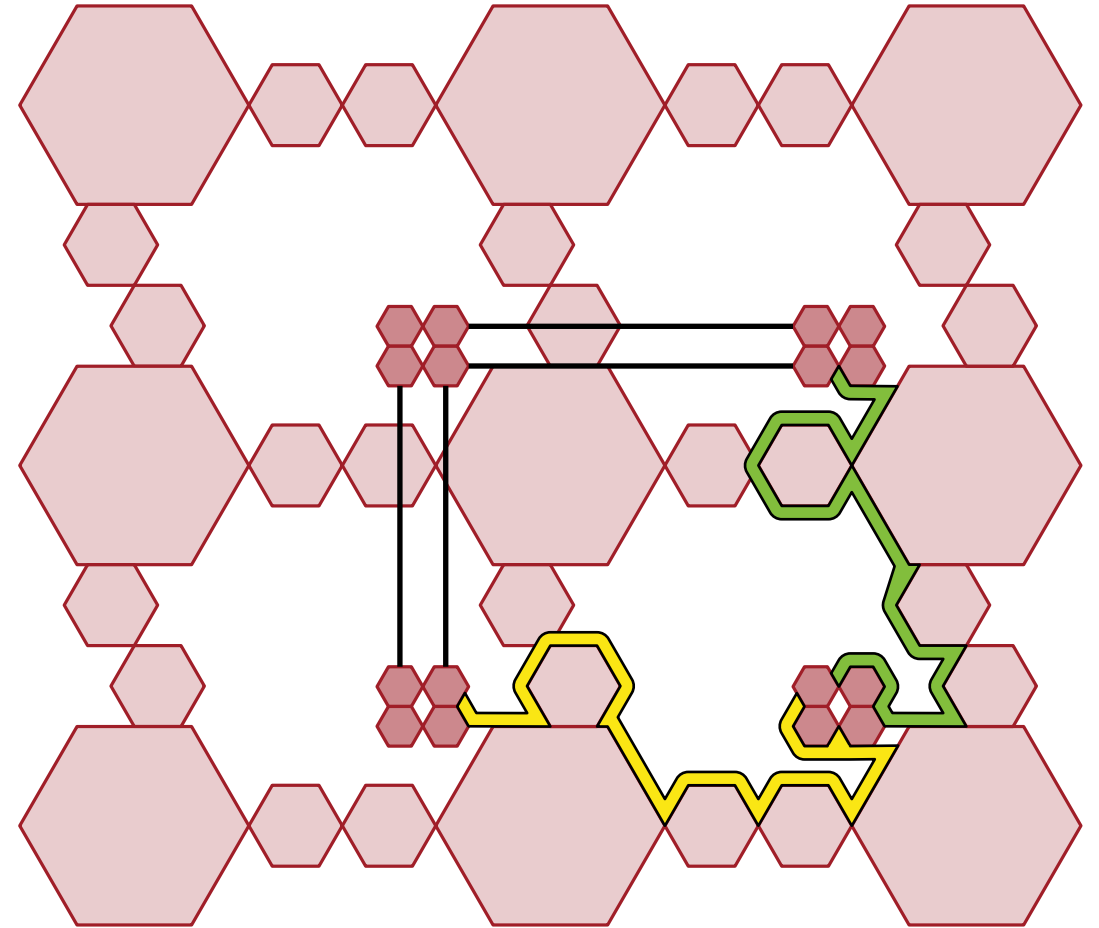
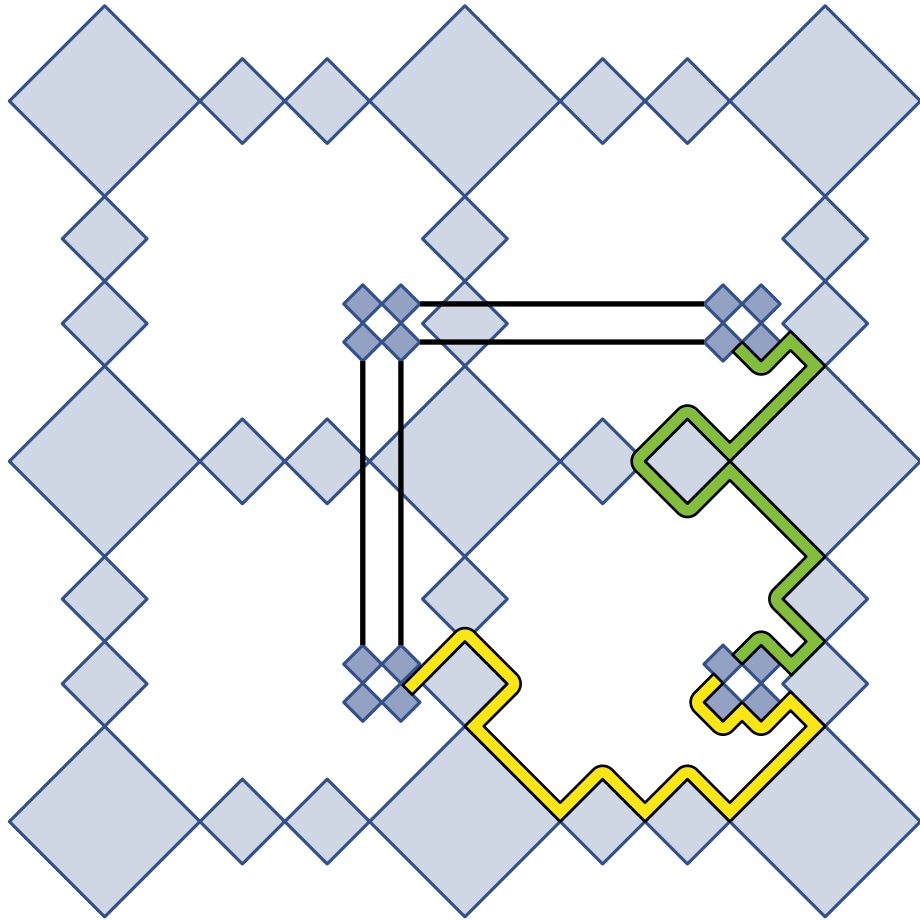
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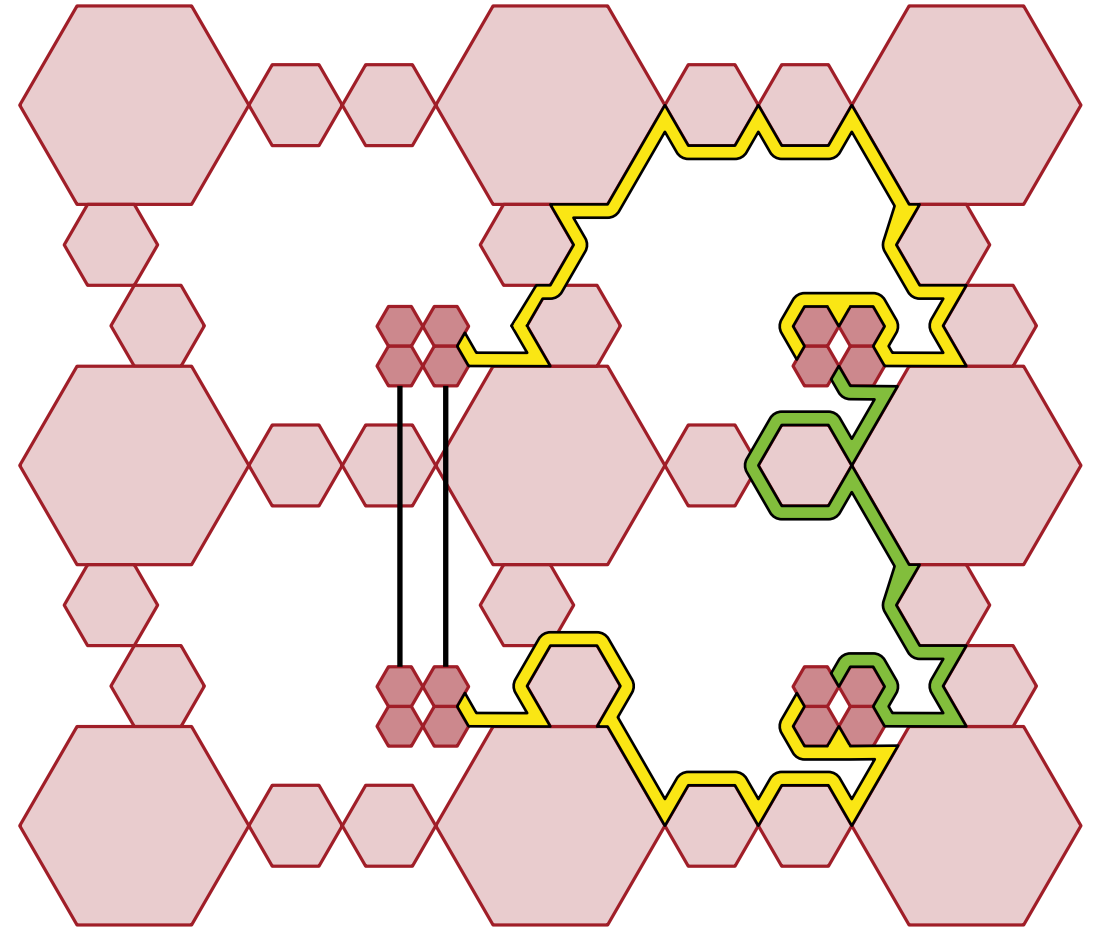
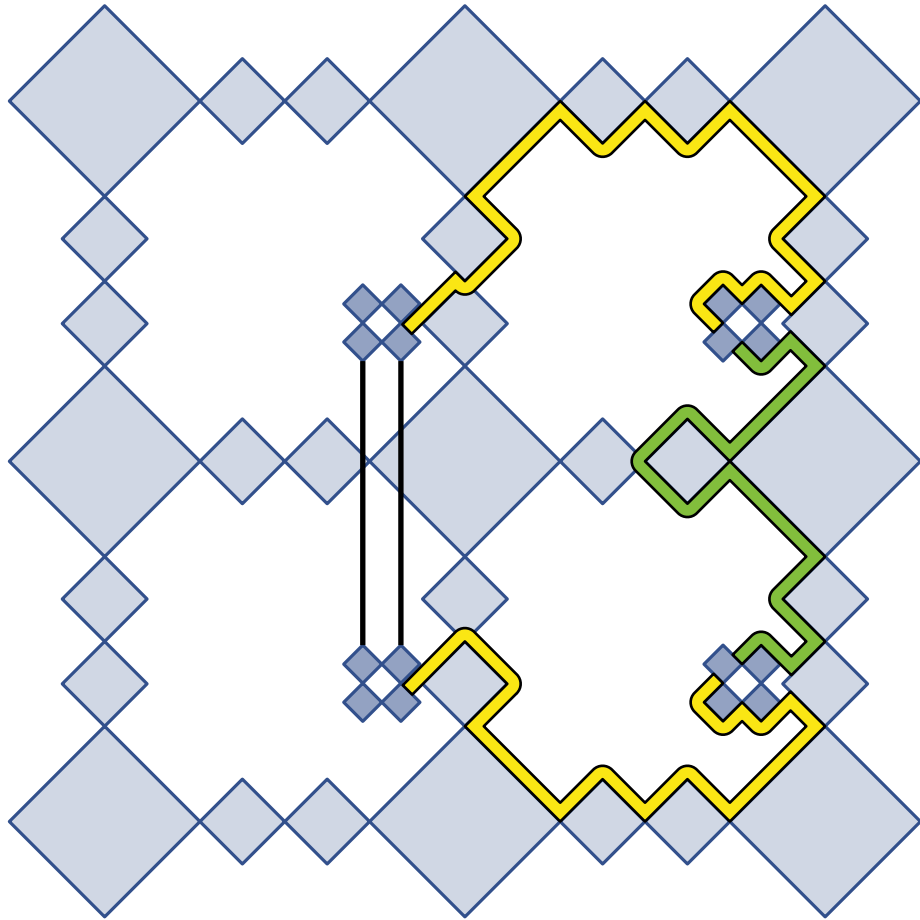
Constructing Graphs with Radius $\mathcal{O}(k)$ and Treewidth $\Omega(k^2)$



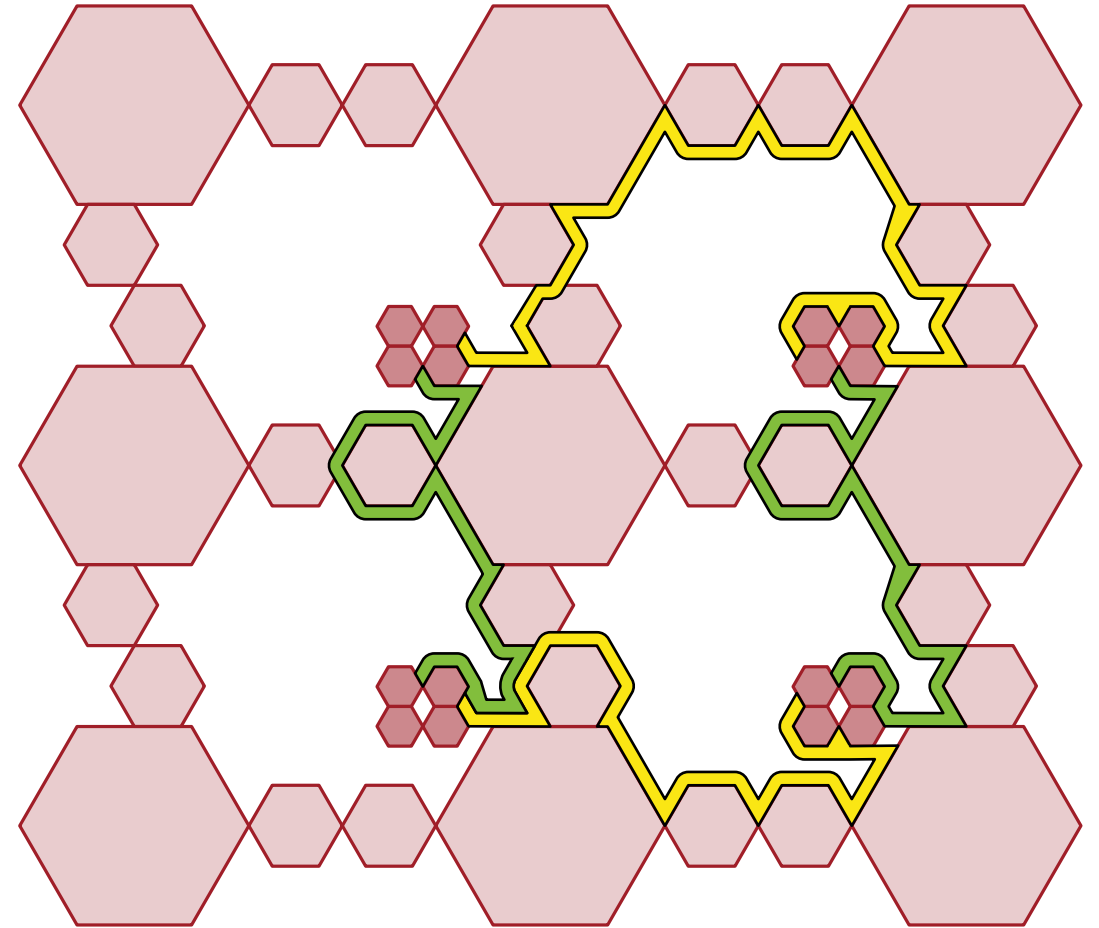
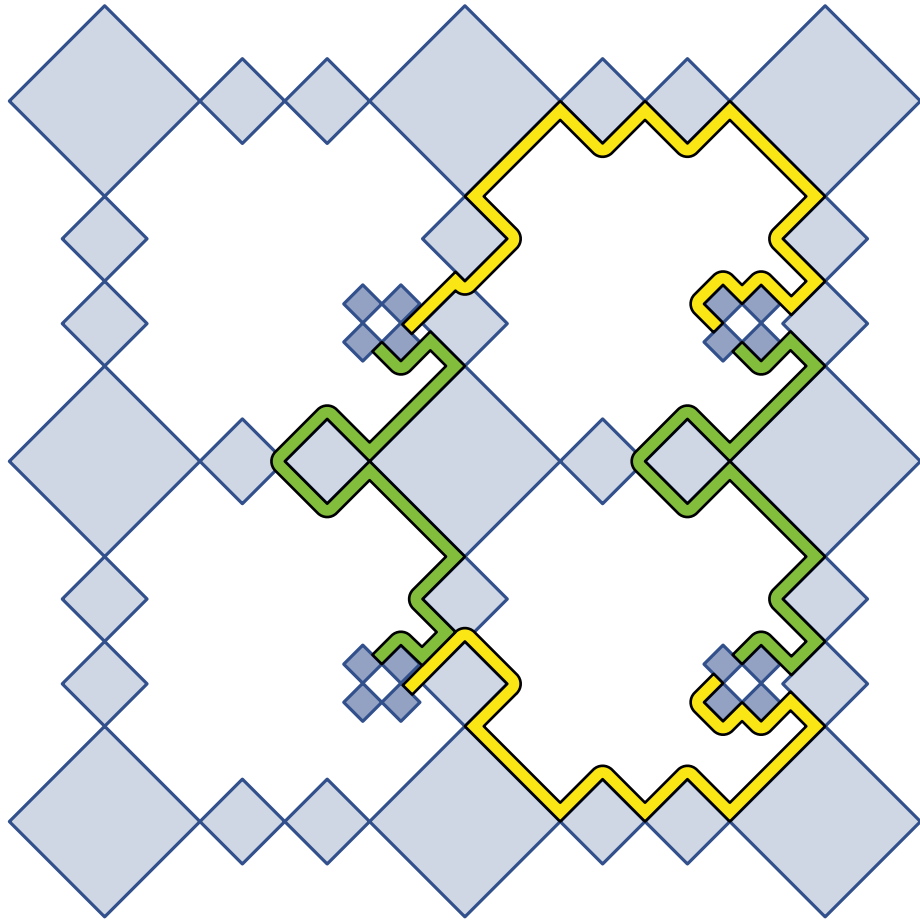
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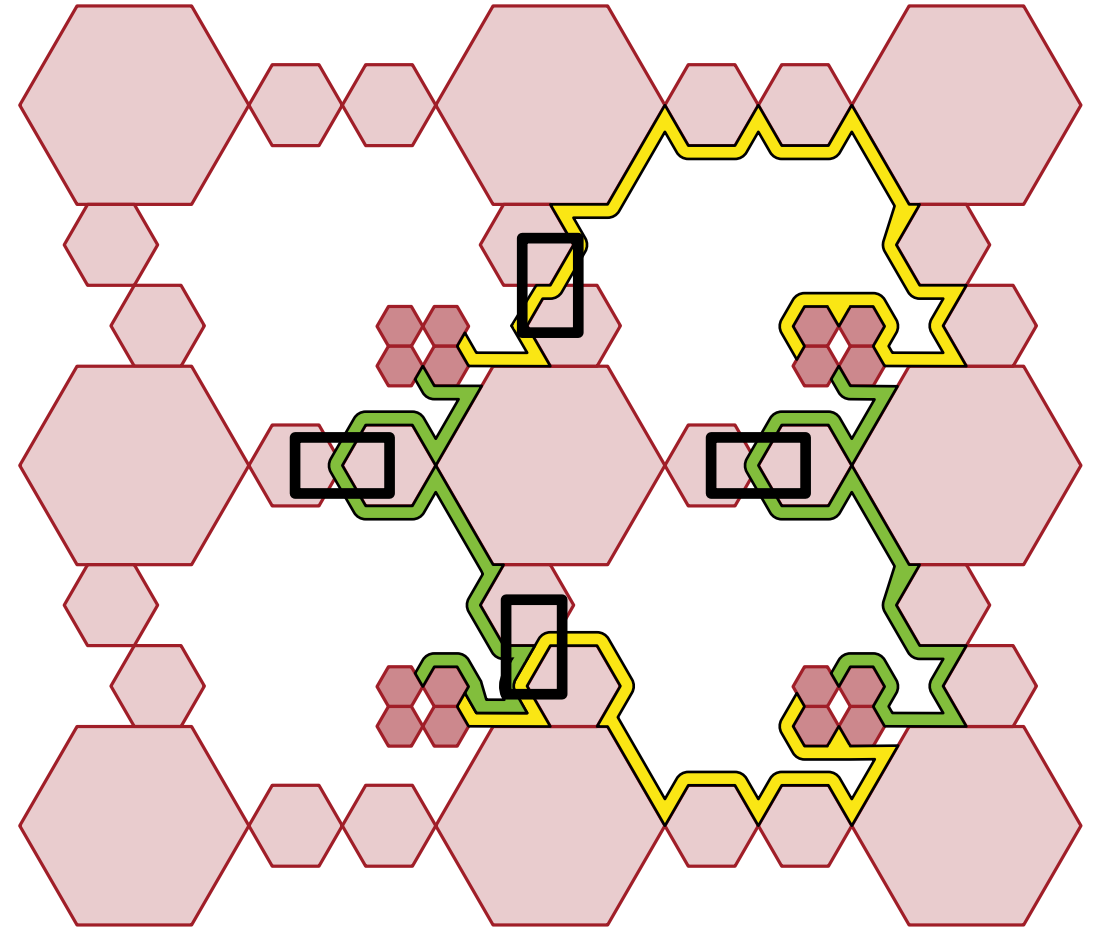
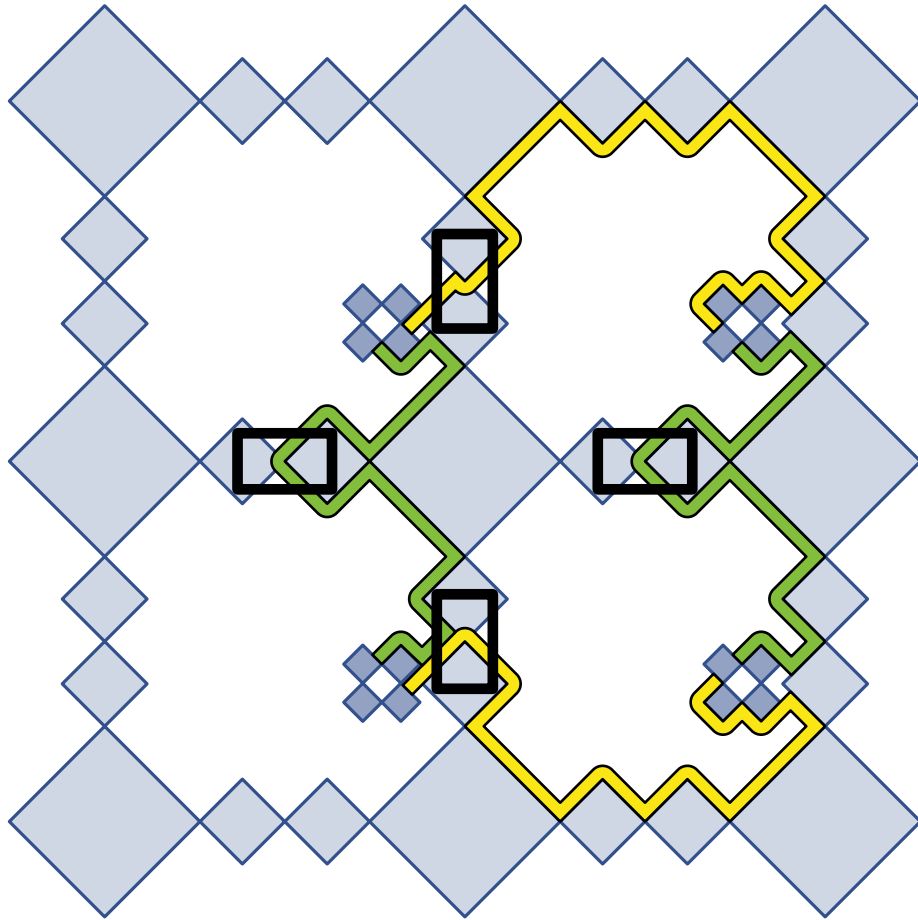
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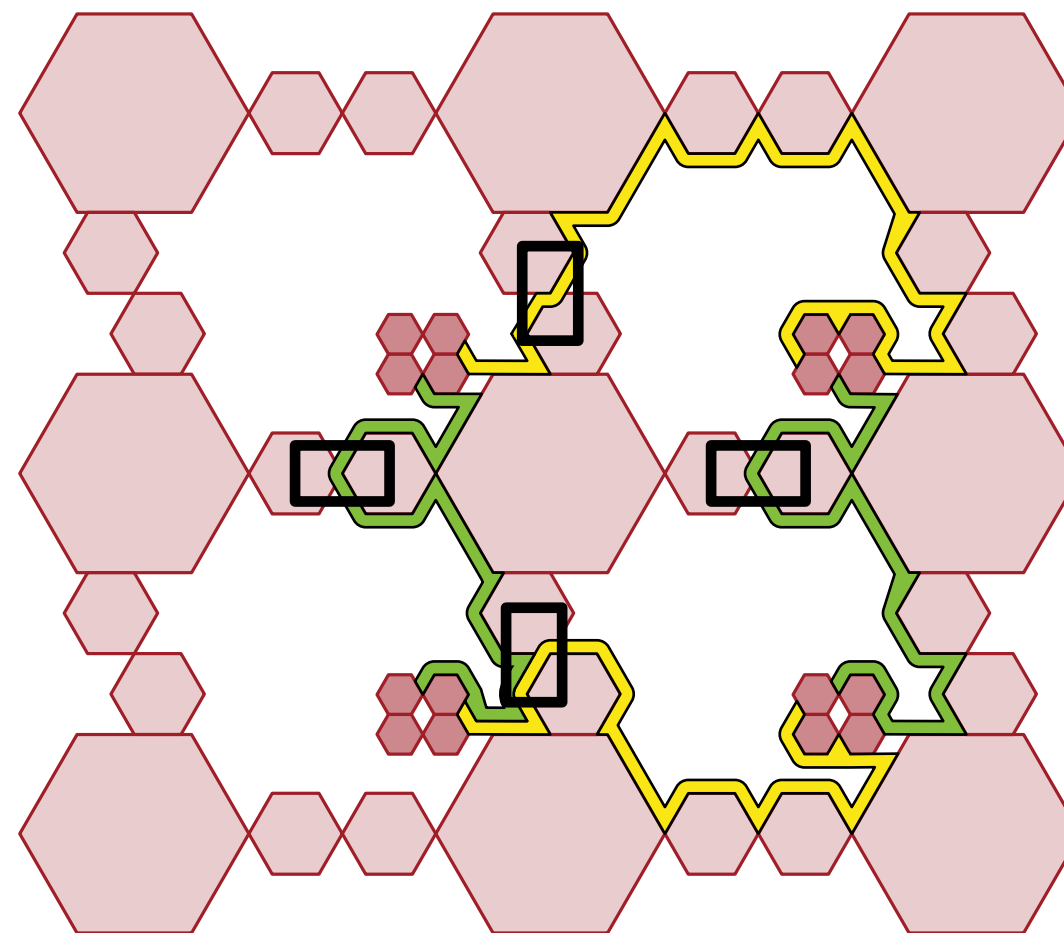
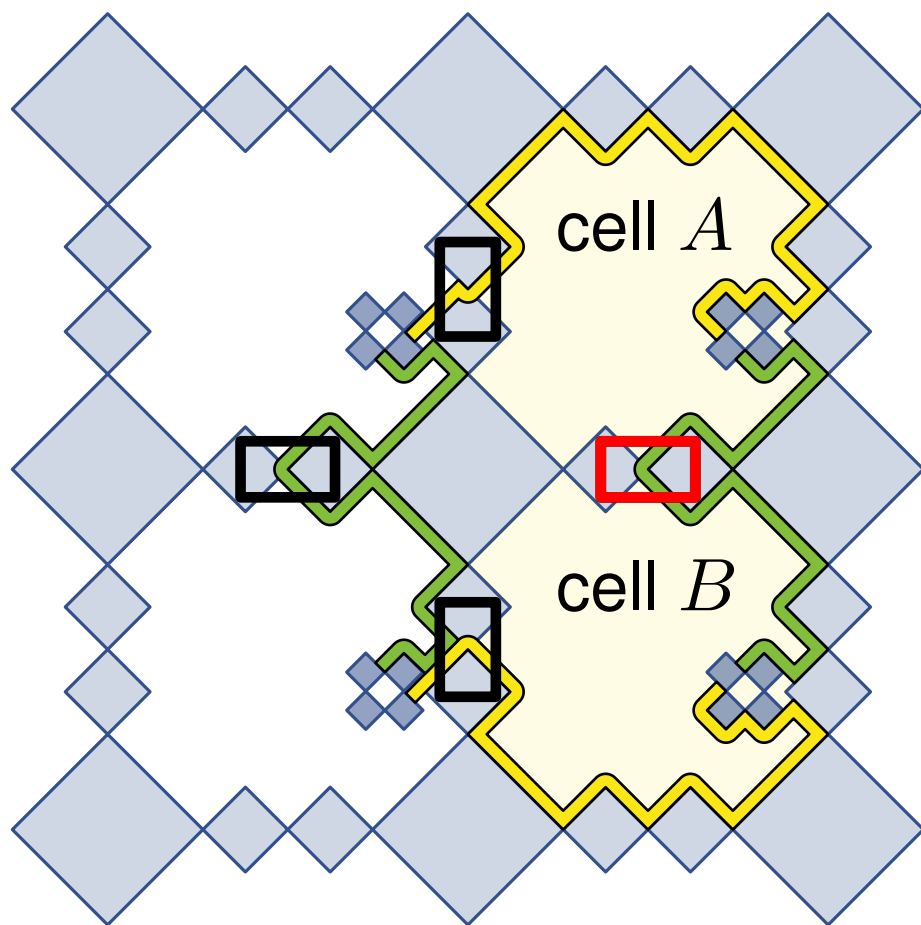
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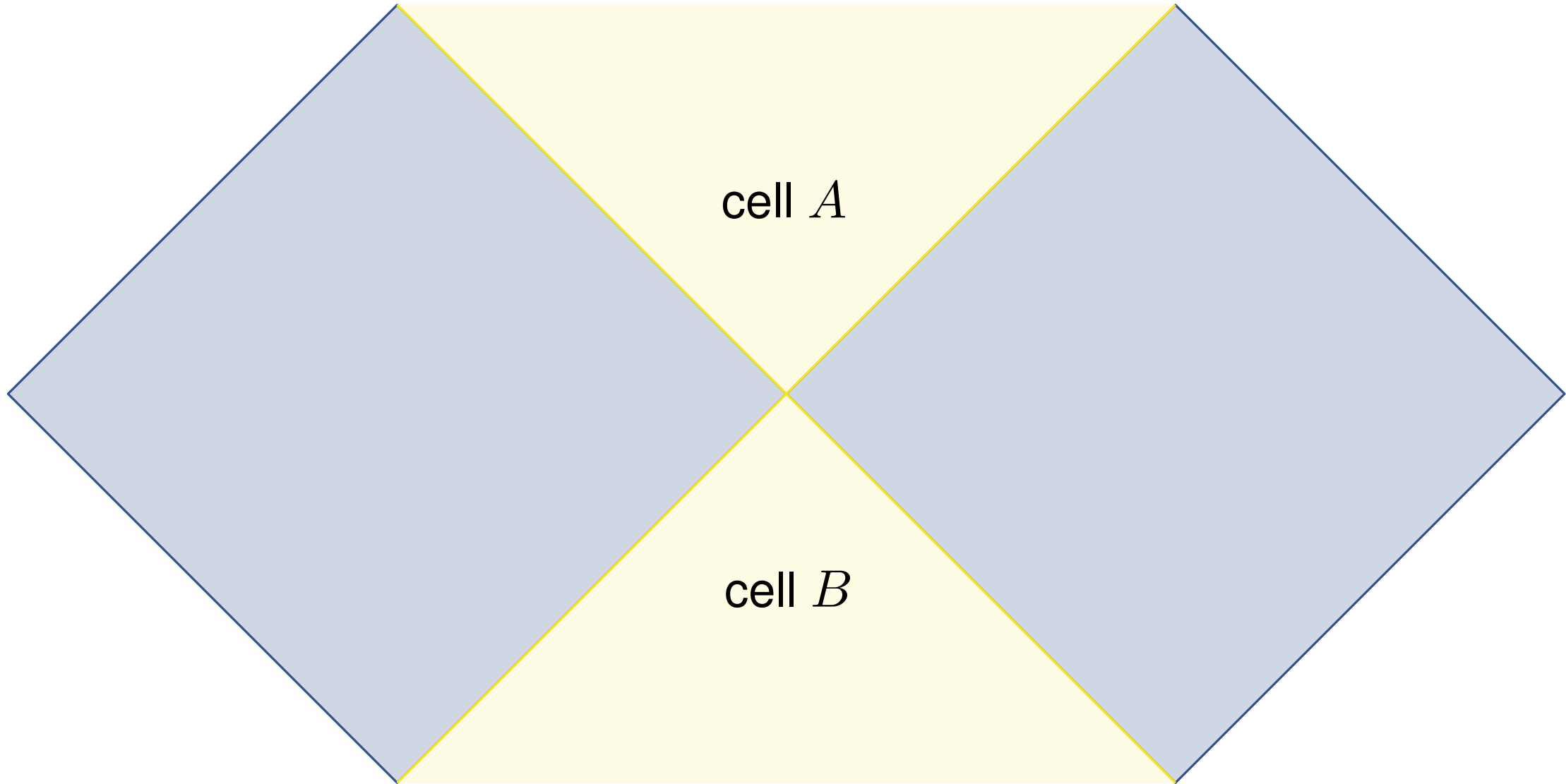
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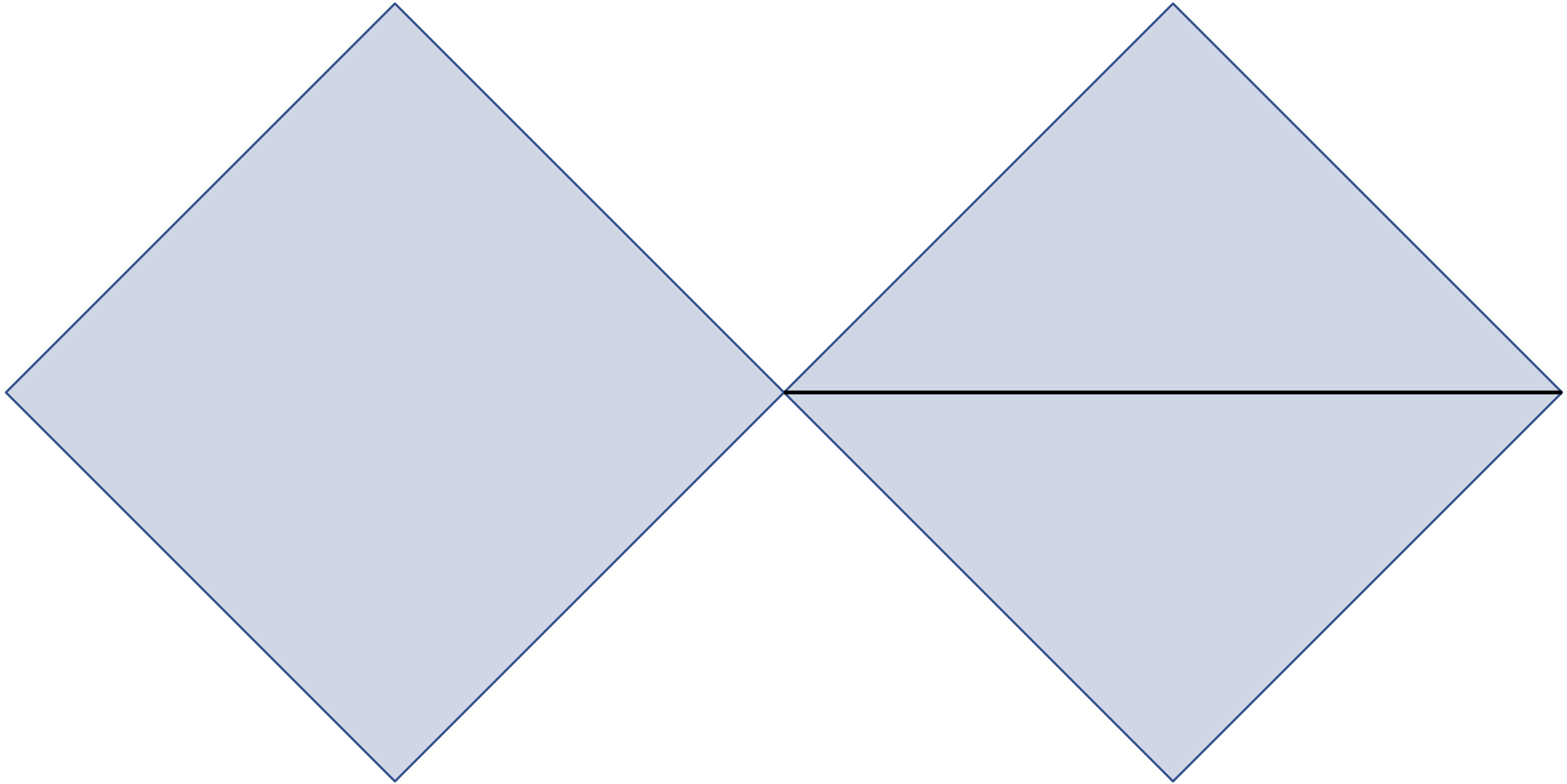
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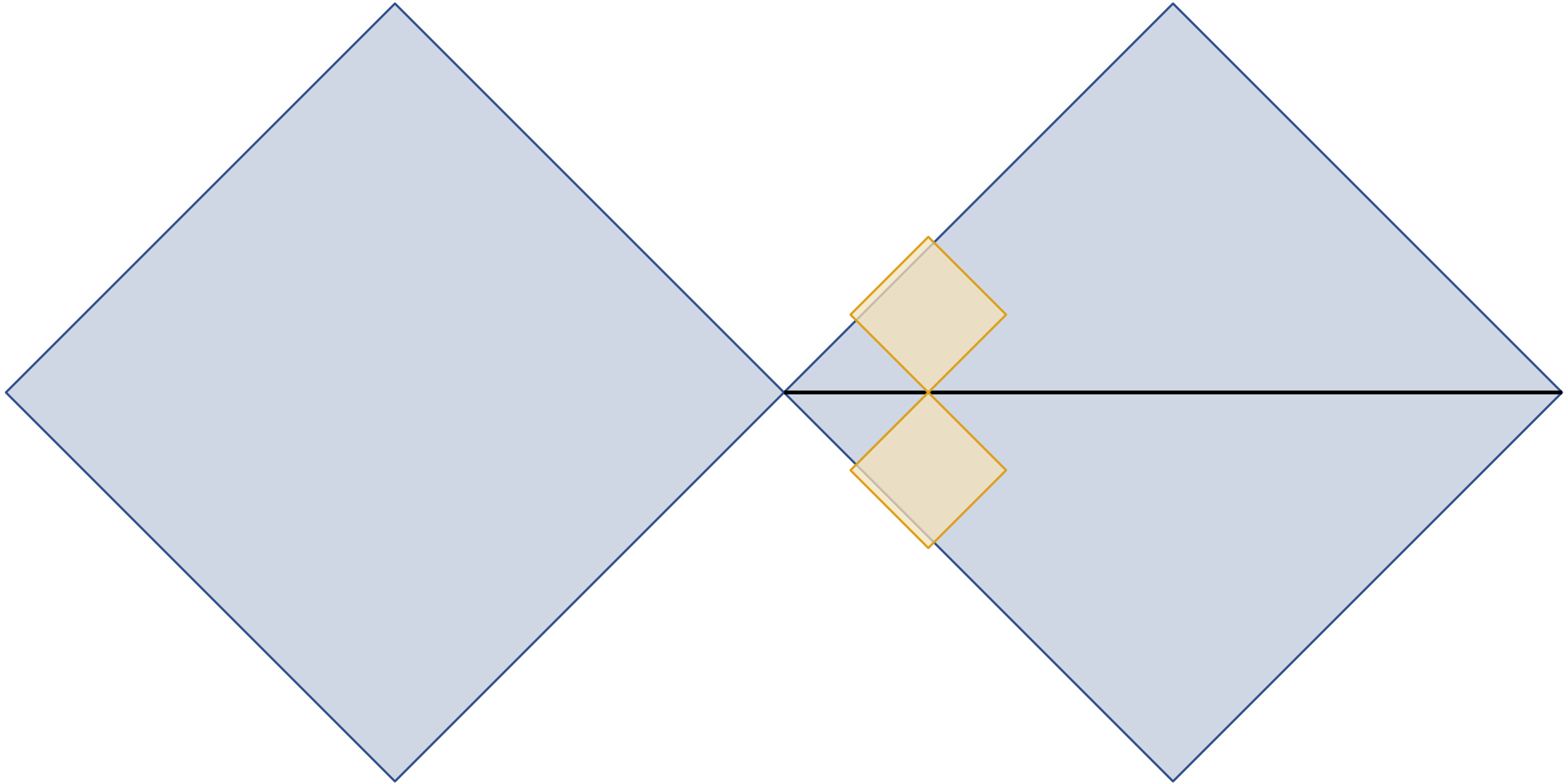
Independent Edges Between Cells: Rectangles



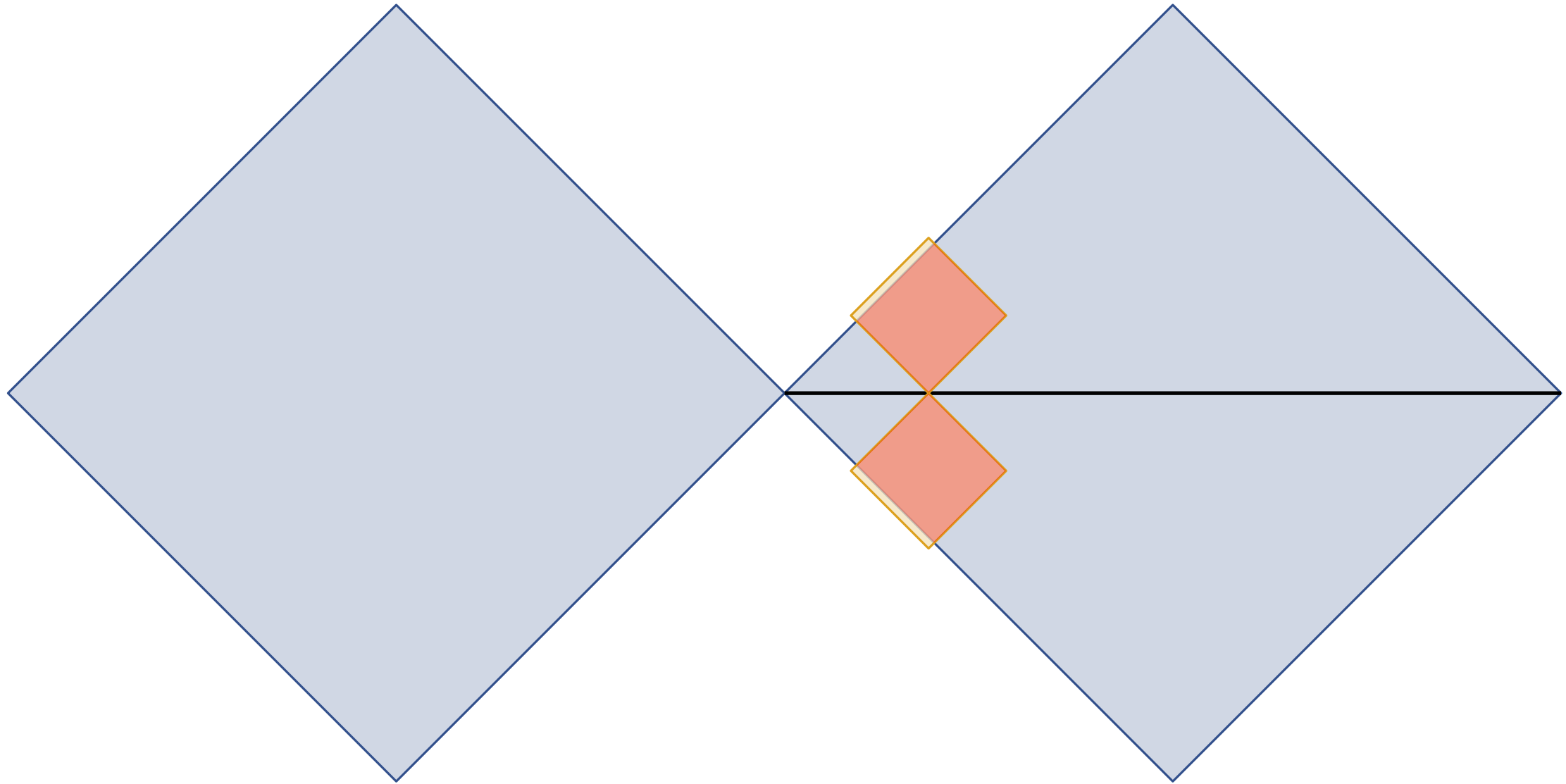
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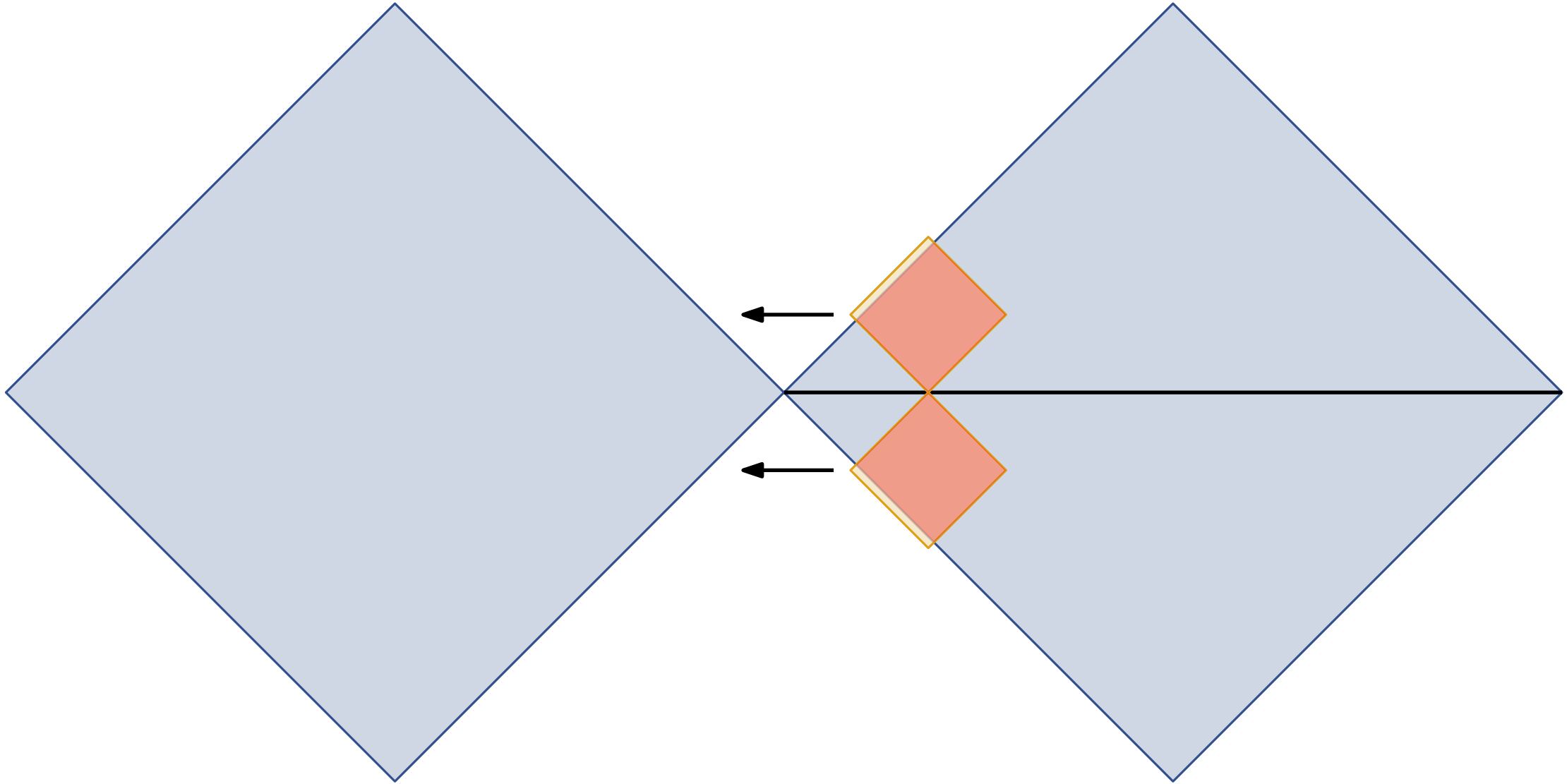
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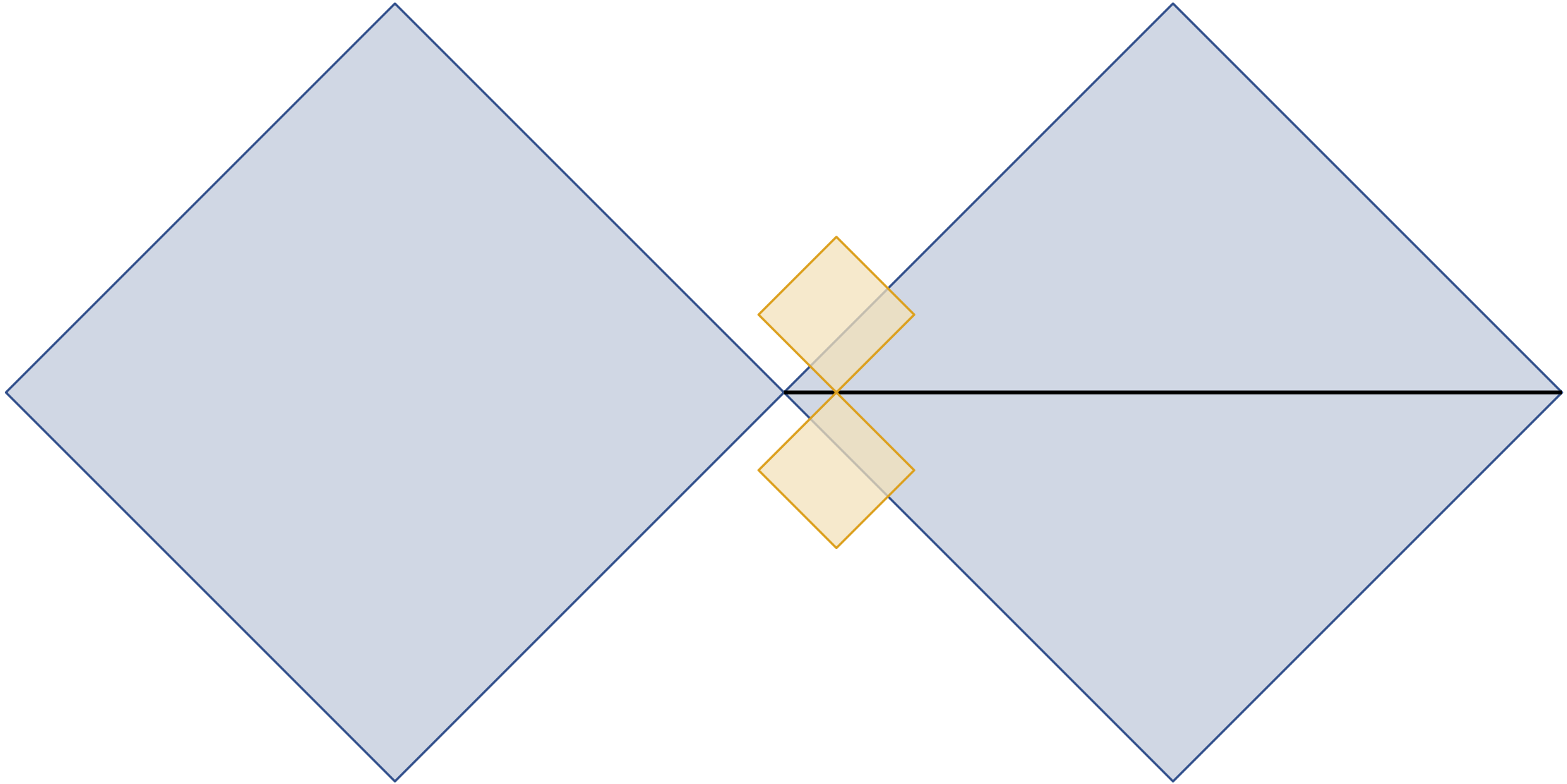
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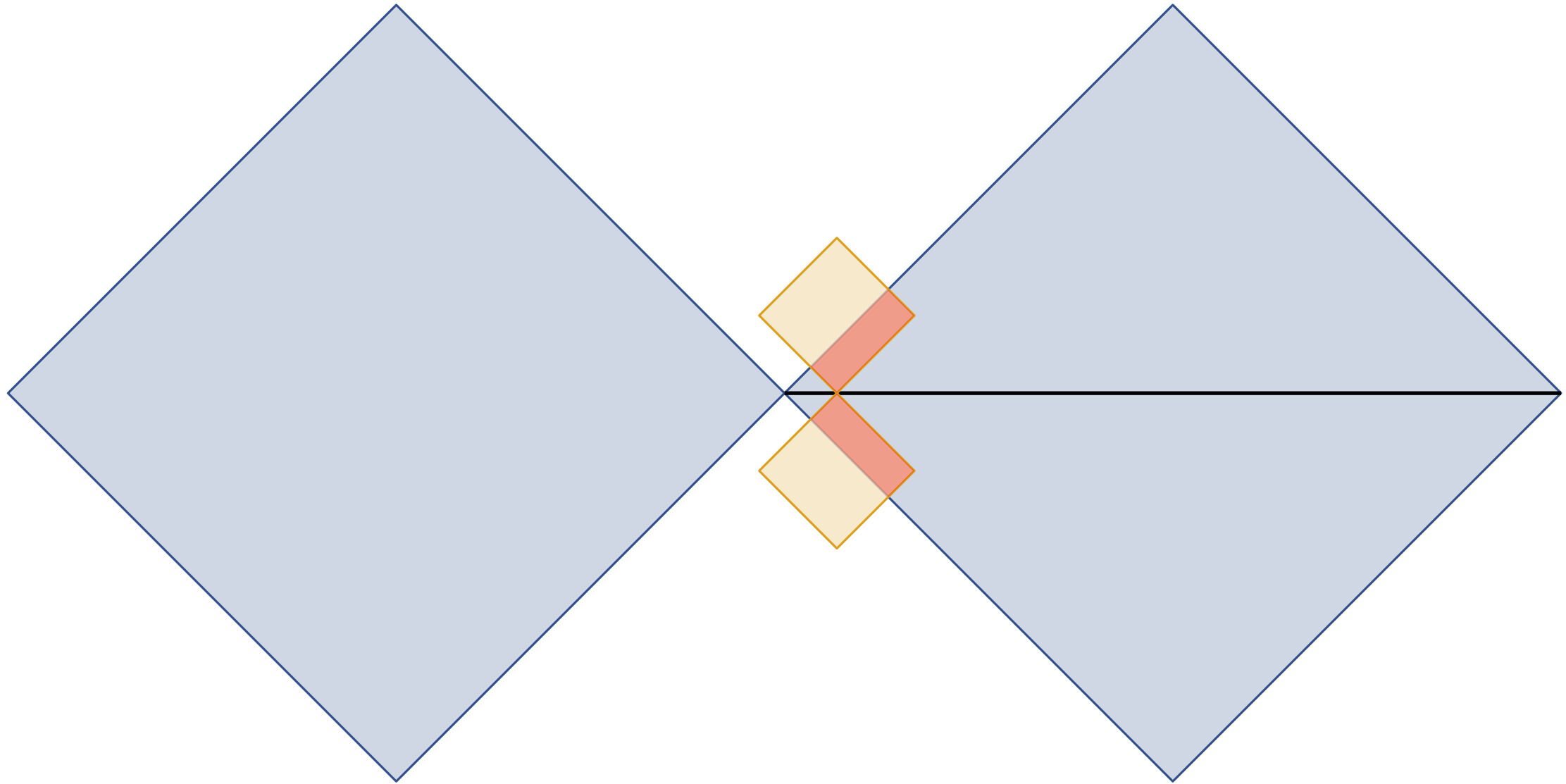
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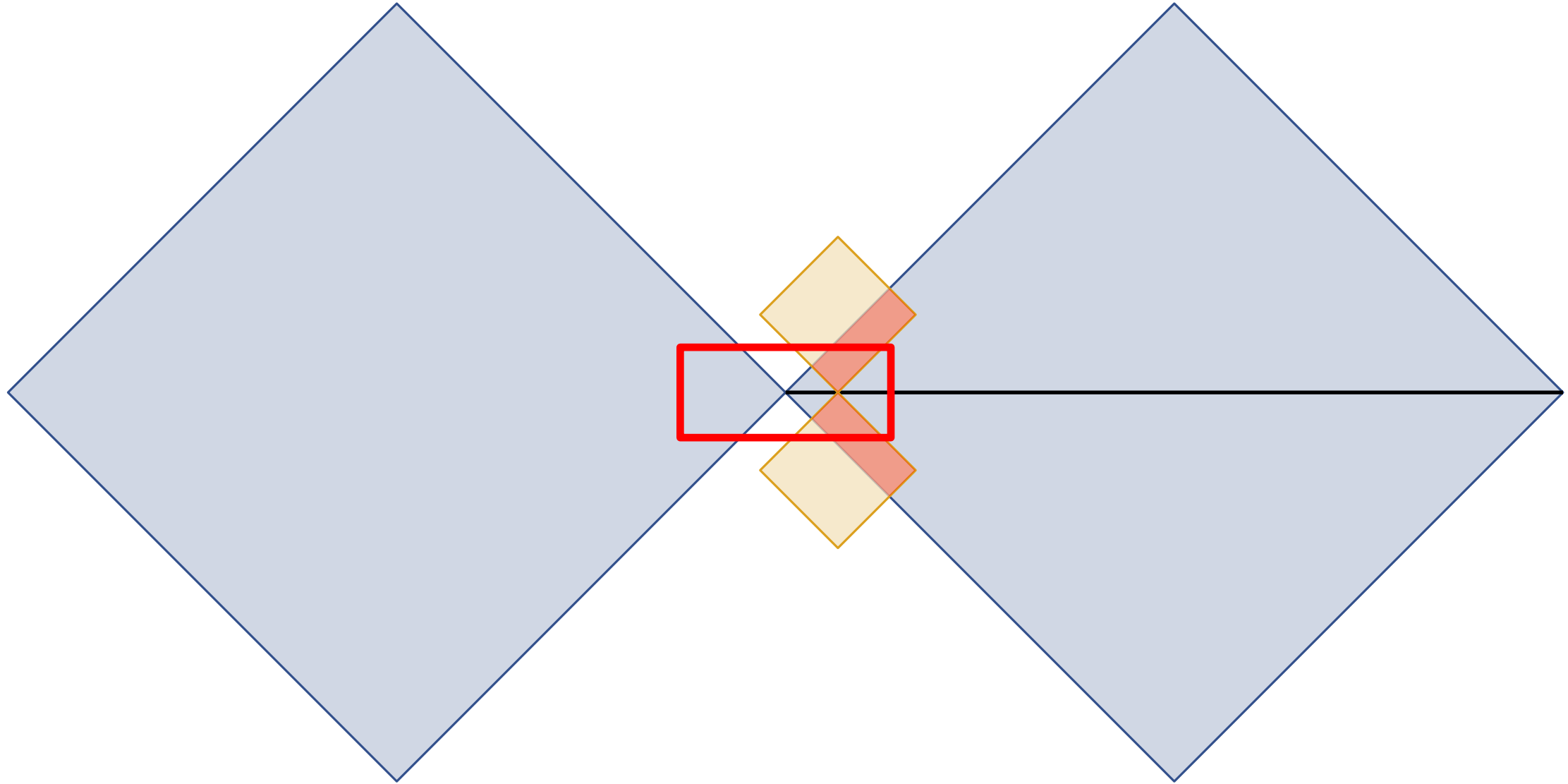
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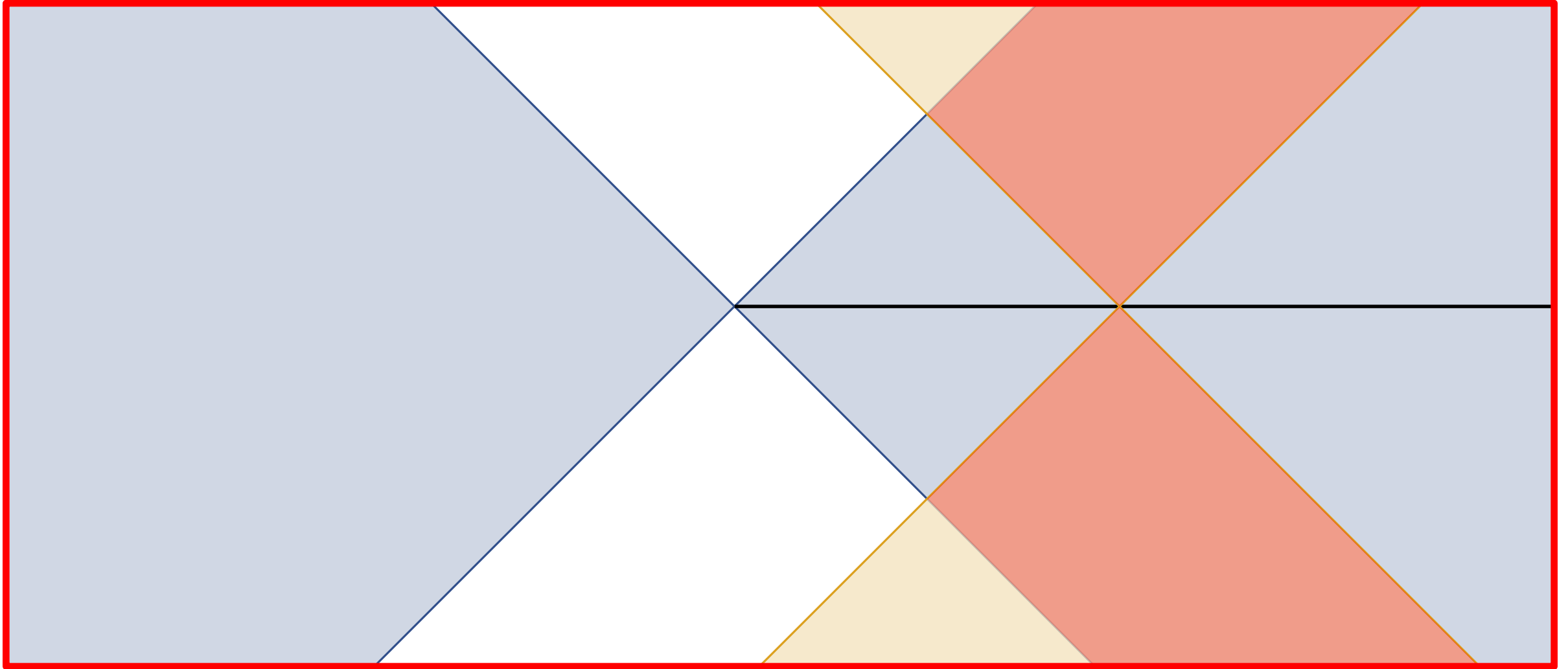
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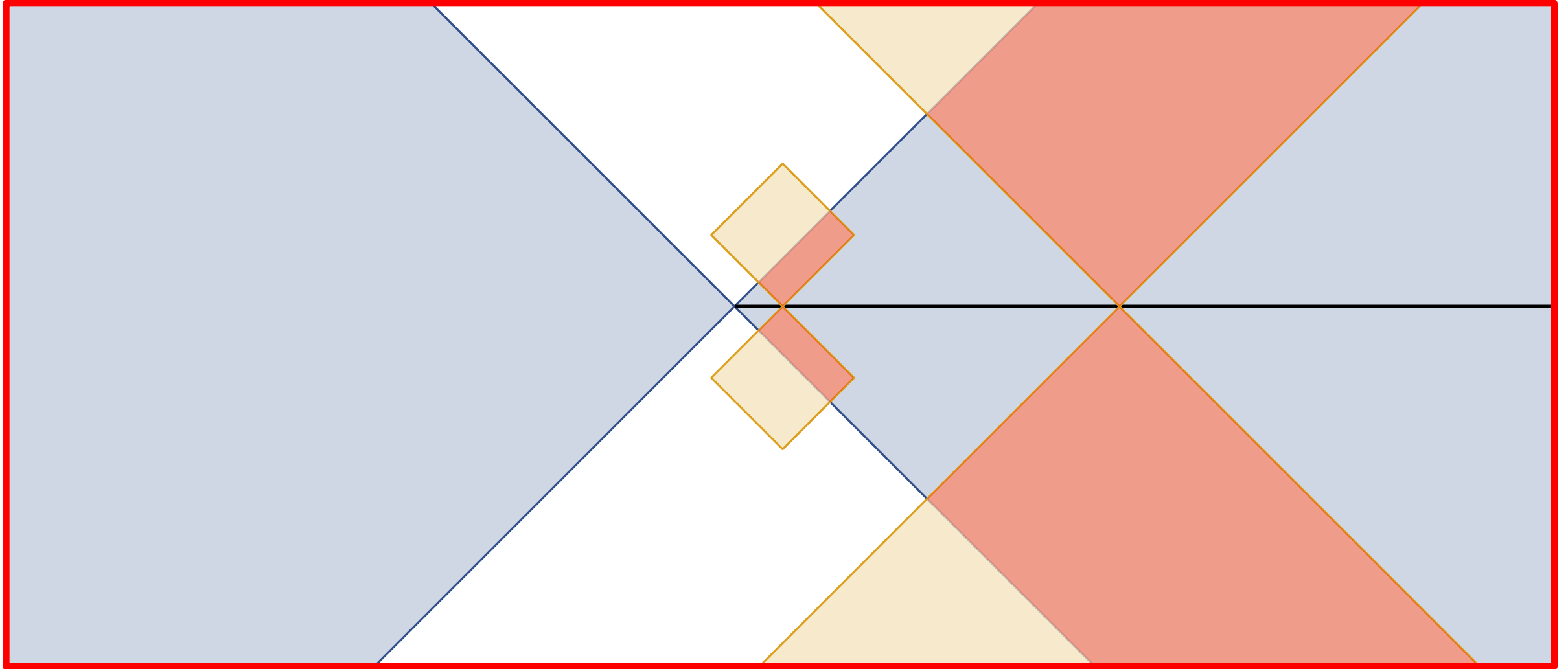
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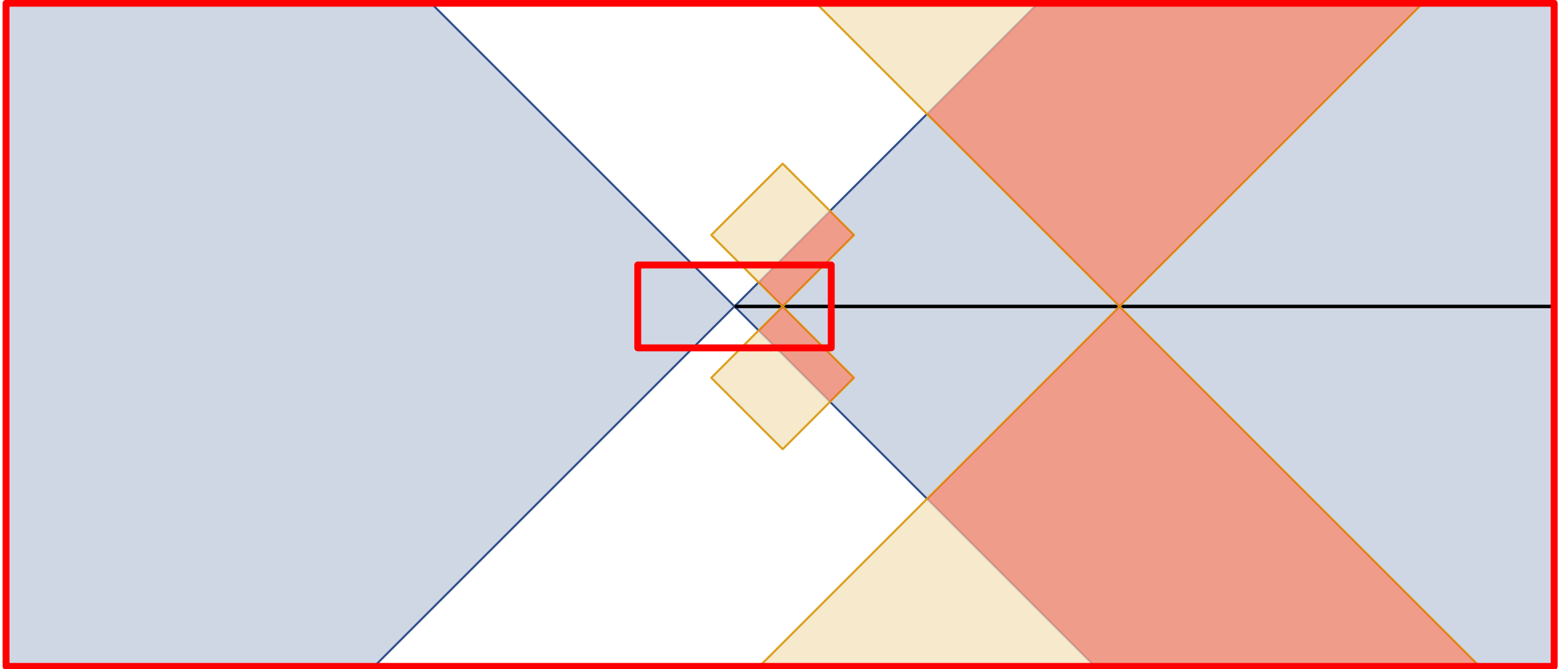
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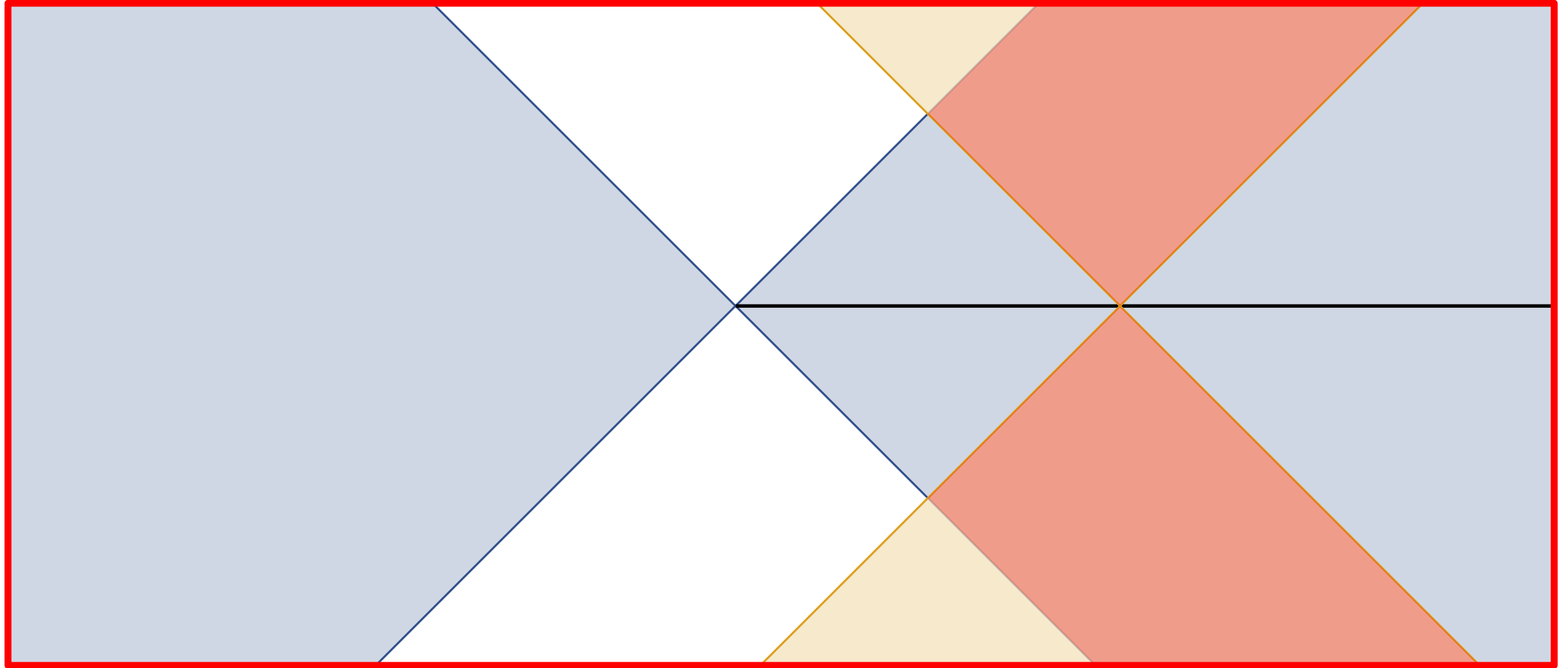
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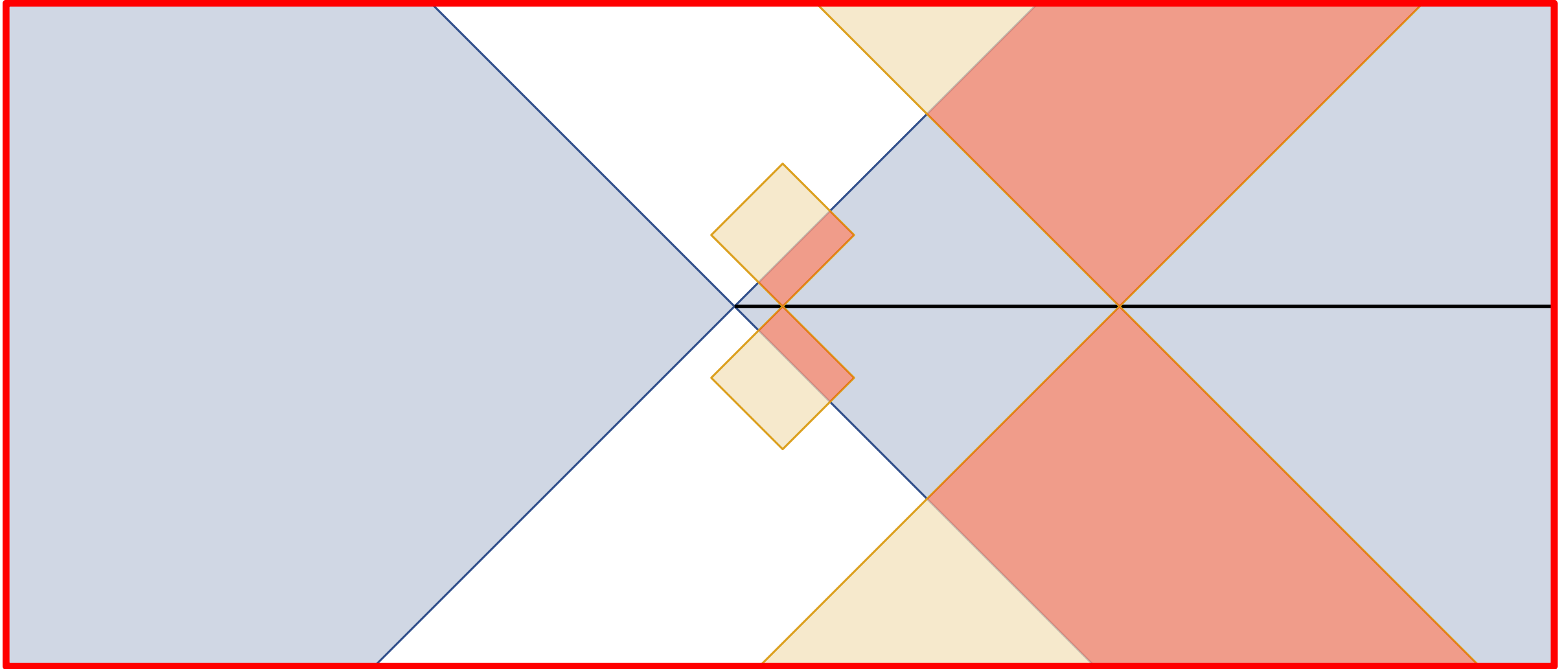
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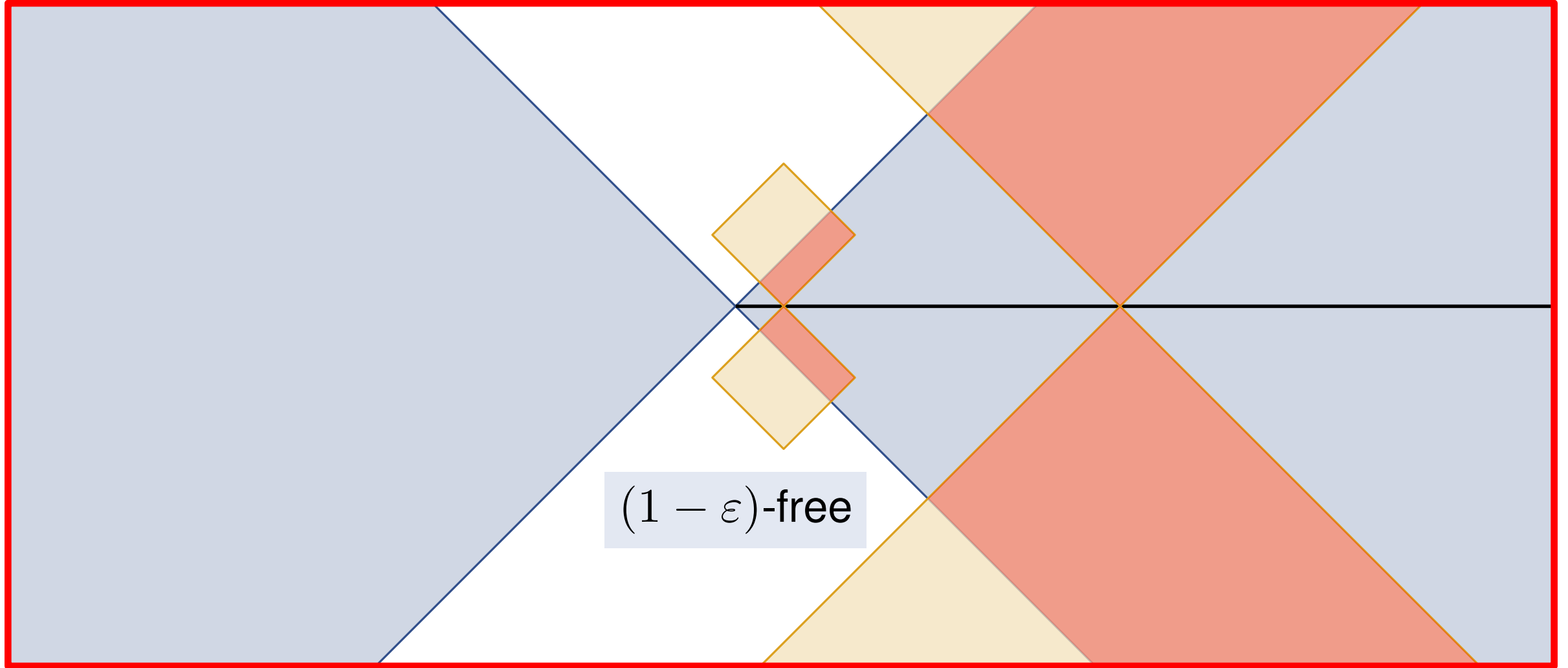
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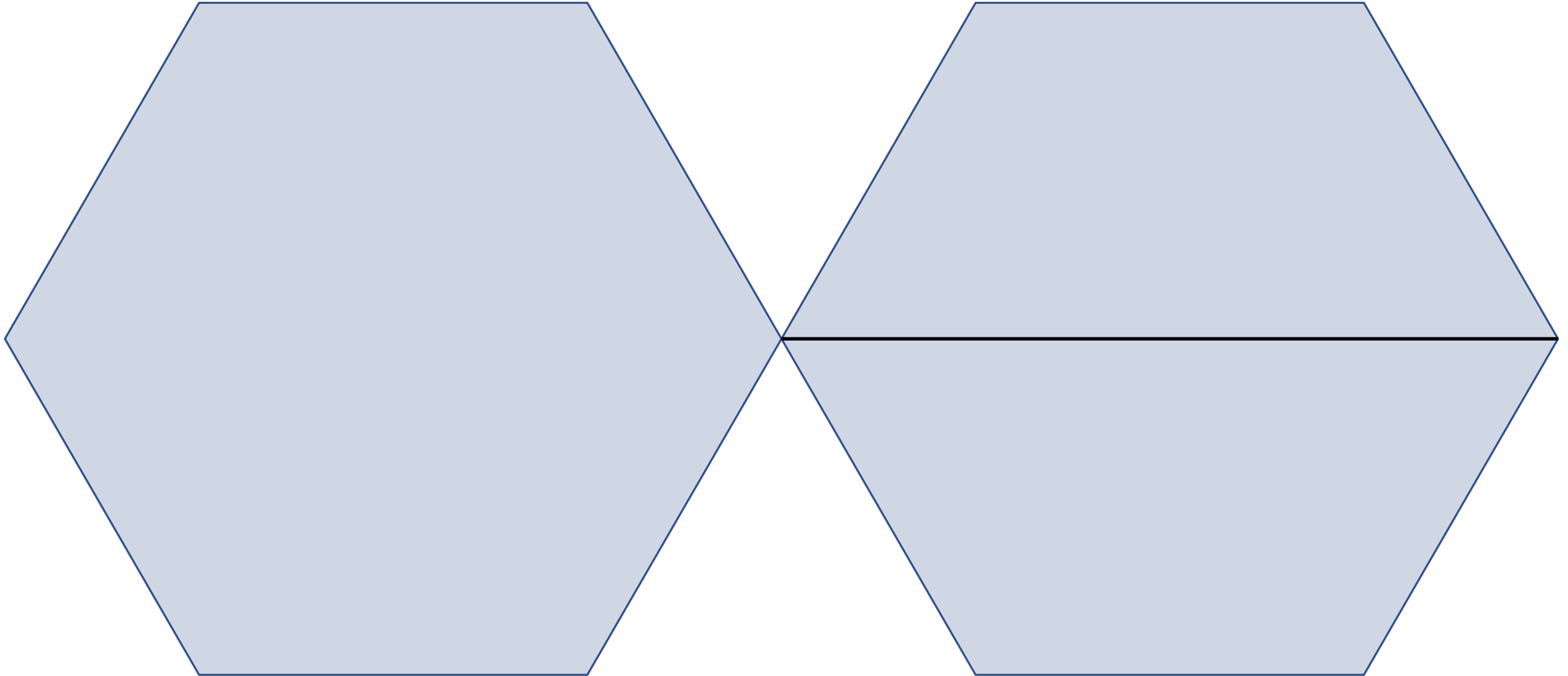
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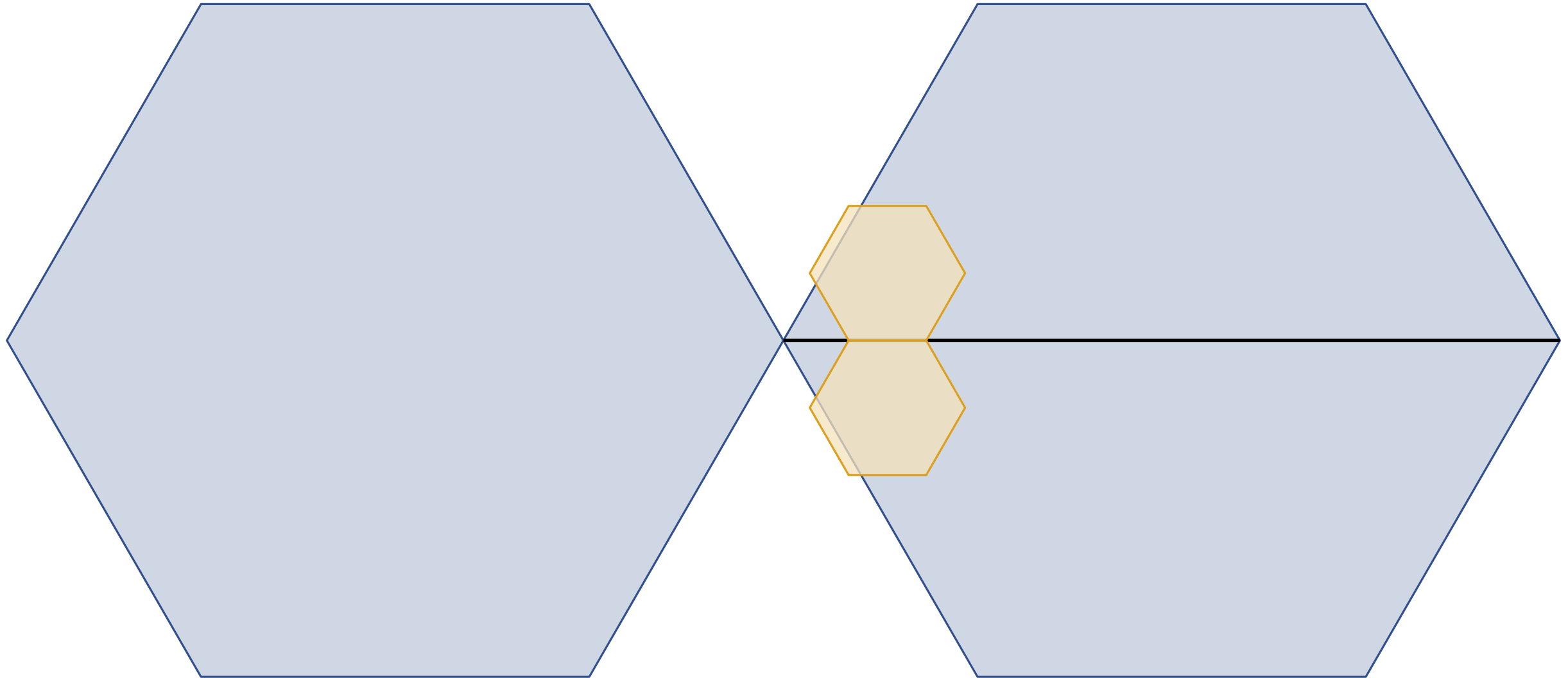
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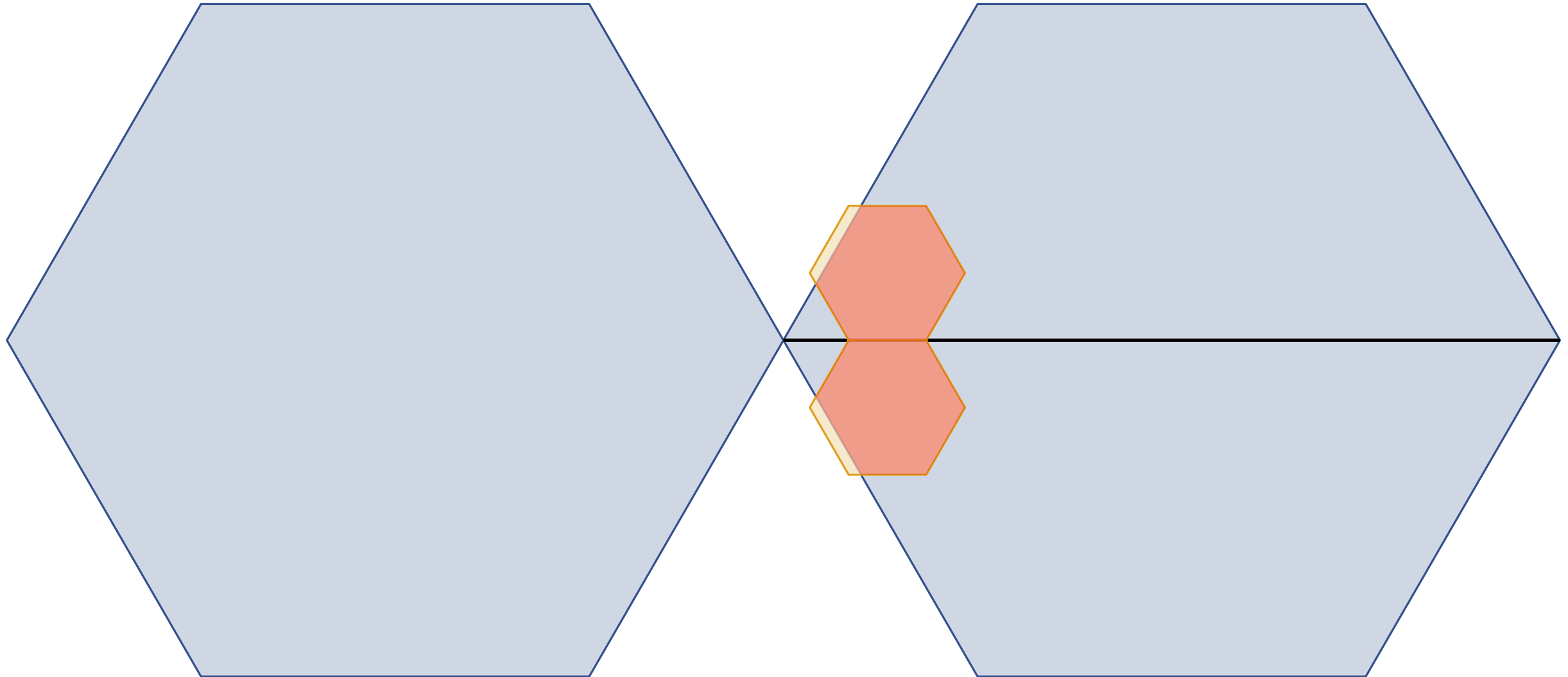
Independent Edges Between Cells: Hexagons



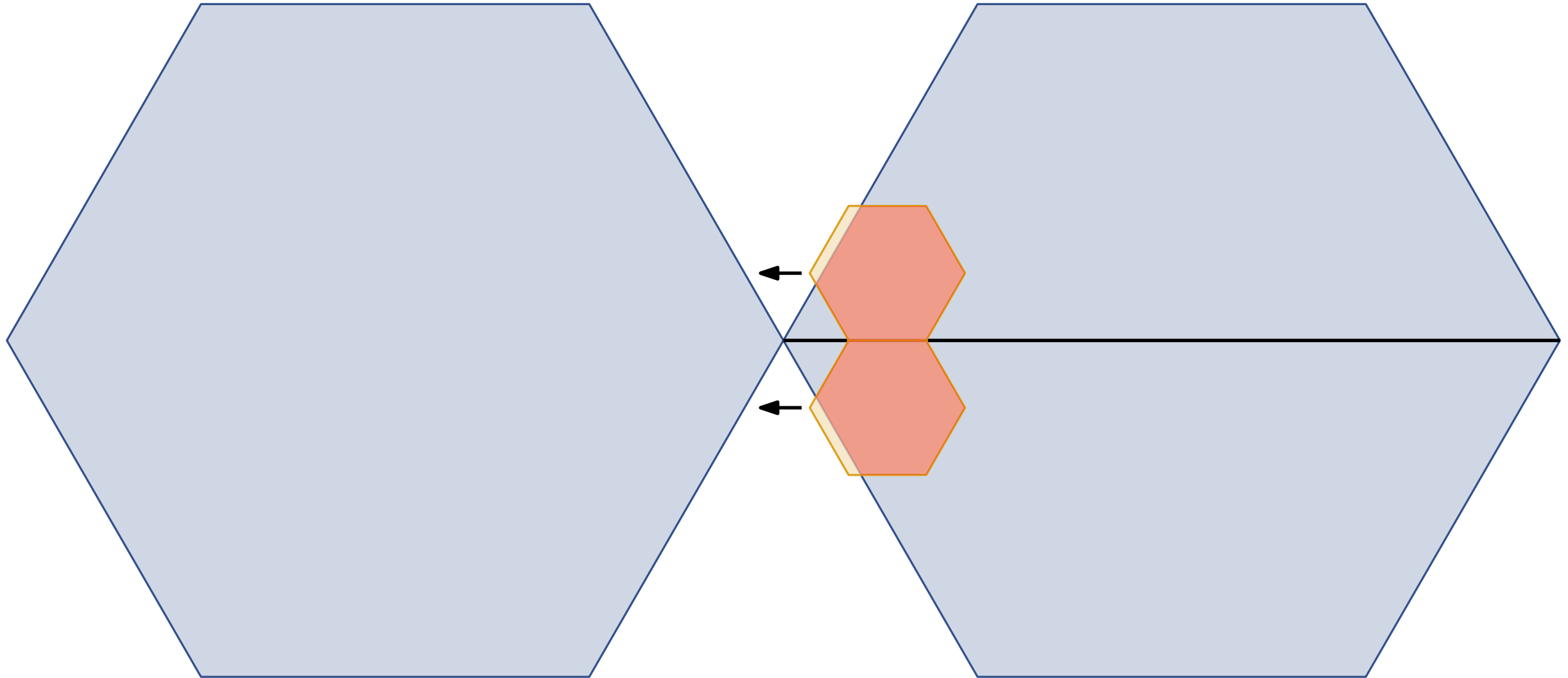
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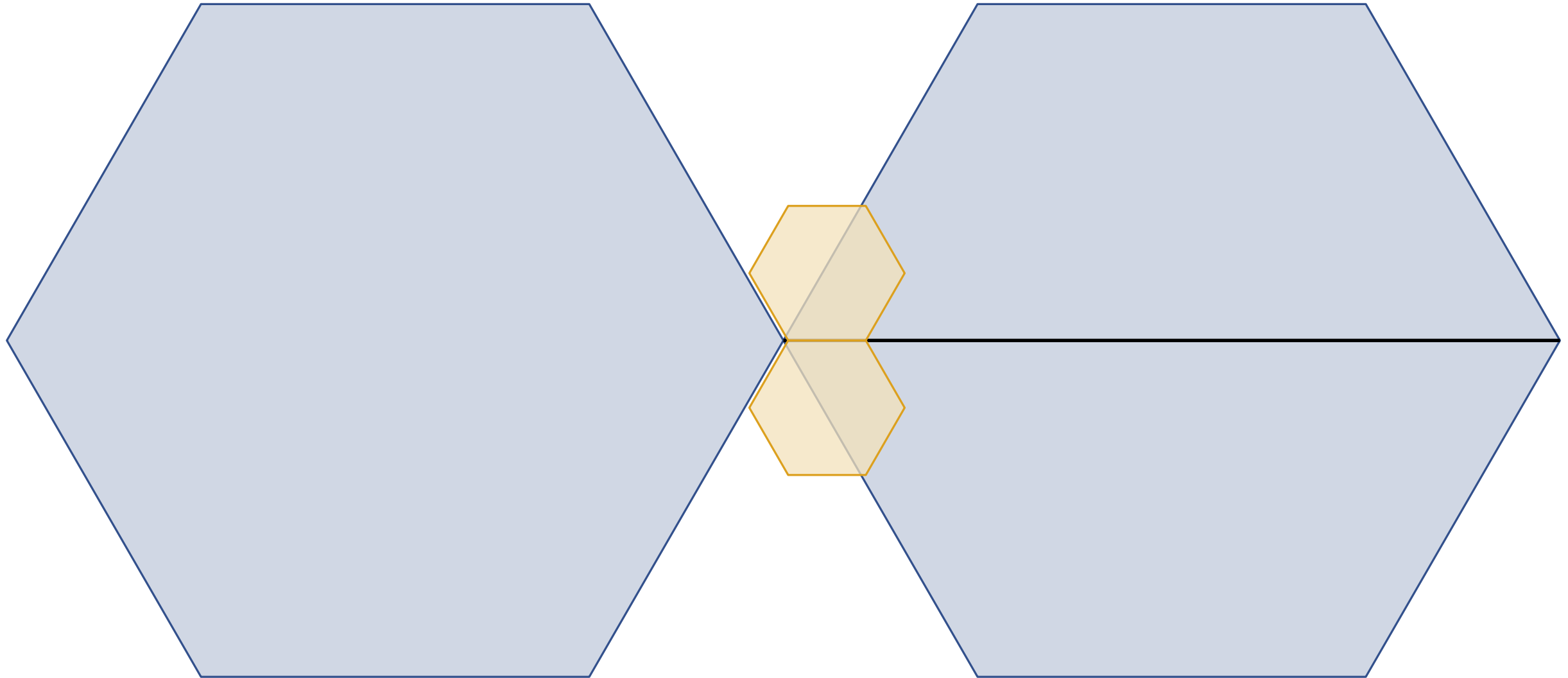
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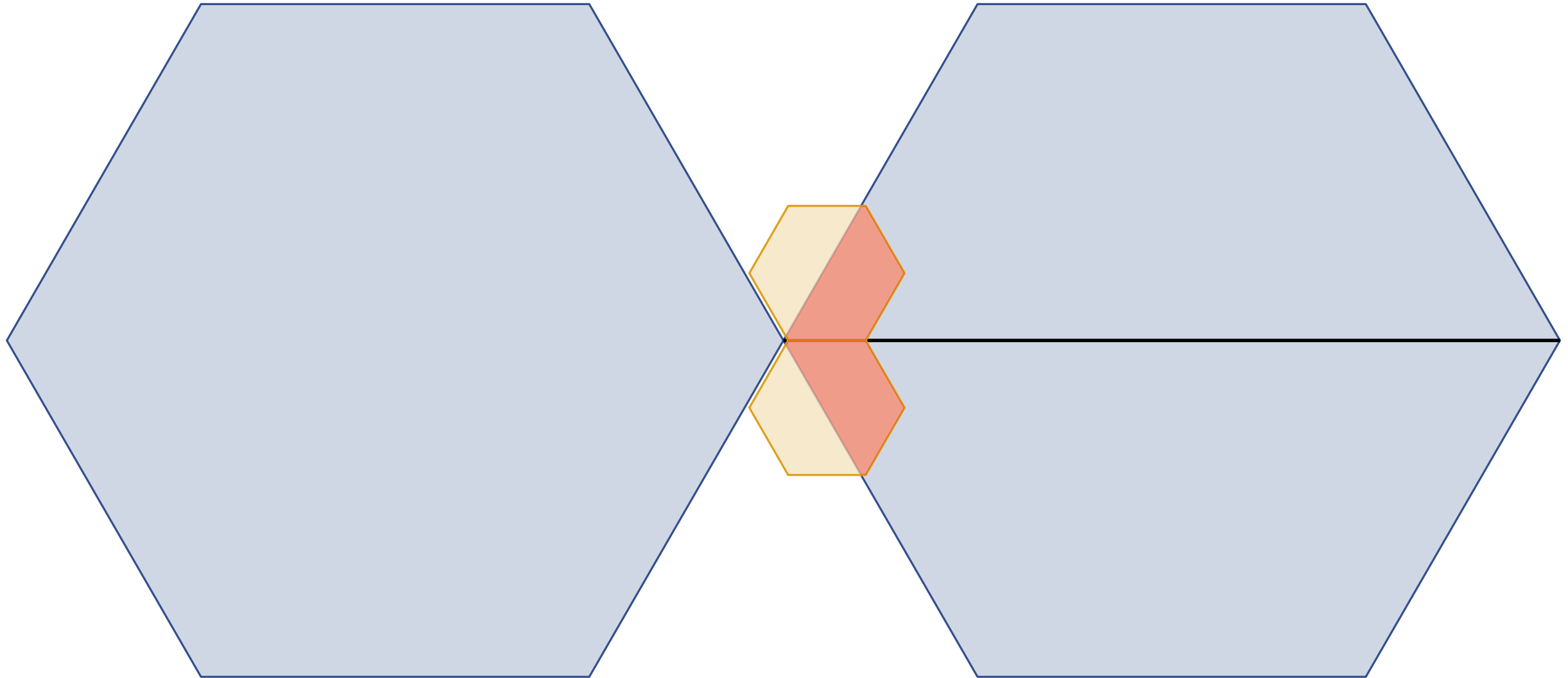
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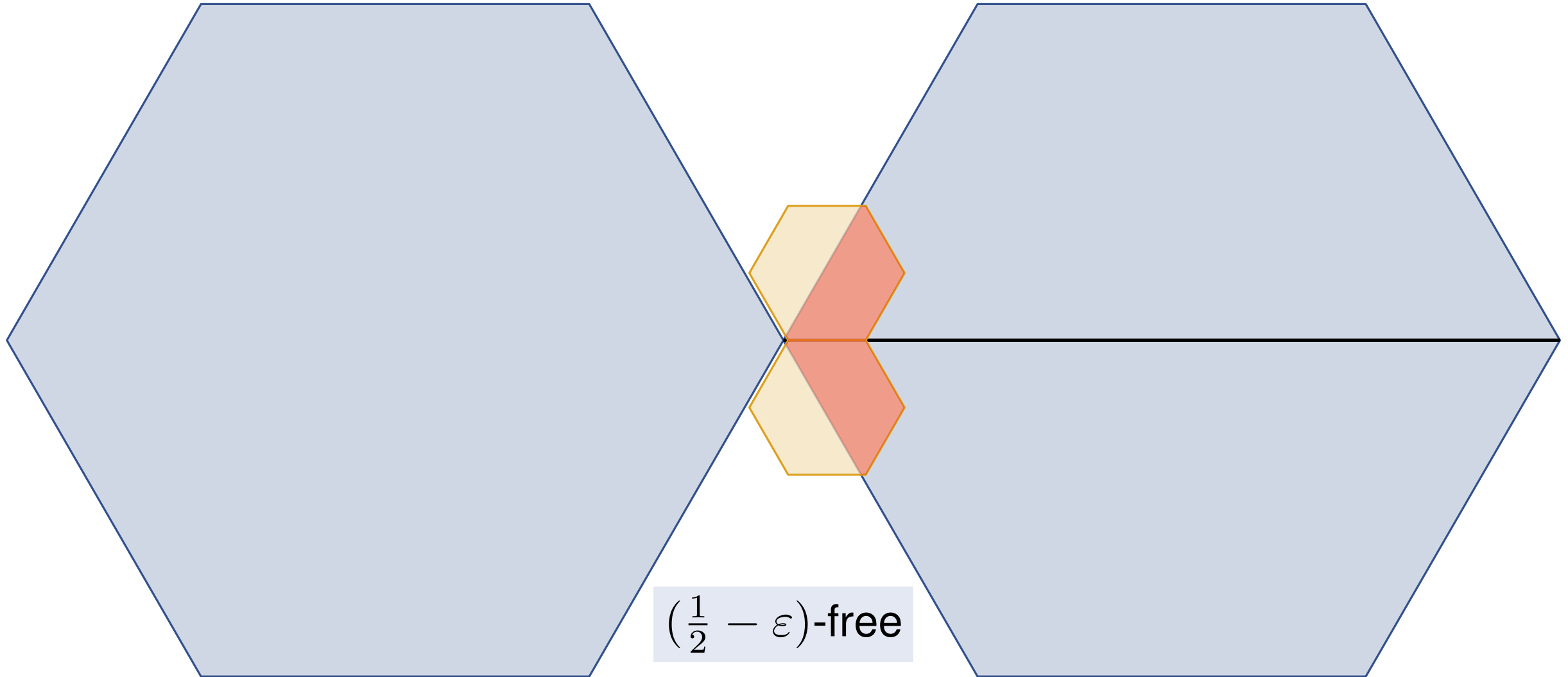
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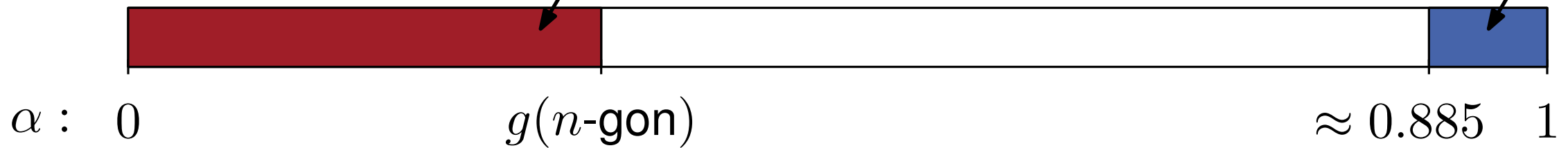
Our Results for Regular n -gons

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■ For even $n \geq 6$:

no linear local treewidth

planar



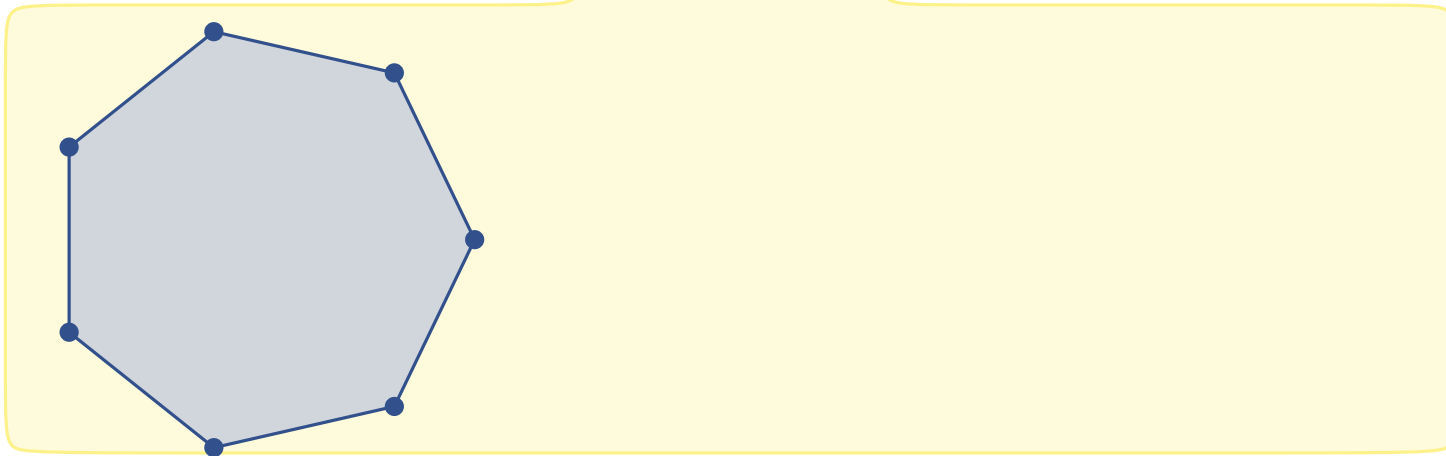
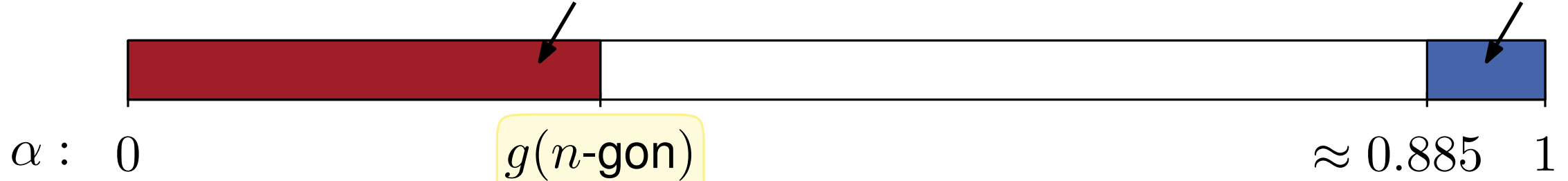
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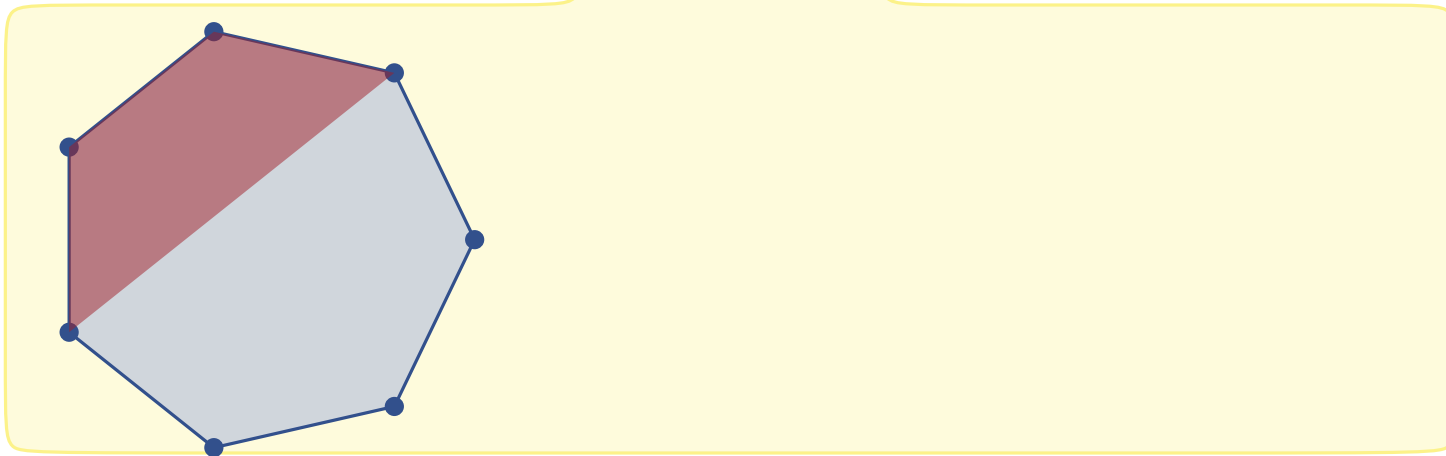
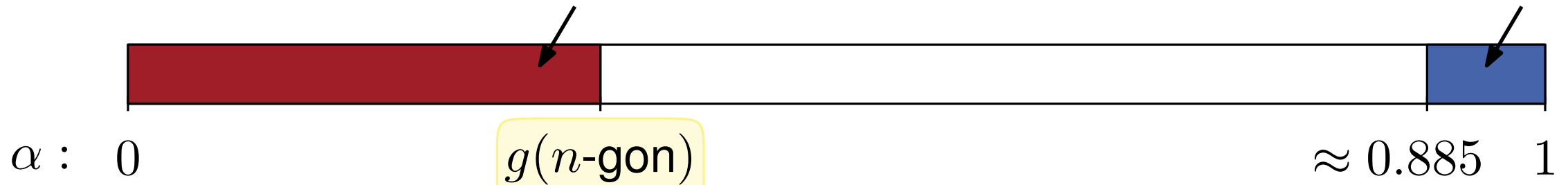
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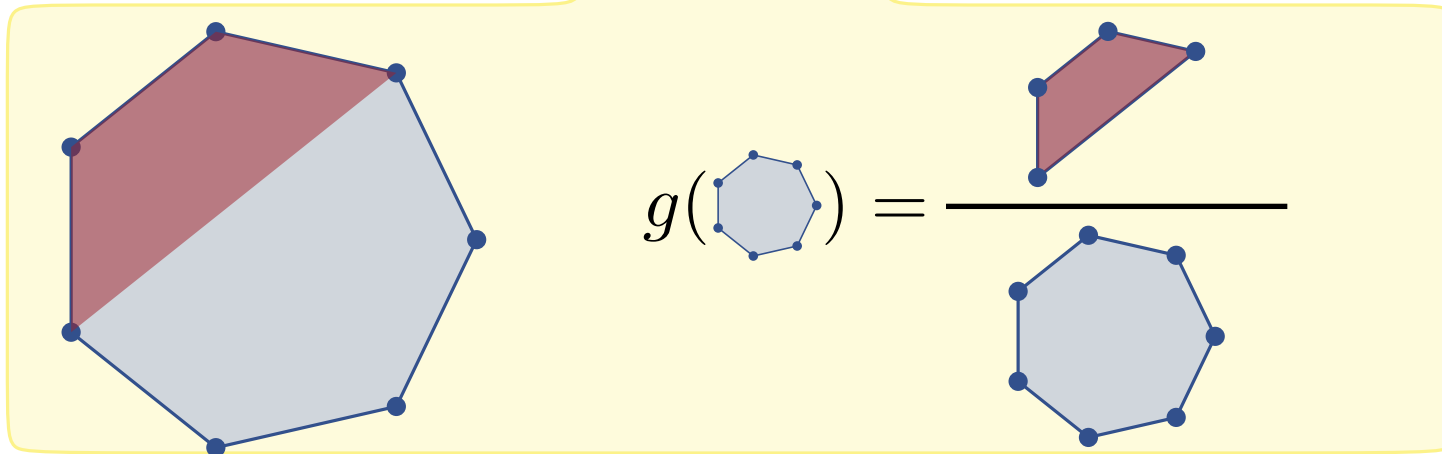
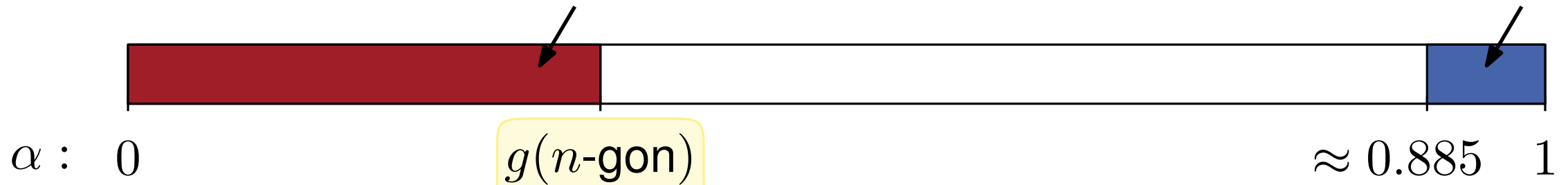


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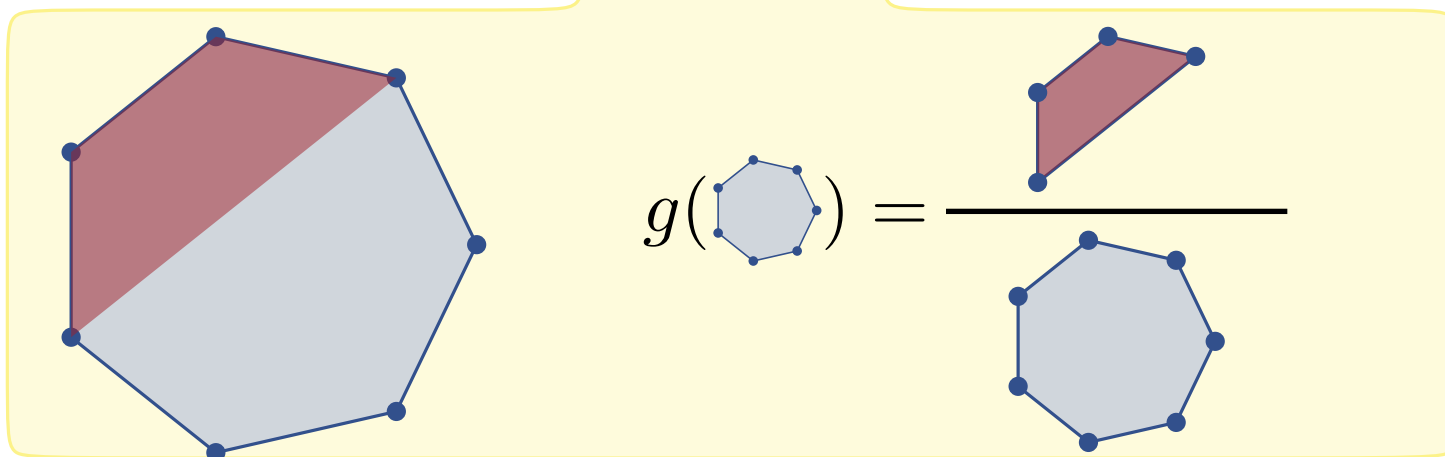
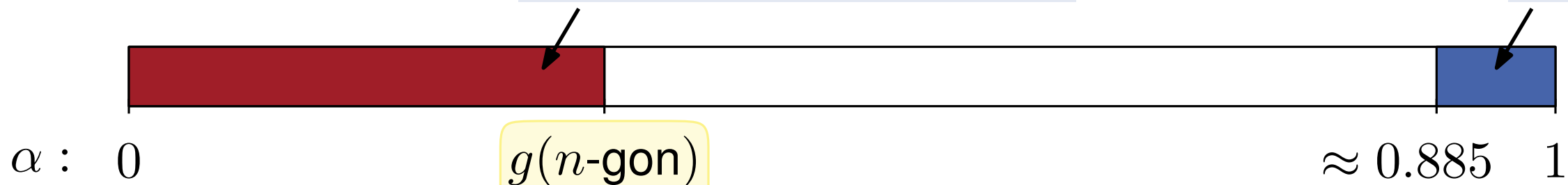
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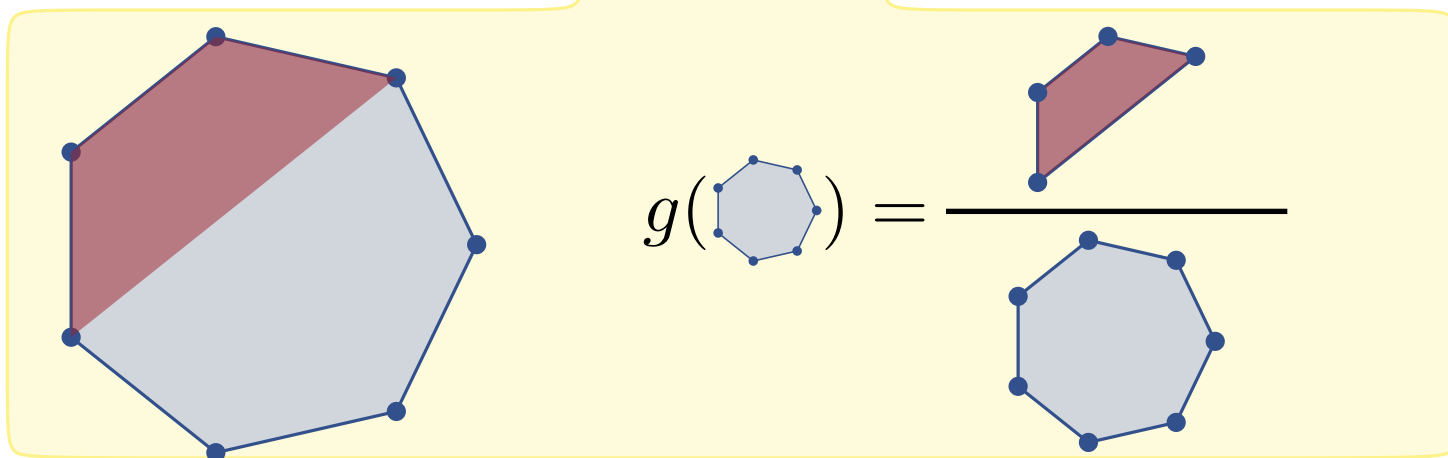
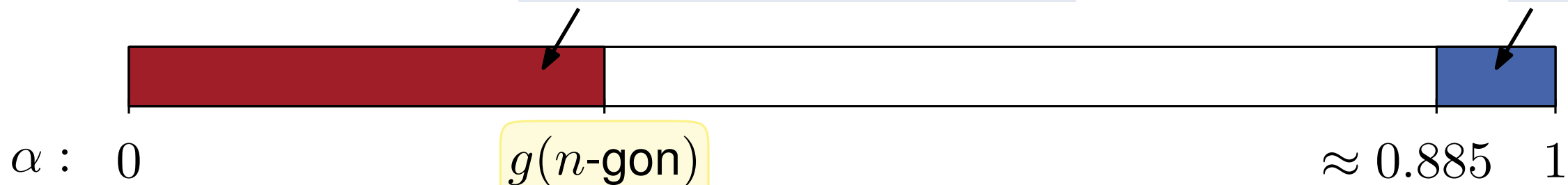
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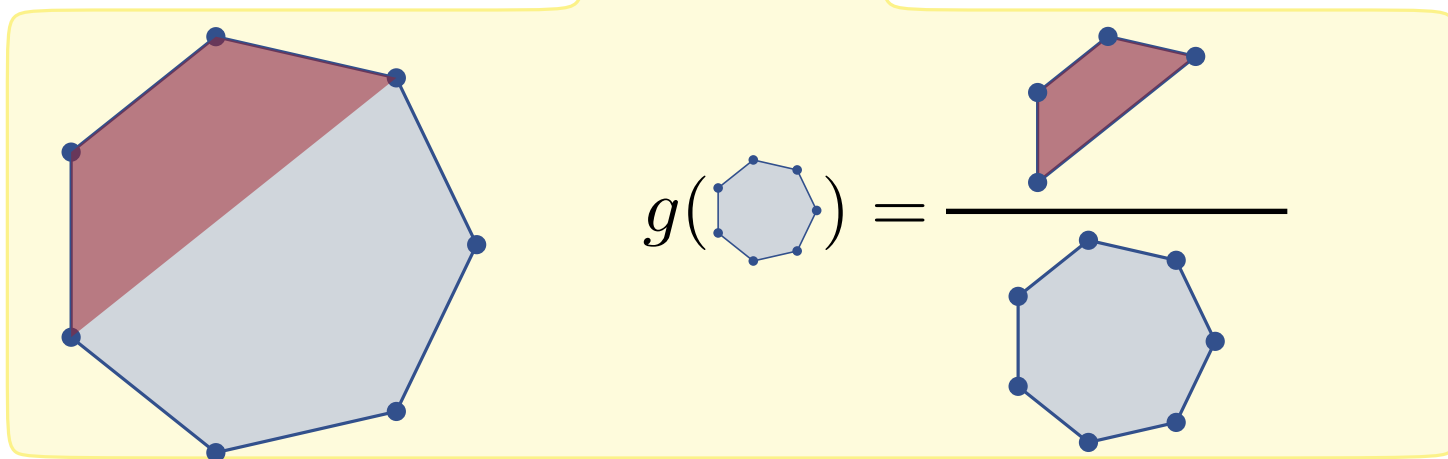
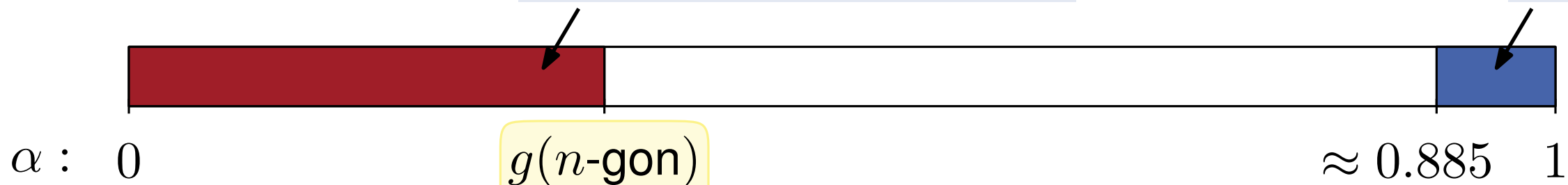
■ $g(\text{hexagon}) = \frac{1}{2}$

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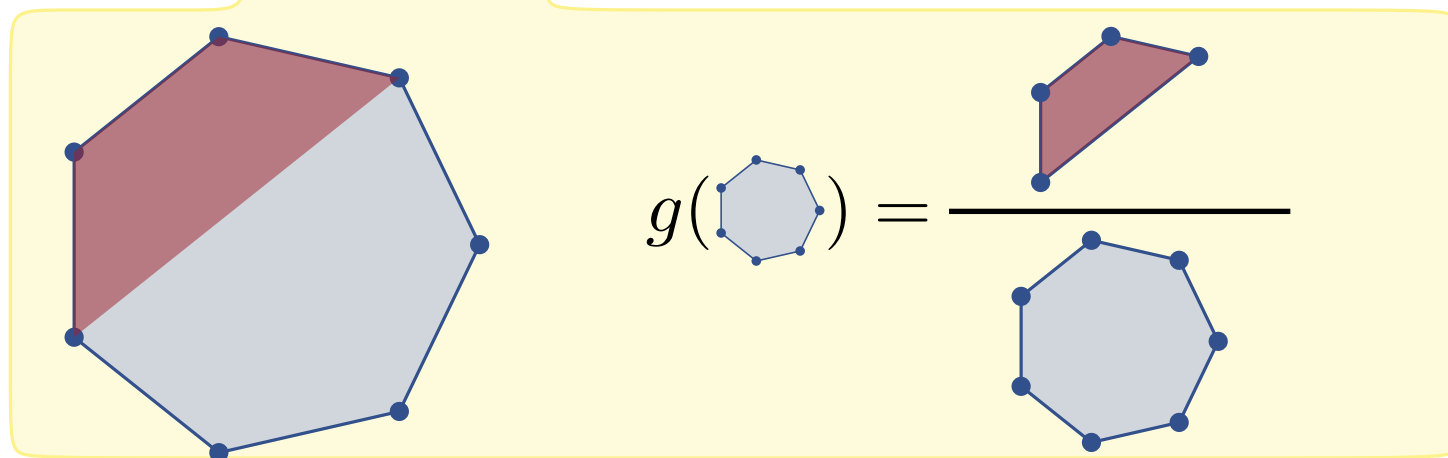
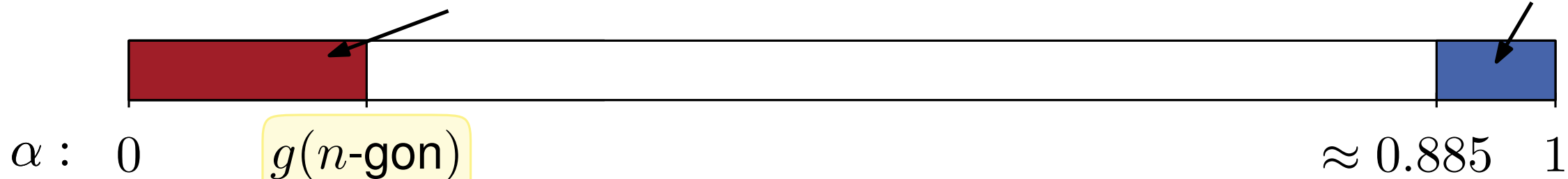
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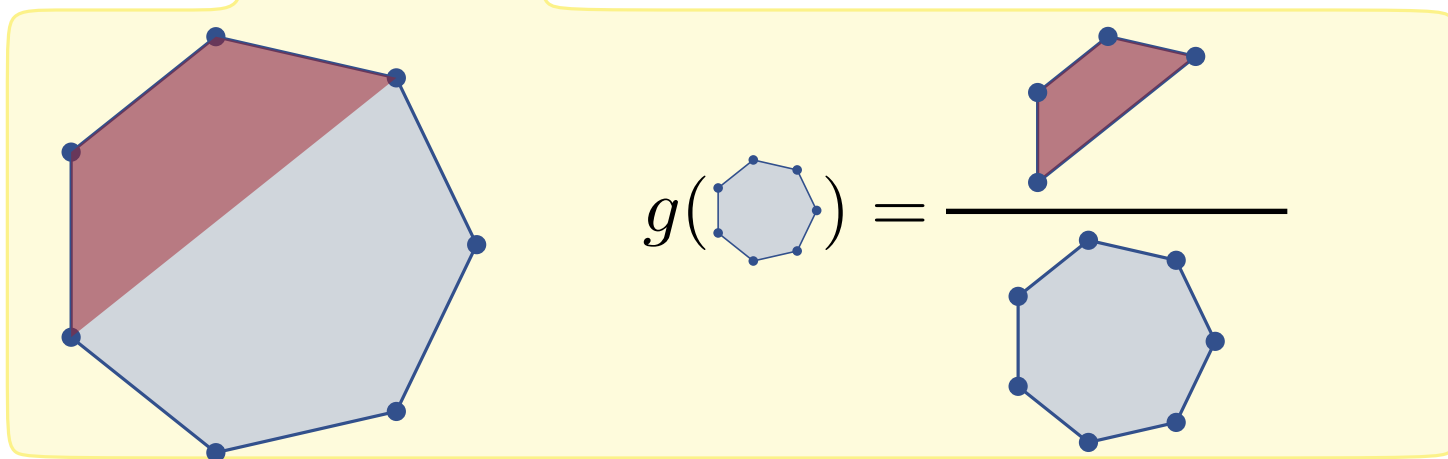
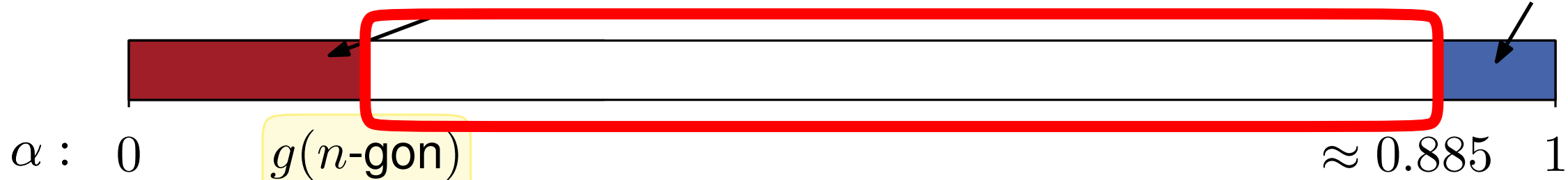
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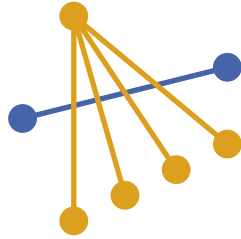
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k -independent crossing: no edge is crossed by more than k independent edges.

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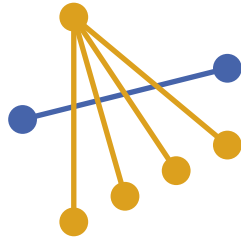
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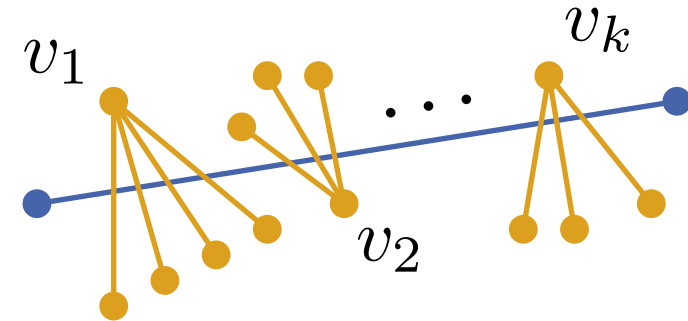
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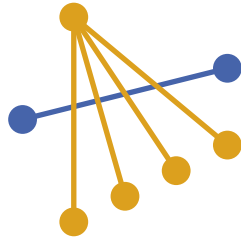
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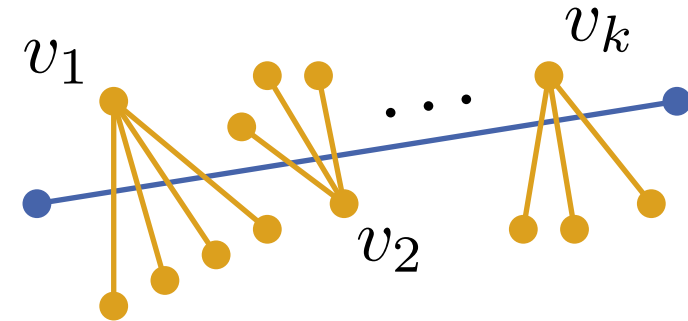
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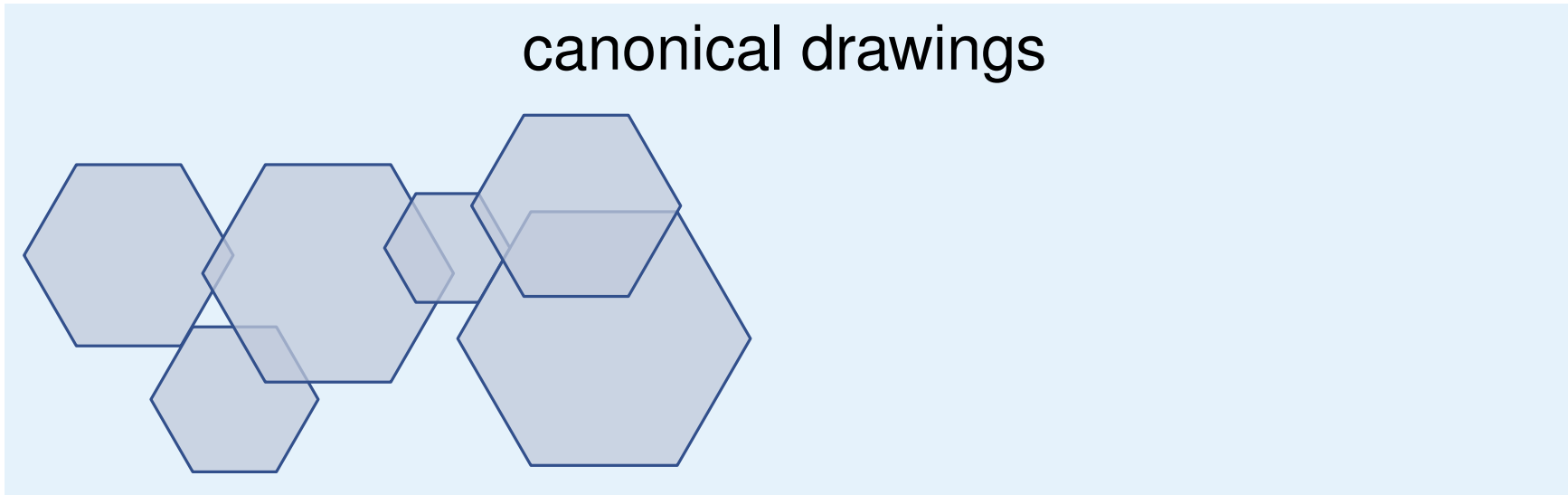


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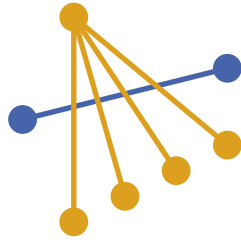
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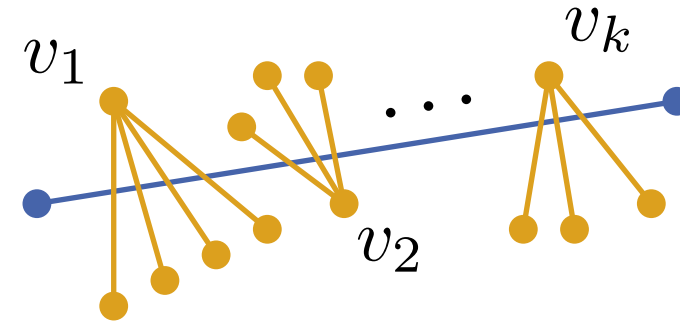


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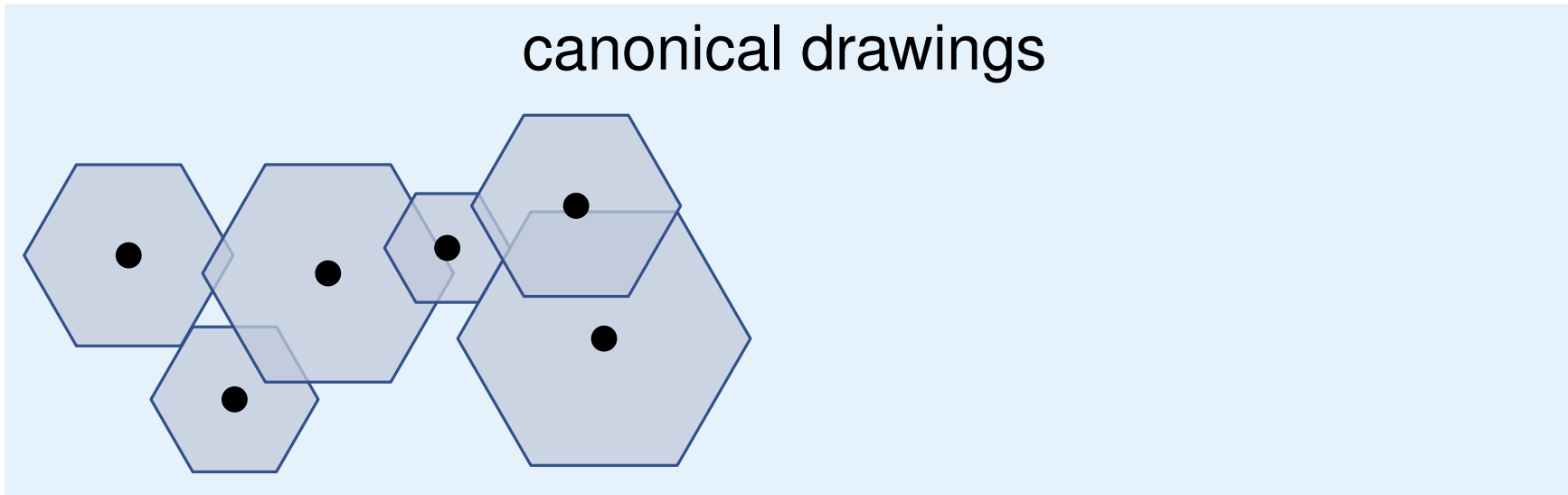


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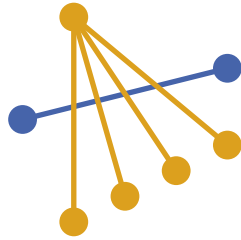
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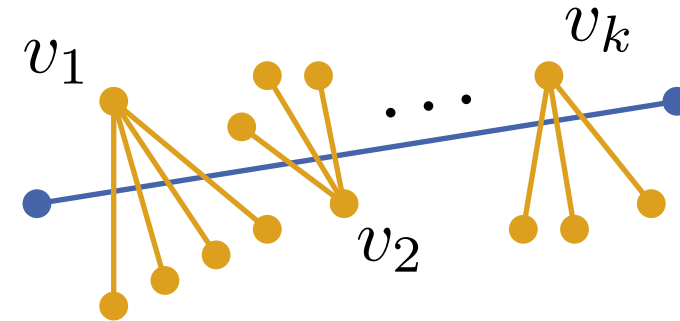


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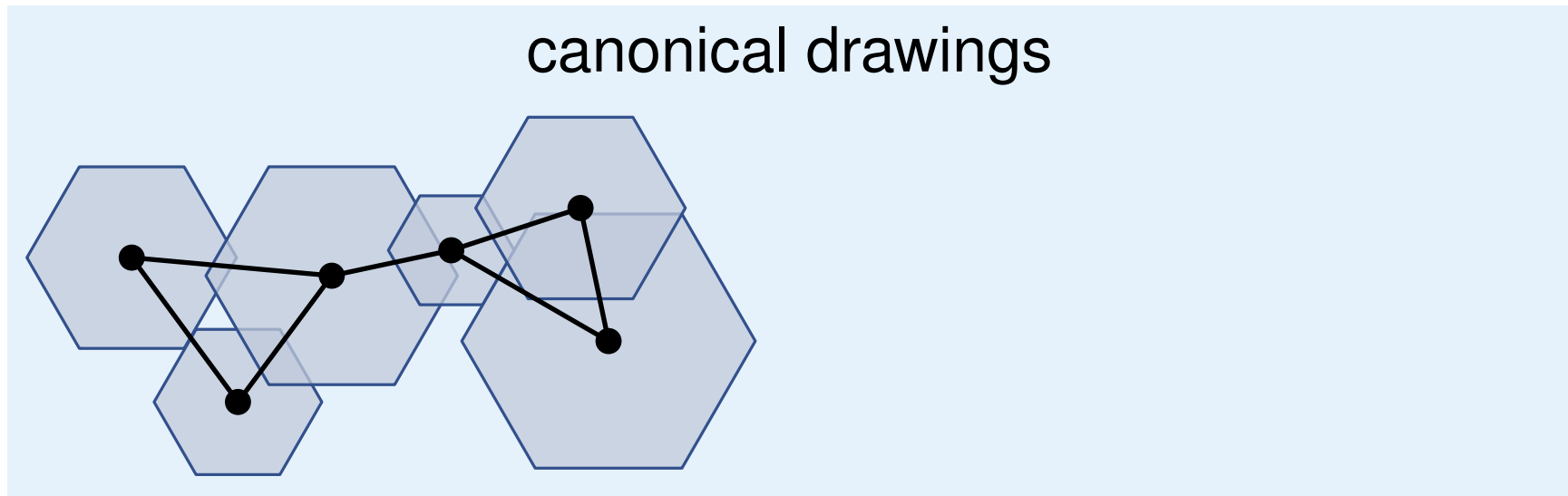
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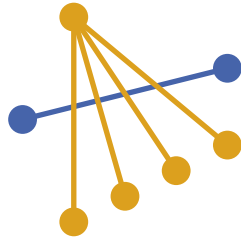


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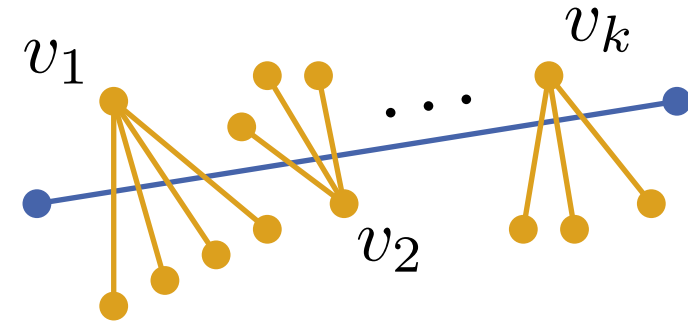


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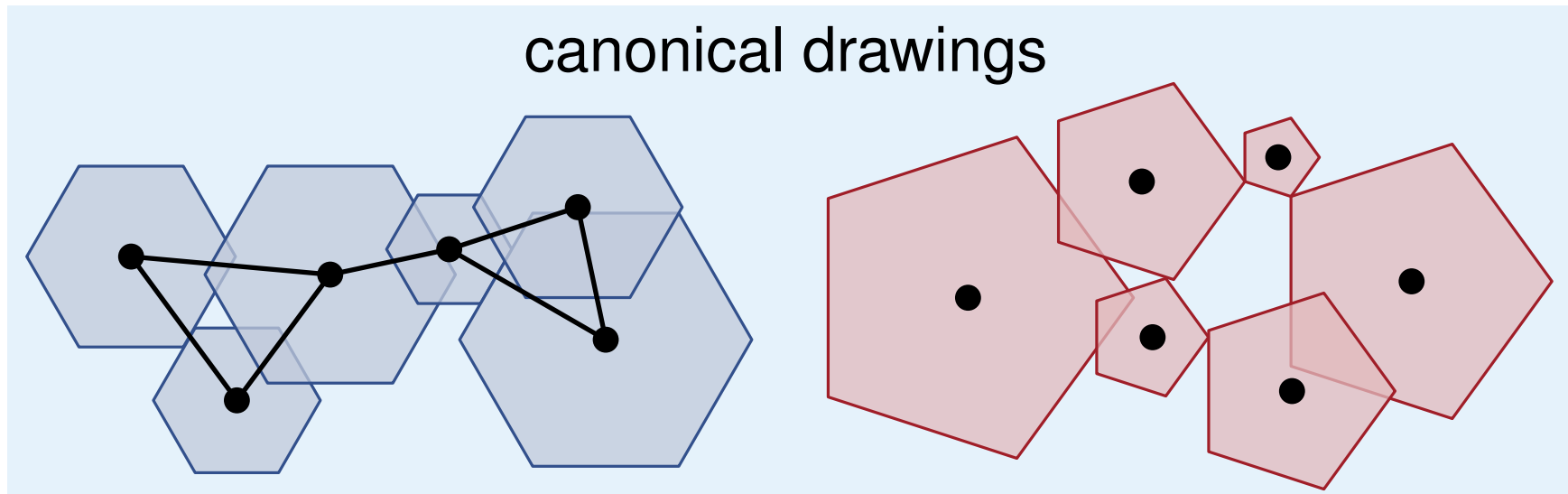
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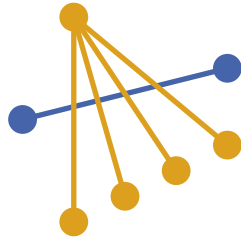


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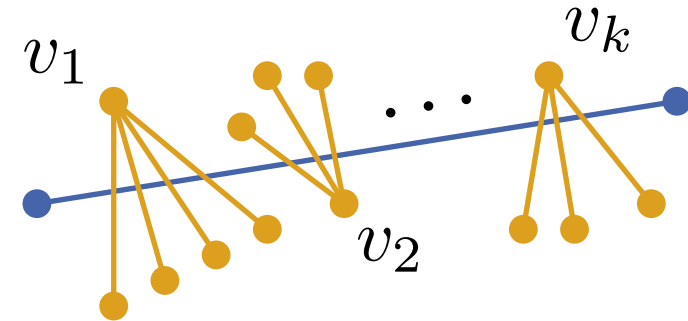


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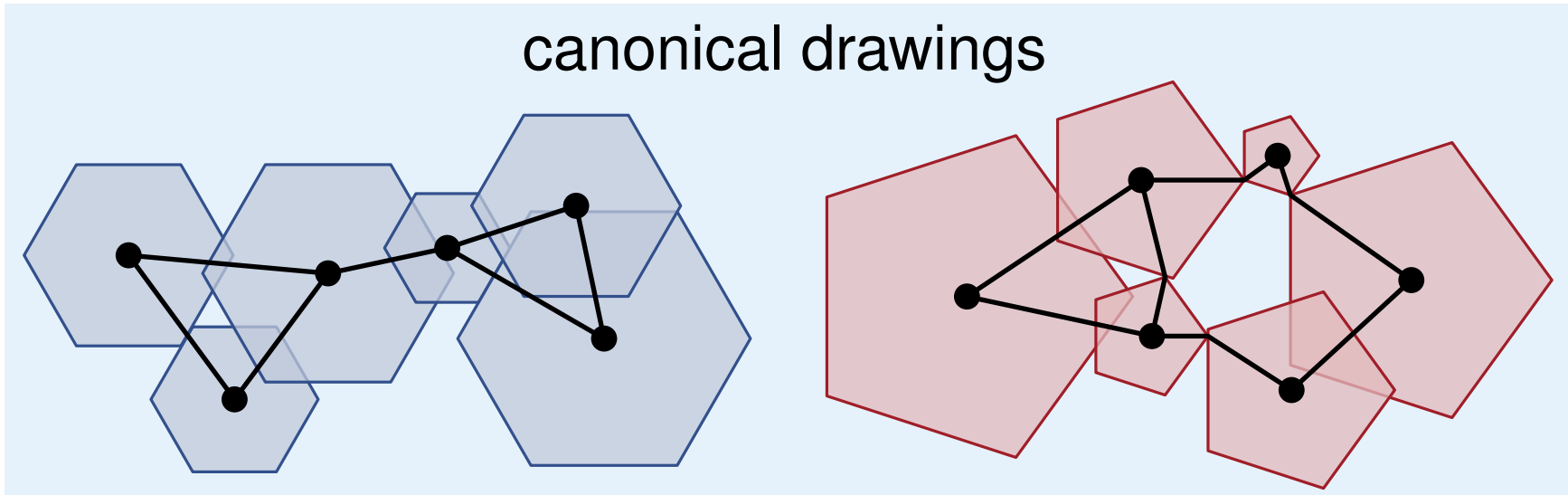


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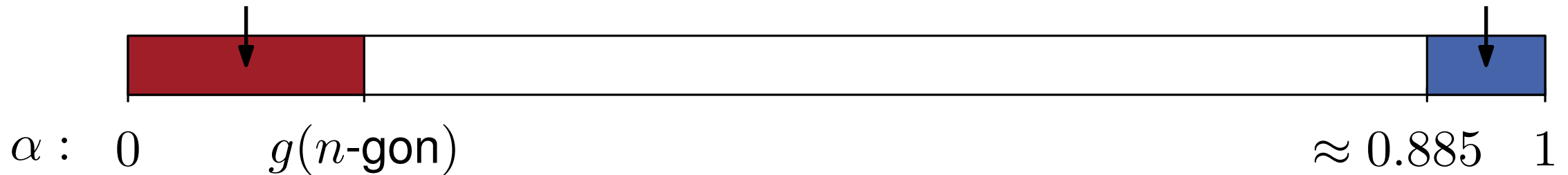
Conclusion

■ For  and  : **no** product structure for $\alpha < 1$.

■ For even $n \geq 6$:

no product structure

product structure



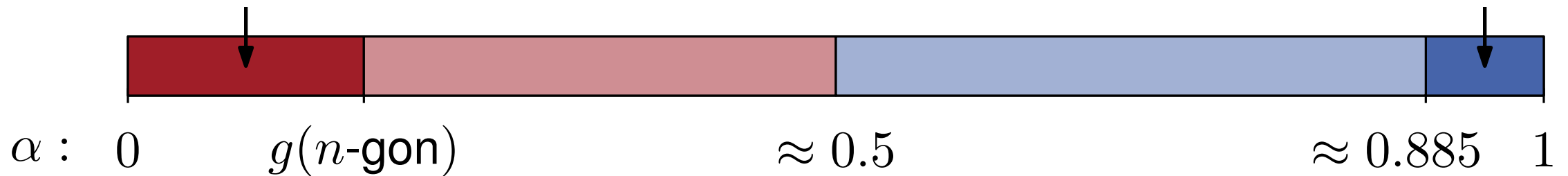
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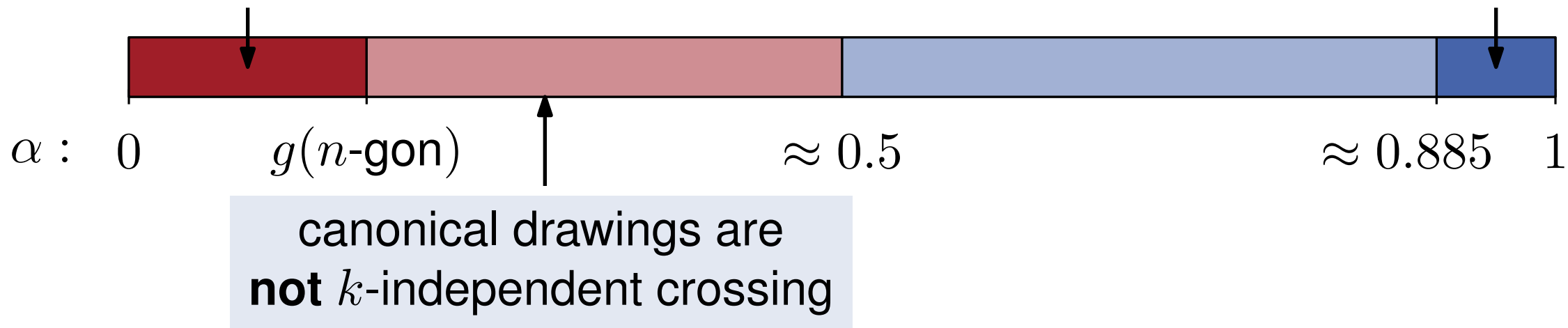
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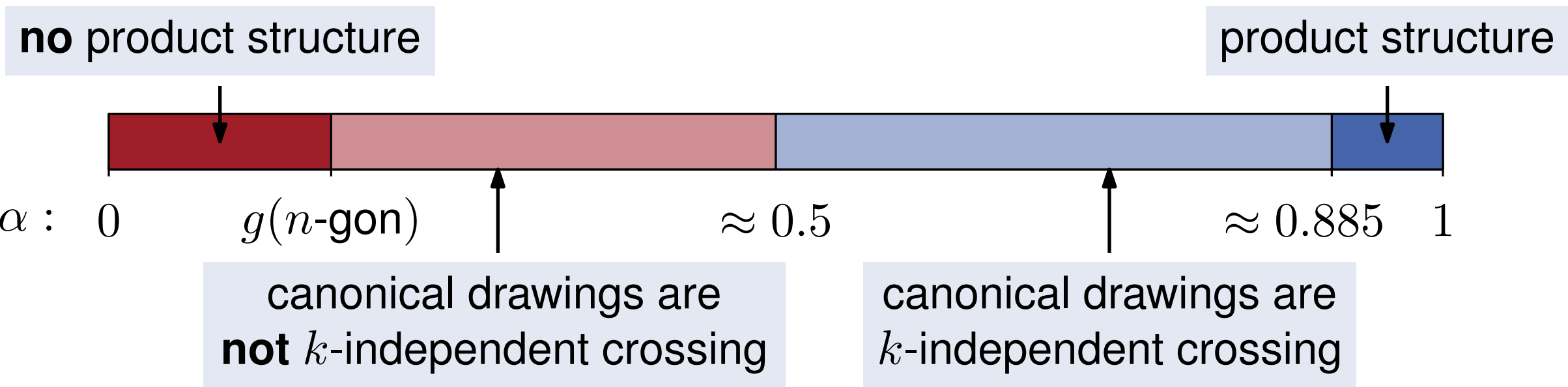
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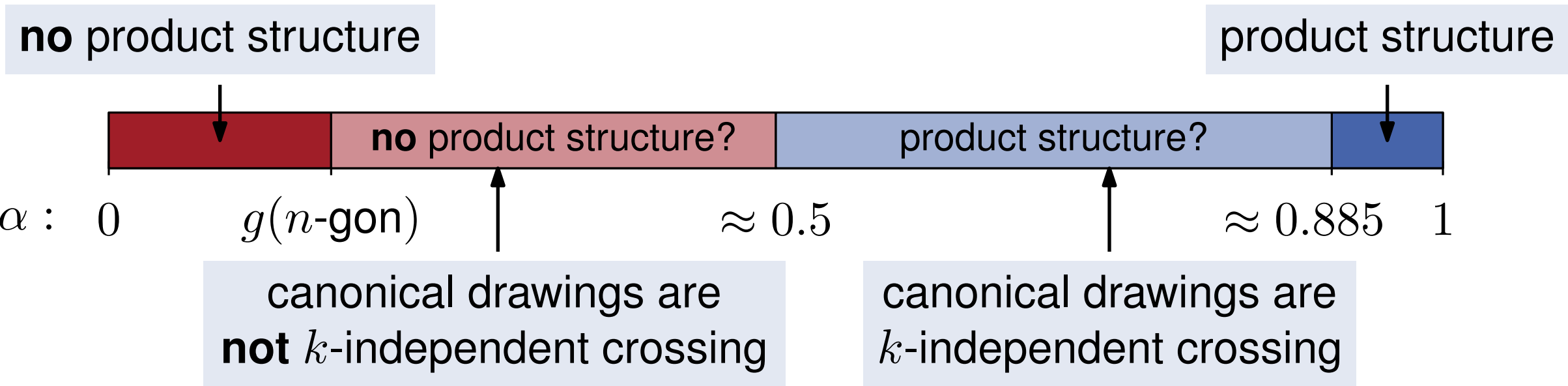
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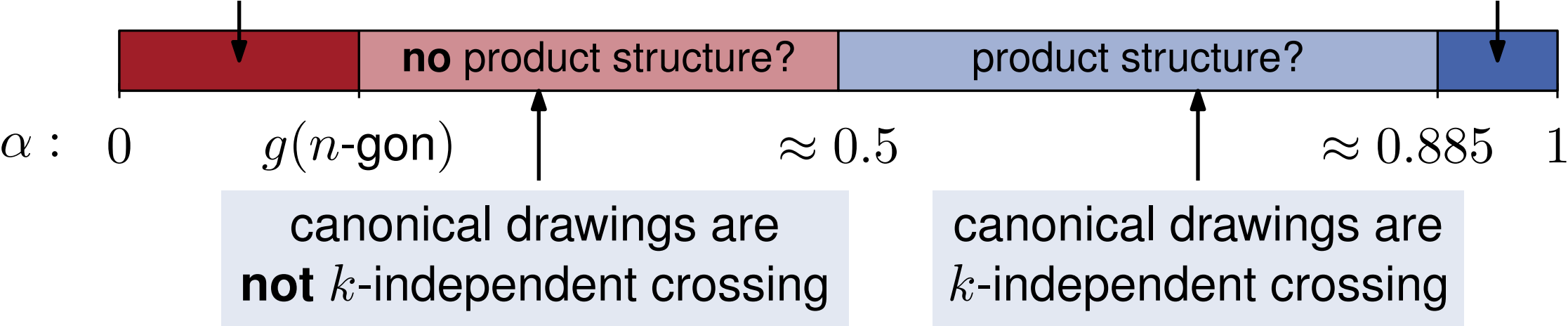
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Question: Do k -independent crossing graphs have product structure?